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SUPERCONVERGENCE OF LOCAL DISCONTINUOUS GALERKIN METHODS FOR ONE-DIMENSIONAL LINEAR PARABOLIC EQUATIONS

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Abstract. In this paper, we study superconvergence properties of the local discontinuous Galerkin method for one-dimensional linear parabolic equations when alternating fluxes are used. We prove, for any polynomial degree k, that the numerical fluxes converge at a rate of $2k+1$ (or $2k+1/2$) for all mesh nodes and the domain average under some suitable initial discretization. We further prove a $k+1$th superconvergence rate for the derivative approximation and a $k+2$th superconvergence rate for the function value approximation at the Radau points. Numerical experiments demonstrate that in most cases, our error estimates are optimal, i.e., the error bounds are sharp.

1. Introduction

The superconvergence behavior of discontinuous Galerkin (DG) and local discontinuous Galerkin (LDG) [10] methods has been studied for some years. Some early results can be found in Thomée's 1997 book [13]. Later, in [1], Adjerid et al. showed a $k+2$th superconvergence rate of the DG solution at the downwind-biased Radau points for some ordinary differential equations; in [6], Celiker and Cockburn studied superconvergence of the numerical traces for DG and hybridizable DG methods in solving some steady state problems. Recently, Yang and Shu investigated superconvergence phenomenon of the DG method for hyperbolic conservation laws [15] and linear parabolic equations [16] in the one-dimensional setting. Superconvergence properties of DG and LDG methods for hyperbolic and parabolic problems based on Fourier approach were studied in [12]. We also refer to [2, 3, 8, 9, 14, 17] for an incomplete list of references. Very recently, in [5], we studied superconvergence properties of a DG method for linear hyperbolic equations when upwind fluxes were used. We proved a $2k+1$th superconvergence rate of the DG approximation at the downwind points (on average) as well as the domain average under suitable initial discretization.

The current work is the second in a series to study superconvergence phenomena of the DG method in solving partial differential equations where parabolic equations are under concern. Our main result is a rigorous mathematical proof of the

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$2k+1$th (or $2k+1/2$th) superconvergence rate for the domain average and numerical fluxes at mesh nodes. To the best of our knowledge, the best rate proved so far in the literature is $k+2$ [16]. As a by-product, we also prove a pointwise $k+2$th superconvergence rate for the function value approximation and $k+1$th superconvergence rate for the derivative approximation at the Radau (left or right) points. By doing so, we paint a full picture for superconvergence properties of the LDG method for linear parabolic equations in one space dimension.

In order to establish the $2k+1$th superconvergence rate, some new analysis tools are needed. At the core of our analysis here is the construction of a correction function. The correction function idea has been successfully applied to finite element methods (FEM) and finite volume methods (FVM) for elliptic equations (see, e.g. [4, 7]), and more recently, to the DG method for hyperbolic equations [5]. However, the construction for parabolic equations is very different from steady state problems using finite element [7] or finite volume methods [4] due to the time dependent effects. Moreover, it is also quite different from the DG method for hyperbolic equations [5] due to the interplay between two correction functions. The main difficulty for parabolic equations lies in that correction functions for both variables (the exact solution u and an auxiliary variable $q = u_x$) have to be constructed simultaneously. To be more precise, we shall correct the error between the LDG solution and the Gauss-Radau projection of the exact solution $(P_h^- u, P_h^+ q)$ or $(P_h^+ u, P_h^- q)$, depending on the choice of numerical fluxes. The construction not only is more complicated than that of hyperbolic equations, but also requires a novel idea to match the two variables.

With help of the correction functions, we prove that the LDG solution (u_h, q_h) is super-close with order $2k+1$ to our specially constructed interpolation function (u_I, q_I) (defined in Section 3). It is this super-closeness that leads to the $2k+1$th superconvergence rate for the numerical fluxes at all nodes (on average) and for the domain average.

To end this introduction, we would like to point out that all superconvergent results here are valid for one-dimensional linear systems, and the proof is along the same line without any difficulty. Our analysis also leads to some interesting new numerical discoveries, which will be reported in the last section.

The rest of the paper is organized as follows. In Section 2, we present the LDG scheme for linear parabolic equations. Section 3 is the most technical part, where we construct some special functions to correct the error between the LDG solution and the Gauss-Radau projection of the exact solution. Section 4 is the main body of the paper, where superconvergence results are proved with suitable initial discretization. In Section 5, we provide some numerical examples to support our theoretical findings. Finally, some possible future works and concluding remarks are presented in Section 6.

Throughout this paper, we adopt standard notation for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D, and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$. Notation"$A \lesssim B$" implies that A can be bounded by B multiplied by a constant independent of the mesh size h and the exact solution u. "$A \sim B$" stands for "$A \lesssim B$" and "$B \lesssim A$".

2. LDG schemes

We consider local discontinuous Galerkin (LDG) methods for the following one-dimensional linear parabolic equation

$$\begin{aligned} &u_t = u_{xx}, && (x,t) \in [0,2\pi] \times (0,T], \\ &u(x,0) = u_0(x), && x \in R, \end{aligned} \tag{2.1}$$

where u_0 is sufficiently smooth. We will consider both the periodic boundary condition $u(0,t) = u(2\pi,t)$ and the mixed boundary condition $u(0,t) = g_0(t), u_x(2\pi,t) = g_1(t)$ or $u_x(0,t) = g_0(t), u(2\pi,t) = g_1(t)$.

Let $\Omega = [0,2\pi]$ and $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}} = 2\pi$ be $N+1$ distinct points on the interval Ω. For any positive integer r, we define $\mathbb{Z}_r = \{1,\ldots,r\}$ and denote by

$$\tau_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}), \ j \in \mathbb{Z}_N$$

the cells and cell centers, respectively. Let $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $\bar{h}_j = h_j/2$ and $h = \max_j h_j$. We assume that the mesh is quasi-uniform, i.e., there exists a constant c such that $h \le ch_j$, $j \in \mathbb{Z}_N$. Define the finite element space

$$V_h = \{v : \ v|_{\tau_j} \in \mathbb{P}_k(\tau_j), \ j \in \mathbb{Z}_N\},$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree at most k with coefficients as functions of t.

To construct the LDG scheme, we introduce an auxiliary variable $q = u_x$, then (2.1) can be rewritten as a first order linear system

$$u_t = q_x, \quad q = u_x. \tag{2.2}$$

The LDG scheme for (2.1) reads as: Find $u_h, q_h \in V_h$ such that for any $v, w \in V_h$,

$$\begin{aligned} (u_{ht}, v)_j &= -(q_h, v_x)_j + \hat{q}_h v^-|_{j+\frac{1}{2}} - \hat{q}_h v^+|_{j-\frac{1}{2}}, \\ (q_h, w)_j &= -(u_h, w_x)_j + \hat{u}_h w^-|_{j+\frac{1}{2}} - \hat{u}_h w^+|_{j-\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Here $(u,v)_j = \int_{\tau_j} uv dx$, $v^-|_{j+\frac{1}{2}}$ and $v^+|_{j+\frac{1}{2}}$ denote the left and right limits of v at the point $x_{j+\frac{1}{2}}$, respectively, and $\hat{u}_h, \hat{q}_h$ are numerical fluxes. For LDG schemes, we consider alternating fluxes

$$\hat{u}_h = u_h^-, \quad \hat{q}_h = q_h^+ \tag{2.4}$$

or

$$\hat{u}_h = u_h^+, \quad \hat{q}_h = q_h^-. \tag{2.5}$$

In this paper, we use both (2.4) and (2.5) as numerical fluxes in the periodic boundary condition, (2.4) in the mixed boundary condition $u(0,t) = g_0(t), u_x(2\pi,t) = g_1(t)$, and (2.5) in the mixed boundary condition $u_x(0,t) = g_0(t), u(2\pi,t) = g_1(t)$.

Define

$$H_h^1 = \{v : \ v|_{\tau_j} \in H^1(\tau_j), \ j \in \mathbb{Z}_N\}$$

and for all $\xi, \eta, v \in H_h^1$, let

$$a^1(\xi,\eta;v) = \sum_{j=1}^N a_j^1(\xi,\eta;v), \quad a^2(\xi,\eta;v) = \sum_{j=1}^N a_j^2(\xi,\eta;v)$$

where

$$a_j^1(\xi,\eta;v) = (\xi_t, v)_j + (\eta, v_x)_j - \hat{\eta}v^-|_{j+\frac{1}{2}} + \hat{\eta}v^+|_{j-\frac{1}{2}},$$
$$a_j^2(\xi,\eta;v) = (\eta, v)_j + (\xi, v_x)_j - \hat{\xi}v^-|_{j+\frac{1}{2}} + \hat{\xi}v^+|_{j-\frac{1}{2}}.$$

Here $\hat{\xi}, \hat{\eta}$ are taken as the alternating fluxes (2.4) or (2.5). Then we deduce from (2.3):

$$a^1(u_h, q_h; v) = 0, \quad a^2(u_h, q_h; w) = 0, \quad \forall v, w \in V_h. \tag{2.6}$$

Obviously, the exact solutions u, q also satisfy

$$a^1(u, q; v) = 0, \quad a^2(u, q; w) = 0, \quad \forall v, w \in V_h. \tag{2.7}$$

By a direct calculation, there hold

$$a^1(v, w; v) + a^2(v, w; w) = (v_t, v) + (w, w) - w^+v^-|_{N+\frac{1}{2}} + w^+v^-|_{\frac{1}{2}} \tag{2.8}$$

for the fluxes choice (2.4) and

$$a^1(v, w; v) + a^2(v, w; w) = (v_t, v) + (w, w) - w^-v^+|_{N+\frac{1}{2}} + w^-v^+|_{\frac{1}{2}} \tag{2.9}$$

for the fluxes choice (2.5).

3. Construction of special interpolation functions

Our goal here is to construct a special interpolation function (u_I, q_I), which is super-close to the LDG solution (u_h, q_h).

