THE CURSE OF DIMENSIONALITY FOR NUMERICAL INTEGRATION OF SMOOTH FUNCTIONS

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ABSTRACT. We prove the curse of dimensionality for multivariate integration of C^r functions: The number of needed function values to achieve an error ϵ is larger than $c_r(1 + \gamma)^d$ for $\epsilon \leq \epsilon_0$, where $c_r, \gamma > 0$. The proofs are based on volume estimates for $r = 1$ together with smoothing by convolution. This allows us to obtain smooth fooling functions for $r > 1$.

1. INTRODUCTION

We study multivariate integration for different classes F_d of smooth functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$. Our emphasis is on large values of $d \in \mathbb{N}$. We want to approximate

$$(1) \quad S_d(f) = \int_{D_d} f(x) \, dx \quad \text{for } f \in F_d$$

up to some error $\varepsilon > 0$, where $D_d \subset \mathbb{R}^d$ has Lebesgue measure 1. The results in this paper hold for arbitrary sets D_d , the standard example of course is $D_d = [0, 1]^d$.

We consider (deterministic) algorithms that use only function values, and classes F_d of functions bounded in absolute value by 1 and containing all constant functions $f(x) \equiv c$ with $|c| \leq 1$. An algorithm that uses no function value at all must be a constant, $A_0(f) \equiv b$, and its error is at least

$$\max_{f \in F_d} |S_d(f)| = 1.$$

We call this the initial error of the problem, it does not depend on d . Hence multivariate integration is well scaled and that is why we consider $\varepsilon < 1$.

Let $n(\varepsilon, F_d)$ denote the minimal number of function values needed for this task in the worst case setting¹. By the *curse of dimensionality* we mean that $n(\varepsilon, F_d)$ is exponentially large in d . That is, there are positive numbers c , ε_0 and γ such that

$$(2) \quad n(\varepsilon, F_d) \geq c(1 + \gamma)^d \quad \text{for all } \varepsilon \leq \varepsilon_0 \quad \text{and infinitely many } d \in \mathbb{N}.$$

For many natural classes F_d the bound in (2) will hold for all $d \in \mathbb{N}$. This applies in particular to the classes considered in this paper.

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¹We add that $n(\varepsilon, F_d)$ is the information complexity of multivariate integration over F_d and is proportional to the (total) complexity as long as F_d is convex and symmetric. The last two assumptions are needed to guarantee that a linear algorithm is optimal and its implementation cost is linear in $n(\varepsilon, F_d)$.

There are many classes F_d for which the curse of dimensionality has been proved; see [5, 7] for such examples. However, it has *not* been known if the curse of dimensionality occurs for probably the most natural class which is the unit ball of r times continuously differentiable functions,

$$\mathcal{C}_d^r = \{f \in C^r(\mathbb{R}^d) \mid \|D^\beta f\| \leq 1 \text{ for all } |\beta| \leq r\},$$

where $\beta = (\beta_1, \beta_2, \dots, \beta_d)$, with nonnegative integers β_j , $|\beta| = \sum_{j=1}^d \beta_j$, and D^β denotes the operator of β_j times differentiation with respect to the j th variable for $j = 1, 2, \dots, d$. By $\|\cdot\|$ we mean the sup norm, $\|D^\beta f\| = \sup_{x \in \mathbb{R}^d} |(D^\beta f)(x)|$.

For $r = 0$, we obviously have $n(\varepsilon, \mathcal{C}_d^0) = \infty$ for all $\varepsilon < 1$ and all $d \in \mathbb{N}$. Therefore, from now on we always assume that $r \geq 1$. For $r = 1$, the curse of dimensionality for $\mathcal{C}_d^1$ follows from the results of Sukharev [8]. Whether the curse holds for $r \geq 2$ has been an open problem for many years.

