EXACT COUNTING OF $D_\ell$ NUMBER FIELDS
WITH GIVEN QUADRATIC RESOLVENT

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ABSTRACT. We give efficient numerical methods for counting exactly the number of $D_\ell$ number fields of degree $\ell$ with given quadratic resolvent, for calculating the constants occurring in their asymptotic expansions, and we give tables for typical cases.

1. INTRODUCTION AND NOTATION

In a previous paper [8], we have given asymptotic formulas for the number of cubic extensions of a number field with given quadratic resolvent, and even exact formulas in many cases, and in [9] these formulas have been made completely explicit in every case. In a more recent paper [10], we have generalized the results of [8] and [9] to $D_\ell$-fields of degree $\ell$ having a given quadratic resolvent. The aim of the present paper is to solve efficiently the following two problems:

• Compute to high accuracy the constants occurring in the asymptotic expansions given in [10].
• Compute the exact number of $D_\ell$ number fields of degree $\ell$ and given quadratic resolvent, with absolute discriminant up to large bounds $X$.

We will give in detail the examples with tables of the first task for $3 \leq \ell \leq 11$, and for the second task for $3 \leq \ell \leq 7$.

We summarize the notation and results of [10] that we need. Let $\ell$ be an odd prime, let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field of discriminant $D \neq 1$.

We let $\mathcal{F}(K)$ be the set of (isomorphism classes of) $D_\ell$ number fields $L$ of degree $\ell$ with quadratic resolvent isomorphic to $K$. We have $\text{Disc}(L) = D^{(\ell-1)/2}f(L)^{\ell-1}$ for some positive integer $f(L)$ which by abuse of language we call the conductor of $L$, and we denote by $M_\ell(D;X)$ the number of such fields $L$ with $f(L) \leq X$.

A special case of the main result proved in [10] is as follows:

Theorem 1.1. Set

$$
\Phi_{\ell,K}(s) = \frac{1}{\ell-1} + \sum_{L \in \mathcal{F}(K)} \frac{1}{f(L)^s}.
$$

(1) Set $r_2(D) = 0$ if $D > 0$ and $r_2(D) = 1$ if $D < 0$. We have

$$
\Phi_{\ell,K}(s) = \phi_{\ell,K}(s) + \frac{1}{(\ell-1)^{1-r_2(D)}L_\ell(s)} \prod_{p \equiv (D/p) \equiv \pm 1 \pmod{\ell}} \left(1 + \frac{\ell-1}{p^s}\right),
$$

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with
\[ L_\ell(s) = \begin{cases} 
1 + (\ell - 1)/\ell^{2s} & \text{if } \ell \nmid D , \\
1 + (\ell - 1)/\ell^s & \text{if } \ell \mid D \text{ and } \ell \geq 5 , \\
1 + (\ell - 1)/\ell^s & \text{if } \ell = 3 \text{ and } D \equiv 3 \pmod{9} , \\
1 + (\ell - 1)/\ell^s + \ell(\ell - 1)/\ell^{2s} & \text{if } \ell = 3 \text{ and } D \equiv 6 \pmod{9} ,
\end{cases} \]

and where \( \phi_{\ell,K}(s) \) can be extended to a holomorphic function in \( \Re(s) > 1/2 \).

(2) If \( D < 0 \) and \( \ell \nmid h(D) \) we have \( \phi_{\ell,K}(s) = 0 \). In particular this is the case when \( \ell \equiv 3 \pmod{4} \) and \( D = (-1)^{(\ell-1)/2}\ell = \ell^* \) (see below).

(3) Write
\[ \Phi_{\ell,K}(s) = \frac{1}{\ell - 1} + \sum_{n \geq 1} \frac{a(n)}{n^s} \]
and \( M_\ell(D;X) = \sum_{1 \leq n \leq X} a(n) \), which counts the number of \( L \) such that \( f(L) \leq X \), and set \( \ell^* = (-1)^{(\ell-1)/2}\ell \). In addition, set
\[ c_1 = 1/((\ell - 1)^{1-r_2(D)}), \]
\[ c_2 = \begin{cases} 
(\ell^2 + \ell - 1)/\ell^2 & \text{if } \ell \nmid D , \\
2 - 1/\ell & \text{if } \ell \mid D \text{ and } \ell \geq 5 , \\
5/3 & \text{if } \ell = 3 \text{ and } D \equiv 3 \pmod{9} , \\
7/5 & \text{if } \ell = 3 \text{ and } D \equiv 6 \pmod{9}.
\end{cases} \]

(a) If \( D \neq \ell^* \), there exists a constant \( C_\ell(D) > 0 \) such that \( M_\ell(D;X) = C_\ell(D)X + O(X^{1-2/(\ell+3)+\varepsilon}) \) for all \( \varepsilon > 0 \), and we have \( C_\ell(D) = c_1c_2c_3 \) with
\[ c_3 = \Res_{s=1} \prod_{e(p)=f(p)=1} (1 + (\ell - 1)/p^s), \]
where \( e(p) \) and \( f(p) \) are the ramification and residual index of a prime number \( p \) in the extension \( K'/\mathbb{Q} \), where \( K' = \mathbb{Q}(\sqrt{D}(\zeta_\ell - \zeta_\ell^{-1})) \).

(b) If \( \ell \equiv 1 \pmod{4} \) and \( D = \ell^* = \ell \), there exists a constant \( C_\ell(D) > 0 \) such that \( M_\ell(D;X) = C_\ell(D)X + O(X^{1-2/(\ell+3)+\varepsilon}) \) for all \( \varepsilon > 0 \), and we have \( C_\ell(D) = c_1c_2c_3 \) with
\[ c_3 = \Res_{s=1} \prod_{p \equiv 1 \pmod{\ell}} (1 + (\ell - 1)/p^s). \]

(c) If \( \ell \equiv 3 \pmod{4} \) and \( D = \ell^* = -\ell \), there exist constants \( C_\ell(D) > 0 \) and \( C'_\ell(D) \) such that \( M_\ell(D;X) = C_\ell(D)(X \log(X) + C'_\ell(D)X) + O(X^{1-2/(\ell+3)+\varepsilon}) \) for all \( \varepsilon > 0 \), and we have \( C_\ell(D) = c_1c_2c_3 \) with
\[ c_3 = \lim_{s \to 1} (s - 1)^2 \prod_{p \equiv 1 \pmod{\ell}} (1 + (\ell - 1)/p^s), \]
and the constant \( C'(D) \) can also be given explicitly if desired.

Note that when \( \ell = 3 \) and \( D \equiv 6 \pmod{9} \) we include \( p = \ell \) in the product for \( c_3 \), and compensate by setting \( c_2 = (7/3)/(1 + 2/3) = 7/5 \). In all other cases we have \( e(\ell) > 1 \) so \( p = \ell \) does not occur in the product for \( c_3 \).
Our goal is to compute for reasonably small $D$ (say $|D| \leq 100$) and small $\ell$ (say $3 \leq \ell \leq 11$) the constants $C_\ell(D)$ to high accuracy (say 60 decimal digits) and the quantities $M_\ell(D; X)$ for rather large $X$ (say $X \leq 10^{12}$, the upper bound here depending on $\ell$). Although there are some similarities between the two problems, we consider them independently.

2. Computation of $C_\ell(D)$ in the general case

In [10], we have (easily) expressed $c_2$ as a product of four other constants, two of them being slowly convergent Euler products. However, the formula given above is the one we will use to compute $c_3$. As can be seen, the case $D = \ell^s$ leads to slightly different formulas, so we will first assume that $D \neq \ell^s$, which we will call the “general case”.

2.1. Splitting of primes in subfields of $K'$. Recall that we have set $K' = \mathbb{Q}(\sqrt{D}(\zeta_{\ell} - \zeta_{\ell}^{-1}))$. It is immediate to see that the extension $K'/\mathbb{Q}$ is cyclic of degree $\ell - 1$, so for any $d \mid (\ell - 1)$ there exists a unique subfield $K'_d$ such that $[K': K'_d] = d$, or equivalently of degree $(\ell - 1)/d$.

