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FOURIER COEFFICIENTS OF SEXTIC THETA SERIES

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Abstract. This article focuses on the theta series on the 6-fold cover of GL_2. We investigate the Fourier coefficients $\tau(r)$ of the theta series, and give partially proven, partially conjectured values for $\tau(\pi)^2$, $\tau(\pi^2)$ and $\tau(\pi^4)$ for prime values π.

1. Introduction

The Jacobi theta function, defined by

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z},$$

for $z = x + iy$ with $y > 0$ is of fundamental importance in many areas of mathematics. In many applications, x is set equal to 0, y is set equal $t/2$ and the relevant property of the theta function is the transformation formula

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} = \sqrt{t} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

However, $\theta(z)$ possesses a more general transformation property. We let

$$\Gamma_0(4) = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) \mid \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{with } c \equiv 0 \bmod 4\}$$

be the usual congruence subgroup. Then, for $\gamma \in \Gamma_0(4)$ we have

$$\theta(\gamma z) = j(\gamma, z)\theta(z),$$

where

$$j(\gamma, z) = \epsilon_d^{-1} \left(\frac{c}{d}\right) \sqrt{cz + d}.$$

Here $\epsilon_d = 1$ if $d \equiv 1 \bmod 4$, and $\epsilon_d = i$ if $d \equiv 3 \bmod 4$, and $\left(\frac{c}{d}\right)$ is the usual quadratic symbol except that we multiply by -1 for $c, d < 0$. The square root $\sqrt{cz + d}$ is chosen to have argument with absolute value less than $\pi/2$.

The theta function has a beautiful connection with Eisenstein series of half-integral weight. One can construct such an Eisenstein series as follows:

$$E^{(2)}(z, s) = \sum_{\Gamma_\infty \backslash \Gamma_0(4)} \mathrm{Im}(\gamma z)^s \frac{\theta(z)}{\theta(\gamma z)} = \sum_{\substack{(c,d)=1, c \geq 0 \\ c \equiv 0 \bmod 4}} \frac{\epsilon_d \left(\frac{c}{d}\right) y^s}{|cz + d|^{2s} \sqrt{cz + d}}.$$

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This converges absolutely for $\mathrm{Re}(s) > 3/4$ and, by construction, satisfies the same transformation property as $\theta(z)$, namely

$$E^{(2)}(\gamma z, s) = j(\gamma, z)E^{(2)}(z, s).$$

The remarkable thing is that $E^{(2)}(z, s)$ has a simple pole at $s = 3/4$, and upon taking the residue, one recovers the original theta function. In other words, the equality

$$\mathrm{Res}_{s=3/4}E^{(2)}(z, s) = c\theta(z),$$

holds for some non-zero constant c.

Weil made the observation that just as an automorphic form on the upper half-plane can be interpreted as an automorphic form on the group $G = \mathrm{GL}_2(\mathbf{R})$, the functions $\theta(z)$ and $E^{(2)}(z, s)$ can be interpreted as functions on $\tilde{G}$, the 2-fold metaplectic cover of G. Here, one has

$$\tilde{G} = \{(g, \epsilon) | g \in G, \epsilon = \pm 1\},$$

and multiplication is defined by

$$(g, \epsilon)(g', \epsilon') = (gg', \epsilon\epsilon'\sigma(g, g')),$$

with $\sigma(g, g')$ a certain explicit 2-cocycle.

Kubota [10] defined Eisenstein series on an n-fold metaplectic cover of GL_2, and observed that these Eisenstein series have simple poles at $s = 1/2 + 1/(2n)$. The residues at this point are automorphic forms on this n-cover of GL_2, and generalize the notion of the quadratic theta function. Unlike the quadratic theta function, however, the Fourier coefficients of the generalized theta function when $n \geq 3$ are very mysterious, and at present are only completely understood in the case $n = 3$. For $n = 3$, Patterson and Heath-Brown used them to prove a modified version of Kummer's conjecture on the equidistribution of the argument of the cubic Gauss sum [15]. In [1], the Fourier coefficients on the n-fold cover were used to show that if one n-th order twist of a GL_2 L-series does not vanish at the center of the critical strip, then an infinite number of n-th order twists must also not vanish at the center. They have, however, sufficiently many mysterious and beautiful properties that are worth studying for their own sake. For example, we will show below that square roots of Gauss sums appear naturally in the case $n = 4$ and very possibly also in the case $n = 6$. There is even reason to speculate that the cube roots of fifth order Gauss sums are present when $n = 5$, but for larger values of n the characteristics and properties of the coefficients remain inscrutable for now. In this introduction we will survey what is known and conjectured about these Fourier coefficients. To make the underlying structure clearer we will restrict our attention *in the introduction* to the class of primes that lie outside a finite set of 'bad primes' and for which reciprocity operates perfectly.

Kubota's Eisenstein series can be defined in the following way in the cases $n = 3, 4, 6$. We let ζ_n be a primitive n-th root of unity, and put $K = \mathbf{Q}(\zeta_n)$ with the ring of integers $\mathbf{Z}[\zeta_n]$. Let $\left(\frac{c}{d}\right)_n$ represent the n-th order residue symbol and, for

$$z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

with $x \in \mathbf{C}$ and $y > 0$, and $k \in U(2, \mathbf{C})$, $\alpha \in \mathbf{C}^*$, let $I(z) = y$. Kubota observed that for suitable N, the n-th power reciprocity law implies that the function

$$\kappa(\gamma) = \left(\frac{c}{d}\right)_n,$$

from $\Gamma(N) = \Gamma_N(\mathrm{SL}_2(\mathbf{Z}[\zeta_n])) \to \mathbf{C}^*$, is a homomorphism. (The choice $N = n^2$ works, but need not be minimal.) He used this to define the Eisenstein series

$$E^{(n)}(z, s) = \sum_{\Gamma_\infty \backslash \Gamma(N)} \kappa(\gamma) I(\gamma z)^{2s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$ and satisfies the automorphic relation

$$E^{(n)}(\gamma z, s) = \overline{\kappa(\gamma)} E^{(n)}(z, s).$$

The series $E^{(n)}(z, s)$ can be expanded in a Fourier series, and the constant coefficient is

$$A_0(s, y) = y^{2s} + \frac{\zeta_K^*(2ns - n)}{\zeta_K^*(2ns - n + 1)} y^{2-2s},$$

where $\zeta_K(2ns - n)$ is the zeta function of the underlying field with completion ζ_K^*. This has a simple pole when $2ns - n = 1$, i.e., at $s = 1/2 + 1/(2n)$. Taking the residue at this point, Kubota defined the theta function on the n-cover of GL_2 by

$$\theta^{(n)}(z) = \mathrm{Res}_{s=1/2+1/(2n)} E^{(n)}(z, s).$$

Ignoring non-generic primes, the series $E^{(n)}(z, s)$ has a Fourier expansion of the form

$$E^{(n)}(z, s) = A_0(s, y) + y \sum_{m \neq 0} A_m(s) N_{K/\mathbf{Q}}(m)^{s-1/2} K_{2s-1}(4\pi |m| y) e(mx).$$

Here $e(x)$ is an additive character with kernel the ring of integers of K. The coefficients are written as an arithmetic part multiplied by a K-Bessel function. For $m \neq 0$, the arithmetic part is

$$A_m(s) = \sum_{d \equiv 1 \bmod N} \frac{g_n(m, d)}{N_{K/\mathbf{Q}}(d)^{2s}}. \tag{1.1}$$

This is a Dirichlet series built from Gauss sums:

$$g_n(m, d) = \sum_{r \bmod d} \left(\frac{r}{d}\right)_n e\left(\frac{rm}{d}\right).$$

If we write the Fourier expansion of $\theta^{(n)}(z)$ as

$$\theta^{(n)}(z) = \tau_n(0) y^{1-1/n} + y \sum_{m \neq 0} \tau_n(m) K_{1/n}(4\pi |m| y),$$

then

$$\tau_n(m) = N_{K/\mathbf{Q}}(m)^{1/(2n)} \mathrm{Res}_{2s=1+1/n} A_m(s).$$

The question facing us is the determination of the nature of the coefficients $\tau_n(m)$ for $m \neq 0$. In the remainder of this section, we normalize the Fourier expansion to have $\tau_n(1) = 1$.

