LINEAR FINITE ELEMENTS MAY BE ONLY FIRST-ORDER POINTWISE ACCURATE ON ANISOTROPIC TRIANGULATIONS

NATALIA KOPTEVA

Abstract. We give a counterexample of an anisotropic triangulation on which the exact solution has a second-order error of linear interpolation, while the computed solution obtained using linear finite elements is only first-order pointwise accurate. Our example is given in the context of a singularly perturbed reaction-diffusion equation, whose exact solution exhibits a sharp boundary layer. Furthermore, we give a theoretical justification of the observed numerical phenomena using a finite-difference representation of the considered finite element methods. Both standard and lumped-mass cases are addressed.

1. Introduction

It appears that there is a perception in the finite-element community that the computed-solution error in the maximum norm is closely related to the corresponding interpolation error. While an almost best approximation property of finite-element solutions in the maximum norm has been rigourously proved (with a logarithmic factor in the case of linear elements) for some equations on quasi-uniform meshes \cite{12}, there is no such result for strongly-anisotropic triangulations. Nevertheless, this perception is frequently considered a reasonable heuristic conjecture to be used in the anisotropic mesh adaptation \cite{5,6–8}.

In this note we give a counterexample of an anisotropic triangulation on which

- the exact solution is in $C^\infty(\bar{\Omega})$ and has a second-order pointwise error of linear interpolation $O(N^{-2})$,
- the computed solution obtained using linear finite elements is only first-order pointwise accurate, i.e., the pointwise error is as large as $O(N^{-1})$.

Here the maximum side length of mesh elements is $O(N^{-1})$ and the global number of mesh nodes does not exceed $O(N^2)$.

Our example is given in the context of singularly perturbed differential equations. Their solutions exhibit sharp boundary and interior layers, so locally anisotropic meshes (fine and anisotropic in layer regions and standard outside) are frequently employed in their numerical solution and, furthermore, have been shown to yield reliable numerical approximations in an efficient way (see, e.g., \cite{5,9,14} and references in \cite{10}).

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Example. Consider an exact solution \( u(x, y) = e^{-x/\varepsilon} \) of the singularly perturbed reaction-diffusion problem

\[
-\varepsilon^2 \Delta u + u = 0 \quad \text{for } (x, y) \in \Omega, \quad u = g \quad \text{for } (x, y) \in \partial \Omega,
\]

where \( \varepsilon \) is a small positive parameter, \( \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 \) is the Laplace operator, and the boundary data \( g \) matches the chosen exact solution. The domain \( \Omega \) is a bounded polygonal domain. In particular we consider the cases of \( \Omega = (0, 1)^2 \) and \( \Omega = (0, 2\varepsilon) \times (0, 1) \), as well as a more general case of \( \Omega \supset \hat{\Omega} := (0, 2\varepsilon) \times (-H, H) \), where \( H = O(N^{-1}) \).

The paper is organized as follows. In §2, problem (1.1) is solved numerically using both standard and lumped-mass linear finite elements. Triangulations of types A, B and C are considered that are obtained from standard layer-adapted tensor-product meshes by drawing diagonals as in Figure 1. In §3, we give a theoretical justification of the observed numerical phenomena using a finite-difference representation of the considered finite element methods.

Notation. We let \( C \) denote a generic positive constant that may take different values in different formulas, but is always independent of the mesh and \( \varepsilon \). A subscripted \( C \) (e.g., \( C_1 \)) denotes a positive constant that is independent of \( N \) and \( \varepsilon \) and takes a fixed value. Notation \( v = O(w) \) will be used for \( C^{-1}w \leq v \leq Cw \), while \( v = O(w) \) will denote \( |v| \leq Cw \), with some constant \( C \).

2. Numerical results

Consider a tensor product of a layer-adapted mesh \( \{x_i\}_{i=0}^N \) in the \( x \)-direction and the uniform mesh \( \{x_j^M\}_{j=0}^M \) in the \( y \)-direction, where \( M = O(N) \). Whenever \( \Omega = (0, 1)^2 \), the mesh \( \{x_i\} \) will be a version of the Bakhvalov mesh [2] or the Shishkin mesh [14] described below. Whenever \( \Omega = (0, 2\varepsilon) \times (0, 1) \), the mesh \( \{x_i\} \) will be uniform.