The class $\mathcal{C}_d^r$ for $D_d = [0, 1]^d$ (and functions and norms restricted to D_d) was already studied in 1959 by Bakhvalov [2]; see also [4]. He proved that there are two positive numbers $a_{d,r}$ and $A_{d,r}$ such that

$$(3) \quad a_{d,r} \varepsilon^{-d/r} \leq n(\varepsilon, \mathcal{C}_d^r) \leq A_{d,r} \varepsilon^{-d/r} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0, 1).$$

This means that for a fixed d and for ε tending to zero, we know that $n(\varepsilon, \mathcal{C}_d^r)$ is of order $\varepsilon^{-d/r}$ and the exponent of ε^{-1} grows linearly in d . Unfortunately, Bakhvalov's result does not allow us to conclude whether the curse of dimensionality holds for the class $\mathcal{C}_d^r$. In fact, if we reverse the roles of d and ε , and consider a fixed ε and d tending to infinity, the bound (3) on $n(\varepsilon, \mathcal{C}_d^r)$ is useless. We prove the following result and hereby solve Open Problem 1 from [5]:

Main Theorem. *The curse of dimensionality holds for the classes $\mathcal{C}_d^r$ with the super-exponential lower bound*

$$n(\varepsilon, \mathcal{C}_d^r) \geq c_r (1 - \varepsilon) d^{d/(2r+3)} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0, 1),$$

where $c_r \in (0, 1]$ depends only on r .

We also prove that the curse of dimensionality holds for even smaller classes of functions F_d for which the norms of arbitrary directional derivatives are bounded proportionally to $1/\sqrt{d}$.

We now discuss how we obtain lower bounds on $n(\varepsilon, F_d)$ for numerical integration defined on convex and symmetric classes F_d . The standard proof technique is to find a fooling function $f \in F_d$ that vanishes at the points $\mathcal{P} = \{x_1, x_2, \dots, x_n\}$ at which we sample functions from F_d , and the integral of f is as large as possible. All algorithms that use function values at x_j 's must give the same approximation of the integral for f and $-f$. Thus, each such algorithm makes an error of at least $|S_d(f) - S_d(-f)|/2 = |S_d(f)|$ for one of the functions. That is why the integral of f is a lower bound on the worst case error of all algorithms using function values at x_j 's. If, for all choices of $x_1, x_2, \dots, x_n$, there are functions $f \in F_d$ vanishing at x_j 's with integrals larger than ε , then $n(\varepsilon, F_d) \geq n$.

We start with the fooling function

$$f_0(x) = \min \left\{ 1, \frac{1}{\delta \sqrt{d}} \text{dist}(x, \mathcal{P}_\delta) \right\} \quad \text{for all } x \in \mathbb{R}^d,$$

where

$$\mathcal{P}_\delta = \bigcup_{i=1}^n B_\delta^d(x_i)$$

and $B_\delta^d(x_i)$ is the ball with center x_i and radius $\delta\sqrt{d}$. The function f_0 is Lipschitz. By a suitable smoothing via convolution we construct a fooling function $f_r \in \mathcal{C}_d^r$ with $f_r|_{\mathcal{P}} = 0$.

2. PRELIMINARIES

In this section, we precisely define our problem. Let F_d be a class of continuous functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $S_d(f)$ (see (1)) exists for every $f \in F_d$. We approximate the integral $S_d(f)$, $f \in F_d$, by algorithms

$$A_{n,d}(f) = \phi_{n,d}(f(x_1), f(x_2), \dots, f(x_n)),$$

where $x_j \in \mathbb{R}^d$ can be chosen adaptively and $\phi_{n,d} : \mathbb{R}^n \rightarrow \mathbb{R}$ is an arbitrary mapping. Adaption means that the selection of x_j may depend on the already computed values $f(x_1), f(x_2), \dots, f(x_{j-1})$. The (worst case) error of the algorithm $A_{n,d}$ is defined as

$$e(A_{n,d}) = \sup_{f \in F_d} |S_d(f) - A_{n,d}(f)|.$$

The minimal number of function values to guarantee that the error is at most ε is defined as

$$n(\varepsilon, F_d) = \min\{n \in \mathbb{N} \mid \exists A_{n,d} \text{ such that } e(A_{n,d}) \leq \varepsilon\}.$$

Hence we minimize n over all choices of adaptive sample points x_j and mappings $\phi_{n,d}$. It is well known that, as long as the class F_d is convex and symmetric, we may restrict the minimization of n by considering only nonadaptive choices of x_j and linear mappings $\phi_{n,d}$. Furthermore,

$$(4) \quad n(\varepsilon, F_d) = \min\left\{n \in \mathbb{N} \mid \inf_{\mathcal{P} \subset \mathbb{R}^d, \# \mathcal{P} = n} \sup_{f \in F_d, f|_{\mathcal{P}} = 0} |S_d(f)| \leq \varepsilon\right\};$$

see [4, Prop. 1.2.6] or [9, Theorem 5.5.1]. In this paper, we always consider convex and symmetric F_d so that we can use the last formula for $n(\varepsilon, F_d)$. For more details see, e.g., Chapter 4 in [5].

