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# $L^1$ ERROR ESTIMATES FOR DIFFERENCE APPROXIMATIONS OF DEGENERATE CONVECTION-DIFFUSION EQUATIONS

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Abstract. We analyze monotone finite difference schemes for strongly degenerate convection-diffusion equations in one spatial dimension. These nonlinear equations are well-posed within a class of (discontinuous) entropy solutions. We prove that the $L^1$ error between the approximate and exact solutions is $\mathcal{O}(\Delta x^{1/3})$, where $\Delta x$ is the spatial grid parameter. This result should be compared with the classical $\mathcal{O}(\Delta x^{1/2})$ error estimate for conservation laws (Kuznecov, 1976), and a recent estimate of $\mathcal{O}(\Delta x^{1/11})$ for degenerate convection-diffusion equations (Karlsen, Koley, Risebro 2012).

## 1. Introduction

Nonlinear convection-dominated flow problems arise in a range of applications, such as fluid dynamics, meteorology, transport of oil and gas in porous media, electro-magnetism, as well as in many other applications. As a consequence it has become a very important undertaking to construct robust, accurate, and efficient methods for the numerical approximation of such problems. Over the years a large number of stable (convergent) numerical methods have been developed for linear and nonlinear convection-diffusion equations in which the "diffusion part" is small, or even vanishing, relative to the "convection part" of the equation. There is a large literature on this topic, and we will provide a few relevant references later.

One central but exceedingly difficult issue relating to numerical methods for convection-diffusion equations, is the derivation of (a priori) error estimates that are robust in the singular limit as the diffusion coefficient vanishes, avoiding the exponential growth of error constants. This problem has been resolved only partly in special situations, such as for linear equations or in the completely degenerate case of no diffusion (scalar conservation laws). For general nonlinear equations containing both convection and (degenerate) diffusion terms this is a long standing open problem in numerical analysis.

This paper makes a small contribution to this general problem by deriving an error estimate for a class of simple difference schemes for nonlinear and strongly degenerate convection-diffusion problems of the form

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x^2 A(u), & (x,t) \in \Pi_T, \\ u(0,x) = u^0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $\Pi_T = \mathbb{R} \times (0,T)$ for some fixed final time $T > 0$, and $u(x,t)$ is the scalar unknown function that is sought. The initial function $u_0 : \mathbb{R} \to \mathbb{R}$ is a given integrable and bounded function, while the convection flux $f : \mathbb{R} \to \mathbb{R}$ and the diffusion function $A : \mathbb{R} \to \mathbb{R}$ are given functions satisfying

$$f, A \text{ locally } C^1;\ A(0) = 0;\ A \text{ nondecreasing.}$$

The moniker *strongly degenerate* means that we allow $A'(u) = 0$ for all $u$ in some interval $[\alpha, \beta] \subset \mathbb{R}$. Thus, the class of equations becomes very general, including purely hyperbolic equations (scalar conservation laws),

$$\partial_t u + \partial_x f(u) = 0, \tag{1.2}$$

as well as nondegenerate (uniformly parabolic) equations, such as the heat equation $\partial_t u = \partial_x^2 u$, and point-degenerate diffusion equations, such as the heat equation with a power-law nonlinearity: $\partial_t u = \partial_x(u^m \partial_x u)$, which is degenerate at $u = 0$.

Whenever the problem (1.1) is uniformly parabolic (i.e., $A' \geq \eta$ for some $\eta > 0$), it is well known that the problem admits a unique classical (smooth) solution. On the other hand, in the strongly degenerate case, (1.1) must be interpreted in the weak sense to account for possibly discontinuous (shock wave) solutions. Regarding weak solutions, it turns out that one needs an additional condition, the so-called *entropy condition*, to ensure that (1.1) is well-posed. More precisely, the following is known: For $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique solution $u \in C((0,T); L^1(\mathbb{R}^d))$, $u \in L^\infty(\Pi_T)$ of (1.1) such that $\partial_x A(u) \in L^2(\Pi_T)$ and for all convex functions $S : \mathbb{R} \to \mathbb{R}$ with $q_S' = f'S'$ and $r_S' = A'S'$,

$$\partial_t S(u) + \partial_x q_S(u) - \partial_x^2 r_S(u) \leq 0 \quad \text{in the weak sense on } [0,T) \times \mathbb{R}. \tag{1.3}$$

The satisfaction of these inequalities for all convex $S$ is the entropy condition, and a weak solution satisfying the entropy condition is called an entropy solution. The well-posedness of entropy solutions is a famous result due to Kružkov [21] for conservation laws (1.2), and a more recent work by Carrillo [5] extends this to degenerate parabolic equations (1.1). These results are available in the multi-dimensional context, and we refer to [1,10] for an overview of the relevant literature. For uniqueness of entropy solutions in the $BV$ class; see [26,28].

One traditional way of constructing entropy solutions is by the vanishing viscosity method, which starts off from classical solutions to the nondegenerate equation

$$\partial_t u_\eta + \partial_x f(u_\eta) = \partial_x^2 A(u_\eta) + \eta \partial_x^2 u_\eta, \qquad \eta > 0,$$

and establishes the strong convergence of $u_\eta$ as $\eta \to 0$ by deriving $BV$ estimates that are independent of $\eta$; see Vol′pert and Hudjaev [27].

Besides proving that $u_\eta$ converges in the $L^1$ norm to the unique entropy solution $u$ of (1.1), it is possible to prove the error estimate

$$\|u_\eta(\cdot,t) - u(\cdot,t)\|_{L^1} \leq C\sqrt{\eta}, \qquad \text{(whenever } u_0 \in BV\text{)}; \tag{1.4}$$

see [14] (cf. also [15]). The error bound (1.4) can also be obtained as a consequence of the more general continuous dependence estimate derived in [9]; see also [6,18].

Herein we are interested in the much more difficult problem of deriving error estimates for numerical approximations of entropy solutions to convection-diffusion equations. Convergence results (without error estimates) have been obtained for finite difference schemes [12] (see also [13,19]); finite volume schemes [16] (see also [2]); operator splitting methods [17]; and BGK approximations [3,4], to mention just a few references. For a posteriori estimates for finite volume schemes, see [24].

To be concrete in what follows, let us for simplicity assume $f' \geq 0$ and consider the semi-discrete difference scheme

$$\frac{d}{dt}u_j(t) + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = \frac{A(u_{j+1}) - 2A(u_j) + A(u_{j-1})}{\Delta x^2}, \tag{1.5}$$

where $u_j(t) \approx u(t, j\Delta x)$ and $\Delta x > 0$ is the spatial mesh size. Convergence of this scheme can be proved as in the works [12, 13], where explicit and implicit time discretizations are treated. Denote by $u_{\Delta x}(x,t)$ the piecewise constant interpolation of $\{u_j(t)\}_j$. The basic question we address in this paper is the following one: Does there exist a number $r \in (0,1)$ and a constant $C$, independent of $\Delta x$, such that

$$\|u_{\Delta x}(\cdot, t) - u(\cdot, t)\|_{L^1} \leq C\,\Delta x^r, \tag{1.6}$$

where $u$ is the unique entropy solution of (1.1). We refer to the number $r$ as the rate of convergence.

In the purely hyperbolic case (1.2) ($A' \equiv 0$), the answer to this question is a classical result due to Kuznetsov [22], who proved that the rate of convergence is $1/2$ for viscous approximations as well as monotone difference schemes, and this is optimal for discontinuous solutions. The work of Kuznetsov [22] turned out to be extremely influential, and by now a large number of related works have been devoted to error estimation theory for conservation laws. We refer to [7] for an overview of the relevant results and literature.

Unfortunately, the situation is unclear in the degenerate parabolic case (1.1). Let us expose some reasons why adding a nonlinear diffusion term to (1.2) can make the error analysis significantly more difficult than in the streamlined Kuznetsov theory. First of all, it is well known that the purely hyperbolic difference scheme

$$\frac{d}{dt}u_j(t) + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = 0 \tag{1.7}$$

has as a model equation, the second order viscous equation

$$\partial_t u + \partial_x f(u) = \frac{\Delta x}{2}\partial_x^2 f(u),$$

an equation that is compatible with the notion of entropy solution for (1.2). Indeed, an error estimate for this viscous equation is highly suggestive for what to expect for the upwind scheme (1.7) (this is of course what Kuznetsov proved). However, for convection-diffusion equations such as (1.1) the situation changes. The model equation for (1.5) is no longer second order but rather fourth order:

$$\partial_t u + \partial_x f(u) = \partial_x^2 A(u) + \frac{\Delta x}{2}\partial_x^2 f(u) - \frac{\Delta x^2}{12}\partial_x^4 A(u);$$

hence the error estimate (1.4) appears no longer so relevant for numerical schemes. Another added difficulty comes from the necessity to work with an explicit form of the parabolic dissipation term associated with (1.1). Indeed, in the analysis one needs to replace (1.3) by the following more precise entropy equation [5],

$$\begin{aligned} &\partial_t |u - c| + \partial_x\big(\operatorname{sign}(u-c)(f(u) - f(c)\big) - \partial_x^2 |A(u) - A(c)| \\ &\qquad = -\operatorname{sign}'(A(u) - A(c))\,|\partial_x A(u)|^2, \qquad c \in \mathbb{R}, \end{aligned} \tag{1.8}$$

which is formally obtained multiplying (1.1) by $\operatorname{sign}(A(u) - A(c))$, assuming for the sake of this discussion that $A'(\cdot) > 0$. The term on the right-hand side is the parabolic dissipation term, which is a finite (signed) measure and thus very singular. To illustrate why the parabolic dissipation term is needed, let $u(y,s)$ and

$v(x,t)$ be two solutions satisfying (1.8). In the entropy equation for $u(y,s)$ one takes $c = v(x,t)$, while in the entropy equation for $v(x,t)$ one takes $c = u(y,s)$. Adding the two resulting equations yields

$$\begin{aligned}&(\partial_t + \partial_s)\left|u - v\right| + (\partial_x + \partial_y)\big(\mathrm{sign}(u-v)(f(u) - f(v))\big)\\ &\quad - (\partial_x^2 + \partial_y^2)\left|A(u) - A(v)\right| = -\mathrm{sign}'(A(u) - A(v))\big(\left|\partial_y A(u)\right|^2 + \left|\partial_x A(v)\right|^2\big),\end{aligned}$$

By adding $-2\partial_{xy}^2 \left|A(u) - A(v)\right|$ to both sides of this equation, noting that

$$-2\partial_{xy}^2 \left|A(u) - A(v)\right| = 2\mathrm{sign}'(A(u) - A(v))\partial_y A(u)\partial_x A(v),$$

we arrive at

$$\begin{aligned}(\partial_t + \partial_s)\left|u - v\right| &+ (\partial_x + \partial_y)\big(\mathrm{sign}(u-v)(f(u) - f(v))\big)\\ &\qquad - (\partial_x^2 - 2\partial_{xy}^2 + \partial_y^2)\left|A(u) - A(v)\right|\\ &= -\mathrm{sign}'(A(u) - A(v))\big(\left|\partial_y A(u)\right| - \left|\partial_x A(v)\right|\big)^2\\ &\le 0,\end{aligned} \tag{1.9}$$

from which the contraction property $\frac{d}{dt}\left\|u(\cdot,t) - v(\cdot,t)\right\|_{L^1} \le 0$ follows [5]. Similarly, to obtain error estimates for numerical methods, it is necessary to derive a "discrete" version of (1.9) with $v$ replaced by $u_{\Delta x}$. The main challenge is to suitably replicate at the discrete level the delicate balance between the two terms in (1.9) involving $A$; the difficulty stems from the lack of a chain rule for finite differences.

Despite the mentioned difficulties, in this paper we will prove that there exists a constant $C$, independent of $\Delta x$, such that for any $t > 0$,

$$\left\|u_{\Delta x}(\cdot,t) - u(\cdot,t)\right\|_{L^1} \le C\,\Delta x^{\frac{1}{3}}.$$

The only other work we are aware of that provides $L^1$ error estimates for numerical approximations of (1.1) is [20]; therein (1.6) is established with $r = \frac{1}{11}$; if $A$ is a linear function, then the convergence rate is the usual one, namely $r = \frac{1}{2}$. In addition to the semi-discrete scheme (1.7), we will prove similar results for fully discrete (implicit and explicit) difference schemes.

Roughly speaking, the reason is two-fold for why we can significantly improve the result in [20]. First, we are herein able to provide a more faithful analog of (1.9) at the discrete level. Second, since $\mathrm{sign}'(\cdot)$ is singular, one has to work with a Lipschitz continuous approximation $\mathrm{sign}_\varepsilon(\cdot)$ of the sign function $\mathrm{sign}(\cdot)$. The use of this approximation breaks the symmetry of the corresponding entropy fluxes, and introduces new error terms that depend on the parameter $\varepsilon$; the process of "balancing" terms involving $\Delta x$ and $\varepsilon$ lowers the convergence rate (to $r = \frac{1}{11}$) [20]. In the present paper we are able to dispense with this balancing act. Indeed, we show that it is possible to send $\varepsilon \to 0$ independently of $\Delta x$.

The remaining part of this paper is organized as follows: In Section 2 we list some relevant a priori estimates satisfied by viscous approximations and entropy solutions, and provide a definition of entropy solutions. The semi-discrete difference scheme is defined and proved to be well-posed in Section 3. We also list several relevant a priori estimates. Section 4 is devoted to the proof of the error estimate. In Section 5 we show that the proof in Section 4 can be adapted to a fully discrete scheme that is implicit in the time variable. In fact, we go through all the steps of the proof and provide the details where there are considerable differences between the two cases. In Section 6 the explicit version of the scheme is treated. We end the paper with a few concluding remarks in Section 7.

## 2. Preliminary material

Set $A^\eta(u) := A(u) + \eta u$ for any fixed $\eta > 0$, and consider the uniformly parabolic problem

$$\begin{cases} u_t^\eta + f(u^\eta)_x = A^\eta(u^\eta)_{xx}, & (x,t) \in \Pi_T, \\ u^\eta(x,0) = u^0(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

It is well known that (2.1) admits a unique classical (smooth) solution.

We collect some relevant (standard) a priori estimates in the next three lemmas.

**Lemma 2.1.** *Suppose $u^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$, and let $u^\eta$ be the unique classical solution of* (2.1)*. Then for any $t > 0$,*

$$\begin{aligned} \|u^\eta(\cdot,t)\|_{L^1(\mathbb{R})} &\leq \left\|u^0\right\|_{L^1(\mathbb{R})}, \\ \|u^\eta(\cdot,t)\|_{L^\infty(\mathbb{R})} &\leq \|u^0\|_{L^\infty(\mathbb{R})}, \\ |u^\eta(\cdot,t)|_{BV(\mathbb{R})} &\leq \left|u^0\right|_{BV(\mathbb{R})}. \end{aligned}$$

For a proof of the previous and next lemmas, see for example [27].

**Lemma 2.2.** *Suppose $u^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $f(u^0) - A(u^0)_x \in BV(\mathbb{R})$. Let $u^\eta$ be the unique classical solution of* (2.1)*. Then for any $t_1, t_2 > 0$,*

$$\|u^\eta(\cdot,t_2) - u^\eta(\cdot,t_1)\|_{L^1(\mathbb{R})} \leq \left|f(u^0) - A(u^0)_x\right|_{BV(\mathbb{R})} |t_2 - t_1|.$$

Regarding the following lemma, see [12, 25].

**Lemma 2.3.** *Suppose $u^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $f(u^0) - A(u^0)_x \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$. Let $u^\eta$ be the unique classical solution of* (2.1)*. Then for any $t > 0$,*

$$\begin{aligned} \|f(u^\eta(\cdot,t)) - A(u^\eta(\cdot,t))_x\|_{L^\infty(\mathbb{R})} &\leq \left\|f(u^0) - A(u^0)_x\right\|_{L^\infty(\mathbb{R})}, \\ |f(u^\eta(\cdot,t)) - A(u^\eta(\cdot,t))_x|_{BV(\mathbb{R})} &\leq \left|f(u^0) - A(u^0)_x\right|_{BV(\mathbb{R})}. \end{aligned}$$

Note that $\|A(u^\eta)_x\|_{L_t^\infty(L_x^\infty)}$ and $\|A(u^\eta)_{xx}\|_{L_t^\infty(L_x^1)}$ are bounded independently of $\eta$ provided that $A(u^0)_x$ is in $BV(\mathbb{R})$.

The results above imply that $\{u^\eta\}_{\eta>0}$ is relatively compact in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$. If $u = \lim_{\eta\to 0} u^\eta$, then

$$\|u^\eta - u\|_{L^1(\Pi_T)} \leq C\eta^{1/2},$$

for some constant $C$ which does not depend on $\eta$; see [14]. Moreover, $u$ is an entropy solution according to the following definition.

**Definition 2.1.** An entropy solution of the Cauchy problem (1.1) is a measurable function $u = u(x,t)$ satisfying:

(D.1) $u \in L^\infty(\Pi_T) \cap C((0,T); L^1(\mathbb{R}))$.
(D.2) $A(u) \in L^2((0,T); H^1(\mathbb{R}))$.
(D.3) For all constants $c \in \mathbb{R}$ and test functions $0 \leq \varphi \in C_0^\infty(\mathbb{R} \times [0,T))$, the following entropy inequality holds:

$$\iint_{\Pi_T} |u-c|\,\varphi_t + \operatorname{sign}(u-c)\,(f(u) - f(c))\varphi_x + |A(u) - A(c)|\,\varphi_{xx}\,dxdt \\ + \int_{\mathbb{R}} |u_0 - c|\,\varphi(x,0)\,dx \geq 0.$$

The uniqueness of entropy solutions follows from the work [5]. Actually, in view of the above a priori estimates, the relevant functional class is $BV(\Pi_T)$, in which case we can replace (D.2) by the condition $A(u)_x \in L^\infty(\Pi_T)$. For a uniqueness result in the $BV$ class, see [28].

## 3. Difference scheme

We start by specifying the numerical flux to be used in the difference scheme.

**Definition 3.1** (Numerical flux)**.** We call a function $F \in C^1(\mathbb{R}^2)$ a two-point numerical flux for $f$ if $F(u,u) = f(u)$ for $u \in \mathbb{R}$. If

$$\frac{\partial}{\partial u} F(u,v) \geq 0 \quad \text{and} \quad \frac{\partial}{\partial v} F(u,v) \leq 0$$

holds for all $u, v \in \mathbb{R}$, we call $F$ monotone.

Let $F_u$ and $F_v$ denote the partial derivatives of $F$ with respect to the first and second variable, respectively. We will also assume that $F$ is Lipschitz continuous.

Let $\Delta x > 0$ and set $x_j = j\Delta x$ for $j \in \mathbb{Z}$, and define

$$D_\pm \sigma_j = \pm \frac{\sigma_{j\pm 1} - \sigma_j}{\Delta x},$$

for any sequence $\{\sigma_j\}$.

We may now define a semi-discrete approximation of the solution to (1.1) as the solution to the (infinite) system of ordinary differential equations

$$\begin{cases} \frac{d}{dt} u_j(t) + D_- F_{j+1/2} = D_- D_+ A(u_j), & t > 0, \\ u_j(0) = \frac{1}{\Delta x} \int_{I_j} u^0(x)\, dx, & \end{cases} \qquad j \in \mathbb{Z}, \tag{3.1}$$

where $F_{j+1/2} = F(u_j, u_{j+1})$ is a numerical flux function and $I_j = (x_{j-1/2}, x_{j+1/2}]$.

The problem above can be viewed as an ordinary differential equation in the Banach space $\ell^1(\mathbb{Z})$ (see, e.g., [23]). To get bounds independent of $\Delta x$ we define

$$\|\sigma\|_1 = \Delta x \sum_j |\sigma_j| \quad \text{and} \quad |\sigma|_{BV} = \sum_j |\sigma_{j+1} - \sigma_j| = \|D_+\sigma\|_1 .$$

If these are bounded we say that $\sigma = \{\sigma_j\}$ is in $\ell^1$ and of bounded variation. Let $u(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$, $u^0 = \{u_j(0)\}_{j\in\mathbb{Z}}$, and define the operator $\mathcal{A} : \ell^1 \to \ell^1$ by $(\mathcal{A}(u))_j := D_-(F(u_j, u_{j+1}) - D_+ A(u_j))$. Then (3.1) takes the following form:

$$\frac{du}{dt} + \mathcal{A}(u) = 0, \quad t > 0, \quad u(0) = u^0.$$

This problem has a unique continuously differentiable solution since $\mathcal{A}$ is Lipschitz continuous for each fixed $\Delta x > 0$. This solution defines a strongly continuous semigroup $\mathcal{S}(t)$ on $\ell^1$. If $\mathcal{S}$ also satisfies

$$\|\mathcal{S}(t)u - \mathcal{S}(t)v\|_1 \leq \|u - v\|_1 \quad \text{for} \quad u, v \in \ell^1,$$

we say that it is nonexpansive. The next lemma sums up some important properties of the solutions to (3.1) (for a proof see [11]).