Bakhvalov mesh [2]. For some \( \gamma \in (0, 1) \), set \( \sigma := 2\varepsilon(\gamma^{-1}|\ln\varepsilon| + 1) \) and assume that \( \varepsilon \) is sufficiently small for \( \sigma \in (0, 1) \). Now define the mesh \( \{x_i\}_{i=0}^{N/2} \) on \([0, \sigma]\) by

\[
x_i := x\left((1 + \gamma - \varepsilon) \frac{2i}{N}\right), \quad x(t) := \begin{cases} 2\varepsilon \gamma^{-1}t, & t \in [0, \gamma], \\ 2\varepsilon \left[1 - \gamma^{-1}\ln(1 + \gamma - t)\right], & t \in [\gamma, 1 + \gamma - \varepsilon]. \end{cases}
\]

The remaining part of the mesh \( \{x_i\}_{i=N/2}^N \) on \([\sigma, 1]\) is uniform. In fact, one can easily see that on \([0, 2\varepsilon]\) this mesh is also uniform, with \( x_i - x_{i-1} = O(\varepsilon N^{-1}) \).
Shishkin mesh [14]. For some $\gamma \in (0, 1)$, set $\sigma = 2\gamma^{-1}\varepsilon \ln \frac{N}{2}$. Now construct a piecewise uniform mesh by dividing the intervals $[0, \sigma]$ and $[\sigma, 1]$ into $N_1$ and $N - N_1$ equal subintervals for some $N_1 = O(N)$.

Remark 2.1 (Interpolation error). A calculation shows that the linear interpolation error of our exact solution $u = e^{-x/\varepsilon}$ on any of the Triangulations A, B or C is $O\left(\left[N^{-1} \ln^p \frac{N}{2}\right]^2\right)$ in the maximum norm, where $p = 0$ if the Bakhvalov mesh is used or the uniform mesh in the domain $\Omega = (0, 2\varepsilon) \times (0, 1)$, and $p = 1$ if the Shishkin mesh is used. Interestingly, one gets a similar second-order bound (with a logarithmic factor in the case of the Shishkin mesh) for the error of the standard five-point difference scheme applied to problem (1.1) on the corresponding tensor-product mesh (see, e.g., [5,9]).

Tables 1–3 give the maximum nodal errors (odd rows) and the computational convergence rates $r$ in $\left(N^{-1} \ln^p \frac{N}{2}\right)^r$ (even rows) for Triangulations A and B obtained from the three tensor-product meshes. For the considered values of $\varepsilon$, these meshes are highly anisotropic; for example, in the case of the Bakhvalov mesh (see Table 1), the mesh aspect ratio changes between 2 away from the layer and $(2.25\varepsilon)^{-1}$ in the layer region. The numerical results for the standard finite elements are quite similar to the case of mass lumping, so we consider both cases only in Table 1.

We observe that whenever Triangulation A is used, one gets a second-order accuracy, with a logarithmic factor in the case of the Shishkin mesh (similar to the accuracy of the five-point finite-difference scheme, which, in fact, is identical with the lumped-mass finite elements on Triangulation A). However, when one switches to Triangulation B, linear finite elements become only (almost) first-order pointwise accurate in contrast to the (almost) second-order accuracy of the interpolation error. Triangulation C also yields only (almost) first-order convergence (see also Remark 3.6 and Figure 3).
Table 2. Shishkin mesh, $\gamma = 0.9$, $M = \frac{1}{4}N$, $N_1 = \frac{3}{4}N$: maximum nodal errors and computational rates $r$ in $(N^{-1} \ln \frac{N}{2})^r$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Triangulation A $\varepsilon = 2^{-8}$ $\varepsilon = 2^{-16}$ $\varepsilon = 2^{-24}$</th>
<th>Triangulation B $\varepsilon = 2^{-8}$ $\varepsilon = 2^{-16}$ $\varepsilon = 2^{-24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1.850e-3 2.109e-3 2.109e-3</td>
<td>1.477e-2 1.478e-2 1.478e-2</td>
</tr>
<tr>
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<td>3.30 3.28 3.28</td>
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</tr>
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<tr>
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</tr>
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<td>1.02 0.97 0.97</td>
</tr>
<tr>
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<td>4.832e-5 4.832e-5 4.832e-5</td>
<td>2.908e-3 3.027e-3 3.027e-3</td>
</tr>
</tbody>
</table>