As already mentioned, our lower bounds are based on a volume estimate of a neighborhood of certain sets in $\mathbb{R}^d$; see also [3]. In the following, we denote by A_δ the $(\delta\sqrt{d})$ -neighborhood of $A \subset \mathbb{R}^d$, which is defined by

$$(5) \quad A_\delta = \{x \in \mathbb{R}^d \mid \text{dist}(x, A) \leq \delta\sqrt{d}\},$$

where $\text{dist}(x, A) = \inf_{a \in A} \|x - a\|_2$ denotes the Euclidean distance of x from A .

Furthermore, we denote by $B_\delta^d(x)$ the d -dimensional ball with center $x \in \mathbb{R}^d$ and radius $\delta\sqrt{d}$, i.e.,

$$B_\delta^d(x) = \{y \in \mathbb{R}^d \mid \|x - y\|_2 \leq \delta\sqrt{d}\}.$$

We will need some standard volume estimates for Euclidean balls. Recall that the volume of a Euclidean ball of radius 1 is given by

$$V_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}.$$

From Stirling's formula for the Γ function, we have

$$\Gamma(x+1) = \sqrt{2\pi x} x^x e^{-x+\frac{\theta_x}{12x}} \quad \text{for all } x > 0,$$

where $\theta_x \in (0, 1)$; see [1, p. 257]. This leads to the estimate

$$\Gamma(x+1) > \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \quad \text{for all } x > 0.$$

Combining this estimate with the volume formula for the ball, we obtain for all $d \in \mathbb{N}$,

$$(6) \quad \lambda_d(B_\delta^d(x)) = (\delta\sqrt{d})^d V_d < (\delta\sqrt{d})^d \frac{\left(\frac{2\pi e}{d}\right)^{d/2}}{\sqrt{\pi d}} = \frac{(\delta\sqrt{2\pi e})^d}{\sqrt{\pi d}} < (\delta\sqrt{2\pi e})^d,$$

where λ_d is the Lebesgue measure. The volume formula for the Euclidean unit ball also shows the recurrence relation

$$\frac{V_{d-1}}{V_d} = \frac{d}{d-1} \frac{V_{d-3}}{V_{d-2}} \quad \text{for all } d \geq 4.$$

This easily implies

$$\frac{2}{\sqrt{d}} \frac{V_{d-1}}{V_d} < \frac{2}{\sqrt{d-2}} \frac{V_{d-3}}{V_{d-2}} \quad \text{for all } d \geq 4.$$

The last inequality can be used in an inductive argument leading to

$$(7) \quad \frac{2}{\sqrt{d}} \frac{V_{d-1}}{V_d} \leq 1 \quad \text{for all } d \geq 2.$$

This will be needed later.

3. CONVOLUTION

In this section, we fix $k \in \mathbb{N}$ and study the convolution

$$f_k := f * g_1 * \dots * g_k$$

of a function f defined on $\mathbb{R}^d$ with (normalized) indicator functions g_j . We are interested in properties of f_k in terms of the properties of the initial function f . Recall that the convolution of two functions f and g on $\mathbb{R}^d$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-t) g(t) dt \quad \text{for all } x \in \mathbb{R}^d.$$

Fix a number $\delta > 0$ and a sequence $(\alpha_j)_{j=1}^k$ with $\alpha_j > 0$ such that

$$\sum_{j=1}^k \alpha_j \leq 1.$$

For example, we may take $\alpha_j = 1/k$ for $j = 1, 2, \dots, k$. For $j = 1, \dots, k$, we define the ball

$$B_j = \left\{ x \in \mathbb{R}^d \mid \|x\|_2 \leq \alpha_j \delta \sqrt{d} \right\}$$

and the function $g_j: \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$(8) \quad g_j(x) = \frac{\mathbb{1}_{B_j}(x)}{\lambda_d(B_j)} = \frac{1}{\lambda_d(B_j)} \begin{cases} 1 & \text{if } x \in B_j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the convolution of a function f with g_j can be written as

$$(f * g_j)(x) = \frac{1}{\lambda_d(B_j)} \int_{B_j} f(x+t) dt \quad \text{for all } x \in \mathbb{R}^d.$$