For a prime $p$, denote by $p_d$ a prime ideal of $K'_d$ above $p$, by $e(p_d), f(p_d)$, and $g(p_d)$ the usual quantities linked to the splitting of primes in $K'_d/\mathbb{Q}$, abbreviate these quantities to $e(p), f(p), g(p)$ when $d = 1$ (this being compatible with our previous notation). We first note the following lemma, which is Proposition 1.3 of [6], but which we prove again here in the form that we need:

**Lemma 2.1.** Let $K'/\mathbb{Q}$ be a cyclic extension of degree $n$, and let $K'_d, p_d$, etc., be defined as above. Assume that $p$ is unramified in $K'/\mathbb{Q}$, i.e., that $e(p) = 1$. We have $f(p_d) = f(p)/(f(p), d)$, where $(u, v)$ is as usual an abbreviation for $\gcd(u, v)$.

**Proof.** Let $\rho$ be the canonical surjection from $\text{Gal}(K'/\mathbb{Q})$ to $\text{Gal}(K'_d/\mathbb{Q})$, and for simplicity write $f$ instead of $f(p)$. We know that $D(p_d/p) = \rho(D(p_1/p))$ and denote by $\sigma$ a generator of the cyclic group $\text{Gal}(K'/\mathbb{Q})$. Since $p$ is unramified in $K'/\mathbb{Q}$ we have

$$f(p_d) = |D(p_d/p)| = \frac{|D(p_1/p)|}{|\ker(\rho) \cap D(p_1/p)|} = \frac{f}{|\langle \sigma^n \rangle \cap \langle \sigma^n/f \rangle|}.$$ 

Now for $a$ and $b$ dividing $n$ we evidently have

$$|\langle \sigma^a \rangle \cap \langle \sigma^b \rangle| = |\langle \sigma^{ab/(a,b)} \rangle| = m(a, b)/ab,$$

so for $a = n/d$ and $b = n/f$, this is equal to

$$\frac{n(n/d, n/f)}{n^2/df} = (d, f),$$

proving the lemma.

2.2. Isolating the condition $e(p) = f(p) = 1$. The following is essentially Theorem 2.16 of [5], which we also prove again in our context.

**Proposition 2.2.** Keep the notation of Lemma 2.1 in particular, $K'/\mathbb{Q}$ is cyclic of degree $n$. Set

$$Z(s) = \prod_{d \mid n} \zeta_{K'_d}(ds)^{\mu(d)}.$$

Then \( Z(s) = Z_R(s)L(s) \), with

\[
Z_R(s) = \prod_{e(p)>1 \atop d | n} (1 - 1/p^{f(p_d)}d_s) - \mu(d)g(p_d),
\]

\[
L(s) = \prod_{e(p)=f(p)=1 \atop d | n} (1 - 1/p^{d_s}) - \mu(d)n/d.
\]

Note that we will apply this proposition with \( n = \ell - 1 \) in the general case and also in the special case when \( \ell \equiv 1 \pmod{4} \), and with \( n = (\ell - 1)/2 \) in the special case when \( \ell \equiv 3 \pmod{4} \).

Proof. Both sides being Euler products, we prove the result for each Euler factor at \( p \). The Euler factor at \( p \) of \( \zeta_{K^*_l}(s) \) is by definition equal to \( (1 - 1/p^{f(p)d_s}) - g(p_d) \). It follows that the result is immediate when \( e(p) > 1 \), and also when \( e(p) = f(p) = 1 \), since for \( d \mid n \) we have \( e(p_d) = f(p_d) = 1 \), hence \( g(p_d) = n/d \).

Thus assume that \( e(p) = 1 \) and \( f(p) > 1 \). If we denote by \( Z_p(s) \) the Euler factor at \( p \) of \( Z(s) \) we must show that \( Z_p(s) = 1 \). Since \( e(p) = 1 \) we have \( g(p_d) = ((\ell - 1)/d)/f(p_d) \), so

\[
Z_p(s) = \prod_{d | n} (1 - q^{df(p_d)}) - \mu(d)(n/d)/f(p_d),
\]

where for simplicity we set \( q = 1/p^s \). Using the above lemma and setting for simplicity \( f = f(p) \), we thus have

\[
Z_p(s) = \prod_{d | n} (1 - q^{df/(d,f)}) - \mu(d)n(d,f)/(df).
\]

By definition of the cyclotomic polynomials \( \Phi_m \) we have

\[
q^{df/(d,f)} - 1 = \prod_{m | df/(d,f)} \Phi_m(q),
\]

so that

\[
Z_p(s) = \pm \prod_{d | n} \prod_{m | df/(d,f)} \Phi_m(q)^{-\mu(d)n(d,f)/(df)} = \pm \prod_{m | n} \Phi_m(q)^{h(m)},
\]

with

\[
h(m) = -n \sum_{d | n, \ m | df/(d,f)} \mu(d)(d,f)/(df).
\]

Setting \( d_1 = df/(d,f) \), which is equal to the LCM of \( d \) and \( f \), hence divides \( n \), we have

\[
h(m) = -n \sum_{f | [d_1] \atop m | d_1} (1/d_1) \sum_{d | d_1 \atop \text{lcm}(d,f) = d_1} \mu(d) = -n \sum_{m_1 | d_1 \atop \text{lcm}(d,f) = d_1} (1/d_1)I(d_1, f),
\]

where we set \( m_1 = \text{lcm}(f, m) \) and

\[
I(d_1, f) = \sum_{d | d_1 \atop \text{lcm}(d,f) = d_1} \mu(d).
\]

Set \( d' = d_1/d \) and \( f' = d_1/f \). We have

\[
(d', f') = (d_1f, d_1d)/(df) = d_1(d, f)/(df),
\]
so the condition \( \text{lcm}(d, f) = d_1 \) is equivalent to \( d' | d_1 \) and \( (d', f') = 1 \). Thus,

\[
I(d_1, f) = \sum_{d'|d_1} \mu(d_1/d') = \sum_{d'|d_1} \mu(d_1/d') \sum_{e|d'(d',f')} \mu(e)
\]

\[
= \sum_{e|f'} \mu(e) \sum_{d'|d_1} \mu(d_1/d') = \sum_{e|f'} \mu(e) \sum_{d'|d_1} \mu((d_1/e)/d'')
\]

since \( f' | d_1 \). The inner sum vanishes unless \( d_1/e = 1 \), so that \( I(d_1, f) = \mu(d_1) \) if \( d_1 | f' \) and \( I(d_1, f) = 0 \) otherwise. However, since \( f' = d_1/f \), we have \( d_1 | f' \) if and only if \( f = 1 \). We deduce that \( I(d_1, f) = 0 \) for all \( d_1 \) when \( f > 1 \), and otherwise \( I(d_1, f) = \mu(d_1) \). Thus \( h(m) = 0 \) if \( f > 1 \), so that \( Z_p(s) = \pm 1 \), and of course since the constant term in \( q = p^{-s} \) of \( Z_p(s) \) is equal to 1 we have \( Z_p(s) = 1 \), proving the result.

\[\square\]

2.3. The folklore trick. See for instance Section 10.3.6 of [4] or [3]. This “trick” is completely general, but we apply it to our case:

**Proposition 2.3.** Set

\[
a(k) = \frac{1}{k} \sum_{n|k} \frac{(-1)^{n-1} \mu(k/n)(\ell - 1)^{n-1}}{(k/n, \ell-1)=1}.
\]

With a slight abuse of notation, we have

\[
c_3(s) := \prod_{e(p)=f(p)=1} (1 + (\ell - 1)/p^s) = \prod_{k \geq 1} L(ks)^{a(k)}.
\]

In particular

\[
c_3 = \text{Res}_{s=1} L(s) \prod_{k \geq 2} L(k)^{a(k)}.
\]

**Proof.** I claim that there exist unique exponents \( a(k) \) such that

\[
c_3(s) = \prod_{k \geq 1} L(ks)^{a(k)},
\]

and that they are given by the formula of the proposition. Indeed, setting \( q = p^{-s} \) and taking logarithms of the Euler factors at \( p \), this is equivalent to the equality

\[
\sum_{n \geq 1} (-1)^{n-1} \frac{1}{n} q^n = \sum_{k \geq 1} a(k) \sum_{d|k} \frac{1}{d} \mu(d) \sum_{m \geq 1} \frac{1}{m} q^{km},
\]

hence setting \( n = kdm \) this gives the identities

\[
(-1)^{n-1} (\ell - 1)^{n-1} = \sum_{kdm=n, \ d|k(\ell - 1)} ka(k) \mu(d) = \sum_{k|n, \ (n/k, \ell-1)=1} ka(k) \sum_{d|n/k, \delta(n/k)} \mu(d)
\]

\[
= \sum_{k|n, \ (n/k, \ell-1)=1} ka(k) = \sum_{k|n} ka(k) \delta(n/k),
\]
where $\delta(n) = 1$ if $(n, \ell - 1) = 1$ and $\delta(n) = 0$ otherwise. Now $\sum_{n \geq 1} \delta(n)/n^s = \sum_{(n, \ell - 1) = 1} 1/n^s$, which implies that $\left( \sum_{n \geq 1} \delta(n)/n^s \right)^{-1} = \sum_{(n, \ell - 1) = 1} \mu(n)/n^s$, so

$$ka(k) = \sum_{n \mid k} (-1)^{n-1} \mu(k/n)(\ell - 1)^{n-1},$$

proving the proposition.

