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FORMULAS FOR CENTRAL VALUES OF TWISTED SPIN L-FUNCTIONS ATTACHED TO PARAMODULAR FORMS

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WITH AN APPENDIX BY RALF SCHMIDT

Abstract. In the 1980s Böcherer formulated a conjecture relating the central values of the imaginary quadratic twists of the spin L-function attached to a Siegel modular form F to the Fourier coefficients of F. This conjecture has been proved when F is a lift. More recently, we formulated an analogous conjecture for paramodular forms F of prime level, even weight and in the plus-space. In this paper, we examine generalizations of this conjecture. In particular, our new formulations relax the conditions on F and allow for twists by real characters. Moreover, these formulations are more explicit than the earlier ones. We prove the conjecture in the case of lifts and provide numerical evidence in the case of nonlifts.

1. Introduction

Many problems in number theory are related to central values of L-functions associated to modular forms, and central values of twisted L-functions are tools used to make progress on these problems. In this paper we focus our attention on paramodular forms of level N and the spin L-functions associated to them.

In the 1980s a conjecture was formulated by Böcherer [Böc86] that relates the coefficients of a Siegel modular form F of degree 2 and the central values of the spin L-function associated to F. One fixes a discriminant D and, roughly speaking, adds up all the coefficients of F indexed by quadratic forms of discriminant D. One also considers the central value of the spin L-function twisted by the quadratic character χ_D. The conjecture asserts that the central value, up to a constant that depends only on F (and not on D) and a suitable normalization, is the square of the sum of coefficients. In Böcherer's original paper [Böc86] it was proved for F that are Saito-Kurokawa lifts and later Böcherer and Schulze-Pillot [BSP92] proved the conjecture in the case when F is a Yoshida lift. Kohnen and Kuss [KK02] gave numerical evidence in the case when F is of level one, degree 2 and is not a Saito-Kurokawa lift (these computations have recently been extended by Raum [Rau10]). A much more general approach to the conjecture has been pursued by Furusawa, Martin and Shalika [Fur93, FM11, FS99, FS00, FS03].

In [RT11] we investigated an analogous conjecture in the setting of paramodular forms and our goal here is to state some generalizations of the conjecture and to point out some subtleties in the statement of the original conjecture. Paramodular forms have been gaining a great deal of attention because, for example, the most explicit analogue of Taniyama-Shimura for abelian surfaces, known as the

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Paramodular Conjecture, has been formulated by Brumer and Kramer [BK10] and has been verified computationally by Poor and Yuen [PY14a].

While the conjectures we formulate are of theoretical interest, at its heart this particular paper is about computation. Our motivation for formulating these conjectures is computational and the tools we used to formulate them were, as well. A formula like the one we conjecture below is a useful computational tool. Similar formulas have, for example, been used to compute millions of central values of twisted L-functions (see, for example, [HTW10]), by computing millions of coefficients of an associated modular form. These computations can, for example, be used to verify and test the predictions made by random matrix theory regarding the distribution of central values. In ongoing work, we are developing tools to compute millions of coefficients of paramodular forms, work that will allow us to carry out experiments similar to ones just described.

Second, such conjectures are notoriously subtle and require a great deal of care to get them exactly right, especially in the case of composite level. So, in addition to proving that our conjectures are true for a certain class of paramodular forms, we carry out and describe extensive and nontrivial calculations of central values of L-functions associated to hyperelliptic curves. In fact, without carrying out these initial computations it is unlikely that we, nor anyone else, would have discovered these formulas in the explicit form stated below. At the end of the paper we include some of the data that we computed.

We mention now, for future reference, the way in which the examples we used to verify our conjecture were computed. We assumed the Paramodular Conjecture and needed to compute central values of the spin L-function associated to several paramodular forms as well as the Fourier coefficients of the forms. The Fourier coefficients of the paramodular forms whose L-functions correspond to the L-functions of the curves listed in Table 1 were computed by Cris Poor and David Yuen. The paramodular forms of prime level between 277 and 587 are publicly available and computed via the methods of [PY14a]. The other four paramodular forms, of composite level, were computed by Poor and Yuen for us, using an as of yet unpublished method. Data for these forms can be found at [PY14b].

To get started, we summarize the main results in [RT11]. Fix a paramodular eigenform F of level N, assumed to be a newform as defined in Section 4 of [RS06], with Fourier coefficients $a(T;F)$ for each positive semidefinite quadratic form T. One defines

$$A(D) = A_F(D) := \frac{1}{2} \sum_{\{T>0\,:\,\operatorname{disc} T = D\}/\Gamma_0(N)} \frac{a(T;F)}{\varepsilon(T)}$$

where $\varepsilon(T) := \#\{U \in \Gamma_0(N)/\{\pm 1\} \;:\; T[U] = T\}$. We provided evidence for a conjecture, which can be considered a generalization of Waldspurger's theorem [Wal81].

Conjecture A (Paramodular Böcherer's Conjecture)**.** *Let p be prime and let $F \in S_k^{\mathrm{new}}(\Gamma^{para}[p])^+$ be a paramodular Hecke eigenform of even weight k. Then, for fundamental discriminants $D < 0$ we have*

$$\star\, A_F(D)^2 = C_F\, L(F, 1/2, \chi_D)\, |D|^{k-1} \tag{1}$$

where C_F is a nonnegative constant that depends only on F, and where $\star = 1$ when $p \nmid D$ and $\star = 2$ when $p \mid D$. Moreover, when F is a Gritsenko lift we have $C_F > 0$, and when F is not a lift, we have $C_F = 0$ if and only if $L(F, 1/2) = 0$.

This conjecture corrects a defect of the corresponding conjecture in [RT11], the defect being that it might be wrong in the case of nonlifts since the conjecture in [RT11] requires the constant to be positive. It is a theorem in [RT11] that $C_F > 0$ when F is a Gritsenko lift and we know that $C_F > 0$ for all the L-functions we computed that are associated with nonlifts. Nevertheless, the Main Conjecture below supports the claim that $C_F = 0$ when $L(F, 1/2) = 0$.

Remark 1. We make the simple observation that if Conjecture A is true and if F is a nonlift for which $L(F, 1/2) = 0$, then the average $A(D)$ of its Fourier coefficients would be zero for all D. This is a step in characterizing the kinds of forms that might violate the conjecture as stated in [RT11].

Remark 2. In the original version of the conjecture first stated by Böcherer in [Böc86], the constant C_F is required to be positive. In [BSP92], where the analogous conjecture for Siegel modular forms of level N was considered, the constant is allowed to be zero. Our Conjecture A only requires the constant to be nonnegative, and we do characterize exactly when we expect the constant to be zero.

Remark 3. In order to verify Conjecture A in a case where F is a nonlift and $L(F, 1/2) = 0$, one would have to compute the Fourier coefficients of a paramodular form whose L-function vanishes to even order greater than zero. Here we note that if the Paramodular Conjecture holds, there should be such a paramodular form with rational eigenvalues of, for example, levels 3319, 3391, 3571, 4021, 4673, 5113, 5209, 5449, 5501, 5599 since there is a hyperelliptic curve for each of these conductors whose Hasse-Weil L-function vanishes to even order at least two [Sto08]. This last assertion about the order of vanishing was verified by directly computing the central value of these L-functions using `lcalc` [Rub08].

1.1. **Two surprises.** After carrying out the computations used to verify the conjecture in [RT11], we made two observations that we describe now. We asked about what happens if one does not restrict the computations to forms in the plus space. To do this, we first noticed that the two sides of (1) do indeed make sense. We also carried out a simple computation [RT11, Section 4] that shows the averages $A(D)$ add to zero when a form is in the minus space. Undaunted, we carried out the computations and tabulated the following data for F_{587}^- using the Hasse-Weil L-function of the hyperelliptic curve associated to the paramodular form F_{587}^- via the Paramodular Conjecture (see Table 1 for a list of all the examples considered in this paper):

D	-4	-7	-31	-40	-43	-47
L_D/L_{-3}	1.0	1.0	4.0	9.0	144.0	1.0

Here $L_D := L(F_{587}^-, 1/2, \chi_D)\,|D|$ and the table shows fundamental discriminants for which $\left(\frac{D}{587}\right) = -1$. The obvious thing to notice is that the numbers in the table appear to be squares and so the natural question to ask is: squares of what?

This first experiment was a natural extension of our previous work in [RT11] as we had a paramodular form (as can be found at [PY14b]) to compute the right-hand side of (1) and both sides of the equation make sense. Emboldened by the results of the first experiment, we decided to change another hypothesis in the conjecture: we decided to look at the case when $D > 0$. This is somewhat unnatural as the sum $A(D)$ is an empty sum in this case. Nevertheless we get the following data for F_{277}:

D	12	13	21	28	29	40
L_D/L_1	225.0	225.0	225.0	225.0	2025.0	900.0

Here $L_D := L(F_{277}, 1/2, \chi_D)\,|D|$ and $\left(\frac{D}{277}\right) = +1$. Again, these seem to be squares, but squares of what? (Also, the observant reader may have noticed that all these squares are divisible by 15^2. See Section 5 for more about this.)