We begin with some preliminaries. First, for any r, we denote by $\lfloor r \rfloor$ the maximal integer no more than r, and $\lceil r \rceil$ the minimal integer no less than r. Next, we define on $v \in H_h^1$, two Gauss-Radau projections P_h^-, P_h^+ by

$$(P_h^- v, w)_j = (v, w)_j, \forall w \in \mathbb{P}_{k-1}(\tau_j) \quad \text{and} \quad P_h^- v(x_{j+\frac{1}{2}}^-) = v(x_{j+\frac{1}{2}}^-),$$
$$(P_h^+ v, w)_j = (v, w)_j, \forall w \in \mathbb{P}_{k-1}(\tau_j) \quad \text{and} \quad P_h^+ v(x_{j-\frac{1}{2}}^+) = v(x_{j-\frac{1}{2}}^+),$$

and an integral operator D_s^{-1} by

$$D_s^{-1}v(x) = \frac{1}{\bar{h}_j}\int_{x_{j-\frac{1}{2}}}^{x} v(x')dx' = \int_{-1}^{s} \hat{v}(s')ds', \quad x \in \tau_j, j \in \mathbb{Z}_N,$$

where

$$s = (x - x_j)/\bar{h}_j \in [-1, 1], \quad \hat{v}(s) = v(x).$$

We have, for any function $v \in H_h^1$, the following Legendre expansion in each element $\tau_j, j \in \mathbb{Z}_N$,

$$v(x,t) = \sum_{m=0}^{\infty} v_{j,m}(t)L_{j,m}(x), \quad v_{j,m} = \frac{2m+1}{h_j}(v, L_{j,m})_j,$$

where $L_{j,m}$ denotes the normalized Legendre polynomial of degree m on τ_j. By the definition of P_h^-, P_h^+,

$$(v - P_h^- v)(x,t) = \bar{v}_{j,k}(t)L_{j,k} + \sum_{m=k+1}^{\infty} v_{j,m}(t)L_{j,m}(x),$$

$$(v - P_h^+ v)(x,t) = \tilde{v}_{j,k}(t)L_{j,k} + \sum_{m=k+1}^{\infty} v_{j,m}(t)L_{j,m}(x),$$

where

$$\bar{v}_{j,k} = -v(x^-_{j+\frac{1}{2}}, t) + \frac{1}{h_j}\int_{\tau_j} v(x,t)\sum_{m=0}^{k}(2m+1)L_{j,m}(x)dx, \tag{3.1}$$

$$\tilde{v}_{j,k} = (-1)^{k+1}v(x^+_{j-\frac{1}{2}}, t) + \frac{1}{h_j}\int_{\tau_j} v(x,t)\sum_{m=0}^{k}(-1)^{k+m}(2m+1)L_{j,m}(x)dx. \tag{3.2}$$

Obviously,

$$(v - P_h^- v, w)_j = \bar{v}_{j,k}(L_{j,k}, w)_j, \quad (v - P_h^+ v, w) = \tilde{v}_{j,k}(L_{j,k}, w)_j, \quad \forall w \in V_h. \tag{3.3}$$

In each element $\tau_j, j \in \mathbb{Z}_N$, we define

$$F_{1,1} = P_h^+ D_s^{-1} L_{j,k}, \quad F_{1,i} = (P_h^+ D_s^{-1} P_h^- D_s^{-1})^{i-1} F_{1,1}, \ i \geq 2, \tag{3.4}$$

$$F_{2,1} = P_h^- D_s^{-1} L_{j,k}, \quad F_{2,i} = (P_h^- D_s^{-1} P_h^+ D_s^{-1})^{i-1} F_{2,1}, \ i \geq 2. \tag{3.5}$$

Lemma 3.1. *For all $1 \leq i \leq \lceil k/2 \rceil, x \in \tau_j, j \in \mathbb{Z}_N$, $F_{1,i}(x), F_{2,i}(x)$ have the following representations*

$$F_{1,i}(x) = \textstyle\sum_{m=k-2i+2}^{k} a_{i,m}(L_{j,m} + L_{j,m-1})(x), \tag{3.6}$$

$$F_{2,i}(x) = \textstyle\sum_{m=k-2i+2}^{k} b_{i,m}(L_{j,m} - L_{j,m-1})(x), \tag{3.7}$$

where the coefficients $a_{i,m}, b_{i,m}$ are some bounded constants independent of the mesh size h_j, and $L_{j,m} + L_{j,m-1}, L_{j,m} - L_{j,m}, m \geq 1$ are the left and right Radau polynomials of degree m on τ_j, respectively. Consequently,

$$F_{1,i}(x^+_{j-\frac{1}{2}}) = 0, \quad \|F_{1,i}\|_{0,\infty,\tau_j} \lesssim 1, \tag{3.8}$$

$$F_{2,i}(x^-_{j+\frac{1}{2}}) = 0, \quad \|F_{2,i}\|_{0,\infty,\tau_j} \lesssim 1. \tag{3.9}$$

Proof. For all $m \geq 1$, noticing that $\|L_{j,m}\|_{0,\infty,\tau_j} = 1$ and

$$(L_{j,m} + L_{j,m-1})(x^+_{j-\frac{1}{2}}) = 0, \quad (L_{j,m} - L_{j,m-1})(x^-_{j+\frac{1}{2}}) = 0,$$

then (3.8)-(3.9) follow directly from (3.6)-(3.7).

In the following, We shall focus our attention on (3.6) since (3.7) can be obtained by following the same line. We show (3.6) by induction. First, by the definition of P_h^+ and the fact that

$$D_s^{-1} L_{j,m} = \frac{1}{2m+1}(L_{j,m+1} - L_{j,m-1}), \ m \geq 1, \tag{3.10}$$

we derive

$$F_{1,1} = -\frac{1}{2k+1}(L_{j,k} + L_{j,k-1}),$$

which implies (3.6) is valid for $i = 1$ with $a_{1,k} = -\frac{1}{2k+1}$. Now we suppose (3.6) is valid for $i, i \leq \lceil k/2 \rceil - 1$. Since

$$P_h^- L_{j,k+1} = L_{j,k}, \quad P_h^+ L_{j,k+1} = -L_{j,k}, \quad P_h L_{j,m} = L_{j,m}, \quad 1 \leq m \leq k,$$

where $P_h = P_h^-$ or P_h^+, it is easy to deduce from (3.10) that

$$P_h^- D_s^{-1} L_{j,k} = \frac{1}{2k+1}(L_{j,k} - L_{j,k-1}), \quad P_h^+ D_s^{-1} L_{j,k} = \frac{-1}{2k+1}(L_{j,k} + L_{j,k-1}),$$

$$P_h D_s^{-1} L_{j,m} = \frac{1}{2m+1}(L_{j,m+1} - L_{j,m-1}), \quad 1 \leq m \leq k-1.$$

Therefore,

$$P_h^- D_s^{-1} F_{1,i} = \sum_{m=k-2i+1}^{k} \beta_{i,m}(L_{j,m} - L_{j,m-1}), \tag{3.11}$$

where

$$\beta_{i,m} = \frac{a_{i,m+1} + a_{i,m}}{2m+1} + \frac{a_{i,m} + a_{i,m-1}}{2m-1}$$

with $a_{i,k+1} = a_{i,k-2i+1} = a_{i,k-2i} = 0$. Now we consider $F_{1,i+1}$. Note that

$$F_{1,i+1} = P_h^+ D_s^{-1} P_h^- D_s^{-1} F_{1,i},$$

and we have from (3.11) that

$$\begin{aligned} F_{1,i+1} &= \sum_{m=k-2i+1}^{k} \beta_{i,m} P_h^+ D_s^{-1}(L_{j,m} - L_{j,m-1}) \\ &= \sum_{m=k-2i}^{k} a_{i+1,m}(L_{j,m} + L_{j,m-1}), \end{aligned}$$

where

$$a_{i+1,m} = \frac{\beta_{i,m+1} - \beta_{i,m}}{2m+1} + \frac{\beta_{i,m-1} - \beta_{i,m}}{2m-1}$$

with $\beta_{i,k+1} = \beta_{i,k-2i} = \beta_{i,k-2i-1} = 0$. Consequently, (3.6) is valid for $i+1$. Then (3.6) follows. This completes our proof. □

With the functions $F_{1,i}, F_{2,i}$, we define in each $\tau_j, j \in \mathbb{Z}_N$ two other functions $\bar{F}_{1,i}, \bar{F}_{2,i}$ as

$$\bar{F}_{1,i} = P_h^- D_s^{-1} F_{1,i}, \quad \bar{F}_{2,i} = P_h^+ D_s^{-1} F_{2,i}, \ 1 \le i \le \lfloor k/2 \rfloor. \tag{3.12}$$

By the same arguments as in Lemma 3.1, we obtain

$$\bar{F}_{1,i} = \sum_{m=k-2i+1}^{k} \beta_{i,m}(L_{j,m} - L_{j,m-1}), \tag{3.13}$$

$$\bar{F}_{2,i} = \sum_{m=k-2i+1}^{k} \gamma_{i,m}(L_{j,m} + L_{j,m-1}), \tag{3.14}$$

where $\beta_{i,m}, \gamma_{i,m}$ are constants independent of h_j. Consequently,

$$\bar{F}_{1,i}(x^-_{j+\frac{1}{2}}) = 0, \quad \|\bar{F}_{1,i}\|_{0,\infty,\tau_j} \lesssim 1, \tag{3.15}$$

$$\bar{F}_{2,i}(x^+_{j-\frac{1}{2}}) = 0, \quad \|\bar{F}_{2,i}\|_{0,\infty,\tau_j} \lesssim 1. \tag{3.16}$$

In addition, a straightforward calculation from (3.4)-(3.5) and (3.12) yields

$$F_{1,i+1} = P_h^+ D_s^{-1} \bar{F}_{1,i}, \qquad F_{2,i+1} = P_h^- D_s^{-1} \bar{F}_{2,i}, \ 1 \le i \le \lfloor k/2 \rfloor. \tag{3.17}$$

3.1. **Correction functions for the fluxes** (2.4)**.** In each element $\tau_j, j \in \mathbb{Z}_N$, we have, from (3.3),

$$(u-P_h^- u, v)_j = \bar{u}_{j,k}(t)(L_{j,k}, v)_j, \ \ (q-P_h^+ q, v)_j = \tilde{q}_{j,k}(t)(L_{j,k}, v)_j, \ \ \forall v \in V_h, \tag{3.18}$$

where the coefficients $\bar{u}_{j,k}, \tilde{q}_{j,k}$ are given by (3.1)-(3.2). Let

$$G_i(t) = D_t^i \bar{u}_{j,k}(t), \quad Q_i(t) = D_t^i \tilde{q}_{j,k}(t), \ \ 0 \le i \le \lceil k/2 \rceil.$$