For any Euler product $P(s)$, write $P_{> B}(s)$ (resp., $P_{\leq B}(s)$) for the Euler product limited to primes $p > B$ (resp., $p \leq B$). In practice, we will use the proposition through the following corollary, whose immediate proof is left to the reader:

**Corollary 2.4.** For all $B \geq 1$ we have

$$c_3 = \prod_{e(p) = f(p) = 1, p \leq B} \prod_{k \geq 2} \left( 1 + (\ell - 1)/p \right) \res_{s = 1} L_{> B}(s) \prod_{k \geq 2} L_{> B}(k)^{a(k)} ,$$

$$L_{> B}(k) = \prod_{e(p) = f(p) = 1, p \leq B} \prod_{d \mid (\ell - 1)} (1 - 1/p^d)^{\mu(d)/d} \zeta(k/Z_R(k) ,$$

$$\res_{s = 1} L_{> B}(s) = \prod_{e(p) = f(p) = 1, p \leq B} \prod_{d \mid (\ell - 1)} (1 - 1/p^d)^{\mu(d)/d} \res_{s = 1} \zeta(s)/Z_R(1) ,$$

$$\res_{s = 1} \zeta(s) = \res_{s = 1} \zeta_{K'_d}(s) \prod_{d \mid (\ell - 1)} \zeta_{K'_d}(d)^{\mu(d)} .$$

### 2.4. Implementation

We will see below how to compute $\zeta_{K'_d}(k)$. The main problem in the use of the above corollary is the computation of $L_{> B}(k)^{a(k)}$: if $k$ is not small, say $k = 100$, then $L_{> B}(k)$ is extremely close to 1, and $a(k)$ is large, so we will have quite a loss of accuracy. Let us quantify this: since primes such that $e(p) = f(p) = 1$ are common, we will have very roughly $L_{> B}(k) \approx (1 - 1/B^k)^{-(\ell - 1)}$. On the other hand, $a(k)$ will be of the order of $(\ell - 1)^{k-1}/k \approx (\ell - 1)^{(k-1)}$. Thus, $|L_{> B}(k)^{a(k)} - 1| \leq e^{-D}$ is essentially equivalent to $a(k) \log(L_{> B}(k)) < e^{-D}$, hence to $(\ell - 1)^{k}/B^k < e^{-D}$; in other words $k > D/\log(B/(\ell - 1))$ (which of course implies that we must take $B$ considerably larger than $\ell - 1$). This gives roughly the value of $k$ that we will need, but is not the whole story: given this value of $k$, we must compute $L_{> B}(k)^{a(k)}$ to accuracy (absolute or relative, here it is the same since this quantity is close to 1) less than $e^{-D}$. Thus, it is necessary to perform the computation of $L_{> B}(k)$ to precision less than $e^{-D}/a(k) = e^{-D'}$, with

$$D' \approx D + k \log(\ell - 1) = D + D \log(\ell - 1)/\log(B/(\ell - 1)) = D \log(B)/\log(B/(\ell - 1)) .$$

For instance, if we choose $B = (\ell - 1)^2$, we simply have $D' = 2D$.

### 2.5. Computation of $\zeta_{K'_d}(k)$

To compute $c_3$, it remains to compute $\res_{s = 1} \zeta_{K'_d}(s)$ and $\zeta_{K'_d}(k)$ for $k \geq 2$. For the first quantity, we can use Dirichlet’s class number formula, and for the others the preprogrammed functions zetakinit and zetak of Pari/GP. However, as mentioned in the manual, these functions are very inefficient especially when the degree $\ell - 1$ is not tiny, so it is highly preferable instead to use Dirichlet characters.
Indeed, if we denote by $\omega$ a generator of the group of characters modulo $\ell$ and by $\chi_D$ the quadratic character $\left(\frac{D}{\cdot}\right)$, we have

$$\zeta_{K'_d}(s) = \prod_{0 \leq j < (\ell-1)/d} L((\omega\chi_D)^{dj}, s) ,$$

where we recall that the product of characters is taken in the group of characters, meaning, in particular, that when $dj$ is even we have $(\omega\chi_D)^{dj} = \omega^{dj}$. In particular, we have

$$\text{Res}_{s=1} \zeta_{K'}(s) = \prod_{1 \leq j < \ell-1} L((\omega\chi_D)^j, 1) .$$

The quantities $L((\omega\chi_D)^{dj}, s)$ can in turn be computed using the approximate functional equation, which here involves the incomplete gamma function. We omit the details, which are classical, but simply note that for a given value of $s$ the quantities $L((\omega\chi_D)^{dj}, s)$ for $1 \leq j < (\ell-1)/d$ can be computed simultaneously for essentially the same cost as a single value, since the incomplete gamma function arguments will be the same.

3. Computation of $C_\ell(D)$ in the special case

We now assume that we are in the “special case”, i.e., that $D = \ell^* = (-1)^{(\ell-1)/2}\ell$. In view of Theorem 1.1 we will need to distinguish $\ell \equiv 1 \pmod{4}$ and $\ell \equiv 3 \pmod{4}$. In fact, nothing much needs to be changed compared to the general case.

Assume first that $\ell \equiv 1 \pmod{4}$. The condition $p \equiv 1 \pmod{\ell}$ is equivalent to $p$ splitting completely in the cyclotomic field $Q_z = \mathbb{Q}(\zeta_\ell)$, which is again cyclic of order $\ell - 1$. If in this case we set $K' = \mathbb{Q}(\zeta_\ell)$, then Lemma 2.1, Proposition 2.2, Proposition 2.3, and Corollary 2.4 are valid as such (with again $n = [K' : \mathbb{Q}] = \ell-1$), and the only change is that the formula for $\zeta_{K'_d}(s)$ is replaced by

$$\zeta_{K'_d}(s) = \prod_{0 \leq j < (\ell-1)/d} L(\omega^{dj}, s) .$$

Assume now that $\ell \equiv 3 \pmod{4}$. The condition $p \equiv \pm 1 \pmod{\ell}$ is equivalent to $p$ splitting completely in the totally real subfield $Q^+_z = \mathbb{Q}(\zeta_\ell + \zeta_\ell^{-1})$ of $Q_z = \mathbb{Q}(\zeta_\ell)$ which is now cyclic of order $(\ell-1)/2$. Thus we set $K' = Q^+_z$. Here the results change slightly. Lemma 2.1 and Proposition 2.2 are now used with $n = (\ell-1)/2$, and the formula for $\zeta_{K'_d}(s)$ is now

$$\zeta_{K'_d}(s) = \prod_{0 \leq j < (\ell-1)/(2d)} L(\omega^{2dj}, s) .$$

However, we must now change Proposition 2.3 (hence also Corollary 2.4). The new formulas are as follows:

Proposition 3.1. Keep the above notation, in particular $K' = Q^+_z$, and set

$$a(k) = \frac{2}{k} \sum_{\substack{n \mid k \\ (n,k/((\ell-1)/2)) = 1}} (-1)^{n-1} \mu(k/n)(\ell-1)^{n-1} .$$
We have
\[ c_3(s) := \prod_{e(p) = f(p) = 1} (1 + (\ell - 1)/p^s) = \prod_{k \geq 1} L(ks)^{a(k)} . \]