The Gauss sums factor in the following way: if $d = d_1 d_2$, with $(d_1, d_2) = 1$, then

$$g_n(m, d_1 d_2) = g_n(m, d_1) g_n(m, d_2) \left(\frac{d_1}{d_2}\right)_n \left(\frac{d_2}{d_1}\right)_n.$$

Thus, if $n = 2$, the two quadratic symbols cancel at all but finitely many places, and the Dirichlet series (1.1) factors into an Euler product which, up to a finite number of factors, equals $L_K(2s - 1/2, \chi_m)$, the Hecke L-series associated to the quadratic extension $K(\sqrt{m})$. This has a pole at $s = 3/4$ when m is a square, explaining why the residue of $E^{(2)}(z, s)$, which is the quadratic theta function over the field K, has a Fourier expansion supported by the square indices.

For $n \geq 3$ the product $\left(\frac{d_1}{d_2}\right)_n \left(\frac{d_2}{d_1}\right)_n$ is not trivial, and the Dirichlet series (1.1) does not factor into an Euler product. This has so far made it impossible to analyze $A_m(s)$ and compute its residue directly. Patterson [12] was able to use a converse theorem to show that in the case $n = 3$ the Mellin transform of $\theta^{(n)}(z)$ essentially equaled $\overline{A}_1(s)$, the conjugate of the first Fourier coefficient of $E^{(3)}(z, s)$. As a consequence, he discovered that the coefficients $\tau_3(m)$ satisfy a periodicity relation:

$$\tau_3(m^3 d) = N_{K/\mathbf{Q}}(m)^{1/2} \tau_3(d).$$

Also, for d cubefree, $\tau_3(d) = 0$ if $p^2 \mid d$ for any prime p, and for d squarefree,

$$\tau_3(d) = \frac{\overline{g_{(3)}(1, d)}}{N_{K/\mathbf{Q}}(d)^{1/2}}.$$

Suzuki [16] attempted to generalize Patterson's method to $n = 4$, but only succeeded in obtaining partial information about the $\tau_4(m)$. Deligne, studying this problem from a representation theoretic point of view, was able to explain that the inaccessibility of the cases $n \geq 4$ was due to a phenomenon of non-uniqueness of Whittaker models. This approach was greatly generalized in a paper of Kazhdan and Patterson [9]. Working in the context of local Whittaker functions on the n-fold cover of GL_r, they showed that the periodicity property held in great generality. They also showed that the theta functions were eigenfunctions of Hecke operators and that a certain subset of the coefficients were determined by these operators. For GL_2, this subset was everything in the case $n = 3$, but for all $n \geq 4$ the coefficients were only partially determined.

For each prime p there is an associated Hecke operator T_{p^n}. The eigenvalue of $\theta^{(n)}(z)$ is

$$\lambda_{p^n} = N_{K/\mathbf{Q}}(p)^{1/2} + N_{K/\mathbf{Q}}(p)^{-1/2}.$$

To describe the effect of T_{p^n} it will be useful to introduce the following notation. For $0 \leq j \leq n-1$,

$$G_j(m, d) = \frac{\sum_{r \bmod d} \left(\frac{r}{d}\right)_n^j e\left(\frac{rm}{d}\right)}{N_{K/\mathbf{Q}}(d)^{1/2}}.$$

This is simply the Gauss sum with numerator m and denominator d, formed with the j-th power of the residue symbol, and normalized to have absolute value 1 when d is squarefree.

Applying T_{p^n} to $\theta^{(n)}(z)$ forces the following relation upon the coefficients $\tau_n(m)$. For $(m, p) = 1$,

$$\lambda_{p^n} \tau_n(mp^j) = \tau_n(mp^{j+n}) + \tau_n(mp^{j-n}) + N_{K/\mathbf{Q}}(p)^{-1/2} G_{j+1}(m, p) \tau_n(mp^{n-2-j}).$$

We adopt the convention that $\tau_n(a)$ vanishes unless a is an integer. The periodicity established in this context in [8] and [9] is

$$\tau_n(mp^n) = \tau_n(m) N_{K/\mathbf{Q}}(p)^{1/2}.$$

For $j = n-1$, the above becomes

$$(N_{K/\mathbf{Q}}(p)^{1/2} + N_{K/\mathbf{Q}}(p)^{-1/2})\tau_n(mp^{n-1}) = \tau_n(mp^{n-1})N_{K/\mathbf{Q}}(p)^{1/2},$$

which forces $\tau_n(mp^{n-1}) = 0$. For $0 \le j \le n-2$, we obtain

$$\tau_n(mp^j) = G_{j+1}(m,p)\tau_n(mp^{n-2-j}).$$

In the case $n = 2$, this means that we have $\tau_2(m) = 0$ if m is not a square, and $\tau_2(m^2) = N_{K/\mathbf{Q}}(m)^{1/2}$, a complete description of $\tau_2(m)$. When $n = 3$, we see that $\tau_3(mp^2) = 0$, and

$$\tau_3(mp) = G_2(m,p)\tau_3(m).$$

Remembering our normalisation $\tau_3(1) = 1$, this yields

$$\tau_3(cd^3) = N_{K/\mathbf{Q}}(d)^{1/2}\overline{G_1(1,c)},$$

for c squarefree, and $\tau_3(m) = 0$ otherwise. This is a complete description of $\theta^{(3)}(z)$, which agrees with that found by Patterson.

When $n = 4$, the first example of undetermined coefficients occurs. We see that for $(m,p) = 1$, $\tau_4(mp^3) = 0$; also,

$$\tau_4(mp^2) = G_3(m,p)\tau_4(m)$$

and

$$\tau_4(mp) = G_2(m,p)\tau_4(mp). \tag{1.2}$$

Taking $m = 1$, we see that although $\tau_4(p^2)$ is determined, $\tau_4(p)$ is not. Interestingly, as the quadratic Gauss sum is trivial in this context, we have

$$G_2(m,p) = \left(\frac{m}{p}\right)_4^2 = \left(\frac{m}{p}\right)_2.$$

It follows then, from the above, that if $\left(\frac{m}{p}\right)_2 = -1$, then $\tau_4(mp) = 0$. More generally, if m possesses any factorization $m = m_1m_2$, with $\left(\frac{m_1}{m_2}\right)_2 = -1$, then $\tau_4(m) = 0$.

When $n = 5$, one finds that $\tau_5(p^4) = 0$, $\tau_5(p^3) = \overline{G_1(1,p)}$, and that

$$\tau_5(p) = G_2(1,p)\tau_5(p^2).$$

This finally leads us to the subject of this paper. When $n = 6$, the Hecke relations imply that $\tau_6(p^5) = 0$, $\tau_6(p^4) = \overline{G_1(1,p)}$, that

$$\tau_6(p) = G_2(1,p)\tau_6(p^3),$$

and that $\tau_6(p^2)$ is related to itself via

$$\tau_6(mp^2) = G_3(m,p)\tau_6(mp^2). \tag{1.3}$$

Interestingly, the Gauss sum appearing in (1.3) is quadratic, as in (1.2), suggesting a possible parallel phenomenon occurring in the cases $n = 4$ and $n = 6$. We will see in Section 5 that the relation $\tau_6(p^4) = \overline{G_1(1,p)}$ almost holds in a more precise setup.