By the standard approximation theory, if $u \in W^{k+2+2i,\infty}(\Omega)$,

$$|G_i| \lesssim h^{k+1} \|\partial_t^i u\|_{k+1,\infty,\tau_j} \lesssim h^{k+1} \|u\|_{k+1+2i,\infty,\tau_j}, \tag{3.19}$$

$$|Q_i| \lesssim h^{k+1} \|\partial_t^i u\|_{k+2,\infty,\tau_j} \lesssim h^{k+1} \|u\|_{k+2+2i,\infty,\tau_j}. \tag{3.20}$$

Now we are ready to construct our correction functions. For all $1 \le l \le k$, we define, first at the boundary points $x = x_{\frac{1}{2}}$ and $x = x_{N+\frac{1}{2}}$,

$$W_1^l(x^+_{N+\frac{1}{2}}, t) = 0, \quad W_2^l(x^-_{\frac{1}{2}}, t) = 0, \ \ \forall t \ge 0, \tag{3.21}$$

and then in each element $\tau_j, j \in \mathbb{Z}_N$,

$$W_1^l(x,t) = \sum_{i=1}^{\lceil l/2 \rceil} w_{1,i} + \sum_{i=1}^{\lfloor l/2 \rfloor} \bar{w}_{2,i}, \ \ W_2^l(x,t) = \sum_{i=1}^{\lfloor l/2 \rfloor} \bar{w}_{1,i} + \sum_{i=1}^{\lceil l/2 \rceil} w_{2,i}, \tag{3.22}$$

where

$$w_{1,i} = (\bar{h}_j)^{2i-1} G_i F_{1,i}, \quad \bar{w}_{1,i} = (\bar{h}_j)^{2i} G_i \bar{F}_{1,i}, \tag{3.23}$$

$$w_{2,i} = (\bar{h}_j)^{2i-1} Q_{i-1} F_{2,i}, \quad \bar{w}_{2,i} = (\bar{h}_j)^{2i} Q_i \bar{F}_{2,i}. \tag{3.24}$$

Lemma 3.2. *Suppose $W_1^l, W_2^l \in V_h$ are defined by* (3.21)-(3.24)*. Then*

$$W_1^l(x^+_{j-\frac{1}{2}}, t) = 0, \quad W_2^l(x^-_{j-\frac{1}{2}}, t) = 0, \ \ \forall j \in \mathbb{Z}_{N+1}. \tag{3.25}$$

Moreover, if $l = 2r$ is even, then

$$((W_2^l)_t, v)_j + (W_1^l, v_x)_j = (w_{1,1}, v_x)_j + ((\bar{w}_{1,rt})_t, v)_j, \tag{3.26}$$

$$(W_2^l, v_x)_j + (W_1^l, v)_j = (w_{2,1}, v_x)_j + (\bar{w}_{2,r}, v)_j. \tag{3.27}$$

If $l = 2r+1$ is odd, then

$$((W_2^l)_t, v)_j + (W_1^l, v_x)_j = (w_{1,1}, v_x)_j + ((w_{2,r+1})_t, v)_j, \tag{3.28}$$

$$(W_2^l, v_x)_j + (W_1^l, v)_j = (w_{2,1}, v_x)_j + (w_{1,r+1}, v)_j. \tag{3.29}$$

Proof. By (3.8)-(3.9) and (3.15)-(3.16),

$$w_{1,i}(x^+_{j-\frac{1}{2}}, t) = \bar{w}_{2,i}(x^+_{j-\frac{1}{2}}, t) = 0, \ \ w_{2,i}(x^-_{j+\frac{1}{2}}, t) = \bar{w}_{1,i}(x^-_{j+\frac{1}{2}}, t) = 0, \ \ j \in \mathbb{Z}_N.$$

Then (3.25) follows from (3.21)-(3.22).

We now show (3.26)-(3.29). For any integer $l, 1 \le l \le k$, a direct calculation from (3.6)-(3.7) and (3.13)-(3.14) gives

$$D_s^{-1} F_{1,i}(x^-_{j+\frac{1}{2}}) = D_s^{-1} F_{1,i}(x^+_{j-\frac{1}{2}}) = 0, \ \ D_s^{-1} F_{2,i}(x^-_{j+\frac{1}{2}}) = D_s^{-1} F_{2,i}(x^+_{j-\frac{1}{2}}) = 0$$

for all $i \in \mathbb{Z}_{\lfloor l/2 \rfloor}$, and

$$D_s^{-1} \bar{F}_{1,i}(x^-_{j+\frac{1}{2}}) = D_s^{-1} \bar{F}_{1,i}(x^+_{j-\frac{1}{2}}) = 0, \ \ D_s^{-1} \bar{F}_{2,i}(x^-_{j+\frac{1}{2}}) = D_s^{-1} \bar{F}_{2,i}(x^+_{j-\frac{1}{2}}) = 0$$

for all $i \in \mathbb{Z}_{\lfloor l/2 \rfloor - 1}$ in case $l = 2r$ and $i \in \mathbb{Z}_{\lfloor l/2 \rfloor}$ in case $l = 2r+1$. Then by integration by parts, (3.12) and (3.17), we obtain

$$\begin{aligned}
(\bar{w}_{1,i}, v_x)_j + (w_{1,i}, v)_j &= (\bar{h}_j)^{2i} G_i(\bar{F}_{1,i}, v_x)_j + (\bar{h}_j)^{2i-1} G_i(F_{1,i}, v)_j \\
&= (\bar{h}_j)^{2i} G_i(\bar{F}_{1,i} - D_s^{-1} F_{1,i}, v_x)_j = 0, \\
((w_{2,i})_t, v)_j + (\bar{w}_{2,i}, v_x)_j &= (\bar{h}_j)^{2i-1} Q_i(F_{2,i}, v)_j + (\bar{h}_j)^{2i} Q_i(\bar{F}_{2,i}, v_x)_j \\
&= (\bar{h}_j)^{2i} Q_i(\bar{F}_{2,i} - D_s^{-1} F_{2,i}, v_x)_j = 0
\end{aligned}$$

for all $i \in \mathbb{Z}_{\lfloor l/2 \rfloor}$, and

$$\begin{aligned}
((\bar{w}_{1,i})_t, v)_j + (w_{1,i+1}, v_x)_j &= (\bar{h}_j)^{2i} G_{i+1}(\bar{F}_{1,i}, v)_j + (\bar{h}_j)^{2i+1} G_{i+1}(F_{1,i+1}, v_x)_j \\
&= (\bar{h}_j)^{2i+1} G_{i+1}(F_{1,i+1} - D_s^{-1} \bar{F}_{1,i}, v_x)_j = 0, \\
(w_{2,i+1}, v_x)_j + (\bar{w}_{2,i}, v)_j &= (\bar{h}_j)^{2i+1} Q_i(F_{2,i+1}, v_x)_j + (\bar{h}_j)^{2i} Q_i(\bar{F}_{2,i}, v)_j \\
&= (\bar{h}_j)^{2i+1} Q_i(F_{2,i+1} - D_s^{-1} \bar{F}_{2,i}, v_x)_j = 0
\end{aligned}$$

for all $i \in \mathbb{Z}_{\lfloor l/2 \rfloor - 1}$ in case $l = 2r$ and $i \in \mathbb{Z}_{\lfloor l/2 \rfloor}$ in case $l = 2r+1$. Then the desired results (3.26)-(3.29) follow by summing over all i. □

With the correction functions $W_1^l, W_2^l, 1 \le l \le k$, we define the special interpolation functions

$$u_I^l = P_h^- u - W_2^l, \quad q_I^l = P_h^+ q - W_1^l. \tag{3.30}$$

By (3.25), we have

$$u_I^l(x_{j-\frac{1}{2}}^-, t) = u(x_{j-\frac{1}{2}}^-, t), \quad q_I^l(x_{j-\frac{1}{2}}^+, t) = q(x_{j-\frac{1}{2}}^+, t), \quad \forall j \in \mathbb{Z}_{N+1}. \tag{3.31}$$

3.2. **Correction functions for the fluxes** (2.5)**.** In this case, we still use the notation

$$G_i(t) = D_t^i \tilde{u}_{j,k}(t), \quad Q_i(t) = D_t^i \bar{q}_{j,k}(t), \quad 0 \le i \le \lceil k/2 \rceil,$$

where $\tilde{u}_{j,k}, \bar{q}_{j,k}$ are defined by (3.1)-(3.2).

Similarly as the fluxes choice (2.4), we construct the correction functions as follows. For all $1 < l \le k, t \ge 0$, we define, at the boundary points $x = x_{\frac{1}{2}}$ and $x = x_{N+\frac{1}{2}}$,

$$W_1^l(x_{\frac{1}{2}}^-, t) = 0, \quad W_2^l(x_{N+\frac{1}{2}}^+, t) = 0, \tag{3.32}$$

and in each element $\tau_j, j \in \mathbb{Z}_N$,

$$W_1^l(x,t) = \sum_{i=1}^{\lfloor l/2 \rfloor} \bar{w}_{1,i} + \sum_{i=1}^{\lceil l/2 \rceil} w_{2,i}, \quad W_2^l(x,t) = \sum_{i=1}^{\lceil l/2 \rceil} w_{1,i} + \sum_{i=1}^{\lfloor l/2 \rfloor} \bar{w}_{2,i}, \tag{3.33}$$

where

$$\begin{aligned}
w_{1,i} &= (\bar{h}_j)^{2i-1} Q_{i-1} F_{1,i}, \quad \bar{w}_{1,i} = (\bar{h}_j)^{2i} Q_i \bar{F}_{1,i}, \\
w_{2,i} &= (\bar{h}_j)^{2i-1} G_i F_{2,i}, \quad \bar{w}_{2,i} = (\bar{h}_j)^{2i} G_i \bar{F}_{2,i}.
\end{aligned}$$

We define the special interpolation functions in each element $\tau_j, j \in \mathbb{Z}_N$ as

$$u_I^l = P_h^+ u - W_2^l, \quad q_I^l = P_h^- q - W_1^l, \quad 1 \le l \le k. \tag{3.34}$$

A direct calculation yields

$$u_I^l(x_{j-\frac{1}{2}}^+, t) = u(x_{j-\frac{1}{2}}^+, t), \quad q_I^l(x_{j-\frac{1}{2}}^-, t) = q(x_{j-\frac{1}{2}}^-, t), \quad \forall j \in \mathbb{Z}_{N+1}. \tag{3.35}$$

3.3. **Estimates.** We shall present in this subsection some estimates for W_1^l, W_2^l and the interpolation function (u_I, q_I), which play important roles in our superconvergence analysis.