In particular,
\[ c_3 = (\text{Res}_{s=1} L(s))^2 \prod_{k \geq 2} L(k)^{a(k)} . \]

**Proof.** The proof is identical to the one in the general case, except that we must now use the formula of Proposition \[2.2\] with \( n = (\ell - 1)/2 \), which implies that the GCD of \( k/n \) is now taken with respect to \( (\ell - 1)/2 \), and we must replace the factor \( 1/k \) by \( 2/k \) in the formula for \( a(k) \). In particular note that \( a(1) = 2 \). \( \square \)

4. Tables of \( C_\ell(D) \)

Using the methods explained above, we have computed to 80 decimal digits a number of values of \( C_\ell(D) \) for \( \ell = 3, 5, 7, \) and 11. The \textsc{Pari/GP} programs used to compute them as well as extensive tables are available from the author. Here is a short sample:

\[
\begin{align*}
C_3(-4) & = 0.1362190676241212841449867354342013681519368439930712 \\
C_3(-15) & = 0.17637191872547206599912366625284592827312349690969698 \\
C_3(5) & = 0.0818840074459636358232037502298557955922438940548390 \\
C_3(12) & = 0.0803828977056554045622405320212726495945956846819225 \\
C_5(-3) & = 0.050785324449780099378201676018829363084151272832622 \\
C_5(-15) & = 0.07880481380138291282282439800292179771995450785328207 \\
C_5(8) & = 0.0134747747475919140437863334316374195138193286084845 \\
C_5(40) & = 0.0209091332290219355851856898077132371766161995648898 \\
C_7(-3) & = 0.0296332163247300745247219282260715878558066725943508 \\
C_7(-56) & = 0.0476482546822232432485663400482260513425675409153078 \\
C_7(5) & = 0.0064676733264714100259068107168272139045182602157945 \\
C_7(21) & = 0.009256138047982484596549623915860255741312857684523 \\
C_{11}(-3) & = 0.0147492212096080611979142320145680269303157592953914 \\
C_{11}(-55) & = 0.0262064374769044759545849551388248215923277698248101 \\
C_{11}(5) & = 0.001506218102961248773863338522967418184629721573614 \\
C_{11}(33) & = 0.0030220320861762508020132258509009135197011333981156 \\
\end{align*}
\]
We also give the constants $C_\ell(D) = C_\ell(\ell^*)$ in the special case, as well as the constants $C_\ell(-\ell)$ when $\ell \equiv 3 \pmod{4}$:

\[
C_3(-3) = 0.066907733013783712918416329842956375013440969559 \cdots \\
C_3'(3) = 2.4502227978305919627907119196711104182688504252980 \cdots \\
C_5(5) = 0.0203781870559037146558936043338516820583779270481 \cdots \\
C_7(-7) = 0.012105263421451229801857880331290147832963231428 \cdots \\
C_7'(7) = 4.5891109397329766650879126190307954015065605836198 \cdots \\
C_{11}(-11) = 0.005935461887438645278306590185602843604168690292 \cdots \\
C_{11}'(-11) = 4.9407586645730224262224577573236175226465489202623178 \cdots 
\]

5. Computation of $M_\ell(D; X)$

5.1. Reduction to $\zeta_{K'}(s)$. Recall from Theorem 1.1 that if $\phi_{\ell,K}(s)$ is identically zero, we have

\[
\Phi_{\ell,K}(s) = \frac{1}{\ell - 1} L_\ell(s) \prod_{\ell(p) = f(p) = 1, \; p \neq \ell} (1 + (\ell - 1)/p^s),
\]

where $L_\ell(s)$ is a polynomial of degree less than or equal to 2 in $1/\ell^s$ given in Theorem 1.1 so that if we write $\Phi_{\ell,K}(s) = 1/(\ell - 1) + \sum_{n \geq 1} a(n)/n^s$, we have $M_\ell(D; X) = \sum_{1 \leq n \leq X} a(n)$. Note that the condition $e(p) = f(p) = 1$, $p \neq \ell$, is equivalent to $p \equiv (D/p) \equiv \pm 1 \pmod{\ell}$ so it is immediate to test without doing any work in the field $K'$.

Now we write

\[
A_1(s) := \prod_{\ell(p) = f(p) = 1, \; p \neq \ell} (1 + (\ell - 1)/p^s) = \sum_{n \geq 1} a_1(n)/n^s,
\]

with $a_1(n) = (\ell - 1)^{\omega_1(n)}$, where $\omega_1(n)$ is the number of distinct prime divisors $p$ of $n$ such that $e(p) = f(p) = 1$ and $p \neq \ell$, and set $A_1(X) = \sum_{1 \leq n \leq X} a_1(n)$; we have

\[
(\ell - 1)M_\ell(D; X) + 1
= \begin{cases}
A_1(X) + (\ell - 1)A_1(X/\ell^2) & \text{if } \ell \nmid D, \\
A_1(X) + (\ell - 1)A_1(X/\ell) & \text{if } \ell \mid D \text{ and } \ell \geq 5, \\
A_1(X) + (\ell - 1)A_1(X/\ell) + 2\ell A_1(X/\ell^2) & \text{if } \ell = 3 \text{ and } D \equiv 3 \pmod{9}, \\
A_1(X) + (\ell - 1)A_1(X/\ell) + 2\ell A_1(X/\ell^2) & \text{if } \ell = 3 \text{ and } D \equiv 6 \pmod{9}.
\end{cases}
\]

We are thus evidently reduced to the computation of $A_1(X)$. A straightforward method to do this is to use the explicit formula for $a_1(n)$: since $\omega_1(n)$ can be computed in $O(n^\varepsilon)$ time for any $\varepsilon > 0$, this gives a $O(X^{1+\varepsilon})$ method for computing $A_1(X)$, hence $M_\ell(D; X)$.

By using once again the Dedekind zeta function of the field $K'$ it is, however, possible to improve this to a $O(X^{1-1/(\ell-1)+\varepsilon})$ method. In particular, this is $O(X^{1/2+\varepsilon})$ for $\ell = 3$ and $O(X^{3/4+\varepsilon})$ for $\ell = 5$.

We will assume from now on that we are in the general case, the modifications that are needed for the special case being immediate and mentioned at the end.
Write $\zeta_{K'}(s) = P_R P_F P_1$, where
\[
P_R(s) = \prod_{e(p) > 1 \text{ or } p = \ell} (1 - 1/p^{\ell+1})^{(\ell-1)/(e(p) - \ell)}.
\]
\[
P_F(s) = \prod_{e(p) = 1, f(p) > 1, p \neq \ell} (1 - 1/p^{\ell+1})^{(\ell-1)/f(p)}.
\]
\[
P_1(s) = \prod_{e(p) = f(p) = 1, p \neq \ell} (1 - 1/p^s)^{-(\ell-1)}.
\]
Thus $A_1(s) = Q_1(s)P_1(s)$ with
\[
Q_1(s) = \prod_{e(p) = f(p) = 1, p \neq \ell} ((1 + (\ell-1)/p^s)(1 - 1/p^s)^{\ell-1}),
\]
hence $A_1(s) = P_R(s)^{-1}A_2(s)$, with
\[
A_2(s) = Q_1(s)P_F(s)^{-1}\zeta_{K'}(s).
\]
Set
\[
A_2(s) = \sum_{n \geq 1} a_2(n)/n^s \quad \text{and} \quad A_2(X) = \sum_{1 \leq n \leq X} a_2(n).
\]
Since $P_R(s)^{-1}$ is a Dirichlet polynomial of reasonably small degree $d$, if we write $P_R(s)^{-1} = \sum_{1 \leq n \leq q} r(n)/n^s$ we have $A_1(X) = \sum_{1 \leq n \leq q} r(n)A_2(X/n)$.