We will frequently use the following probabilistic interpretation. Let Y_j be a random variable that is uniformly distributed on B_j . Then the convolution of f with g_j can be written as the expected value

$$(f * g_j)(x) = \mathbb{E}[f(x + Y_j)].$$

The next theorem is the basis for the induction steps of the proofs of our main results. For $f: \mathbb{R}^d \rightarrow \mathbb{R}$, we use the Lipschitz constant

$$\text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_2}.$$

Define

$$C^r = \left\{ f: \mathbb{R}^d \rightarrow \mathbb{R} \mid D^{\theta_\ell} \dots D^{\theta_1} f \text{ is continuous for all } \ell \leq r \right. \\ \left. \text{and all } \theta_1, \dots, \theta_r \in \mathbb{S}^{d-1} \right\},$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and $D^{\theta_1} f(x) = \lim_{h \rightarrow 0} \frac{1}{h} (f(x + h\theta_1) - f(x))$ is the derivative in the direction of θ_1 .

Theorem 1. For $k \in \mathbb{N}$ and $f \in C^r$, define

$$f_k = f * g_1 * \dots * g_k \quad \text{with } g_j \text{ from (8).}$$

For $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ and let Ω_δ be its neighborhood defined as in (5). Then:

- (i) if $f(x) = 0$ for all $x \in \Omega_\delta$, then $f_k(x) = 0$ for all $x \in \Omega$,
- (ii) $\text{Lip}(f_k) \leq \text{Lip}(f)$,
- (iii) if $\int_{D_a} f(x+t) dx \geq \varepsilon$ for all $t \in \mathbb{R}^d$ with $\|t\|_2 \leq \delta\sqrt{d}$, then $\int_{D_a} f_k(x) dx \geq \varepsilon$,
- (iv) for all $\ell \leq r$ and all $\theta_1, \theta_2, \dots, \theta_\ell \in \mathbb{S}^{d-1}$,

$$\text{Lip}\left(D^{\theta_\ell} D^{\theta_{\ell-1}} \dots D^{\theta_1} f_k\right) \leq \text{Lip}\left(D^{\theta_\ell} D^{\theta_{\ell-1}} \dots D^{\theta_1} f\right),$$

- (v) $f_k \in C^{r+k}$, and for all $\ell \leq r$, all $j = 1, \dots, k$ and all $\theta_1, \theta_2, \dots, \theta_{\ell+j} \in \mathbb{S}^{d-1}$,

$$\text{Lip}\left(D^{\theta_{\ell+j}} D^{\theta_{\ell+j-1}} \dots D^{\theta_1} f_k\right) \leq \left(\prod_{i=1}^j \frac{1}{\delta\alpha_i}\right) \text{Lip}\left(D^{\theta_\ell} D^{\theta_{\ell-1}} \dots D^{\theta_1} f\right).$$

The parts (i)–(iv) of this theorem show that some properties of the initial function f are preserved by convolutions. Part (v) states that we gain one “degree of smoothness” with every convolution, losing only a multiplicative constant for its Lipschitz constant.

Proof. First note that we can write f_k as

$$f_k(x) = \mathbb{E}[f(x + Y)], \quad \text{for all } x \in \mathbb{R}^d,$$

where Y is a random variable with probability density function $g_1 * \dots * g_k$. By construction of g_j ’s which are the indicator functions of the balls whose sum of the radii is at most $\delta\sqrt{d}$, we have

$$\{t \in \mathbb{R}^d \mid g_1 * \dots * g_k(t) > 0\} \subset \{t \in \mathbb{R}^d \mid \|t\|_2 \leq \delta\sqrt{d}\},$$

which implies that $x + Y \in \Omega_\delta$ almost surely for every $x \in \Omega$. Thus, $f(x) = 0$ for all $x \in \Omega_\delta$ implies that $f_k(x) = 0$ for all $x \in \Omega$, which is property (i).

Property (ii) is proven by

$$\begin{aligned} |f_k(x) - f_k(y)| &= |\mathbb{E}[f(x+Y) - f(y+Y)]| \leq \mathbb{E}[|f(x+Y) - f(y+Y)|] \\ &\leq \text{Lip}(f) \mathbb{E}[\|(x+Y) - (y+Y)\|_2] = \text{Lip}(f) \|x - y\|_2. \end{aligned}$$

To prove (iii), we use Fubini's theorem and we obtain

$$\int_{D_d} f_k(x) \, dx = \int_{D_d} \mathbb{E}[f(x+Y)] \, dx = \mathbb{E}\left[\int_{D_d} f(x+Y) \, dx\right] \geq \varepsilon$$

by assumption.