What rule or pattern, if any, governs the undetermined coefficients? One striking observation and conjecture was made by Patterson in the case $n = 4$. Recall that the first Fourier coefficient of $E^{(4)}(z, s)$ was

$$A_1(s) = \sum \frac{G_1(1, m)}{N_{K/\mathbf{Q}}(m)^{2s-1/2}}.$$

As $A_1(s)$ is a Fourier coefficient of $E^{(4)}(z, s)$, which possesses a functional equation as $s \to 1 - s$, $A_1(s)$ inherits the same functional equation. Change the variable, rename this series as

$$\psi(w) = \sum \frac{G_1(1, m)}{N_{K/\mathbf{Q}}(m)^{w}},$$

and consider the Dirichlet series $D_1(w) = \zeta_K(4w - 1)\psi(w)$. This has a functional equation as $w \to 1 - w$, and a simple pole at $w = 3/4$. On the other hand, the Dirichlet series

$$D_2(w) = \zeta_K(4w - 1) \sum \frac{\tau_4(m)^2}{N_{K/\mathbf{Q}}(m)^{w}}$$

is the Rankin-Selberg convolution of $\theta^{(4)}(z)$ with itself and can be easily seen to have a functional equation as $w \to 1 - w$, and a double pole at $w = 3/4$. Patterson observed that the gamma factors occurring in the functional equations of $D_1(w)^2$ and $D_2(w)$ were identical, and conjectured that the series $\overline{D}_1$ obtained by conjugating the Gauss sums in D_1 satisfies

$$\overline{D_1(w)}^2 = D_2(w).$$

This conjectured equality can be seen to be consistent with all the information provided by periodicity and the Hecke relations. Dividing by an extra $\zeta_K(4w - 1)$, the conjecture states that

$$\sum \frac{\tau_4(m)^2}{N_{K/\mathbf{Q}}(m)^{w}} = \zeta_K(4w - 1) \left(\sum \frac{\overline{G_1(1, m)}}{N_{K/\mathbf{Q}}(m)^{w}} \right)^2 .$$

In other words, the conjecture predicts the values of $\tau_4(m)$ up to sign. Checking the coefficients of $m = p^2$, we see that on the left-hand side we have $\tau_4(p^2)^2 = \overline{G_1(1, p)}^2$, while on the right-hand side, as $G_1(1, p^2) = 0$, the only contribution comes from the square of the $m = p$ term, namely $\overline{G_1(1, p)}^2$. Checking further, for m squarefree, on the right-hand side we have

$$\sum_{m=m_1 m_2} \overline{G_1(1, m_1)G_1(1, m_2)} = \overline{G_1(1, m)} \sum_{m=m_1 m_2} \left(\frac{m_1}{m_2} \right)_2,$$

which does indeed vanish if m possesses any factorization $m = m_1 m_2$, with $\left(\frac{m_1}{m_2} \right)_2$ $= -1$. Most interestingly, looking at the prime indices, the conjecture predicts that

$$\tau_4(p)^2 = 2\overline{G_1(1, p)}.$$

In [3] a conjecture was made about the $n = 6$ case that was weaker than the $n = 4$ conjecture, in that it did not quite pin down all of the coefficients. This conjecture was that

$$\sum \frac{\tau_6(m^2)}{Nm^u} = \sum \frac{\overline{\tau_3(m)}}{Nm^u} \cdot \sum \frac{G_1^{(3)}(1, d)}{Nd^u},$$

where the superscript (3) indicates that we are considering the *cubic* Gauss sum. The left-hand side is the convolution of the theta function on the 6-cover of $\mathrm{GL}_2(\mathbf{C})$ with the theta function on the 2-cover of $\mathrm{GL}_2(\mathbf{C})$. This has the effect of picking from Fourier coefficients with square indices. The right-hand side is the product of the Mellin transform of the theta function on the 3-cover of $\mathrm{GL}_2(\mathbf{C})$, with the first coefficient of the cubic Eisenstein series. The two, however, are equal in this cubic case, up to a zeta function factor. Writing $m = m_1 m_2^2 m_3^3$, with m_1, m_2 squarefree and relatively prime, m_3 unrestricted we see by the periodicity properties of τ_6 and the known value of τ_3 that after canceling a zeta factor on both sides this relation translates to

$$\sum \frac{\tau_6(m_1^2 m_2^4)}{Nm_1^u Nm_2^{2u}} = \left(\sum \frac{G_1^{(3)}(1,d)}{Nd^u} \right)^2 ;$$

another curious identity involving the square of a series without an Euler product. Note that the Gauss sums $G_1^{(3)}(1,d)$ on the right-hand side vanish unless d is squarefree.

Equating corresponding coefficients we have the following predicted behavior for the coefficients $\tau_6(m_1^2 m_2^4)$:

$$\tau_6(m_1^2 m_2^4) = G_1^{(3)}(1,m_2)^2 G_1^{(3)}(1,m_1) \left(\frac{m_2}{m_1}\right)_3^2 \sum_{m_1 = d_1 d_2} \left(\frac{d_2}{d_1}\right)_3 .$$

In particular, when $m_1 = p$ and $m_2 = 1$, this reduces to the relation $\tau_6(p^2) = 2G_1^{(3)}(1,p)$. This is the fundamental relation which is being tested in this paper.

We will see in Section 5 that computational evidence overwhelmingly supports the conjecture for $|p| \equiv 7 \bmod 12$. Indeed, our computations suggest that, apart from a 12-th root of unity,

$$\tau_6(p^2) = 2\frac{G_1^{(3)}(1,p)}{\sqrt{3}}.$$

We recall the conjecture from [3] was made disregarding the prime 3, so it should come as no surprise that an additional power of 3 occurs in the actual coefficients. We remark that care should be made in comparing the current article and [3], since the definition of the sixth order symbol in the two articles are *conjugates* of each other.

We will give a conjecture for $\tau(p^2)$ for the other congruence classes of $|p|$ in Section 5. We give a proof of certain special cases as well. Finally, we examine the square $\tau(p)^2$ and give a conjectured value for this coefficient.

Remark 1.1. The conjecture $\tau_6(p^2) = 2G_1^{(3)}(1,p)$ is proven in [3] in the case that the base-field is a rational function field. The techniques used in that proof do not carry over to the number field case we are working with in this paper. Recently, Friedberg and Ginzburg [7] proved that the p^2-coefficients are arithmetic for infinitely many primes p. The results in [7] require selecting a certain vector in a representation space; this is consistent with the fact that our conjectures in Section 5 require a special set of coset representatives.

2. Theta series

Throughout this section, we fix an integer $n > 2$. We let $K = \mathbf{Q}(\zeta_n)$ be the cyclotomic field obtained by adjoining a primitive n-th root of unity ζ_n. Later on, we will focus on $n = 6$, and to make the exposition easier, we restrict ourselves to the case that K has class number one in this article. We define the set

$$S = \{v_\pi \text{ with } \pi \mid n\infty\}$$

to be the places dividing n together with the infinite places. We note that since K is totally imaginary, all infinite places are complex. The set of all finite places dividing n is denoted by S_f. We let

$$K_S = \prod_{v \in S} K_v$$

be the product of the completions at all the places in S. We embed K into the product K_S along the diagonal. Our first goal in this section is to define a *Gauss sum* on the ring of S-integers $\mathcal{O}_S = \mathcal{O}[\pi^{-1} \mid v_\pi \in S_f]$.

2.1 Gauss sums. For $v \in S$, the localization K_v admits a generalized Hilbert symbol. We recall its construction here. We let L be a local field of characteristic zero with $\zeta_n \in L$, and we let $M = L(\sqrt[n]{L^*})$. By local Artin reciprocity, we have

$$L^*/N_{M/L}(M^*) \cong \mathrm{Gal}(M/L)$$

via the Artin map. The equality $L^{*n} = N_{M/L}(M^*)$ and Kummer theory give a map

$$L^*/L^{*n} \cong \mathrm{Hom}(\mathrm{Gal}(M/K), \mu_n).$$

Combining both displayed equations gives the *Hilbert symbol*

$$(x, y) : L/L^{*n} \times L/L^{*n} \to \mu_n$$

as $(x, y) = \chi_y((x, M/L))$. Here, χ_y is the Kummer character of y, and $(\cdot, M/L)$ is the Artin symbol. We combine the local Hilbert symbols to get a symbol on K_S via

$$(x, y)_S = \prod_{v \in S} (x, y)_v.$$

For coprime $a, b \in \mathcal{O}_S$, we let

$$\left(\frac{a}{b}\right)_S = \prod_{v \notin S, v \mid b} (a, b)_v$$

be the generalized Legendre symbol. The Hilbert symbol and the Legendre symbol satisfy a *reciprocity law*

$$\left(\frac{a}{b}\right)_S = (a, b)_S \left(\frac{b}{a}\right)_S$$

that will be useful for explicit computations; see Section 3. We note that the Hilbert symbol is local and defined for all $a, b \in K_S$, whereas the Legendre symbol is global and more restricted.