In Section 2 we will define, for an auxiliary discriminant ℓ, a twisted average $B_\ell(D)$. When ℓ is properly chosen, the squares of $B_\ell(D)$ are exactly the squares we see in the previous two tables.

Given a discriminant D, we put

$$\nu(D) = \nu_N(D) := \#\{\text{primes } p \mid \gcd(N, D)\}\,.$$

Conjecture B. *Let N be squarefree. Suppose $F \in S_k^{\text{new}}(\Gamma^{para}[N])$ is a Hecke eigenform and not a Gritsenko lift. Let ℓ and d be fundamental discriminants such that $\ell d < 0$ and such that ℓd is a square modulo $4N$. Then*

$$B_{\ell,F}(\ell d)^2 = C_{\ell,F}\, 2^{\nu_N(d)}\, L(F, 1/2, \chi_d)\, |d|^{k-1}\,,$$

where $C_{\ell,F}$ is a nonnegative constant independent of d. Moreover, $C_{\ell,F} = 0$ if and only if $L(F, 1/2, \chi_\ell) = 0$.

The notation in this conjecture is further explained in Section 2 but we define some of it now. Fix ρ such that $\rho^2 \equiv \ell d \pmod{4N}$. Then

$$B_{\ell,F}(\ell d) = \left| 2^{\nu_N(\gcd(\ell,d))} \cdot \sum \psi_\ell(T)\, \frac{a(T;F)}{\varepsilon(T)} \right|$$

where the sum is over $\{T = [Nm, r, n] > 0 \ : \ \operatorname{disc} T = \ell d,\ r \equiv \rho \pmod{2N}\}/\Gamma_0(N)$ and where $\psi_\ell(T)$ is the genus character corresponding to $\ell \mid \operatorname{disc} T$. This is independent of the choice of ρ. Essentially, $B_{\ell,F}(\ell d)$ is the same sum as $A_F(\ell d)$, but appropriately twisted by the genus character ψ_ℓ.

Now, the analogy of this conjecture with Conjecture A will be made clear. First, note that $B_\ell(D)$ is a twisted average of the Fourier coefficients of F indexed by quadratic forms of discriminant $D = \ell d$. When k is even and N is prime, in the case $\ell = 1$ we have $D = d$ and $B_1(D) = 2^{\nu(D)}\,|A(D)|$ so we recover Conjecture A with $C_F = C_{1,F}$. Note that χ_1 is the trivial character, hence $L(F, 1/2, \chi_1) = L(F, 1/2)$.

In a later section, we are interested in verifying Conjecture B in the case of nonlifts. To do this, we attempt to understand the constant $C_{\ell,F}$ a little better. In Conjecture B, one can think of the discriminant ℓ as being fixed. In this next conjecture, we think of it as a parameter.

Main Conjecture. *Let N be squarefree. Suppose $F \in S_k^{\text{new}}(\Gamma^{para}[N])$ is a Hecke eigenform and not a Gritsenko lift. Let ℓ and d be fundamental discriminants such that $\ell d < 0$ and such that ℓd is a square modulo $4N$. Then*

$$B_{\ell,F}(\ell d)^2 = k_F \cdot \left\{ 2^{\nu_N(\ell)}\, L(F, 1/2, \chi_\ell)\, |\ell|^{k-1} \right\} \cdot \left\{ 2^{\nu_N(d)}\, L(F, 1/2, \chi_d)\, |d|^{k-1} \right\}$$

for some positive constant k_F independent of ℓ and d.

This gives us a very explicit statement of a conjecture for forms that are not Gritsenko lifts. It is this formula that we verify in Section 3. We observe that the Main Conjecture implies Conjecture B with $C_{\ell,F} = k_F\, 2^{\nu(\ell)}\, L(F, 1/2, \chi_\ell)\, |\ell|^{k-1}$.

We note that when F is a Gritsenko lift the formula of Conjecture B is valid in the case $\ell = 1$ with $C_{\ell,F} > 0$, as shown in Theorem 4.1 below; the formula of the

Main Conjecture is valid provided $\ell \neq 1$ and $d \neq 1$, but uninteresting with both sides being zero for trivial reasons (see Propositions 4.2 and 4.3).

1.2. **Notation.** The main objects of study in this paper are paramodular forms of level N and their L-functions.

Suppose R is a commutative ring with identity. Then, we define the symplectic group $\mathrm{Sp}(4,R) := \{x \in \mathrm{GL}(4,R) \ : \ x' J_2 x = J_2\}$, where the transpose of matrix x is denoted x' and for the $n \times n$ identity matrix I_n we set $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$.

The paramodular group of level N is

$$\Gamma^{\mathrm{para}}[N] := \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{pmatrix} * & * & */N & * \\ N* & * & * & * \\ N* & N* & * & N* \\ N* & * & * & * \end{pmatrix}, \text{ where } * \in \mathbb{Z}.$$

Paramodular forms of degree 2, level N and weight k are modular forms with respect to the group $\Gamma^{\mathrm{para}}[N]$. We denote the space of such modular forms by $M_k(\Gamma^{\mathrm{para}}[N])$ and the space of cuspforms by $S_k(\Gamma^{\mathrm{para}}[N])$. The space $S_k(\Gamma^{\mathrm{para}}[N])$ can be split into a plus space and a minus space according to the action of the canonical involution

$$\mu = \mu_N := \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -N & 0 & 0 & 0 \\ 0 & 0 & 0 & N \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

In particular, $S_k^{\pm}(\Gamma^{\mathrm{para}}[N]) := \{F \in S_k(\Gamma^{\mathrm{para}}[N]) \ : \ F \mid \mu = \pm F\}$.

Every $F \in M_k(\Gamma^{\mathrm{para}}[N])$ has a Fourier expansion of the form

$$F(Z) = \sum_{T=[Nm,r,n] \in \mathcal{Q}_N} a(T;F)\, q^{Nm}\, \zeta^r\, {q'}^n$$

where $q := e^{2\pi i z}$, $q' := e^{2\pi i z'}$ $(z, z' \in \mathcal{H}_1)$, $\zeta := e^{2\pi i \tau}$ $(\tau \in \mathbb{C})$ and

$$\mathcal{Q}_N := \{[Nm,r,n] \geq 0 \ : \ m,r,n \in \mathbb{Z}\};$$

here we use Gauss's notation for binary quadratic forms.

We will want to decompose $\mathcal{Q}_N$ by discriminant $D < 0$ so we also define

$$\mathcal{Q}_{N,D} = \{T \in \mathcal{Q}_N \ : \ \operatorname{disc} T = D\}.$$

This is useful, for example, so that we can write

$$A_F(D) := \frac{1}{2} \sum_{T \in \mathcal{Q}_{N,D}/\Gamma_0(N)} \frac{a(T;F)}{\varepsilon(T)}.$$

For $F \in S_k(\Gamma^{\mathrm{para}}[N])$, we have $a(T[U];F) = a(T;F)$ for every $U \in \Gamma_0(N)$, where $\Gamma_0(N)$ is the congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ with lower left-hand entry congruent to 0 mod N, and $a(T[\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)];F) = (-1)^k a(T;F)$. Moreover, cusp forms are supported on the positive definite matrices in $\mathcal{Q}_N$.

The space of newforms $S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}}[N])$ is defined in Section 4 of [RS06]. Suppose we are given a paramodular form $F \in S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}}[N])$ so that for all Hecke operators $T(n)$, $F|T(n) = \lambda_{F,n} F = \lambda_n F$. Then we can define the spin L-series by the Euler product

$$L(F,s) := \prod_{p \text{ prime}} L_p\big(p^{-s-k+3/2}\big)^{-1},$$

where the local Euler factors are given by

$$L_p(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3} X^3 + p^{4k-6} X^4$$

for $p \nmid N$. The local Euler factors for $p \parallel N$ are given by

$$\begin{aligned} L_p(X) &:= \left(1 - \alpha_p p^{k-3/2} X\right)\left(1 - \alpha_p^{-1} p^{k-3/2} X\right)\left(1 + \epsilon_p\, p^{k-2} X\right) \\ &= 1 - \lambda_p X + (p^{2k-3} - \epsilon_p \lambda_p p^{k-2} - p^{2k-4})X^2 + \epsilon_p p^{3k-5} X^3 \end{aligned}$$

as outlined in the Appendix; observe that here, as there, $|\alpha_p| = 1$. We observe that in both the Euler factors at a good prime p and at a bad prime p, the coefficient λ_p corresponds to the coefficient a_p in the Dirichlet series.