Table 3. Uniform mesh in $(0, 2\varepsilon) \times (0, 1)$, $M = \frac{1}{4}N$: maximum nodal errors and computational rates $r$ in $(N^{-1})^r$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Triangulation A $\varepsilon = 2^{-8}$ $\varepsilon = 2^{-16}$ $\varepsilon = 2^{-24}$</th>
<th>Triangulation B $\varepsilon = 2^{-8}$ $\varepsilon = 2^{-16}$ $\varepsilon = 2^{-24}$</th>
</tr>
</thead>
<tbody>
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<td>1.02 1.02 1.02</td>
</tr>
<tr>
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<td>1.193e-5 1.193e-5 1.193e-5</td>
<td>1.474e-3 1.476e-3 1.476e-3</td>
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<td>1.02 1.01 1.01</td>
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<tr>
<td>128</td>
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<td>7.288e-4 7.339e-4 7.339e-4</td>
</tr>
<tr>
<td></td>
<td>2.00 2.00 2.00</td>
<td>1.03 1.00 1.00</td>
</tr>
<tr>
<td>256</td>
<td>7.458e-7 7.458e-7 7.458e-7</td>
<td>3.561e-4 3.659e-4 3.659e-4</td>
</tr>
</tbody>
</table>

In summary, we conclude that

- when Triangulations B and C are used, linear finite elements are only first-order pointwise accurate;
- the order of convergence dramatically deteriorates from 2 to 1 as one switches from Triangulation A to Triangulation B or C.

Remark 2.2 (Comparison with the $L_2$ projection error). The $L_2$ projection errors are less understood on general anisotropic meshes. But one can easily estimate the pointwise error of this projection for the triangulations addressed in Table 3, i.e., obtained from the uniform tensor-product mesh in the domain $(0, 2\varepsilon) \times (0, 1)$. Indeed, the stretching transformation $\hat{x} := x/\varepsilon$ maps any of Triangulations A, B or C of the domain $(0, 2\varepsilon) \times (0, 1)$ into a quasi-uniform triangulation in the domain $(0, 2) \times (0, 1)$. Denoting the $L_2$ projection operators on the original and stretched triangulations by $P$ and $\hat{P}$, respectively, one gets $Pu(x,y) = \hat{Pu}(\varepsilon \hat{x}, y)$. As on the quasi-uniform stretched triangulation $\hat{Pu}(\varepsilon \hat{x}, y) - u(\varepsilon \hat{x}, y) = O(N^{-2})$, the $L_2$ projection error bound $Pu(x,y) - u(x,y) = O(N^{-2})$ follows. So in this case, the exact solution has a second-order pointwise error of the $L_2$ projection, while, as Table 3 demonstrates, the pointwise error of the computed solution obtained using linear finite elements may be only first-order pointwise accurate.
Errors in the $L_2$ norm. Although our main interest in this paper is in the pointwise accuracy of computed solutions, in Table 4 we also list the normalized $L_2$ norm errors $\varepsilon^{-1/2} \| U - u^I \|_{L_2(\Omega)}$ (odd rows) and the computational convergence rates $r$ in $(N^{-1})^r$, where $u^I$ is the piecewise-linear interpolant of $u$, for the Bakhvalov mesh.

The table suggests that $\| U - u^I \|_{L_2(\Omega)} \approx O(\sqrt{\varepsilon} N^{-q})$, where $q$ is 2, $\frac{3}{2}$ and 1 for Triangulations A, B and C, respectively. Thus, although Triangulations B and C yield very similar maximum nodal errors, the latter is superior when the errors are computed in the $L_2$ norm (see also Figure 3, and Remark 3.6 for a theoretical explanation). The numerical results for the standard finite elements are quite similar to the case of mass lumping, so are not presented here.

When we switch to the uniform tensor-product mesh in the domain $(0, 2\varepsilon) \times (0, 1)$ (as used in Table 3) to obtain triangulations of type A, B and C, the numerical results again suggest that $\| U - u^I \|_{L_2(\Omega)} \approx O(\sqrt{\varepsilon} N^{-q})$. Finally, for the Shishkin mesh (as used in Table 2), one computationally observes that $\| U - u^I \|_{L_2(\Omega_\sigma)} \approx O\left(\sqrt{\varepsilon} N^{-q}[\ln N]^{\max\{1, 2q - 2\}}\right)$, where $\Omega_\sigma := (0, \sigma) \times (0, 1)$ is essentially the boundary-layer region (while the $L_2(\Omega_\sigma)$ norm of the error is quite small $\varepsilon$-independently).

