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LOCAL BOUNDED COCHAIN PROJECTIONS

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ABSTRACT. We construct projections from $H\Lambda^k(\Omega)$, the space of differential k forms on Ω which belong to $L^2(\Omega)$ and whose exterior derivative also belongs to $L^2(\Omega)$, to finite dimensional subspaces of $H\Lambda^k(\Omega)$ consisting of piecewise polynomial differential forms defined on a simplicial mesh of Ω. Thus, their definition requires less smoothness than assumed for the definition of the canonical interpolants based on the degrees of freedom. Moreover, these projections have the properties that they commute with the exterior derivative and are bounded in the $H\Lambda^k(\Omega)$ norm independent of the mesh size h. Unlike some other recent work in this direction, the projections are also locally defined in the sense that they are defined by local operators on overlapping macroelements, in the spirit of the Clément interpolant. A double complex structure is introduced as a key tool to carry out the construction.

1. Introduction

Projection operators which commute with the governing differential operators are key tools for the stability analysis of finite element methods associated to a differential complex. In fact, such projections have been a central feature of the analysis of mixed finite element methods since the beginning of such analysis; cf. [5, 6]. However, a key difficulty is that, for most of the standard finite element spaces, the canonical projection operators defined from the degrees of freedom are not well defined on the appropriate function spaces. This is the case for the Lagrange finite elements, considered as a subspace of the Sobolev space H^1, and for the Raviart-Thomas [20], Brezzi-Douglas-Marini [7], and Nédélec [18, 19] finite element spaces considered as subspaces of $H(\mathrm{div})$ or $H(\mathrm{curl})$. For example, the classical continuous piecewise linear interpolant, based on the values at the vertices of the mesh, is not defined for functions in H^1 in dimensions higher than one. Therefore, even if the canonical projections commute with the governing differential operators on smooth functions, these operators cannot be directly used in a stability argument for the associated finite element method due to the lack of boundedness of the projections in the proper operator norms. In addition to the canonical projection operators, it is worth mentioning another family of projection operators that commute with the exterior derivative. This approach, usually referred to as projection based interpolation, is detailed in the work of Demkowicz and collaborators (cf. [8], [12], [13], [14], [15]). The main motivation for the construction of these operators was the analysis of the so-called p-version of the finite element method, i.e., the focus is on the dependence of the polynomial degree of the finite element spaces.

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However, as in the case of the canonical projection operators, the definition of these operators requires some additional smoothness of the underlying functions, so again they cannot be used directly in the standard stability arguments. On the other hand, the classical Clément interpolant [11] is a local operator, and it is well defined for functions in L^2. However, the Clément interpolant is not a projection, and the obvious extensions of the Clément operator to higher order finite element differential forms (cf. [1,3]) do not commute with the exterior derivative. Therefore, these operators are not directly suitable for a stability analysis.

Bounded commuting projections have been constructed in previous work. The first such construction was given by Schöberl in [21]. The idea is to compose a smoothing operator and the unbounded canonical projection to obtain a bounded operator which maps the proper function space into the finite element space. In order to obtain a projection, one composes the resulting operator with the inverse of this operator restricted to the finite element space. In [21], a perturbation of the finite element space itself was used to construct the proper smoother. In a related paper, Christiansen [9] proposed to use a more standard smoothing operator defined by a mollifier function. Using this idea, variants of Schöberl's construction are analyzed in [1, Section 5], [3, Section 5], and [10]. The constructed projections, frequently referred to as "smoothed projections," commute with the exterior derivative and they are bounded in L^2. Therefore, they can be used to establish stability of finite element methods. However, these projections lack another key property of the canonical projections; they are not locally defined. In fact, up to now it has been an open question if it is possible to construct bounded and commuting projections which are locally defined. The projections defined in this paper have all these properties. The construction presented below resembles the construction of the Clément operator in the sense that it is based on local operators on overlapping macroelements. The discussion here is performed in the setting of no boundary conditions, but the construction of the projections also adapt naturally to homogeneous essential boundary conditions; cf. [10] for a corresponding discussion in the setting of smoothed projections.

We will adopt the language of finite element exterior calculus as in [1,3]. The theory presented in these papers may be described as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral domain, and let $H\Lambda^k(\Omega)$ be the space of differential k forms u on Ω, which is in L^2, and where its exterior derivative, $du = d^k u$, is also in L^2. This space is a Hilbert space. The L^2 version of the de Rham complex then takes the form

$$H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega).$$

The basic construction in finite element exterior calculus is of a corresponding subcomplex

$$\Lambda_h^0 \xrightarrow{d} \Lambda_h^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda_h^n,$$

where the spaces Λ_h^k are finite dimensional subspaces of $H\Lambda^k(\Omega)$ consisting of piecewise polynomial differential forms with respect to a partition, $\mathcal{T}_h$, of the domain Ω. In the theoretical analysis of the stability of numerical methods constructed from this discrete complex, bounded projections $\pi_h^k : H\Lambda^k(\Omega) \to \Lambda_h^k$ are utilized, such that the following diagram commutes.

$$\begin{array}{ccccccc}
H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & H\Lambda^n(\Omega) \\
\Big\downarrow \pi_h^0 & & \Big\downarrow \pi_h^1 & & & & \Big\downarrow \pi_h^n \\
\Lambda_h^0 & \xrightarrow{d} & \Lambda_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda_h^n
\end{array}$$

Such commuting projections are referred to as cochain projections. The importance of bounded cochain projections is immediately seen from the analysis of the mixed finite element approximation of the associated Hodge Laplacian. In fact, it follows from the results of [3, Section 3.3] that the existence of bounded cochain projections is equivalent to stability of the associated finite element method. Furthermore, if these projections are local, like the ones we construct here, then improved properties with respect to error estimates and adaptivity may be obtained; cf. [16, 17].

For a general reference to finite element exterior calculus, we refer to the survey papers [1, 3], and references given therein. As is shown there, the spaces Λ_h^k are taken from two main families. Either Λ_h^k is of the form $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$, consisting of all elements of $H\Lambda^k(\Omega)$ which restrict to polynomial k-forms of degree at most r on each simplex T in the partition $\mathcal{T}_h$, or $\Lambda_h^k = \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$, which is a space which sits between $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h)$ (the exact definition will be recalled below). These spaces are generalizations of the Raviart-Thomas and Brezzi-Douglas-Marini spaces, used to discretize $H(\mathrm{div})$ and $H(\mathrm{rot})$ in two space dimensions, and the Nédélec edge and face spaces of the first and second kind, used to discretize $H(\mathrm{curl})$ and $H(\mathrm{div})$ in three space dimensions.

A main feature of the construction of the projections given below is that they are based on a direct sum geometrical decomposition of the finite element space. In the general case of finite element differential forms, such a decomposition was constructed in [2]. However, this is a standard concept in the case of Lagrange finite elements. Let $\mathcal{T}_h$ be a simplicial triangulation of a polyhedral domain $\Omega \in \mathbb{R}^n$. If T is a simplex we let $\Delta(T)$ be the set of all subsimplexes of T, and by $\Delta_m(T)$ all subsimplexes of dimension m. So if T is a tetrahedron in $\mathbb{R}^3$, then $\Delta_m(T)$ are the set of vertices, edges, and faces of T for $m = 0, 1, 2$, respectively. We further denote by $\Delta(\mathcal{T}_h)$ the set of all subsimplices of all dimensions of the triangulation $\mathcal{T}_h$, and correspondingly by $\Delta_m(\mathcal{T}_h)$ the set of all subsimplices of dimension m. The desired geometric decomposition of the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ is based on the property that the elements of these spaces are uniquely determined by their trace, tr_f, for all f of $\Delta(\mathcal{T}_h)$ with dimension greater or equal to k. The decompositions of the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ established in [2] is then of the form

$$P_r\Lambda^k(\mathcal{T}_h) = \bigoplus_{\substack{f \in \Delta(\mathcal{T}_h) \\ \dim f \geq k}} E_{f,r}^k(\mathring{\mathcal{P}}_r\Lambda^k(f)). \tag{1.1}$$

Here $\mathring{\mathcal{P}}_r\Lambda^k(f)$ is the subspace of $\mathcal{P}_r\Lambda^k(f)$ consisting of elements with vanishing trace on the boundary of f. The operator $E_{f,r}^k : \mathring{\mathcal{P}}_r\Lambda^k(f) \to P_r\Lambda^k(\mathcal{T}_h)$ is an extension operator in the sense that $\mathrm{tr}_f \circ E_{f,r}^k$ is the identity operator on $\mathring{\mathcal{P}}_r\Lambda^k(f)$. Furthermore, $E_{f,r}^k$ is local in the sense that the support of functions in $E_{f,r}^k(\mathring{\mathcal{P}}_r\Lambda^k(f))$ is restricted to the union of the elements of $\mathcal{T}_h$ which have f as a subsimplex. A completely analogous decomposition

$$P_r^-\Lambda^k(\mathcal{T}_h) = \bigoplus_{\substack{f\in\Delta(\mathcal{T}_h)\\ \dim f\geq k}} E_{f,r}^{k-}(\mathring{\mathcal{P}}_r^-\Lambda^k(f)) \tag{1.2}$$

exists for the space $P_r^-\Lambda^k(\mathcal{T}_h)$.

We will utilize modifications of the decompositions (1.1) and (1.2) to construct local bounded cochain projections onto the finite element spaces $P_r\Lambda^k(\mathcal{T}_h)$ and $P_r^-\Lambda^k(\mathcal{T}_h)$. In the spirit of the Clément operator, we will use local projections to define the operators $\mathrm{tr}_f \circ\pi_h^k$ for each $f \in \Delta(\mathcal{T}_h)$ with dimension greater or equal to k. To make sure that the projections π_h^k commute with the exterior derivative, we will use a local Hodge Laplace problem to define the local projections, while the extension operators will be of the form of harmonic extension operators.

This paper is organized as follows. In Section 2 we introduce some basic notation, and we show how to construct the new projection in the case of scalar valued functions, or zero forms. We also review some basic results on differential forms and their finite element approximations. A key step of the theory below is to construct a special projection into the space of Whitney forms [22], i.e., the space $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$. In fact, in the present setting the construction in this lowest order case is in some sense the most difficult part of the theory, since here we need to relate local operators defined on different subdomains. To achieve this we utilize a structure which resembles the Čech-de Rham double complex; cf. [4]. In addition to being a projection onto the Whitney forms, the special projection constructed in Section 3 will also satisfy a mean value property with respect to higher order finite element spaces; cf. equation (3.1) below. The general construction of the cochain projections, covering all spaces of the form $P_r\Lambda^k(\mathcal{T}_h)$ or $P_r^-\Lambda^k(\mathcal{T}_h)$, is then performed in Section 4. Finally, in Section 5 we derive precise local bounds for the constructed projections.

2. Notation and preliminaries

We will use $\langle\cdot,\cdot\rangle$ to denote L^2 inner products on the domain Ω. For subdomains $D \subset \Omega$ we will use a subscript to indicate the domain, i.e., we write $\langle u, v\rangle_D$ to denote the L^2 inner product on the domain D.

We will assume that $\{\mathcal{T}_h\}$ is a family of simplicial triangulations of $\Omega \in \mathbb{R}^n$, indexed by the mesh parameter $h = \max_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of T. In fact, h_f will be used to denote the diameter of any $f \in \Delta(\mathcal{T}_h)$. We will assume throughout that the triangulation is shape regular, i.e., the ratio $h_T^n/|T|$ is uniformly bounded for all the simplices $T \in \mathcal{T}_h$ and all triangulations of the family. Here $|T|$ denotes the volume of T. Note that it is a simple consequence of shape regularity that the ratio h_T/h_f, for $f \in \Delta(T)$ with $\dim f \geq 1$ is also uniformly bounded. We will use $[x_0, x_1, \dots x_k]$ to denote the convex combination of the points $x_0, x_1, \dots, x_k \in \Omega$. Hence, any $f \in \Delta_k(\mathcal{T}_h)$ is of the form $f = [x_0, x_1, \dots x_k]$, where $x_0, x_1, \dots, x_k \in \Delta_0(\mathcal{T}_h)$. Furthermore, the order of the points x_j reflects the orientation of the manifold f. We will let $f_j \in \Delta_{k-1}(\mathcal{T}_h)$ denote the subcomplex of f obtained by deleting the vertex x_j, i.e., $f_j = [x_0, \dots, x_{j-1}, \hat{x}_j, x_{j+1}, \dots, x_k]$. Here the symbol $\hat{}$ over a term means that the term is omitted. Hence, if j is even, then f_j has the orientation induced from f, while the orientation is reversed if j is odd.