The Paramodular Conjecture [BK10] asserts that the L-function of a paramodular form is the same as the L-function of an associated abelian surface. In all the examples we consider, the paramodular forms have corresponding abelian surfaces isogenous to Jacobians of hyperelliptic curves as listed in Table 1; thus we compute hyperelliptic curve L-functions when we carry out our computations. A table in [Dok04] summarizes the data that we use to write down the functional equation of the L-function of an hyperelliptic curve:

$$L^*(F,s) := \left(\tfrac{\sqrt{N}}{4\pi^2}\right)^s \Gamma(s+1/2)\,\Gamma(s+1/2)\,L(F,s) \tag{2}$$

so that, assuming the Paramodular Conjecture, it is expected that, as outlined in the Appendix,

$$L^*(F,s) = \epsilon\, L^*(F,1-s),$$

when $F \in S_2(\Gamma^{\mathrm{para}}[N])^\epsilon$.

Let D be a fundamental discriminant, and denote by χ_D the unique quadratic character of conductor D. For the spin L-series $L(F,s) = \sum_{n\geq 1} a_n\, n^{-s}$ of a paramodular form F, we define the quadratic twist

$$L(F,s,\chi_D) := \sum_{n\geq 1} \chi_D(n)\, a_n\, n^{-s},$$

which, as described in the Appendix, is expected to have an analytic continuation and satisfy a functional equation. Suppose N is squarefree, let $N_0 = N/\gcd(N,D)$, and define

$$L^*(F,s,\chi_D) := \left(\tfrac{\sqrt{N_0\, D^4}}{4\pi^2}\right)^s \Gamma(s+1/2)\,\Gamma(s+1/2)\,L(F,s,\chi_D)$$

so that

$$L^*(F,s,\chi_D) = \epsilon'\, L^*(F,1-s,\chi_D).$$

The global root number ϵ' of the functional equation for $L(F,s,\chi_D)$ is given in terms of the local root numbers ϵ_p of $L(F,s)$ by the following lemma whose proof can be found in the Appendix. Part (4) of the lemma assumes that F is a *newform*, as defined in Sect. 4 of [RS06]. In the squarefree case, this means that F does not arise from a paramodular form of lower level via one of the level raising operators θ_p or θ'_p defined in Sect. 3 of [RS06]. It is also shown in the Appendix that, in the case of paramodular newforms, the local root numbers are exactly the eigenvalues for the action of the Atkin-Lehner operators μ_p on F (see (3) in the proof of Lemma 2.1, and the references around it, for a description of the action of μ_p on the coefficients of a paramodular form F).

Lemma 1.1. *Let $F \in S_k(\Gamma^{para}[N])$ be a Hecke eigenform, with N squarefree. Denote the local root numbers of, respectively, $L(F,s,\chi_D)$ and $L(F,s)$, as follows: $\epsilon' = \prod_{p\le\infty} \epsilon'_p$, and $\epsilon = \prod_{p\le\infty}\epsilon_p$. Then:*

(1) At the infinite place, $\epsilon'_\infty = \epsilon_\infty = (-1)^k$.
(2) Assume $p \nmid N$, then $\epsilon'_p = \epsilon_p = +1$.
(3) Assume $p \mid N$ and $p \mid D$, then $\epsilon'_p = +1$.
(4) Assume $p \mid N$ and $p \nmid D$, and that F is a newform. Then $\epsilon'_p = \epsilon_p\, \chi_D(p)$.

In particular, if F is a newform,

$$\epsilon' = \epsilon\, \chi_D(N_0) \prod_{p \mid \gcd(D,N)} \epsilon_p,$$

where $N_0 = N/\gcd(N,D)$.

Remark 4. For a cusp form of weight 2 and prime level, the newform hypothesis is automatically satisfied, since there are no cusp forms of weight 2 and full level. All of the nonprime level examples we consider are also conjecturally newforms.

2. Generalizations of the Paramodular Böcherer's Conjecture

In this section, we motivate Conjecture B and the Main Conjecture. We do it by describing what happens for particular F that are not Gritsenko lifts. We place particular emphasis on the transition from the hypotheses in Conjecture A (namely, F in the plus-space, of prime level and even weight) to Conjecture B and to the Main Conjecture, which have no such hypotheses.

2.1. **A simple case.** For $F = F_{249}$, the conjectured unique Hecke eigenform that is not a lift, of weight 2 and level $249 = 3 \cdot 83$, we have $A(D) = 0$ for all D (see Lemma 2.1, below) since its eigenvalues under the Atkin-Lehner operators μ_3 and μ_{83} are $\epsilon_3 = \epsilon_{83} = -1$ (as mentioned above, these eigenvalues can be computed from the Fourier coefficients of F). However, we have

D	-7	-8	-20	-31	-35	-40	-47	-56	-71
L_D/L_{-4}	1.0	1.0	4.0	1.0	16.0	4.0	1.0	4.0	0.0

where $L_D := L(F_{249}, 1/2, \chi_D)\,|D|$ and $\left(\frac{D}{249}\right) = +1$.

In this and the next section we will show where the squares in this table come from. Before we do that, we show that for F_{249} (as well as F^-_{587} and F^-_{713}), we really do get $A(D) = 0$ for all D.

Lemma 2.1. *Let F be a paramodular form of weight k and squarefree level N. Assume F is an eigenform under the Atkin-Lehner operators μ_p for every $p \mid N$, so that $F\,|\,\mu_p = \epsilon_p\, F$. If $\epsilon_p = -1$ for any $p \mid N$ or if k is odd, then $A_F(D) = 0$ for all D.*

Proof. For $N' \parallel N$, one can define an involution $W_{N'}$ over the set $\mathcal{Q}_{N,D}/\Gamma_0(N)$ (see [GKZ87, p. 507]). This involution is related to the Atkin-Lehner operators in the following way:

$$a(W_{p^i}(T); F) = a(T; F\,|\,\mu_p) = \epsilon_p\, a(T;F) \tag{3}$$

where p^i is the largest power of p dividing N. Taking the sum over all classes $T \in \mathcal{Q}_{N,D}/\Gamma_0(N)$ shows that $A(D) = \epsilon_p\, A(D)$, and it follows that $A(D) = 0$ if $\epsilon_p = -1$. The case of odd k is similar using $a(T[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}]; F) = (-1)^k\, a(T;F)$. □

We will show now how to define a more refined average $B(D)$ on the coefficients of F for which Lemma 2.1 does not apply. In order to do that, we further decompose $\mathcal{Q}_{N,D}$ as follows. Note that for any $T = [Nm, r, n] \in \mathcal{Q}_{N,D}$ we have $r^2 \equiv D \pmod{4N}$ and we can thus define

$$R_{N,D} := \left\{\rho \bmod 2N \;:\; \rho^2 \equiv D \pmod{4N}\right\}.$$

For each $\rho \in R_{N,D}$ we set

$$\mathcal{Q}_{N,D,\rho} := \{T = [Nm, r, n] \in \mathcal{Q}_{N,D} \;:\; r \equiv \rho \pmod{2N}\}.$$

We observe that $\mathcal{Q}_{N,D}$ is the disjoint union of $\mathcal{Q}_{N,D,\rho}$ for $\rho \in R_{N,D}$. Now for each $\rho \in R_{N,D}$ we put

$$B(D, \rho) = B_F(D, \rho) := \sum_{T \in \mathcal{Q}_{N,D,\rho}/\Gamma_0(N)} \frac{a(T; F)}{\varepsilon(T)}.$$

Lemma 2.2. *Let F be as in 2.1. We note the following:*

(1) $A_F(D) = \frac{1}{2} \sum_{\rho \in R_{N,D}} B_F(D, \rho)$,
(2) $B_F(D, -\rho) = (-1)^k B_F(D, \rho)$, *and*
(3) $|B_F(D, \rho)|$ *is independent of* ρ.

Proof. The first statement is obvious, and the second follows from the fact that $a(T[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}]; F) = (-1)^k a(T; F)$. The last statement follows by noting that the Atkin-Lehner involutions $W_{N'}$ mentioned in the proof of Lemma 2.1 transitively permute the sets $\mathcal{Q}_{N,D,\rho}$. □

Now we can finally define the new average:

$$B(D) = B_F(D) := |B_F(D, \rho)|.$$

This is well defined by Lemma 2.2. We also note that when k is even and N is prime, we have $B(D) = 2^{\nu(D)} |A(D)|$ since $R_{N,D} = \{\pm\rho\}$ and $B(D, -\rho) = B(D, \rho)$ in this case.