**Lemma 3.1.** *Suppose that $F$ is monotone. Then there exists a unique solution $u = \{u_j\}$ to (3.1) on $[0,T]$ with the following properties:*

(a) $\|u(t)\|_1 \le \left\|u^0\right\|_1$.
(b) *For every $j \in \mathbb{Z}$ and $t \in [0,T]$,*

$$\inf_k \left\{u_k^0\right\} \le u_j(t) \le \sup_k \left\{u_k^0\right\}.$$

(c) $|u(t)|_{BV} \le \left|u^0\right|_{BV}$.
(d) *If $v = \{v_j\}$ is a another solution with initial data $v^0$, then*

$$\|u(t) - v(t)\|_1 \le \left\|u^0 - v^0\right\|_1.$$

**Lemma 3.2.** *If $F$ is monotone, then*

$$\|F(u_j,u_{j+1}) - D_+A(u_j)\|_{\ell^\infty} \le \left\|F(u_j^0,u_{j+1}^0) - D_+A(u_j^0)\right\|_{\ell^\infty}, \tag{3.2}$$

$$|F(u_j,u_{j+1}) - D_+A(u_j)|_{BV} \le \left|F(u_j^0,u_{j+1}^0) - D_+A(u_j^0)\right|_{BV}. \tag{3.3}$$

*Furthermore, $t \mapsto \{u_j(t)\}_{j\in\mathbb{Z}}$ is $\ell^1$ Lipschitz continuous.*

*Proof.* The proof follows [12]. Let $v_j = \Delta x \sum_{k\le j} \frac{du_k}{dt}$. Then $v_j$ satisfies

$$v_j = \Delta x \sum_{k=-\infty}^{j} D_-\left(D_+A(u_k) - F(u_k,u_{k+1})\right) = D_+A(u_j) - F(u_j,u_{j+1}), \tag{3.4}$$

and we may define $v_j$ for all $t \in [0,T]$. Note that $\{v_j(t)\}$ is in $\ell^1$ for all $t$ by Lemma 3.1. Differentiating (3.4) with respect to $t$ we obtain

$$\begin{aligned}\frac{dv_j}{dt} = \frac{1}{\Delta x}&\left[a(u_{j+1})\frac{du_{j+1}}{dt} - a(u_j)\frac{du_j}{dt}\right]\\ &- F_u(u_j,u_{j+1})\frac{du_j}{dt} - F_v(u_j,u_{j+1})\frac{du_{j+1}}{dt},\end{aligned}$$

where $a(u) = A'(u)$. Note that $D_-v_j = \frac{du_j}{dt}$ and $D_+v_j = \frac{du_{j+1}}{dt}$. Therefore

$$\begin{aligned}\frac{dv_j}{dt} = &\left(\frac{1}{\Delta x}a(u_{j+1}) - F_v(u_j,u_{j+1})\right)D_+v_j\\ &- \left(\frac{1}{\Delta x}a(u_j) + F_u(u_j,u_{j+1})\right)D_-v_j.\end{aligned} \tag{3.5}$$

Assume $v_{j_0}(t_0)$ is a local maximum in $j$. Then $D_+v_{j_0}(t_0) \le 0$ and $D_-v_{j_0}(t_0) \ge 0$ so $\frac{v_{j_0}}{dt}(t_0) \le 0$ since $F$ is monotone. Similarly, if $v_{j_0}(t_0)$ is a local minimum in $j$, then $\frac{v_{j_0}}{dt}(t_0) \ge 0$. Then inequality (3.2) follows by the fact that $\{v_j(t)\} \in \ell^1$. Consider (3.3). We want to show that $\frac{d}{dt}\left(|v(t)|_{BV}\right) \le 0$. Now,

$$\frac{d}{dt}\Big(\sum_j |v_{j+1} - v_j|\Big) = \sum_j \operatorname{sign}\left(v_{j+1} - v_j\right)\frac{d}{dt}\left(v_{j+1} - v_j\right),$$

so we may use (3.5). Thus

$$
\begin{aligned}
&\frac{d}{dt}\left|v(t)\right|_{BV} \\
&\quad= \sum_j \left(\frac{1}{\Delta x}a(u_{j+2}) - F_v(u_{j+1},u_{j+2})\right)(D_+v_{j+1})\,\mathrm{sign}(v_{j+1}-v_j) \\
&\qquad - \sum_j \left(\frac{1}{\Delta x}a(u_{j+1}) + F_u(u_{j+1},u_{j+2})\right)\left|D_+v_j\right| \\
&\qquad - \sum_j \left(\frac{1}{\Delta x}a(u_{j+1}) - F_v(u_j,u_{j+1})\right)\left|D_+v_j\right| \\
&\qquad + \sum_j \left(\frac{1}{\Delta x}a(u_j) + F_u(u_j,u_{j+1})\right)((D_-v_j)\mathrm{sign}(v_{j+1}-v_j)) \\
&\quad= \sum_j \left(\frac{1}{\Delta x}a(u_{j+1}) - F_v(u_j,u_{j+1})\right)\left[(D_+v_j)\,\mathrm{sign}\,(v_j-v_{j-1}) - \left|D_+v_j\right|\right] \\
&\qquad + \sum_j \left(\frac{1}{\Delta x}a(u_j) + F_u(u_j,u_{j+1})\right)\left[(D_-v_j)\mathrm{sign}(v_{j+1}-v_j) - \left|D_-v_j\right|\right] \\
&\quad\le 0,
\end{aligned}
$$

since $a(u) > 0$, $F_v \le 0$, and $F_u \ge 0$. Given the preceding estimates, the $\ell^1$ Lipschitz continuity is straightforward to prove. ☐

It turns out that we need more conditions on $F$ than mere monotonicity.

**Definition 3.2.** Given an entropy pair $(\psi, q)$ and a numerical flux $F$, we define $Q \in C^1(\mathbb{R}^2)$ by

$$
\begin{aligned}
Q(u,u) &= q(u), \\
\frac{\partial}{\partial v}Q(v,w) &= \psi'(v)\frac{\partial}{\partial v}F(v,w), \\
\frac{\partial}{\partial w}Q(v,w) &= \psi'(w)\frac{\partial}{\partial w}F(v,w).
\end{aligned}
$$

We call $Q$ a numerical entropy flux.

The next lemma gives a sufficient condition on the numerical flux to ensure that there exists a numerical entropy flux.

**Lemma 3.3.** *Given a two-point numerical flux $F$, assume that there exist $C^1$ functions $F_1, F_2$ such that*

$$
F(u,v) = F_1(u) + F_2(v), \qquad F_1'(u) + F_2'(u) = f'(u), \tag{3.6}
$$

*for all relevant $u$ and $v$. Then there exists a numerical entropy flux $Q$ for any entropy flux pair $(\psi, q)$.*

*Proof.* Let $(\psi, q)$ be an entropy pair. Then $q$ has the form

$$
q(u) = \int_c^u \psi'(z) f'(z)\,dz + C,
$$

for some constant $C$. Define $Q$ by

$$Q(u,v) = \int_c^u \psi'(z)F_1'(z)\,dz + \int_c^v \psi'(z)F_2'(z)\,dz + C. \tag{3.7}$$

It is easily verified that $Q$ is a numerical entropy flux. □

Let us list a few numerical flux functions to which Lemma 3.3 applies.

**Example 3.1** (Engquist-Osher flux)**.** Let

$$f_+'(s) = \max(f'(s), 0) \qquad \text{and} \qquad f_-'(s) = \min(f'(s), 0).$$

Then, in the terminology of Lemma 3.3, let $F(u,v) = F_1(u) + F_2(v)$ with

$$F_1(u) = f(0) + \int_0^u f_+'(s)\,ds \qquad \text{and} \qquad F_2(v) = \int_0^v f_-'(s)\,ds.$$

It is easily seen that the criteria given in Lemma 3.3 are satisfied, and $F$ is also clearly monotone.

**Example 3.2.** Let $a, b \in \mathbb{R}$ and define

$$F_1(u) = af(u) + bu \qquad \text{and} \qquad F_2(v) = (1-a)f(v) - bv.$$

Note that $F(u,v) = F_1(u) + F_2(v)$ is monotone if

$$a \inf_u\{f'(u)\} \geq -b \quad \text{and} \quad (1-a)\sup_u\{f'(u)\} \leq b.$$

This example includes both the upwind scheme and the Lax-Friedrichs scheme.

From a more general point of view we may consider any flux splitting, that is, $f(u) = f^+(u) + f^-(u)$ with $(f^+)'(u) \geq 0$ and $(f^-)'(u) \leq 0$ for all $u \in \mathbb{R}$. Then the numerical flux

$$F(u,v) = f^+(u) + f^-(v)$$

satisfies the assumptions of Lemma 3.3. Note also that any convex combination of numerical flux functions which satisfy the hypothesis of Lemma 3.3, itself satisfies the assumptions of the lemma.

If (3.6) holds, then we have a representation of $Q$ given by (3.7). It follows that

$$Q(u,v) = q(u) + \int_u^v \psi'(z)F_2'(z)\,dz. \tag{3.8}$$

Note that we may obtain another representation depending on $F_1$ by splitting up the first integral.

## 4. Error estimate

Let $\{u_j\}_{j\in\mathbb{Z}}$ be the solution to (3.1). We associate with it the piecewise constant function

$$u_{\Delta x}(x,t) = u_j(t) \quad \text{for } x \in I_j. \tag{4.1}$$

To derive the error estimate we need many of the uniform bounds from Sections 2 and 3. For these estimates to hold independently of $\Delta x$, we make the following assumptions on the initial data $u^0$:

(i) $u^0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$.
(ii) $A(u^0)_x \in BV(\mathbb{R})$.

We may now state the theorem.

**Theorem 4.1.** *Let $u$ be the entropy solution to* (1.1) *and $\{u_j(t)\}_{j\in\mathbb{Z}}$ solve the semi-discrete difference scheme* (3.1)*. If $u^0$ satisfies* (i) *and* (ii) *above, then for all sufficiently small $\Delta x$,*

$$\|u_{\Delta x}(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbb{R})}\le \left\|u^0_{\Delta x}-u^0\right\|_{L^1(\mathbb{R})}+C_T\Delta x^{\frac{1}{3}},\qquad t\in[0,T],$$

*where the constant $C_T$ depends on $A$, $f$, $u^0$, and $T$, but not on $\Delta x$.*

Let us define some of the functions we are going to work with. First, we will use the following approximation of the sign function:

$$\operatorname{sign}_\varepsilon(\sigma)=\begin{cases}\sin(\frac{\pi\sigma}{2\varepsilon}) & \text{for } |\sigma|<\varepsilon,\\ \operatorname{sign}(\sigma) & \text{otherwise,}\end{cases}$$

where $\varepsilon>0$. Note that $\operatorname{sign}_\varepsilon$ is continuously differentiable and nondecreasing. We define

$$|u|_\varepsilon=\int_0^u \operatorname{sign}_\varepsilon(z)\,dz.$$

Furthermore, we introduce an entropy pair $(\psi_\varepsilon,q_\varepsilon)$ defined by

$$\begin{aligned}\psi_\varepsilon(u,c)&=\int_c^u \operatorname{sign}_\varepsilon(A(z)-A(c))\,dz,\\ q_\varepsilon(u,c)&=\int_c^u \psi_\varepsilon'(z,c)f'(z)\,dz=\int_c^u \operatorname{sign}_\varepsilon(A(z)-A(c))f'(z)\,dz,\end{aligned}$$

where $\psi_\varepsilon'$ is the derivative with respect to the first variable.

**Lemma 4.1.** *Suppose $A'>0$. Let $u=u(y,s)$ be the classical solution of* (1.1)*. Then for any constant $c\in\mathbb{R}$,*

$$\partial_s\psi_\varepsilon(u,c)+\partial_y q_\varepsilon(u,c)-\partial_y^2|A(u)-A(c)|_\varepsilon=-\partial_y\psi_\varepsilon'(u,c)\partial_y A(u).$$

*Proof.* Multiply equation (1.1) by $\psi_\varepsilon'(u,c)$ to obtain

$$\partial_s\psi_\varepsilon(u,c)+\partial_y q_\varepsilon(u,c)=\psi_\varepsilon'(u,c)\partial_y^2A(u).$$

The term on the right may be rewritten according to

$$\partial_y(\psi_\varepsilon'(u,c)\partial_yA(u))=\partial_y\psi_\varepsilon'(u,c)\partial_yA(u)+\psi_\varepsilon'(u,c)\partial_y^2A(u).$$

By the chain rule,

$$\partial_y(\psi_\varepsilon'(u,c)\partial_yA(u))=\partial_y^2|A(u)-A(c)|_\varepsilon.$$

Combining these equalities proves the lemma. ☐

The next lemma is a simple identity taken from [8].

**Lemma 4.2.** *For any differentiable function $g$ and all real numbers $a,b,c$,*

$$\begin{aligned}\psi_\varepsilon'(a,c)(g(b)-g(a))=&\int_c^b \psi_\varepsilon'(z,c)g'(z)\,dz-\int_c^a \psi_\varepsilon'(z,c)g'(z)\,dz\\ &+\int_a^b \psi_\varepsilon''(z,c)(g(z)-g(b))\,dz.\end{aligned}$$

*Proof.* Integration by parts yields

$$\psi_\varepsilon'(\zeta,c)(g(\zeta)-g(b)) = \int_c^\zeta \psi_\varepsilon'(z,c)g'(z)\,dz + \int_c^\zeta \psi_\varepsilon''(z,c)(g(z)-g(b))\,dz,$$

for any $\zeta \in \mathbb{R}$. Take the two equations obtained by taking $\zeta = a$ and $\zeta = b$ and subtract one from the other. □

**Lemma 4.3.** *Let $u_j$ be the solution to* (3.1)*. Then for all $c \in \mathbb{R}$,*

$$\begin{aligned}\partial_t\psi_\varepsilon(u_j,c) + D_-Q^c(u_j,u_{j+1}) - D_-D_+|A(u_j)-A(c)|_\varepsilon&\\ \leq -\frac{1}{(\Delta x)^2}\int_{u_{j+1}}^{u_j}\psi_\varepsilon''(z,c)(A(z)-A(u_{j+1}))\,dz&\\ -\frac{1}{(\Delta x)^2}\int_{u_{j-1}}^{u_j}\psi_\varepsilon''(z,c)(A(z)-A(u_{j-1}))\,dz,&\end{aligned}$$

*where $Q^c(u,v) := Q_1^c(u) + Q_2^c(v)$,*

$$Q_1^c(u) := \int_c^u \psi_\varepsilon'(z,c)F_1'(z)\,dz, \quad Q_2^c(v) := \int_c^v \psi_\varepsilon'(z,c)F_2'(z)\,dz,$$

*for all real numbers $u$ and $v$.*

*Proof.* From (3.1) it follows that

$$\psi_\varepsilon'(u_j,c)\partial_t u_j + \psi_\varepsilon'(u_j,c)D_-F(u_j,u_{j+1}) = \psi_\varepsilon'(u_j,c)D_-D_+A(u_j).$$

Note that

$$\psi_\varepsilon'(u_j,c)D_-F(u_j,u_{j+1}) = \psi_\varepsilon'(u_j,c)D_-F_1(u_j) + \psi_\varepsilon'(u_j,c)D_+F_2(u_j),$$

and so we may apply Lemma 4.2. Let $g = F_1$. Then we obtain

$$\psi_\varepsilon'(u_j,c)D_-F_1(u_j) = D_-Q_1^c(u_j) - \frac{1}{\Delta x}\int_{u_j}^{u_{j-1}}\psi_\varepsilon''(z,c)(F_1(z)-F_1(u_{j-1}))\,dz.$$

Similarily, let $g = F_2$ to obtain

$$\psi_\varepsilon'(u_j,c)D_+F_2(u_j) = D_+Q_2^c(u_j) + \frac{1}{\Delta x}\int_{u_j}^{u_{j+1}}\psi_\varepsilon''(z,c)(F_2(z)-F_2(u_{j+1}))\,dz.$$

Finally, apply Lemma 4.2 twice with $g = A$. Adding the equations we obtain

$$\begin{aligned}&\psi_\varepsilon'(u_j,c)\left(A(u_{j-1}) - 2A(u_j) + A(u_{j+1})\right)\\ &\quad= \int_{u_j}^{u_{j+1}}\psi_\varepsilon'(z,c)A'(z)\,dz + \int_{u_j}^{u_{j-1}}\psi_\varepsilon'(z,c)A'(z)\,dz\\ &\qquad+ \int_{u_j}^{u_{j+1}}\psi_\varepsilon''(z,c)(A(z)-A(u_{j+1}))\,dz\\ &\qquad+ \int_{u_j}^{u_{j-1}}\psi_\varepsilon''(z,c)(A(z)-A(u_{j-1}))\,dz.\end{aligned}$$

Note that

$$
\begin{aligned}
&\left[\int_{u_j}^{u_{j+1}} \psi_\varepsilon'(z,c)A'(z)\,dz + \int_{u_j}^{u_{j-1}} \psi_\varepsilon'(z,c)A'(z)\,dz\right] \\
&= \left[\int_{u_j}^{u_{j+1}} \frac{\partial}{\partial z}|A(z)-A(c)|_\varepsilon\,dz + \int_{u_j}^{u_{j-1}} \frac{\partial}{\partial z}|A(z)-A(c)|_\varepsilon\,dz\right] \\
&= \left[|A(u_{j-1})-A(c)|_\varepsilon - 2|A(u_j)-A(c)|_\varepsilon + |A(u_{j+1})-A(c)|_\varepsilon\right].
\end{aligned}
$$

Combining the above computations we obtain

$$
\partial_t\psi_\varepsilon(u_j,c) + D_-Q^c(u_j,u_{j+1}) - D_-D_+|A(u_j)-A(u)|_\varepsilon = -E^c(u_{j-1},u_j,u_{j+1}),
$$

where

$$
\begin{aligned}
E^c(u_{j-1},u_j,u_{j+1}) = \frac{1}{\Delta x}\int_{u_{j-1}}^{u_j} &\psi_\varepsilon''(z,c)(F_1(z)-F_1(u_{j-1}))\,dz \\
&- \frac{1}{\Delta x}\int_{u_{j+1}}^{u_j} \psi_\varepsilon''(z,c)(F_2(z)-F_2(u_{j+1}))\,dz \\
&+ \frac{1}{(\Delta x)^2}\int_{u_{j+1}}^{u_j} \psi_\varepsilon''(z,c)(A(z)-A(u_{j+1}))\,dz \\
&+ \frac{1}{(\Delta x)^2}\int_{u_{j-1}}^{u_j} \psi_\varepsilon''(z,c)(A(z)-A(u_{j-1}))\,dz.
\end{aligned}
$$

The result follows from the monotonicity of $F$. ☐

We shall need the next lemma, which deals with a mixed term, in order to carry out the "second order" doubling-of-the-variables argument.

**Lemma 4.4.** *Let $\{u_j\}$ be some sequence and $u$ some differentiable function of $y$. Then*

$$
\begin{aligned}
&\left(\frac{1}{\Delta x}\int_{u_{j-1}}^{u_j} \psi_\varepsilon''(z,u)\,dz + \frac{1}{\Delta x}\int_{u_j}^{u_{j+1}} \psi_\varepsilon''(z,u)\,dz\right)\partial_y A(u) \\
&\qquad = -\partial_y(D_- + D_+)|A(u_j)-A(u)|_\varepsilon.
\end{aligned}
$$

*Proof.* Let $a,b$ be fixed real numbers. Then

$$
\begin{aligned}
&\int_a^b \psi_\varepsilon''(z,u)\,dz A(u)_y \\
&\quad = \int_a^b \operatorname{sign}_\varepsilon'(A(z)-A(u))A(u)_y A'(z)\,dz \\
&\quad = -\frac{\partial}{\partial y}\left(\int_a^b \operatorname{sign}_\varepsilon(A(z)-A(u))A'(z)\,dz\right) \\
&\quad = -\frac{\partial}{\partial y}\left(|A(b)-A(u)|_\varepsilon - |A(a)-A(u)|_\varepsilon\right).
\end{aligned}
$$

Let $a = u_{j-1}$, $b = u_j$ and $a = u_j$, $b = u_{j+1}$. Then add up the resulting equations and divide by $\Delta x$. ☐

We are now in a position to carry out the doubling-of-the-variables argument.