Now note that $Q_1(s)$ and $P_F(s)^{-1}$ are Euler products of power series in $1/p^s$ of the form $1 + O(1/p^2s)$. Thus, we can set
\[
Q_1(s)P_F(s)^{-1} = \sum_{n \geq 1} c(n)/n^s,
\]
and $n$ such that $c(n) \neq 0$ are powerful numbers, i.e., such that if a prime $p$ divides $n$, then $p^2 \mid n$ also. Hence if we set $\zeta_{K'}(s) = \sum_{n \geq 1} a_3(n)/n^s$ and $A_3(X) = \sum_{1 \leq n \leq X} a_3(n)$, we have
\[
A_2(X) = \sum_{1 \leq n \leq X, c(n) \neq 0} c(n)A_3(X/n).
\]
The point of this formula is that the number of powerful $n \leq X$ is asymptotic to $c_pX^{1/2}$ (with $c_p = \zeta(2)\zeta(3)/\zeta(6)$), so that the above sum involves only $O(X^{1/2})$ terms. More precisely, if $A_3(X)$ can be computed in time $O(X^{\alpha + \varepsilon})$ for all $\varepsilon > 0$ with $\alpha \geq 1/2$, then $A_2(X)$ can be computed in time $O(X^{\alpha + \varepsilon}S)$, with $S = \sum_{n \text{ powerful}} X^{-\alpha + \varepsilon}$, which is a convergent series for $\alpha \geq 1/2$.

Remarks 5.1.  
(1) The same reasoning shows that if $\alpha < 1/2$ the computation time is $O(X^{1/2+\varepsilon})$, so there is no real improvement compared to the case $\alpha = 1/2$. This is because we use Euler products in $1 + O(1/p^{2s})$, and not $1 + O(1/p^{\alpha s})$ for some $m \geq 3$. Using these more rapidly convergent Euler products would considerably complicate the combinatorics and not gain any time in practice.

(2) The formula given above for $A_2(X)$ must of course not be used as written, since it would involve factoring all $n \leq X$, and even if factoring could be done in time $O(1)$, this would take time $O(X)$. Instead, note that the Euler factors of $Q_1(s)$ and of $P_F^{-1}$ have degree less than or equal to $\ell (\ell - 1$ for
$P_{\ell}^{-1}$, so any $n$ with $c(n) \neq 0$ can be written $n = \prod_{2 \leq k \leq \ell} x_k^k$ for some integers $x_k$, and the summation over $n \leq X$ is written as

$$\sum_{x_\ell \leq X^{1/\ell}} \ldots \sum_{x_{\ell-1} \leq (X/x_\ell^{1/(\ell-1)})} \ldots$$

The number of terms in this multiple sum is less than or equal to the number of powerfull numbers up to $X$, hence $O(X^{1/2})$.

Thus the computation time of $A_2(X)$ is of the same order of magnitude as that of $A_3(X)$ (for $\alpha \geq 1/2$, which is what we will have in practice), so we are reduced to studying the summatory function of the Dirichlet series coefficients of $\zeta_{K'}(s)$. In actual practice, the computation of this latter function takes more than 99% of the time.

### 5.2. Summatory function of $\zeta_{K'}(s)$

Recall from Section 2.5 that in the general case we have

$$\zeta_{K'}(s) = \prod_{0 \leq j < \ell-1} L((\omega\chi_D)^j, s) = \sum_{n \geq 1} a_3(n)/n^s,$$

and we want to compute $A_3(X) = \sum_{1 \leq n \leq X} a_3(n)$ (in the special case the formulas are slightly different (see above), but the method will be identical).

For this, we will make use of the method of the hyperbola in the following form:

#### Proposition 5.2

Assume that $c$ is the arithmetic convolution of $a$ and $b$, and denote by $A(X), B(X),$ and $C(X)$ the summatory functions of $a, b,$ and $c$, respectively. For any $E \in \mathbb{R}$ such that $1 \leq E \leq X$ we have

$$C(X) = \sum_{1 \leq n \leq E} a(n)B(X/n) + \sum_{1 \leq n \leq X/E} b(n)A(X/n) - A(E)B(X/E).$$

#### Corollary 5.3

Assume that the functions $a(n)$ and $b(n)$ can be computed in time $O(n^\varepsilon)$, and that $A(X)$ and $B(X)$ can be computed in time $O(X^{\alpha+\varepsilon})$ and $O(X^{\beta+\varepsilon})$ respectively, with $\alpha < 1$ and $\beta < 1$. Then, using the proposition, $C(X)$ can be computed in time $O(X^{\gamma+\varepsilon})$ with $\gamma = (1 - \alpha\beta)/(2 - \alpha - \beta)$ by choosing $E = X^{(1-\beta)/(2-\alpha-\beta)}$.

#### Proof

Immediate and left to the reader. \hfill $\square$

Now note that for $j \not\equiv 0 \pmod{\ell - 1}$ the functions $(\omega\chi_D)^j$ as well as their summatory functions are periodic of period dividing $(\ell - 1)|D|$, and since we assume $\ell$ and $D$ small they can be tabulated once and for all. Although not periodic, for $j = 0$ the function 1 and its summatory function $[X]$ also take negligible time to compute. Thus, splitting the product of $L$-functions by pairs, we have to compute the product of $(\ell - 1)/2$ Dirichlet series, whose coefficients take negligible time to compute, and summatory functions taking time $O(X^{1/2+\varepsilon})$. Continuing by taking products by pairs, it is easy to see that the global time will be $O(X^{1-1/(\ell - 1)+\varepsilon})$.

### 5.3. The Case $\ell = 3$

We summarize the reductions made above in this case:

$$2M_3(D; X) + 1 = \begin{cases} A_1(X) + 2A_1(X/9) & \text{if } 3 \nmid D, \\ A_1(X) + 2A_1(X/3) & \text{if } D \equiv 3 \pmod{9}, \\ A_1(X) + 2A_1(X/3) + 6A_1(X/9) & \text{if } D \equiv 6 \pmod{9}. \end{cases}$$
Since
\[ P_R(s)^{-1} = L_3(s) \prod_{p \mid D, \ p \neq 3} (1 - 1/p^s) = L_3(s) \sum_{n \mid D, \ 3 \nmid n} \mu(n)/n^s \]
with \( L_3(s) = 1 - 1/3^s, 1 - 1/3^{2s}, \) or \( 1 - 2/3^s + 1/3^{2s} \) according to the three cases, we have
\[
A_1(X) = \begin{cases} 
A'_2(X) - A'_2(X/3) & \text{if } 3 \nmid D, \\
A'_2(X) - A'_2(X/9) & \text{if } D \equiv 3 \pmod{9}, \\
A'_2(X) - 2A'_2(X/3) + A'_2(X/9) & \text{if } D \equiv 6 \pmod{9},
\end{cases}
\]
and
\[ A'_2(X) = \sum_{n \mid D, \ 3 \nmid n} \mu(n)A_2(X/n). \]

We have
\[
Q_1(s) = \prod_{\frac{-3D}{p} = 1} (1 - 3/p^{2s} + 2/p^{3s}) \quad \text{and} \quad P_F(s)^{-1} = \prod_{\frac{-3D}{p} = -1} (1 - 1/p^{2s}).
\]
Thus \( Q_1(s)P_F(s)^{-1} = \sum_{n \geq 1} c(n)/n^s, \) and we have \( c(n) \neq 0 \) if and only if \( 3 \nmid n \) and \( n = x^2y^3, \) where \( x \) and \( y \) are coprime and squarefree and \( p \mid y \) implies \( \left( \frac{-3D}{p} \right) = 1, \)
in which case
\[
c(n) = 2^{\omega(y)}(-3)^{\omega^+(x)}(-1)^{\omega^-(x)},
\]
where \( \omega^\pm(m) \) is the number of \( p \mid m \) with \( \left( \frac{-3D}{p} \right) = \pm 1. \) It follows that
\[
A_2(X) = \sum_{y \leq X^{1/3}, \ y \ n \mid y \Rightarrow \left( \frac{-3D}{p} \right) = 1} 2^{\omega(y)} \sum_{x \leq (X/y^3)^{1/2}, \ 3 \mid x, \ x \ n \mid x \ n \mid x, \ x \ \text{squarefree}} (-3)^{\omega^+(x)}(-1)^{\omega^-(x)}A_3(X/(x^2y^3)).
\]