We return to the example F_{249}. We get the following table computing L_D using the L-function of the hyperelliptic curve associated to F_{249} via the Paramodular Conjecture and computing the averages $B(D)$ using the Fourier coefficients of F_{249}:

D	-4	-7	-8	-20	-31	-35	-40	-47	-56	-71
L_D/L_{-4}	1.0	1.0	1.0	4.0	1.0	16.0	4.0	1.0	4.0	0.0
$B(D)/B(-8)$			1	2		4		1	2	0

where again $L_D := L(F_{249}, 1/2, \chi_D)\,|D|$ and $\left(\frac{D}{249}\right) = +1$. When $\left(\frac{D}{3}\right) = \left(\frac{D}{83}\right) = -1$ the definition of $B(D)$ gives an empty sum; this is indicated in the table above with an empty space (for instance, $B(-4) = 0$ for this reason). In the next section we describe where the remaining squares come from.

Remark 5. Note that the value of $B(-71) = 0$ is a nontrivial zero average, predicting the vanishing of the twisted L-function at the center. We will investigate such phenomena in a future paper.

2.2. **A general case.** In the previous section we looked at a form F_{249} of composite level for which the averages $A(D)$ vanish for trivial reasons. We introduced a refined average $B(D)$ that explained some of the data in the tables, but in the case $\left(\frac{D}{3}\right) = \left(\frac{D}{83}\right) = -1$ the sum $B(D)$ is empty, although the central values L_D/L_{-4} are (nonzero) squares.

Consider the form F^-_{587} as described in the introduction. It was shown in [RT11, Section 4] that for the discriminants D so that $\left(\frac{D}{587}\right) = -1$ the sum $A(D)$ was empty and so, in particular, our new sum $B(D)$ is also empty, and cannot explain the fact that its normalized twisted central values are (nonzero) squares.

Also, in the definition of $A(D)$ and of $B(D)$ we require that D be negative, so neither average can make sense of the data in the introduction related to real quadratic twists of the L-functions of F_{277}. In this section, using the genus theory for $\Gamma_0(N)$-classes of quadratic forms, we fully explain these examples by defining another new average $B_\ell(D)$ weighted by a genus character. We define this now.

Let $T \in \mathcal{Q}_{N,D}$. Given a fundamental discriminant $\ell \mid D$ such that $D/\ell \equiv 0, 1 \pmod 4$, the genus character $\psi_\ell(T)$ is defined as usual by

$$\psi_\ell(T) := \begin{cases} \left(\frac{\ell}{x}\right) & \text{if } T \text{ represents an integer } x \text{ relatively prime to } \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Then we define

$$B_\ell(D,\rho) = B_{\ell,F}(D,\rho) := 2^{\nu_N(\gcd(\ell, D/\ell))} \cdot \sum \psi_\ell(T) \, \frac{a(T;F)}{\varepsilon(T)}$$

where the sum is over all $T \in \mathcal{Q}_{N,D,\rho}/\Gamma_0(N)$. As in Lemma 2.2 the absolute value is independent of ρ, hence we can define

$$B_\ell(D) = B_{\ell,F}(D) := |B_{\ell,F}(D,\rho)| \,.$$

We note that $B_1(D) = B(D)$ as defined in the previous section. One can also prove, using quadratic reciprocity, that $B_\ell(D) = B_{D/\ell}(D)$.

Remark 6. If D is not a square modulo $4N$, then $B_\ell(D) = 0$, because $\mathcal{Q}_{N,D}$ is empty in this case. Nevertheless, for any fundamental discriminant d, there exists some ℓ for which $D = \ell d$ is a square modulo $4N$ and $B_\ell(D)$ is not an empty sum.

We will now complete the explanation of the examples we have discussed so far. We start with the form F_{249}. In the previous section we were able to explain the case $\left(\frac{D}{3}\right) = \left(\frac{D}{83}\right) = +1$ with auxiliary discriminant $\ell = 1$ (implicitly). We can explain the other case, $\left(\frac{D}{3}\right) = \left(\frac{D}{83}\right) = -1$, by choosing $\ell = 5$:

D	-4	-7	-8	-20	-31	-35	-40	-47	-56	-71
L_D/L_{-4}	1.0	1.0	1.0	4.0	1.0	16.0	4.0	1.0	4.0	0.0
$B_1(D)/B_1(-8)$			1	2		4		1	2	0
$B_5(5D)/B_5(-20)$	1	1			1		2			

where $L_D := L(F_{249}, 1/2, \chi_D)\,|D|$ and $\left(\frac{D}{249}\right) = +1$. The empty entries correspond to empty sums as noted in the remark above.

In order to explain the case of the form F^-_{587}, in the minus space, we need to use an auxiliary discriminant $\ell > 0$ such that $\left(\frac{\ell}{587}\right) = -1$. Using $\ell = 5$:

D	-4	-7	-31	-40	-43	-47
L_D/L_{-3}	1.0	1.0	4.0	9.0	144.0	1.0
$B_5(5D)/B_5(-15)$	1	1	2	3	12	1

where $L_D := L(F^-_{587}, 1/2, \chi_D)\,|D|$ and the table shows fundamental discriminants for which $\left(\frac{D}{587}\right) = -1$.

TABLE 1. Hyperelliptic curves C used to compute the L-series of the paramodular form F associated to C via the Paramodular Conjecture. Here T denotes the torsion of the abelian surface $\mathrm{Jac}(C)$. The data in every row in the table except the one labelled F^+_{587} can be found in [Sto08] while for the row labelled F^+_{587} the model can be found in [BK10] and the torsion can be computed via MAGMA [BCP97]; we also observe this hyperelliptic curve's Jacobian has good reduction at 3, even though the curve does not.

F	N	C	T
F_{249}	$249 = 3 \cdot 83$	$y^2 + (x^3 + 1)y = x^2 + x$	14
F_{277}	277	$y^2 + y = x^5 - 2x^3 + 2x^2 - x$	15
F_{295}	$295 = 5 \cdot 59$	$y^2 + (x^3 + 1)y = -x^4 - x^3$	14
F_{349}	349	$y^2 + y = -x^5 - 2x^4 - x^3 + x^2 + x$	13
F_{353}	353	$y^2 + (x^3 + x + 1)y = x^2$	11
F_{389}	389	$y^2 + xy = -x^5 - 3x^4 - 4x^3 - 3x^2 - x$	10
F_{461}	461	$y^2 + y = -2x^6 + 3x^5 - 3x^3 + x$	7
F_{523}	523	$y^2 + xy = -x^5 + 4x^4 - 5x^3 + x^2 + x$	10
F^+_{587}	587	$y^2 = -3x^6 + 18x^4 + 6x^3 + 9x^2 - 54x + 57$	11
F^-_{587}	587	$y^2 + (x^3 + x + 1)y = -x^3 - x^2$	1
F^+_{713}	$713 = 23 \cdot 31$	$y^2 + (x^3 + x + 1)y = -x^4$	9
F^-_{713}	$713 = 23 \cdot 31$	$y^2 + (x^3 + x + 1)y = x^5 - x^3$	1

Finally, in order to handle positive discriminants D, we can choose a negative discriminant ℓ. In the example of F_{277} we choose $\ell = -3$:

D	12	13	21	28	29	40
L_D/L_1	225.0	225.0	225.0	225.0	2025.0	900.0
$B_{-3}(-3D)/B_{-3}(-3)$	15	15	15	15	45	30

where $L_D := L(F_{277}, 1/2, \chi_D)\,|D|$ and $\left(\frac{D}{277}\right) = +1$.

3. The case of nonlifts

In the previous section, we highlighted some tables that give evidence for the Main Conjecture. Now we describe how those tables were computed and how the tables in Section 6 were computed.

The Paramodular Conjecture asserts that for each rational Hecke eigenform F that is not a Gritsenko lift, there is an abelian surface $\mathcal{A}$ so that the Hasse-Weil L-function of $\mathcal{A}$ and the spin L-function of F are the same. Suppose we have such an F and such an $\mathcal{A}$. In all our examples, $\mathcal{A}$ is isogenous to the Jacobian of a hyperelliptic curve C. In Table 1 we list the curves we use in our computations and point out that extensive data about these curves can be found at [RT13].

Consider such a C. Then the Euler product of C can be found as in [RT11, Section 3] and the functional equation can be found, for example, in [Dok04], though we give it the analytic normalization. The central values were then computed using Michael Rubinstein's `lcalc` [Rub08].

Using data for Fourier coefficients found at [PY14b], the sum $B_\ell(D)$ is computed using a combination of Sage [Ste11] code and custom-written Python code. In particular, we implemented a class that represents binary quadratic forms modulo

TABLE 2. Summary of forms and discriminants for which we have computed both twisted averages $B_\ell(D)$ and the corresponding twisted central values, and for which the Main Conjecture has been numerically verified. The discriminants for which we computed satisfy $0 > D \geq D_{\min}$, with the following exceptions:
(a) If a discriminant D is in the last column, it means that we did not have all the Fourier coefficients necessary to compute the averages.
(b) In the case of F_{277}, we have the further restriction $|\ell| \leq 500$ and $|d| \leq 500$ due to loss of precision in computing $L(F, 1/2, \chi_\ell)$ and $L(F, 1/2, \chi_d)$.