**Lemma 4.5.** *Suppose $A' > 0$. Let $u = u(y,s)$ be the classical solution to* (1.1) *and let $\{u_j\} = \{u_j(t)\}$ be the solution to* (3.1). *Then*

$$\begin{aligned}&\partial_t \psi_\varepsilon(u_j,u) + \partial_s \psi_\varepsilon(u,u_j) + \partial_y q_\varepsilon(u,u_j) + D_- Q^u(u_j,u_{j+1})\\&\qquad - (\partial_y^2 + \partial_y(D_- + D_+) + D_-D_+)|A(u_j) - A(u)|_\varepsilon) \le -E_j^\varepsilon,\end{aligned}$$

*where $E_j^\varepsilon := E^\varepsilon[u](u_{j-1},u_j,u_{j+1})$ with*

$$\begin{aligned}E^\varepsilon[u](u_{j-1},u_j,u_{j+1}) := &\frac{1}{(\Delta x)^2}\int_{u_{j+1}}^{u_j} \psi_\varepsilon''(z,u)(A(z) - A(u_{j+1}))\,dz\\&+ \frac{1}{(\Delta x)^2}\int_{u_{j-1}}^{u_j} \psi_\varepsilon''(z,u)(A(z) - A(u_{j-1}))\,dz\\&- \frac{1}{\Delta x}\int_{u_{j-1}}^{u_j} \psi_\varepsilon''(z,u)\,dz\,\partial_y A(u)\\&- \frac{1}{\Delta x}\int_{u_j}^{u_{j+1}} \psi_\varepsilon''(z,u)\,dz\,\partial_y A(u)\\&+ \partial_y \psi_\varepsilon'(u,u_j)\partial_y A(u).\end{aligned}$$

*Proof.* Let $c = u_j$ in Lemma 4.1 and $c = u$ in Lemma 4.3. Then add up the equations together with Lemma 4.4. □

*Remark* 4.1. Note that $E_j^\varepsilon$ is a function of $y, s, t$.

In what follows it will be necessary to work with the piecewice constant approximation defined in (4.1). To do this we introduce some new notation. Let the shift operator $S_\sigma$ be defined for any $\varphi : \Pi_T \to \mathbb{R}$ by

$$S_\sigma \varphi(x,t) = \varphi(x+\sigma,t),$$

and let the difference quotient be defined by

$$D_\pm \varphi = \pm \frac{S_{\pm\Delta x}\varphi - \varphi}{\Delta x}.$$

Note that for any two functions $u, v$ of $x$ we have $D_+(uv) = S_{\Delta x} u D_+ v + (D_+ u)v$. If $uv$ has compact support it follows that

$$\int_{\mathbb{R}} (D_+ u)v\,dx = -\int_{\mathbb{R}} u D_- v\,dx.$$

We will refer to these identities as the Leibniz rule for difference quotients and integration by parts for difference quotients. We will frequently integrate over the domain $\Pi_T^2$. To avoid writing four integral signs we will in general write one for each domain $\Pi_T$ and let $dX = dxdtdyds$.

**Lemma 4.6.** *Suppose $A' > 0$. Let $u_{\Delta x} = u_{\Delta x}(x,t)$ be defined by* (4.1), *and let $u = u(y,s)$ be the classical solution of* (1.1). *Let $\rho \in C_0^\infty(\mathbb{R})$ satisfy*

$$\operatorname{supp}(\rho) \subset [-1,1], \quad \rho(-\sigma) = \rho(\sigma), \quad \rho(\sigma) \ge 0, \quad \int_{\mathbb{R}} \rho(\sigma)\,d\sigma = 1,$$

*and set*

$$\omega_r(x) = \frac{1}{r}\rho\left(\frac{x}{r}\right), \quad \rho_\alpha(\xi) = \frac{1}{\alpha}\rho\left(\frac{\xi}{\alpha}\right), \quad \rho_{r_0}(t) = \frac{1}{r_0}\rho\left(\frac{t}{r_0}\right),$$

*for positive (small) $r$, $\alpha$ and $r_0$. Let $\nu$ and $\tau$ be such that $0 < \nu < \tau < T$ and define*

$$\psi^\alpha(t) := H_\alpha(t-\nu) - H_\alpha(t-\tau), \quad H_\alpha(t) = \int_{-\infty}^{t} \rho_\alpha(\xi)\, d\xi.$$

*Let*

$$\varphi(x,t,y,s) = \psi^\alpha(t)\omega_r(x-y)\rho_{r_0}(t-s).$$

*To ensure $\varphi_{|t=0} \equiv 0$, $\varphi_{|s=0} \equiv 0$, we choose $\nu$ and $\tau$ such that $0 < r_0 < \min(\nu, T-\tau)$ and $0 < \alpha < \min(\nu - r_0, T - \tau - r_0)$. Then*

$$\begin{aligned}
&\iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\nu)\omega_r \rho_{r_0}\, dX \\
&\qquad + \iint_{\Pi_T^2} \operatorname{sign}(u_{\Delta x} - u)\, (f(u_{\Delta x}) - f(u))\, (D_+\varphi + \varphi_y)\, dX \\
&\qquad + \iint_{\Pi_T^2} \left( \int_{u_{\Delta x}}^{S_{\Delta x} u_{\Delta x}} \operatorname{sign}(z-u)\, F_2'(z)\, dz \right) D_+\varphi\, dX \\
&\qquad + \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\, (D_-D_+\varphi + (D_+ + D_-)\varphi_y + \varphi_{yy})\, dX \\
&\qquad\quad \geq \iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\tau)\omega_r\rho_{r_0}\, dX + \liminf_{\varepsilon \downarrow 0} \iint_{\Pi_T^2} E_{\Delta x}^\varepsilon \varphi\, dX,
\end{aligned} \tag{4.2}$$

*where $E_{\Delta x}^\varepsilon(x,t,y,s) = E_j^\varepsilon(t,y,s)$ for $x \in I_j$.*

*Remark* 4.2. Note that both

$$\varphi_x + \varphi_y = 0 \quad \text{and} \quad \varphi_{xx} + 2\varphi_{xy} + \varphi_{yy} = 0.$$

In equation (4.2) these expressions appear with difference quotients instead of $x$-derivatives. We expect that these equalities turn into good approximations as long as $\Delta x$ tends relatively fast to zero compared to $r$. We will show that this is the case in what follows.

*Proof.* By Lemma 4.5 it follows that

$$\begin{aligned}
&\partial_t \psi_\varepsilon(u_{\Delta x}, u) + \partial_s \psi_\varepsilon(u, u_{\Delta x}) + \partial_y q_\varepsilon(u, u_{\Delta x}) + D_- Q^u(u_{\Delta x}, S_{\Delta x} u_{\Delta x}) \\
&\qquad - (\partial_y^2 + \partial_y(D_- + D_+) + D_-D_+)|A(u_{\Delta x}) - A(u)|_\varepsilon \leq -E_{\Delta x}^\varepsilon,
\end{aligned}$$

for all $(x,t,y,s) \in \Pi_T^2$. Let us multiply with $\varphi$ and integrate over $\Pi_T^2$. Using both ordinary integration by parts and integration by parts for difference quotients, we obtain

$$\begin{aligned}
&\iint_{\Pi_T^2} \psi_\varepsilon(u_{\Delta x}, u)\varphi_t + \psi_\varepsilon(u, u_{\Delta x})\varphi_s\, dX \\
&\qquad + \iint_{\Pi_T^2} q_\varepsilon(u, u_{\Delta x})\varphi_y + Q^u(u_{\Delta x}, S_{\Delta x}u_{\Delta x}) D_+\varphi\, dX \\
&\qquad + \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|_\varepsilon (\varphi_{yy} + (D_- + D_+)\varphi_y + D_-D_+\varphi)\, dX \\
&\qquad\quad \geq \iint_{\Pi_T^2} E_{\Delta x}^\varepsilon \varphi\, dX.
\end{aligned}$$

We want to take the limit as $\varepsilon \downarrow 0$. Consider the first term on the left. By the dominated convergence theorem, for any $a, b \in \mathbb{R}$,

$$\lim_{\varepsilon\downarrow 0} \psi_\varepsilon(a,b) = \lim_{\varepsilon\downarrow 0} \int_b^a \operatorname{sign}_\varepsilon(A(z) - A(b))\, dz = |a-b|\,,$$

since $A' > 0$. It follows that

$$\lim_{\varepsilon\downarrow 0} \psi_\varepsilon(u_{\Delta x}, u) = \lim_{\varepsilon\downarrow 0} \psi_\varepsilon(u, u_{\Delta x}) = |u_{\Delta x} - u|.$$

Furthermore,

$$(\varphi_t + \varphi_s)(x,t,y,s) = (\rho_\alpha(t-\nu) - \rho_\alpha(t-\tau))\omega_r(x-y)\rho_{r_0}(t-s),$$

so by the dominated convergence theorem

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} \psi_\varepsilon(u_{\Delta x}, u)\varphi_t + \psi_\varepsilon(u, u_{\Delta x})\varphi_s \, dX \\
&\quad = \iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\nu)\omega_r \rho_{r_0}\, dX - \iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\tau)\omega_r \rho_{r_0}\, dX.
\end{aligned}$$

Consider the second term on the left. By (3.8) we obtain

$$Q^u(u_{\Delta x}, S_{\Delta x} u_{\Delta x}) = q_\varepsilon(u_{\Delta x}, u) + \int_{u_{\Delta x}}^{S_{\Delta x} u_{\Delta x}} \operatorname{sign}_\varepsilon(A(z) - A(u)) F_2'(z)\, dz.$$

Since $A' > 0$,

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} q_\varepsilon(u_{\Delta x}, u) &= \lim_{\varepsilon\downarrow 0} \int_u^{u_{\Delta x}} \operatorname{sign}_\varepsilon(A(z) - A(u_{\Delta x})) f'(z)\, dz \\
&= \operatorname{sign}(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u)).
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} Q^u(u_{\Delta x}, S_{\Delta x} u_{\Delta x}) &= \operatorname{sign}(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u)) \\
&\qquad + \int_{u_{\Delta x}}^{S_{\Delta x} u_{\Delta x}} \operatorname{sign}(z-u)\, F_2'(z)\, dz.
\end{aligned}$$

As above

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} q_\varepsilon(u, u_{\Delta x}) &= \operatorname{sign}(u - u_{\Delta x})\,(f(u) - f(u_{\Delta x})) \\
&= \operatorname{sign}(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u)).
\end{aligned}$$

Hence, again by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} q_\varepsilon(u, u_{\Delta x})\varphi_y + Q^u(u_{\Delta x}, S_{\Delta x} u_{\Delta x}) D_+\varphi\, dX \\
&\quad = \iint_{\Pi_T^2} \operatorname{sign}(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))(\varphi_y + D_+\varphi)\, dX \\
&\qquad + \iint_{\Pi_T^2} \left( \int_{u_{\Delta x}}^{S_{\Delta x} u_{\Delta x}} \operatorname{sign}(z-u)\, F_2'(z)\, dz \right) D_+\varphi\, dX.
\end{aligned}$$

□

**Lemma 4.7.** *Let $E^\varepsilon_{\Delta x}$ and $\varphi$ be defined in Lemma* 4.6. *Then*

$$\liminf_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} E^\varepsilon_{\Delta x} \varphi\, dX \geq \int_{\Pi_T} \liminf_{\varepsilon\downarrow 0} \left( \int_{\Pi_T} E^\varepsilon_{\Delta x} \varphi\, dy ds \right) dx dt.$$

*Proof.* Let

$$f_\varepsilon(x,t) := \int_{\Pi_T} E^\varepsilon_{\Delta x}\varphi \, dyds$$

and

$$h_\varepsilon(x,t) := \int_{\Pi_T} \partial_y(D_- + D_+)\left|A(u_{\Delta x}) - A(u)\right|_\varepsilon \varphi \, dyds.$$

Recall that $E^\varepsilon_{\Delta x}(x,t,y,s) = E^\varepsilon_j(t,y,s)$ for $x \in I_j$, where $E^\varepsilon_j$ is defined in Lemma 4.5. Note that

$$\begin{aligned} E^\varepsilon_j \geq &- \frac{1}{\Delta x}\int_{u_{j-1}}^{u_j} \psi''_\varepsilon(z,u)\, dz\, \partial_y A(u) \\ &- \frac{1}{\Delta x}\int_{u_j}^{u_{j+1}} \psi''_\varepsilon(z,u)\, dz\, \partial_y A(u), \end{aligned}$$

so by Lemma 4.4 it follows that $f_\varepsilon \geq h_\varepsilon$. Using integration by parts and the triangle inequality we obtain the bound

$$|h_\varepsilon| \leq \left(|D_- A(u_{\Delta x})| + |D_+ A(u_{\Delta x})|\right)\left(\int_{\Pi_T} |\varphi_y| \; dyds\right) =: h.$$

It follows by Lemma 3.1 that $h$ is an integrable nonnegative function such that $-h \leq f_\varepsilon$. By Fatou's lemma we obtain

$$\liminf_{\varepsilon \downarrow 0} \int_{\Pi_T} f_\varepsilon \, dxdt \geq \int_{\Pi_T} \liminf_{\varepsilon\downarrow 0} f_\varepsilon \, dxdt. \qquad \square$$

Note that as $\varepsilon \downarrow 0$ the terms in $E^\varepsilon_j$ concentrate on the domains specified by $u \in \operatorname{int}(u_j, u_{j+1})$, $u \in \operatorname{int}(u_{j-1}, u_j)$, or $u = u_j$. In order to analyze this limit we will need the following elementary lemma:

**Lemma 4.8.** *Let $\{u_j\}_{j\in\mathbb{Z}}$ be some sequence in $\mathbb{R}$ and let $A : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuously differentiable function. For any $u \in \mathbb{R}$ there exist sequences $\{\tau_j^\pm\}_{j\in\mathbb{Z}}, \{\theta_j^\pm\}_{j\in\mathbb{Z}}$ such that for each $j \in \mathbb{Z}$ both $\tau_j^\pm$ and $\theta_j^\pm$ are in $\operatorname{int}(u_j, u_{j\pm1})$ and*

$$\begin{aligned} D_\pm \operatorname{sign}_\varepsilon(A(u_j) - A(u)) &= \operatorname{sign}'_\varepsilon(A(\tau_j^\pm) - A(u)) D_\pm A(u_j), \\ D_\pm |A(u_j) - A(u)|_\varepsilon &= \operatorname{sign}_\varepsilon(A(\theta_j^\pm) - A(u)) D_\pm A(u_j). \end{aligned}$$

*If $u$ is a differentiable function of $y$, then for each $j \in \mathbb{Z}$,*

$$\operatorname{sign}'_\varepsilon(A(\tau_j^\pm) - A(u))A(u)_y = -(\operatorname{sign}_\varepsilon(A(\theta_j^\pm) - A(u)))_y. \tag{4.3}$$

*Both $\{\tau_j^\pm\}_{j\in\mathbb{Z}}$ and $\{\theta_j^\pm\}_{j\in\mathbb{Z}}$ depend on $u$ and $\varepsilon$.*

*Proof.* The first statement is a direct consequence of the mean value theorem. Consider (4.3). First note that $\tau_j^- = \tau_{j-1}^+$ and $\theta_j^- = \theta_{j-1}^+$, so it suffices to consider $\tau_j^+$ and $\theta_j^+$. If $u_j = u_{j+1}$, then $\theta_j = \tau_j$ is independent of $u$ and hence of $y$, so (4.3) follows by the chain rule. In general,

$$\begin{aligned} \operatorname{sign}'_\varepsilon(A(\tau_j) - A(u))A(u)_y D_+ A(u_j) &= D_+ \operatorname{sign}_\varepsilon(A(u_j) - A(u))A(u)_y \\ &= -D_+(|A(u_j) - A(u)|_\varepsilon)_y \\ &= -\operatorname{sign}_\varepsilon(A(\theta_j) - A(u))_y D_+ A(u_j). \end{aligned}$$

In the case $u_j \neq u_{j+1}$ we have $D_+ A(u_j) \neq 0$ and (4.3) follows. $\square$

The following result is concerned with the pointwise limit of $\mathrm{sign}_\varepsilon(A(\theta_j^\pm)-A(u))$ as $\varepsilon\downarrow 0$. The explicit formula for this limit, which will be used later, shows that the limit is in fact a Lipschitz continuous function in the case that $A(u_j)\neq A(u_{j\pm1})$.

**Lemma 4.9.** *Let*

$$\mathrm{sg}_{(a,b)}(\sigma) := \begin{cases} \frac{|a-\sigma|-|b-\sigma|}{b-a} & \text{if } a\neq b,\\ \mathrm{sign}(a-\sigma) & \text{if } a=b,\ \sigma\neq b,\\ 0 & \text{if } a=b=\sigma, \end{cases}$$

*for any real numbers $a$ and $b$. Under the same assumptions as in Lemma* 4.8,

$$\lim_{\varepsilon\downarrow 0}\mathrm{sign}_\varepsilon(A(\theta_j^\pm)-A(u)) = -\mathrm{sg}_{(A(u_j),A(u_{j\pm1}))}(A(u)).$$

*Furthermore, if $a\neq b$, then*

$$\mathrm{sg}_{(a,b)}(\sigma) = \begin{cases} -1 & \text{if } \sigma\le \min\{a,b\},\\ \frac{2}{|a-b|}\left(\sigma-\frac{1}{2}(b+a)\right) & \text{if } \sigma\in \mathrm{int}(a,b),\\ 1 & \text{if } \sigma\ge\max\{a,b\}. \end{cases}$$

*Proof.* To prove the first statement we consider the case of $\theta_j^+$. The same argument applies to $\theta_j^-$. Recall the definition of $\theta_j^+$:

$$\mathrm{sign}_\varepsilon\left(A(\theta_j^+)-A(u)\right)\left(A(u_{j+1})-A(u_j)\right) = \left|A(u_{j+1})-A(u)\right|_\varepsilon - \left|A(u_j)-A(u)\right|_\varepsilon.$$

If $u_{j+1}=u_j$, then $\theta_j^+=u_j$ for all $u$ and $\varepsilon$, since $\theta_j^+\in\mathrm{int}(u_j,u_{j+1})$. Thus in this case

$$\lim_{\varepsilon\downarrow 0}\mathrm{sign}_\varepsilon\left(A(\theta_j^+)-A(u)\right) = \begin{cases} 0 & \text{if } u=u_j,\\ \mathrm{sign}\left(A(u_j)-A(u)\right) & \text{otherwise.}\end{cases}$$

Assume that $D_+A(u_j)\neq 0$. Then

$$\mathrm{sign}_\varepsilon(A(\theta_j^+)-A(u)) = \frac{\left|A(u_{j+1})-A(u)\right|_\varepsilon - \left|A(u_j)-A(u)\right|_\varepsilon}{A(u_{j+1})-A(u_j)},$$

and the result follows by letting $\varepsilon\downarrow 0$. Let us prove the second statement. First observe that all expressions are symmetric in $a$ and $b$, so we may assume that $a<b$. Under this assumption we have

$$\begin{aligned}(b-a)\mathrm{sg}_{(a,b)}(\sigma) &= |a-\sigma|-|b-\sigma|\\ &= \mathrm{sign}(a-\sigma)(a-\sigma)-\mathrm{sign}(b-\sigma)(b-\sigma)\\ &= \begin{cases}\mathrm{sign}(b-\sigma)(a-b) & \text{if } \sigma\notin(a,b),\\ 2\sigma-(b+a) & \text{if } \sigma\in(a,b).\end{cases}\end{aligned}$$

Dividing by $(b-a)$ concludes the proof. ∎

**Lemma 4.10.** *Let $E^\varepsilon_{\Delta x}$ and $\varphi$ be defined as in Lemma* 4.6. *For each $(x,t) \in \Pi_T$,*

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0} &\int_{\Pi_T} E^\varepsilon_{\Delta x}\varphi \, dyds \\
\geq &\int_{\Pi_T} D_- \Bigg( D_+ \operatorname{sign}(A(u_{\Delta x}) - A(u)) \\
&\qquad\qquad \times \left[\frac{1}{2}(A(u_{\Delta x}) + A(S_{\Delta x}u_{\Delta x})) - A(u)\right]\Bigg)\varphi \, dyds \\
&+ \liminf_{\varepsilon\downarrow 0} \frac{1}{2}\int_{\Pi_T} \left(\zeta^\varepsilon(u_{\Delta x}, \tau^-_{\Delta x}, u) + \zeta^\varepsilon(u_{\Delta x}, \tau^+_{\Delta x}, u)\right)(A(u)_y)^2\varphi \, dyds,
\end{aligned}$$

*where*

$$\zeta^\varepsilon(a,b,c) := \operatorname{sign}'_\varepsilon(A(a) - A(c)) - \operatorname{sign}'_\varepsilon(A(b) - A(c)), \qquad \forall a,b,c \in \mathbb{R}.$$

*Proof.* We split the proof into two claims.