Theorem 3.3. *Let $u \in W^{k+l+3,\infty}(\Omega), 1 \le l \le k$ be the solution of* (2.1). *Suppose W_1^l, W_2^l are defined by* (3.21)-(3.22) *for fluxes* (2.4) *or* (3.32)-(3.33) *for fluxes* (2.5). *Then for all $j \in \mathbb{Z}_N$,*

$$\|W_1^l\|_{0,\infty,\tau_j} + \|W_2^l\|_{0,\infty,\tau_j} \lesssim h^{k+2}\|u\|_{k+l+2,\infty,\tau_j}. \tag{3.36}$$

Moreover, if $(u_I^l, q_I^l) \in V_h$ is the corresponding interpolation function defined by (3.30) *and* (3.34) *for fluxes* (2.4) *and* (2.5), *respectively, then*

$$\left|((u_I^l - u)_t, v)_j - (W_1^l, v_x)_j\right| \lesssim h^{k+l+1}\|u\|_{k+l+3,\infty,\tau_j}\|v\|_{0,1,\tau_j}, \tag{3.37}$$

$$\left|(q_I^l - q, v)_j - (W_2^l, v_x)_j\right| \lesssim h^{k+l+1}\|u\|_{k+l+3,\infty,\tau_j}\|v\|_{0,1,\tau_j}. \tag{3.38}$$

Proof. We only consider the fluxes (2.4), since the the proof for fluxes (2.5) is similar. For all $i \ge 1$, as direct consequences of the second inequality of (3.8)-(3.9) and (3.15)-(3.16), and (3.19)-(3.20), we obtain

$$\|w_{1,i}\|_{0,\infty,\tau_j} \lesssim h^{k+2i}\|u\|_{k+1+2i,\infty,\tau_j}, \|\bar{w}_{1,i}\|_{0,\infty,\tau_j} \lesssim h^{k+2i+1}\|u\|_{k+1+2i,\infty,\tau_j}, \tag{3.39}$$

$$\|w_{2,i}\|_{0,\infty,\tau_j} \lesssim h^{k+2i}\|u\|_{k+2i,\infty,\tau_j}, \quad \|\bar{w}_{2,i}\|_{0,\infty,\tau_j} \lesssim h^{k+2i+1}\|u\|_{k+2+2i,\infty,\tau_j}. \tag{3.40}$$

Then (3.36) follows.

We now show (3.37)-(3.38). By (3.18), integration by parts, and the first formula of (3.4)-(3.5),

$$(P_h^- u_t - u_t, v)_j = -G_1(L_{j,k}, v)_j = \bar{h}_j G_1(F_{1,1}, v_x)_j = (w_{1,1}, v_x)_j,$$
$$(P_h^+ q - q, v)_j = -Q_0(L_{j,k}, v)_j = \bar{h}_j Q_0(F_{2,1}, v_x)_j = (w_{2,1}, v_x)_j.$$

Then

$$((u_I^l - u)_t, v)_j - (W_1^l, v_x)_j = (w_{1,1}, v_x)_j - ((W_2^l)_t, v)_j - (W_1^l, v_x)_j,$$
$$(q_I^l - q, v)_j - (W_2^l, v_x)_j = (w_{2,1}, v_x)_j - (W_1^l, v)_j - (W_2^l, v_x)_j.$$

In light of (3.26)-(3.29), we have

$$((u_I^l - u)_t, v)_j - (W_1^l, v_x)_j = ((\bar{w}_{1,r})_t, v)_j, \quad (q_I^l - q, v)_j - (W_2^l, v_x)_j = (\bar{w}_{2,r}, v)_j$$

for $l = 2r$ and

$$((u_I^l - u)_t, v)_j - (W_1^l, v_x)_j = ((w_{2,r+1})_t, v)_j, \quad (q_I^l - q, v)_j - (W_2^l, v_x)_j = (w_{1,r+1}, v)_j$$

for $l = 2r + 1$. By (3.39)-(3.40) and the fact that $u_t = u_{xx}$, we have for all $l \ge 1$,

$$|((u_I^l - u)_t, v)_j - (W_1^l, v_x)_j| \lesssim h^{k+l+1}\|u\|_{k+l+3,\infty,\tau_j}\|v\|_{0,1,\tau_j},$$
$$|(q_I^l - q, v)_j - (W_2^l, v_x)_j| \lesssim h^{k+l+1}\|u\|_{k+l+3,\infty,\tau_j}\|v\|_{0,1,\tau_j}.$$

The proof is completed. □

4. Superconvergence

In this section, we shall study superconvergence properties of the LDG solution at some special points: nodes, left and right Radau points, and superconvergence for the domain and cell average. We denote by $R_{j,m}^l, R_{j,m}^r, m \in \mathbb{Z}_k$ the k interior left and right Radau points in the interval $\tau_j, j \in \mathbb{Z}_N$, respectively. Namely, $R_{j,m}^l, m \in \mathbb{Z}_k$ are zeros of $L_{j,k+1} + L_{j,k}$ except the point $x = x_{j-\frac{1}{2}}$, and $R_{j,m}^r, m \in \mathbb{Z}_k$ are zeros of $L_{j,k+1} - L_{j,k}$ except the point $x = x_{j+\frac{1}{2}}$.

We begin with a study of the error between the LDG solution (u_h, q_h) and the interpolation function $(u_I^l, q_I^l), 1 \le l \le k$ defined in (3.30) or (3.34).

Theorem 4.1. *Let $u \in W^{k+l+3,\infty}(\Omega), 1 \le l \le k$ be the solution of* (2.1)*, and $u_h, q_h \in V_h$ the solution of* (2.3)*. Let $u_I^l, q_I^l \in V_h$ be defined by* (3.30) *for fluxes* (2.4) *or* (3.34) *for fluxes* (2.5)*. Suppose the initial solution $u_h(\cdot, 0) = u_I^l(\cdot, 0)$. Then for both the periodic and mixed boundary conditions,*

$$\begin{aligned} \|u_I^l - u_h\|_0(t) &\lesssim (1+t)h^{k+l+1}\|u\|_{k+l+3,\infty}, \\ \|q_I^l - q_h\|_0(t) &\lesssim (1+t^{\frac{1}{2}})h^{k+l+1}\|u\|_{k+l+3,\infty}. \end{aligned} \tag{4.1}$$

Proof. Let $\eta_u = u_I^l - u_h, \eta_q = q_I^l - q_h$. Recall the definition of $a^1(\cdot,\cdot;\cdot), a^2(\cdot,\cdot;\cdot)$ and (2.6)-(2.7), we have for all $v, w \in V_h$,

$$\begin{aligned} a^1(\eta_u, \eta_q; v) &= (u_{It}^l - u_t, v) - (W_1^l, v_x), \\ a^2(\eta_u, \eta_q; w) &= (q_I^l - q, w) - (W_2^l, w_x). \end{aligned}$$

By Theorem 3.3, the inequalities (3.37)-(3.38) hold for both the fluxes (2.4) and (2.5), then

$$\begin{aligned} |a^1(\eta_u, \eta_q; v)| &\lesssim h^{k+l+1}\|u\|_{k+l+3,\infty}\|v\|_{0,1}, \\ |a^2(\eta_u, \eta_q; w)| &\lesssim h^{k+l+1}\|u\|_{k+l+3,\infty}\|w\|_{0,1}. \end{aligned}$$

We now show (4.1). We first consider the periodic boundary condition. Since

$$\begin{aligned} (u_I^l - u_h)^-_{N+\frac{1}{2}} &= (u_I^l - u_h)^-_{\frac{1}{2}}, \quad (u_I^l - u_h)^+_{N+\frac{1}{2}} = (u_I^l - u_h)^+_{\frac{1}{2}}, \\ (q_I^l - q_h)^+_{N+\frac{1}{2}} &= (q_I^l - q_h)^+_{\frac{1}{2}}, \quad (q_I^l - q_h)^-_{N+\frac{1}{2}} = (q_I^l - q_h)^-_{\frac{1}{2}}, \end{aligned}$$

by choosing $v = \eta_u, w = \eta_q$ in (2.8) for fluxes (2.4), or in (2.9) for fluxes(2.5), we obtain for both fluxes choice

$$\begin{aligned} |(\eta_{ut}, \eta_u) + (\eta_q, \eta_q)| &= |a^1(\eta_u, \eta_q; \eta_u) + a^2(\eta_u, \eta_q; \eta_q)| \\ &\lesssim h^{k+l+1}\|u\|_{k+l+3,\infty}(\|\eta_u\|_{0,1} + \|\eta_q\|_{0,1}). \end{aligned}$$

By Cauchy-Schwarz inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|\eta_u\|_0^2 = (\eta_{ut}, \eta_u) \lesssim h^{k+l+1}\|u\|_{k+l+3,\infty}(\|\eta_u\|_0 + h^{k+l+1}\|u\|_{k+l+3,\infty}). \tag{4.2}$$

Due to the special choice of initial condition, we have $\|\eta_u\|_0(0) = 0$, which yields

$$\|\eta_u\|_0^2(t) = \int_0^t \frac{d}{dt}\|\eta_u\|_0^2 dt \lesssim th^{k+l+1}\|u\|_{k+l+3,\infty}(\|\eta_u\|_0(t) + h^{k+l+1}\|u\|_{k+l+3,\infty}).$$

Then the first inequality of (4.1) follows from a direct calculation. Note that

$$\|\eta_q\|_0^2 \lesssim h^{k+l+1}\|u\|_{k+l+3,\infty}\|\eta_u\|_0 + h^{k+l+1}\|u\|_{k+l+3,\infty}\|\eta_q\|_0,$$

we obtain

$$\|\eta_q\|_0 \lesssim (1+t^{\frac{1}{2}})h^{k+l+1}\|u\|_{k+l+3,\infty}.$$

This finishes the second inequality of (4.1) for the periodic boundary condition.