It should also be noted that these results are obtained for the exact solution $e^{-x/\varepsilon}$, which is negligible away from the boundary-layer region. In general, the errors in the $L_2$ norm may involve an additional component $O(N^{-2})$ induced by a nontrivial smooth part of the exact solution.

3. Theoretical justification

To understand the numerical phenomena described in the previous section, we represent the considered finite elements as finite-difference schemes on the underlying rectangular tensor-product meshes.

Suppose $\Omega \supset \hat{\Omega}$, where the subdomain $\hat{\Omega}$ and the tensor-product mesh $\hat{\omega}_h$ in this subdomain are defined by

$$\hat{\Omega} := (0, 2\varepsilon) \times (-H, H), \quad \hat{\omega}_h := \{ x_i = hi \}_{i=0}^{2N_0} \times \{ -H, 0, H \}, \quad h = \frac{\varepsilon}{N_0}. $$

The triangulation $\hat{T}$ in $\hat{\Omega}$ is obtained by drawing diagonals in each rectangle as shown on Figure 2 using the mesh transition point $(\varepsilon, 0)$. 

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Table 4. Normalized $L_2$ norm errors $\varepsilon^{-1/2} \| U - u^I \|_{L_2(\Omega)}$ and computational rates $r$ in $(N^{-1})^r$: Bakhvalov tensor-product mesh, $\gamma = 0.8$, $M = \frac{1}{2} N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Triangulation A</th>
<th>Triangulation B</th>
<th>Triangulation C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon = 2^{-16}$</td>
<td>$\varepsilon = 2^{-24}$</td>
<td>$\varepsilon = 2^{-16}$</td>
</tr>
<tr>
<td>32</td>
<td>1.660e-3 2.000e-3</td>
<td>4.551e-3 4.679e-3</td>
<td>7.985e-3 8.058e-3</td>
</tr>
<tr>
<td>64</td>
<td>2.14 2.22</td>
<td>1.62 1.65</td>
<td>1.05 1.06</td>
</tr>
<tr>
<td>128</td>
<td>3.775e-4 4.300e-4</td>
<td>1.478e-3 1.492e-3</td>
<td>3.861e-3 3.866e-3</td>
</tr>
<tr>
<td>256</td>
<td>2.09 2.16</td>
<td>1.51 1.52</td>
<td>0.96 0.96</td>
</tr>
<tr>
<td></td>
<td>8.874e-5 9.618e-5</td>
<td>5.179e-4 5.192e-4</td>
<td>1.988e-3 1.989e-3</td>
</tr>
<tr>
<td></td>
<td>2.05 2.10</td>
<td>1.51 1.51</td>
<td>0.98 0.98</td>
</tr>
<tr>
<td></td>
<td>2.148e-5 2.247e-5</td>
<td>1.823e-4 1.824e-4</td>
<td>1.010e-3 1.010e-3</td>
</tr>
</tbody>
</table>
Next, let $U$ be the piecewise-linear finite-element solution obtained on some triangulation $\mathcal{T} \supset \hat{T}$ in the global domain $\Omega$, for whose nodal values in $\hat{\Omega}$ we use the notation

$$U_i := U(x_i, 0), \quad U_i^{\pm} := U(x_i, \pm H).$$

Now a calculation yields a finite-difference representation in the lumped-mass case:

$$L^h U_i(x_i, 0) := \frac{\varepsilon^2}{H^2} [U_{i-1} - 2U_i + U_{i+1}] + \frac{\varepsilon^2}{H^2} [U_0 - 2U_i + U_{i+1}] + \gamma_i U_i = 0$$

for $i = 1, \ldots, 2N_0 - 1$, where

$$\gamma_i = 1 \quad \text{for} \quad i \neq N_0, \quad \gamma_{N_0} = \frac{2}{3}.$$

In the standard (non-lumped-mass) case, one gets a similar finite-difference representation for $i = 1, \ldots, 2N_0 - 1$:

$$\tilde{L}^h U_i(x_i, 0) := L^h U_i - \frac{1}{12} \sum_{(x', y') \in S_i} [U_i - U(x', y')] = 0,$$

where $S_i$ denotes the set of meshnodes that have a common edge with $(x_i, 0)$.