Having defined a multiplicative character on K_S, we now proceed with defining an additive character e. As before, we will do so by defining a local character e_v for each $v \in S$. The desired character e is then simply the product of the local characters. First assume that L is $\mathfrak{p}$-adic, i.e., a finite extension of the p-adic field $\mathbf{Q}_p$. We define e_v as the composition of the maps

$$L \xrightarrow{\mathrm{Tr}} \mathbf{Q}_p \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p \xrightarrow{\lambda} \mathbf{Q}/\mathbf{Z} \xrightarrow{\exp(-2\pi i \cdot)} \mathbf{C}^*.$$

Here, the map $\lambda : \mathbf{Q}_p \mapsto \mathbf{Q}/\mathbf{Z}$ satisfies

$$\lambda\left(\sum_{j=-N}^{\infty} x_j p^j\right) = \sum_{j=-N}^{-1} x_j p^j,$$

i.e., it is the 'tail' of the p-adic extension of x, so that $\ker(\lambda) = \mathbf{Z}_p$. For $L = \mathbf{C}$, we put $e_\infty(x) = \exp(-2\pi i \mathrm{Tr}(x))$.

For a homomorphism $\varepsilon : \mu_n(K) \to \mathbf{C}^*$, we can now define the *Gauss sum*

$$g_n(r, \varepsilon, c) = \sum_{x \bmod c} \varepsilon\left(\left(\frac{x}{c}\right)_S\right) e(rx/c)$$

for $r, c \in \mathcal{O}_S$. We will present an algorithm to compute $g_n(r, \varepsilon, c)$ in Section 3.

2.2. Dirichlet series. The Dirichlet series we will be working with are indexed by the group $K_S^*/(K_S^{*n}\mathcal{O}_S^*)$. We first explain the structure of this group.

Lemma 2.1. *We have $\mathcal{O}_S^* \cap K_S^{*n} = \mathcal{O}_S^{*n}$.*

Proof. Let $x \in \mathcal{O}_S^* \cap K_S^{*n}$. Since x is locally an n-th power, we have $(x, c)_S = 1$ for all $c \in \mathcal{O}_S$ by the properties of the Hilbert symbol. This means that $\left(\frac{x}{c}\right)_S = 1$ holds, which implies that x arises from a global n-th power. Hence, $x \in \mathcal{O}_S^{*n}$. □

Lemma 2.2. *The following equality holds:*

$$[K_S^* : K_S^{*n}\mathcal{O}_S^*] = n^{\#S}.$$

Proof. This is proven in [13, Section 3] We give a slightly modified proof here for convenience. By standard group theory, we have

$$[K_S^* : K_S^{*n}\mathcal{O}_S^*] = [K_S^* : K_S^{*n}]/[\mathcal{O}_S^* K_S^{*n} : K_S^{*n}] = [K_S^* : K_S^{*n}]/[\mathcal{O}_S^* : \mathcal{O}_S^* \cap K_S^{*n}],$$

and by Lemma 2.1 we have $[\mathcal{O}_S^* : \mathcal{O}_S^* \cap K_S^{*n}] = [\mathcal{O}_S^* : \mathcal{O}_S^{*n}]$. Using Dirichlet's unit theorem, we compute this last index to be $n^{\#S}$. It remains to compute $[K_S^* : K_S^{*n}]$. For a finite place $v_p \in S$, we have

$$[K_{v_p}^* : K_{v_p}^{*n}] = \frac{n^2}{p^{-v_p(n)}}$$

by [11, Corollary II.5.8]. We let S_∞ be the set of infinite places in S. By the product formula [11, Proposition III.1.3], we have

$$\prod_{v_p \in S_f} p^{-v_p(n)} = \left(\prod_{v_p \in S_\infty} N_{\mathbf{C}/\mathbf{R}}(n)\right)^{-1} = n^{-2\#S_\infty},$$

and we conclude that we have $[K_S^* : K_S^{*n}] = n^{2\#S}$. The lemma follows. □

We pick a coset η of K_S^*/K_S^{*n}. For $r \in K_S$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1/2$, we define the Dirichlet series

$$\psi(r, s, \eta) = \sum_{c \in (\eta K_S^{*n} \cap \mathcal{O}_S)/\mathcal{O}_S^{*n}} g_n(r, \varepsilon, c)|c|_S^{-s-1} L_S(|\cdot|_S^{ns+1}),$$

where

$$L_S(|\cdot|_S^s) = \prod_{v \in S_f} (1 - |\pi_v|_v^s)\zeta_K(s).$$

In the last expression, π_v is a uniformizer for K_v. The norm $|c|_S$ appearing in ψ is the 'S-norm' $|c|_S = \prod_{v\in S} |c_v|_v$. The S-norm coincides with the regular norm $N_{K/\mathbf{Q}}$ for $\gcd(c, n) = 1$.

Remark 2.3. For $u \in \mathcal{O}_S^*$, we have

$$\psi(r, s, u\eta) = \varepsilon(u, \eta)_S \psi(r, s, \eta).$$

From Remark 2.3, we see that for understanding $\psi(r, s, \eta)$ it suffices to pick a coset η for $K_S^*/(K_S^{*n}\mathcal{O}_S^*)$. The following theorem describes the analytic continuation of ψ.

Theorem 2.4. *The function ψ defined above converges absolutely for* $\mathrm{Re}(s) > 1/2$. *It admits a memorphic extension to* $\mathbf{C}$. *This extension has for* $\mathrm{Re}(s) \geq 0$ *at most a single pole at $s = 1/n$.*

Proof. See [6, Section 5]. □

The residue at $s = 1/n$ of $\psi(r, s, \eta)$ is related to the Fourier coefficient $\tau_6(r)$ from the introduction in the following way. We pick a full set of coset representatives V for $K_S^*/(K_S^{*n}\mathcal{O}_S^*)$; this set has cardinality $n^{\#S}$ by Lemma 2.2. After possibly multiplying by some element of $\mathcal{O}_S^*$, we assume that all $\eta = (\eta_1, \ldots, \eta_{\#S}) \in V$ are integral at each component.

We now look at the series

$$A(r, s, V) = \sum_{\eta\in V} \psi(r, s, \eta) = \sum_{\substack{c\in\mathcal{O}/\mathcal{O}^{*n} \\ c=(c_1,\ldots,c_{\#S})\in V}} g_n(r, \varepsilon, c)|c|^{-s-1} L_S(|\cdot|^{ns+1}).$$

The series A has a pole at $s = 1/n$. By comparing the formula above with (1.1), we see that this is in fact the same place as the pole for $A_m(s)$ from the introduction.

The last sum in the equation above is a sum over *ideals* $I \subset \mathcal{O}_K$ coprime to n with the convention that we pick a generator c of I with $c \in \eta k_S^{*n}$ for some $\eta \in V$. The quantity

$$\tau(r, V) = N_{K/\mathbf{Q}}(r)^{1/2n} \mathrm{Res}_{s=1/n} A(r, s, V)$$

is the main object of study in this paper. Different choices for V yield different Fourier coefficients $\tau(r, V)$. The introduction takes $d \equiv 1 \bmod N$ and uses $\tau(r)$ as a shorthand notation, but there are other choices one can make. As we will see in Section 5, selecting a convenient V is part of our conjecture.

3. Computing Fourier coefficients

The functions $\psi(r, s, \eta)$ satisfy a functional equation in $s \to -s$. To state the equation, we modify ψ slightly and define

$$\Psi(r, s, \eta) = y_1^{-s} \left(\frac{\Gamma(ns)}{\Gamma(s)}\right)^{N/2} \psi(r, s, \eta),$$

where $N = [K : \mathbf{Q}]$ and

$$y_1 = (2\pi)^{\frac{N(n-1)}{2}} |r|_S^{-1/2} \prod_{v\in S_f} |\pi_v|_v^{nd_v/2},$$

with d_v the local difference; see below for its definition.

Functional Equation 3.1 ([6, Section 5]). *The function* Ψ *satisfies a functional equation*

$$\Psi(r,s,\eta_i) = n^{N/2} G_f(s) \prod_{v\in S_f} |\pi_v|_v^{-d_v/2} \sum_{j=1}^{n^{\#S}} T_{ij}(r,s)\Psi(r,-s,\eta_j), \tag{3.1}$$

where $\{\eta_i\}_i$ *is a full set of representatives for* $K_S^*/(K_S^{*n}\mathcal{O}_S^*)$.

Before we explain the notation in equation (3.1) above, we note that it translates a *vector* $\Psi(r,s,\eta_i)_i$ to $\Psi(r,-s,\eta_i)_i$. This means that although we are interested in computing a sum over all η_i, we do have to work with the individual $\Psi(r,s,\eta_i)$'s.