Now we consider the mixed boundary condition. Noticing that

$$(u_I^l - u_h)^-_{\frac{1}{2}} = 0, \quad (q_I^l - q_h)^+_{N+\frac{1}{2}} = 0$$

for the condition $u(0,t) = g_0(t), u_x(2\pi, t) = g_1(t)$ and

$$(q_I^l - q_h)^-_{\frac{1}{2}} = 0, \quad (u_I^l - u_h)^+_{N+\frac{1}{2}} = 0$$

for the condition $u_x(0,t) = g_0(t), u(2\pi,t) = g_1(t)$, by choosing $v = \eta_u, w = \eta_q$ in (2.8) and (2.9), respectively, we derive in both cases

$$(\eta_{ut}, \eta_u) + (\eta_q, \eta_q) = a^1(\eta_u, \eta_q; \eta_u) + a^2(\eta_u, \eta_q; \eta_q).$$

Following the same line as in the periodic case, we obtain (4.1) directly for the mixed boundary condition. □

Remark 4.2. By choosing $l = k$ in Theorem 4.1, the special interpolation function (u_I^k, q_I^k) is super-close to the LDG solution (u_h, q_h), with a superconvergence rate $2k+1$. It is the super-closeness that leads to the $2k+1$ superconvergence rate at nodes as well as the domain average.

As direct consequences of (4.1) and (3.36), we have the following superconvergence results.

Corollary 4.3. *Let $u \in W^{k+5,\infty}(\Omega)$ be the solution of* (2.1)*, and $u_h, q_h \in V_h$, the solution of* (2.3)*. Suppose the initial solution $u_h(\cdot,0) = u_I^l(\cdot,0), l = 2$ with u_I^l defined by* (3.30) *for fluxes* (2.4) *or* (3.34) *for fluxes* (2.5)*. Then for both the periodic and mixed boundary conditions,*

$$\|\xi_u\|_0 \lesssim (1+th)h^{k+2}\|u\|_{k+5,\infty}, \quad \|\xi_q\|_0 \lesssim (1+t^{\frac{1}{2}}h)h^{k+2}\|u\|_{k+5,\infty}, \tag{4.3}$$

where $\xi_u = P_h^- u - u_h, \xi_q = P_h^+ q - q_h$ for fluxes (2.4) and $\xi_u = P_h^+ u - u_h, \xi_q = P_h^- q - q_h$ for fluxes (2.5).

Remark 4.4. In Theorem 4.1 and Corollary 4.3, if we choose the initial solution $u_h(\cdot,0) = u_I^k(\cdot,0)$ instead of $u_h(\cdot,0) = u_I^l(\cdot,0), 1 \le l \le k$, we have

$$\|u_h - u_I^l\|_0(0) = \|u_I^k - u_I^l\|_0(0) \lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}.$$

Following the same line of argument, estimates in (4.1) and (4.3) are still valid.

4.1. **Superconvergence of the numerical fluxes at nodal points.** We are now ready to present our superconvergence results of the numerical fluxes at nodes.

Theorem 4.5. *Let $u \in W^{2k+3,\infty}(\Omega)$ be the solution of* (2.1)*, and u_h, q_h the solution of* (2.3)*. Suppose the initial solution $u_h(\cdot,0) = u_I^k(\cdot,0)$ with $u_I^k(\cdot,0)$ defined by* (3.30) *for fluxes* (2.4)*, or* (3.34) *for fluxes* (2.5)*. Then for both the periodic and mixed boundary conditions,*

$$e_{u,n} \lesssim (1+t)h^{2k+\frac{1}{2}}\|u\|_{2k+3,\infty}, \quad e_{q,n} \lesssim (1+t^{\frac{1}{2}})h^{2k+\frac{1}{2}}\|u\|_{2k+3,\infty}, \tag{4.4}$$

$$\|e_u\|_* \lesssim (1+t)h^{2k+1}\|u\|_{2k+3,\infty}, \quad \|e_q\|_* \lesssim (1+t^{\frac{1}{2}})h^{2k+1}\|u\|_{2k+3,\infty}, \tag{4.5}$$

where

$$e_{u,n} = \max_{j\in\mathbb{Z}_{N+1}} \left|(u-\hat{u}_h)(x_{j-\frac{1}{2}},t)\right|, \quad \|e_u\|_* = \left(\frac{1}{N+1}\sum_{j=1}^{N+1}\left(u-\hat{u}_h\right)^2(x_{j-\frac{1}{2}},t)\right)^{\frac{1}{2}},$$

$$e_{q,n} = \max_{j\in\mathbb{Z}_{N+1}} \left|(q-\hat{q}_h)(x_{j-\frac{1}{2}},t)\right|, \quad \|e_q\|_* = \left(\frac{1}{N+1}\sum_{j=1}^{N+1}\left(q-\hat{q}_h\right)^2(x_{j-\frac{1}{2}},t)\right)^{\frac{1}{2}},$$

with the numerical fluxes $\hat{u}_h, \hat{q}_h$ taken as (2.4) *or* (2.5)*.*

Proof. Let $(u_I, q_I) = (u_I^k, q_I^k)$. By (3.31) and (3.35),

$$u(x_{j-\frac{1}{2}},t) = \hat{u}_I(x_{j-\frac{1}{2}},t), \quad q(x_{j-\frac{1}{2}},t) = \hat{q}_I(x_{j-\frac{1}{2}},t), \quad j \in \mathbb{Z}_{N+1}.$$

For any fixed t, $u_I - u_h \in \mathbb{P}_k$ in each $\tau_j, j \in \mathbb{Z}_N$. Then the inverse inequality holds and thus,

$$\left|(\hat{u}_I - \hat{u}_h)(x_{j+\frac{1}{2}}, t)\right| \leq \|u_I - u_h\|_{0,\infty,\Omega_j}(t) \lesssim h^{-\frac{1}{2}}\|u_I - u_h\|_{0,\Omega_j}(t), \tag{4.6}$$

$$\left|(\hat{q}_I - \hat{q}_h)(x_{j+\frac{1}{2}}, t)\right| \leq \|q_I - q_h\|_{0,\infty,\Omega_j}(t) \lesssim h^{-\frac{1}{2}}\|q_I - q_h\|_{0,\Omega_j}(t). \tag{4.7}$$

Here $\Omega_j = \tau_j \cup \tau_{j+1}, j \in \mathbb{Z}_{N-1}$ and $\Omega_j = \tau_1 \cup \tau_N, j = 0, N$. By (4.1), the desired result (4.4) follows.

We next show (4.5). Again by the inverse inequality,

$$\frac{1}{N}\sum_{j=1}^{N}\|v\|^2_{0,\infty,\tau_j} \lesssim \frac{1}{N}\sum_{j=1}^{N} h_j^{-1}\|v\|^2_{0,\tau_j} \lesssim \|v\|_0^2, \quad \forall v \in V_h.$$

Then

$$\begin{aligned}
\frac{1}{N+1}\sum_{j=1}^{N+1}\left(\hat{u}_I - \hat{u}_h\right)^2\left(x_{j-\frac{1}{2}}, t\right) &\lesssim \|u_I - u_h\|_0^2(t),\\
\frac{1}{N+1}\sum_{j=1}^{N+1}\left(\hat{q}_I - \hat{q}_h\right)^2\left(x_{j-\frac{1}{2}}, t\right) &\lesssim \|q_I - q_h\|_0^2(t).
\end{aligned}$$

The inequality (4.5) follows directly from the estimate (4.1). □

4.2. **Superconvegence for the domain and cell averages.** We first denote by $\|e_u\|_d$ and $\|e_u\|_c$ the domain average and the cell average of $u - u_h$, respectively. Precisely,

$$\|e_u\|_d = \left|\frac{1}{2\pi}\int_0^{2\pi}(u - u_h)(x,t)dx\right|,$$

$$\|e_u\|_c = \left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}(u - u_h)(x,t)dx\right)^2\right)^{\frac{1}{2}}.$$

Similarly, the domain average $\|e_q\|_d$ and the cell average $\|e_q\|_c$ of $q - q_h$ can be defined in the same way.

We have the following superconvergence results for the domain and cell averages.

Theorem 4.6. *Suppose all the conditions of Theorem* 4.5 *hold. Then*

$$\|e_u\|_c \lesssim (h + t^{\frac{3}{2}} + t)h^{2k}\|u\|_{2k+3,\infty}, \quad \|e_q\|_c \lesssim (1+t)h^{2k}\|u\|_{2k+3,\infty}. \tag{4.8}$$

In addition, for the periodic boundary condition, there holds

$$\|e_u\|_d \lesssim h^{2k+1}\|u\|_{2k+3,\infty}, \quad \|e_q\|_d = 0, \tag{4.9}$$

and for the mixed boundary condition

$$\|e_u\|_d \lesssim (h^{\frac{1}{2}} + t^{\frac{3}{2}} + t)h^{2k+\frac{1}{2}}\|u\|_{2k+3,\infty}, \quad \|e_q\|_d \lesssim (1+t)h^{2k+\frac{1}{2}}\|u\|_{2k+3,\infty}. \tag{4.10}$$

Proof. Note that $a_j^i(u-u_h, q-q_h; v) = 0, \forall v \in V_h, i = 1,2, j \in \mathbb{Z}_N$. By taking $v = 1$, we obtain

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (q-q_h)(x,t)dx = (u-\hat{u}_h)(x_{j+\frac{1}{2}},t) - (u-\hat{u}_h)(x_{j-\frac{1}{2}},t),$$
$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u-u_h)_t(x,t)dx = (q-\hat{q}_h)(x_{j+\frac{1}{2}},t) - (q-\hat{q}_h)(x_{j-\frac{1}{2}},t).$$

In light of (4.6)-(4.7),

$$\left|\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (q-q_h)(x,t)dx\right| \lesssim h^{-\frac{1}{2}}\|u_I^k - u_h\|_{0,\Omega_j},$$
$$\left|\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u-u_h)_t(x,t)dx\right| \lesssim h^{-\frac{1}{2}}\|q_I^k - q_h\|_{0,\Omega_j}.$$

Then

$$\|e_q\|_c \lesssim \left(\frac{1}{N}\sum_{j=1}^{N} h^{-3}\|u_I^k - u_h\|_{0,\Omega_j}^2\right)^{\frac{1}{2}} \lesssim h^{-1}\|u_I^k - u_h\|_0.$$

The second inequality of (4.8) follows directly from the estimate (4.1). On the other hand, since

$$\begin{aligned}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u-u_h)(x,t)dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u-u_h)(x,0)dx + \int_0^t \frac{d}{dt}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u-u_h)(x,t)dxdt \\ &\lesssim \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u-u_h)(x,0)dx + th^{-\frac{1}{2}}\|q_I^k - q_I\|_{0,\Omega_j},\end{aligned}$$

then the estimate for the cell average of $u-u_h$ at $\tau_j, j \in \mathbb{Z}_N$ at any time $t > 0$ is reduced to the estimate at $t = 0$. By the special initial condition,