For the proof of properties (iv) and (v), let $\theta = (\theta_1, \dots, \theta_\ell) \in (\mathbb{S}^{d-1})^\ell$. We write D^θ for $D^{\theta_\ell} \dots D^{\theta_1}$. Clearly, $f \in C^r$ and $\ell \leq r$ implies that $D^\theta f \in C^{r-\ell} \subseteq C$. Since f_k is at least as smooth as f , both $D^\theta f$ and $D^\theta f_k$ are well defined.

We need the well-known fact that $D^\theta(f * g) = (D^\theta f) * g$ if $f \in C^\ell$ and g has compact support. For $g = g_1 * \dots * g_k$, we have

$$\begin{aligned} |D^\theta f_k(x) - D^\theta f_k(y)| &= |((D^\theta f) * g)(x) - ((D^\theta f) * g)(y)| \\ &= \left| \int_{\mathbb{R}^d} [(D^\theta f)(x+t) - (D^\theta f)(y+t)] g(t) \, dt \right| \\ &\leq \text{Lip}(D^\theta f) \|x - y\|_2 \int_{\mathbb{R}^d} g(t) \, dt \\ &= \text{Lip}(D^\theta f) \|x - y\|_2 \end{aligned}$$

for all $x, y \in \mathbb{R}^d$. The last equality follows since the g_k is normalized. This proves (iv).

For (v), we need to prove that $f_k \in C^{r+k}$ with $f_0 = f \in C^r$ by assumption, and then it is enough to show that for all $m \leq r+k$ and all $\theta = (\theta_m, \dots, \theta_1) \in (\mathbb{S}^{d-1})^m$,

$$\text{Lip}(D^\theta f_k) \leq \frac{1}{\delta \alpha_k} \text{Lip}(D^{\bar{\theta}} f_{k-1}),$$

where $\bar{\theta} = (\theta_{m-1}, \dots, \theta_1) \in (\mathbb{S}^{d-1})^{m-1}$.

Assume inductively that $f_{k-1} \in C^{m-1}$, which holds for $k = 1$. This implies $D^{\bar{\theta}}(f_{k-1} * g_k) = (D^{\bar{\theta}} f_{k-1}) * g_k$, and

$$\begin{aligned} D^\theta f_k(x) &= D^{\theta_m}((D^{\bar{\theta}} f_{k-1}) * g_k)(x) \\ &= D^{\theta_m} \left(\frac{1}{\lambda_d(B_k)} \int_{\mathbb{R}^d} D^{\bar{\theta}} f_{k-1}(x+t) \mathbb{1}_{B_k}(t) \, dt \right) \\ &= \frac{1}{\lambda_d(B_k)} D^{\theta_m} \left(\int_{\theta_m^\perp} \int_{\mathbb{R}} D^{\bar{\theta}} f_{k-1}(x+s+h\theta_m) \mathbb{1}_{B_k}(s+h\theta_m) \, dh \, ds \right) \\ &= \frac{1}{\lambda_d(B_k)} \int_{\theta_m^\perp} D^{\theta_m} \left(\int_{\mathbb{R}} D^{\bar{\theta}} f_{k-1}(x+s+h\theta_m) \mathbb{1}_{B_k}(s+h\theta_m) \, dh \right) \, ds, \end{aligned}$$

where $\theta_m^\perp$ is the hyperplane orthogonal to θ_m . For any function f on $\mathbb{R}$ of the form

$$f(x) = \int_{x-a}^{x+a} g(y) \, dy$$

with some continuous function g we have

$$f'(x) = g(x+a) - g(x-a).$$

Therefore, we obtain

$$D^\theta f_k(x) = \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta_m^\perp} \left[D^{\bar{\theta}} f_{k-1}(x + s + h_{\max}(s) \theta_m) - D^{\bar{\theta}} f_{k-1}(x + s - h_{\max}(s) \theta_m) \right] ds$$

with

$$h_{\max}(s) = \max\{h \geq 0 \mid s + h\theta_m \in B_k\}.$$

For each $s \in B_k \cap \theta_m^\perp$, define the points $s_1 = s + h_{\max}(s) \theta_m \in B_k$ and $s_2 = s - h_{\max}(s) \theta_m \in B_k$. Then