Finally, if we set \( D^* = -3D \) if \( 3 \nmid D \) and \( D^* = -D/3 \) if \( 3 \mid D, \) we have
\[
\zeta_K^*(s) = \zeta_{\mathbb{Q}(\sqrt{D^*})}(s) = \zeta(s)L\left(\frac{D^*}{n}\right), s = \sum_{n \geq 1} a_3(n)/n^s.
\]

To compute \( A_3(X), \) we use the method of the hyperbola (Proposition 5.2), where, as mentioned above, we set \( a(n) = 1 \) and \( b(n) = \left( \frac{D^*}{n} \right). \) Thus \( A(X) = |X|, \) and \( B(X) = B(X - |D^*||\lfloor X/|D^*|\rfloor|) \) is a periodic function of period \( |D^*| \) which we tabulate, since \( D \) is quite small. Thus
\[
A_3(X) = \sum_{1 \leq n \leq E} B(X/n) + \sum_{1 \leq n \leq X/E} \left( \frac{D^*}{n} \right) \lfloor X/n \rfloor - [E]B(X/E).
\]
The optimal value of \( E \) in this formula is close to \( E = X^{1/2}, \) so as claimed above \( A_3(X) \) is computed in \( O(X^{1/2+\varepsilon}) \) time, hence \( M_3(D; X) \) also.

Thanks to these formulas, we can compute the following tables, where the given quantities are as follows: \( M_3(D; X) \) is as above, \( P_2(D; X) \) is the nearest integer to \( C_3(D) \cdot X, \) where \( C_3(D) \) is given above, and \( E_3(D; X) \) is an approximation to \( (M_3(D; X) - P_3(D; X))/X^{1/4}; \) indeed, even though we have only proved that the error is at most \( O(X^{2/3}) \), it seems to be much closer to \( O(X^{1/4+\varepsilon}) \). In this table, as in all that follow, the values for \( X = 10^n \) for small \( n \) are much easier to compute and are available on the author’s website.
Table 1. Number of cubic fields $L$ with $K = \mathbb{Q}(\sqrt{-4})$ and $f(L) \leq X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_3(-4; X)$</th>
<th>$P_3(-4; X)$</th>
<th>$E_3(-4; X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{10}$</td>
<td>1362190594</td>
<td>1362190676</td>
<td>-0.26</td>
</tr>
<tr>
<td>$10^{11}$</td>
<td>13621906974</td>
<td>13621906762</td>
<td>0.38</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>136219069065</td>
<td>136219067624</td>
<td>1.44</td>
</tr>
<tr>
<td>$10^{13}$</td>
<td>1362190679530</td>
<td>1362190676241</td>
<td>1.85</td>
</tr>
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<td>13621906762412</td>
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</tr>
<tr>
<td>$10^{15}$</td>
<td>136219067626288</td>
<td>136219067624121</td>
<td>0.39</td>
</tr>
<tr>
<td>$10^{16}$</td>
<td>1362190676222935</td>
<td>1362190676241213</td>
<td>-1.83</td>
</tr>
<tr>
<td>$10^{17}$</td>
<td>13621906762416611</td>
<td>13621906762412128</td>
<td>0.25</td>
</tr>
<tr>
<td>$10^{18}$</td>
<td>136219067623987308</td>
<td>136219067624121284</td>
<td>-4.23</td>
</tr>
<tr>
<td>$10^{19}$</td>
<td>1362190676241140759</td>
<td>1362190676241212841</td>
<td>-1.28</td>
</tr>
</tbody>
</table>

Table 2. Number of cubic fields $L$ with $K = \mathbb{Q}(\sqrt{-15})$ and $f(L) \leq X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_3(-15; X)$</th>
<th>$P_3(-15; X)$</th>
<th>$E_3(-15; X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{10}$</td>
<td>1763718442</td>
<td>1763719187</td>
<td>-2.36</td>
</tr>
<tr>
<td>$10^{11}$</td>
<td>17637191093</td>
<td>17637191873</td>
<td>-1.39</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>176371916883</td>
<td>176371918725</td>
<td>-1.84</td>
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<td>$10^{13}$</td>
<td>1763719181402</td>
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</tr>
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<td>$10^{14}$</td>
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</tr>
</tbody>
</table>

Table 3. Number of cubic fields $L$ with $K = \mathbb{Q}(\sqrt{-39})$ and $f(L) \leq X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_3(-39; X)$</th>
<th>$P_3(-39; X)$</th>
<th>$E_3(-39; X)$</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>$10^{11}$</td>
<td>21450797505</td>
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<td>-1.85</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>214507985582</td>
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<td>0.13</td>
</tr>
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<td>$10^{13}$</td>
<td>2145079851212</td>
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<td>$10^{14}$</td>
<td>21450798558854</td>
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<td>4.43</td>
</tr>
<tr>
<td>$10^{15}$</td>
<td>214507985447211</td>
<td>214507985448322</td>
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</tr>
<tr>
<td>$10^{16}$</td>
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</tr>
<tr>
<td>$10^{17}$</td>
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<td>-12.30</td>
</tr>
</tbody>
</table>

5.4. The special case for $\ell = 3$. The special case $D = -3$ for $\ell = 3$ corresponds to the enumeration of pure cubic fields $\mathbb{Q}(\sqrt{m})$. There are a few additional difficulties
compared to the general case which are easily taken care of, so we will simply give
the algorithm and the corresponding table, leaving the details of the proofs to the
reader. The formula for the Dirichlet series, proved in [8], is as follows:

**Proposition 5.4.** We have

\[
\Phi_{5,\mathbb{Q}(\sqrt{-3})}(s) = \frac{1}{6} \left( 1 + \frac{2}{3^s} + \frac{6}{3^{2s}} \right) \prod_{p \neq 3} \left( 1 + \frac{2}{p^s} \right) + \frac{1}{3} \left( 1 - \frac{1}{3^s} \right) \prod_{p \equiv \pm 1 \pmod{9}} \left( 1 + \frac{2}{p^s} \right) \prod_{p \equiv \pm 2, \pm 4 \pmod{9}} \left( 1 - \frac{1}{p^s} \right).
\]

From this we deduce the following:

\[6M_3(-3; X) = A_1(X) + 2A_1(X/3) + 6A_1(X/9) + 2U_1(X)/3 - 3,\]

\[A_1(X) = \sum_{y \leq X^{1/3}} 2^{\omega(y)} \sum_{p \mid y \Rightarrow \left( \frac{D}{p} \right) = 1} \mu(x)3^{\omega(x)} A_2(X/(x^2y^3)),\]

\[A_2(X) = A_3(X) - 2A_3(X/3) + A_3(X/9),\]

\[A_3(X) = 2 \sum_{1 \leq n \leq X^{1/2}} \left[ X/n \right] - \left[ X^{1/2} \right]^2,\]

\[U_1(X) = U_2(X) - U_2(X/3),\]

\[U_2(X) = \sum_{y \leq X^{1/3}} \frac{\mu(y)(-2)^{\omega(y)}}{|\omega(y)| = 1} \sum_{x \leq (X/y)^{1/2}} \left( -3 \right)^{\omega(x)} U_3(X/(x^2y^3)).\]

Finally, let \( \rho = (-1 + \sqrt{-3})/2 \) be a primitive cube root of unity, set \( \chi_9(n) = 0, 1, \rho, 0, \rho^2 \) for \( n \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4 \pmod{9} \) respectively, and let \( \psi_9(n) \) be its
summatory function, equal to \( 0, 1, -\rho^2, -\rho^2, 0, \rho^2, \rho^2, -1, 0 \) for \( n \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8 \pmod{9} \). We have

\[U_3(X) = 2 \Re \left( \sum_{1 \leq n \leq X^{1/2}} \chi_9(n)\overline{\psi_9(X/n)} \right) - |\psi_9(X^{1/2})|^2,\]

from which we deduce Table 4.