*Claim* 1.

$$\begin{aligned}
E^\varepsilon_j \geq &\frac{1}{2(\Delta x)^2}\int_{u_{j+1}}^{u_j} \zeta^\varepsilon(z, \tau^+_j, u)\partial_z(A(z) - A(u_{j+1}))^2 \, dz \\
&+ \frac{1}{2(\Delta x)^2}\int_{u_{j-1}}^{u_j} \zeta^\varepsilon(z, \tau^-_j, u)\partial_z(A(z) - A(u_{j-1}))^2 \, dz \\
&+ \frac{1}{2}\left(\zeta^\varepsilon(u_j, \tau^-_j, u) + \zeta^\varepsilon(u_j, \tau^+_j, u)\right)(A(u)_y)^2.
\end{aligned}$$

*Proof of Claim* 1. Let

$$\begin{aligned}
T^- := &\frac{1}{(\Delta x)^2}\int_{u_{j-1}}^{u_j} \psi''_\varepsilon(z,u)(A(z) - A(u_{j-1})) \, dz \\
&- \frac{1}{\Delta x}\int_{u_{j-1}}^{u_j} \psi''_\varepsilon(z,u)\, dz A(u)_y + \frac{1}{2}\partial_y \psi'_\varepsilon(u, u_j)A(u)_y.
\end{aligned}$$

We start by rewriting the first term as follows:

$$\begin{aligned}
\frac{1}{(\Delta x)^2}&\int_{u_{j-1}}^{u_j} \psi''_\varepsilon(z,u)(A(z) - A(u_{j-1})) \, dz \\
&= \frac{1}{2(\Delta x)^2}\int_{u_{j-1}}^{u_j} \operatorname{sign}'_\varepsilon(A(z) - A(u))\partial_z(A(z) - A(u_{j-1}))^2 \, dz \\
&= \frac{1}{2(\Delta x)^2}\int_{u_{j-1}}^{u_j} \operatorname{sign}'_\varepsilon(A(\tau^-_j) - A(u))\partial_z(A(z) - A(u_{j-1}))^2 \, dz \\
&\quad + \frac{1}{2(\Delta x)^2}\int_{u_{j-1}}^{u_j} \zeta^\varepsilon(z, \tau^-_j, u)\partial_z(A(z) - A(u_{j-1}))^2 \, dz \\
&= \frac{1}{2}\operatorname{sign}'_\varepsilon(A(\tau^-_j) - A(u))(D_-A(u_j))^2 \\
&\quad + \frac{1}{2(\Delta x)^2}\int_{u_{j-1}}^{u_j} \zeta^\varepsilon(z, \tau^-_j, u)\partial_z(A(z) - A(u_{j-1}))^2 \, dz.
\end{aligned}$$

Concerning the second term in the definition of $T^-$, Lemma 4.8 gives

$$-\frac{1}{\Delta x}\int_{u_{j-1}}^{u_j} \psi''_\varepsilon(z,u)\,dz A(u)_y = -D_-\mathrm{sign}_\varepsilon(A(u_j)-A(u))A(u)_y$$
$$= -\mathrm{sign}'_\varepsilon(A(\tau_j^-)-A(u))D_-A(u_j)A(u)_y.$$

For the last term we simply add and subtract to obtain

$$\frac{1}{2}\partial_y\psi'_\varepsilon(u,u_j)A(u)_y = \frac{1}{2}\mathrm{sign}'_\varepsilon(A(\tau_j^-)-A(u))(A(u)_y)^2 + \frac{1}{2}\zeta^\varepsilon(u_j,\tau_j^-,u)(A(u)_y)^2.$$

Hence,

$$\begin{aligned} T^- &= \frac{1}{2}\mathrm{sign}'_\varepsilon(A(\tau_j^-)-A(u))(D_-A(u_j)-A(u)_y)^2 \\ &\quad + \frac{1}{2(\Delta x)^2}\int_{u_{j-1}}^{u_j}\zeta^\varepsilon(z,\tau_j^-,u)\partial_z(A(z)-A(u_{j-1}))^2\,dz \\ &\quad + \frac{1}{2}\zeta^\varepsilon(u_j,\tau_j^-,u)(A(u)_y)^2. \end{aligned}$$

Define

$$\begin{aligned} T^+ :=& \frac{1}{(\Delta x)^2}\int_{u_{j+1}}^{u_j}\psi''_\varepsilon(z,u)(A(z)-A(u_{j+1}))\,dz \\ &- \frac{1}{\Delta x}\int_{u_{j+1}}^{u_j}\psi''_\varepsilon(z,u)\,dz A(u)_y + \frac{1}{2}\partial_y\psi'_\varepsilon(u,u_j)A(u)_y. \end{aligned}$$

Using the same strategy as above we arrive at

$$\begin{aligned} T^+ &= \frac{1}{2}\mathrm{sign}'_\varepsilon(A(\tau_j^+)-A(u))\left(D_+A(u_j)-A(u)_y\right)^2 \\ &\quad + \frac{1}{2(\Delta x)^2}\int_{u_{j+1}}^{u_j}\zeta^\varepsilon(z,\tau_j^+,u)\partial_z(A(z)-A(u_{j+1}))^2\,dz \\ &\quad + \frac{1}{2}\zeta^\varepsilon(u_j,\tau_j^+,u)(A(u)_y)^2. \end{aligned}$$

Note that $E_j^\varepsilon = T^- + T^+$, so Claim 1 follows by removing the nonnegative terms on the righthand side. □

*Claim* 2. Suppose that $x \in I_j$. Then

(4.4)
$$\begin{aligned} \liminf_{\varepsilon\downarrow 0}\frac{1}{2(\Delta x)^2}\int_{\Pi_T}\Bigg[&\int_{u_{j-1}}^{u_j}\zeta^\varepsilon(z,\tau_j^-,u)\frac{d}{dz}(A(z)-A(u_{j-1}))^2\,dz \\ &+ \int_{u_{j+1}}^{u_j}\zeta^\varepsilon(z,\tau_j^+,u)\frac{d}{dz}(A(z)-A(u_{j+1}))^2\,dz\Bigg]\varphi\,dyds \\ = \int_{\Pi_T} D_-\Bigg(D_+&\mathrm{sign}(A(u_j)-A(u))\left[\frac{1}{2}(A(u_j)+A(u_{j+1}))-A(u)\right]\Bigg)\varphi\,dyds. \end{aligned}$$

*Proof of Claim* 2. Let

$$
\begin{aligned}
T_1^\varepsilon &:= \frac{1}{2(\Delta x)^2} \int_{u_{j-1}}^{u_j} \operatorname{sign}_\varepsilon'(A(z) - A(u)) \frac{d}{dz} (A(z) - A(u_{j-1}))^2 \, dz, \\
K_1^\varepsilon &:= \frac{1}{2(\Delta x)^2} \int_{u_{j-1}}^{u_j} \operatorname{sign}_\varepsilon'(A(\tau_j^-) - A(u)) \frac{d}{dz} (A(z) - A(u_{j-1}))^2 \, dz, \\
T_2^\varepsilon &:= \frac{1}{2(\Delta x)^2} \int_{u_{j+1}}^{u_j} \operatorname{sign}_\varepsilon'(A(z) - A(u)) \frac{d}{dz} (A(z) - A(u_{j+1}))^2 \, dz, \\
K_2^\varepsilon &:= \frac{1}{2(\Delta x)^2} \int_{u_{j+1}}^{u_j} \operatorname{sign}_\varepsilon'(A(\tau_j^+) - A(u)) \frac{d}{dz} (A(z) - A(u_{j+1}))^2 \, dz,
\end{aligned}
$$

and note that the left-hand side of (4.4) may be written as

$$
\liminf_{\varepsilon \downarrow 0} \int_{\Pi_T} \Big( (T_1^\varepsilon - K_1^\varepsilon) + (T_2^\varepsilon - K_2^\varepsilon) \Big) \varphi \, dy ds.
$$

Let us rewrite $T_1^\varepsilon$ as follows:

$$
\begin{aligned}
T_1^\varepsilon &= \frac{1}{(\Delta x)^2} \int_{u_{j-1}}^{u_j} \operatorname{sign}_\varepsilon'(A(z) - A(u)) A'(z) (A(z) - A(u_{j-1})) \, dz \\
&= \frac{1}{(\Delta x)^2} \int_{u_{j-1}}^{u_j} \operatorname{sign}_\varepsilon'(A(z) - A(u)) A'(z) (A(u) - A(u_{j-1})) \, dz \\
&\qquad + \frac{1}{(\Delta x)^2} \int_{u_{j-1}}^{u_j} \operatorname{sign}_\varepsilon'(A(z) - A(u)) A'(z) (A(z) - A(u)) \, dz \\
&= D_- \operatorname{sign}_\varepsilon(A(u_j) - A(u)) \frac{(A(u) - A(u_{j-1}))}{\Delta x} + R_1^\varepsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
R_1^\varepsilon := \frac{1}{\Delta x^2} \Bigg[ & \operatorname{sign}_\varepsilon(A(z) - A(u))(A(z) - A(u)) \Big|_{z = u_{j-1}}^{z = u_j} \\
& - \int_{u_{j-1}}^{u_j} \frac{d}{dz} |A(z) - A(u)|_\varepsilon \, dz \Bigg].
\end{aligned}
$$

Concerning $K_1^\varepsilon$, we apply Lemma 4.8 to obtain

$$
\begin{aligned}
K_1^\varepsilon &= \frac{1}{2(\Delta x)^2} \int_{u_{j-1}}^{u_j} \operatorname{sign}_\varepsilon'(A(\tau_j^-) - A(u)) \partial_z (A(z) - A(u_{j-1}))^2 \, dz \\
&= \frac{1}{2} \operatorname{sign}_\varepsilon'(A(\tau_j^-) - A(u)) (D_- A(u_j))^2 \, dz \\
&= \frac{1}{2} D_- \operatorname{sign}_\varepsilon(A(u_j) - A(u)) D_- A(u_j).
\end{aligned}
$$

It now follows that

$$
T_1^\varepsilon - K_1^\varepsilon = -\frac{1}{\Delta x} D_- \operatorname{sign}_\varepsilon(A(u_j) - A(u)) \left[ \frac{1}{2} \left( A(u_j) + A(u_{j-1}) \right) - A(u) \right] + R_1^\varepsilon.
$$

Performing the same type of computations as above yields

$$
T_2^\varepsilon - K_2^\varepsilon = \frac{1}{\Delta x} D_+ \operatorname{sign}_\varepsilon(A(u_j) - A(u)) \left[ \frac{1}{2} \left( A(u_{j+1}) + A(u_j) \right) - A(u) \right] + R_2^\varepsilon,
$$

where

$$R_2^\varepsilon := \frac{1}{\Delta x^2}\left[\int_{u_j}^{u_{j+1}} \frac{d}{dz}\left|A(z)-A(u)\right|_\varepsilon dz \right.$$
$$\left. - \operatorname{sign}_\varepsilon(A(z)-A(u))(A(z)-A(u))\Big|_{z=u_j}^{z=u_{j+1}}\right].$$

Next, observe that

$$R_1^\varepsilon = \frac{1}{\Delta x^2}\left[\operatorname{sign}_\varepsilon(A(z)-A(u))(A(z)-A(u)) - \left|A(z)-A(u)\right|_\varepsilon\right]_{z=u_{j-1}}^{z=u_j},$$

so $\lim_{\varepsilon\downarrow 0} R_1^\varepsilon = 0$. The same considerations apply to $R_2^\varepsilon$ so $\lim_{\varepsilon\downarrow 0} R_2^\varepsilon = 0$ also. Claim 2 follows from an application of the dominated convergence theorem. Finally, combining Claim 1 and Claim 2 finishes the proof. $\square$

4.1. **Estimates.** The purpose of this section is to find bounds on the "unwanted" terms in inequality (4.2) and Lemma 4.10. Throughout this section the notation is the one given in Lemma 4.6. We let $C$ denote a generic constant. By constant it is meant that it does not depend on the "small" variables but it might depend on $T$ and the initial data. For any set $A$, let $\mathbb{1}_A$ denote its characteristic function.

For future reference we collect some standard estimates in a lemma.

**Lemma 4.11.** *Let $\varphi$ be defined in Lemma 4.6. Then*

$$\left|\frac{\partial^k}{\partial x^k}\varphi(x,t,y,s)\right| \le \psi(t)\frac{\left\|\rho^{(k)}\right\|_{L^\infty}}{r^{k+1}}\mathbb{1}_{\{|x-y|\le r\}}(x,y)\rho_{r_0}(t-s).$$

*Recall that $S_\sigma\varphi(x,t,y,s) = \varphi(x+\sigma,t,y,s)$. If $|\sigma|\le\Delta x$, then*

$$\left|\frac{\partial^k}{\partial x^k}S_\sigma\varphi(x,t,y,s)\right| \le \psi(t)\frac{\left\|\rho^{(k)}\right\|_{L^\infty}}{r^{k+1}}\mathbb{1}_{\{|x-y|\le r+\Delta x\}}(x,y)\rho_{r_0}(t-s).$$

*Considering the difference quotient applied to $\omega_r$ we have*

$$|D_+\omega_r(x-y)| \le \frac{\|\rho'\|_{L^\infty}}{r^2}\mathbb{1}_{\{|x-y|\le r+\Delta x\}}(x,y).$$

*Proof.* Note that

$$\frac{\partial^k}{\partial x^k}\omega_r(x) = \frac{1}{r^{k+1}}\rho^{(k)}\left(\frac{x}{r}\right).$$

Since $\operatorname{supp}(\rho)\subset[-1,1]$ we have

$$\left|\frac{\partial^k}{\partial x^k}\omega_r(x)\right| \le \frac{\left\|\rho^{(k)}\right\|_{L^\infty}}{r^{k+1}}\mathbb{1}_{\{|x|\le r\}}(x),$$

which proves the first statement.

Consider the second statement. If $|x-y|\ge r+\Delta x$, then

$$|x+\sigma-y| \ge |x-y|-|\sigma| \ge r+\Delta x-\Delta x = r,$$

so it follows that $\mathbb{1}_{\{|x+\sigma-y|\le r\}}(x,y)\le \mathbb{1}_{\{|x-y|\le r+\Delta x\}}(x,y)$; this proves the second statement.

To prove the last statement, recall that

$$D_+\omega_r(x) = \frac{\omega_r(x+\Delta x)-\omega_r(x)}{\Delta x}.$$

If $|x| \geq r+\Delta x$, then $\omega_r(x+\Delta x) = \omega_r(x) = 0$, so $\mathrm{supp}(D_+(\omega_r)) \subset [-r-\Delta x, r+\Delta x]$. By the mean value theorem and the fact that $\|\omega_r'\|_{L^\infty} = \|\rho'\|_{L^\infty} r^{-2}$ we get

$$|\omega_r(x+\Delta x) - \omega_r(x)| \leq \frac{\|\rho'\|_{L^\infty}}{r^2}\Delta x.$$

The last statement follows from this. ☐

**Estimate 4.1.**

$$\left| \iint_{\Pi_T^2} \mathrm{sign}\,(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))\,(D_+\varphi + \varphi_y)\; dX \right| \leq C\frac{\Delta x}{r}\left(1 + \frac{\Delta x}{r}\right).$$

*Proof.* Let

$$\beta := \iint_{\Pi_T^2} \mathrm{sign}\,(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))\,(D_+\varphi + \varphi_y)\; dX.$$

First note that

$$D_+\varphi + \varphi_y = D_+\varphi - \varphi_x.$$

We claim that

$$(D_+\varphi - \varphi_x)\,(x,t,y,s) = \frac{1}{\Delta x}\int_0^{\Delta x} (\Delta x - \sigma)\varphi_{xx}(x+\sigma, t, y, s)\, d\sigma. \tag{4.5}$$

Hence

$$\beta = \frac{1}{\Delta x}\iint_{\Pi_T^2}\int_0^{\Delta x} \mathrm{sign}_\varepsilon\,(A(u_{\Delta x}) - A(u))\,(f(u_{\Delta x}) - f(u))\,(\Delta x - \sigma)S_\sigma\varphi_{xx}\, d\sigma\, dX.$$

We can write

$$\begin{aligned}
&\mathrm{sign}\,(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))\,(x,t,y,s) \\
&\quad = \sum_j \underbrace{\mathrm{sign}\,(u_j - u)\,(f(u_j) - f(u))\,(t,y,s)}_{\Theta_j}\, \mathbb{1}_{\{I_j\}}(x).
\end{aligned}$$

Using summation by parts we get

$$\begin{aligned}
\frac{1}{\Delta x}&\int_{\mathbb{R}}\int_0^{\Delta x} \mathrm{sign}\,(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))\,(\Delta x - \sigma)S_\sigma\varphi_{xx}\, d\sigma dx \\
&= \frac{1}{\Delta x}\int_0^{\Delta x}\sum_j \Theta_j \int_{\mathbb{R}} \mathbb{1}_{\{I_j\}}(x)(\Delta x - \sigma)S_\sigma\varphi_{xx}\, dx d\sigma \\
&= \frac{1}{\Delta x}\int_0^{\Delta x}\sum_j \Theta_j \int_{I_j} \varphi_{xx}(x+\sigma, t, y, s)\, dx(\Delta x - \sigma)\, d\sigma \\
&= \int_0^{\Delta x}\sum_j \Theta_j \left(D_- S_\sigma\varphi_{x,j+1/2}\right)(\Delta x - \sigma)\, d\sigma \\
&= -\sum_j D_+\Theta_j \int_0^{\Delta x} S_\sigma\varphi_{x,j+1/2}(\Delta x - \sigma)\, d\sigma,
\end{aligned}$$

where $S_\sigma\varphi_{x,j+1/2} = \varphi_x(x_{j+1/2} + \sigma, t, y, s)$. By Lemma 4.11 we have

$$|\varphi_x(x+\sigma, t, y, s)| \leq C\frac{1}{r^2}\mathbb{1}_{\{|x-y| \leq r+\Delta x\}}(x,y)\rho_{r_0}(t-s).$$

Hence,

$$\Big|\int_0^{\Delta x} S_\sigma \varphi_{x,j+1/2}(\Delta x-\sigma)\,d\sigma\Big| \le C\frac{\Delta x^2}{r^2}\,\mathbb{1}_{\left\{|x_{j+1/2}-y|\le r+\Delta x\right\}}(y)\rho_{r_0}(t-s).$$

Now

$$|D_+\Theta_j| \le \|f\|_{\mathrm{Lip}}\,|D_+u_j|\,.$$

Therefore,

$$\begin{aligned}
&\Big|\frac{1}{\Delta x}\int_{\mathbb{R}}\int_0^{\Delta x}\operatorname{sign}(u_{\Delta x}-u)\,(f(u_{\Delta x})-f(u))\,(\Delta x-\sigma)\varphi^\sigma_{xx}\,d\sigma dx\Big| \\
&\qquad\le \sum_j |D_+\Theta_j|\,\Big|\int_0^{\Delta x} S_\sigma\varphi_{x,j+1/2}(\Delta x-\sigma)\,d\sigma\Big| \\
&\qquad\le C\,\|f\|_{\mathrm{Lip}}\sum_j |D_+u_j|\,\frac{\Delta x^2}{r^2}\,\mathbb{1}_{\left\{|x_{j+1/2}-y|\le r+\Delta x\right\}}(y)\rho_{r_0}(t-s).
\end{aligned}$$