In the functional equation, we have

$$G_f(s) = \prod_{v\in S_f} \frac{1}{1-|\pi_v|_v^{1-ns}}.$$

The integer d_v for a place v over p is related to the *different* of the extension $\mathbf{Q}_p(\zeta_n)/\mathbf{Q}_p$ in the following way. The maximal order of $\mathbf{Q}_p(\zeta_n)$ equals $\mathbf{Z}_p[\zeta_n]$ and we let $f \in \mathbf{Z}_p[x]$ be the minimal polynomial of ζ_n. The different of $\mathbf{Q}_p(\zeta_n)/\mathbf{Q}_p$ equals $(f'(\zeta_n))$ and we have

$$(f'(\zeta_n)) = (\pi_v)^{d_v}.$$

In particular, $d_v = 0$ if $\mathbf{Q}_p(\zeta_n)$ is unramified.

Finally, the $T_{ij}(r,s)$ occurring in (3.1) are coefficients of an $n^{\#S}\times n^{\#S}$-matrix $T(r,s)$. The matrix T is defined over $\overline{\mathbf{Q}}(|\pi_v|_v^s \mid v\in S_f)$. We will give a method to compute $T_{ij}(r,s)$ below; see formula (3.9).

Knowing the functional equation that $\Psi(r,s,\eta)$ satisfies, the idea is to compute its residue at $s=1/n$ using contour integration. A careful analysis of the proof of functional equation (3.1) shows that $\Psi(r,s,\eta)$ only has a pole at $s=1/n$ in the domain $\Re(s) > -1/2$; see [6, p. 242]. Since the function $\Psi(r,s,\eta)$ decays exponentially fast for $\mathrm{Im}(s)\to\infty$ by [5, Theorem 2.4], we therefore have

$$\mathrm{Res}_{s=1/n}\Psi(r,s,\eta) = \frac{1}{2\pi i}\int_{(\sigma)} \Psi(r,s,\eta)\mathrm{d}s - \frac{1}{2\pi i}\int_{(-\sigma)} \Psi(r,s,\eta)\mathrm{d}s, \tag{3.2}$$

where (σ) denotes the vertical line $\mathrm{Re}(s)=\sigma>1/2$. It will be convenient for our computations to modify (3.2) slightly. If f is a holomorphic function such that $f(s)(\Gamma(ns)/\Gamma(s))^{N/2}$ decays exponentially for $\mathrm{Im}(s)\to\infty$, then we also have

$$f(1/n)x^{-1/n}\mathrm{Res}_{s=1/n}\Psi(r,s,\eta) = \frac{1}{2\pi i}\int_{(\sigma)-(-\sigma)} \Psi(r,s,\eta)f(s)x^{-s}\mathrm{d}s \tag{3.3}$$

for any $x>0$, with $(\sigma)-(-\sigma)$ the union of the two vertical lines. Not only does (3.3) allow for greater flexibility in computing the two integrals by selecting an appropriate f, but it also serves as a check on our computations by letting the parameter $x>0$ vary.

3.1. Integral with $\sigma > 0$. In this subsection we explain how to approximate

$$\frac{1}{2\pi i}\sum_{\eta\in V}\int_{(\sigma)} \Psi(r,s,\eta)f(s)x^{-s}\mathrm{d}s \tag{3.4}$$

for $\sigma > 1/2$. Using the formula for $A(r,s,V)$, we are interested in computing the sum

$$\frac{1}{2\pi i}\sum_{\substack{c\in\mathcal{O}/\mathcal{O}^{*n}\\ c\in V}}\int_{(\sigma)}\frac{g_n(r,\varepsilon,c)}{|c|}(y_1|c|x)^{-s}L_S(|\cdot|^{ns+1})\left(\frac{\Gamma(ns)}{\Gamma(s)}\right)^{N/2}f(s)\mathrm{d}s.$$

In order to compute this sum, we first write $L_S(|\cdot|^{ns+1})=\sum_{m=1}^{\infty}a(m)m^{-s}$. Since L_S is basically the Dedekind zeta-function of K, the coefficients $a(m)$ are easily determined. Indeed, we have

$$a(m)=\begin{cases} I(k)/k, & \text{if } m=k^n \text{ and if } m_v=1 \text{ for } v\in S_f,\\ 0, & \text{otherwise,}\end{cases} \tag{3.5}$$

with $I(k)$ the number of $\mathcal{O}$-ideals of norm k. We note that the coefficients $a(m)$ decay quite rapidly since they are only supported on n-th powers. If we now put

$$F_1(x)=\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{\Gamma(ns)}{\Gamma(s)}\right)^{N/2}f(s)x^{-s}\mathrm{d}s,$$

then the sum in (3.4) is equal to

$$\sum_{\substack{c\in\mathcal{O}/\mathcal{O}^{*n}\\ c\in V}}\frac{g_n(r,\varepsilon,c)}{|c|}\sum_{m=1}^{\infty}a(m)F_1(xy_1m|c|). \tag{3.6}$$

A key idea to approximating the sum above is to loop over all *ideals* of $\mathcal{O}$ that are coprime to n. For every ideal I, we only consider the generators c that lie in V when viewed as elements of K_S. If we can efficiently compute the function F_1, then for every such c, we approximate the sum $\sum_m a(m)F_1(xy_1m|c|)$ to high precision and get the contribution coming from c to the sum (3.6).

The inclusion of the function $f(s)$ in the integral gives us many choices for the function $F_1(x)$. There is a trade off in picking f so that F_1 converges fast, and is easy to evaluate. We refer to Section 4 for an example.

3.2. Integral with $\sigma < 0$. The main idea behind evaluating (3.4) for $\sigma < 0$ is the same as for $\sigma > 0$. The only technical difficulty is that since Ψ does not admit a Dirichlet expansion for $\sigma < 1/2$, we will map s to $-s$ and use the functional equation. After replacing s by $-s$ in expression (3.4), we see that we have to evaluate

$$\frac{1}{2\pi i}\sum_{\eta_i\in V}\int_{(-\sigma)}G_f(-s)f(-s)\left(\frac{1}{x}\right)^{-s}\left(\sum_{j=1}^{n^{\#S}}T_{ij}(r,-s)\Psi(r,s,\eta_j)\right)\mathrm{d}s. \tag{3.7}$$

We note that $G_f(-s)L_S(|\cdot|^{ns+1})=\zeta_K(ns+1)=\sum_m b(m)m^{-s}$ holds. The coefficients $b(m)$ satisfy $b(m)=I(k)/k$ if $m=k^n$, and $b(m)=0$ otherwise. Just like in the previous subsection, these coefficients decay rapidly. Similar to the case $\sigma > 0$, we put

$$F_2(x)=\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{\Gamma(ns)}{\Gamma(s)}\right)^{N/2}f(-s)x^{-s}\mathrm{d}s.$$

The only difference with the previous subsection is that we have to deal with the sum $\sum_i T_{ij}(r,-s)$ of the coefficients in the j-th column of the transition matrix.

Because each coefficient T_{ij} lives in $\overline{\mathbf{Q}}(|\pi_v|_v^s \mid v \in S_f)$, we can write

$$\sum_{i=1}^{n^{\#S}} T_{ij}(r,-s) = \sum_{w\in \mathbf{Z}^{\#S_f}} c_{jw}|\pi|^{ws} \qquad \text{with} \qquad |\pi|^w = \prod_{v\in S_f} |\pi_v|_v$$

for certain algebraic numbers c_{jw}. Putting everything together, we can write

$$\frac{1}{2\pi i}\sum_{\eta\in V}\int_{(\sigma)} \Psi(r,s,\eta) f(s) x^{-s} \mathrm{d}s$$

$$= y_2 \sum_{\substack{c\in \mathcal{O}/\mathcal{O}^{*n}\\ c\in V}} \frac{g_n(r,\varepsilon,c)}{|c|} \sum_{w\in \mathbf{Z}^{\#S_f}} c_{jw} \sum_{m=1}^{\infty} b(m) F_2(x^{-1}y_1|c||\pi_v|_v^w m), \tag{3.8}$$

with $y_2 = n^{N/2}\prod_v |\pi_v|_v^{d_v/2}$.

Just as for the integral with $\sigma > 0$, we will loop over all *ideals* of $\mathcal{O}$ that are coprime to n and for every ideal I, we only consider the generators c that lie in V when viewed as elements of K_S.