For each $f \in \Delta(\mathcal{T}_h)$, we let Ω_f be the associated macroelement consisting of the union of the elements of $\mathcal{T}_h$ containing f, i.e.,

$$\Omega_f = \bigcup \{ T \,|\, T \in \mathcal{T}_h,\ f \in \Delta(T) \}.$$

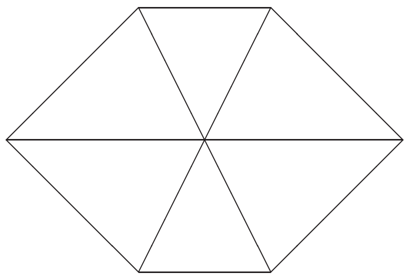

Figure 1. Vertex macroelement, $n = 2$.

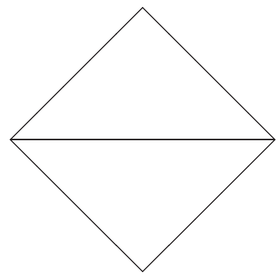

Figure 2. Edge macroelement, $n = 2$.

In addition to macroelements Ω_f, we will also find it convenient to introduce the notion of an extended macroelement Ω_f^e defined for $f \in \Delta(\mathcal{T}_h)$ by

$$\Omega_f^e = \bigcup_{g \in \Delta_0(f)} \Omega_g.$$

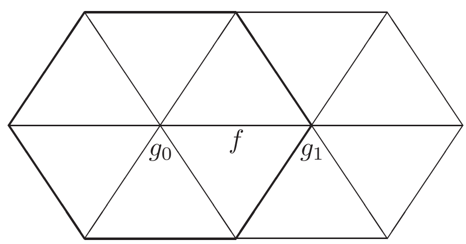

Figure 3. The extended macroelement Ω_f^e corresponding to the union of the two macroelements Ω_{g_0} (outlined by the thick lines) and Ω_{g_1}, $n = 2$.

In the special case that $\dim f = 0$, i.e., f is a vertex, then $\Omega_f^e = \Omega_f$. In general, if $f, g \in \Delta(\mathcal{T}_h)$ with $g \in \Delta(f)$, then

$$\Omega_f \subset \Omega_g \quad \text{and } \Omega_g^e \subset \Omega_f^e.$$

We shall assume throughout that all the macroelements of the form Ω_f and Ω_f^e, for $f \in \Delta(\mathcal{T}_h)$, are contractive. We let $\mathcal{T}_{f,h}$ denote the restriction of $\mathcal{T}_h$ to Ω_f, while $\mathcal{T}_{f,h}^e$ is the corresponding restriction of $\mathcal{T}_h$ to Ω_f^e. It is straightforward to check that a consequence of the shape regularity of the family $\{\mathcal{T}_h\}$ is that the ratio $|\Omega_f^e|/|\Omega_f|$ is uniformly bounded. Furthermore, the coverings $\{\Omega_f\}_{f \in \Delta(\mathcal{T}_h)}$ and $\{\Omega_f^e\}_{f \in \Delta(\mathcal{T}_h)}$ of the domain Ω both have the bounded overlap property, i.e., the sum of the characteristic functions is bounded uniformly in h. Finally, although the projections π_h^k that we construct clearly depend on h, we will simplify notation by dropping the subscript h, referring to them as π^k. The subscript h will also be dropped on other operators whose dependence on h is clear.

2.1. **Construction of the projection for scalar valued functions.** To motivate the construction for the general case of k forms given below, we will first give an outline of how the projection is constructed for zero forms, i.e., for scalar valued functions. The projection π^0 will map the space $H^1(\Omega) = H\Lambda^0(\Omega)$ into $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$, the space of continuous piecewise polynomials of degree r with respect to the partition $\mathcal{T}_h$. The space $\mathcal{P}_r\Lambda^0(\mathcal{T}_{f,h})$ is the restriction of the space $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ to $\mathcal{T}_{f,h}$, and $\mathring{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h})$ is the subspace of $\mathcal{P}_r\Lambda^0(\mathcal{T}_{f,h})$ of functions which vanish on the boundary, $\partial\Omega_f$, of Ω_f. Of course, by the zero extension the space $\mathring{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h})$ can also be considered as a subspace of $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$.

A key tool for the construction is the local projection $P_f^0 : H^1(\Omega_f) \to \mathcal{P}_r(\mathcal{T}_{f,h})$, associated to each $f \in \Delta(\mathcal{T}_h)$. If $\dim f = 0$, such that f is a vertex, we define P_f^0 by $P_f^0 u \in \mathcal{P}_r\Lambda^0(\mathcal{T}_{f,h})$ as the H^1 projection of u, i.e., $P_f^0 u$ is the solution of

$$\begin{aligned} \langle P_f^0 u, 1\rangle_{\Omega_f} &= \langle u, 1\rangle_{\Omega_f}, \\ \langle dP_f^0 u, dv\rangle_{\Omega_f} &= \langle du, dv\rangle_{\Omega_f}, \quad v \in \mathcal{P}_r(\mathcal{T}_{f,h}). \end{aligned}$$

Of course, for zero forms, the exterior derivative, d, can be identified with the ordinary gradient operator. When $1 \le \dim f \le n$, we first define the space

$$\breve{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h}) = \{u \in \mathcal{P}_r\Lambda^0(\mathcal{T}_{f,h}) \mid \operatorname{tr}_f u \in \mathring{\mathcal{P}}_r(f)\}.$$

We then define $P_f^0 u \in \breve{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h})$ as the solution of

$$\langle dP_f^0 u, dv\rangle_{\Omega_f} = \langle du, dv\rangle_{\Omega_f}, \quad v \in \breve{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h}).$$

The projection π^0 will be defined recursively with respect to the dimensions of the subsimplices of the triangulation $\mathcal{T}_h$. More precisely, we will utilize a sequence of local operators $\{\pi_m^0\}_{m=0}^n$, and define $\pi^0 = \pi_n^0$. The operators π_m^0 are defined recursively by

$$\pi_m^0 u = \pi_{m-1}^0 u + \sum_{f\in\Delta_m(\mathcal{T}_h)} E_f^0 \operatorname{tr}_f P_f^0(u - \pi_{m-1}^0 u), \quad 1 \le m \le n. \tag{2.1}$$

Here $E_f^0 : \mathring{\mathcal{P}}_r(f) \to \mathring{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h}) \subset \mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ is the harmonic extension operator determined by

$$\langle dE_f^0\phi, dv\rangle_{\Omega_f} = 0, \quad v \in \mathring{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h}), \operatorname{tr}_f v = 0,$$

and that $\operatorname{tr}_f E_f^0$ is the identity on $\mathring{\mathcal{P}}_r(f)$. To simplify notation, we have suppressed the dependency of the operator E_f^0 on the degree r. It is a key property that $\operatorname{tr}_g E_f^0\phi = 0$ for all $g \in \Delta(\mathcal{T}_h)$, $\dim g \le \dim f$, and $g \ne f$. For the vertex degrees of freedom we will use an alternative extension operator. We simply define π_0^0 by

$$\pi_0^0 u = \sum_{f\in\Delta_0(\mathcal{T}_h)} \mathcal{E}_f^0 \operatorname{tr}_f P_f^0 u = \sum_{f\in\Delta_0(\mathcal{T}_h)} \mathcal{E}_f^0 (P_f^0 u)(f)$$

where, for any $\alpha \in \mathbb{R}$, $\mathcal{E}_f^0\alpha$ is the piecewise linear function with value α at the vertex f and value zero at all other vertices. Hence, for $f \in \Delta_0(\mathcal{T}_h)$ we have $\mathcal{E}_f^0 = E_f^0$ if $r = 1$. The reason for choosing the special low order extension operator for vertices is not essential at this point, but will be needed later to make sure that the projections π^k commute with the exterior derivative.

The key result for the construction above is the following lemma.

Lemma 2.1. *The operator π^0 is a projection onto $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$.*

Proof. To see that π^0 is a projection, we only need to check that if $u \in \mathcal{P}_r\Lambda^0(\mathcal{T}_h)$, then for all $f \in \Delta(\mathcal{T}_h)$, $\operatorname{tr}_f \pi^0 u = \operatorname{tr}_f u$. We do this by induction on m, where m corresponds to the dimension of the face $f \in \Delta(\mathcal{T}_h)$. We assume throughout that $u \in \mathcal{P}_r\Lambda^0(\mathcal{T}_h)$. We will show that the operator π^0_m has the property that

$$\operatorname{tr}_f \pi^0_m u = \operatorname{tr}_f u \quad \text{if } f \in \Delta(\mathcal{T}_h) \quad \text{with } \dim f \le m, \tag{2.2}$$

and since $\pi^0 = \pi^0_n$ this will establish the desired result. If $f \in \Delta_0(\mathcal{T}_h)$, then $P^0_h u = u|_{\Omega_f}$. By construction, it therefore follows that (2.2) holds for $m = 0$. Assume next that (2.2) holds for $m-1$, where $1 \le m \le n$. It follows that for any $f \in \Delta_m(\mathcal{T}_h)$, we have $\operatorname{tr}_f(u - \pi^0_{m-1}u) \in \mathring{\mathcal{P}}_r(f)$, and therefore $P^0_f(u - \pi^0_{m-1}u) = u - \pi^0_{m-1}u$. It follows by construction that $\operatorname{tr}_g \pi^0_m u = \operatorname{tr}_g \pi^0_{m-1}u = \operatorname{tr}_g u$ for $g \in \Delta(\mathcal{T}_h)$, with $\dim g < m$, while for $f \in \Delta_m(\mathcal{T}_h)$ we have

$$\operatorname{tr}_f \pi^0_m u = \operatorname{tr}_f(\pi^0_{m-1}u + P^0_f(u - \pi^0_{m-1}u)) = \operatorname{tr}_f u.$$

Therefore, (2.2) holds for m and the proof is completed. □

It follows from the construction above that the operator π^0 is local. For example, for any $T \in \mathcal{T}_h$ we have that $(\pi^0_0 u)_T$ depends only on u restricted to the extended macroelement Ω^e_T. Define $D_{m,T} \subset \Omega$ by

$$D_{m,T} = \bigcup \{\, D_{m-1,T'} \mid T' \in \mathcal{T}_{f,h},\ f \in \Delta_m(T)\,\}, \quad D_{0,T} = \Omega^e_T. \tag{2.3}$$

It follows from (2.1) that $(\pi_m u)|_T$ depends only on $u|_{D_{m,T}}$. In particular, $(\pi^0 u)|_T$ depends only on $u|_{D_T}$, where $D_T = D_{n,T}$.

The operator π^0 satisfies the following local estimate.

Theorem 2.2. *Let $T \in \mathcal{T}_h$. The operator π^0 satisfies the bounds*

$$\|\pi^0 u\|_{L^2(T)} \le C(\|u\|_{L^2(D_T)} + h_T\|du\|_{L^2(D_T)})$$

and

$$\|d\pi^0 u\|_{L^2(T)} \le C\|du\|_{L^2(D_T)},$$

where the constant C is independent of h and $T \in \mathcal{T}_h$.

In fact, this result is just a special case of Theorem 5.2 below, so we omit the proof here. Of course, due to the bounded overlap property of the covering $\{D_T\}_{T\in\mathcal{T}_h}$ of Ω, derived from the corresponding property of $\{\Omega^e_f\}$, global estimates follow directly from the local estimates above.

2.2. **Differential forms and finite element spaces.** We will basically adopt the notation from [3]. The spaces $\mathcal{P}_r\Lambda^k(\mathcal{T}_h) \subset H\Lambda^k(\Omega)$ can be characterized as the space of piecewise polynomial k forms u of degree less than or equal to r, such that the trace, $\operatorname{tr}_f u$, is continuous for all $f \in \Delta(\mathcal{T}_h)$, with $\dim f \ge k$, where we recall that the trace, tr_f, of a differential form is defined by restricting to f and applying the form only to tangent vectors. The space $\mathcal{P}^-_r\Lambda^k(\mathcal{T}_h) \subset H\Lambda^k(\Omega)$ is defined similarly, but on each element $T \in \mathcal{T}_h$, u is restricted to be in $\mathcal{P}^-_r\Lambda^k \subset \mathcal{P}_r\Lambda^k$. Here, the polynomial class $\mathcal{P}^-_r\Lambda^k$ consists of all elements u of $\mathcal{P}_r\Lambda^k$ such that u contracted with the position vector x, $u \lrcorner x$, is in $\mathcal{P}_r\Lambda^{k-1}$. Hence, for each k we have a sequence of nested spaces

$$\mathcal{P}^-_1\Lambda^k(\mathcal{T}_h) \subset \mathcal{P}_1\Lambda^k(\mathcal{T}_h) \subset \mathcal{P}^-_2\Lambda^k(\mathcal{T}_h) \subset \dots H\Lambda^k(\Omega).$$

In particular, $\mathcal{P}_r^-\Lambda^0(\mathcal{T}_h) = \mathcal{P}_r\Lambda^0(\mathcal{T}_h)$, and $\mathcal{P}_r^-\Lambda^n(\mathcal{T}_h) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}_h)$.