**Remark 3.1.** Note that if $\gamma_i$ in (3.2a) is replaced by 1, one gets the standard five-point difference scheme for equation (1.1), for which, using the discrete maximum principle, one can easily show that if a tensor-product mesh of type $\omega_h$ is used in the entire domain $\Omega$, then the nodal error is $O\left(\frac{h^2}{\varepsilon^2}\right)$. Our method differs from this difference scheme at one point $(x_{N_0}, 0)$, where $\gamma_i = \frac{2}{3}$, i.e., at this point, compared to the second-order finite-difference method, we have a truncation error $O(1)$. As we show in Lemma 3.3 below, this results in the deterioration of pointwise accuracy of the computed solution to $O\left(\frac{h}{\varepsilon}\right)$.

**Remark 3.2.** A triangulation, similar to Triangulation B on Figure 1 (and also to triangulation $\hat{T}$ on Figure 2 (right)), but obtained from a uniform tensor-product mesh, has been used to show that the logarithmic factor in the $L_\infty$-norm error estimate for linear finite elements applied to the Laplace equation is sharp [1,3]. Our approach is, in fact, similar to that in [1] in that we represent a finite element method as a finite-difference scheme to get a lower bound for the error.

We shall consider the lumped-mass and non-lumped-mass cases separately.
3.1. Lumped-mass linear finite elements.

Lemma 3.3 (Lumped-mass case). Let \( u = e^{-x/\varepsilon} \) be the exact solution of problem (3.1) posed in a domain \( \Omega \supset \bar{\Omega} \), a triangulation \( \mathcal{T} \) in \( \Omega \) be such that \( \mathcal{T} \supset \bar{\mathcal{T}} \) subject to the definitions (3.1), and let \( U \) be the computed solution obtained using lumped-mass linear finite elements. For any positive constant \( C_2 \), there exist sufficiently small constants \( C_0 \) and \( C_1 \) such that if \( N_{0}^{-1} \leq C_1 \) and \( \varepsilon \leq C_2 H \), then

\[
\max_{\bar{\Omega}} |U - u| \geq C_0 N_0^{-1}.
\]

Proof. (i) First, consider the auxiliary piecewise-linear computed solution \( \hat{U} \) obtained on the triangulation \( \bar{\mathcal{T}} \) in the subdomain \( \bar{\Omega} \), subject to the boundary conditions \( \hat{U}(x', y') = u(x', y') \) at each meshnode \( (x', y') \in \partial \bar{\Omega} \). At the interior meshnodes, \( \hat{U} \) satisfies (3.2a), i.e., \( L^h \hat{U}(x_i, 0) = 0 \) for \( i = 1, \ldots, 2N_0 - 1 \). We shall now prove that, for a sufficiently small constant \( C_0 \), one has

\[
\hat{U} - u(x_N, 0) \geq 2C_0 N_0^{-1}.
\]

Let \( e := \hat{U} - u \) and \( e_i := e(x_i, 0) \). Recall that \( e(x_i, \pm H) = 0 \), so \( L^h e(x_i, 0) \) can be rewritten using a one-dimensional discrete operator \( L^h_x \) applied to the unknown vector \( \{e_i\}_{i=0}^{2N_0} \) as

\[
L^h e(x_i, 0) = L^h_x e_i = \frac{\varepsilon^2}{h^2}[-e_{i-1}^2 + 2e_i - e_{i+1}] + [\frac{1}{\gamma_i} + \frac{2\varepsilon^2}{\gamma_i}] e_i.
\]

On the other hand, \( L^h e(x_i, 0) = -L^h u(x_i, 0) \), so the standard truncation error estimation (using \( u(x_i, \pm H) = u(x_i, 0) \)) yields

\[
\left| L^h e(x_i, 0) \right| \leq |1 - \gamma_i| u(x_i, 0) + O(\frac{h^2}{\varepsilon^2}).
\]