It follows by the above and Lemma 3.1 that

$$\begin{aligned}
|\beta| &\le C\Delta x^2\frac{r+\Delta x}{r^2}\int_0^T\sum_j |D_+u_j|\;dt \\
&= C\frac{r+\Delta x}{r^2}\int_{\Pi_T}|u_{\Delta x}(x+\Delta x,t)-u_{\Delta x}(x,t)|\;dxdt \\
&= CT\frac{1}{r}\left(1+\frac{\Delta x}{r}\right)\Delta x\left|u^0_{\Delta x}\right|_{BV(\mathbb{R})}.
\end{aligned}$$

This concludes the proof. $\square$

**Estimate 4.2.**

$$\begin{aligned}
&\Big|\iint_{\Pi_T^2}|A(u_{\Delta x})-A(u)|\left(D_-D_+\varphi+(D_++D_-)\varphi_y+\varphi_{yy}\right)\,dX\Big| \\
&\qquad\le C\frac{\Delta x^2}{r^3}\left(1+\frac{\Delta x}{r}\right).
\end{aligned}$$

*Proof.* Since $\varphi_{xx}+2\varphi_{xy}+\varphi_{yy}=0$ it follows that

$$D_-D_+\varphi+(D_++D_-)\varphi_y+\varphi_{yy} = (D_-D_+\varphi-\varphi_{xx})+\left((D_++D_-)\varphi-2\varphi_x\right)_y.$$

Thus

$$\begin{aligned}
&\iint_{\Pi_T^2}|A(u_{\Delta x})-A(u)|\left(D_-D_+\varphi+(D_++D_-)\varphi_y+\varphi_{yy}\right)\,dX \\
&\qquad= \iint_{\Pi_T^2}|A(u_{\Delta x})-A(u)|\left(D_-D_+\varphi-\varphi_{xx}\right)\,dX \\
&\qquad\quad+\iint_{\Pi_T^2}|A(u_{\Delta x})-A(u)|\left((D_++D_-)\varphi-2\varphi_x\right)_y\,dX \\
&\qquad=: \zeta_1+\zeta_2.
\end{aligned}$$

Consider the term $\zeta_1$. We use the same strategy as in Estimate 4.1. Writing $\mu(\sigma)=\varphi(x+\sigma,t,y,s)$, a Taylor expansion gives

$$\mu(z)-\mu(0) = z\mu'(0)+\frac{1}{2}z^2\mu''(0)+\frac{1}{6}z^3\mu^{(3)}(0)-\frac{1}{6}\int_0^z(\sigma-z)^3\mu^{(4)}(\sigma)\,d\sigma.$$

Using this, we get

$$\begin{aligned}&\mu(\Delta x) - 2\mu(0) + \mu(-\Delta x) - \Delta x^2 \mu''(0)\\&\qquad = -\frac{1}{6}\int_0^{\Delta x} (\sigma - \Delta x)^3 \mu^{(4)}(\sigma)\,d\sigma + \frac{1}{6}\int_{-\Delta x}^0 (\sigma + \Delta x)^3 \mu^{(4)}(\sigma)\,d\sigma.\end{aligned}$$

It follows that

$$\begin{aligned}D_+D_-\varphi - \varphi_{xx} = &-\frac{1}{6\Delta x^2}\int_0^{\Delta x} (\sigma - \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\,d\sigma\\&+ \frac{1}{6\Delta x^2}\int_{-\Delta x}^0 (\sigma + \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\,d\sigma.\end{aligned}$$

Splitting $\zeta_1$ according to this equality we get

$$\begin{aligned}\zeta_1 &= \iint_{\Pi_T^2} \left|A(u_{\Delta x}) - A(u)\right| \left(D_-D_+\varphi - \varphi_{xx}\right)\,dX\\&= -\frac{1}{6\Delta x^2}\iint_{\Pi_T^2}\int_0^{\Delta x} \left|A(u_{\Delta x}) - A(u)\right| (\sigma - \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\,d\sigma\,dX\\&\quad + \frac{1}{6\Delta x^2}\iint_{\Pi_T^2}\int_{-\Delta x}^0 \left|A(u_{\Delta x}) - A(u)\right| (\sigma + \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\,d\sigma\,dX\\&=: \zeta_{1,1} + \zeta_{1,2}.\end{aligned}$$

We also have that

$$\left|A(u_{\Delta x}) - A(u)\right|(x, t, y, s) = \sum_j \underbrace{\left|A(u_j) - A(u)\right|(t, y, s)}_{\Phi_j}\,\mathbb{1}_{\{I_j\}}(x).$$

Consider $\zeta_{1,1}$,

$$\begin{aligned}&-\int_0^{\Delta x}\int_{\mathbb{R}} \left|A(u_{\Delta x}) - A(u)\right| (\sigma - \Delta x)^3 \frac{\partial^4}{\partial x^4} S_\sigma\varphi\,dx d\sigma\\&\qquad = -\sum_j \left|A(u_j) - A(u)\right|(t, y, s)\int_0^{\Delta x} (\sigma - \Delta x)^3 \int_{\mathbb{R}} \mathbb{1}_{\{I_j\}}(x)\frac{\partial^4}{\partial x^4} S_\sigma\varphi\,dx d\sigma\\&\qquad = -\Delta x\int_0^{\Delta x} (\sigma - \Delta x)^3 \sum_j \Phi_j D_-\varphi^\sigma_{xxx,j+1/2}\,d\sigma\\&\qquad = \Delta x\sum_j D_+\Phi_j \int_0^{\Delta x} (\sigma - \Delta x)^3 S_\sigma\varphi_{xxx,j+1/2}\,d\sigma,\end{aligned}$$

where

$$S_\sigma\varphi_{xxx,j+1/2}(t, y, s) = \frac{\partial^3}{\partial x^3}\varphi(x_{j+1/2} + \sigma, t, y, s).$$

Now we use Lemma 4.11 to estimate this term as follows:

$$
\begin{aligned}
|\zeta_{1,1}| &= \Big|\frac{1}{6\Delta x^2}\iint_{\Pi_T^2}\int_0^{\Delta x} |A(u_{\Delta x})-A(u)|\,(\sigma-\Delta x)^3\frac{\partial^4}{\partial x^4}\varphi(x+\sigma,t,y,s)\,d\sigma\,dX\Big| \\
&= \Big|\frac{1}{6\Delta x}\int_{\Pi_T}\int_0^T\sum_j D_+\Phi_j\int_0^{\Delta x}(\sigma-\Delta x)^3 S_\sigma\varphi_{xxx,j+1/2}\,d\sigma\;dt\,dyds\Big| \\
&\le C\frac{r+\Delta x}{\Delta x^2 r^4}\int_{\Pi_T}|D_+A(u_{\Delta x})|\Big(\int_0^{\Delta x}(\sigma-\Delta x)^3\,d\sigma\Big)\,dxdt \\
&\le C\Delta x^2\frac{r+\Delta x}{r^4} = C\frac{\Delta x^2}{r^3}\left(1+\frac{\Delta x}{r}\right),
\end{aligned}
$$

where we have used that $|A(u_{\Delta x}(\cdot,t))|_{BV(\mathbb{R})}$ is bounded independently of $\Delta x, t, \eta$ by Lemma 3.1. The term $\zeta_{1,2}$ is estimated in a similar way.

Consider $\zeta_2$. Again, let $\mu(\sigma)=\varphi(x+\sigma,t,y,s)$. Then

$$(D_+ + D_-)\varphi - 2\varphi_x = \frac{1}{\Delta x}\left[\mu(\Delta x)-\mu(-\Delta x)-2\Delta x\mu'(0)\right].$$

By a Taylor expansion

$$\mu(z)-\mu(0) = z\mu'(0)+\frac{1}{2}z^2\mu''(0)+\frac{1}{2}\int_0^z(\sigma-z)^2\mu^{(3)}(\sigma)\,d\sigma.$$

Writing $z=\pm\Delta x$ and subtracting the corresponding equations we obtain

$$
\begin{aligned}
(D_+ + D_-)\varphi - 2\varphi_x = \frac{1}{2\Delta x}\int_0^{\Delta x}(\sigma-\Delta x)^2\frac{\partial^3}{\partial x^3}\varphi(x+\sigma,t,y,s)\,d\sigma \\
+\frac{1}{2\Delta x}\int_{-\Delta x}^0(\sigma+\Delta x)^2\frac{\partial^3}{\partial x^3}\varphi(x+\sigma,t,y,s)\,d\sigma.
\end{aligned}
$$

We may split $\zeta_2$ into the two terms

$$
\begin{aligned}
\zeta_2 &= \frac{1}{2\Delta x}\iint_{\Pi_T^2}\int_0^{\Delta x}|A(u_{\Delta x})-A(u)|\,(\sigma-\Delta x)^2\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial y}\varphi(x+\sigma,t,y,s)\,d\sigma\,dX \\
&\quad+\frac{1}{2\Delta x}\iint_{\Pi_T^2}\int_{-\Delta x}^0|A(u_{\Delta x})-A(u)|\,(\sigma+\Delta x)^2\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial y}\varphi(x+\sigma,t,y,s)\,d\sigma\,dX \\
&=: \zeta_{2,1}+\zeta_{2,2}.
\end{aligned}
$$

Performing integration by parts, $\zeta_{2,1}$ becomes

$$\frac{1}{2\Delta x}\iint_{\Pi_T^2}\int_0^{\Delta x}\operatorname{sign}\left(A(u_{\Delta x})-A(u)\right)A(u)_y(\sigma-\Delta x)^2\frac{\partial^3}{\partial x^3}\varphi(x+\sigma,t,y,s)\,d\sigma\,dX.$$

Thus, by Lemma 4.11,

$$
\begin{aligned}
|\zeta_{2,1}| &\le \frac{1}{2\Delta x}\iint_{\Pi_T^2}|A(u)_y|\left|\int_0^{\Delta x}(\sigma-\Delta x)^2\frac{\partial^3}{\partial x^3}\varphi(x+\sigma,t,y,s)\,d\sigma\right|\,dX \\
&\le TC\frac{r+\Delta x}{r^4\Delta x}\left(\int_0^{\Delta x}(\sigma-\Delta x)^2\,d\sigma\right)\int_{\Pi_T}|A(u)_y|\,dyds \\
&\le C\frac{\Delta x^2}{r^3}\left(1+\frac{\Delta x}{r}\right),
\end{aligned}
$$

as $|A(u(\cdot,s))|_{BV(\mathbb{R})}\le\left|A(u^0(\cdot))\right|_{BV(\mathbb{R})}$ for all $s$. The same estimate holds for $\zeta_{2,2}$. $\square$

**Estimate 4.3.**

$$\left|\iint_{\Pi_T^2}\left(\int_{u_{\Delta x}}^{S_{\Delta x}u_{\Delta x}}\operatorname{sign}(z-u)\,F_2'(z)\,dz\right)D_+\varphi\,dX\right|\le C\frac{\Delta x}{r}\left(1+\frac{\Delta x}{r}\right).$$

*Proof.* By definition $F_2'$ is bounded. Hence,

$$\left|\int_{u_j}^{u_{j+1}}\operatorname{sign}(z-u)\,F_2'(z)\,dz\right|\le\|F_2\|_{\mathrm{Lip}}\,\Delta x\,|D_+u_j|\,.$$

Note that $|u_{\Delta x}(\cdot,t)|_{BV(\mathbb{R})}$ is bounded independently of $\Delta x,t,\eta$ by Lemma 3.1, so we may apply Lemma 4.11 to obtain the result. □

Next, we consider the terms from Lemma 4.10.

**Estimate 4.4.**

$$\iint_{\Pi_T^2}D_-\left(D_+\operatorname{sign}(A(u_j)-A(u))\left[\frac{1}{2}(A(u_j)+A(u_{j+1}))-A(u)\right]\right)\varphi\,dX$$
$$\ge -C(1+r+\Delta x)\frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right)^3.$$

*Proof.* Let us first show that

$$\left|D_+\operatorname{sign}(A(u_j)-A(u))\left[\frac{1}{2}(A(u_j)+A(u_{j+1}))-A(u)\right]\right|\le\Delta x D_+\operatorname{sign}(A(u_j)-A(u))D_+(A(u_j)).\tag{4.6}$$

First note that

$$D_+\operatorname{sign}(A(u_j)-A(u))=\frac{2}{\Delta x}\operatorname{sign}(A(u_j)-A(u_{j+1}))\,\mathbb{1}_{\{A(u)\in\operatorname{int}(A(u_j),A(u_{j+1}))\}},$$

so the left-hand side of (4.6) is zero whenever $A(u)\notin\operatorname{int}(A(u_j),A(u_{j+1}))$. Second, if $c\in\operatorname{int}(a,b)$, then it follows that

$$\left|\frac{1}{2}(a+b)-c\right|=\frac{1}{2}(|b-c|+|a-c|)\le|b-a|.$$

Since $z\mapsto\operatorname{sign}(A(z)-A(u))$ is increasing, the right-hand side is positive. This proves (4.6).

Performing integration by parts we obtain

$$\begin{aligned}&\left|\iint_{\Pi_T^2}D_-\left(D_+\operatorname{sign}(A(u_j)-A(u))\left[\frac{1}{2}(A(u_j)+A(u_{j+1}))-A(u)\right]\right)\varphi\,dX\right|\\&\quad\le\iint_{\Pi_T^2}\left|D_+\operatorname{sign}(A(u_j)-A(u))\left[\frac{1}{2}(A(u_j)+A(u_{j+1}))-A(u)\right]\right|\,|D_+\varphi|\,dX\\&\quad\le\Delta x\iint_{\Pi_T^2}D_+\operatorname{sign}(A(u_j)-A(u))D_+(A(u_j))\,|D_+\varphi|\,dX.\end{aligned}$$

Using integration by parts for difference quotients and the Leibniz rule for difference quotients, we obtain

$$\begin{aligned}
&\iint_{\Pi_T^2} D_+\operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u))D_+A(u_{\Delta x})\left|D_+\varphi\right| \, dX \\
&\quad = -\iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u))D_+A(u_{\Delta x})D_-\left|D_+\varphi\right| \, dX \\
&\qquad - \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u))D_-D_+A(u_{\Delta x})\left|D_-\varphi\right| \, dX \\
&\quad =: \zeta_1 + \zeta_2.
\end{aligned}$$

To estimate $\zeta_1$ we first observe that $D_-\left|D_+\varphi\right| \leq \left|D_+D_-\varphi\right|$. Furthermore, when proving Estimate 4.2, we established that

$$\begin{aligned}
&D_+D_-\varphi(x,t,y,s) \\
&\quad = \varphi_{xx}(x,t,y,s) - \frac{1}{6\Delta x^2}\int_0^{\Delta x}(\sigma - \Delta x)^3\frac{\partial^4}{\partial x^4}\varphi(x+\sigma,t,y,s)\, d\sigma \\
&\qquad + \frac{1}{6\Delta x^2}\int_{-\Delta x}^{0}(\sigma + \Delta x)^3\frac{\partial^4}{\partial x^4}\varphi(x+\sigma,t,y,s)\, d\sigma.
\end{aligned}$$

By Lemma 4.11,

$$\begin{aligned}
&\Big|\int_0^{\pm\Delta x}(\sigma \mp \Delta x)^3\frac{\partial^4}{\partial x^4}\varphi(x+\sigma,t,y,s)\, d\sigma\Big| \\
&\qquad \leq C\frac{(\Delta x)^4}{r^5}\mathbb{1}_{\{|x-y|\leq r+\Delta x\}}(x,y)\rho_{r_0}(t-s).
\end{aligned}$$

Using Lemma 4.11 once more, the above implies that

$$\begin{aligned}
\int_{\Pi_T}\left|D_+D_-\varphi\right| \, dyds &\leq \int_{\Pi_T}\left|\varphi_{xx}\right| \, dyds + C\frac{\Delta x^2}{r^4}\left(1+\frac{\Delta x}{r}\right) \\
&\leq C\left(\frac{1}{r^2}+\frac{\Delta x^2}{r^4}\right)\left(1+\frac{\Delta x}{r}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\zeta_1| &= \Big|\iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u))D_+A(u_{\Delta x})D_+\left|D_-\varphi\right| \, dX\Big| \\
&\leq \int_{\Pi_T}\left|D_+A(u_{\Delta x})\right|\Big(\int_{\Pi_T}\left|D_+D_-\varphi\right| \, dyds\Big)\, dxdt \\
&\leq C\left(\frac{1}{r^2}+\frac{\Delta x^2}{r^4}\right)\left(1+\frac{\Delta x}{r}\right)\int_{\Pi_T}\left|D_+A(u_{\Delta x})\right| \, dxdt.
\end{aligned}$$

Recall that $\left|A(u_{\Delta x}(\cdot,t))\right|_{BV(\mathbb{R})}$ is bounded independently of $\Delta x, t, \eta$ by Lemma 3.1.

Concerning $\zeta_2$ we have

$$\begin{aligned}|\zeta_2| &= \left|\iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u))\,(D_-D_+A(u_{\Delta x}))\,|D_-\varphi|\,dX\right| \\ &\le \iint_{\Pi_T^2} |D_-D_+A(u_{\Delta x})|\,|D_-\varphi|\,dX \\ &\le C\frac{r+\Delta x}{r^2}\int_{\Pi_T} |D_-D_+A(u_{\Delta x})|\,dxdt.\end{aligned}$$

Note that it follows from (3.1) and Lemma 3.2 that $\|D_-D_+A(u_{\Delta x}(\cdot,t))\|_{L^1(\mathbb{R})}$ is bounded independently of $\Delta x, t, \eta$. Hence,

$$\begin{aligned}\Delta x &\iint_{\Pi_T^2} D_+\operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u))D_+A(u_{\Delta x})\,|D_+\varphi|\,dX \\ &\le \Delta x\,(|\zeta_1| + |\zeta_2|) \\ &\le C(1+r+\Delta x)\left(\frac{\Delta x}{r^2} + \frac{\Delta x^3}{r^4}\right)\left(1 + \frac{\Delta x}{r}\right) \\ &\le C(1+r+\Delta x)\frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right)^3. \qquad \square\end{aligned}$$

**Estimate 4.5.**

$$\begin{aligned}\int_{\Pi_T}&\left(\liminf_{\varepsilon\downarrow 0}\int_{\Pi_T}\frac{1}{2}\left(\zeta^\varepsilon(u_{\Delta x},\tau_{\Delta x}^-,u) + \zeta^\varepsilon(u_{\Delta x},\tau_{\Delta x}^+,u)\right)(A(u)_y)^2\varphi\,dyds\right)dxdt \\ &\ge -C\left(\frac{\Delta x}{r_0} + \frac{\Delta x}{r} + \frac{\Delta x}{r^2}\right).\end{aligned}$$

*Proof.* Set

$$\begin{aligned}\mathcal{R}_j^\varepsilon &:= \left(\zeta^\varepsilon(u_j,\tau_j^-,u) + \zeta^\varepsilon(u_j,\tau_j^+,u)\right)(A(u)_y)^2 \\ &= (\operatorname{sign}_\varepsilon'(A(u_j)-A(u)) - \operatorname{sign}_\varepsilon'(A(\tau_j^-)-A(u)))(A(u)_y)^2 \\ &\quad + (\operatorname{sign}_\varepsilon'(A(u_j)-A(u)) - \operatorname{sign}_\varepsilon'(A(\tau_j^+)-A(u)))(A(u)_y)^2,\end{aligned}$$

and $\mathcal{R}_{\Delta x}^\varepsilon(x,t,y,s) = \mathcal{R}_j^\varepsilon(y,t,s)$ for $x \in I_j$. Note that the term we want to estimate may be written as

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}&\int_{\Pi_T}\mathcal{R}_{\Delta x}^\varepsilon(x,t,y,s)\varphi(x,t,y,s)\,dyds \\ &= \sum_j \liminf_{\varepsilon\downarrow 0}\left(\int_{\Pi_T}\mathcal{R}_j^\varepsilon(t,y,s)\varphi(x,t,y,s)\,dyds\right)\mathbb{1}_{\{I_j\}}(x).\end{aligned}$$

Let us define an entropy function by

$$\begin{aligned}&\partial_u\Psi_\varepsilon(u,u_{j-1},u_j,u_{j+1}) \\ &\quad := \operatorname{sign}_\varepsilon(A(\theta_j^-)-A(u)) - 2\operatorname{sign}_\varepsilon(A(u_j)-A(u)) + \operatorname{sign}_\varepsilon(A(\theta_j^+)-A(u)).\end{aligned}$$

Recall that $\theta_j^\pm = \theta_j^\pm(u)$, so the above function is not as explicit as it appears. However, by Lemma 4.9 we are able to obtain an explicit expression for the limit as $\varepsilon \to 0$. To simplify the notation we write $\Psi_{\varepsilon,j}'(u)$ for $\partial_u\Psi_\varepsilon(u,u_{j-1},u_j,u_{j+1})$. Let us also define the entropy flux functions

$$\Xi_{\varepsilon,j}'(u) = \Psi_{\varepsilon,j}'(u)f'(u), \qquad \Phi_{\varepsilon,j}'(u) = \Psi_{\varepsilon,j}'(u)A'(u).$$

That is, $(\Psi_{\varepsilon,j}, \Xi_{\varepsilon,j}, \Phi_{\varepsilon,j})$ is an entropy-entropy flux triple.