Instead of distinguishing the theory for the spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$, we will use the simplified notation $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ to denote either a space of the family $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ or $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$. More precisely, we assume that we are given a sequence of spaces $\mathcal{P}\Lambda^k(\mathcal{T}_h)$, for $k = 0, 1, \ldots, n$, such that the corresponding polynomial sequence $(\mathcal{P}\Lambda, d)$, given by

$$\mathbb{R} \to \mathcal{P}\Lambda^0(\mathbb{R}^n) \xrightarrow{d} \mathcal{P}\Lambda^1(\mathbb{R}^n) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}\Lambda^n(\mathbb{R}^n) \to 0 \tag{2.4}$$

is an exact complex (cf. Section 5.1.4 of [3]). In particular, this allows for combinations of spaces taken from the two families $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$. For any $f \in \Delta(\mathcal{T}_h)$, with $\dim f \geq k$, the space $\mathcal{P}\Lambda^k(f) = \operatorname{tr}_f \mathcal{P}\Lambda^k(\mathcal{T}_h)$, while $\mathring{\mathcal{P}}\Lambda^k(f) = \{v \in \mathcal{P}\Lambda^k(f) \mid \operatorname{tr}_{\partial f} v = 0\}$. The corresponding polynomial complexes of the form $(\mathcal{P}\Lambda(f), d)$ are all exact. Furthermore, the complexes with homogeneous boundary conditions, $(\mathring{\mathcal{P}}\Lambda(f), d)$, given by

$$\mathring{\mathcal{P}}\Lambda^0(f) \xrightarrow{d} \mathring{\mathcal{P}}\Lambda^1(f) \xrightarrow{d} \cdots \xrightarrow{d} \mathring{\mathcal{P}}\Lambda^{\dim f}(f) \to \mathbb{R} \tag{2.5}$$

are also exact.

We recall that the spaces $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ admit degrees of freedom of the form

$$\int_f \operatorname{tr}_f u \wedge \eta, \quad \eta \in \mathcal{P}'(f,k), \quad f \in \Delta(\mathcal{T}_h), \tag{2.6}$$

where $\mathcal{P}'(f,k) \subset \Lambda^{\dim f - k}(f)$ is a polynomial space of differential forms and the symbol $\wedge$ is used to denote the exterior product. These degrees of freedom uniquely determine an element in $\mathcal{P}\Lambda^k(\mathcal{T}_h)$, (cf. Theorem 5.5 of [3]). In fact, if

$$\mathcal{P}\Lambda^k(\mathcal{T}_h) = \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h), \quad \text{then } \mathcal{P}'(f,k) = \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f - k}(f),$$

while if

$$\mathcal{P}\Lambda^k(\mathcal{T}_h) = \mathcal{P}_r\Lambda^k(\mathcal{T}_h), \quad \text{then } \mathcal{P}'(f,k) = \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f).$$

If $v \in \mathring{\mathcal{P}}\Lambda^k(f)$, then v is uniquely determined by the functionals derived from $\mathcal{P}'(f,k)$. Furthermore, any $v \in \mathcal{P}\Lambda^k(f)$ is uniquely determined by $\mathcal{P}'(g,k)$ for all $g \in \Delta(f)$. In particular, if $\dim f < k$, then $\mathcal{P}'(f,k)$ is empty, while $\mathcal{P}'(f,k)$ is always nonempty if $\dim f = k$. For $\dim f > k$ the set $\mathcal{P}'(f,k)$ can also be empty if the polynomial degree r is sufficiently low.

The local spaces $\mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$ and $\mathcal{P}\Lambda^k(\mathcal{T}^e_{f,h})$ are defined by restricting the space $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ to the macroelements Ω_f or Ω_f^e. It follows from the assumption that Ω_f and Ω_f^e are contractive, that all the local complexes $(\mathcal{P}\Lambda(\mathcal{T}_{f,h}), d)$ and $(\mathcal{P}\Lambda(\mathcal{T}^e_{f,h}), d)$ are exact. The same holds for the subcomplexes $(\mathring{\mathcal{P}}\Lambda(\mathcal{T}_{f,h}), d)$ and $(\mathring{\mathcal{P}}\Lambda(\mathcal{T}^e_{f,h}), d)$, corresponding to the subspaces of functions with zero trace on the boundary of the macroelements.

For a given triangulation $\mathcal{T}_h$, the spaces of lowest order polynomial degree, $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, i.e., the space of Whitney forms, will play a special role in our construction. The dimension of this space is equal to the number of elements in $\Delta_k(\mathcal{T}_h)$, and the properties of these spaces will in some sense reflect the properties of the triangulation. Therefore, this space will be used to transfer information between different macroelements; cf. Section 3 below. For $k = 0$ this space is just $\mathcal{P}_1\Lambda^k(\mathcal{T}_h)$, the space of continuous piecewise linear functions. The natural basis for this space is the set of generalized barycentric coordinates, defined to be one at one vertex, and zero at all other vertices. It follows from the discussion above that the degrees of freedom for the space $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, $0 \le k \le n$, are $\int_f u$ for all $f \in \Delta_k(\mathcal{T}_h)$. In fact, if $f = [x_0, x_1, \dots x_k] \in \Delta_k(\mathcal{T}_h)$, we define the Whitney form associated to f, $\phi_f^k \in \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, by

$$\phi_f^k = \sum_{i=0}^{k} (-1)^i \lambda_i d\lambda_0 \wedge \cdots \wedge \widehat{d\lambda_i} \wedge \cdots \wedge d\lambda_k,$$

where $\lambda_0, \lambda_1, \dots, \lambda_k$ are the barycentric coordinates associated to the vertices x_i. The basis function ϕ_f^k reduces to a constant k form on f, i.e., $\operatorname{tr}_f \phi_f^k \in \mathcal{P}_0\Lambda^k(f)$, and it has the property that $\operatorname{tr}_g \phi_f^k = 0$ for $g \in \Delta_k(\mathcal{T}_h)$, $g \ne f$. In fact, if $\mathsf{vol}_f \in \mathcal{P}_0\Lambda^k(f)$ is the volume form on f, scaled such that $\int_f \mathsf{vol}_f = 1$, then

$$\operatorname{tr}_f \phi_f^k = (k!)^{-1} \mathsf{vol}_f;$$

cf. [1, Section 4.1]. Furthermore, the map $\mathsf{vol}_f \to \mathcal{E}_f^k \mathsf{vol}_f = k! \phi_f^k$ defines an extension operator $\mathcal{E}_f^k : \mathcal{P}_0\Lambda^k(f) \to \mathring{\mathcal{P}}_1^-\Lambda^k(\mathcal{T}_{f,h})$ for any $f \in \Delta_k(\mathcal{T}_h)$. We observe that the operators $\mathcal{E}_f^k$ are natural generalizations of the piecewise linear extension operators $\mathcal{E}_f^0$, introduced above for scalar valued functions. In fact, any element u of $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ admits the representation

$$u = \sum_{f \in \Delta_k(\mathcal{T}_h)} \left(\int_f \operatorname{tr}_f u \right) \mathcal{E}_f^k \mathsf{vol}_f. \tag{2.7}$$

We finally note that it follows from Stokes' theorem that if $f = [x_0, x_1, \dots, x_{k+1}]$ and u is a sufficiently smooth k form on f, then

$$\int_f du = \sum_{j=0}^{k+1} (-1)^j \int_{f_j} \operatorname{tr}_{f_j} u, \tag{2.8}$$

where $f_j = [x_0, \dots, x_{j-1}, \hat{x}_j, x_{j+1}, \cdots x_{k+1}]$. Here the factor $(-1)^j$ enters as a consequence of orientation.

3. A special projection onto the Whitney forms

Recall that the purpose of this paper is to construct local cochain projections π^k which map $H\Lambda^k(\Omega)$ boundedly onto the piecewise polynomial space $\mathcal{P}\Lambda^k(\mathcal{T}_h)$. Furthermore, in the construction of π^0 given above, the construction of $\operatorname{tr}_f \circ \pi^0$ is based on a local projection, P_f^0, defined with respect to the associated macroelement Ω_f. Therefore one might hope that all the projections π^k have the property that $\operatorname{tr}_f \circ \pi^k$ is defined from a local projection operator defined on Ω_f for $f \in \Delta(\mathcal{T}_h)$, $\dim f \ge k$. However, a simple computation in two space dimensions, and with $\mathcal{P}\Lambda^k(\mathcal{T}_h) = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, will convince the reader that if $f = [x_0, x_1] \in \Delta_1(\mathcal{T}_h)$, then

$$\int_f \operatorname{tr}_f d\pi^0 u = \int_{x_0}^{x_1} \frac{d}{ds} \pi^0 u \, ds = (\pi^0 u)(x_1) - (\pi^0 u)(x_0),$$

and the right-hand side here clearly depends on u restricted to the union of the macroelements associated to the vertices x_0 and x_1. Therefore, $\int_f \operatorname{tr}_f \pi^1 du = \int_f \operatorname{tr}_f d\pi^0 u$ must also depend on u restricted to the union of these macroelements, and this domain is exactly equal to the extended macroelement Ω_f^e. This motivates why the extended macroelements, Ω_f^e, for $f \in \Delta_k(\mathcal{T}_h)$, will appear in the construction below. In fact, a special projection operator, $R^k : H\Lambda^k(\Omega) \to \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h) \subset \mathcal{P}\Lambda^k(\mathcal{T}_h)$, will be utilized in the construction of π^k to make sure that

$$\int_f \operatorname{tr}_f \pi^k du = \int_f \operatorname{tr}_f d\pi^{k-1} u = \int_{\partial f} \operatorname{tr}_{\partial f} \pi^{k-1} u,$$

for all $f \in \Delta_k(\mathcal{T}_h)$.

The operator R^k will commute with the exterior derivative, and it is a projection onto $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$. Therefore, in the case of lowest polynomial degree, when $\mathcal{P}\Lambda^k(\mathcal{T}_h) = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, we will take $\pi^k = R^k$. However, another key property of the operator R^k is that in the general case, when $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ is only contained in $\mathcal{P}\Lambda^k(\mathcal{T}_h)$, we will have

$$\int_f \operatorname{tr}_f R^k u = \int_f \operatorname{tr}_f u, \quad f \in \Delta_k(\mathcal{T}_h),\ u \in \mathcal{P}\Lambda^k(\mathcal{T}_h), \tag{3.1}$$

i.e., the operator R^k preserves the mean values of the traces of function in $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ on subsimplexes f of dimension k. The rest of this section is devoted to the construction of the operator R^k, and the derivation of the key properties given in Theorem 3.6 below.

3.1. **Tools for the construction.** Following our convention, we have suppressed the dependence on the mesh parameter h of the operator R^k and the other operators defined in this section. To define the special projection R^k onto the Whitney forms, $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, we will use local projections, Q_f^k, defined with respect to the extended macroelements Ω_f^e. We define the projection $Q_f^k : H\Lambda^k(\Omega_f^e) \to \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}^e)$ by the system

$$\langle Q_f^k u, d\tau\rangle_{\Omega_f^e} = \langle u, d\tau\rangle_{\Omega_f^e}, \quad \tau \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h}^e),$$
$$\langle dQ_f^k u, dv\rangle_{\Omega_f^e} = \langle du, dv\rangle_{\Omega_f^e}, \quad v \in \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}^e).$$

For $k = 0$, the first equation should be replaced by a mean value condition, so that $Q_f^0 = P_f^0$. This system has a unique solution due to the exactness of the complex $(\mathcal{P}\Lambda(\mathcal{T}_{f,h}^e), d)$. Furthermore, by construction we have

$$Q_f^k du = dQ_f^{k-1} u, \quad 0 < k \le n. \tag{3.2}$$

We will also find it useful to introduce the operator $Q_{f,-}^k : H\Lambda^k(\Omega_f^e) \to \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h}^e)$ defined by the corresponding reduced system

$$\langle Q_{f,-}^k u, d\tau\rangle_{\Omega_f^e} = 0, \quad \tau \in \mathcal{P}\Lambda^{k-2}(\mathcal{T}_{f,h}^e),$$
$$\langle dQ_{f,-}^k u, dv\rangle_{\Omega_f^e} = \langle u, dv\rangle_{\Omega_f^e}, \quad v \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h}^e).$$

As a consequence, the projection Q_f^k can be expressed as

$$Q_f^k = dQ_{f,-}^k + Q_{f,-}^{k+1} d. \tag{3.3}$$

To make this relation true also in the case when $k = 0$ and $f \in \Delta_0(\mathcal{T}_h)$, the operator $dQ_{f,-}^0$ should have the interpretation that $dQ_{f,-}^0 u$ is the constant $\int_{\Omega_f} u \wedge \mathsf{vol}_{\Omega_f}$ on Ω_f, where vol_{Ω_f} is the volume form on Ω, restricted to Ω_f and scaled such that $\int_{\Omega_f} \mathsf{vol}_{\Omega_f} = 1$.