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u-u_h)(x,0)dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u-u_I^k)(x,0)dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_2^k(x,0)dx,$$

where W_2^k is defined by (3.22) for fluxes (2.4) or (3.33) for fluxes (2.5). When W_2^k is defined by (3.22), we have, from (3.8)-(3.9), (3.15)-(3.16) and the orthogonal properties of Legendre polynomials,

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_2^k(x,t)dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{w}_{1,r}(x,t)dx, \quad k = 2r,$$
$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_2^k(x,t)dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w_{2,r+1}(x,t)dx, \quad k = 2r+1.$$

Recall the estimates for $\bar{w}_{1,r}$ and $\bar{w}_{2,r+1}$ in (3.39)-(3.40), we obtain for all $k \geq 1$,

$$\left|\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_2^k(x,t)dx\right| \lesssim h^{2k+2}\|u\|_{2k+2,\infty,\tau_j}, \quad \forall j \in \mathbb{Z}_N.$$

Similarly, when W_2^k is defined by (3.33), the above inequality still holds true. Then,

$$\left|\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}(u-u_h)(x,0)dx\right| \lesssim h^{2k+2}\|u\|_{2k+2,\infty,\tau_j}, \tag{4.11}$$

which yields

$$\left|\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}(u-u_h)(x,t)dx\right| \lesssim h^{2k+2}\|u\|_{2k+2,\infty,\tau_j} + th^{-\frac{1}{2}}\|q_I^k - q_I\|_{0,\Omega_j}.$$

Then a direct calculation and the estimate (4.1) yield the first inequality of (4.8).

Now we move on to the domain average. Noticing that

$$\begin{aligned}\int_0^{2\pi}(q-q_h)(x,t)dx &= (u-\hat{u}_h)(x_{N+\frac{1}{2}},t) - (u-\hat{u}_h)(x_{\frac{1}{2}},t),\\ \int_0^{2\pi}(u-u_h)_t(x,t)dx &= (q-\hat{q}_h)(x_{N+\frac{1}{2}},t) - (q-\hat{q}_h)(x_{\frac{1}{2}},t),\end{aligned}$$

the second inequalities of (4.9) and (4.10) follow from the fact $(u-\hat{u}_h)(x_{N+\frac{1}{2}},t) = (u-\hat{u}_h)(x_{\frac{1}{2}},t)$ for the periodic boundary condition and (4.4) for the mixed boundary condition, respectively. As for the domain average of $u-u_h$, by (4.4), the fact that $(q-\hat{q}_h)(x_{N+\frac{1}{2}},t) = (q-\hat{q}_h)(x_{\frac{1}{2}},t)$ for the periodic boundary condition, we obtain for the periodic boundary condition

$$\int_0^{2\pi}(u-u_h)(x,t)dx = \int_0^{2\pi}(u-u_h)(x,0)dx,$$

and for the mixed boundary condition

$$\left|\int_0^{2\pi}(u-u_h)(x,t)dx\right| \lesssim \left|\int_0^{2\pi}(u-u_h)(x,0)dx\right| + (t^{\frac{3}{2}}+t)h^{2k+\frac{1}{2}}\|u\|_{2k+3,\infty}.$$

In light of (4.11), the first inequalities of (4.9) and (4.10) follow. □

4.3. **Superconvergence of the function value approximation at Radau points.** As a by-product of (4.1), we have the following superconvergence results of the function value approximation at Radau points.

Theorem 4.7. *Suppose all the conditions of Corollary* 4.3 *hold. For both the periodic and mixed boundary conditions, there hold,*

$$e_{u,r} \lesssim (1+t\sqrt{h})h^{k+2}\|u\|_{k+5,\infty}, \quad e_{q,l} \lesssim (1+\sqrt{th})h^{k+2}\|u\|_{k+5,\infty} \tag{4.12}$$

for fluxes (2.4) *and*

$$e_{u,l} \lesssim (1+t\sqrt{h})h^{k+2}\|u\|_{k+5,\infty}, \quad e_{q,r} \lesssim (1+\sqrt{th})h^{k+2}\|u\|_{k+5,\infty} \tag{4.13}$$

for fluxes (2.5). *Here*

$$\begin{aligned}e_{u,r} &= \max_{(j,m)\in\mathbb{Z}_N\times\mathbb{Z}_k}\left|(u-u_h)(R_{j,m}^r,t)\right|, \quad e_{u,l} = \max_{(j,m)\in\mathbb{Z}_N\times\mathbb{Z}_k}\left|(u-u_h)(R_{j,m}^l,t)\right|,\\ e_{q,r} &= \max_{(j,m)\in\mathbb{Z}_N\times\mathbb{Z}_k}\left|(q-q_h)(R_{j,m}^r,t)\right|, \quad e_{q,l} = \max_{(j,m)\in\mathbb{Z}_N\times\mathbb{Z}_k}\left|(q-q_h)(R_{j,m}^l,t)\right|.\end{aligned}$$

Proof. We first consider (4.12). By using the inverse inequality and choosing $l = 2$ in (4.1), we obtain

$$\|u_I^2 - u_h\|_{0,\infty} \lesssim h^{-\frac{1}{2}}\|u_I^2 - u_h\|_0 \lesssim (1+t)h^{k+\frac{5}{2}}\|u\|_{k+5,\infty},$$
$$\|q_I^2 - q_h\|_{0,\infty} \lesssim h^{-\frac{1}{2}}\|q_I^2 - q_h\|_0 \lesssim (1+t^{\frac{1}{2}})h^{k+\frac{5}{2}}\|u\|_{k+5,\infty}.$$

By (3.36) and the triangular inequality,

$$\|u_h - P_h^- u\|_{0,\infty} \lesssim \|W_2^2\|_{0,\infty} + \|u_I^2 - u_h\|_{0,\infty} \lesssim (1+t\sqrt{h})h^{k+2}\|u\|_{k+5,\infty}, \tag{4.14}$$
$$\|q_h - P_h^+ q\|_{0,\infty} \lesssim \|W_1^2\|_{0,\infty} + \|q_I^2 - q_h\|_{0,\infty} \lesssim (1+\sqrt{th})h^{k+2}\|u\|_{k+5,\infty}. \tag{4.15}$$

For all $v \in W^{k+2,\infty}(\Omega)$, the standard approximation theory gives

$$\left|(v - P_h^- v)(R_{j,m}^r, t)\right| \lesssim h^{k+2}\|v\|_{k+2,\infty}, \quad \left|(v - P_h^+ v)(R_{j,m}^l, t)\right| \lesssim h^{k+2}\|v\|_{k+2,\infty}.$$

Then (4.12) follows. The proof of (4.13) can be obtained by the same arguments. □

4.4. **Superconvergence of the derivative approximation at Radau points.** For all $v \in W^{k+2,\infty}(\Omega)$, it is shown in [5] that

$$\left|\partial_x(v - P_h^- v)(R_{j,m}^l, t)\right| \lesssim h^{k+1}\|v\|_{k+2,\infty}, \quad \forall (j,m) \in \mathbb{Z}_N \times \mathbb{Z}_k. \tag{4.16}$$

Similarly, we can obtain

$$\left|\partial_x(v - P_h^+ v)(R_{j,m}^r, t)\right| \lesssim h^{k+1}\|v\|_{k+2,\infty}, \quad \forall (j,m) \in \mathbb{Z}_N \times \mathbb{Z}_k. \tag{4.17}$$

We have the following superconvergence results.

Theorem 4.8. *Suppose all the conditions of Corollary* 4.3 *hold. Let*

$$e_{ux,l} = \max_{j,m}\left|\partial_x(u - u_h)(R_{j,m}^l, t)\right|, \quad e_{ux,r} = \max_{j,m}\left|\partial_x(u - u_h)(R_{j,m}^r, t)\right|,$$
$$e_{qx,l} = \max_{j,m}\left|\partial_x(q - q_h)(R_{j,m}^l, t)\right|, \quad e_{qx,r} = \max_{j,m}\left|\partial_x(q - q_h)(R_{j,m}^r, t)\right|.$$

For both the periodic and mixed boundary conditions, there hold,

$$e_{ux,l} \lesssim (1+t\sqrt{h})h^{k+1}\|u\|_{k+5,\infty}, \quad e_{qx,r} \lesssim (1+\sqrt{th})h^{k+1}\|u\|_{k+5,\infty} \tag{4.18}$$

for fluxes (2.4) *and*

$$e_{ux,r} \lesssim (1+t\sqrt{h})h^{k+1}\|u\|_{k+5,\infty}, \quad e_{qx,l} \lesssim (1+\sqrt{th})h^{k+1}\|u\|_{k+5,\infty} \tag{4.19}$$

for fluxes (2.5).

Proof. Using the inverse inequality in (4.14)-(4.15) gives

$$|P_h^- u - u_h|_{1,\infty} \lesssim (1+t\sqrt{h})h^{k+1}\|u\|_{k+5,\infty},$$
$$|P_h^+ q - q_h|_{1,\infty} \lesssim (1+\sqrt{th})h^{k+1}\|u\|_{k+5,\infty}.$$

Then the desired result (4.18) follows from (4.16)-(4.17) and the triangular inequality. The proof of (4.19) follows the same line. □

To end this section, we would like to demonstrate how to calculate $u_I^l(x,0), 1 \le l \le k$ using only the information of the initial value $u_0(x)$. Without loss of generality, we consider the fluxes choice (2.4). Since $u_t = u_{xx}$, we have for all integers $i \ge 1$,

$$\partial_t^i u(x,0) = D_x^{2i} u_0(x), \quad \forall x \in \Omega.$$

Therefore, by (3.1)-(3.2), we have the derivatives at $t = 0$,

$$D_t^i \bar{u}_{j,k} = -D_x^{2i} u_0(x^-_{j+\frac{1}{2}}) + \frac{1}{h_j}\int_{\tau_j} D_x^{2i} u_0 \sum_{m=0}^{k}(2m+1)L_{j,m} \tag{4.20}$$

and

$$\begin{aligned} D_t^i \tilde{q}_{j,k} = {} & (-1)^{k+1} D_x^{2i+1} u_0(x^+_{j-\frac{1}{2}}) \\ & + \frac{1}{h_j}\int_{\tau_j} D_x^{2i+1} u_0 \sum_{m=0}^{k}(-1)^{k+m}(2m+1)L_{j,m}. \end{aligned} \tag{4.21}$$

Now we divide the process into the following steps:

1. In each element τ_j, calculate $G_i = D_t^i \bar{u}_{j,k}, Q_i = D_t^i \tilde{q}_{j,k}$ by (4.20)-(4.21).
2. Compute $F_{1,i}, F_{2,i}$ from (3.4) and (3.5).
3. Calculate $\bar{F}_{1,i}$ by $F_{1,i}$ and (3.12).
4. Choose $w^l = \sum_{i=1}^{\lfloor l/2 \rfloor} \bar{w}_{1,i} + \sum_{i=1}^{\lceil l/2 \rceil} w_{2,i}$ with $\bar{w}_{1,i} = (\bar{h}_j)^{2i} G_i \bar{F}_{1,i}, w_{2,i} = (\bar{h}_j)^{2i-1} Q_{i-1} F_{2,i}$.
5. Figure out $u_I^l = P_h^- u_0 - w^l$.