F	k_F	$D_{\min}$	except these D
F_{249}	0.831968	−295	∅
F_{277}	0.537716	−2435	$\{-2191, -2200, -2212\}$
F_{295}	0.224745	−276	$\{-200, -211, -231, -259\}$
F_{349}	0.381237	−300	$\{-276, -279\}$
F_{353}	0.283530	−300	$\{-267, -295\}$
F_{389}	0.234013	−300	∅
F_{461}	0.168613	−300	$\{-263, -295, -296\}$
F_{523}	0.295058	−200	$\{-199\}$
F_{587}^+	0.478986	−300	∅
F_{587}^-	0.002681	−1108	$\{-927\}$
F_{713}^+	0.422122	−260	∅
F_{713}^-	0.005249	−260	∅

$\Gamma_0(N)$. In Table 2 we summarize the forms and discriminants for which we have computed both twisted averages and twisted central values.

The following theorem summarizes the cases in which the Main Conjecture has been verified.

Theorem 3.1. *Let F be one of the paramodular forms of weight 2 listed in Table 2. Let ℓ and d be fundamental discriminants such that $D = \ell d < 0$ and D is a square modulo $4N$, satisfying the constraints described in the same table. Then*

$$B_{\ell,F}(D)^2 \approx k_F \cdot \left\{2^{\nu_N(\ell)}\, L(F, 1/2, \chi_\ell)\, |\ell|\right\} \cdot \left\{2^{\nu_N(d)}\, L(F, 1/2, \chi_d)\, |d|\right\}$$

numerically, with k_F a positive constant listed in the table.

We point out that the data in Table 2 that we used to prove Theorem 3.1 can be downloaded from [RT13].

4. The case of lifts

A Gritsenko lift F [Gri95] is a paramodular form that comes from a Jacobi form ϕ which in turn corresponds to an elliptic modular form f. The standard reference for Jacobi forms is [EZ85] and we refer the reader to [PY14a] for background on the Gritsenko lift. We will now state and prove a theorem that gives evidence for Conjecture B in the case of lifts.

Theorem 4.1. *Let N be squarefree. Suppose $F \in S_k(\Gamma^{para}[N])$ is a Hecke eigenform and a Gritsenko lift. Let $D < 0$ be a fundamental discriminant such that D is a square modulo $4N$. Then*

$$B_F(D)^2 = C_F\, 2^{\nu_N(D)}\, L(F, 1/2, \chi_D)\, |D|^{k-1}$$

where C_F is a positive constant independent of D.

Let $F = \mathrm{Grit}(\phi)$ where

$$\phi(\tau, z) = \sum_{n \geq 0} \sum_{r^2 \leq 4nN} c(n, r)\, q^n\, \zeta^r$$

is a Jacobi form of weight k and index N. We note [EZ85, Theorem 2.2, p. 23] that $c(n, r)$ depends only on $D = r^2 - 4nN$ and $r \mod 2N$; for each $\rho \in R_{N,D}$ we let

$$c_\rho(D) := c\left(\frac{\rho^2 - D}{4N}, \rho\right) \qquad \text{and} \qquad c^*(D) := |c_\rho(D)|\,.$$

We remark that $|c_\rho(D)|$ is independent of ρ and that $c^*(D)$ is, up to sign, the coefficient of a weight $k - 1/2$ modular form [EZ85, Theorem 5.6, p. 69].

Proposition 4.2. *If $D < 0$ is a fundamental discriminant, then*

$$B_F(D) = c^*(D) \frac{h(D)}{w_D}\,,$$

where $h(D)$ and w_D are the class number and half the number of units of the quadratic order of discriminant D, respectively. If $D = \ell d$ with $\ell \neq 1$ and $d \neq 1$ fundamental discriminants, we have $B_{\ell,F}(D) = 0$.

Proof. By the definition of the Gritsenko lift, we know that $a(T; F) = c(mn, r) = c_r(\operatorname{disc} T)$ for $T = [Nm, r, n] \in \mathcal{Q}_N$, provided T is primitive. If $\operatorname{disc} T = \ell d$ with ℓ and d fundamental discriminants, the content of T must be a divisor of ℓ. Hence, T nonprimitive implies $\psi_\ell(T) = 0$. Thus

$$\begin{aligned} |B_\ell(D, \rho)| &= \left| \sum_{T \in \mathcal{Q}_{N,D,\rho}/\Gamma_0(N)} \psi_\ell(T)\, \frac{a(T; F)}{\varepsilon(T)} \right| \\ &= |c_\rho(D)| \left| \sum_{T \in \mathcal{Q}_{N,D,\rho}/\Gamma_0(N)} \psi_\ell(T)\, \frac{1}{\varepsilon(T)} \right|. \end{aligned}$$

When $\ell = 1$ the sum in the last term is $\sum \frac{1}{\varepsilon(T)} = \frac{h(D)}{w_D}$, since $|\mathcal{Q}_{N,D,\rho}/\Gamma_0(N)| = h(D)$ for fundamental D and $\varepsilon(T) = w_D$. On the other hand, if $l \neq 1$ and $d \neq 1$, then ψ_ℓ is a nontrivial character on $\mathcal{Q}_{N,D,\rho}/\Gamma_0(N)$, hence the sum vanishes. ☐

Let f be the elliptic modular form corresponding to the Jacobi form ϕ as in [SZ88, Theorem 5]. It is a standard fact that

$$L(F, s) = \zeta(s + 1/2)\, \zeta(s - 1/2)\, L(f, s)$$

(for instance, see [Sch07]; here we are using the analytic normalization, so that the center is at $s = 1/2$). Twisting by χ_D we obtain

$$L(F, s, \chi_D) = L(s + 1/2, \chi_D)\, L(s - 1/2, \chi_D)\, L(f, s, \chi_D)$$

valid on the region of convergence. It follows from this that $L(F, s, \chi_D)$ has an analytic continuation (with a pole at $s = 3/2$ for $D = 1$) and, using Dirichlet's class number formula for the special values $L(0, \chi_D)$ and $L(1, \chi_D)$, we have

Proposition 4.3.

$$L(F, 1/2, \chi_D) = \begin{cases} \pi \, \frac{h(D)^2}{w_D^2} \, L(f, 1/2, \chi_D) \, |D|^{-1/2} & \text{if } D < 0, \\ 0 & \text{if } D > 1, \\ -\frac{1}{2} \, L'(f, 1/2) & \text{if } D = 1. \end{cases}$$

Proof of Theorem 4.1. By Waldspurger's formula [Wal81, Koh85], we have

$$c^*(D)^2 = k_f \, 2^{\nu_N(D)} \, L(f, 1/2, \chi_D) \, |D|^{k-3/2}$$

with $k_f > 0$. The theorem thus follows from Proposition 4.2 and Proposition 4.3, with $k_F = k_f/\pi$. □

5. Torsion

In the introduction, we observed that $L(F_{277}, 1/2, \chi_d) \, |d| / L(F_{277}, 1/2)$ is divisible by 15^2 when $d > 1$:

Proposition 5.1. *Let $d > 1$ and assume Conjecture B. Then, the ratio of special values $2^{\nu_{277}(d)} \, L(F_{277}, 1/2, \chi_d) \, |d| / L(F_{277}, 1/2)$ is divisible by 15^2.*

Proof. We recall [PY14a, Theorem 7.3] which asserts: suppose ϕ is the first Fourier-Jacobi coefficient of F_{277} and let $G = \text{Grit}(\phi)$. Then, for all $T \in \mathcal{Q}_N$,

$$a(T; F) \equiv a(T; G) \pmod{15}.$$

In Proposition 4.2 we observed that $B_{\ell,G}(\ell d) = 0$ when $\ell, d \neq 1$; hence, it follows that $B_{\ell,F}(\ell d) \equiv 0 \pmod{15}$. Finally, Conjecture B implies that

$$\frac{2^{\nu_{277}(d)} \, L(F, 1/2, \chi_d) \, |d|}{L(F, 1/2)} = \frac{B_{-3,F}(-3d)^2}{B_{-3,F}(-3)^2} \, .$$

Since $B_{-3,F}(-3) = 1$ (see Table 4) the right-hand side is $B_{-3,F}(-3d)^2 \equiv 0 \pmod{15^2}$. □

We recall (see Table 1) that the Jacobian of the abelian surface associated to F_{277} by the Paramodular Conjecture has torsion of size 15. In [BK10], it is suggested that a congruence between Gritsenko lifts and nonlifts modulo the torsion of the abelian surface associated to the nonlift might hold in general. So if this phenomenon occurs, in general, an analogous argument to the one used in the proof of Proposition 5.1 can be used to prove congruences between special values. Observe in Tables 3–8 each entry (both the integers $B_\ell(\ell d)$ and the normalized central values) is divisible by the odd part of the corresponding curve's torsion unless $\ell = 1$ or $d = 1$.