To motivate the rest of the tools we need for our construction, consider again $d\pi^0 u$ in the special case when $\mathcal{P}\Lambda^k(\mathcal{T}_h) = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$. To obtain a commuting relation of the form $d\pi^0 u = \pi^1 du$, we have to be able to express $d\pi^0 u$ in terms of du. However, using the notation just introduced, we have

$$d\pi^0 u = \sum_{g\in\Delta_0(\mathcal{T}_h)} \Big[\Big(\int_{\Omega_g} u\wedge \mathsf{vol}_{\Omega_g}\Big) + \mathrm{tr}_g(Q^1_{g,-}du)\Big] d\mathcal{E}_g \mathsf{vol}_g.$$

The second part of this sum is already expressed in terms of du. By combining the contributions from neighboring macroelements we will see that the first part of the right-hand side can also be expressed in terms of du. If $f=[x_0,x_1]\in\Delta_1(\mathcal{T}_h)$, we have

$$\int_f \mathrm{tr}_f \sum_{g\in\Delta_0(\mathcal{T}_h)} \Big(\int_{\Omega_g} u\wedge\mathsf{vol}_{\Omega_g}\Big) d\mathcal{E}_g\mathsf{vol}_g = \int_{\Omega_f^e} u\wedge(\mathsf{vol}_{\Omega_{g_1}} - \mathsf{vol}_{\Omega_{g_0}}),$$

where $g_i=[x_i]$. Furthermore, $\mathsf{vol}_{\Omega_{g_1}} - \mathsf{vol}_{\Omega_{g_0}} \in \mathcal{P}_0\Lambda^n(\mathcal{T}^e_{f,h}) = \mathcal{P}_1^-\Lambda^n(\mathcal{T}^e_{f,h})$, and with vanishing integral. As a consequence, there exists $z^1_f\in \mathring{\mathcal{P}}_1^-\Lambda^{n-1}(\mathcal{T}^e_{f,h})$ such that $dz^1_f = \mathsf{vol}_{\Omega_{g_0}} - \mathsf{vol}_{\Omega_{g_1}}$, and by integration by parts

$$\int_f \mathrm{tr}_f \sum_{g\in\Delta_0(\mathcal{T}_h)} \Big(\int_{\Omega_g} u\wedge\mathsf{vol}_{\Omega_g}\Big) d\mathcal{E}_g\mathsf{vol}_g = -\int_{\Omega_f^e} u\wedge dz^1_f = \int_{\Omega^e_f} du\wedge z^1_f.$$

By utilizing the representation (2.7), we therefore obtain

$$\sum_{g\in\Delta_0(\mathcal{T}_h)} \Big(\int_{\Omega_g} u\wedge\mathsf{vol}_{\Omega_g}\Big) d\mathcal{E}_g\mathsf{vol}_g = \sum_{f\in\Delta_1(\mathcal{T}_h)} \Big(\int_{\Omega^e_f} du\wedge z^1_f\Big)\mathcal{E}^k_f\mathsf{vol}_f.$$

This discussion shows that to construct local cochain projections, we must utilize relations between local operators defined on different macroelements. To derive the proper relations, we introduce an operator

$$\delta : \bigoplus_{g\in\Delta_m(\mathcal{T}_h)} \mathring{\mathcal{P}}_1^-\Lambda^k(\mathcal{T}^e_{g,h}) \to \bigoplus_{f\in\Delta_{m+1}(\mathcal{T}_h)} \mathring{\mathcal{P}}_1^-\Lambda^k(\mathcal{T}^e_{f,h}).$$

If $f=[x_0,\dots,x_{m+1}]\in\Delta_{m+1}(\mathcal{T}_h)$, then the component $(\delta u)_f$ of δu is defined by

$$(\delta u)_f = \sum_{j=0}^{m+1}(-1)^j u_{f_j},$$

where, as above, $f_j=[x_0,\dots,x_{j-1},\hat{x}_j,x_{j+1},\dots,x_{m+1}]$, and u_{f_j} the corresponding component of u. We will also consider the exterior derivative d as an operator mapping $\bigoplus_{g\in\Delta_m(\mathcal{T}_h)}\mathring{\mathcal{P}}_1^-\Lambda^k(\mathcal{T}^e_{g,h})$ to $\bigoplus_{g\in\Delta_m(\mathcal{T}_h)}\mathring{\mathcal{P}}_1^-\Lambda^{k+1}(\mathcal{T}^e_{g,h})$ by applying it to each component. Hence, the two operators $d\circ\delta$ and $\delta\circ d$ both map $\bigoplus_{g\in\Delta_m(\mathcal{T}_h)}\mathring{\mathcal{P}}_1^-\Lambda^k(\mathcal{T}^e_{g,h})$ into $\bigoplus_{f\in\Delta_{m+1}(\mathcal{T}_h)}\mathring{\mathcal{P}}_1^-\Lambda^{k+1}(\mathcal{T}^e_{f,h})$. In fact, we have the stucture of a double complex which resembles the well-known Čech–de Rham complex; cf. [4]. The following two properties of the operator δ are crucial.

Lemma 3.1.

$$d\circ\delta = \delta\circ d, \quad \textit{and} \quad \delta\circ\delta = 0.$$

Proof. It follows directly from the definition of δ that for $f=[x_0,\dots,x_{m+1}]\in\Delta_{m+1}(\mathcal{T}_h)$,

$$(d\circ\delta u)_f = (\delta\circ du)_f = \sum_{j=0}^{m+1}(-1)^j du_{f_j}.$$

If we further denote by f_{ij} the subsimplex of f obtained by deleting both x_i and x_j, then

$$\begin{aligned}(\delta \circ \delta u)_f &= \sum_{j=0}^{m+1}(-1)^j(\delta u)_{f_j} \\ &= \sum_{j=0}^{m+1}(-1)^j\Big[\sum_{i=0}^{j-1}(-1)^i u_{f_{ij}} - \sum_{i=j+1}^{m+1}(-1)^i u_{f_{ij}}\Big] = 0,\end{aligned}$$

since for each $i, j = 0, \dots, m+1$, with $i \neq j$, the term $u_{f_{ij}}$ appears exactly twice with opposite signs. $\square$

The construction of the projection R^k will depend on local weight functions, $z_f^k \in \mathring{\mathcal{P}}_1^-\Lambda^{n-k}(\mathcal{T}_{f,h}^e)$ for $f \in \Delta_k(\mathcal{T}_h)$. In particular, the function $z_f^0 \in \mathcal{P}_0\Lambda^n(\mathcal{T}_{f,h})$ for $f \in \Delta_0(\mathcal{T}_h)$ will be given by $z_f^0 = \mathsf{vol}_{\Omega_f}$. For $k = 1, 2, \dots, n$, the functions $z_f^k \in \mathring{\mathcal{P}}_1^-\Lambda^{n-k}(\mathcal{T}_{f,h}^e)$ are defined recursively to satisfy the conditions

$$dz_f^k = (-1)^k(\delta z^{k-1})_f \tag{3.4}$$

and

$$\langle z_f^k, d\tau\rangle_{\Omega_f^e} = 0, \qquad \tau \in \mathring{\mathcal{P}}_1^-\Lambda^{n-k-1}(\mathcal{T}_{f,h}^e), \tag{3.5}$$

for any $f \in \Delta_k(\mathcal{T}_h)$. We will not give an explicit construction of the functions z_f^k. However, we have the following basic result.

Lemma 3.2. *The weight functions $z_f^k \in \mathring{\mathcal{P}}_1^-\Lambda^{n-k}(\mathcal{T}_{f,h}^e)$ exist and are uniquely determined by z_f^0 and the conditions* (3.4) *and* (3.5)*.*

Proof. We establish the existence of the functions z_f^k by induction on k. Let $f = [x_0, x_1] \in \Delta_1(\mathcal{T}_h)$. Then

$$(\delta z^0)_f = (z_{f_1}^0 - z_{f_0}^0) = \mathsf{vol}_{\Omega_{f_1}} - \mathsf{vol}_{\Omega_{f_0}},$$

which implies that $\int_{\Omega_f^e}(\delta z^0)_f = 0$. Hence, by the exactness of the complex

$$\mathring{\mathcal{P}}_1^-\Lambda^{n-1}(\mathcal{T}_{f,h}^e) \xrightarrow{d} \mathcal{P}_0\Lambda^n(\mathcal{T}_{f,h}^e) \to \mathbb{R},$$

there exists $z_f^1 \in \mathring{\mathcal{P}}_1^-\Lambda^{n-1}(\mathcal{T}_{f,h}^e)$ satisfying $dz_f^1 = -(\delta z^0)_f$. Next, assume we have constructed

$$z^{k-1} \in \bigoplus_{f\in\Delta_{k-1}(\mathcal{T}_h)} \mathring{\mathcal{P}}_1^-\Lambda^{n-k+1}(\mathcal{T}_{f,h}^e)$$

such that $dz_f^{k-1} = (-1)^{k-1}(\delta z^{k-2})_f$ for all $f \in \Delta_{k-1}(\mathcal{T}_h)$. From Lemma 3.1 we obtain

$$(d \circ \delta)z^{k-1} = (\delta \circ d)z^{k-1} = (-1)^{k-1}(\delta \circ \delta)z^{k-2} = 0,$$

and for each $f \in \Delta_k(\mathcal{T}_h)$ the complex $(d, \mathring{\mathcal{P}}_1^-\Lambda(\mathcal{T}_{f,h}^e))$ is exact. Therefore, we can conclude that there is a $z_f^k \in \mathring{\mathcal{P}}_1^-\Lambda^{n-k}(\Delta_k(\mathcal{T}_{f,h}^e))$ such that (3.4) holds. This completes the induction argument. Finally, we observe that it is a consequence of the exactness of the complex $(d, \mathring{\mathcal{P}}_1^-\Lambda(\mathcal{T}_{f,h}^e))$ and (3.5) of the definition of z_f^k that these functions are uniquely determined. $\square$

We will use the functions z_f^k to define the operator $M^k : L^2\Lambda^k(\Omega) \to \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ by

$$M^k u = \sum_{f\in\Delta_k(\mathcal{T}_h)} \Big(\int_{\Omega_f^e} u \wedge z_f^k\Big)\mathcal{E}_f^k \mathsf{vol}_f.$$

Note that $M^k u$ is a generalization for k-forms of the expression

$$\sum_{f\in\Delta_0(\mathcal{T}_h)} \Big(\int_{\Omega_f} u \wedge \mathsf{vol}_{\Omega_f}\Big)\mathcal{E}_f \mathsf{vol}_f$$

appearing above in the case of zero-forms. It follows from the construction of the functions z_f^k that the operator M^k commutes with the exterior derivative.

Lemma 3.3. *For any $v \in H\Lambda^{k-1}(\Omega)$ the identity $dM^{k-1}v = M^k dv$ holds.*

Proof. We have to show that

$$\sum_{g\in\Delta_{k-1}(\mathcal{T}_h)} \Big(\int_{\Omega_g^e} v \wedge z_g^{k-1}\Big) d\mathcal{E}_g^{k-1}\mathsf{vol}_g = \sum_{f\in\Delta_k(\mathcal{T}_h)} \Big(\int_{\Omega_f^e} dv \wedge z_f^k\Big)\mathcal{E}_f^k \mathsf{vol}_f$$

for any $v \in H\Lambda^{k-1}(\Omega)$. Since both sides of this equation are elements of $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, we need only check that the integrals of their traces are the same over each $f = [x_0, x_1, \ldots, x_k] \in \Delta_k(\mathcal{T}_h)$. Now it follows from the properties of the extension operators $\mathcal{E}_f^k$ that the integral of the right-hand side is simply $\int_{\Omega_f^e} dv \wedge z_f^k$, while (2.8) implies that the corresponding integral of the left-hand side is

$$\sum_{j=0}^k (-1)^j \int_{\Omega_{f_j}^e} v \wedge z_{f_j}^{k-1} = \int_{\Omega_f^e} v \wedge (\delta z^{k-1})_f = (-1)^k \int_{\Omega_f^e} v \wedge dz_f^k,$$

where the last identity follows by (3.4). However, by integration by parts (cf. [1, Section 2.2]), utilizing that $\mathrm{tr}_{\partial\Omega_f^e} z_f^k = 0$, we have

$$\int_{\Omega_f^e} v \wedge dz_f^k = (-1)^k \int_{\Omega_f^e} dv \wedge z_f^k,$$

and this completes the proof. □

The operator M^k will be a key tool for the construction of the special projection R^k onto the Whitney forms. However, the operator M^k itself is not a projection since it is not equal to the identity on the Whitney forms. The next step towards the final construction of R^k is to define operators $S^k : H\Lambda^k(\Omega) \to \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ recursively by $S^0 = M^0$ and

$$S^k u = M^k u + \sum_{g\in\Delta_{k-1}(\mathcal{T}_h)} \Big(\int_g \mathrm{tr}_g[I - S^{k-1}]Q_{g,-}^k u\Big) d\mathcal{E}_g^{k-1}\mathsf{vol}_g, \quad 1 \le k \le n.$$

We recall that the operator $Q_{g,-}^k$ is a local operator with range $\mathcal{P}\Lambda(\mathcal{T}_{f,h}^e)$. However, by an inductive argument, it follows that the composition $\mathrm{tr}_f \circ S^k$ is a local operator mapping $H\Lambda^k(\Omega_f^e)$ into $\mathcal{P}_0\Lambda^k(f)$. Therefore, the operators S^k are indeed well defined.