Here, by virtue of (3.2b), \( |1 - \gamma_i| u(x_i, 0) = \left[ \frac{1}{2} \delta_i N_0 \right] e^{-x_i/\varepsilon} = \frac{1}{2} \varepsilon^{-1} \delta_i N_0 \), where \( \delta_i N_0 \) denotes the Kronecker delta. So combining (3.6) and (3.7) yields

\[
L^h_x e_i = \frac{1}{3} e^{-1} \delta_i N_0 + O(\frac{h^2}{\varepsilon^2}).
\]

As \( L^h_x \) satisfies the discrete maximum principle [11], one gets the representation

\[
\hat{U} - u(x_i, 0) = e_i = \frac{1}{3} e^{-1} h G^h_i + O(\frac{h^2}{\varepsilon^2}).
\]

Here \( G^h_i \) is the discrete Green’s function of the one-dimensional operator \( L^h \) that satisfies \( L^h G^h_i = h^{-1} \delta_{iN_0} \) for \( i = 1, \ldots, 2N_0 - 1 \), subject to \( G^h_0 = G^h_{2N_0} = 0 \). To complete the proof of (3.5), it now suffices to show that \( G^h_{N_0} \geq C_3 \varepsilon^{-1} \) for some constant \( C_3 \); then one simply needs to choose sufficiently small constants \( C_0 \) and \( C_1 \) such that \( \frac{1}{3} e^{-1} C_3 \frac{h}{\varepsilon} + O(\frac{h^2}{\varepsilon^2}) \geq 2C_0 \frac{h}{\varepsilon} \) (To be more precise, \( C_0 := \frac{1}{12} e^{-1} C_3 \), while by virtue of \( \frac{1}{3} \varepsilon = N_0^{-1} \leq C_1 \), we choose \( C_1 \) such that \( \left| O(\frac{h^2}{\varepsilon^2}) \right| \leq \frac{1}{3} e^{-1} C_3 \frac{h}{\varepsilon} \)).

To bound \( G^h_{N_0} \), note that \( G^h_i = G^h_{2N_0 - i} \) and \( G^h_i = G^h_{N_0} [w(x_i/\varepsilon) + O(\frac{h^2}{\varepsilon^2})] \) for \( i \leq N_0 \), where \( w(x) \) solves the equation \(-w'' + (1 + 2\varepsilon^2)w = 0\) subject to \( w(0) = 0 \) and \( w(1) = 1 \). This implies that \( \rho := \frac{1}{h} [G^h_{N_0} - G^h_{N_0 + 1}]/G^h_{N_0} \leq C_4 \) for some constant \( C_4 = C_4(C_2) \). (Roughly speaking, \( \rho \approx w'(1) \leq 1.32 \sqrt{1 + 2C_2^2} \). Now, \( L^h G^h_{N_0} = h^{-1} \) can be rewritten as \( G^h_{2N_0} [2\frac{h}{\varepsilon} + O(\frac{h^2}{\varepsilon^2})] = h^{-1} \) or, equivalently, \( G^h_{N_0} \left[ 2\rho + O(\frac{h}{\varepsilon}) \right] = \varepsilon^{-1} \). As \( \frac{h}{\varepsilon} \leq C_1 \) and \( \rho \geq C_4 \), choosing \( C_1 \) sufficiently small we get \( G^h_{N_0} \geq C_3 \varepsilon^{-1} \) with \( C_3 := (3C_4)^{-1} \), and, hence, the desired bound (3.5).
(ii) In view of (3.5), to establish (3.4), it suffices to show that
\[
\max_{\bar{\Omega}} |U - u^I| \geq \frac{1}{2} \max_{\Omega} |\bar{U} - u^I| =: \frac{1}{2} \hat{e},
\]
where \( u^I \) is the standard piecewise-linear interpolant of \( u \).

Let \( Z := U - \bar{U} \) in \( \hat{\Omega} \) and \( Z_{\max} := \sup_{\hat{\Omega}} |Z| \). Note that \( L^h Z(x,0) = 0 \) and \( Z = U - u^I \) on \( \partial \hat{\Omega} \). As, by the discrete maximum principle, \( |Z| \) attains its maximum on \( \partial \hat{\Omega} \), so \( \max_{\partial \hat{\Omega}} |U - u^I| = Z_{\max} \). On the other hand, \( U - u^I = (\bar{U} - u^I) + Z \) yields \( \max_{\hat{\Omega}} |U - u^I| \geq \hat{e} - Z_{\max} \). As the maximum of the two values \( \hat{e} - Z_{\max} \) and \( Z_{\max} \) exceeds their average, the desired relation (3.9) follows. \( \square \)

**Remark 3.4.** An inspection of the proof of Lemma 3.3 shows that if the assumption \( \varepsilon \leq C_2 H \) is replaced by a weaker \( h \ll H \), one gets a version of (3.4) with \( C_0 \) replaced by \( C_0' / \sqrt{1 + 2 \varepsilon^2 H^2} \).