We observe that for each $c \in \mathcal{O}$, we need to evaluate the function F_2 many more times than F_1. Indeed, whereas we only need to evaluate F_1 once for every $m \geq 1$, we need to evaluate F_2 for every w such that c_{jw} is non-zero. This means that we need to pick our function f so that F_2 is particulary easy to evaluate.

3.3. The transition matrix. We recall that for every choice V of cosets for $K_S^*/(K_S^{*n}\mathcal{O}_S^*)$, there exists a matrix

$$T(r,s) \in \overline{\mathbf{Q}}(|\pi_v|_v^s \mid v \in S_f)$$

such that $\Psi(r,s,\eta_i)_{\eta_i\in V}$ satisfies the functional equation (3.1). In this subsection we explain how to compute the coefficients of this matrix. We will only give the results needed for actual computations, and refer to [6, Section 5] for the underlying theory.

We fix a choice of coset representatives V. One can show that T_{ij} satisfies

$$T_{ij}(s,r) = \left(\frac{r}{\eta_i\eta_j}\right)_{S_f}^{s} \frac{(\eta_i,-\eta_j)_S}{n^{2\#S}} \prod_{v\in S_f}(1-|\pi_v|_v^{ns})$$

$$\cdot \sum_{h\in \mathcal{O}_S/\mathcal{O}_S^{*n}} \left(h, -\frac{\eta_j}{\eta_i}\right)_S \prod_{v\in S_f}\sum_{y\in K_v^*/K_v^{*n}} \left(-\frac{r_v}{h\eta_i\eta_j}, y\right)_S \Gamma_v(|\cdot|_v^s \varepsilon(y,\cdot)). \tag{3.9}$$

As indicated in the previous Subsection 3.2, we will view $|\pi_v|_v$ as indeterminates and compute the coefficients T_{ij} as elements of an $\#S_f$-dimensional function field over $\overline{\mathbf{Q}}$. To make the computations as fast as possible, we should be careful to select the number field L over which the coefficients of T_{ij} are defined. We will see in the next section that we can take the Euclidean field $L = \mathbf{Q}(\zeta_{36})$ for $n = 6$ for instance.

Before we detail the computation of the local Gamma function Γ_v, we explain an idea from [18] to speed up the computations of all the T_{ij}. By examining (3.9) closely, we see that except for a factor $(\eta_i, -\eta_j)_S$, it only depends on η_i^2 and $\eta_i\eta_j$. Hence, for even n, we can save time by only computing the coefficients in (3.9) for η_i^2 and η_j.

The factor Γ_v occurring in (3.9) was introduced by Tate in his thesis [17], where it is called $\rho(c)$. We recall some of the basic theory here. We let $\chi : K_v^* \to \mathbf{C}^*$ be a quasi-character, i.e., a continuous multiplicative (but not necessarily of absolute value 1) map. We have $K_v^* \cong (\pi_v) \times U$, and χ equals $\widetilde{\chi}|\cdot|^s$ by [17, Theorem 2.3.1], where $\widetilde{\chi}$ is a *character* on the unit group U of the maximal order of K_v. We furthermore have $U = \mu_{q-1} \times U^{(1)}$ with q the characteristic of the residue field, and $U^{(1)} = 1 + \mathfrak{p} = 1 + (\pi_v)$. If $\widetilde{\chi}$ is trivial on U, we call $\widetilde{\chi}$ *unramified*. Otherwise, since the subgroups $1 + \mathfrak{p}^n$ form a filtration of U, there exists a minimal $f > 0$ with with $\widetilde{\chi}(1 + \mathfrak{p}^f) = 1$. In this ramified case, we call the integer f the *conductor* of $\widetilde{\chi}$.

The character we are interested in for (3.9) is $\widetilde{\chi} = \varepsilon(y, \cdot)$. We compute its conductor in the following way. We have an isomorphism $U^{(n)}/U^{(n+1)} \cong \mathcal{O}_v/\pi_v$ for all n, and we first compute a set of representatives R_1 for $\mathcal{O}_v/(\pi_v)$. We now *guess* that $\varepsilon(y, \cdot)$ is trivial on $\mathfrak{p}^g$ for some g, like $g = n$. We check that our guess is correct by computing

$$\varepsilon(y, 1 + r\pi_v^g) \qquad \text{for all} \qquad r \in R_1.$$

If the computation above does not yield 1 for all r, then we replace g by $2g$ and repeat the check until we do get 1 for all $r \in R_1$. If we do get 1 for all r, we replace g by $g - 1$ and repeat the check. We continue doing the latter until we do *not* get 1. The last $g > 1$ for which we get 1 for all r is the conductor of $\varepsilon(y, \cdot)$. If we have $\varepsilon(y, 1 + r\pi) = 1$ for all r, then we do a last check to see if $\varepsilon(y, x) = 1$ for all $x \in \mu_{q-1}$. If this is the case, then $\varepsilon(y, \cdot)$ is unramified, otherwise it has conductor 1.

In the unramified case, we have

$$\Gamma_v(\chi) = \chi(\pi_v)^{-d} \frac{1 - |\pi|_v \chi(\pi_v)^{-1}}{1 - \chi(\pi)} |\pi_v|_v^{d_v/2} \in \overline{\mathbf{Q}}(|\pi_v|_v^s)$$

for $\chi = |\cdot|_v^s \varepsilon(y, \cdot)_S$. Here, d_v is as before the local different of $\mathbf{Q}_p(\zeta_n)/\mathbf{Q}_p$, and we pick the totally positive square root of $|\pi_v|_v^{d_v}$. We note that we view $|\pi_v|_v^s$ as an indeterminate, and view the image of χ in $\overline{\mathbf{Q}}$.

In the ramified case, we compute a set of representatives R_2 for

$$U/(1 + \mathfrak{p}^f) \cong (U/\mathfrak{p}^f)^*,$$

and compute the sum

$$W(\chi) = |\pi_v|_v^{f/2} \sum_{x \in R_2} \chi(x) e_v\left(\frac{x}{\pi_v^{d+f}}\right)$$

for $\chi = |\cdot|_v^s \varepsilon(y, \cdot)$. As before, we take the totally positive square root of $|\pi_v|_v^f$. The map e_v is the same character as in Section 2.1, except that we identify its image in $\overline{\mathbf{Q}}$. A good check for the computations is that $W(\chi)$ is a root of unity. We now have

$$\Gamma_v(\chi) = \chi(\pi_v)^{-d_v - f} |\pi_v|_v^{\frac{d_v + f}{2}} W(\chi),$$

where we again view $|\pi_v|_v$ as an indeterminate.

3.4. Gauss sums. The last subsection deals with the computation of the Gauss sum

$$g_n(r, \varepsilon, c) = \sum_{x \bmod c} \varepsilon\left(\left(\frac{x}{c}\right)_S\right) e(rx/c)$$

that occurs in the sums (3.6) and (3.8). The following lemma reduces the computation to the case $g_n(1, \varepsilon, \pi)$ for a prime element $\pi \in \mathcal{O}_S$.

Lemma 3.2. *Let $r, c \in \mathcal{O}_S$ be non-zero with* $\gcd(r,c) = 1$. *Then we have*

$$g_n(r, \varepsilon, c) = \varepsilon\left(\left(\frac{r}{c}\right)_S^{-1}\right) g_n(1, \varepsilon, c).$$

Let $c_1, c_2 \in \mathcal{O}_S$ be non-zero with $\gcd(c_1, c_2) = 1$. *Then we have*

$$g_n(r, \varepsilon, c_1c_2) = \varepsilon\left(\left(\frac{c_1}{c_2}\right)_S\right)\varepsilon\left(\left(\frac{c_2}{c_1}\right)_S\right) g_n(r, \varepsilon, c_1) g_n(r, \varepsilon, c_2).$$

Let $\pi \in \mathcal{O}_S$ be prime. Then we have

$$g_n(r, \varepsilon, \pi^k) = 0 \qquad \textit{for} \qquad \gcd(r, \pi) = 1 \qquad \textit{and} \qquad k \geq 2.$$

Finally, we have

$$g_n(\pi^k, \varepsilon, \pi^l) = \begin{cases} 0 & \textit{for } k \neq l-1, \\ N_{K/\mathbf{Q}}(\pi)^k g_n(1, \varepsilon^l, \pi) & \textit{otherwise,} \end{cases}$$

in the relevant case $\varepsilon^l \neq 1$.