$$\begin{aligned} |D^\theta f_k(x) - D^\theta f_k(y)| &\leq \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta_m^\perp} \left[\left| D^{\bar{\theta}} f_{k-1}(x + s_1) - D^{\bar{\theta}} f_{k-1}(x + s_2) \right. \right. \\ &\quad \left. \left. - D^{\bar{\theta}} f_{k-1}(y + s_1) + D^{\bar{\theta}} f_{k-1}(y + s_2) \right| \right] ds \\ &\leq \frac{1}{\lambda_d(B_k)} \int_{B_k \cap \theta_m^\perp} \left[\left| D^{\bar{\theta}} f_{k-1}(x + s_1) - D^{\bar{\theta}} f_{k-1}(y + s_1) \right| \right. \\ &\quad \left. + \left| D^{\bar{\theta}} f_{k-1}(x + s_2) - D^{\bar{\theta}} f_{k-1}(y + s_2) \right| \right] ds \\ &\leq \frac{2\lambda_{d-1}(B_k \cap \theta_m^\perp)}{\lambda_d(B_k)} \text{Lip}(D^{\bar{\theta}} f_{k-1}) \|x - y\|_2. \end{aligned}$$

In particular, this shows the implication

$$f_{k-1} \in C^{m-1} \implies f_k \in C^m$$

for all $k \in \mathbb{N}$. Taking $m = r + k$ we have $f_k \in C^{r+k}$, as claimed.

For $m \leq r + k$, it remains to bound $2\lambda_{d-1}(B_k \cap \theta_m^\perp)/\lambda_d(B_k)$. Recall that B_k is a ball with radius $\delta\alpha_k\sqrt{d}$ and that V_d is the volume of the Euclidean unit ball in $\mathbb{R}^d$. We obtain from (7) that

$$\frac{2\lambda_{d-1}(B_k \cap \theta_m^\perp)}{\lambda_d(B_k)} = \frac{2(\delta\alpha_k\sqrt{d})^{d-1} V_{d-1}}{(\delta\alpha_k\sqrt{d})^d V_d} = \frac{2}{\delta\alpha_k\sqrt{d}} \frac{V_{d-1}}{V_d} \leq \frac{1}{\delta\alpha_k}.$$

This concludes the proof that

$$\text{Lip}(D^{\theta_{\ell+j}} D^{\theta_{\ell+j-1}} \dots D^{\theta_1} f_k) \leq \left(\prod_{i=1}^j \frac{1}{\delta\alpha_{k+1-i}} \right) \text{Lip}(D^{\theta_\ell} D^{\theta_{\ell-1}} \dots D^{\theta_1} f),$$

but since the order of convolution is arbitrary, we obtain in the same way,

$$\text{Lip}(D^{\theta_{\ell+j}} D^{\theta_{\ell+j-1}} \dots D^{\theta_1} f_k) \leq \left(\prod_{i \in J} \frac{1}{\delta\alpha_i} \right) \text{Lip}(D^{\theta_\ell} D^{\theta_{\ell-1}} \dots D^{\theta_1} f),$$

for all $J \subset \{1, \dots, k\}$ with $\#J = j$. In particular, this implies (v). $\square$

4. MAIN RESULTS

Let $\mathcal{P} = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ be a collection of n points. As pointed out in the introduction, we want to construct functions that vanish at $\mathcal{P}$ and have a large integral. For this, we choose

$$f_0(x) = \min \left\{ 1, \frac{1}{\delta\sqrt{d}} \operatorname{dist}(x, \mathcal{P}_\delta) \right\} \quad \text{for all } x \in \mathbb{R}^d,$$

where

$$\mathcal{P}_\delta = \bigcup_{i=1}^n B_\delta^d(x_i)$$

and $B_\delta^d(x_i)$ is the ball with center x_i and radius $\delta\sqrt{d}$.

The function $\operatorname{dist}(\cdot, \mathcal{P}_\delta)$ is Lipschitz with constant 1. Hence, for $\delta \leq 1$,

$$(9) \quad \operatorname{Lip}(f_0) = \frac{1}{\delta\sqrt{d}}.$$

Additionally, $f_0(x) = 0$ for all $x \in \mathcal{P}_\delta$ by definition.