**Remark 3.5.** Lemma 3.3 applies to all triangulations of type B used in Tables 1–3, with \( N_0^{-1} = O(N^{-1} \ln^p N) \), where \( p = 0 \) if the Bakhvalov mesh is used or the uniform mesh in the domain \( \Omega = (0, 2\varepsilon) \times (0, 1) \), and \( p = 1 \) if the Shishkin mesh is used. Furthermore, this lemma still applies if one switches from Triangulation B to Triangulation C.

**Remark 3.6.** For all triangulations considered in Section 2 one has \( \varepsilon \ll M^{-1} \). So a version of part (i) of the proof of Lemma 3.3 (with negligible \( \varepsilon^2 H^2 \)) can be applied to the global Triangulation B in \( \Omega \) to show that \( \|U - u|(x,y)\| \leq C(N^{-1} \ln^p N)^2 \) whenever \( |y - \frac{1}{2}| \geq M^{-1} \), i.e., the deterioration in the accuracy occurs only near \( y = \frac{1}{2} \); see Figure 3 (left). However, if one switches to Triangulation C, a similar analysis yields \( \|U - u|(x, \frac{1}{M}) = (-1)^j e_{ij} \) with \( e_{ij} \) from (3.8) and a slightly different \( G_{ij}^h \) (now associated with the mesh \( \{x_i\}_{i=0}^N \) ). Again, \( \max e_{ij} = O(N_0^{-1}) = O(N^{-1} \ln^p N) \).

So, as Figure 3 (right) demonstrates, the error of order (almost) 1 becomes spread throughout the boundary-layer region. Consequently, when the errors are computed in the \( L^2 \) norm, Triangulation B is superior to C (as shown in Table 4).

The computational convergence rates in Table 4 can be explained by a similar, but more detailed error analysis. To be more precise, with the notation \( \mathcal{E}_s := (N^{-1} \ln^p N)^s e^{-C_2 x/\varepsilon} \), it suffices to show that the error \( \|U - u|(x, \frac{1}{M}) \) roughly
behaves like $E_2$ for Triangulation A, and like $E_1$ for Triangulation C, while for Triangulation B it behaves like $E_1$ when $j = \frac{M}{2}$ and like $E_2$ otherwise.

3.2. Standard linear finite elements.

**Lemma 3.7** (Non-lumped-mass case). The statement of Lemma 3.3 remains valid for the computed solution $U$ obtained using standard linear finite elements.

**Proof.** We imitate the proof of Lemma 3.3 with a few changes described below.

(i) This part of the proof is devoted to establishing (3.5), only now $\tilde{U}(x_1,0)$ satisfies (3.3), i.e., $\tilde{\mathcal{L}}^h \tilde{U}(x_1,0) = 0$ for $i = 1, \ldots, 2N_0 - 1$. So a version of (3.6) for $e := \tilde{U} - u$ is obtained using the definition of $\tilde{\mathcal{L}}^h$ in (3.3) and the structure of the mesh (see Figure 2):

\[(3.10) \quad \tilde{\mathcal{L}}^h e(x_1,0) = \tilde{L}_e^h e_i := \frac{e_i^2}{h_i} (1 - \frac{h_i^2}{12\gamma_i}) [-e_i - 2\varepsilon_i - e_{i+1}] + [\tilde{\gamma}_i + \frac{2e_i^2}{h_i^2}] e_i.
\]

Here $\tilde{\gamma}_i := \gamma_i - \frac{4}{12} = \frac{2}{3}$ for $i \neq N_0$ and $\tilde{\gamma}_N_0 := \gamma_{N_0} - \frac{2}{12} = \frac{1}{2}$, while $\frac{h_i^2}{\varepsilon} = N_0^{-1}$ is assumed sufficiently small, so the one-dimensional discrete operator $\tilde{L}_e^h$ satisfies the discrete maximum principle and is handled similarly to $L_e^h$ of (3.6).