5. Numerical results

In this section, we present numerical examples to verify our theoretical findings. We shall measure various errors, including ξ_u, ξ_q, the numerical fluxes at nodes, interior left and right Radau points, and the domain and cell averages, which are defined in Corollary 4.3 and Theorems 4.5–4.8, respectively.

Example 1. We consider the following problem:

$$\begin{aligned} & u_t = u_{xx}, && (x,t) \in [0, 2\pi] \times (0, 1], \\ & u(x, 0) = \sin(x), && x \in [0, 2\pi] \end{aligned}$$

with periodic boundary condition $u(0, t) = u(2\pi, t)$. The exact solution is

$$u(x, t) = e^{-t}\sin(x).$$

We solve this problem by the LDG scheme (2.3) with $k = 3, 4$, respectively. The numerical fluxes are chosen as (2.4), and the initial solution $u_h(x, 0) = u_I^k(x, 0)$ with u_I^k defined by (3.30). We construct our meshes by equally dividing each interval, $[0, \frac{3\pi}{4}]$ and $[\frac{3\pi}{4}, 2\pi]$, into $N/2$ subintervals, $N = 2^m$, $m = 2, 3, \ldots, 7$. To reduce the time discretization error, we use the ninth order strong-stability preserving (SSP) Runge-Kutta method [11] with time step $\triangle t = 0.01h_{min}^2$ in $k = 3$ and $\triangle t = 0.001h_{min}^2$ in $k = 4$, where $h_{min} = 3\pi/2N$.

Numerical data are demonstrated in Tables 5.1–5.2, and corresponding error curves are depicted in Figures 5.1–5.2 on the log-log scale.

We observe from Figures 5.1–5.2 a convergence rate $k + 2$ for $\|\xi_u\|_0, \|\xi_q\|_0$, $e_{u,r}, e_{q,l}$, $e_{ux,l}$ and $e_{qx,r}$, and $2k + 1$ for $\|e_u\|_*, \|e_q\|_*$ and $e_{u,n}, e_{q,n}$. These results confirm our theoretical findings in Corollary 4.3, Theorem 4.5, and Theorems 4.7–4.8 : for fluxes choice (2.4), the LDG solution (u_h, q_h) is $k + 2$th order superconvergent to the Gauss-Radau projection of the exact solution $(P_h^- u, P_h^+ q)$; the function value error $u - u_h$ at right Radau points and its derivative error $\partial_x(u - u_h)$ at interior left Radau points, and $q - q_h$ at left Radau points and $\partial_x(q - q_h)$ at interior right Radau points, all converge with the same rate $k + 2$; the maximum

TABLE 5.1. Various errors in the periodic boundary condition for $k = 3$.

N	$\|\xi_u\|_0$	$e_{u,r}$	$e_{ux,l}$	$e_{u,n}$	$\|e_u\|_*$	$\|e_u\|_c$	$\|e_u\|_d$
4	5.06e-04	2.82e-04	1.72e-04	1.04e-04	5.92e-05	7.57e-05	3.12e-04
8	1.29e-05	5.13e-06	4.14e-06	7.19e-07	4.17e-07	5.11e-07	2.00e-06
16	3.92e-07	1.35e-07	1.25e-07	5.65e-09	3.16e-09	3.89e-09	1.44e-08
32	1.22e-08	3.98e-09	3.92e-09	4.37e-11	2.44e-11	3.01e-11	1.08e-10
64	3.80e-10	1.24e-10	1.23e-10	3.40e-13	1.89e-13	2.34e-13	8.30e-13
128	1.19e-11	3.85e-12	3.85e-12	2.65e-15	1.47e-15	1.83e-15	6.43e-15
N	$\|\xi_q\|_0$	$e_{q,l}$	$e_{qx,r}$	$e_{q,n}$	$\|e_q\|_*$	$\|e_q\|_c$	$\|e_q\|_d$
4	4.01e-04	1.72e-04	2.41e-04	1.12e-04	6.67e-05	3.74e-05	1.84e-13
8	1.11e-05	4.14e-06	4.76e-06	6.86e-07	4.12e-07	3.18e-07	4.05e-17
16	3.44e-07	1.25e-07	1.32e-07	5.24e-09	3.17e-09	2.56e-09	9.64e-21
32	1.07e-08	3.92e-09	3.98e-09	4.14e-11	2.47e-11	2.02e-11	2.34e-24
64	3.35e-10	1.23e-10	1.24e-10	3.24e-13	1.93e-13	1.59e-13	5.70e-28
128	1.05e-11	3.85e-12	3.85e-12	2.54e-15	1.51e-15	1.24e-15	1.39e-31

TABLE 5.2. Various errors in the periodic boundary condition for $k = 4$

N	$\|\xi_u\|_0$	$e_{u,r}$	$e_{ux,l}$	$e_{u,n}$	$\|e_u\|_*$	$\|e_u\|_c$	$\|e_u\|_d$
4	2.34e-05	8.40e-06	1.06e-05	1.47e-06	7.96e-07	1.04e-06	2.94e-06
8	3.92e-07	1.32e-07	1.28e-07	2.62e-09	1.47e-09	1.74e-09	5.96e-09
16	6.21e-09	2.02e-09	2.05e-09	5.06e-12	2.85e-12	3.37e-12	1.21e-11
32	9.73e-11	3.18e-11	3.18e-11	1.01e-14	5.59e-15	6.60e-15	2.41e-14
64	1.52e-12	4.97e-13	4.97e-13	1.97e-17	1.09e-17	1.29e-17	4.77e-17
128	2.38e-14	7.76e-15	7.76e-15	3.85e-20	2.14e-20	2.53e-20	9.37e-20
N	$\|\xi_q\|_0$	$e_{q,l}$	$e_{qx,r}$	$e_{q,n}$	$\|e_q\|_*$	$\|e_q\|_c$	$\|e_q\|_d$
4	3.06e-05	1.07e-05	9.30e-06	1.92e-06	1.14e-06	6.52e-07	1.86e-13
8	4.56e-07	1.28e-07	1.33e-07	2.41e-09	1.50e-09	1.23e-09	4.06e-17
16	7.09e-09	2.05e-09	2.03e-09	4.44e-12	2.78e-12	2.37e-12	9.64e-21
32	1.11e-10	3.18e-11	3.18e-11	8.55e-15	5.36e-15	4.60e-15	2.34e-24
64	1.73e-12	4.97e-13	4.96e-13	1.66e-17	1.04e-17	8.97e-18	5.70e-28
128	2.70e-14	7.76e-15	7.76e-15	3.25e-20	2.03e-20	1.75e-20	1.39e-31

and average errors of $u - u_h$ and $q - q_h$ are supercovergent at downwind points and upwind points, respectively, with the same rate $2k + 1$. Moreover, our numerical results demonstrate that the superconvergence rates in (4.3), (4.5) and (4.12) are optimal; while the convergence rate for the derivative approximation at Radau points is one order better than the estimate provided in (4.18).

For the domain and cell averages, we first observe, from Tables 5.1–5.2, that the error for the domain average of $q - q_h$ reaches the machine precision at the initial mesh, which indicates the equality in (4.9) is true. Then from Figures 5.1–5.2, we observe a $2k + 1$th superconvergence rate for the domain average of $u - u_h$, as predicted in (4.9). Furthermore, we also observe $2k + 1$th superconvergence rates for the cell average of $u - u_h$ and $q - q_h$, one order higher than the one given in (4.8).

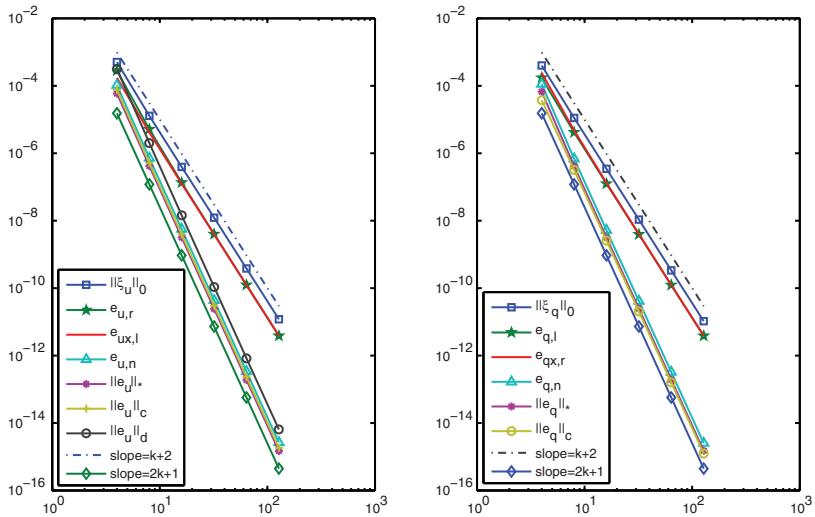

FIGURE 5.1. Error curves in the periodic boundary condition for $k = 3$.