Multiplying equation (1.1) by $\Psi'_{\varepsilon,j}(u)$ yields

$$\Psi_{\varepsilon,j}(u)_s + \Xi_{\varepsilon,j}(u)_y = \Phi_{\varepsilon,j}(u)_{yy} - \partial_y \Psi'_{\varepsilon,j}(u) A(u)_y.$$

By Lemma 4.8 we see that

$$\begin{aligned}
\partial_y \Psi'_{\varepsilon,j}(u) A(u)_y &= \Big[\mathrm{sign}_\varepsilon(A(\theta_j^-) - A(u))_y - 2\mathrm{sign}_\varepsilon(A(u_j) - A(u))_y \\
&\qquad + \mathrm{sign}_\varepsilon(A(\theta_j^+) - A(u))_y\Big] A(u)_y \\
&= -\Big[\mathrm{sign}'_\varepsilon(A(\tau_j^-) - A(u)) - 2\mathrm{sign}'_\varepsilon(A(u_j) - A(u)) \\
&\qquad + \mathrm{sign}'_\varepsilon(A(\tau_j^+) - A(u))\Big] \left(A(u)_y\right)^2 \\
&= \mathcal{R}_j^\varepsilon. \qquad (4.7)
\end{aligned}$$

It follows that we can write

$$\begin{aligned}
\int_{\Pi_T} \mathcal{R}_j^\varepsilon \varphi \, dyds &= \int_{\Pi_T} \Psi_{\varepsilon,j}(u)\varphi_s + \Xi_{\varepsilon,j}(u)\varphi_y + \Phi_{\varepsilon,j}(u)\varphi_{yy} \, dyds \\
&=: T_1^\varepsilon + T_2^\varepsilon + T_3^\varepsilon. \qquad (4.8)
\end{aligned}$$

Let us consider the three terms separately.

By Lemma 4.9,

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \Psi_{\varepsilon,j}(u) &= \lim_{\varepsilon \downarrow 0} \int_{u_j}^{u} \Psi'_{\varepsilon,j}(z)\, dz = \int_{u_j}^{u} \lim_{\varepsilon \downarrow 0} \Psi'_{\varepsilon,j}(z)\, dz \\
&= \int_{u_j}^{u} \mathrm{sign}(A(z) - A(u_j)) - \mathrm{sg}_{j-1}\left(A(z)\right) dz \\
&\quad + \int_{u_j}^{u} \mathrm{sign}(A(z) - A(u_j)) - \mathrm{sg}_{j}\left(A(z)\right) dz,
\end{aligned}$$

where $\mathrm{sg}_j(\sigma) := \mathrm{sg}_{(A(u_j), A(u_{j+1}))}(\sigma)$. Again by Lemma 4.9, the mapping

$$z \mapsto \mathrm{sign}(A(z) - A(u_j)) - \mathrm{sg}_{j-1}\left(A(z)\right)$$

has support in $\mathrm{int}(u_j, u_{j-1})$. Similar considerations apply to the second term. Hence,

$$\begin{aligned}
\left|\lim_{\varepsilon \downarrow 0} \Psi_{\varepsilon,j}(u)\right| &\le \left|\int_{u_j}^{u_{j-1}} \mathrm{sign}(A(z) - A(u_j)) - \mathrm{sg}_{j-1}\left(A(z)\right) dz\right| \\
&\quad + \left|\int_{u_j}^{u_{j+1}} \mathrm{sign}(A(z) - A(u_j)) - \mathrm{sg}_{j}\left(A(z)\right) dz\right| \\
&\le 2\left|u_j - u_{j-1}\right| + 2\left|u_{j+1} - u_j\right|.
\end{aligned}$$

By the same type of reasoning we obtain the bound

$$\begin{aligned}
\left|\lim_{\varepsilon \downarrow 0} \Xi_j^\varepsilon(u)\right| &\le \left|\int_{u_j}^{u_{j-1}} \left[\mathrm{sign}(A(z) - A(u_j)) - \mathrm{sg}_{j-1}\left(A(z)\right)\right] f'(z)\, dz\right| \\
&\quad + \left|\int_{u_j}^{u_{j+1}} \left[\mathrm{sign}(A(z) - A(u_j)) - \mathrm{sg}_{j}\left(A(z)\right)\right] f'(z)\, dz\right| \\
&\le 2\left\|f'\right\|_{L^\infty}\left(\left|u_j - u_{j-1}\right| + \left|u_{j+1} - u_j\right|\right).
\end{aligned}$$

Concerning $\Phi_j^\varepsilon$ we use substitution and the explicit expression given in Lemma 4.9. This leads to

$$
\begin{aligned}
\left|\lim_{\varepsilon\downarrow 0}\Phi_j^\varepsilon(u)\right| &= \left|\int_{u_j}^{u}\left(-\operatorname{sg}_{j-1}(A(z)) + 2\operatorname{sign}(A(z)-A(u_j)) - \operatorname{sg}_j(A(z))\right)A'(z)\,dz\right| \\
&= \left|\int_{A(u_j)}^{A(u)} -\operatorname{sg}_{j-1}(\sigma) + 2\operatorname{sign}(\sigma - A(u_j)) - \operatorname{sg}_j(\sigma)\,d\sigma\right| \\
&\le \left|\int_{A(u_j)}^{A(u_{j-1})} \operatorname{sign}(\sigma - A(u_j)) - \operatorname{sg}_{j-1}(\sigma)\,d\sigma\right| \\
&\qquad + \left|\int_{A(u_j)}^{A(u_{j+1})} \operatorname{sign}(\sigma - A(u_j)) - \operatorname{sg}_{j}(\sigma)\,d\sigma\right| \\
&\le \left|A(u_j) - A(u_{j-1})\right| + \left|A(u_{j+1}) - A(u_j)\right|.
\end{aligned}
$$

Let us return to equation (4.8). By the dominated convergence theorem and the above computations we have

$$
\begin{aligned}
\left|\lim_{\varepsilon\downarrow 0} T_1^\varepsilon\right| &\le \left\|\lim_{\varepsilon\downarrow 0}\Psi_j^\varepsilon\right\|_{L^\infty}\int_{\Pi_T}|\varphi_s|\,dyds \le C\left(|D_-u_j| + |D_+u_j|\right), \\
\left|\lim_{\varepsilon\downarrow 0} T_2^\varepsilon\right| &\le \left\|\lim_{\varepsilon\downarrow 0}\Xi_j^\varepsilon\right\|_{L^\infty}\int_{\Pi_T}|\varphi_y|\,dyds \le C\frac{\Delta x}{r}\left(|D_-u_j| + |D_+u_j|\right), \\
\left|\lim_{\varepsilon\downarrow 0} T_3^\varepsilon\right| &\le \left\|\lim_{\varepsilon\downarrow 0}\Phi_j^\varepsilon\right\|_{L^\infty}\int_{\Pi_T}|\varphi_{yy}|\,dyds \le C\frac{\Delta x}{r^2}\left(|D_-A(u_j)| + |D_+A(u_j)|\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{\Pi_T}\sum_j \liminf_{\varepsilon\downarrow 0}&\left(\int_{\Pi_T}\mathcal{R}_j^\varepsilon(t,y,s)\varphi(x,t,y,s)\,dyds\right)\mathbb{1}_{\{I_j\}}(x)\,dxdt \\
&\ge -C\left(\frac{\Delta x}{r_0} + \frac{\Delta x}{r}\right)\int_{\Pi_T}|D_-u_{\Delta x}| + |D_+u_{\Delta x}|\,dxdt \\
&\qquad - C\frac{\Delta x}{r^2}\int_{\Pi_T}|D_-A(u_{\Delta x})| + |D_+A(u_{\Delta x})|\,dxdt.
\end{aligned}
$$

The desired estimate now follows from the uniform bounds in Lemma 3.1. □

4.2. **Proof of Theorem 4.1.** Let us now combine the previous results to conclude the proof of Theorem 4.1. We begin by stating a rather standard lemma.

**Lemma 4.12.** *Set*

$$
\kappa(t) := \int_{\mathbb{R}}\int_{\Pi_T}|u_{\Delta x}(x,t) - u(y,s)|\,\omega_r(x-y)\rho_{r_0}(t-s)\,dydsdx.
$$

*Let $t \ge r_0$, and denote by $L_c$ the Lipschitz constant of $t \mapsto \|u(\cdot,t)\|_{L^1(\mathbb{R})}$. Then*

$$
\left|\kappa(t) - \|u_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})}\right| \le |u(\cdot,t)|_{BV(\mathbb{R})}\, r + L_c r_0.
$$

*Proof.* By the reverse triangle inequality,

$$\begin{aligned}
&\left|\kappa(t) - \|u_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})}\right| \\
&\quad\le \int_{\mathbb{R}} \int_{\Pi_T} |u(y,s) - u(x,t)|\, \omega_r(x-y)\rho_{r_0}(t-s)\, dydsdx \\
&\quad\le \int_0^T \left(\int_{\mathbb{R}} |u(y,s) - u(t,y)|\, dy\right) \rho_{r_0}(t-s)\, ds \\
&\qquad + \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t,y) - u(x,t)| \omega_r(x-y) dydx \\
&\quad\le L_c r_0 + |u(\cdot,t)|_{BV(\mathbb{R})}\, r.
\end{aligned}$$

□

*Proof of Theorem* 4.1. Our starting point is Lemma 4.6. Let $A(\sigma) = \hat{A}(\sigma) + \eta\sigma$, where $\hat{A}$ is the original degenerate diffusion function. Let

$$\begin{aligned}
\Xi = &\iint_{\Pi_T^2} \operatorname{sign}(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))\,(D_+\varphi + \varphi_y)\, dX \\
&+ \iint_{\Pi_T^2} \left(\int_{u_{\Delta x}}^{S_{\Delta x} u_{\Delta x}} \operatorname{sign}(z-u)\, F_2'(z)\, dz\right) D_+\varphi\, dX \\
&+ \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\,(D_- D_+\varphi + (D_+ + D_-)\varphi_y + \varphi_{yy})\, dX.
\end{aligned}$$

By Estimate 4.1, Estimate 4.2, and Estimate 4.3, it follows that

$$|\Xi| \le C\frac{\Delta x}{r}\left(1 + \frac{\Delta x}{r^2}\right)\left(1 + \frac{\Delta x}{r}\right) =: E_1. \tag{4.9}$$

Furthermore, by Lemma 4.7, Lemma 4.10, Estimate 4.4, and Estimate 4.5, it follows that

$$\liminf_{\varepsilon \downarrow 0} \iint_{\Pi_T^2} E^{\varepsilon}_{\Delta x}\varphi\, dX \ge -C(1 + r + \Delta x)\frac{\Delta x}{r^2}\left(1 + \frac{\Delta x}{r}\right)^3 - C\frac{\Delta x}{r_0} =: -E_2. \tag{4.10}$$

Applying the estimates (4.9) and (4.10), the inequality (4.2) becomes

$$\begin{aligned}
&\iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\tau)\omega_r(x-y)\rho_{r_0}(t-s)\, dX \\
&\quad\le \iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\nu)\omega_r(x-y)\rho_{r_0}(t-s)\, dX + E_1 + E_2.
\end{aligned}$$

Note that both $E_1$ and $E_2$ are independent of $\alpha$. Thus, we can send $\alpha$ to zero, arriving at

$$\kappa(\tau) \le \kappa(\nu) + E_1 + E_2,$$

where $\kappa$ is defined as in Lemma 4.12.

By Lemma 4.12 it follows that

$$\begin{aligned}
&\|u_{\Delta x}(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(\mathbb{R})} \\
&\quad\le \|u_{\Delta x}(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbb{R})} + 2\left(L_c r_0 + \left|u^0\right|_{BV(\mathbb{R})} r\right) + E_1 + E_2.
\end{aligned}$$

Recall that we had to pick $\nu > r_0$. Denote by $L_d$ the $L^1$ Lipschitz constant of $t \mapsto u_{\Delta x}(\cdot, t)$. By the triangle inequality

$$\begin{aligned}
&\|u_{\Delta x}(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbb{R})} \\
&\quad \le \left\|u_{\Delta x}(\cdot,\nu) - u^0_{\Delta x}\right\|_{L^1(\mathbb{R})} + \left\|u^0_{\Delta x} - u^0\right\|_{L^1(\mathbb{R})} + \left\|u^0 - u(\cdot,\nu)\right\|_{L^1(\mathbb{R})} \\
&\quad \le L_d\nu + \left\|u^0_{\Delta x} - u^0\right\|_{L^1(\mathbb{R})} + L_c\nu.
\end{aligned}$$

This means that

$$\begin{aligned}
\|u_{\Delta x}(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(\mathbb{R})} \le \left\|u^0_{\Delta x} - u^0\right\|_{L^1(\mathbb{R})} &+ (L_c + L_d)\,\nu \\
&+ 2\left(L_c r_0 + \left|u^0\right|_{BV(\mathbb{R})} r\right) + E_1 + E_2.
\end{aligned}$$

Choose $r^3 = r_0^2 = \Delta x$ and $\nu = 2r_0$. Then there exists a constant $C$ such that

$$\|u_{\Delta x}(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(\mathbb{R})} \le \left\|u^0_{\Delta x} - u^0\right\| + C\Delta x^{\frac{1}{3}}.$$

Now recall that $A(\sigma) = \hat{A}(\sigma) + \eta\sigma$ and so we need to send $\eta$ to zero to finish the proof. If $u_\eta$ is the classical solution of the regularized equation and $u$ is the entropy solution of the nonregularized equation, then it is well known that $u_\eta(\cdot,t) \to u(\cdot,t)$ in $L^1(\mathbb{R})$ as $\eta \to 0$ (see Section 2). Concerning the scheme, one may prove continuous dependence in $\ell^1$ on $\eta$ using Gronwall's inequality. Hence, we can also send $\eta$ to zero in the scheme. This finishes the proof of Theorem 4.1. $\square$

## 5. Implicit difference schemes

In this section we show that the arguments presented in the previous sections carry through for implicit schemes. Fix a time step $\Delta t > 0$. We consider implicit difference schemes of the form

$$D^t_- u^n_j + D_- F(u^n_j, u^n_{j+1}) = D_- D_+ A(u^n_j) \qquad n \ge 1,\ j \in \mathbb{Z}, \tag{5.1}$$

where

$$D^t_- u^n_j = \frac{u^n_j - u^{n-1}_j}{\Delta t}.$$

Let $t_n = n\Delta t$ and $x_j = j\Delta x$. We define the grid cells

$$I^n_j = [x_{j-1/2}, x_{j+1/2}) \times (t_{n-1}, t_n] \quad \text{for } n \ge 0 \text{ and } j \in \mathbb{Z}.$$

The piecewise constant approximation is defined for all $(x,t) \in \mathbb{R} \times (-\Delta t, T]$ by

$$u_\Delta(x,t) = u^n_j \ \text{ for } (x,t) \in I^n_j. \tag{5.2}$$

The domain is chosen so that $D^t_- u_\Delta$ is defined for all $(x,t) \in \mathbb{R} \times (0,T)$. For the existence of a unique solution $u^n_j$ to the nonlinear equation (5.1) and the convergence of $u_\Delta$ to an entropy solution; see [11].

We now state the main theorem.

**Theorem 5.1.** *Let $u$ be the entropy solution to* (1.1)*, and let $u_\Delta$ be defined via $u^n_j$ by* (5.2)*, where $u^n_j$ solves* (5.1)*. If $u^0$ satisfies the same assumptions as in Theorem* 4.1*, then for all sufficiently small $\Delta x$ and $\Delta t$, and for all $n \in \mathbb{N}$ such that $t_n \in [0,T]$,*

$$\|u_\Delta(\cdot,t_n) - u(\cdot,t_n)\|_{L^1(\mathbb{R})} \le \left\|u^0_\Delta - u^0\right\|_{L^1(\mathbb{R})} + C\left(\Delta x^{1/3} + \Delta t^{1/2}\right),$$

*where the constant $C_T$ depends on $u_0, A, f, T$, but not on $\Delta x, \Delta t$.*

To prove this theorem we will follow step-by-step the proof of Theorem 4.1 and present the details whenever there is a significant difference between the two cases.

Thanks to [11, Lemma 2.4], we have the following $L^1$ Lipschitz continuity result:

**Lemma 5.1.** *Let $m$ and $n$ be two nonnegative integers. Then*

$$\|u_\Delta(\cdot,t_n)-u_\Delta(\cdot,t_m)\|_{L^1(\mathbb{R})}\le L_d\,|t_n-t_m|\,,$$

*where $L_d=\left|F(u_j^0,u_{j+1}^0)-D_+A(u_j^0)\right|_{BV}$.*

Next, let us prove an implicit version of Lemma 4.3.

**Lemma 5.2.** *Let $u_j^n$ be the solution to* (5.1)*. Then for all $c\in\mathbb{R}$,*

$$\begin{aligned}
D_-^t\psi_\varepsilon(u_j^n,c)&+D_-Q^c(u_j^n,u_{j+1}^n)-D_-D_+\left|A(u_j^n)-A(c)\right|_\varepsilon\\
&\le-\frac{1}{(\Delta x)^2}\int_{u_{j+1}^n}^{u_j^n}\psi_\varepsilon''(z,c)(A(z)-A(u_{j+1}^n))\,dz\\
&\quad-\frac{1}{(\Delta x)^2}\int_{u_{j-1}^n}^{u_j^n}\psi_\varepsilon''(z,c)(A(z)-A(u_{j-1}^n))\,dz,
\end{aligned}$$

*where $Q^c(u,v)$ is defined in Lemma 4.3.*

*Proof.* From (5.1) it follows that

$$\psi_\varepsilon'(u_j^n,c)D_-^tu_j^n+\psi_\varepsilon'(u_j^n,c)D_-F(u_j^n,u_{j+1}^n)=\psi_\varepsilon'(u_j^n,c)D_-D_+A(u_j^n).$$

Apply Lemma 4.2 with $g(\sigma)=\sigma$, $a=u_j^n$, and $b=u_j^{n-1}$ to obtain

$$\begin{aligned}
\psi_\varepsilon'(u_j^n,c)D_-^tu_j^n&=D_-^t\psi_\varepsilon(u_j^n,c)-\frac{1}{\Delta t}\int_{u_j^n}^{u_j^{n-1}}\psi_\varepsilon''(z,c)(z-u_j^{n-1})\,dz\\
&\ge D_-^t\psi_\varepsilon(u_j^n,c).
\end{aligned}$$

The remaining part of the proof follows exactly as in the proof of Lemma 4.3. $\square$

Let us define the time shift operator

$$S_{\Delta t}^t\sigma(t)=\sigma(t+\Delta t),$$

for any function $\sigma=\sigma(t)$.