Proof. See [14]. □

We see from this lemma that for computing the residue of $A(r, s, V)$ we can, except for the case $\gcd(r, c) \neq 1$, restrict our attention to c being *squarefree*. Furthermore, we note that it suffices to make a list of Gauss sums $g_n(1, \varepsilon, \pi)$ for prime elements π of norm up to some bound. In the remainder of this section we detail a method to compute $g_n(1, \varepsilon, \pi)$ in many cases.

The Gauss sum $g_n(1, \varepsilon, \pi)$ is closely related to the 'ordinary' Gauss sum

$$g(\chi) = \sum_{x=1}^{p-1} \chi(x) \exp(2\pi i x/p) \tag{3.10}$$

in case π has prime norm p. Here, χ is the character $\varepsilon\left(\left(\frac{x}{\pi}\right)_S\right)$ and the π in the exponential is of course the complex number $\pi \approx 3.14$. The exact relationship between $g_n(1, \varepsilon, \pi)$ and $g(\chi)$ depends on n, we refer to Section 4 for the case $n = 6$.

The naive way of computing (3.10) by evaluating the sum directly takes $\widetilde{O}(p)$ operations, and this run time can be a bottleneck for the computations. It is well known that we can do much better, at least heuristically. We define the *root number* $W_D(\chi)$ as

$$W_D(\chi) = \begin{cases} \frac{g(\chi)}{m^{1/2}} & \text{if } \chi(-1) = 1, \\ \frac{g(\chi)}{m^{1/2} i}, & \text{if } \chi(-1) = -1, \end{cases}$$

where we remark that the subscript D serves to distinguish $W_D(\chi)$ from the root number in Subsection 3.3. We define $e \in \{0, 1\}$ so that $\chi(-1) = (-1)^e$ holds. Poisson summation now gives

$$\theta\left(\chi, \frac{-1}{\tau}\right) = W_D(\chi)\left(\frac{\tau}{i}\right)^{2e+1} \theta(\overline{\chi}, \tau),$$

with θ the series

$$\theta(\chi, \tau) = \sum_{n \in \mathbf{Z}} n^e \chi(n) \exp\left(\frac{i\pi n^2 \tau}{p}\right)$$

introduced by Shimura; see e.g. [4, Corollary 10.2.12]. The relation above is most useful for $\tau = it$ with $\theta(\overline{\chi}, it) \neq 0$. For us, the choice $t = 1$ has always worked, and this leads to the formula

$$W_D(\chi) = \frac{\theta(\chi, i)}{\theta(\overline{\chi}, i)}. \tag{3.11}$$

The key point is that we don't need that many terms of $\theta(\chi, i)$ to use (3.11) since the theta-series decays very rapidly. For a rigorous analysis we need to take possible rounding errors into account and we have to analyze when $\theta(\overline{\chi}, i) = 0$. We have not done this, and have only done the analysis without these technical difficulties. One shows in this case that we get an approximation to $g(\chi)$ by taking the first

$$\lceil \sqrt{p \log p} \rceil$$

terms of the theta-series.

We remark that only a 'rough' approximation to $g(\chi)$ is needed. Indeed, the n-th power of $g(\chi)$ is known by the theorem of Eisenstein and Weil, so that we only need to approximate $g(\chi)$ with error in the argument less than π/n.

4. The case $n = 6$

We now restrict to $n = 6$, and fix $K = \mathbf{Q}(\zeta_6)$. The extension $K/\mathbf{Q}$ has degree two, and since K is norm-Euclidean, we can take $S = \{2, 3, \infty\}$. The localization $K_2 = \mathbf{Q}_2(\zeta_6)$ is the unique *unramified* degree two extension of $\mathbf{Q}_2$, and the localization $K_3 = \mathbf{Q}_3(\zeta_6)$ is totally ramified of degree two.

It is a standard computation to compute the local unit groups U_2, U_3; see e.g. [11] for an algorithm. We have

$$U_2/U_2^6 \cong \mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z},$$

and for convenience we take the same generators $\alpha_1 = \pi_2 = 2, \alpha_2 = 3 + 2\zeta_6, \alpha_3 = 5 + 3\zeta_6, \alpha_4 = 1 + 2\zeta_6$ as in [18]. For the other localization, we have

$$U_3/U_3^6 \cong \mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$$

with generators $\beta_1 = \pi_3 = 2\zeta_6 - 1 = \sqrt{-3}, \beta_2 = (1 + \pi_3)^2, \beta_3 = 2, \beta_4 = 1 + \pi_3^3$.

4.1 Hilbert symbol. In this subsection we detail the computation of the Hilbert symbol $(x, y)_S$ on $K_S = K_2 \times K_3 \times \mathbf{C}$. First, the symbol is trivial on $\mathbf{C}$, so we restrict ourselves to the non-archimedean case. The basic idea in computing $(x, y)_v$ is to write

$$(x, y)_v = \prod_{w \neq v} (x, y)_w^{-1}$$

by the product formula. Now, for $v \notin S \cup \{\pi_w \mid x\}$, the Hilbert symbol basically equals the power residue symbol which we can compute by Euler's criterion; see [2, Exercise 1]. The trouble lies in computation of $(x, y)_w$ for $w \in S$ and for $w \mid x$. We will follow an idea from [5] to make those remaining cases easy to compute as well.

We fix a place $v \in \{2, 3\}$, and let $x_1, \ldots, x_4$ be a basis for U_v/U_v^6. We let $w \neq v$ be the other divisor of 6, and write $x_i = y_{i,w} a_{i,w}^6$ with $y_{i,w} \in \mathcal{O}$. Without loss of generality, we assume that x_i is chosen so that $y_{i,w}$ has w-adic valuation 0. We claim that we may assume that

$$x_i \equiv 1 \bmod K_w^{*6}$$

holds. To see this, we write $x_i = y_{i,w} a_{i,w}^6$ with $y_{i,w} \in \mathcal{O}$. Furthermore, we let $m_w \in \mathbf{Z}_{>0}$ be such that $U_w^{m_w} \subset K_w^{*6}$ holds. Using the Chinese Remainder Theorem, we choose $z_{i,w} \in \mathcal{O}$ with $z_{i,w} \equiv y_{i,w}^{-1} \bmod P_w^{m_w}$ and with $z_{i,w} \equiv 1 \bmod P_v^{m_v}$, where P_w denotes the $\mathcal{O}$-ideal corresponding to the valuation w. In the formula, the inverse of $y_{i,w}$ is taken in the group K_w^*/K_w^{*6}. The element

$$z_{i,w} x_i$$

now has the desired property. We note that multiplication by $z_{i,w}$ has not changed $x_i \bmod K_v^{*6}$.

By construction, we have $(x_i, x_j)_w = 1$. For a place $s \notin S$, we have

$$(x_i, x_j)_s = \left(\frac{c}{s}\right) \qquad \text{for} \qquad c = (-1)^{s(x_i)s(x_j)} u_i^{-s(x_j)} u_j^{s(x_i)} \tag{4.1}$$

with $x_i = \pi_s^{s(x_i)} u_i$ and $x_j = \pi_s^{s(x_j)} u_j$ by [2, Exercise 2]. In particular, if $s(x_i) = s(x_j) = 0$, then we have $(x_i, x_j)_s = 1$. Summarizing, we have

$$(x_i, x_j)_v = \prod_s (x_i, x_j)_s^{-1} \tag{4.2}$$

where the product is over those places $s \notin S$ with either $s(x_i) \neq 0$ or $s(x_j) \neq 0$. The symbols in (4.2) are easily computed using (4.1) and the generalized Euler criterion: $\left(\frac{c}{s}\right)$ is the unique 6-th root of unity with

$$\left(\frac{c}{s}\right) \equiv c^{\frac{N_{K/\mathbf{Q}}(P_s)-1}{6}} \bmod P_s.$$

For our choice of basis, we get $(\alpha_i, \alpha_j)_2 = \zeta_6^{a_{ij}}$ with the matrix $A = (a_{ij})$ given by

$$A = \begin{pmatrix} 0 & 3 & 4 & 0 \\ 3 & 3 & 3 & 0 \\ 2 & 3 & 0 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}.$$

For the localization at 3, we get $(\beta_i, \beta_j)_3 = \zeta_6^{b_{ij}}$ with $B = (b_{ij})$ given by

$$B = \begin{pmatrix} 3 & 0 & 3 & 2 \\ 0 & 0 & 4 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices satisfy

$$A \equiv -A^T \bmod 6 \quad \text{and} \quad B \equiv -B^T \bmod 6.$$

This property, which follows from $(x,y)_v (y,x)_v = 1$, is a good check on the computation.