6. Tables

Table 3. Data for the modular form F_{249} based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + \left(x^3+1\right) y = x^2 + x$, whose Jacobian has 14-torsion. The constant $C_\ell := k_{249}\, 2^{\nu(\ell)}\, L(F_{249}, 1/2, \chi_\ell)\,|\ell|$ with $k_{249} = 0.831968$, and the value $L_d := 2^{\nu(d)}\, L(F_{249}, 1/2, \chi_d)\,|d|$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT13].

d	-8	-20	-35	-47	-56	-71
$C_1\, L_d$	1.0	4.0	16.0	1.0	4.0	0.0
$B(d)$	1	2	4	1	2	0

d	-3	-4	-7	-31	-40	-51
$C_5\, L_d$	196.0	196.0	196.0	196.0	784.0	19600.0
$B_5(5d)$	14	14	14	14	28	140

d	-3	-4	-7	-31	-40	-51
$C_8\, L_d$	196.0	196.0	196.0	196.0	784.0	19600.0
$B_8(8d)$	14	14	14	14	–	–

d	5	8	24	53	56	57
$C_{-3}\, L_d$	196.0	196.0	784.0	3136.0	3136.0	0.0
$B_{-3}(-3d)$	14	14	28	56	56	0

d	5	8	24	53	56	57
$C_{-4}\, L_d$	196.0	196.0	784.0	3136.0	3136.0	0.0
$B_{-4}(-4d)$	14	14	28	56	56	0

d	1	28	37	40	61	109
$C_{-8}\, L_d$	1.0	0.0	784.0	784.0	784.0	7056.0
$B_{-8}(-8d)$	1	0	–	–	–	–

TABLE 4. Data for the modular form F_{277} based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + y = x^5 - 2\,x^3 + 2\,x^2 - x$, whose Jacobian has 15-torsion. The constant $C_\ell := k_{277}\, 2^{\nu(\ell)}\, L(F_{277}, 1/2, \chi_\ell)\, |\ell|$ with $k_{277} = 1.075431$, and the value $L_d := 2^{\nu(d)}\, L(F_{277}, 1/2, \chi_d)\, |d|$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT13].

d	-3	-4	-7	-19	-23	-39
$C_1\, L_d$	1.0	1.0	1.0	4.0	-0.0	1.0
$B(d)$	1	1	1	2	0	1

d	-3	-4	-7	-19	-23	-39
$C_{12}\, L_d$	225.0	225.0	225.0	900.0	-0.0	225.0
$B_{12}(12d)$	15	15	15	30	0	15

d	-3	-4	-7	-19	-23	-39
$C_{13}\, L_d$	225.0	225.0	225.0	900.0	-0.0	225.0
$B_{13}(13d)$	15	15	15	30	0	15

d	1	12	13	21	28	29
$C_{-3}\, L_d$	1.0	225.0	225.0	225.0	225.0	2025.0
$B_{-3}(-3d)$	1	15	15	15	15	45

d	1	12	13	21	28	29
$C_{-4}\, L_d$	1.0	225.0	225.0	225.0	225.0	2025.0
$B_{-4}(-4d)$	1	15	15	15	15	45

d	1	12	13	21	28	29
$C_{-7}\, L_d$	1.0	225.0	225.0	225.0	225.0	2025.0
$B_{-7}(-7d)$	1	15	15	15	15	45

TABLE 5. Data for the modular form F_{295} based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + \left(x^3+1\right)y = -x^4 - x^3$, whose Jacobian has 14-torsion. The constant $C_\ell := k_{295}\, 2^{\nu(\ell)}\, L(F_{295}, 1/2, \chi_\ell)\, |\ell|$ with $k_{295} = 0.224745$, and the value $L_d := 2^{\nu(d)}\, L(F_{295}, 1/2, \chi_d)\, |d|$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT13].

d	-11	-24	-31	-39	-40	-55
$C_1\, L_d$	1.0	1.0	1.0	1.0	16.0	4.0
$B(d)$	1	1	1	1	4	2

d	-11	-24	-31	-39	-40	-55
$C_5\, L_d$	196.0	196.0	196.0	196.0	3136.0	784.0
$B_5(5d)$	14	14	14	14	–	28

d	-3	-7	-68	-87	-88	-107
$C_8\, L_d$	49.0	49.0	784.0	196.0	784.0	3969.0
$B_8(8d)$	7	7	–	–	–	–

d	8	13	33	37	73	77
$C_{-3}\, L_d$	49.0	49.0	-0.0	441.0	-0.0	196.0
$B_{-3}(-3d)$	7	7	0	21	0	–

d	8	13	33	37	73	77
$C_{-7}\, L_d$	49.0	49.0	-0.0	441.0	-0.0	196.0
$B_{-7}(-7d)$	7	7	–	–	–	–

d	1	5	21	29	41	60
$C_{-11}\, L_d$	1.0	196.0	196.0	784.0	196.0	3136.0
$B_{-11}(-11d)$	1	14	–	–	–	–

TABLE 6. Data for the modular form F^-_{587} based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + \left(x^3 + x + 1\right) y = -x^3 - x^2$, whose Jacobian has 1-torsion. The constant $C_\ell := k^-_{587}\, 2^{\nu(\ell)}\, L(F^-_{587}, 1/2, \chi_\ell)\, |\ell|$ with $k^-_{587} = 0.005361$, and the value $L_d := 2^{\nu(d)}\, L(F^-_{587}, 1/2, \chi_d)\, |d|$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT13].

d	-3	-4	-7	-31	-40	-43
$C_5\, L_d$	4.0	4.0	4.0	16.0	36.0	576.0
$B_5(5d)$	2	2	2	4	6	24

d	-3	-4	-7	-31	-40	-43
$C_8\, L_d$	4.0	4.0	4.0	16.0	36.0	576.0
$B_8(8d)$	2	2	2	4	6	24

d	-3	-4	-7	-31	-40	-43
$C_{13}\, L_d$	4.0	4.0	4.0	16.0	36.0	576.0
$B_{13}(13d)$	2	2	2	4	6	24

d	5	8	13	24	33	37
$C_{-3}\, L_d$	4.0	4.0	4.0	4.0	4.0	16.0
$B_{-3}(-3d)$	2	2	2	2	2	4

d	5	8	13	24	33	37
$C_{-4}\, L_d$	4.0	4.0	4.0	4.0	4.0	16.0
$B_{-4}(-4d)$	2	2	2	2	2	4

d	5	8	13	24	33	37
$C_{-7}\, L_d$	4.0	4.0	4.0	4.0	4.0	16.0
$B_{-7}(-7d)$	2	2	2	2	2	4

TABLE 7. Data for the modular form F^+_{713} based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + \left(x^3 + x + 1\right) y = -x^4$, whose Jacobian has 9-torsion. The constant $C_\ell := k^+_{713}\, 2^{\nu(\ell)}\, L(F^+_{713}, 1/2, \chi_\ell)\, |\ell|$ with $k^+_{713} = 0.422122$, and the value $L_d := 2^{\nu(d)}\, L(F^+_{713}, 1/2, \chi_d)\, |d|$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT13].

d	-11	-15	-23	-43	-68	-79
$C_1\, L_d$	4.0	4.0	-0.0	36.0	16.0	16.0
$B(d)$	2	2	0	6	4	4

d	-11	-15	-23	-43	-68	-79
$C_8\, L_d$	324.0	324.0	-0.0	2916.0	1296.0	1296.0
$B_8(8d)$	18	18	0	–	–	–

d	-4	-8	-35	-39	-47	-59
$C_{17}\, L_d$	0.0	0.0	0.0	-0.0	0.0	-0.0
$B_{17}(17d)$	0	0	–	–	–	–

d	17	21	37	44	53	57
$C_{-4}\, L_d$	0.0	324.0	-0.0	324.0	324.0	-0.0
$B_{-4}(-4d)$	0	18	0	18	18	0

d	17	21	37	44	53	57
$C_{-8}\, L_d$	0.0	324.0	-0.0	324.0	324.0	-0.0
$B_{-8}(-8d)$	0	18	–	–	–	–

d	1	8	41	69	93	101
$C_{-11}\, L_d$	4.0	324.0	324.0	20736.0	20736.0	5184.0
$B_{-11}(-11d)$	2	18	–	–	–	–