5.5. **The case** \( \ell = 5 \). Again we summarize the reductions made above in this case:

\[4M_5(D; X) + 1 = \begin{cases} A_1(X) + 4A_1(X/25) & \text{if } 5 \nmid D, \\ A_1(X) + 4A_1(X/5) & \text{if } 5 \mid D. \end{cases}\]

Here

\[P_{R}(s)^{-1} = \prod_{p \mid D\ell} \left( 1 - 1/p^{f(p)s} \right)^{(\ell-1)/(e(p)f(p))}.\]

It is immediate to check that \( p = \ell = 5 \) is always totally ramified in \( K' \), while if
\( p \mid D \) but \( p \neq 5 \), we always have \( e(p) = 2 \), and \( f(p) = 1 \) if \( p \equiv \pm 1 \pmod{5} \), \( f(p) = 2 \)
Table 4. Number of pure cubic fields $L$ with $f(L) \leq X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_3(-3; X)$</th>
<th>$P_3(-3; X)$</th>
<th>$E_3(-3; X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{10}$</td>
<td>17045461815</td>
<td>17045463465</td>
<td>-5.22</td>
</tr>
<tr>
<td>$10^{11}$</td>
<td>185860709664</td>
<td>185860709585</td>
<td>0.14</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>2012667847513</td>
<td>2012667845156</td>
<td>2.36</td>
</tr>
<tr>
<td>$10^{13}$</td>
<td>21667285929697</td>
<td>21667285944613</td>
<td>-8.39</td>
</tr>
<tr>
<td>$10^{14}$</td>
<td>23207893410572</td>
<td>232078934376706</td>
<td>10.71</td>
</tr>
<tr>
<td>$10^{15}$</td>
<td>2474850093045781</td>
<td>2474850093072833</td>
<td>-4.81</td>
</tr>
<tr>
<td>$10^{16}$</td>
<td>26289108423790515</td>
<td>26289108423786084</td>
<td>0.44</td>
</tr>
<tr>
<td>$10^{17}$</td>
<td>278297159168325245</td>
<td>278297159168438350</td>
<td>-6.36</td>
</tr>
<tr>
<td>$10^{18}$</td>
<td>293703234099044425</td>
<td>2937032340990158620</td>
<td>9.04</td>
</tr>
</tbody>
</table>

if $p \equiv \pm 2 \pmod{5}$, so the corresponding Euler factors of $P_R(s)^{-1}$ are $(1 - 1/p^s)^2$ and $(1 - 1/p^{2s})$, hence

$$P_R(s)^{-1} = \prod_{p|5D} \left( (1 - 1/p^s) \left( 1 - \left( \frac{p}{5} \right) / p^s \right) \right).$$

If as above we write $P_R(s)^{-1} = \sum_{n \geq 1} r(n)/n^s$, then if $r(n) \neq 0$ we must have $n = xy^2$ with $x$ and $y$ coprime, squarefree and dividing $5D$, and in that case

$$r(n) = \left( \frac{y}{5} \right) \prod_{p|x} \left( 1 + \left( \frac{p}{5} \right) \right).$$

Thus

$$A_1(X) = \sum_{x, y|5D \text{ squarefree, coprime}} \left( \frac{y}{5} \right) \prod_{p|x} \left( - \left( 1 + \left( \frac{p}{5} \right) \right) \right) A_2(X/(xy^2)).$$

To treat the other two factors $Q_1(s)$ and $P_F(s)^{-1}$ it is preferable to separate them.

When $p \nmid 5D$ we have $e(p) = 1$, and when $f(p) > 1$ we have $f(p) = 2$ if $p \equiv -\left( \frac{D}{p} \right) \pmod{5}$, and $f(p) = 4$ when $p \equiv \pm 2 \pmod{5}$. Thus,

$$P_F(s)^{-1} = \prod_{p\equiv -\left( \frac{D}{p} \right) \pmod{5}} (1 - 1/p^{2s})^2 \prod_{p\equiv \pm 2 \pmod{5}} (1 - 1/p^{4s}) = \sum_{n \geq 1} f(n)/n^s,$$

where $f(n) \neq 0$ implies that $n = x^2y^4$ with $x$ and $y$ coprime, squarefree, and coprime to $5D$, $p \mid x$ implies $p \equiv -\left( \frac{D}{p} \right) \pmod{5}$, $p \mid y$ implies $p \equiv \left( \frac{D}{p} \right) \pmod{5}$, and $f(n) = (-2)^{\omega(x)} (-1)^{\omega_2(y)}$, where $\omega_2(y)$ is the number of prime divisors of $y$ congruent to $\pm 2$ modulo 5. Hence

$$A_2(X) = \sum_{y \leq X^{1/4} \text{ cond}} (-1)^{\omega_2(y)} \sum_{x \leq (X/y^4)^{1/2} \text{ cond}} (-2)^{\omega(x)} A_3^*(X/(x^2y^4)).$$
for some intermediate counting function $A'_3(X)$, where “cond” are the conditions given above. Furthermore,

$$Q_1(s) = \prod_{p \equiv \left( \frac{D}{p} \right) \equiv \pm 1 (\text{mod } 5)} (1 - 10/p^{2s} + 20/p^{3s} - 15/p^{4s} + 4/p^{5s}) = \sum_{n \geq 1} g(n)/n^s,$$

where $g(n) \neq 0$ if and only if $n$ is of the form $n = x_2^2 x_3^3 x_4^4 x_5^5$ with the $x_i$ squarefree, pairwise coprime, and divisible only by primes $p$ such that $p \equiv \left( \frac{D}{p} \right) \equiv \pm 1 (\text{mod } 5)$, in which case

$$g(n) = (-10)^{\omega(x_2)}(20)^{\omega(x_3)}(-15)^{\omega(x_4)}(4)^{\omega(x_5)},$$

so that

$$A'_3(X) = \sum_{\text{cond } x_5 \leq X^{1/5}} 4^{\omega(x_5)} \sum_{\text{cond } x_4 \leq (X/x_5^{1/4})} (-15)^{\omega(x_4)} \sum_{\text{cond } x_3 \leq (X/(x_4^{1/3} x_5^{1/4}))} 20^{\omega(x_3)} \sum_{\text{cond } x_2 \leq (X/(x_3^{1/2} x_4^{1/3} x_5^{1/4}))} (-10)^{\omega(x_2)} A_3(X/(x_2^2 x_3^3 x_4^4 x_5^5)),$$

where as above $A_3(X)$ is the summatory function of the Dirichlet coefficients of the Dedekind zeta function $\zeta_{K'}(s)$.

**Remark 5.5.** The above formulas may look incredibly messy, but first they are immediate to program, and second as explained above, the running time will be dominated by the time needed to compute $A_3(X)$, and not by the multiple summations, which will only multiply the time by a small constant factor.

As mentioned above, to compute $A_3(X)$ we use the method of the hyperbola (Proposition 5.2) three times.

In our case $\ell = 5$, since $\omega^2 = \chi_5$, we have

$$\zeta_{K'}(s) = \zeta(s)L(\chi_5, s)L(\omega\chi_D, s)L(\overline{\omega}\chi_D, s).$$

Thus, we write

$$\zeta(s)L(\chi_5, s) = \sum_{n \geq 1} a_{3,1}(n)/n^s, \quad L(\omega\chi_D, s)L(\overline{\omega}\chi_D, s) = \sum_{n \geq 1} a_{3,2}(n)/n^s,$$