The following result shows that S^k, restricted to the space $d\mathcal{P}_1^-\Lambda^{k-1}(\mathcal{T}_h)$, preserves the degrees of freedom of the space $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$.

Lemma 3.4. *For any $v \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_h)$ the following identity holds:*

$$\int_f \operatorname{tr}_f S^k dv = \int_f \operatorname{tr}_f dv, \quad f \in \Delta_k(\mathcal{T}_h).$$

Proof. The proof goes by induction on k. For $k = 0$ the space $d\mathcal{P}\Lambda^{k-1}(\mathcal{T}_h)$ should be interpreted as the space of constants on Ω, and since $S^0 = M^0$ reproduces constants the desired identity holds.

Assume next that $k \geq 1$ and that the desired identity holds for $k-1$. By utilizing the result of Lemma 3.3, we obtain that the commutator $S^k d - dS^{k-1}$ has the representation

$$S^k dv - dS^{k-1}v = \sum_{g\in\Delta_{k-1}(\mathcal{T}_h)} \Big(\int_g \operatorname{tr}_g[I - S^{k-1}]Q^k_{g,-}dv\Big) d\mathcal{E}^{k-1}_g \mathsf{vol}_g, \quad v \in H\Lambda^{k-1}(\Omega).$$

We recall that $Q^k_{g,-}dv = Q^{k-1}_g v - dQ^{k-1}_{g,-}v$, and by the induction hypothesis $\int_g \operatorname{tr}_g(I - S^{k-1})dQ^{k-1}_{g,-}v = 0$. Therefore, for any $v \in H\Lambda^{k-1}(\Omega)$ the commutator above can be expressed as

$$S^k dv - dS^{k-1}v = \sum_{g\in\Delta_{k-1}(\mathcal{T}_h)} \Big(\int_g \operatorname{tr}_g[I - S^{k-1}]Q^{k-1}_g v\Big) d\mathcal{E}^{k-1}_g \mathsf{vol}_g. \tag{3.6}$$

However, since Q^{k-1}_g is a projection onto $\mathcal{P}\Lambda^{k-1}(\mathcal{T}^e_{g,h})$, it follows from (2.7) that

$$dS^{k-1}v = \sum_{g\in\Delta_{k-1}(\mathcal{T}_h)} \Big(\int_g \operatorname{tr}_g S^{k-1}Q^{k-1}_g v\Big) d\mathcal{E}^{k-1}_g \mathsf{vol}_g, \quad v \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_h).$$

By restricting to a function $v \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_h)$, equation (3.6) therefore reduces to

$$S^k dv = \sum_{g\in\Delta_{k-1}(\mathcal{T}_h)} \Big(\int_g \operatorname{tr}_g v\Big) d\mathcal{E}^{k-1}_g \mathsf{vol}_g.$$

By integrating this representation over any $f = [x_0, x_1, \dots, x_k] \in \Delta_k(\mathcal{T}_h)$, we obtain

$$\int_f S^k dv = \sum_{j=0}^{k}(-1)^j \int_{f_j} \operatorname{tr}_{f_j} v = \int_f dv,$$

where the final identity follows from (2.8). This completes the induction argument. ∎

In general, the operators S^k will not commute with the exterior derivative. However, as a direct consequence of the proof above, we have the following result.

Lemma 3.5. *The identity* (3.6) *holds for $k = 1, 2, \dots, n$ and all $v \in H\Lambda^{k-1}(\Omega)$.*

3.2. **The projection R^k.** In order to obtain an operator R^k which is the identity on all of $\mathcal{P}^-_1\Lambda^k(\mathcal{T}_h)$, and hence a projection, the operator S^k will be modified. For each k, $0 \leq k \leq n$, the operator $R^k : H\Lambda^k(\Omega) \to \mathcal{P}^-_1\Lambda^k(\mathcal{T}_h)$ is defined by

$$R^k u = S^k u + \sum_{f\in\Delta_k(\mathcal{T}_h)} \Big(\int_f \operatorname{tr}_f[I - S^k]Q^k_f u\Big)\mathcal{E}^k_f \mathsf{vol}_f.$$

Recall that the operator S^k is local in the sense that $\operatorname{tr}_f \circ S^k$ can be seen as a local operator mapping $H\Lambda^k(\Omega^e_f)$ onto $\mathcal{P}\Lambda^k(f)$. It is immediate from this and the properties of the projection Q^k_f that $\operatorname{tr}_f \circ R^k$ also is local. In fact, for any $T \in \mathcal{T}_h$,

$(R^k u)|_T$ only depends on $u|_{\Omega_T^e}$. Furthermore, if $f \in \Delta_0(\mathcal{T}_h)$, then $Q_f^0 = P_f^0$. Therefore, it follows that for $k = 0$ the operator R^0 is identical to the operator π_0^0, used in the construction of the projection π^0 in Section 2.1 above.

The key properties of the operator R^k are given in the theorem below.

Theorem 3.6. *The operators $R^k : H\Lambda^k(\Omega) \to \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ are cochain projections. Furthermore, they satisfy property* (3.1)*, i.e.,*

$$\int_f \operatorname{tr}_f R^k u = \int_f \operatorname{tr}_f u, \quad f \in \Delta_k(\mathcal{T}_h),\ u \in \mathcal{P}\Lambda^k(\mathcal{T}_h).$$

Proof. As above we use a notation where we suppress the dependence on h. It is a consequence of the projection property of the operators Q_f^k that if $u \in \mathcal{P}\Lambda^k(\mathcal{T}_h)$, then

$$R^k u = \sum_{f \in \Delta_k(\mathcal{T}_h)} \Big(\int_f \operatorname{tr}_f u \Big) \mathcal{E}_f^k \mathsf{vol}_f.$$

However, this implies the identity (3.1), and an immediate further consequence is that R^k is a projection onto $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$.

It remains to show that R^k commutes with the exterior derivative. From the definition of R^k and Lemma 3.5 we have

$$dR^k u = dS^k u + \sum_{f \in \Delta_k(\mathcal{T}_h)} \Big(\int_f \operatorname{tr}_f [I - S^k] Q_f^k u \Big) d\mathcal{E}_f^k \mathsf{vol}_f = S^{k+1} du.$$

However, $S^{k+1} du = R^{k+1} du$ since

$$\begin{aligned} R^{k+1} du - S^{k+1} du &= \sum_{f \in \Delta_{k+1}(\mathcal{T}_h)} \Big(\int_f \operatorname{tr}_f [I - S^{k+1}] Q_f^{k+1} du \Big) \mathcal{E}_f^{k+1} \mathsf{vol}_f \\ &= \sum_{f \in \Delta_{k+1}(\mathcal{T}_h)} \Big(\int_f \operatorname{tr}_f [I - S^{k+1}] d Q_{f,-}^{k+1} du \Big) \mathcal{E}_f^{k+1} \mathsf{vol}_f = 0, \end{aligned}$$

where the last identity follows from Lemma 3.4. ☐

The operators R^k introduced above are local operators in the sense that $(R^k u)|_T$ only depends on $u|_{\Omega_T^e}$ for any $T \in \mathcal{T}_h$. Furthermore, for any fixed h the operator R^k is a bounded operator on $H\Lambda^k(\Omega)$. The discussion of more precise local bounds is delayed until the final section of the paper.

4. Construction of the Projection: The General Case

We finally turn to the construction of the projections π^k in the general case, in which $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ denotes any family of spaces of the form $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ or $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$, such that the corresponding polynomial sequence $(\mathcal{P}\Lambda^k, d)$, given by (2.4) is an exact complex. In particular, the Whitney forms, $\mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$, are a subset of $\mathcal{P}\Lambda^k(\mathcal{T}_h)$, and in the special case when $\mathcal{P}\Lambda^k(\mathcal{T}_h) = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ we will take π^k to be the operator R^k constructed above.

In the construction we will utilize a decomposition of $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ of the form

$$\mathcal{P}\Lambda^k(\mathcal{T}_h) = \bigoplus_{f \in \Delta_k(\mathcal{T}_h)} \mathcal{E}_f^k(\mathcal{P}_0\Lambda^k(f)) + \bigoplus_{\substack{f \in \Delta(\mathcal{T}_h) \\ \dim f \ge k}} E_f^k(\breve{\mathcal{P}}\Lambda^k(f)), \tag{4.1}$$

where $\mathcal{E}_f^k$ is the extension operator defined in the previous section, mapping into the space of Whitney forms, while E_f^k is an harmonic extension operator mapping into $\mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$. Furthermore, the space $\breve{\mathcal{P}}\Lambda^k(f) = \mathring{\mathcal{P}}\Lambda^k(f)$ if $\dim f > k$, while

$$\breve{\mathcal{P}}\Lambda^k(f) = \{u \in \mathcal{P}\Lambda^k(f) \,|\, \int_f u = 0\}, \quad \text{if } \dim f = k.$$

The decomposition (4.1) can be seen as a modification of the more standard decompositions (1.1) and (1.2) in the sense that we are utilizing the special extension, $\mathcal{E}_f^k$, for the constant term of the traces on f, when $\dim f = k$. The existence of such a decomposition of the space $\mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$ is an immediate consequence of the degrees of freedom (2.6).

As in the case $k = 0$ (cf. Section 2.1), the projection π^k will be constructed from a sequence of operators π_m^k, where $\pi^k = \pi_n^k$. The operators π_m^k are defined by a recursion of the form

$$\pi_m^k = \pi_{m-1}^k + \sum_{f\in\Delta_m(\mathcal{T}_h)} E_f^k \circ \mathrm{tr}_f \circ P_f^k[I - \pi_{m-1}^k], \qquad k \le m \le n, \tag{4.2}$$

where the operators P_f^k are local projections defined with respect to the macroelements Ω_f, generalizing the operators P_f^0 introduced in Section 2.1. Furthermore, the operator π_{k-1}^k will be taken to be the operator R^k defined in Section 3 above. Hence, to complete the definition of π^k, it remains to give precise definitions of the local operators E_f^k and P_f^k.

4.1. **Extension operators.** The extension operators E_f^k are generalizations of the harmonic extension operators E_f^0 used for zero forms in Section 2.1. Let us first assume that $f \in \Delta(\mathcal{T}_h)$ such that f is not a subset of the boundary of Ω_f. In this case, the harmonic extension E_f^k maps $\mathring{\mathcal{P}}\Lambda^k(f)$ to $\mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$, where $0 \le k \le \dim f$. More specifically, we let $E_f^k\phi$ be characterized by

$$\|dE_f^k\phi\|_{L^2(\Omega_f)} = \inf\{\|dv\|_{L^2(\Omega_f)} \,|\, v \in \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}),\ \mathrm{tr}_f\, v = \phi\}.$$

We should note that it is a consequence of the degrees of freedom of the spaces $\mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$ and $\mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ that there are feasible solutions to this optimization problem. As a consequence, an optimal solution exists. However, the solution is in general not unique. The solution is only determined up to adding functions w in $\mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ satisfying $dw = 0$ on Ω_f and $\mathrm{tr}_f\, w = 0$. Therefore, to obtain a well-defined extension operator, we need to introduce a corresponding gauge condition. Hence, for any $\phi \in \mathring{\mathcal{P}}\Lambda^k(f)$ we let $E_f^k\phi \in \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ be the solution of the system

$$\begin{aligned} \langle E_f^k\phi, d\tau\rangle_{\Omega_f} &= 0, \quad \tau \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h})),\\ \langle dE_f^k\phi, dv\rangle_{\Omega_f} &= 0, \quad v \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^{k}(\mathcal{T}_{f,h})), \end{aligned} \tag{4.3}$$

and such that $\mathrm{tr}_f \circ E_f^k$ is the identity on $\mathring{\mathcal{P}}\Lambda^k(f)$. Here $N(\mathrm{tr}_f; X)$ denotes the kernel of the operator tr_f restricted to the function space X. A key property of the extension operators E_f^k is that they commute with the exterior derivative.