Next, by virtue of $\tilde{\mathcal{L}}^h e(x_1,0) = -\tilde{\mathcal{L}}^h u(x_1,0)$, a version of (3.7) is given by

\[(3.11) \quad \tilde{\mathcal{L}}^h e(x_1,0) = [\mathcal{L}^h - \tilde{\mathcal{L}}^h] u(x_1,0) + [1 - \gamma_i] u(x_1,0) + O\left(\frac{h_i^2}{\varepsilon}\right).
\]

Note that the definition of $\tilde{\mathcal{L}}^h$ in (3.3) implies that

\[[\mathcal{L}^h - \tilde{\mathcal{L}}^h] u(x_1,0) = \frac{1}{12} \sum_{(x',y') \in S_i} [u(x',0) - u(x',y')],
\]

where $S_i$ is the set of mesh nodes that have a common edge with $(x_1,0)$. The right-hand side here involves $u(x', \pm H) = u(x',0)$, so, with the notation $u_i := u(x_1,0)$, a calculation using the structure of the mesh (see Figure 2) yields

\[(3.12) \quad [\mathcal{L}^h - \tilde{\mathcal{L}}^h] u(x_1,0) = \frac{1}{12} [-u_{i-1} + 2u_i - u_{i+1}] + \frac{1}{12} F_i = O\left(\frac{h_i^2}{\varepsilon}\right) + \frac{1}{12} F_i,
\]

where

\[F_i := \tilde{\alpha}_i(u_i - u_{i+1}) + \tilde{\alpha}_i(u_i - u_{i-1}), \quad \tilde{\alpha}_i := \begin{cases} 2, & i < N_0, \\ 0, & i \geq N_0, \end{cases} \quad \tilde{\alpha}_i := \begin{cases} 0, & i \leq N_0, \\ 2, & i > N_0.
\]

Next, define a decomposition $F_i = F_i' + F_i''$ with

\[
F_i' := \begin{cases} -F^i_{2N_0 - i}, & i \leq N_0 \\ F^i_i, & i \geq N_0 \end{cases} \Rightarrow F_i'' = \tilde{\alpha}_i [(u_i - u_{i+1}) + (u_{2N_0 - i} - u_{2N_0 - i-1})].
\]

Here, by virtue of $u_j = e^{-x_j/\varepsilon}$, one has $u_j - u_{j+1} = u_j(1 - e^{-h_j/\varepsilon})$ for any $j$, so one can easily check that $F_i'' \geq 0$. Combining this with (3.10), (3.11) and (3.12) yields

\[
\tilde{L}_e^h e_i = \tilde{\mathcal{L}}^h e(x_1,0) = \frac{1}{12} F_i' + [1 - \gamma_i] u(x_1,0) + O\left(\frac{h_i^2}{\varepsilon}\right).
\]

To deal with $F_i'$, let $\tilde{L}_e^h e_i' = \frac{1}{12} F_i'$ subject to $e_0' = e_{2N_0}' = 0$. The symmetry of this problem combined with the symmetry of $F_i'$ immediately implies that $e_{N_0}' = 0$.

Now $\tilde{L}_e^h [e_i - e_i'] \geq [1 - \gamma_i] u(x_1,0) + O\left(\frac{h_i^2}{\varepsilon}\right)$ yields a version of (3.5) for this equation $e_{N_0} - e_{N_0}' \geq 2C_0N_0^{-1}$ (with a different $C_0$). As $e_{N_0}' = 0$, the desired bound (3.5) follows.

(ii) This part of the proof is identical with part (ii) in the proof of Lemma 3.3.

To show that $|Z|$ attains its maximum on $\partial\Omega$, move the terms $Z_i^\pm$, available from the boundary data, to the right-hand side in the discrete equation $\tilde{\mathcal{L}}^h Z = 0$. The
resulting nonhomogeneous one-dimensional discrete equation for the unknown vector \( \{Z_i\}_{i=0}^{12N_0} \) satisfies the discrete maximum principle, which can be used to deduce that the maximum of \(|Z|\) indeed occurs on \( \partial \Omega \).

\[\square\]

**References**