Using these facts we can apply Theorem 1 to prove the curse of dimensionality for the following class of functions that are defined on $\mathbb{R}^d$. For a fixed $r \in \mathbb{N}$, we now take $\alpha_1 = \dots = \alpha_r = \frac{1}{r}$ and define

$$F_{d,r,\delta} = \{f: \mathbb{R}^d \rightarrow \mathbb{R} \mid f \in C^r \text{ satisfies (10)–(12)}\},$$

where

$$(10) \quad \|f\| \leq 1,$$

$$(11) \quad \operatorname{Lip}(f) \leq \frac{1}{\delta\sqrt{d}},$$

$$(12) \quad \forall k \leq r: \max_{\theta_1, \dots, \theta_k \in \mathbb{S}^{d-1}} \operatorname{Lip}(D^{\theta_1} \dots D^{\theta_k} f) \leq \frac{1}{\delta\sqrt{d}} \left(\frac{r}{\delta}\right)^k.$$

Theorem 2. *For any $r \in \mathbb{N}$ and $\delta \in (0, 1]$,*

$$n(\varepsilon, F_{d,r,\delta}) \geq (1 - \varepsilon) \begin{cases} 1 & \text{for } d = 1, \\ (\delta\sqrt{18e\pi})^{-d} & \text{for } d \geq 2, \end{cases} \quad \text{for all } \varepsilon \in (0, 1).$$

Hence the curse of dimensionality holds for the class $F_{d,r,\delta}$ for $\delta < 1/\sqrt{18e\pi}$.

This result shows that the growth rate of $n(\varepsilon, F_{d,r,\delta})$ in d can be arbitrarily large if we choose δ small enough.

Proof. Since the initial error for the classes $F_{d,r,\delta}$ is 1, we obtain $n(\varepsilon, F_{d,r,\delta}) \geq 1$ for all $\varepsilon \in (0, 1)$. This proves the statement for $d = 1$.

For $d \geq 2$, we use Theorem 1 with $k = r$, $\Omega = \mathcal{P}$ and $f_r(x) = f_0 * g_1 * \dots * g_r(x)$. Here, the g_j 's are as in Theorem 1. Recall that we have chosen $\alpha_1 = \dots = \alpha_r = 1/r$ and $\alpha_j = 0$ for $j > r$. The properties of the initial function f_0 and Theorem 1 immediately imply that f_r satisfies (10)–(12). It remains to bound its integral. Note that $f_0(x) = 1$ for all $x \notin \mathcal{P}_{2\delta}$. Clearly, $f_r(x) \geq 0$ for all $x \in \mathbb{R}^d$. Since $f_r(x)$ depends only on the values $f_0(x + t)$ for $t \in \mathbb{R}^d$ with $\|t\|_2 \leq \delta\sqrt{d}$, it follows that

$f_r(x) = 1$ for $x \notin \mathcal{P}_{3\delta}$. We thus obtain

$$\begin{aligned} \int_{D_d} f_r(x) \, dx &\geq \int_{D_d \setminus \mathcal{P}_{3\delta}} f_r(x) \, dx = 1 - \lambda_d(\mathcal{P}_{3\delta} \cap D_d) \\ &\geq 1 - \lambda_d(\mathcal{P}_{3\delta}) \geq 1 - n\lambda_d(B_{3\delta}^d) \\ &> 1 - \frac{n(3\delta\sqrt{2e\pi})^d}{\sqrt{\pi d}} \\ &> 1 - n(3\delta\sqrt{2e\pi})^d, \end{aligned}$$

where the next to last inequality follows from the bound in (6). Hence $\int_{D_d} f_r(x) \, dx \leq \varepsilon$ implies that

$$n \geq (1 - \varepsilon) (\delta\sqrt{18e\pi})^{-d}.$$

Since this holds for arbitrary $\mathcal{P}$, the result follows. $\square$

By Theorem 2, we know how the parameter δ comes into play. For $p > 0$, let

$$\delta = \frac{1}{\sqrt{18e\pi}} d^{-p/(r+1)}.$$

For this δ , we obtain a somehow stronger form of the curse of dimensionality for the class

$$\tilde{F}_{d,r,p} = \{f: \mathbb{R}^d \rightarrow \mathbb{R} \mid f \in C^r \text{ satisfies (13)–(15)}\},$$

where

$$(13) \quad \|f\| \leq 1,$$

$$(14) \quad \text{Lip}(f) \leq d^{-\frac{1}{2} + \frac{p}{r+1}} \sqrt{18e\pi},$$

$$(15) \quad \forall k \leq r: \max_{\theta_1, \dots, \theta_k \in \mathbb{S}^{d-1}} \text{Lip}(D^{\theta_1} \dots D^{\theta_k} f) \leq d^{-\frac{1}{2} + \frac{p(k+1)}{r+1}} r^k (\sqrt{18e\pi})^{k+1}.$$