Lemma 4.1. *Let $f \in \Delta(\mathcal{T}_h)$. The extension operators $E_f^k : \mathring{\mathcal{P}}\Lambda^k(f) \to \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ are well defined by the system* (4.3) *for $k = 0, 1, \ldots, \dim f$, and for $k \ge 1$ we have* *the identity*

$$E_f^k d\phi = dE_f^{k-1}\phi, \quad \phi \in \mathring{\mathcal{P}}\Lambda^{k-1}(f). \tag{4.4}$$

Moreover, the kernel of d *restricted to* $N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}))$ *is* $dN(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h}))$.

Proof. For $k = 0$ the first equation in the system (4.3) should be omitted. The kernel of d restricted to $N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^0(\mathcal{T}_{f,h}))$ is just the zero function, and $E_f^0\phi$ is clearly uniquely determined by the second equation and the property that $\mathrm{tr}_f \circ E_f^k$ is the identity. We proceed by induction on k.

Assume that the statement of the lemma holds for all levels less than k. We first establish the characterization of the kernel of d, restricted to $N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}))$. Assume that $u \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}))$ satisfies $du = 0$. Then, by the exactness of the complex $(\mathring{\mathcal{P}}\Lambda(\mathcal{T}_{f,h}), d)$, $u = d\tau$ for some $\tau \in \mathring{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h})$. Furthermore, $d\,\mathrm{tr}_f\,\tau = \mathrm{tr}_f\,d\tau = \mathrm{tr}_f\,u = 0$. If $k = 1$ this implies that $\tau \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^0(\mathcal{T}_{f,h}))$. For $k > 1$ it follows from the exactness of $(\mathring{\mathcal{P}}\Lambda(f), d)$ that there is a $\phi \in \mathring{\mathcal{P}}\Lambda^{k-2}(f)$ such that $d\phi = \mathrm{tr}_f\,\tau$. However, the function

$$\sigma = \tau - dE^{k-2}\phi = \tau - E^{k-1}d\phi \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h}))$$

and satisfies $d\sigma = u$. Hence the complex $(N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda(\mathcal{T}_{f,h})), d)$ is exact at level k in the sense that $dN(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h}))$ is the kernel of d restricted to $N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}))$.

Consider a local Hodge Laplace problem of the form

$$\begin{aligned} \langle \sigma, \tau\rangle_{\Omega_f} - \langle u, d\tau\rangle_{\Omega_f} &= 0, \quad \tau \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h})), \\ \langle d\sigma, v\rangle_{\Omega_f} + \langle du, dv\rangle_{\Omega_f} &= 0, \quad v \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^{k}(\mathcal{T}_{f,h})), \end{aligned} \tag{4.5}$$

where the unknown $(\sigma, u) \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h})) \times \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$, and with $\mathrm{tr}_f\,u = \phi \in \mathring{\mathcal{P}}\Lambda^k(f)$. Since the complex $(N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda(\mathcal{T}_{f,h})), d)$ is exact at level k, it follows from the abstract theory of Hodge Laplace problems (cf. for example [3, Section 3]), that the system (4.5) has a unique solution. Furthermore, by the exactness of the same complex at level $k - 1$, $\sigma = 0$. Hence, u and $E_f^k\phi$ satisfy the same conditions, and the uniqueness of $E_f^k\phi$ follows by the uniqueness of u.

Finally, to establish the identity (4.4), we just observe that for any $\phi \in \mathring{\mathcal{P}}\Lambda^{k-1}(f)$, the pair (σ, u), with $\sigma = 0$ and $u = dE_f^{k-1}\phi \in \mathring{\mathcal{P}}\Lambda(\mathcal{T}_{f,h})$, satisfies the system (4.5) with $\mathrm{tr}_f\,dE_f^{k-1}\phi = d\,\mathrm{tr}_f\,E_f^{k-1}\phi = d\phi$. By uniqueness of such solutions we conclude that $dE_f^{k-1}\phi = E_f^k d\phi$. This completes the induction argument and the proof of the lemma. ∎

If $g \in \Delta(\mathcal{T}_{f,h})$, with $k \le \dim g \le \dim f$ and $g \ne f$, then $\mathrm{tr}_g\,E_f^k\phi = 0$. In the case that $f \subset \partial\Omega$, we will also have that $f \subset \partial\Omega_f$. In this case, the definition of the operator E_f^k should be properly modified, such that $E_f^k\phi$ is not required to be in $\mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$, but only required to be zero on the interior part of $\partial\Omega_f$. The key desired property is that the extension of $E_f^k\phi$ from Ω_f to Ω, by zero outside Ω_f, is in the global space $\mathcal{P}\Lambda^k(\mathcal{T}_h)$.

It is a consequence of the decomposition (4.1) that any element u of $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ is uniquely determined by its trace on f, $\mathrm{tr}_f\,u$, for all $f \in \Delta(\mathcal{T}_h)$ with $\dim f \ge k$.

Furthermore, if u is an element of the subspace given by

$$\bigoplus_{f\in\Delta_k(\mathcal{T}_h)} \mathcal{E}_f^k(\mathcal{P}_0\Lambda^k(f)) + \bigoplus_{\substack{f\in\Delta(\mathcal{T}_h)\\ k\le \dim f\le m}} E_f^k(\breve{\mathcal{P}}\Lambda^k(f)), \tag{4.6}$$

then u is determined by $\mathrm{tr}_f u$ for all $f \in \Delta(\mathcal{T}_h)$ with $k \le \dim f \le m$. A key observation is the following.

Lemma 4.2. *Assume that $u \in \mathcal{P}\Lambda^k(\mathcal{T}_h)$ belongs to the subspace given by (4.6), where $k < m \le n$. Then its exterior derivative, du, belongs to the corresponding space*

$$\bigoplus_{f\in\Delta_{k+1}(\mathcal{T}_h)} \mathcal{E}_f^{k+1}(\mathcal{P}_0\Lambda^{k+1}(f)) + \bigoplus_{\substack{f\in\Delta(\mathcal{T}_h)\\ k+1\le \dim f\le m}} E_f^{k+1}(\breve{\mathcal{P}}\Lambda^{k+1}(f)).$$

Proof. It follows from the fact that $(\mathcal{P}_1^-\Lambda(\mathcal{T}_h), d)$ is a complex for which $d\mathcal{E}_g^k\mathsf{vol}_g \in \bigoplus_{f\in\Delta_{k+1}(\mathcal{T}_h)} \mathcal{E}_f^{k+1}(\mathcal{P}_0\Lambda^{k+1}(f))$ for any $g \in \Delta_k(\mathcal{T}_h)$. Furthermore, if $g \in \Delta(\mathcal{T}_h)$ and $\dim g > k$, then (4.4) implies that $dE_g^k\phi = E_g^{k+1}d\phi$ for any $\phi \in \breve{\mathcal{P}}\Lambda^k(g)$. As a consequence, it only remains to check terms of the form $dE_g^k\phi$, where $\phi \in \breve{\mathcal{P}}\Lambda^k(g)$ and $\dim g = k$.

Note that $dE_g^k\phi$ is identically zero outside Ω_g. Furthermore, consider any $f \in \Delta_{k+1}(\mathcal{T}_h)$, with $g \in \Delta_k(f)$. Then $\Omega_f \subset \Omega_g$ and the space $N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}))$ can be identified with a subspace of $N(\mathrm{tr}_g; \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{g,h}))$. Therefore, it follows from the definition of $E_g^k\phi$ that

$$\langle dE_g^k\phi, dv\rangle_{\Omega_f} = 0, \quad v \in N(\mathrm{tr}_f; \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})),\ f \in \Delta_{k+1}(\mathcal{T}_h),\ g \in \Delta_k(f).$$

However, this implies that

$$\begin{aligned} dE_g^k\phi &\in \bigoplus_{\substack{f\in\Delta_{k+1}(\mathcal{T}_h)\\ g\in\Delta_k(f)}} E_f^{k+1}(\mathcal{P}\Lambda^{k+1}(f)) \\ &= \bigoplus_{\substack{f\in\Delta_{k+1}(\mathcal{T}_h)\\ g\in\Delta_k(f)}} \mathcal{E}_f^{k+1}(\mathcal{P}_0\Lambda^{k+1}(f)) + \bigoplus_{\substack{f\in\Delta_{k+1}(\mathcal{T}_h)\\ g\in\Delta_k(f)}} E_f^{k+1}(\breve{\mathcal{P}}\Lambda^{k+1}(f)). \end{aligned}$$

This completes the proof. $\square$

The harmonic extension operator discussed above is the one we will use in the construction of the local cochain projection π^k; cf. (4.2). However, in the theory below we will also utilize an alternative local extension, defined with respect to spaces $\mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$ instead of $\mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$. For $0 \le k \le n$ the operator $\tilde{E}_f^k : \mathcal{P}\Lambda^k(f) \to \mathcal{P}\Lambda^k(\mathcal{T}_h)$ is defined by the conditions

$$\begin{aligned} \langle \tilde{E}_f^k\phi, d\tau\rangle_{\Omega_f} &= 0, \quad \tau \in N(\mathrm{tr}_f; \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h})), \\ \langle d\tilde{E}_f^k\phi, dv\rangle_{\Omega_f} &= 0, \quad v \in N(\mathrm{tr}_f; \mathcal{P}\Lambda^k(\mathcal{T}_{f,h})), \end{aligned} \tag{4.7}$$

in addition to the extension property $\mathrm{tr}_f \circ \tilde{E}_f^k\phi = \phi$ for all $\phi \in \mathcal{P}\Lambda^k(f)$. In complete analogy with the discussion for the operators E_f^k above, by utilizing the exactness of the complex $(\mathcal{P}\Lambda(\mathcal{T}_{f,h}), d)$ instead of the exactness of $(\mathring{\mathcal{P}}\Lambda(\mathcal{T}_{f,h}), d)$, we can conclude with the following analog of Lemma 4.1.

Lemma 4.3. *Let $f \in \Delta(\mathcal{T}_h)$. The extension operators $\tilde{E}_f^k : \mathcal{P}\Lambda^k(f) \to \mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$ are well defined by the system* (4.7) *for $k = 0, 1, \dots, \dim f$, and for $k \geq 1$ we have the identity*

$$\tilde{E}_f^k d\phi = d\tilde{E}_f^{k-1}\phi, \quad \phi \in \mathcal{P}\Lambda^{k-1}(f).$$

Moreover, the kernel of d restricted to $N(\mathrm{tr}_f; \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}))$ is $dN(\mathrm{tr}_f; \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h}))$.

4.2. **Local projections.** Let $f \in \Delta(\mathcal{T}_h)$ and recall the definition of the spaces $\breve{\mathcal{P}}\Lambda^k(f)$ given above, as $\mathring{\mathcal{P}}\Lambda^k(f)$ if k is less than the dimension of f, and as the subspace of $\mathcal{P}\Lambda^k(f)$ consisting of functions with zero mean value if $k = \dim f$. Hence, as an alternative to (2.5), we can state that the complex

$$0 \to \breve{\mathcal{P}}\Lambda^0(f) \xrightarrow{d} \breve{\mathcal{P}}\Lambda^1(f) \xrightarrow{d} \cdots \xrightarrow{d} \breve{\mathcal{P}}\Lambda^{\dim f}(f) \to 0$$

is exact. In particular, this means that the first operator, $d = d^0$, is one-to-one and the last operator, $d = d^{\dim f - 1}$, is onto. In order to define the local projections P_f^k, appearing in (4.2), we will use the spaces $\breve{\mathcal{P}}(f)$ to introduce proper local spaces, $\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$. For $0 \leq k < \dim f$ these spaces lie between $\mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$ and $\mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$, i.e.,

$$\mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) \subset \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) \subset \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}).$$

More precisely, for $0 \leq k \leq \dim f$, the space $\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ is defined by

$$\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) = \{ u \in \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) \mid \mathrm{tr}_f \in \breve{\mathcal{P}}\Lambda^k(f) \},$$

while we let $\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) = \mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$ for $\dim f < k \leq n$. We note that for $k = 0$ this definition is consistent with the definition of the space $\breve{\mathcal{P}}_r\Lambda^0(\mathcal{T}_{f,h})$ used in Section 2.1.