TABLE 8. Data for the modular form F^-_{713} based on the Hasse-Weil L-series for the hyperelliptic curve $y^2 + \left(x^3 + x + 1\right) y = x^5 - x^3$, whose Jacobian has 1-torsion. The constant $C_\ell := k^-_{713}\, 2^{\nu(\ell)}\, L(F^-_{713}, 1/2, \chi_\ell)\, |\ell|$ with $k^-_{713} = 0.005249$, and the value $L_d := 2^{\nu(d)}\, L(F^-_{713}, 1/2, \chi_d)\, |d|$. The table displays the first few twists by real and by imaginary characters. More comprehensive data for this curve can be found at [RT13].

d	-3	-24	-52	-55	-104	-116
$C_5\, L_d$	4.0	4.0	100.0	16.0	4.0	36.0
$B_5(5d)$	2	2	10	–	–	–

d	-7	-19	-20	-40	-51	-56
$C_{12}\, L_d$	4.0	64.0	4.0	100.0	144.0	4.0
$B_{12}(12d)$	2	8	2	–	–	–

d	-7	-19	-20	-40	-51	-56
$C_{13}\, L_d$	4.0	64.0	4.0	100.0	144.0	4.0
$B_{13}(13d)$	2	8	2	–	–	–

d	5	28	33	40	56	76
$C_{-3}\, L_d$	4.0	4.0	0.0	4.0	4.0	-0.0
$B_{-3}(-3d)$	2	2	0	2	2	0

d	12	13	24	29	73	77
$C_{-7}\, L_d$	4.0	4.0	4.0	36.0	0.0	144.0
$B_{-7}(-7d)$	2	2	2	6	–	–

d	12	13	24	29	73	77
$C_{-19}\, L_d$	64.0	64.0	64.0	576.0	0.0	2304.0
$B_{-19}(-19d)$	8	8	–	–	–	–

APPENDIX A. PROOF OF LEMMA 1.1

In this appendix we provide a proof of Lemma 1.1 using some local and global representation theory. The idea is that the L- and ϵ-factors attached to a modular form F are, by definition, the same as the L- and ϵ-factors attached to the automorphic representation π generated by F, and on the level of representations it is easy to control how these factors behave under twisting.

Since the functional equations for the L-functions considered in this paper are (in most cases) conjectural anyway, we may as well assume an additional convenient conjecture. Namely, we will assume *paramodular strong multiplicity one*; see Conjecture 4.4 of [RS06] for the precise formulation. Under this assumption, eigenforms generate irreducible representations. (Even without this assumption there is a basis of eigenforms with this property.)

For a positive integer N, let $F \in S_k(\Gamma^{\mathrm{para}}[N])$ be a Hecke eigenform. Just as in the full level case explained in [AS01], there is an adelic function $\Phi : \mathrm{GSp}(4, \mathbb{A}) \to \mathbb{C}$

attached to F; here $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. Let π be the representation generated by the right translates of Φ. Using a standard argument, paramodular strong multiplicity one implies that π is irreducible. Hence, π is a cuspidal, automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$. Just as in [AS01], π has trivial central character.

Being irreducible, π is isomorphic to a restricted tensor product $\otimes_{p\le\infty}\pi_p$, where π_p is an irreducible, admissible representation of $\mathrm{GSp}(4,\mathbb{Q}_p)$. For $p=\infty$ we understand $\mathbb{Q}_\infty=\mathbb{R}$. Via the local Langlands correspondence (see [Bor79] for the real case, and [GT11] for the p-adic case), to each π_p there is attached a local L-factor $L(\pi_p,s)$, and a local ϵ-factor $\epsilon(\pi_p,s)$ (the latter also depend on the choice of an additive character, but in this classical situation we may make a standard choice and suppress it from the notation). By definition,

$$L(F,s)=\prod_{p<\infty}L(\pi_p,s).$$

(This definition is compatible with the one given in Section 1.2, but includes all finite places.) The root number ϵ_p appearing in Lemma 1.1 is, by definition, $\epsilon(\pi_p,1/2)$. Conjecturally, $\epsilon=\prod_{p\le\infty}\epsilon_p$ is the sign in the functional equation of $L^*(F,s)$.

The L-function $L(F,s,\chi_D)$ is just the L-function associated to the twisted automorphic representation $\pi\otimes\chi$, where χ is the character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ corresponding to the Dirichlet character χ_D. Decomposing χ into local components $\bigotimes_{p\le\infty}\chi_p$, it is easy to see that χ_p is the quadratic character of $\mathbb{Q}_p^\times$ associated to the quadratic extension $\mathbb{Q}_p(\sqrt{D})$ via local class field theory (hence, χ_p is trivial if this extension is trivial). The proof of Lemma 1.1 consists in comparing $\epsilon_p=\epsilon(\pi_p,1/2)$ with $\epsilon'_p=\epsilon(\pi_p\otimes\chi_p,1/2)$.

Arguing as in [AS01], one knows that the archimedean component π_∞ is a holomorphic discrete series representation with scalar minimal K-type of weight k. By Sect. 10.5 of [Bor79] and Sect. 3 of [Tat79], $\epsilon(\pi_\infty,1/2)=(-1)^k$. Since π_∞ is invariant under twisting by quadratic characters of $\mathbb{R}^\times$, this proves (1) of Lemma 1.1. (The factor $L(s,\pi_\infty)$ equals $(2\pi)^{-2s-k+1}\Gamma(s+k-3/2)\Gamma(s+1/2)$, which appears in the definition (2) of $L^*(F,s)$ in the case $k=2$.)

Next consider a finite place $p\nmid N$. Then π_p is a spherical representation and hence $\epsilon_p=1$. If $p\nmid D$, then χ_p is unramified. In this case $\pi_p\otimes\chi_p$ is also a spherical representation, so that $\epsilon'_p=1$. But even if $p|D$, so that χ_p is ramified, one can go through the list of ϵ-factors in Table A.9 of [RS07] and verify that $\epsilon'_p=1$ whenever π_p is spherical. This proves (2) of Lemma 1.1.

Now consider a finite place $p|N$. Only from now on will we assume N to be squarefree. This assumption implies that π_p contains a paramodular vector of (local) level p. We have not excluded the possibility that π_p is spherical (since we have not assumed F to be a newform). In any case π_p is Iwahori-spherical, and hence is one of the representations appearing in Table A.15 of [RS07].

Assume first that $p|D$, so that χ_p is ramified. Walking through those representations that have a paramodular vector of level p and looking up their ϵ-factors from Table A.9 of [RS07], we find that $\epsilon'_p=1$ in each case. This proves (3) of Lemma 1.1.

Finally, assume that $p\nmid D$ (but still $p|N$). Then we are twisting π_p by the unramified character χ_p. Now we have to assume in addition that F is a newform as defined in Sect. 4 of [RS06]. This assumption implies that π_p is not spherical, but has minimal paramodular level p. Looking at Table A.15 of [RS07], we find that the possibilities are representations of type IIa, IVc, Vb, Vc, or VIc (type IVc cannot actually occur since it is not unitary). From the same table we see that the root number of the twist is $\chi_p(p)$ times the root number of the untwisted representation. Since $\chi_p(p) = \chi_D(p)$, this proves part (4) of Lemma 1.1. The proof of the lemma is now complete.

Remark 7. Without the newform assumption in part (4), it could happen that π_p is spherical. In this case we would have $\epsilon_p = \epsilon'_p = 1$, but it is possible that $\chi_D(p) = -1$.

Similar representation-theoretic considerations also lead us to the "correct" definition of the Euler factors $L(\pi_p, s)$ at places $p|N$. Here, we continue to assume that N is squarefree, and that F is a newform. We will also assume a version of the *Ramanujan conjecture* that states that a cuspidal automorphic representation $\pi = \bigotimes \pi_p$ of $\mathrm{GSp}(4, \mathbb{A})$ is either CAP (cuspidal associated to parabolics), or all the π_p are tempered.

Assume first that F is not a Gritsenko lift. Then the associated automorphic representation is not CAP. By the Ramanujan conjecture, $\pi = \bigotimes \pi_p$ is everywhere tempered (Weissauer proved in [Wei09], Theorem 3.3, that the π_p at all *good* places are tempered, but we will need the statement specifically for the *bad* places). Consider a place $p|N$. Since F is a newform, π_p is not spherical, but has minimal paramodular level p. Again looking at Table A.15 of [RS07], we see that the only tempered representations with this property are of type IIa. More precisely,

$$\pi_p = \chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$$

where χ and σ are unramified characters of $\mathbb{Q}_p^\times$ such that $(\chi\sigma)^2 = 1$ (this last condition is equivalent to the central character being trivial). The root number of this representation is $\epsilon_p = -(\chi\sigma)(p)$; see Table A.9 of [RS07]. Abbreviating $\alpha = \sigma(p)$, the L-factor of this representation is

$$L(\pi_p, s) = \frac{1}{(1 - \alpha p^{-s})(1 - \alpha^{-1} p^{-s})(1 + \epsilon_p p^{-s-1/2})}$$

by Table A.8 of [RS07]. This is the correct L-factor to be used in the definition of $L(F, s)$ at the place p. By the Ramanujan conjecture, α is a complex number of absolute value 1.