**Lemma 5.3.** *Suppose $A'>0$. Let $u_\Delta=u_\Delta(x,t)$ be defined by* (5.2)*, and let $u=u(y,s)$ be the classical solution of* (1.1)*. Let $\psi(t):=\mathbb{1}_{\{[\nu,\tau)\}}(t)$ and define*

$$\varphi(x,t,y,s)=\psi(t)\omega_r(x-y)\rho_{r_0}(t-s),$$

*where* $\omega_r, \rho_{r_0}, \nu, \tau$ *are chosen as in Lemma* 4.6*. Then*

$$
\begin{aligned}
&\iint_{\Pi_T^2} |u_\Delta - u|\, \delta_{\Delta t}^-(t-\nu)\omega_r \rho_{r_0}\, dX \\
&\quad + \iint_{\Pi_T^2} |u_\Delta - u|\, S_{\Delta t}^t \psi \omega_r (D_+^t \rho_{r_0} - \partial_t \rho_{r_0})\, dX \\
&\quad + \Delta t \iint_{\Pi_T^2} |u_\Delta - u|\, D_+^t \psi \omega_r \partial_s \rho_{r_0}\, dX \\
&\quad + \iint_{\Pi_T^2} \operatorname{sign}(u_\Delta - u)\left(f(u_\Delta) - f(u)\right)(D_+\varphi + \varphi_y)\, dX \\
&\quad + \iint_{\Pi_T^2} \left( \int_{u_{\Delta x}}^{S_{\Delta x} u_\Delta} \operatorname{sign}(z-u)\, F_2'(z)\, dz \right) D_+\varphi\, dX \\
&\quad + \iint_{\Pi_T^2} |A(u_\Delta) - A(u)|\, (D_- D_+ \varphi + (D_+ + D_-)\varphi_y + \varphi_{yy})\, dX \\
&\qquad \ge \iint_{\Pi_T^2} |u_\Delta - u|\, \delta_{\Delta t}^-(t-\tau)\omega_r \rho_{r_0}\, dX + \liminf_{\varepsilon \downarrow 0} \iint_{\Pi_T^2} E_\Delta^\varepsilon \varphi\, dX,
\end{aligned}
$$

*where*

$$
\delta_{\Delta t}^-(t) = \frac{1}{\Delta t} \mathbb{1}_{\{[-\Delta t, 0)\}}(t),
$$

*and* $E_\Delta^\varepsilon(x,t,y,s) = E^\varepsilon[u](u_{j-1}^n, u_j^n, u_{j+1}^n)(y,s)$ *for* $(x,t) \in I_j^n$*.*

*Proof.* As in Lemma 4.5, we obtain by Lemma 5.2 the inequality

$$
\begin{aligned}
&D_-^t \psi_\varepsilon(u_j^n, u) + \partial_s \psi_\varepsilon(u, u_j^n) + \partial_y q_\varepsilon(u, u_j^n) + D_- Q^u(u_j^n, u_{j+1}^n) \\
&\qquad - (\partial_y^2 + \partial_y(D_- + D_+) + D_- D_+)\big|A(u_j^n) - A(u)\big|_\varepsilon) \le -E_{j,n}^\varepsilon,
\end{aligned}
$$

where $E_{j,n}^\varepsilon := E^\varepsilon[u](u_{j-1}^n, u_j^n, u_{j+1}^n)$ is defined in Lemma 4.5. Let us multiply by $\varphi$ and integrate over $\Pi_T^2$. Integration by parts for difference quotients and ordinary integration by parts gives

$$
\begin{aligned}
&\iint_{\Pi_T^2} \psi_\varepsilon(u_\Delta, u) D_+^t \varphi + \psi_\varepsilon(u, u_\Delta)\varphi_s\, dX \\
&\quad + \iint_{\Pi_T^2} q_\varepsilon(u, u_\Delta)\varphi_y + Q^u(u_\Delta, S_{\Delta x} u_\Delta) D_+ \varphi\, dX \\
&\quad + \iint_{\Pi_T^2} |A(u_\Delta) - A(u)|_\varepsilon (\varphi_{yy} + (D_- + D_+)\varphi_y + D_- D_+ \varphi)\, dX \\
&\qquad \ge \iint_{\Pi_T^2} E_\Delta^\varepsilon \varphi\, dX.
\end{aligned}
$$

Consider the first term on the left. Let $\varepsilon$ tend to zero as in the proof of Lemma 4.6. Using the Leibniz rule for difference quotients and adding and subtracting we obtain

$$
D_+^t \varphi = S_{\Delta t}^t \psi \omega_r D_+^t \rho_{r_0} + D_+^t \psi \omega_r \rho_{r_0}.
$$

Furthermore,

$$
\varphi_s = -S_{\Delta t}^t \psi \omega_r \partial_t \rho_{r_0} + \Delta t D_+^t \psi \omega_r \partial_s \rho_{r_0}.
$$

Hence,

$$
\begin{aligned}
&\iint_{\Pi_T^2} |u_\Delta - u| \left(D_+^t \varphi + \varphi_s\right) \, dX \\
&\quad = \iint_{\Pi_T^2} |u_\Delta - u| \, S_{\Delta t}^t \psi \omega_r (D_+^t \rho_{r_0} - \partial_t \rho_{r_0}) \, dX \\
&\qquad + \Delta t \iint_{\Pi_T^2} |u_\Delta - u| \, D_+^t \psi \omega_r \partial_s \rho_{r_0} \, dX + \iint_{\Pi_T^2} |u_\Delta - u| \, D_+^t \psi \omega_r \rho_{r_0} \, dX.
\end{aligned}
$$

Finally, we use that

$$
D_+^t \psi = \delta_{\Delta t}^-(t-\nu) - \delta_{\Delta t}^-(t-\tau). \tag{5.3}
$$

The lemma now follows, as in the proof of Lemma 4.6, by letting $\varepsilon$ tend to zero. $\square$

Comparing the terms in Lemma 4.6 with the terms in Lemma 5.3 we recognize all but two terms.

**Estimate 5.1.**

$$
\left| \iint_{\Pi_T^2} |u - u_\Delta| \, S_{\Delta t}^t \psi \omega_r (D_+^t \rho_{r_0} - \partial_t \rho_{r_0}) \, dX \right| \le C \frac{\Delta t}{r_0} \left(1 + \frac{\Delta t}{r_0}\right).
$$

*Proof.* To show this we use a Taylor expansion:

$$
\begin{aligned}
&\rho_{r_0}(t + \Delta t - s) - \rho_{r_0}(t - s) \\
&\quad = \int_t^{t+\Delta t} \frac{\partial}{\partial z} \rho_{r_0}(z - s) \, dz \\
&\quad = \left.\frac{\partial}{\partial z}\right|_{z=t} \rho_{r_0}(z - s) \Delta t - \int_t^{t+\Delta t} \frac{\partial^2}{\partial z^2} \rho_{r_0}(z - s)(z - (t + \Delta t)) \, dz.
\end{aligned}
$$

It follows that

$$
D_+^t \rho_{r_0} - \partial_t \rho_{r_0} = -\frac{1}{\Delta t} \int_t^{t+\Delta t} \frac{\partial^2}{\partial z^2} \rho_{r_0}(z - s)(z - (t + \Delta t)) \, dz. \tag{5.4}
$$

Integration by parts yields

$$
\begin{aligned}
&-\frac{1}{\Delta t} \int_t^{t+\Delta t} \int_0^T |u - u_\Delta|_\varepsilon \frac{\partial^2}{\partial z^2} \rho_{r_0}(z - s)(z - (t + \Delta t)) \, ds dz \\
&\quad = \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_0^T |u - u_\Delta|_\varepsilon \frac{\partial}{\partial s} \frac{\partial}{\partial z} \rho_{r_0}(z - s)(z - (t + \Delta t)) \, ds dz \\
&\quad = -\frac{1}{\Delta t} \int_t^{t+\Delta t} \int_0^T \operatorname{sign}_\varepsilon(u - u_\Delta) u_s \frac{\partial}{\partial z} \rho_{r_0}(z - s)(z - (t + \Delta t)) \, ds dz.
\end{aligned}
$$

Since

$$
\frac{\partial}{\partial z} \rho_{r_0}(z - s) = \frac{1}{r_0} \frac{\partial}{\partial z} \rho\left(\frac{z - s}{r_0}\right) = \frac{1}{r_0^2} \rho'\left(\frac{z - s}{r_0}\right)
$$

and $\rho_{r_0}$ has support in $[-r_0, r_0]$, it follows that

$$\begin{aligned}
&\frac{1}{\Delta t}\left|\int_t^{t+\Delta t}\int_0^T |u(y,s)-u_\Delta(x,t)|_\varepsilon \frac{\partial^2}{\partial z^2}\rho_{r_0}(z-s)(z-(t+\Delta t))\,dsdz\right| \\
&\qquad \le \frac{C}{r_0^2\Delta t}\int_0^T |u_s(y,s)|\int_t^{t+\Delta t} \mathbb{1}_{\{|z-s|\le r_0\}}|z-(t+\Delta t)|\,dzds \\
&\qquad \le C\frac{\Delta t}{r_0^2}\int_0^T |u_s|\mathbb{1}_{\{|t-s|\le r_0+\Delta t\}}\,ds.
\end{aligned}$$

Multiply the above inequality by $\psi(t)\omega_r(x-y)$ and integrate in $x, y, t$. From the resulting inequality and (5.4), we arrive at the estimate:

$$\begin{aligned}
&\left|\iint_{\Pi_T^2} |u-u_\Delta|_\varepsilon S_{\Delta t}^t\psi\omega_r(D_+^t\rho_{r_0}-\partial_t\rho_{r_0})\,dX\right| \\
&\qquad \le C\frac{\Delta t}{r_0^2}\iint_{\Pi_T^2}|u_s|S_{\Delta t}^t\psi\omega_r\mathbb{1}_{\{|t-s|\le r_0+\Delta t\}}\,dX \le C\frac{\Delta t}{r_0}\left(1+\frac{\Delta t}{r_0}\right)\|u_s\|_{L^1(\Pi_T)}.
\end{aligned}$$

Since $\|u_s(\cdot,s)\|_{L^1(\mathbb{R})}$ is uniformly bounded on $[0,T]$, the estimate follows from the dominated convergence theorem. □

**Estimate 5.2.**

$$\left|\Delta t\iint_{\Pi_T^2}|u-u_\Delta|\,D_+^t\psi\omega_r\partial_s\rho_{r_0}\,dX\right| \le C\frac{\Delta t}{r_0}.$$

*Proof.* Integration by parts yields

$$\left|\iint_{\Pi_T^2}|u-u_\Delta|\,D_+^t\psi\omega_r\partial_s\rho_{r_0}\,dX\right| \le \iint_{\Pi_T^2}|u_s|\left|D_+^t\psi\right|\omega_r\rho_{r_0}\,dX.$$

Because of (5.3) and since

$$\|\rho_{r_0}\|_{L^\infty} \le \frac{\|\rho\|_{L^\infty}}{r_0},$$

it follows that

$$\iint_{\Pi_T^2}|u_s|\left|D_+^t\psi\right|\omega_r\rho_{r_0}\,dX \le 2\frac{\|\rho\|_{L^\infty}}{r_0}\|u_s\|_{L^1(\Pi_T)}.$$ □

*Proof of Theorem* 5.1. We start out from Lemma 5.3 with $A(\sigma)=\hat{A}(\sigma)+\eta\sigma$, where $\hat{A}$ is the original degenerate diffusion function. By Estimate 5.1 and Estimate 5.2,

$$\begin{aligned}
&\iint_{\Pi_T^2}|u-u_\Delta|\,S_{\Delta t}^t\psi\omega_r(D_+^t\rho_{r_0}-\partial_t\rho_{r_0})\,dX \\
&\qquad + \Delta t\iint_{\Pi_T^2}|u-u_\Delta|\,D_+^t\psi\omega_r\partial_s\rho_{r_0}\,dX \le C\frac{\Delta t}{r_0}\left(1+\frac{\Delta t}{r_0}\right) =: E_3.
\end{aligned} \tag{5.5}$$

Since all the estimates from Section 4.1 apply, we obtain

$$\begin{aligned}
&\iint_{\Pi_T^2}|u_\Delta-u|\,\delta^{\Delta t}(t-\tau)\omega_r(x-y)\rho_{r_0}(t-s)\,dX \\
&\qquad \le \iint_{\Pi_T^2}|u_{\Delta x}-u|\,\delta^{\Delta t}(t-\nu)\omega_r(x-y)\rho_{r_0}(t-s)\,dX + E_1+E_2+E_3,
\end{aligned}$$

where $E_1$ and $E_2$ are defined respectively in (4.9) and (4.10).

Let us make the simplifying assumption that $\nu = t_m$ and $\tau = t_n$ for some $m, n \in \mathbb{N}$. Then the above inequality is rewritten as

$$\kappa(t_n) \leq \kappa(t_m) + E_1 + E_2 + E_3,$$

where

$$\kappa(t) = \int_{\mathbb{R}} \int_{\Pi_T} |u_\Delta(x,t) - u(y,s)|\, \omega_r(x-y)\rho_{r_0}(t-s)\, dydsdx.$$

Applying Lemmas 4.12 and 5.1, and following the reasoning given in the semi-discrete case, we arrive at

$$\begin{aligned}
&\|u_\Delta(\cdot,t_n) - u(\cdot,t_n)\|_{L^1(\mathbb{R})} \\
&\quad \leq \left\|u_\Delta^0 - u^0\right\|_{L^1(\mathbb{R})} \\
&\qquad\quad + (L_c + L_d)\, t_m + 2\left(L_c r_0 + \left|u^0\right|_{BV(\mathbb{R})} r\right) \\
&\qquad\quad + C(1 + r + \Delta x)^2 \left(1 + \frac{\Delta x}{r}\right)^3 \frac{\Delta x}{r^2} + C\frac{\Delta x}{r_0} + C\frac{\Delta t}{r_0}\left(1 + \frac{\Delta t}{r_0}\right) \\
&\quad \leq \left\|u_\Delta^0 - u^0\right\|_{L^1(\mathbb{R})} + C\left(\frac{\Delta x}{r^2} + \frac{\Delta x + \Delta t}{r_0} + r + r_0\right),
\end{aligned}$$

where $L_d$ is the constant in Lemma 5.1 and $L_c$ is the constant from Lemma 4.12. Minimizing over $r$ and $r_0$, it is straightforward to see that for sufficiently small $\Delta t$, the minimum of the last term is dominated by

$$C\left(\Delta x^{1/3} + \Delta t^{1/2}\right).$$

This proves the theorem. □

## 6. Explicit difference schemes

In this section we use the techniques developed in the previous section to provide a similar result concerning the explicit scheme. Fix a time step $\Delta t > 0$. We consider explicit schemes of the form

$$D_+^t u_j^n + D_- F(u_j^n, u_{j+1}^n) = D_- D_+ A(u_j^n) \qquad n \geq 1,\ j \in \mathbb{Z}, \tag{6.1}$$

where

$$D_+^t u_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t}.$$

The relevant a priori estimates and convergence to an entropy solution is proved in [12] under the hypothesis

$$1 - \frac{\Delta t}{\Delta x}(F_1'(z) - F_2'(z)) - 2\frac{\Delta t}{\Delta x^2} A'(w) \geq 0, \quad \forall (z,w) \in \mathbb{R}^2. \tag{6.2}$$

Let $t_n = n\Delta t$ and $x_j = j\Delta x$. We define the grid cells

$$I_j^n = [x_{j-1/2}, x_{j+1/2}) \times [t_n, t_{n+1}), \quad \text{for } n \geq 0 \text{ and } j \in \mathbb{Z}.$$

The piecewise constant approximation is defined for all $(x,t) \in \Pi_T$ by

$$u_\Delta(x,t) = u_j^n \ \text{ for } (x,t) \in I_j^n. \tag{6.3}$$

**Theorem 6.1.** *Let $u$ be the entropy solution to* (1.1), *and let $u_\Delta$ be defined by* (6.3) *via $u_j^n$, where $u_j^n$ solves* (6.1). *Suppose $\Delta t$ and $\Delta x$ are chosen such that* (6.2) *and the strengthened condition $\Delta t \leq C\Delta x^{8/3}$ hold. If $u^0$ satisfies the same assumptions as in Theorem* 4.1, *then for all sufficiently small $\Delta x$, and for all $n \in \mathbb{N}$ such that $t_n \in [0, T]$,*

$$\|u_\Delta(\cdot, t_n) - u(\cdot, t_n)\|_{L^1(\mathbb{R})} \leq \left\|u_\Delta^0 - u^0\right\|_{L^1(\mathbb{R})} + C\Delta x^{1/3},$$

*where the constant $C_T$ depends on $A, f, u^0, T$, but not on $\Delta x$.*

We begin by proving the following lemma.

**Lemma 6.1.** *Let $\{u_j^n\}$ be the solution to* (6.1). *Suppose that*

$$1 - \frac{\Delta t}{\Delta x}(F_1'(z) - F_2'(z)) \geq 0, \tag{6.4}$$

*for all $z \in \mathbb{R}$. Then for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$,*

$$\begin{aligned}
D_+^t \psi_\varepsilon(u_j^n, c) &+ D_- Q^c(u_j^n, u_{j+1}^n) - D_- D_+ \left|A(u_j^n) - A(c)\right|_\varepsilon \\
&\leq -\frac{1}{(\Delta x)^2} \int_{u_{j-1}^n}^{u_j^n} \psi_\varepsilon''(z, c)(A(z) - A(u_{j-1}^n))\, dz \\
&\quad - \frac{1}{(\Delta x)^2} \int_{u_{j+1}^n}^{u_j^n} \psi_\varepsilon''(z, c)(A(z) - A(u_{j+1}^n))\, dz \\
&\quad + \left( \int_{u_j^n}^{u_j^{n+1}} \psi_\varepsilon''(z, c)\, dz \right) D_- D_+ A(u_j^n),
\end{aligned}$$

*where $Q^c(u, v)$ is defined by*

$$Q^c(u, v) = \int_c^u \psi_\varepsilon'(z, c) F_1'(z)\, dz + \int_c^v \psi_\varepsilon'(z, c) F_2'(z)\, dz, \qquad u, v \in \mathbb{R}.$$

*Proof.* We divide the proof into two steps.

*Claim* 1. Let $\{u_j^n\}$ be a solution to (6.1). Then

$$\begin{aligned}
D_+^t \psi_\varepsilon(u_j^n, c) &+ D_- Q^c(u_j^n, u_{j+1}^n) - D_- D_+ \left|A(u_j^n) - A(c)\right|_\varepsilon \\
&= -E^c(u_j^n, u_j^{n+1}, u_{j+1}^n, u_{j-1}^n),
\end{aligned}$$

where

$$\begin{aligned}
E^c&(u_j^n, u_j^{n+1}, u_{j+1}^n, u_{j-1}^n) \\
&= \frac{1}{\Delta t} \int_{u_j^n}^{u_j^{n+1}} \psi_\varepsilon''(z, c)(z - u_j^{n+1})\, dz \\
&\quad + \frac{1}{\Delta x} \int_{u_{j-1}^n}^{u_j^n} \psi_\varepsilon''(z, c) \left[ (F_1(z) - F_1(u_{j-1}^n)) + \frac{1}{\Delta x}(A(z) - A(u_{j-1}^n)) \right] dz \\
&\quad + \frac{1}{\Delta x} \int_{u_j^n}^{u_{j+1}^n} \psi_\varepsilon''(z, c) \left[ (F_2(z) - F_2(u_{j+1}^n)) - \frac{1}{\Delta x}(A(z) - A(u_{j+1}^n)) \right] dz.
\end{aligned}$$

*Proof of Claim* 1. By definition (6.1) of $\{u_j^n\}$ it follows that

$$\psi_\varepsilon'(u_j^n, c) \left[ D_+^t u_j^n + D_- F(u_j^n, u_{j+1}^n) - D_- D_+ A(u_j^n) \right] = 0.$$

Let $g(z) = z$ in Lemma 4.2. It follows that

$$\psi'_\varepsilon(u_j^n, c) D_+^t u_j^n = D_+^t \psi_\varepsilon(u_j^n, c) + \frac{1}{\Delta t} \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c)(z - u_j^{n+1})\, dz.$$

The remaining terms can be treated as in Lemma 4.3. □

*Claim* 2. Suppose (6.4) holds. Then

$$\begin{aligned}
&\frac{1}{\Delta t} \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c)(z - u_j^{n+1})\, dz \\
&\quad + \frac{1}{\Delta x} \int_{u_{j-1}^n}^{u_j^n} \psi''_\varepsilon(z, c)(F_1(z) - F_1(u_{j-1}^n))\, dz \\
&\quad + \frac{1}{\Delta x} \int_{u_j^n}^{u_{j+1}^n} \psi''_\varepsilon(z, c)(F_2(z) - F_2(u_{j+1}^n))\, dz \\
&\qquad \geq - \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c) D_- D_+ A(u_j^n)\, dz.
\end{aligned} \tag{6.5}$$

*Proof of Claim* 2. Consider the first term on the left-hand side of (6.5). By definition, $u_j^{n+1} = u_j^n - \Delta t D_- F(u_j^n, u_{j+1}^n) + \Delta t D_- D_+ A(u_j^n)$, and so

$$\begin{aligned}
&\frac{1}{\Delta t} \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c)(z - u_j^{n+1})\, dz \\
&\quad = \frac{1}{\Delta t} \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c)(z - u_j^n)\, dz + \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c) D_- F(u_j^n, u_{j+1}^n)\, dz \\
&\qquad - \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c) D_- D_+ A(u_j^n)\, dz =: T_1 + T_2 + T_3.
\end{aligned}$$

Note that $T_1$ is positive.