We observe that $d\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) \subset \breve{\mathcal{P}}\Lambda^{k+1}(\mathcal{T}_{f,h})$. In other words, $(\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}), d)$, given by

$$0 \to \breve{\mathcal{P}}\Lambda^0(\mathcal{T}_{f,h}) \xrightarrow{d} \breve{\mathcal{P}}\Lambda^1(\mathcal{T}_{f,h}) \xrightarrow{d} \cdots \xrightarrow{d} \breve{\mathcal{P}}\Lambda^n(\mathcal{T}_{f,h}) \to 0,$$

is a complex. We also have the following:

Lemma 4.4. *The complex $(\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}), d)$ is exact.*

Proof. Let $m = \dim f$, and assume that $u \in \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ satisfies $du = 0$. We need to show that there is a $\sigma \in \breve{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h})$ such that $d\sigma = u$. For $k > m+1$ this follows from the exactness of the complex $(\mathcal{P}\Lambda(\mathcal{T}_{f,h}), d)$. Assume next that $k \leq m$. Since $d\,\mathrm{tr}_f\, u = \mathrm{tr}_f\, du = 0$, it follows from the exactness of the complex $(\breve{\mathcal{P}}\Lambda(f), d)$ that there is $\phi \in \breve{\mathcal{P}}\Lambda^{k-1}(f)$ such that $d\phi = \mathrm{tr}_f\, u$. Therefore $u - d\tilde{E}_f^{k-1}\phi$ is in $N(\mathrm{tr}_f, \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}))$ and $d(u - d\tilde{E}_f^{k-1}\phi) = 0$. By Lemma 4.3, there is a $\tau \in N(\mathrm{tr}_f, \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h}))$ such that $d\tau = u - d\tilde{E}_f^{k-1}\phi$. Hence, the function $\sigma = \tau + \tilde{E}_f^{k-1}\phi$ satisfies $\mathrm{tr}_f\, \sigma = \phi \in \breve{\mathcal{P}}\Lambda^{k-1}(f)$. So $\sigma \in \breve{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h})$ and $d\sigma = u$.

Finally, we have to consider the case when $k = m+1$. The exactness of the complex $(\mathcal{P}\Lambda(\mathcal{T}_{f,h}), d)$ and the assumption $du = 0$ implies that there is $\tau \in \mathcal{P}\Lambda^m(\mathcal{T}_{f,h})$ such that $d\tau = u$. Furthermore, the exactness of $(\mathcal{P}\Lambda(f), d)$ implies that there is a $\phi \in \mathcal{P}\Lambda^{m-1}(f)$ such that $d\phi = \mathrm{tr}_f\, \tau$. The function $\sigma = \tau - d\tilde{E}_f^{m-1}\phi$ has vanishing trace on f. Therefore, it is in $\breve{\mathcal{P}}\Lambda^m(\mathcal{T}_{f,h})$, and $d\sigma = u$. $\square$

We are now ready to define a local projection $P_f^k : H\Lambda^k(\Omega_f) \to \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ satisfying

$$\langle P_f^k u, d\tau\rangle_{\Omega_f} = \langle u, d\tau\rangle_{\Omega_f}, \quad \tau \in \breve{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h}),$$
$$\langle dP_f^k u, dv\rangle_{\Omega_f} = \langle du, dv\rangle_{\Omega_f}, \quad v \in \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}).$$

The operator P_f^k is a well defined projection onto $\breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ as a consequence of Lemma 4.4. When $k = 0$, the space $d\mathcal{P}\Lambda^{-1}(\mathcal{T}_{f,h})$ should be interpreted as the space of constants on Ω_f, such that P_f^0 is exactly the projection defined in Section 2.1. With this definition it is straightforward to check that the projections $\mathcal{P}_f^k$ commute with the exterior derivative, i.e.,

$$P_f^k du = dP_f^{k-1} u, \quad 0 < k \le n. \tag{4.8}$$

4.3. **Properties of the Operators π^k.** The definitions of the operators E_f^k and $\mathcal{P}_f^k$ given above complete the construction of the operators π^k given by the recursion (4.2). Here we shall derive two key properties of these operators, namely that they are projections onto $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ and that they commute with the exterior derivative. It is also clear from the construction that the operator π^k is local, and, for each triangulation $\mathcal{T}_h$, π^k is well defined as an operator on $H\Lambda^k(\Omega)$. However, the derivation of more precise bounds will be delayed until the next section.

We recall that the recursion (4.2) is initialized by choosing $\pi_{k-1}^k = R^k$, i.e., the special projection onto the Whitney forms constructed in Section 3 above. Therefore, we obtain from Theorem 3.6 that

$$d\pi_{k-1}^k u = \pi_k^{k+1} du, \quad k = 0, 1, \dots, n-1, \tag{4.9}$$

and for $k = 0$ the two operators π_{-1}^0 and π_0^0 are the same. Furthermore, for functions in $\mathcal{P}\Lambda^k(\mathcal{T}_h)$, the operator π_{k-1}^k preserves the integral of the trace over all subsimplexes of dimension k, i.e.,

$$\int_f \operatorname{tr}_f \pi_{k-1}^k u = \int_f \operatorname{tr}_f u, \quad f \in \Delta_k(\mathcal{T}_h),\ u \in \mathcal{P}\Lambda^k(\mathcal{T}_h). \tag{4.10}$$

In other words, if $u \in \mathcal{P}\Lambda^k(\mathcal{T}_h)$, then $(u - \pi_{k-1}^k u)|_{\Omega_f} \in \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ for $f \in \Delta_k(\mathcal{T}_h)$ and $k \ge 1$.

We observe that it follows from (4.2) and the properties of the extension operators E_f^k, that if $f \in \Delta_m(\mathcal{T}_h)$, with $m \ge k$, then

$$\operatorname{tr}_f \pi_m^k u = \operatorname{tr}_f (\pi_{m-1}^k u + P_f^k [u - \pi_{m-1}^k u]). \tag{4.11}$$

On the other hand,

$$\operatorname{tr}_g \pi_m^k u = \operatorname{tr}_g \pi_{m-1}^k u, \quad g \in \Delta(\mathcal{T}_h),\ k \le \dim g < m. \tag{4.12}$$

These observations are the key tools to obtain the following result.

Theorem 4.5. *The operators π^k are projections onto $\mathcal{P}\Lambda^k(\mathcal{T}_h)$.*

Proof. Assume throughout that $u \in \mathcal{P}\Lambda^k(\mathcal{T}_h)$. We have to show that $\pi^k u = u$. We will argue that

$$\operatorname{tr}_f \pi_m^k u = \operatorname{tr}_f u, \quad \text{if } f \in \Delta(\mathcal{T}_h), \quad k \le \dim f \le m, \tag{4.13}$$

for $m = k, k+1, \dots, n$. This will imply the desired result, since functions in $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ are uniquely determined by their traces on $f \in \Delta(\mathcal{T}_h)$. We will prove (4.13) by induction on m. Recall that $u - \pi_{k-1}^k u \in \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ for any $f \in \Delta_k(\mathcal{T}_h)$.

As a consequence, $P_f^k(u - \pi_{k-1}^k u) = u - \pi_{k-1}^k u$, and therefore (4.13), with $m = k$, follows from (4.11).

Next, if (4.13) holds for m replaced by $m-1$, then (4.12) implies that $\operatorname{tr}_g \pi_m^k u = \operatorname{tr}_g \pi_{m-1}^k u = \operatorname{tr}_g u$ for all $g \in \Delta(\mathcal{T}_h)$, with $k \le \dim f < m$. So it only remains to show the identity (4.13) for $f \in \Delta_m(\mathcal{T}_h)$. However, for each $f \in \Delta_m(\mathcal{T}_h)$, we have $(u - \pi_{m-1}^k u)|_{\Omega_f} \in \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$. Hence $P_f^k(u - \pi_{m-1}^k u) = (u - \pi_{m-1}^k u)|_{\Omega_f}$, and then (4.11) implies that $\operatorname{tr}_f \pi_m^k u = \operatorname{tr}_f u$. We have therefore verified that the operator π_m^k satisfies property (4.13), which completes the proof. $\square$

To show that the projections π^k are cochain projections, the following observation is useful.

Lemma 4.6. *Assume that $0 < k \le n$ and that $u \in H\Lambda^{k-1}(\Omega)$. For any $f \in \Delta_k(\mathcal{T}_h)$ the function $d(\pi_{k-1}^{k-1} u - \pi_{k-2}^{k-1} u)|_{\Omega_f} \in \breve{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$.*

Proof. The function $e \equiv d(\pi_{k-1}^{k-1} u - \pi_{k-2}^{k-1} u)$ is obviously in $\mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$. Therefore, it only remains to show that $\int_f \operatorname{tr}_f e = 0$. If $f = [x_0, x_1, \dots, x_k]$, then it follows from the definition of π_{k-1}^{k-1} and (2.8) that

$$\int_f \operatorname{tr}_f e = \sum_{j=0}^{k} (-1)^j \int_{f_j} \operatorname{tr}_{f_j} P_{f_j}^{k-1}(u - \pi_{k-2}^{k-1} u) = 0.$$

Here the last identity follows since for $\dim f_j = k-1$, the projection $P_{f_j}^{k-1}$ projects into a space of functions of mean value zero on f_j. $\square$

We conclude with the final result of this section.

Theorem 4.7. *The operators π^k are cochain projections, i.e., $d\pi^{k-1} = \pi^k d$ for $k = 1, 2, \dots, n$.*

Proof. We will prove that for $u \in H\Lambda^{k-1}(\Omega)$, and $1 \le k \le n$,

$$\operatorname{tr}_f \pi_m^k du = \operatorname{tr}_f d\pi_m^{k-1} u, \quad \text{if } f \in \Delta(\mathcal{T}_h),\ k \le \dim f \le m, \tag{4.14}$$

for $m = k, k+1, \dots, n$. As above, the case $m = n$ implies the desired result. We note that it follows from Lemma 4.2 that if (4.14) holds for any $k \le m \le n$, then $\pi_m^k du = d\pi_m^{k-1} u$.

The identity (4.14) will be established by induction on m, starting from $m = k$. By (4.8) and (4.11) we have, for any $f \in \Delta_k(\mathcal{T}_h)$,

$$\operatorname{tr}_f \pi_k^k du = \operatorname{tr}_f[\pi_{k-1}^k du + P_f^k(du - \pi_{k-1}^k du)] = d \operatorname{tr}_f P_f^{k-1} u + \operatorname{tr}_f (I - P_f^k)\pi_{k-1}^k du.$$

On the other hand,

$$\begin{aligned} d \operatorname{tr}_f \pi_k^{k-1} u &= d \operatorname{tr}_f P_f^{k-1} u + d \operatorname{tr}_f (I - P_f^{k-1}) \pi_{k-1}^{k-1} u \\ &= d \operatorname{tr}_f P_f^{k-1} u + \operatorname{tr}_f (I - P_f^k) d\pi_{k-1}^{k-1} u. \end{aligned}$$

By comparing the two expressions, and utilizing (4.9), we obtain

$$\begin{aligned} \operatorname{tr}_f(\pi_k^k du - d\pi_k^{k-1} u) &= \operatorname{tr}_f (I - P_f^k)(\pi_{k-1}^k du - d\pi_{k-1}^{k-1} u) \\ &= \operatorname{tr}_f (I - P_f^k)(d\pi_{k-2}^{k-1} u - d\pi_{k-1}^{k-1} u) = 0, \end{aligned}$$

where the last identity is a consequence of Lemma 4.6. So (4.14) holds for $m = k$.

Assume next that (4.14) holds for m replaced by $m-1$. As we observed above, this implies that $\pi_{m-1}^k du = d\pi_{m-1}^{k-1} u$. Furthermore, by (4.12) it follows that the operators π_m^{k-1} and π_m^k satisfy (4.14) for all $f \in \Delta(\mathcal{T}_h)$ with $k \le \dim f \le m-1$. Finally, for $f \in \Delta_m(\mathcal{T}_h)$ we have by (4.8) and (4.11) that

$$\begin{aligned}\operatorname{tr}_f \pi_m^k du &= \operatorname{tr}_f [P_f^k(du - \pi_{m-1}^k du) + \pi_{m-1}^k du] \\ &= \operatorname{tr}_f d[P_f^{k-1}(u - \pi_{m-1}^{k-1} u) + \pi_{m-1}^{k-1} u] = \operatorname{tr}_f d\pi_m^{k-1} u.\end{aligned}$$

This completes the proof. □

Remark 4.1. Recall that the operator M^k, introduced in Section 3, is defined on all $L^2\Lambda^k(\Omega)$. However, the domain of π^k has to be restricted to $H\Lambda^k(\Omega)$, due to the appearance of the projections Q_f^k and P_f^k in the construction. In fact, it may be possible to modify the construction given above to obtain local cochain projections defined on L^2, by replacing Q_f^k and P_f^k by proper local operators defined on L^2. A possibility is to use L^2 bounded cochain projections constructed by following the path used for the nonlocal smoothed projections (cf. [1, 3, 10, 21]), but now restricted to a suitable macroelement. On the other hand, the local projections used above, essentially constructed by local Hodge Laplace problems, may seem more natural.