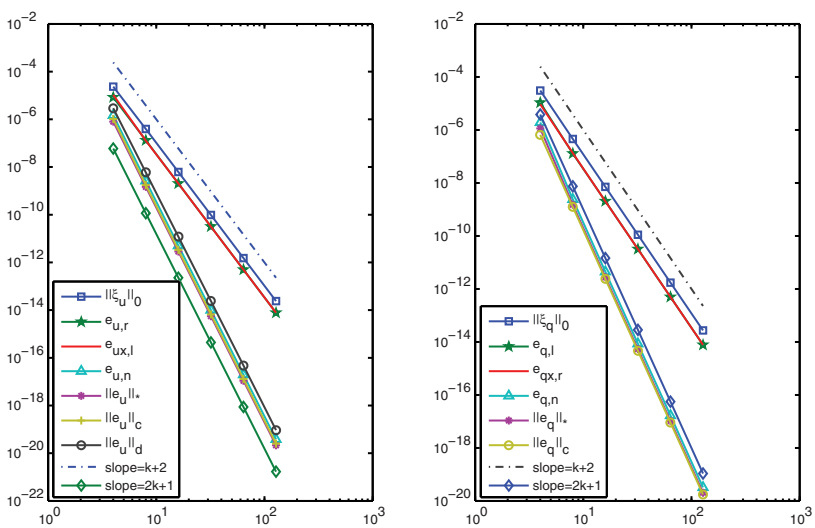

FIGURE 5.2. Error curves in the periodic boundary condition for $k = 4$

Example 2. We consider the following problem

$$u_t = u_{xx}, \qquad (x,t) \in [0, 2\pi] \times (0,1],$$
$$u(x,0) = \cos(x) + e^{x+1}$$

with mixed boundary condition

$$u_x(0,t) = e^{t+1}, \quad u(2\pi, t) = e^{-t} + e^{2\pi+t+1}.$$

The exact solution to this problem is

$$u(x,t) = e^{-t}\cos(x) + e^{x+t+1}.$$

The problem is solved by the LDG scheme (2.3) with $k = 3, 4$, respectively. The numerical fluxes are chosen as (2.5), and the initial solution $u_h(x,0) = u_I^k(x,0)$ with u_I^k defined by (3.34). Uniform meshes are used, which are constructed by dividing the interval $[0, 2\pi]$ into $N = 2^m$ $(m = 2, 3, \ldots, 6)$ equal subintervals. The fourth order Runge-Kutta method is used to diminish the time discretization error with time step $\triangle t = T/n$ using $n = 1000N^2$ in $k = 3$, and $n = 5000N^2$ in $k = 4$.

Listed in Tables 5.3–5.4 are numerical data for various errors in cases $k = 3, 4$. Depicted in Figures 5.3–5.4 are corresponding error curves with log-log scale.

Again, we observe similar superconvergence phenomena as in the periodic case. To be more precise, if we choose the numerical fluxes (2.5), the LDG solution u_h converges to the Gauss-Radau projection $P_h^+ u$ with a rate of $k+2$, as well as the derivative approximation at all interior right Radau points and the function value approximation at all left Radau points; as for the domain and cell averages, along with the maximum and average errors at upwinding points, the convergent rate is $2k+1$; while for the solution q_h, it is convergent to the Gauss-Radau projection $P_h^- q$ with a rate of $k+2$, the same rate for the derivative approximation at all interior left Radau points and the function value approximation at all right Radau points; finally, convergence rates of the maximum and average errors at downwind points as well as the domain and cell averages are all $2k+1$. These results confirm our theoretical findings in Corollary 4.3 and Theorems 4.5–4.8. Note that the $2k+1$th superconvergence rate for the domain average is 1/2 order higher than the one given in (4.10), and the $k+2$th superconvergence rate for the derivative approximation is one order better than the estimate provided in (4.19).

TABLE 5.3. Various errors in the mixed boundary condition for $k = 3$.

N	$\|\xi_u\|_0$	$e_{u,l}$	$e_{ux,r}$	$e_{u,n}$	$\|e_u\|_*$	$\|e_u\|_c$	$\|e_u\|_d$
4	5.08e-01	2.63e-01	2.33e-01	5.50e-03	3.64e-03	1.46e-02	6.67e-02
8	1.89e-02	1.06e-02	1.05e-02	2.57e-05	1.63e-05	1.36e-04	5.59e-04
16	6.28e-04	3.73e-04	3.85e-04	1.62e-07	9.89e-08	1.13e-06	4.46e-06
32	2.00e-05	1.24e-05	1.29e-05	1.16e-09	7.00e-10	8.99e-09	3.49e-08
64	6.31e-07	4.07e-07	4.19e-07	8.80e-12	5.25e-12	7.05e-11	2.73e-10
N	$\|\xi_q\|_0$	$e_{q,r}$	$e_{qx,l}$	$e_{q,n}$	$\|e_q\|_*$	$\|e_q\|_c$	$\|e_q\|_d$
4	5.99e-01	2.33e-01	2.62e-01	4.48e-02	2.30e-02	2.02e-03	7.30e-04
8	2.07e-02	1.05e-02	1.06e-02	4.68e-04	1.85e-04	8.39e-06	4.90e-06
16	6.57e-04	3.85e-04	3.73e-04	3.95e-06	1.32e-06	5.22e-08	3.62e-08
32	2.05e-05	1.29e-05	1.24e-05	3.14e-08	9.63e-09	3.88e-10	2.76e-10
64	6.38e-07	4.19e-07	4.07e-07	2.47e-10	7.21e-11	3.00e-12	2.14e-12

TABLE 5.4. Various errors in the mixed boundary condition for $k = 4$.

N	$\|\xi_u\|_0$	$e_{u,l}$	$e_{ux,r}$	$e_{u,n}$	$\|e_u\|_*$	$\|e_u\|_c$	$\|e_u\|_d$
4	3.05e-02	1.14e-02	1.12e-02	2.59e-05	1.61e-05	1.03e-04	4.55e-04
8	5.61e-04	2.40e-04	2.47e-04	3.60e-08	2.12e-08	2.55e-07	1.01e-06
16	9.23e-06	4.67e-06	4.72e-06	6.42e-11	3.77e-11	5.36e-10	2.07e-09
32	1.47e-07	8.09e-08	8.14e-08	1.27e-13	7.36e-14	1.07e-12	4.10e-12
64	2.31e-09	1.33e-09	1.34e-09	2.50e-16	1.46e-16	2.10e-15	8.06e-15
N	$\|\xi_q\|_0$	$e_{q,r}$	$e_{qx,l}$	$e_{q,n}$	$\|e_q\|_*$	$\|e_q\|_c$	$\|e_q\|_d$
4	3.46e-02	1.12e-02	1.14e-02	3.64e-04	1.85e-04	8.32e-06	3.43e-06
8	5.99e-04	2.47e-04	2.40e-04	9.08e-07	3.57e-07	1.34e-08	6.60e-09
16	9.54e-06	4.72e-06	4.67e-06	1.89e-09	6.33e-10	2.42e-11	1.30e-11
32	1.49e-07	8.14e-08	8.09e-08	3.75e-12	1.15e-12	4.63e-14	2.58e-14
64	2.33e-09	1.34e-09	1.33e-09	7.64e-15	2.16e-15	8.99e-17	5.07e-17

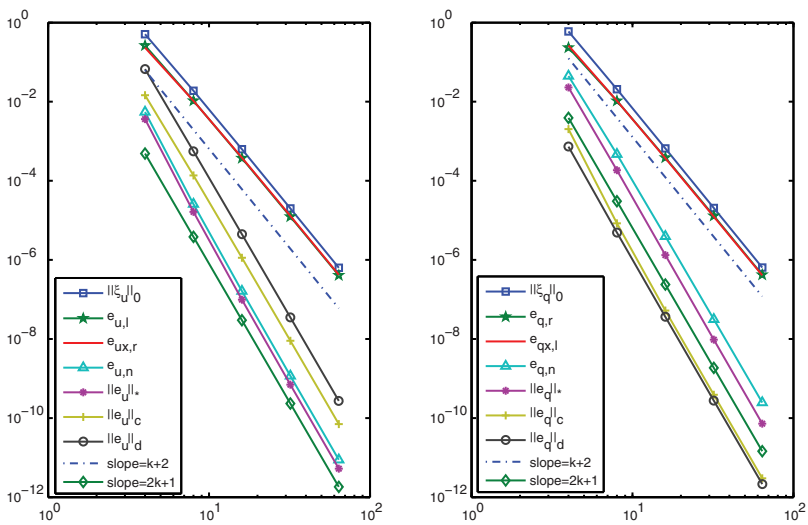

FIGURE 5.3. Error curves in the mixed boundary condition for $k = 3$.

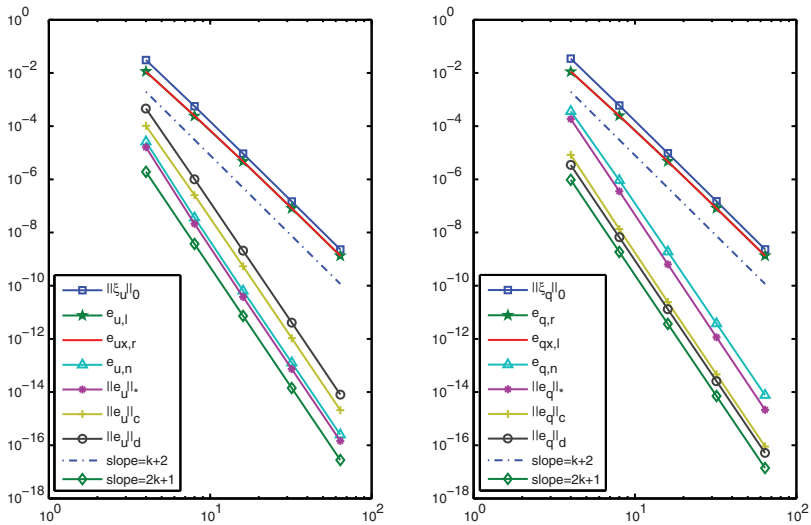

FIGURE 5.4. Error curves in the mixed boundary condition for $k = 4$.

6. Concluding remarks

To summarize, we have established a $2k + 1$th superconvergence rate for the domain average and numerical fluxes at all nodes (on average). As a direct consequence, we obtain a $k+1$th superconvergence rate for the derivative approximation and a $k + 2$th superconvergence rate for the function value approximation of the LDG solution at the Radau points. In addition, we also prove that the LDG solution is superconvergent with a $k + 2$th rate to the Gauss-Radau projection of the exact solution, and a $2k$th rate to the exact solution in the cell average sense. Numerical test data demonstrates that most of our error bounds are sharp, and to the best of our knowledge, the $k + 2$th derivative superconvergence rate at the Radau points is reported for the first time in the literature. Our current and future works include convection-diffusion equations and 2-D problems, which would be more challenging and interesting.

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