We note that, by Corollary 7.5.5 of [RS07], the sign ϵ_p is also the eigenvalue of the Atkin-Lehner involution at p on the newform F.

Now assume that F is a Gritsenko lift. In this case $\pi = \bigotimes \pi_p$ is a CAP representation. By [Sch07] it is known that π_p for $p|N$ is of type Vb,c or type VIc. Looking at the relevant tables, we find that

$$L(\pi_p, s) = \frac{1}{(1 - \delta\epsilon_p(p^{1/2} + p^{-1/2})p^{-s} + p^{-2s})(1 + \epsilon_p p^{-s-1/2})},$$

where $\delta = 1$ for Vb,c, and $\delta = -1$ for VIc. Here again, the root number ϵ_p is the eigenvalue of the Atkin-Lehner involution at p on the newform F.

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References

[AS01] M. Asgari and R. Schmidt, *Siegel modular forms and representations*, Manuscripta Math. **104** (2001), no. 2, 173–200, DOI 10.1007/PL00005869. MR1821182 (2002a:11044)

[BCP97] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, Computational algebra and number theory (London, 1993), J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, DOI 10.1006/jsco.1996.0125. MR1484478

[BK10] A. Brumer and K. Kramer, *Paramodular abelian varieties of odd conductor*, Trans. Amer. Math. Soc. **366** (2014), no. 5, 2463–2516, DOI 10.1090/S0002-9947-2013-05909-0. MR3165645

[Böc86] S. Böcherer, *Bemerkungen über die Dirichletreihen von Koecher und Maaß*. (Remarks on the Dirichlet series of Koecher and Maaß). Technical report, Math. Gottingensis, Schriftenr. Sonderforschungsbereichs Geom. Anal. 68, 36 S., 1986.

[Bor79] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979.

[BSP92] S. Böcherer and R. Schulze-Pillot, *The Dirichlet series of Koecher and Maass and modular forms of weight* $\frac{3}{2}$, Math. Z. **209** (1992), no. 2, 273–287, DOI 10.1007/BF02570834. MR1147818 (93b:11053)

[Dok04] T. Dokchitser, *Computing special values of motivic L-functions*, Experiment. Math. **13** (2004), no. 2, 137–149. MR2068888 (2005f:11128)

[EZ85] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, vol. 55, Birkhäuser Boston, Inc., Boston, MA, 1985. MR781735 (86j:11043)

[FM11] M. Furusawa and K. Martin, *On central critical values of the degree four L-functions for* GSp(4)*: the fundamental lemma, II*, Amer. J. Math. **133** (2011), no. 1, 197–233, DOI 10.1353/ajm.2011.0007. MR2752939 (2012d:11117)

[FS99] M. Furusawa and J. A. Shalika, *The fundamental lemma for the Bessel and Novodvorsky subgroups of* GSp(4) (English, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 2, 105–110, DOI 10.1016/S0764-4442(99)80146-3. MR1669031 (99k:11075)

[FS00] M. Furusawa and J. A. Shalika, *The fundamental lemma for the Bessel and Novodvorsky subgroups of* GSp(4)*. II* (English, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 8, 593–598, DOI 10.1016/S0764-4442(00)01683-9. MR1799095 (2001k:11088)

[FS03] M. Furusawa and J. A. Shalika, *On central critical values of the degree four L-functions for GSp(4): the fundamental lemma*, Mem. Amer. Math. Soc. **164** (2003), no. 782, x+139, DOI 10.1090/memo/0782. MR1983033 (2004h:11046)

[Fur93] M. Furusawa, *On L-functions for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ *and their special values*, J. Reine Angew. Math. **438** (1993), 187–218, DOI 10.1515/crll.1993.438.187. MR1215654 (94e:11057)

[GKZ87] B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series. II*, Math. Ann. **278** (1987), no. 1-4, 497–562, DOI 10.1007/BF01458081. MR909238 (89i:11069)

[Gri95] V. Gritsenko, *Arithmetical lifting and its applications*, Number theory (Paris, 1992), London Math. Soc. Lecture Note Ser., vol. 215, Cambridge Univ. Press, Cambridge, 1995, pp. 103–126, DOI 10.1017/CBO9780511661990.008. MR1345176 (96d:11049)

[GT11] W. T. Gan and S. Takeda, *The local Langlands conjecture for* GSp(4), Ann. of Math. (2) **173** (2011), no. 3, 1841–1882, DOI 10.4007/annals.2011.173.3.12. MR2800725 (2012c:22019)

[HTW10] W. B. Hart, G. Tornaría, and M. Watkins, *Congruent number theta coefficients to* 10^{12}, Algorithmic number theory, Lecture Notes in Comput. Sci., vol. 6197, Springer, Berlin, 2010, pp. 186–200, DOI 10.1007/978-3-642-14518-6_17. MR2721421 (2011m:11063)

[KK02] W. Kohnen and M. Kuss, *Some numerical computations concerning spinor zeta functions in genus 2 at the central point*, Math. Comp. **71** (2002), no. 240, 1597–1607, DOI 10.1090/S0025-5718-01-01399-0. MR1933046 (2003i:11057)

[Koh85] W. Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), no. 2, 237–268, DOI 10.1007/BF01455989. MR783554 (86i:11018)

[PY14a] C. Poor and D. S. Yuen, *Paramodular cusp forms*, Math. Comp. **84** (2015), no. 293, 1401–1438, DOI 10.1090/S0025-5718-2014-02870-6. MR3315514

[PY14b] Cris Poor and David Yuen, *The paramodular forms website*, `http://math.lfc.edu/~yuen/paramodular`, 2014.

[Rau10] M. Raum, *Efficiently generated spaces of classical Siegel modular forms and the Böcherer conjecture*, J. Aust. Math. Soc. **89** (2010), no. 3, 393–405, DOI 10.1017/S1446788711001121. MR2785912 (2012d:11111)

[RS06] B. Roberts and R. Schmidt, *On modular forms for the paramodular groups*, Automorphic forms and zeta functions, World Sci. Publ., Hackensack, NJ, 2006, pp. 334–364, DOI 10.1142/9789812774415_0015. MR2208781 (2006m:11065)

[RS07] B. Roberts and R. Schmidt, *Local Newforms for GSp(4)*, Lecture Notes in Mathematics, vol. 1918, Springer, Berlin, 2007. MR2344630 (2008g:11080)

[RT11] N. C. Ryan and G. Tornaría, *A Böcherer-type conjecture for paramodular forms*, Int. J. Number Theory **7** (2011), no. 5, 1395–1411, DOI 10.1142/S1793042111004629. MR2825979 (2012j:11109)

[RT13] N. C. Ryan and G. Tornaría, *Central values of paramodular form L-functions*, `http://www.ma.utexas.edu/users/tornaria/cnt/paramodular`, 2013.

[Rub08] M. O. Rubinstein, *lcalc: The L-function calculator, a C++ class library and command line program*, `http://www.math.uwaterloo.ca/~mrubinst`, 2008.

[Sch07] R. Schmidt, *On classical Saito-Kurokawa liftings*, J. Reine Angew. Math. **604** (2007), 211–236, DOI 10.1515/CRELLE.2007.024. MR2320318 (2008i:11066)

[Ste11] W. Stein, *Sage: Open Source Mathematical Software (Version 4.8).* The Sage Group, 2011.

[Sto08] M. Stoll, *Genus 2 curves with small odd discriminant*, `http://www.faculty.iu-bremen.de/stoll/data/`, 2008.

[SZ88] N.-P. Skoruppa and D. Zagier, *Jacobi forms and a certain space of modular forms*, Invent. Math. **94** (1988), no. 1, 113–146, DOI 10.1007/BF01394347. MR958592 (89k:11029)

[Tat79] J. Tate, *Number theoretic background*, Automorphic Forms, Representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26. MR546607 (80m:12009)

[Wal81] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier* (French), J. Math. Pures Appl. (9) **60** (1981), no. 4, 375–484. MR646366 (83h:10061)

[Wei09] R. Weissauer, *Endoscopy for* GSp(4) *and the Cohomology of Siegel Modular Threefolds*, Lecture Notes in Mathematics, vol. 1968, Springer-Verlag, Berlin, 2009. MR2498783 (2010h:11086)

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