5. Local bounds

The purpose of this section is to derive local bounds for the projections π^k constructed above. The main technique we will use is scaling, a standard technique in the analysis of finite element methods. The arguments below resemble parts of the discussion given in [1, Section 5.4], where scaling is used in a slightly different setting.

From the construction above, it follows that the operators π^k are local operators. In fact, we observed in Section 3 that the operator $\pi_{k-1}^k = R^k$ has the property that $\operatorname{tr}_f \circ \pi_{k-1}^k u$ only depends on $u|_{\Omega_f^e}$. As a consequence, $(\pi_{k-1}^k u)|_T$ only depends on u restricted to

$$\bigcup_{f \in \Delta_k(T)} \Omega_f^e \subset \Omega_T^e = D_{0,T} \subset D_{k-1,T}$$

for $T \in \mathcal{T}_h$ and $0 < k \le n$. Here we recall that the local domains $D_{m,T}$ and $D_T = D_{n,T}$ are defined by (2.3). Therefore it follows by (2.3), (4.11), and the local properties of the operators P_f^k and E_f^k, that the operator π^k has the property that $(\pi^k u)|_T$ only depends on $u|_{D_T}$ for any $T \in \mathcal{T}_h$, $0 \le k \le n$. Furthermore, for each h the operator π^k is a bounded operator in $H\Lambda^k(\Omega)$. Hence, for each h and each $T \in \mathcal{T}_h$ there is a constant $c = c(h,T)$ such that

$$\|\pi^k u\|_{L^2\Lambda^k(T)} \le c(h,T)\,(\|u\|_{L^2\Lambda^k(D_T)} + \|du\|_{L^2\Lambda^{k+1}(D_T)}), \quad u \in H\Lambda^k(D_T). \tag{5.1}$$

Our goal in this section is to improve this result by establishing the uniform bound

$$\|\pi^k u\|_{L^2\Lambda^k(T)} \le C\,(\|u\|_{L^2\Lambda^k(D_t)} + h_T\|du\|_{L^2\Lambda^{k+1}(D_T)}), \quad u \in H\Lambda^k(D_T), \tag{5.2}$$

for $0 \le k \le n$, where the constant C is independent of h and T. Since the operators π^k commute with the exterior derivative, the estimate (5.2) will also imply that

$$\|d\pi^k u\|_{L^2\Lambda^k(T)} \le C\,\|du\|_{L^2\Lambda^k(D_T)}, \quad u \in H\Lambda^k(D_T), \tag{5.3}$$

for $0 \le k < n$, with the same constant C as in (5.2). Therefore, the estimate (5.2) will, in particular, imply the bounds given in Theorem 2.2.

The rest of this section will be used to prove the estimate (5.2). For any fixed $T \in \mathcal{T}_h$, we introduce the scaling $\Phi_T(x) = (x - x_0)/h_T$, where x_0 is a vertex of T. We let $\hat{T} = \Phi_T(T)$ and $\hat{D}_T = \Phi_T(D_T)$ be the corresponding reference domains with size of order one. The restriction of the triangulation $\mathcal{T}_h$ to D_T will be denoted $\mathcal{T}_h(D_T)$, and $\hat{\mathcal{T}}_h(D_T)$ the induced triangulation on $\hat{D}_T$. In general we will use the hat notation to denote scaled versions of domains and local triangulations, e.g., $\hat{f} = \Phi_T(f)$, $f \in \Delta(\mathcal{T}_h)$. We note that the pullback, Φ_T^* maps $H\Lambda^k(\hat{D}_T)$ to $H\Lambda^k(D_T)$. Furthermore, it follows from the definition of pullbacks that

$$\|\Phi_T^* u\|_{L^2\Lambda^k(D)} = h_T^{-k+n/2}\|u\|_{L^2\Lambda^k(\hat{D})}, \quad u \in L^2\Lambda^k(\hat{D}), \tag{5.4}$$

where $D \subset D_T$ and $\hat{D} = \Phi_T(D)$. We will obtain bounds for the operator π^k, considered as a local operator mapping $H\Lambda^k(D_T)$ to $H\Lambda^k(T)$, by studying the operator $\Phi_T^{*-1}\pi^k\Phi_T^*$ as an operator mapping $H\Lambda^k(\hat{D}_T)$ to $H\Lambda^k(\hat{T})$. In fact, since the since the pullbacks commute with the exterior derivative, it follows from (5.4) that

$$\begin{aligned}
\|\pi^k u\|_{L^2\Lambda^k(T)} &= \|\Phi_T^{*-1}\pi^k u\|_{L^2\Lambda^k(\hat{T})} h_T^{-k+n/2} \\
&\le \|\Phi_T^{*-1}\pi^k\Phi_T^*\|\, h_T^{-k+n/2}(\|\Phi_T^{*-1}u\|_{L^2\Lambda^k(\hat{T})} + \|\Phi_T^{*-1}du\|_{L^2\Lambda^{k+1}(\hat{T})}) \\
&\le \|\Phi_T^{*-1}\pi^k\Phi_T^*\|\,(\|u\|_{L^2\Lambda^k(D_T)} + h_T\|du\|_{L^2\Lambda^{k+1}(D_T)}),
\end{aligned} \tag{5.5}$$

where $\|\Phi_T^{*-1}\pi^k\Phi_T^*\|$ denotes the operator norm in $\mathcal{L}(H\Lambda^k(\hat{D}_T), L^2\Lambda^k(\hat{T}))$. Note that if we can show that this operator norm is uniformly bounded with respect to h and $T \in \mathcal{T}_h$, then (5.5) will imply the desired bound (5.2). The following result is the key tool for this verification.

Lemma 5.1. *The operator $\Phi_T^{*-1}\pi^k\Phi_T^*$ can be identified with the operator $\hat{\pi}^k \in \mathcal{L}(H\Lambda^k(\hat{D}_T), H\Lambda^k(\hat{T}))$ obtained by constructing the operator π^k with respect to the triangulation $\hat{\mathcal{T}}_h(D_T)$ of $\hat{D}_T$.*

Proof. We have to show that the operators π^k and $\hat{\pi}^k$ satisfy $\pi^k\Phi_T^* = \Phi_T^*\hat{\pi}^k$. In fact, the proof just consists of checking that the pullback Φ_T^* commutes properly with the operators used to construct π^k. A key property of the polynomial spaces $\mathcal{P}\Lambda^k$ is that they are affine invariant. Therefore, in particular, we will have that the spaces $\mathcal{P}\Lambda^k(\mathcal{T}_h(D_T)) = \Phi_T^*\mathcal{P}\Lambda^k(\hat{\mathcal{T}}_h(D_T))$. As a consequence of this, we also obtain that the local projections Q_f^k, defined with respect to the extended macroelements Ω_f^e, satisfies

$$\Phi_T^* Q_f^k = \hat{Q}_f^k\Phi_T^*, \quad f \in \Delta_k(\mathcal{T}_h(D_T)), \tag{5.6}$$

with the obvious interpretation of $\hat{Q}_f^k$ as the corresponding projections defined with respect to the domain $\hat{\Omega}_f^e = \Phi_T^*(\Omega_f^e)$. A corresponding property holds for the extension operators $\mathcal{E}_f^k$, i.e., $\mathcal{E}_f^k\phi_T^* = \Phi_T^*\hat{\mathcal{E}}_f^k$, where $\hat{\mathcal{E}}_f^k$ maps $\mathcal{P}_0\Lambda^k(\hat{f})$ to $\mathring{\mathcal{P}}_1^-\Lambda^k(\hat{\mathcal{T}}_{f,h})$. In particular,

$$\mathcal{E}_f^k \mathsf{vol}_f = \Phi_T^*\hat{\mathcal{E}}_f^k\Phi_T^{*-1}\mathsf{vol}_f = \Phi_T^*\hat{\mathcal{E}}_f^k\mathsf{vol}_{\hat{f}}. \tag{5.7}$$

Consider the operator $S^0\Phi_T^*$, where S^k are the operators introduced in Section 3 above. By (5.7) we have, for any $u \in H\Lambda^k(\hat{D}_T)$,

$$\begin{aligned} S^0\Phi_T^* u &= \sum_{f\in\Delta_0(\mathcal{T}_h(D_T))} \left(\int_{\Omega_f} \Phi_T^* u \wedge \mathsf{vol}_{\Omega_f}\right) \mathcal{E}_f^0 \mathsf{vol}_f \\ &= \sum_{f\in\Delta_0(\mathcal{T}_h(D_T))} \left(\int_{\Omega_f} \Phi_T^*(u \wedge \Phi_T^{*-1}\mathsf{vol}_{\Omega_f}\right) \mathcal{E}_f^0 \mathsf{vol}_f \\ &= \sum_{f\in\Delta_0(\mathcal{T}_h(D_T))} \left(\int_{\hat{\Omega}_f} (u \wedge \mathsf{vol}_{\hat{\Omega}_f}\right) \Phi_T^* \hat{\mathcal{E}}_f^0 \mathsf{vol}_{\hat{f}} = \Phi_T^* \hat{S}^0 u. \end{aligned} \tag{5.8}$$

In general, we define the operators $\hat{S}^k$ with respect to the reference domain $\hat{D}_T$ as outlined in Section 3. In particular, the weight functions $\hat{z}_f^k$ are taken to be $\Phi_T^{*-1} z_f^k$. It follows essentially from (5.6), and an argument similar to one leading to (5.8), that $S^k\Phi_T^* = \Phi_T^*\hat{S}^k$, and this further leads to

$$\pi_{k-1}^k \Phi_T^* = R^k \Phi_T^* = \Phi_T^* \hat{R}^k = \Phi_T^* \hat{\pi}_{k-1}^k. \tag{5.9}$$

It is also straightforward to check that the local projections P_f^k and the extension operators E_f^k satisfy the corresponding properties $P_f^k\Phi_T^* = \Phi_T^*\hat{P}_f^k$ and $E_f^k\Phi_T^* = \Phi_T^*\hat{E}_f^k$, which implies that

$$E_f^k \operatorname{tr}_f P_f^k \Phi_T^* = \Phi_T^* \hat{E}_f^k \operatorname{tr}_{\hat{f}} \hat{P}_f^k.$$

By combining this with the recursion (4.11) and (5.9), we obtain the relation $\pi_m^k\Phi_T^* = \Phi_T^*\hat{\pi}_m^k$ for $k \le m \le n$. In particular, the desired relation $\pi^k\Phi_T^* = \Phi_T^*\hat{\pi}^k$ is obtained for $m = n$. ☐

We now have the following main result of this section.

Theorem 5.2. *The operators π^k satisfy the bounds* (5.2) *and* (5.3)*, where the constant C is independent of h and $T \in \mathcal{T}_h$.*

Proof. It follows from (5.1) that for each h and T, there is constant $C(h,T)$ such that

$$\|\hat{\pi}^k u\|_{L^2\Lambda^k(\hat{T})} \le C(h,T)\|u\|_{H\Lambda^k(\hat{D}_T)}, \quad u \in H\Lambda^k(\hat{D}_T), \tag{5.10}$$

where, as above, $\hat{\pi}^k$ is obtained by constructing the operator π^k with respect to the triangulation $\hat{\mathcal{T}}_h(D_T)$ of $\hat{D}_T$. However, due to the assumption of shape regularity of the family $\{\mathcal{T}_h\}$, it follows that the induced triangulations $\hat{\mathcal{T}}_h(D_T)$ vary over a compact set. Therefore, the constant $C(h,T)$ is uniformly bounded with respect to h and $T \in \mathcal{T}_h$. The desired estimate (5.2) now follows from Lemma 5.1, combined with (5.5) and (5.10). As we observed above, (5.3) follows from (5.2) and the fact that the projections π^k commute with d. ☐

Finally, we observe that since the shape regularity of the triangulation $\{\mathcal{T}_h\}$ implies that the covering $\{D_T\}$ of Ω has a bounded overlap property, it follows from the bounds (5.2) and (5.3) that the global estimates

$$\|\pi^k u\|_{L^2(\Omega)} \le C\left(\|u\|_{L^2(\Omega)} + h\|du\|_{L^2(\Omega)}\right)$$

and

$$\|d\pi^k u\|_{L^2(\Omega)} \leq C \, \|du\|_{L^2(\Omega)}, \quad u \in H\Lambda^k(\Omega),$$

also hold, where C is independent of h.

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