and the corresponding summatory functions $A_{3,1}(X)$ and $A_{3,2}(X)$ are computed thanks to Proposition 5.2 (as usual the functions $\chi_5$, $\omega\chi_D$, and $\overline{\omega}\chi_D$ as well as their summatory functions are periodic with small period, so they are tabulated), and using Proposition 5.2 once more, we obtain the desired summatory function $A_3(X)$ since $a_3(n)$ is the arithmetic convolution of $a_{3,1}(n)$ and $a_{3,2}(n)$, for a total running time in $O(X^{3/4+\epsilon})$. Here it seems that the error in the asymptotic estimate proved to be $O(X^{4/5+\epsilon})$ is closer to $O(X^{3/8})$, so we set $E_5(D; X) = (M_5(D; X) - P_5(D; X))/X^{3/8}$. (See Tables 5 and 6.)
Table 5. Number of $D_5$ quintic fields $L$ with $K = \mathbb{Q}(\sqrt{-3})$ and $f(L) \leq X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_5(-3; X)$</th>
<th>$P_5(-3; X)$</th>
<th>$E_5(-3; X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^6$</td>
<td>50817</td>
<td>50785</td>
<td>0.18</td>
</tr>
<tr>
<td>$10^7$</td>
<td>508038</td>
<td>507853</td>
<td>0.44</td>
</tr>
<tr>
<td>$10^8$</td>
<td>5078754</td>
<td>5078532</td>
<td>0.22</td>
</tr>
<tr>
<td>$10^9$</td>
<td>50784256</td>
<td>50785324</td>
<td>-0.45</td>
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<tr>
<td>$10^{10}$</td>
<td>507849182</td>
<td>5078532444</td>
<td>-0.72</td>
</tr>
<tr>
<td>$10^{11}$</td>
<td>5078531182</td>
<td>5078532445</td>
<td>-0.09</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>50785334021</td>
<td>50785324450</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Table 6. Number of $D_5$ quintic fields $L$ with $K = \mathbb{Q}(\sqrt{-15})$ and $f(L) \leq X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_5(-15; X)$</th>
<th>$P_5(-15; X)$</th>
<th>$E_5(-15; X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^6$</td>
<td>78897</td>
<td>78805</td>
<td>0.52</td>
</tr>
<tr>
<td>$10^7$</td>
<td>787986</td>
<td>788048</td>
<td>-0.15</td>
</tr>
<tr>
<td>$10^8$</td>
<td>7879958</td>
<td>7880481</td>
<td>-0.52</td>
</tr>
<tr>
<td>$10^9$</td>
<td>78805084</td>
<td>78804814</td>
<td>0.11</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>788050642</td>
<td>788048138</td>
<td>0.45</td>
</tr>
<tr>
<td>$10^{11}$</td>
<td>7880487110</td>
<td>7880481380</td>
<td>0.43</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>78804743357</td>
<td>78804813801</td>
<td>-2.23</td>
</tr>
</tbody>
</table>

5.6. The special case for $\ell = 5$. Once again, we only give the algorithm and the corresponding table. We first give a preliminary result and then the formula for the Dirichlet series, all proved in [10].

**Proposition 5.6.** Let $E$ be the quintic field defined by $x^5 + 5x^3 + 5x - 1 = 0$, with discriminant $5^7$ and Galois group $C_5 \times C_4$, and let $p$ be a prime such that $p \equiv 1 \pmod{5}$. The following are equivalent:

1. $p$ is totally split in $E$.
2. $\varepsilon = (-1 + \sqrt{5})/2$ is a fifth power modulo $p$, in other words $\varepsilon^{(p-1)/5} \equiv 1 \pmod{p}$.

In practice, testing the second condition is faster.

**Theorem 5.7.** Keep the above notation, and set

$$\omega_E(p) = \begin{cases} -1 & \text{if } p \text{ is inert or totally ramified in } E, \\ 4 & \text{if } p \text{ is totally split in } E, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\Phi_{5, \mathbb{Q}(\sqrt{5})}(s) = \frac{1}{20} \left( \frac{1 + 4}{5^s} \right) \prod_{p \equiv 1 \pmod{5}} \left( 1 + \frac{4}{p^s} \right) + \frac{1}{5} \prod_p \left( 1 + \frac{\omega_E(p)}{p^s} \right).$$

It is immediate to see that $\omega_E(p) = 0$ when $p \not\equiv 1 \pmod{5}$, that the only totally ramified (or ramified) prime is $p = 5$, so the only computation that must be
done is to test whether \( p \equiv 1 \pmod{5} \) satisfies the second condition of the above proposition. Unfortunately, this is a nonabelian condition, so I do not see any way to compute \( M(D; X) \) with an algorithm faster than \( O(X) \). Thus, using rather naive methods we obtain Table 7.

### Table 7. Number of \( D_5 \) quintic fields \( L \) in the special case \( K = \mathbb{Q}(\sqrt{5}) \) and \( f(L) \leq X \)

<table>
<thead>
<tr>
<th>( X )</th>
<th>( M_5(5; X) )</th>
<th>( P_5(5; X) )</th>
<th>( E_5(5; X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^6 )</td>
<td>20426</td>
<td>20378</td>
<td>0.27</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>203938</td>
<td>203782</td>
<td>0.37</td>
</tr>
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<td>( 10^8 )</td>
<td>2037874</td>
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<td>( 10^9 )</td>
<td>20378156</td>
<td>20378187</td>
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</tr>
<tr>
<td>( 10^{10} )</td>
<td>203782163</td>
<td>203781871</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Note that because of the nonabelian second term, with a similar amount of computation than the general case, we reach a much smaller value of \( X \). This does not happen when \( \ell \equiv 3 \pmod{4} \) (see e.g., \( \ell = 7 \) below) since there is no nonabelian term.

### 5.7. The case \( \ell = 7 \).

The case \( \ell = 7 \) (and the case of larger \( \ell \)) is similar but slightly more complicated than the case \( \ell = 5 \), and the tedious details are left to the reader. We give the tables below for \( D = -3 \) and \( D = -35 \), where we set \( E_7(D; X) = (M_7(D; X) - P_7(D; X))/X^{5/12} \) (it would seem that the error is \( O(X^{(\ell-2)/(2(\ell-1))}+\varepsilon) \)).

### Table 8. Number of \( D_7 \) septic fields \( L \) with \( K = \mathbb{Q}(\sqrt{-3}) \) and \( f(L) \leq X \)

<table>
<thead>
<tr>
<th>( X )</th>
<th>( M_7(-3; X) )</th>
<th>( P_7(-3; X) )</th>
<th>( E_7(-3; X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^6 )</td>
<td>29689</td>
<td>29633</td>
<td>0.18</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>296453</td>
<td>296332</td>
<td>0.15</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>2962722</td>
<td>2963322</td>
<td>-0.28</td>
</tr>
<tr>
<td>( 10^9 )</td>
<td>29634095</td>
<td>29633216</td>
<td>0.16</td>
</tr>
<tr>
<td>( 10^{10} )</td>
<td>296332445</td>
<td>296332163</td>
<td>0.02</td>
</tr>
</tbody>
</table>

### Table 9. Number of \( D_7 \) septic fields \( L \) with \( K = \mathbb{Q}(\sqrt{-35}) \) and \( f(L) \leq X \)

<table>
<thead>
<tr>
<th>( X )</th>
<th>( M_7(-35; X) )</th>
<th>( P_7(-35; X) )</th>
<th>( E_7(-35; X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^6 )</td>
<td>52918</td>
<td>53000</td>
<td>-0.26</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>529927</td>
<td>530000</td>
<td>-0.09</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>5300615</td>
<td>5300003</td>
<td>0.28</td>
</tr>
<tr>
<td>( 10^9 )</td>
<td>53000137</td>
<td>53000025</td>
<td>0.02</td>
</tr>
<tr>
<td>( 10^{10} )</td>
<td>530024447</td>
<td>530000255</td>
<td>1.65</td>
</tr>
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</table>
5.8. The special case for $\ell = 7$. The special case for $\ell = 7$ is simpler than that for $\ell = 5$ since (as for all $\ell \equiv 3 \pmod{4}$, $\ell > 3$), there is only the main term, in other words by Theorem 1.1, we have

$$\Phi_{7, \mathbb{Q}(\sqrt{-7})}(s) = \frac{1}{6} \left(1 + \frac{6}{7^s}\right) \prod_{p \equiv \pm 1 \pmod{7}} \left(1 + \frac{6}{p^s}\right).$$

Note that the condition $p \equiv \pm 1 \pmod{7}$ can simply be tested by characters, and so using similar methods to the above, we obtain the following table:

**Table 10.** Number of $D_7$ septic fields $L$ in the special case $K = \mathbb{Q}(\sqrt{-7})$ and $f(L) \leq X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_7(-7; X)$</th>
<th>$P_7(-7; X)$</th>
<th>$E_7(-7; X)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>222793</td>
<td>0.60</td>
</tr>
<tr>
<td>$10^7$</td>
<td>2506500</td>
<td>2506662</td>
<td>-0.20</td>
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<tr>
<td>$10^8$</td>
<td>27858604</td>
<td>27853959</td>
<td>2.16</td>
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<td>306400832</td>
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<td>3342863878</td>
<td>2.47</td>
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</tbody>
</table>

**References**