Theorem 3. For any $r \in \mathbb{N}$ and $p > 0$,

$$n(\varepsilon, \tilde{F}_{d,r,p}) \geq (1 - \varepsilon) d^{p d/(r+1)} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0, 1).$$

Hence the curse of dimensionality holds for the class $\tilde{F}_{d,r,p}$.

Note that the classes $\tilde{F}_{d,r,p}$ are contained in the classes

$$\mathcal{C}_d^r = \{f \in C^r \mid \|D^\beta f\| \leq 1 \text{ for all } |\beta| \leq r\},$$

if $p < 1/2$ and d is large enough. This holds if

$$(16) \quad d \geq \left(r^r (18e\pi)^{(r+1)/2}\right)^{1/(1/2-p)}.$$

From this we easily obtain the main result already stated in the introduction.

Main Theorem. For any $r \in \mathbb{N}$, there exists a constant $c_r \in (0, 1]$ such that

$$n(\varepsilon, \mathcal{C}_d^r) \geq c_r (1 - \varepsilon) d^{d/(2r+3)} \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon \in (0, 1).$$

Hence the curse of dimensionality holds for the class $\mathcal{C}_d^r$.

Proof. The case $d = 1$ is trivial since the initial error for the classes $\mathcal{C}_d^r$ is again 1.

For $d \geq 2$, we know from Theorem 3 and the discussion thereafter that $n(\varepsilon, \mathcal{C}_d^r) \geq (1 - \varepsilon) d^{pd/(r+1)}$ for all $p < 1/2$ if $d \geq d_0$, where $d_0 = d_0(r, p)$ is the right-hand side of (16). This implies

$$n(\varepsilon, \mathcal{C}_d^r) \geq \tilde{c}_{r,p} (1 - \varepsilon) d^{pd/(r+1)} \quad \text{for all } d \geq 2,$$

with

$$\tilde{c}_{r,p} = d_0^{-pd_0/(r+1)},$$

which depends only on r and p . The choice $p^* = (r + 1)/(2r + 3)$ yields the result with $c_r = \tilde{c}_{r,p^*}$. $\square$

Note that c_r in the last theorem is super-exponentially small in r .

Remark 1. The reader might find it more natural to define classes of functions $F_{d,r}(D_d)$ that are defined only on $D_d \subset \mathbb{R}^d$. Not all such functions can be extended to smooth functions on $\mathbb{R}^d$, and even if they can be extended then the norm of the extended function could be much larger. Our lower bound results for functions defined on $\mathbb{R}^d$ can be also applied for functions defined on $D_d \subset \mathbb{R}^d$ and this makes them even stronger.

Remark 2. Note that the possibility of super-exponential lower bounds on the complexity depends on the definition of the Lipschitz constant. For the class

$$F_d = \left\{ f: [0, 1]^d \rightarrow \mathbb{R} \mid \sup_{x, y \in [0, 1]^d} \frac{|f(x) - f(y)|}{\|x - y\|_\infty} \leq 1 \right\},$$

Sukharev [8] proved that the product mid-point rule is optimal with error $e_n = \frac{d}{2d+2} n^{-1/d}$ for $n = m^d$. Hence, roughly, $n(\varepsilon, F_d) \approx 2^{-d} \varepsilon^{-d}$ and the complexity is “only” exponential in d for $\varepsilon < 1/2$.

Remark 3. We mention two results for the very small class

$$F_d = C_d^\infty = \{f \in C^\infty([0, 1]^d) \mid \|D^\beta f\| \leq 1 \text{ for all } \beta \in \mathbb{N}_0^d\}.$$

O. Wojtaszczyk [10] proved that $\lim_{d \rightarrow \infty} n(\varepsilon, F_d) = \infty$ for every $\varepsilon < 1$, hence the problem is not strongly polynomially tractable. It is still open whether the curse of dimensionality holds for this class F_d . The same class F_d was studied for the approximation problem in [6]. For this problem the curse of dimensionality is present even if we allow algorithms that use arbitrary linear functionals.

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