Let us split $T_2$ according to

$$D_- F(u_j^n, u_{j+1}^n) = \frac{1}{\Delta x} \left( F_1(u_j^n) - F_1(u_{j-1}^n) \right) + \frac{1}{\Delta x} \left( F_2(u_{j+1}^n) - F_2(u_j^n) \right),$$

and thus

$$\begin{aligned}
T_2 = &\frac{1}{\Delta x} \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c) \left( F_1(u_j^n) - F_1(u_{j-1}^n) \right)\, dz \\
&+ \frac{1}{\Delta x} \int_{u_j^n}^{u_j^{n+1}} \psi''_\varepsilon(z, c) \left( F_2(u_{j+1}^n) - F_2(u_j^n) \right)\, dz.
\end{aligned}$$

Now, let us split the two other terms appearing in equation (6.5):

$$
\begin{aligned}
S_1 &:= \frac{1}{\Delta x}\int_{u_{j-1}^n}^{u_j^n} \psi_\varepsilon''(z,c)(F_1(z)-F_1(u_{j-1}^n))\,dz \\
&= \frac{1}{\Delta x}\int_{u_{j-1}^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)(F_1(z)-F_1(u_{j-1}^n))\,dz \\
&\quad - \frac{1}{\Delta x}\int_{u_j^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)(F_1(z)-F_1(u_{j-1}^n))\,dz
\end{aligned}
$$

and

$$
\begin{aligned}
S_2 &:= \frac{1}{\Delta x}\int_{u_j^n}^{u_{j+1}^n} \psi_\varepsilon''(z,c)(F_2(z)-F_2(u_{j+1}^n))\,dz \\
&= -\frac{1}{\Delta x}\int_{u_{j+1}^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)(F_2(z)-F_2(u_{j+1}^n))\,dz \\
&\quad + \frac{1}{\Delta x}\int_{u_j^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)(F_2(z)-F_2(u_{j+1}^n))\,dz.
\end{aligned}
$$

Combining the above expressions we obtain

$$
\begin{aligned}
T_2+S_1+S_2 = &-\frac{1}{\Delta x}\int_{u_j^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)(F_1(z)-F_1(u_j^n))\,dz \\
&+ \frac{1}{\Delta x}\int_{u_j^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)(F_2(z)-F_2(u_j^n))\,dz \\
&+ \frac{1}{\Delta x}\int_{u_{j-1}^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)(F_1(z)-F_1(u_{j-1}^n))\,dz \\
&- \frac{1}{\Delta x}\int_{u_{j+1}^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)(F_2(z)-F_2(u_{j+1}^n))\,dz.
\end{aligned}
$$

The two last terms on the right-hand side are positive as $F$ is monotone. Let

$$
H(z) = z - \frac{\Delta t}{\Delta x}\left(F_1(z)-F_2(z)\right).
$$

Then, by assumption (6.4),

$$
T_1+T_2+S_1+S_2 \geq \frac{1}{\Delta t}\int_{u_j^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c)\left[H(z)-H(u_j^n)\right]\,dz \geq 0.
$$

Adding $T_3$ to both sides proves Claim 2. $\square$

By Claim 2,

$$
\begin{aligned}
&E^c(u_j^n, u_j^{n+1}, u_{j+1}^n, u_{j-1}^n) \\
&\quad \geq \frac{1}{\Delta x^2} \int_{u_{j-1}^n}^{u_j^n} \psi_\varepsilon''(z,c)(A(z) - A(u_{j-1}^n))\, dz \\
&\qquad + \frac{1}{\Delta x^2} \int_{u_{j+1}^n}^{u_j^n} \psi_\varepsilon''(z,c)(A(z) - A(u_{j+1}^n))\, dz \\
&\qquad - \int_{u_j^n}^{u_j^{n+1}} \psi_\varepsilon''(z,c) D_- D_+ A(u_j^n)\, dz.
\end{aligned}
$$

Combining this with Claim 1 proves the lemma. □

**Lemma 6.2.** *Suppose $A' > 0$, and (6.4) applies. Let $u_\Delta = u_\Delta(x,t)$ be defined by (6.3), and let $u = u(y,s)$ be the classical solution of (1.1). Set $\psi(t) := \mathbb{1}_{\{[\nu,\tau)\}}(t)$ and define*

$$
\varphi(x,t,y,s) = \psi(t)\omega_r(x-y)\rho_{r_0}(t-s),
$$

*where $\omega_r, \rho_{r_0}, \nu, \tau$ are chosen as in Lemma 4.6. Then*

$$
\begin{aligned}
&\iint_{\Pi_T^2} |u_\Delta - u|\, \delta_{\Delta t}^+(t-\nu)\omega_r \rho_{r_0}\, dX \\
&\quad + \iint_{\Pi_T^2} |u_\Delta - u|\, S_{-\Delta t}^t \psi \omega_r (D_-^t \rho_{r_0} - \partial_t \rho_{r_0})\, dX \\
&\quad + \Delta t \iint_{\Pi_T^2} |u_\Delta - u|\, D_-^t \psi \omega_r \partial_s \rho_{r_0}\, dX \\
&\quad + \iint_{\Pi_T^2} \operatorname{sign}(u_\Delta - u)\,(f(u_\Delta) - f(u))\,(D_+\varphi + \varphi_y)\; dX \\
&\quad + \iint_{\Pi_T^2} \left( \int_{u_{\Delta x}}^{S_{\Delta x} u_\Delta} \operatorname{sign}(z-u)\, F_2'(z)\, dz \right) D_+ \varphi\, dX \\
&\quad + \iint_{\Pi_T^2} |A(u_\Delta) - A(u)|\,(D_- D_+ \varphi + (D_+ + D_-)\varphi_y + \varphi_{yy})\; dX \\
&\qquad \geq \iint_{\Pi_T^2} |u_\Delta - u|\, \delta_{\Delta t}^+(t-\tau)\omega_r \rho_{r_0}\, dX \\
&\qquad\quad + \liminf_{\varepsilon \downarrow 0} \iint_{\Pi_T^2} E_\Delta^\varepsilon \varphi\, dX \\
&\qquad\quad - \Delta t \iint_{\Pi_T^2} D_+^t \operatorname{sign}_\varepsilon(A(u_\Delta) - A(u)) D_- D_+ A(u_\Delta) \varphi\, dX,
\end{aligned}
$$

*where*

$$
\delta_{\Delta t}^+(t) = \frac{1}{\Delta t} \mathbb{1}_{\{[0,\Delta t)\}}(t),
$$

*and $E_\Delta^\varepsilon(x,t,y,s) = E^\varepsilon[u](u_{j-1}^n, u_j^n, u_{j+1}^n)(y,s)$ for $(x,t) \in I_j^n$.*

*Proof.* By Lemma 6.1, we obtain as in Lemma 4.5 the following inequality:

$$
\begin{aligned}
D^t_+\psi_\varepsilon(u^n_j,u) + \partial_s\psi_\varepsilon(u,u^n_j) + \partial_y q_\varepsilon(u,u^n_j) + D_-Q^u(u^n_j,u^n_{j+1}) \\
- (\partial_y^2 + \partial_y(D_- + D_+) + D_-D_+)\left|A(u^n_j) - A(u)\right|_\varepsilon) \\
\le -E^\varepsilon_{j,n} + \left(\int_{u^n_j}^{u^{n+1}_j} \psi''_\varepsilon(z,u)\,dz\right) D_-D_+A(u^n_j),
\end{aligned}
$$

where $E^\varepsilon_{j,n} := E^\varepsilon[u](u^n_{j-1},u^n_j,u^n_{j+1})$ is defined in Lemma 4.5. Note that

$$
\int_{u^n_j}^{u^{n+1}_j} \psi''_\varepsilon(z,u)\,dz = \Delta t\, D^t_+\mathrm{sign}_\varepsilon(A(u^n_j) - A(u)).
$$

Integration by parts for difference quotients and ordinary integration by parts gives

$$
\begin{aligned}
&\iint_{\Pi^2_T} \psi_\varepsilon(u_\Delta,u)D^t_-\varphi + \psi_\varepsilon(u,u_\Delta)\varphi_s\,dX \\
&\quad + \iint_{\Pi^2_T} q_\varepsilon(u,u_\Delta)\varphi_y + Q^u(u_\Delta,S_{\Delta x}u_\Delta)D_+\varphi\,dX \\
&\quad + \iint_{\Pi^2_T} |A(u_\Delta) - A(u)|_\varepsilon(\varphi_{yy} + (D_- + D_+)\varphi_y + D_-D_+\varphi)\,dX \\
&\quad \ge \iint_{\Pi^2_T} E^\varepsilon_\Delta\varphi\,dX - \Delta t\iint_{\Pi^2_T} D^t_+\mathrm{sign}_\varepsilon(A(u_{\Delta t}) - A(u))D_-D_+A(u_\Delta)\varphi\,dX.
\end{aligned}
$$

Consider the first term on the left-hand side. Let $\varepsilon$ tend to zero as in the proof of Lemma 4.6. Using the Leibniz rule for difference quotients we obtain

$$
D^t_-\varphi = S^t_{-\Delta t}\psi\omega_r D^t_-\rho_{r_0} + D^t_-\psi\omega_r\rho_{r_0}.
$$

Recall that $\partial_s\rho_{r_0} = -\partial_t\rho_{r_0}$, so adding and subtracting gives

$$
\varphi_s = -S^t_{-\Delta t}\psi\omega_r\partial_t\rho_{r_0} + \Delta t D^t_-\psi\omega_r\partial_s\rho_{r_0}.
$$

Hence,

$$
\begin{aligned}
&\iint_{\Pi^2_T} |u_\Delta - u|\left(D^t_-\varphi + \varphi_s\right)\,dX \\
&\quad = \iint_{\Pi^2_T} |u_\Delta - u|\,S^t_{-\Delta t}\psi\omega_r(D^t_-\rho_{r_0} - \partial_t\rho_{r_0})\,dX \\
&\qquad + \Delta t\iint_{\Pi^2_T} |u_\Delta - u|\,D^t_-\psi\omega_r\partial_s\rho_{r_0}\,dX + \iint_{\Pi^2_T} |u_\Delta - u|\,D^t_-\psi\omega_r\rho_{r_0}\,dX.
\end{aligned}
$$

Finally, we use that

$$
D^t_-\psi = \delta^+_{\Delta t}(t-\nu) - \delta^+_{\Delta t}(t-\tau). \tag{6.6}
$$

Concerning the second term on the left-hand side, we apply (3.8). The lemma now follows by sending $\varepsilon$ to zero, as in the proof of Lemma 4.6. ◻

As seen by comparing Lemmas 6.2 and 5.3, there is one new term. To estimate this term we will use a result from [12, p. 1853].

**Lemma 6.3.** *Let $u_j^n$ be the solution to* (6.1). *Suppose the CFL condition* (6.2) *is satisfied. Then there exists a constant $L$ such that*

$$\Delta x \sum_j \left|D_+A(u_j^m) - D_+A(u_j^n)\right| \leq L\sqrt{(m-n)\Delta t}, \qquad \textit{for all } m \geq n.$$

**Estimate 6.1.** Suppose (6.2) is satisfied. Then

$$\left|\Delta t \iint_{\Pi_T^2} D_+^t \operatorname{sign}_\varepsilon(A(u_\Delta) - A(u))D_-D_+A(u_\Delta)\varphi\, dX\right| \\ \leq C\frac{\sqrt{\Delta t}}{\Delta x} + C\frac{\Delta t}{r_0}\left(1 + \frac{\Delta t}{r_0}\right) + C\Delta t.$$

*Proof.* Integration by parts for difference quotients gives

$$\begin{aligned}
&\Delta t \iint_{\Pi_T^2} D_+^t \operatorname{sign}_\varepsilon(A(u_\Delta) - A(u))D_-D_+A(u_\Delta)\varphi\, dX \\
&\quad = -\Delta t \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_\Delta) - A(u))S_{-\Delta t}^t D_-D_+A(u_\Delta)D_-^t\varphi\, dX \\
&\qquad - \Delta t \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_\Delta) - A(u))D_-^t D_-D_+A(u_\Delta)\varphi\, dX =: T_1 + T_2,
\end{aligned}$$

where we have used that

$$D_-^t(D_-D_+A(u_\Delta)\varphi) = S_{-\Delta t}^t D_-D_+A(u_\Delta)D_-^t\varphi + D_-^t D_-D_+A(u_\Delta)\varphi.$$

Let us consider $T_1$ first. By the Leibniz rule for difference quotients,

$$\begin{aligned}
T_1 &= \Delta t \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_\Delta) - A(u))S_{-\Delta t}^t D_-D_+A(u_\Delta)S_{-\Delta t}^t\psi\omega_r D_-^t\rho_{r_0}\, dX \\
&\qquad + \Delta t \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_\Delta) - A(u))S_{-\Delta t}^t D_-D_+A(u_\Delta)D_-^t\psi\omega_r\rho_{r_0}\, dX \\
&=: T_{1,1} + T_{1,2}.
\end{aligned}$$

Using equation (6.6),

$$|T_{1,2}| \leq \Delta t \int_{\Pi_T} \left|S_{-\Delta t}^t D_-D_+A(u_\Delta)\right| \left(\left|\delta_{\Delta t}^+(t-\nu)\right| + \left|\delta_{\Delta t}^+(t-\tau)\right|\right) dxdt \leq C\Delta t,$$

as $\|D_-D_+A(u_\Delta(\cdot,t))\|_{L^1(\mathbb{R})}$ is bounded independent of $\Delta$ and $t$ ([13, Lemma 3.4]). Now, as in Lemma 4.11,

$$\left|D_-^t\rho_{r_0}\right| \leq \frac{\|\rho'\|_{L^\infty}}{r_0^2}\mathbb{1}_{\{|t-s|\leq r_0+\Delta t\}}(t,s)$$

and, therefore,

$$|T_{1,1}| \leq C\Delta t\frac{r_0 + \Delta t}{r_0^2}\int_{\Pi_T} \left|S_{-\Delta t}^t D_-D_+A(u_\Delta)\right| dxdt \leq C\frac{\Delta t}{r_0}\left(1 + \frac{\Delta t}{r_0}\right).$$

Next, we consider $T_2$. By Lemma 6.3,

$$
\begin{aligned}
|T_2| &\leq \Delta t \int_{\nu}^{\tau} \int_{\mathbb{R}} \left| D_-^t D_- D_+ A(u_\Delta) \right| \, dxdt \\
&\leq 2\frac{\Delta t}{\Delta x} \int_{\nu}^{\tau} \left\| D_-^t D_+ A(u_\Delta(\cdot, t)) \right\|_{L^1(\mathbb{R})} \, dt \\
&\leq 2TL \frac{\sqrt{\Delta t}}{\Delta x}.
\end{aligned}
$$

□

*Proof of Theorem* 6.1. We start out from Lemma 6.2 with $A(\sigma) = \hat{A}(\sigma) + \eta\sigma$, where $\hat{A}$ is the original degenerate diffusion function. By Estimate 6.1,

$$
\begin{aligned}
&\left| \Delta t \iint_{\Pi_T^2} D_+^t \mathrm{sign}_\varepsilon (A(u_\Delta) - A(u)) D_- D_+ A(u_\Delta) \varphi \, dX \right| \\
&\qquad \leq C \frac{\sqrt{\Delta t}}{\Delta x} + C \frac{\Delta t}{r_0} \left( 1 + \frac{\Delta t}{r_0} \right) + C\Delta t =: E_4.
\end{aligned}
$$

Since all the estimates from Section 4.1 apply, we obtain

$$
\begin{aligned}
&\iint_{\Pi_T^2} |u_\Delta - u| \, \delta^{\Delta t}(t - \tau) \omega_r(x - y) \rho_{r_0}(t - s) \, dX \\
&\qquad \leq \iint_{\Pi_T^2} |u_{\Delta x} - u| \, \delta^{\Delta t}(t - \nu) \omega_r(x - y) \rho_{r_0}(t - s) \, dX \\
&\qquad\qquad + E_1 + E_2 + E_3 + E_4,
\end{aligned}
$$

where $E_1, E_2, E_3$ are defined respectively in (4.9), (4.10), and (5.5). Let us make the assumption that $\nu = t_m$ and $\tau = t_n$ for some $m, n \in \mathbb{N}$. Then the above inequality takes the form

$$
\kappa(t_n) \leq \kappa(t_m) + E_1 + E_2 + E_3 + E_4,
$$

where

$$
\kappa(t) := \int_{\mathbb{R}} \int_{\Pi_T} |u_\Delta(x, t) - u(y, s)| \, \omega_r(x - y) \rho_{r_0}(t - s) \, dydsdx.
$$

Applying Lemmas 4.12 and 5.1, and following the reasoning given in the semi-discrete case, we deduce

$$
\begin{aligned}
&\|u_\Delta(\cdot, t_n) - u(\cdot, t_n)\|_{L^1(\mathbb{R})} \\
&\quad \leq \left\| u_\Delta^0 - u^0 \right\|_{L^1(\mathbb{R})} + (L_c + L_d) \, t_m \\
&\qquad + 2 \left( L_c r_0 + \left| u^0 \right|_{BV(\mathbb{R})} r \right) + C(1 + r + \Delta x)^2 \left( 1 + \frac{\Delta x}{r} \right)^3 \frac{\Delta x}{r^2} \\
&\qquad + C \frac{\Delta x}{r_0} + C \frac{\Delta t}{r_0} \left( 1 + r_0 + \frac{\Delta t}{r_0} \right) + C \frac{\sqrt{\Delta t}}{\Delta x} \\
&\quad \leq \left\| u_\Delta^0 - u^0 \right\|_{L^1(\mathbb{R})} + C \left( \frac{\Delta x}{r^2} + \frac{\Delta x + \Delta t}{r_0} + \frac{\sqrt{\Delta t}}{\Delta x} + r + r_0 \right),
\end{aligned}
$$

where $L_d$ is the constant in Lemma 5.1 and $L_c$ is the constant from Lemma 4.12. Let $r = r_0, \Delta x = r^3$ and $\Delta t = r^8$. It follows that

$$
\|u_\Delta(\cdot, t_n) - u(\cdot, t_n)\|_{L^1(\mathbb{R})} \leq \left\| u_\Delta^0 - u^0 \right\|_{L^1(\mathbb{R})} + C\Delta x^{1/3}.
$$

Finally, we send $\eta \to 0$ to conclude the proof of the theorem. □

## 7. Concluding remarks

The added complexity of convection-diffusion equations versus conservation laws [22] arises as a result of the need to work with an explicit form of the parabolic dissipation term. This is reflected in the fact that the rate of convergence is lowered to 1/3 (from 1/2 for conservation laws) due to Estimate 4.4 and Estimate 4.5. The optimality of the $\frac{1}{3}$ rate is an open problem. Concerning Section 6 (explicit schemes), one may wonder if it is possible to remove the strengthened CFL condition $\Delta t \sim \Delta x^{8/3}$ (the usual one demands $\Delta t \sim \Delta x^2$). The difficulty is that the parabolic dissipation term is needed to balance the temporal error contribution as well as to carry out the doubling-of-the-variables argument, and this forces us to impose a stronger relation between $\Delta t$ and $\Delta x$ in order to appropriately control the temporal error contribution. We do not know if the condition $\Delta t \sim \Delta x^{8/3}$ is genuinely needed or is simply an artifact of our method of proof. Finally, we are currently investigating the multidimensional case. For the semi-discrete scheme the main challenge seems to be the adaptation of Estimate 4.5, or more precisely to produce a multidimensional analogue of (4.7). As an additional difficulty, Lemma 6.3 is not available in several space dimensions; see [12]. At the moment our multidimensional convergence rates are lower than in the one-dimensional case.

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