4.2 Gauss sums. In this subsection we give the details on the computation of $g_6(1, \varepsilon, \pi)$ for a prime $\pi \in \mathcal{O}_S$. Assume first that $p = \pi \in \mathcal{O}_S$ is an inert prime. In this case, we have

$$\begin{aligned} g_6(1,\varepsilon,\pi) &= \sum_{x,y \bmod p\mathbf{Z}} \varepsilon\left(\frac{x+\zeta_6 y}{p}\right)_S e\left(\frac{x+\zeta_6 y}{p}\right) \\ &= \sum_{x\neq 0} \varepsilon\left(\frac{x+\zeta_6 y}{p}\right)_S e\left(\frac{x+\zeta_6 y}{p}\right)_S + \sum_{y \bmod p\mathbf{Z}} \varepsilon\left(\frac{\zeta_6 y}{p}\right)_S e\left(\frac{\zeta_6 y}{p}\right). \\ &= \sum_{y \bmod p\mathbf{Z}} \varepsilon\left(\frac{1+\zeta_6 y}{p}\right)_S \sum_{x\neq 0 \bmod p} e\left(\frac{x+\zeta_6 xy}{p}\right) - \varepsilon\left(\frac{\zeta_6}{p}\right)_S, \end{aligned}$$

where we have made the substitution $y \to xy$ and used the equality $\left(\frac{y}{p}\right)_S = 1$ in the last line. We compute $\varepsilon\left(\frac{x+\zeta_6 xy}{p}\right) = \exp(2\pi i \frac{x(2+y)}{p})$, and derive that

$$\begin{aligned} g_6(1,\varepsilon,\pi) &= -\varepsilon\left(\frac{\zeta_6}{p}\right)_S + (p-1)\varepsilon\left(\frac{1-2\zeta_6}{p}\right)_S + \sum_{y\neq -2} \varepsilon\left(\frac{1-\zeta_6 y}{p}\right)_S \\ &= p\varepsilon\left(\frac{1-2\zeta_6}{p}\right)_S. \end{aligned}$$

Here, the sum over all $y \neq -2$ is computed by expanding the sum in the equality

$$0 = \sum_{x,y \bmod p\mathbf{Z}} \varepsilon\left(\frac{1-2\zeta_6 y}{p}\right)_S$$

and rearranging terms. We conclude that

$$g_6(1,\varepsilon,\pi) = p\varepsilon\left(\frac{1-2\zeta_6}{p}\right)_S = \begin{cases} p & \text{for } p \equiv 3 \bmod 4, \\ -p & \text{for } p \equiv 1 \bmod 4 \end{cases}$$

holds for inert primes.

For a split prime π of norm p, we have

$$\begin{aligned} g_6(1,\varepsilon,\pi) &= \sum_{x=0}^{p-1} \varepsilon\left(\frac{x}{\pi}\right)_S \exp(2\pi i \frac{x}{\pi}\mathrm{Tr}(\pi)) \\ &= \varepsilon\left(\frac{-\overline{\pi}}{\pi}\right)_S^{-1} g\left(\varepsilon\left(\frac{\cdot}{\pi}\right)_S\right), \end{aligned}$$

with g as in (3.10). The methods from Subsection 3.4 can be used to approximate g. In this case, the Eisenstein-Weil theorem tells us that

$$g^3 = \pi^2\sqrt{p} \cdot \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ 3 & \text{if } p \equiv 3 \bmod 4 \end{cases} \tag{4.3}$$

holds, and we only need to approximate g with enough accuracy to select the right cubic root of the right-hand side of (4.3).

To compute $g_6(\pi^{k-1},\varepsilon,\pi^k)$, we need to know $g_6(1,\varepsilon^k,\pi)$ for $k = 2,3,4,5$. The value for these quantities follows directly from the definition. Indeed, we have

$$\begin{aligned} g_6(1,\varepsilon^5,\pi) &= \overline{g_6(1,\varepsilon,\pi)}\varepsilon\left(\tfrac{-1}{\pi}\right)_S, \\ g_6(1,\varepsilon^4,\pi) &= \overline{g_6(1,\varepsilon^2,\pi)}\varepsilon\left(\tfrac{-1}{\pi}\right)_S, \\ g_6(1,\varepsilon^3,\pi) &= g_2(1,\varepsilon^3,\pi), \\ g_6(1,\varepsilon^2,\pi) &= g_3(1,\varepsilon^2,\pi), \end{aligned}$$

with g_2, g_3 the quadratic and cubic Gauss sum respectively. For the quadratic Gauss sum we have

$$g_2(1,\varepsilon^3,\pi) = \begin{cases} \varepsilon^3 \left(\frac{\pi}{\pi}\right)_S \sqrt{N_{K/\mathbf{Q}}(\pi)} & \text{if } |\pi| \equiv 1 \bmod 4, \\ \varepsilon^3 \left(\frac{-\overline{\pi}}{\pi}\right)_S i\sqrt{N_{K/\mathbf{Q}}(\pi)} & \text{if } |\pi| \equiv 3 \bmod 4, \end{cases}$$

and for the cubic Gauss sum we have

$$g_3(1,\varepsilon^2,\pi) = g\left(\varepsilon^2 \left(\frac{\cdot}{\pi}\right)_S\right).$$

This last Gauss sum can be computed as in Subsection 3.4 using the relation

$$g^3 = N_{K/\mathbf{Q}}(\pi)\pi \qquad \text{for} \qquad \pi \equiv -1 \bmod 3.$$

4.3. Hypergeometric function. We recall that we need to evaluate the functions

$$F_{1,2}(x) = \frac{1}{2\pi i}\int_\sigma \left(\frac{\Gamma(6s)}{\Gamma(s)}\right) f(\pm x) x^{-s} \mathrm{d}s \tag{4.4}$$

many times. However, we have to evaluate the function F_2 roughly $6^3 = 216$ times more often than F_1. Hence, we will pick the function f so that F_2 is especially easy to evaluate.

Following [18], we propose to take $f(s) = \Gamma(1+s)^{-1}$. It can be easily checked that

$$F_2(x) = \frac{1}{6\pi}\exp\left(-\sqrt{3}\frac{x^{1/6}}{2}\right)\sin\left(\frac{x^{1/6}}{2}\right)$$

holds. We believe that the fact that F_2 is numerically easy to evaluate makes up for the fact that (4.4) converges slower for this f than for $f = 1$.

The 'price we pay' for the easy formula for F_2 is that F_1 is harder to compute. Using the residue theorem, one computes that

$$F_1(x) = \sum_{k=1}^{5} \frac{-z^k}{\Gamma(-k/6)^2}\left(\frac{1}{k\cdot k!} + \sum_{i=1}^{\infty}\left(\frac{z^6}{36}\right)^i \frac{\prod_{l=1}^{i-1}(k+6l)^2}{(k+6i-1)!}\right) \quad \text{with} \quad z = -x^{1/6}$$

holds. Some remarks about this formula are in order. First, although this series expansion for F_1 converges best for small $|x|$, we have found that even for moderately large $x \approx 10^6$ it is an efficient way to compute $F_1(x)$. However, care must be taken to perform all computations with high precision. For $x \approx 10^6$, a precision of 200 bits sufficed for us. Secondly, it is best to make a table of the quotients

$$\frac{\prod_{l=1}^{i-1}(k+6l)^2}{(k+6i-1)!}$$

instead of computing them on the fly. The first few hundred values for i suffice. Finally, we have found that the series expansion $\sum_{i=1}^{\infty}$ that occurs converges smoothly, so that simply checking that the summand is less than some chosen bound suffices to approximate this series by a partial sum.

4.4. Transition matrix. We use formula (3.9) to compute the coefficients c_{jw}. This is relatively straightforward, albeit technical. Since mistakes are easy to make in this part, we give some details on the computation and on checks one can do to make sure the matrix T is correct.

The local differents are $d_3 = 1$ and $d_2 = 0$. The local Gamma functions Γ_v have coefficients in the ring $\mathbf{Z}[\zeta_{72}]$. It turns out that the coefficients T_{ij} themselves have coefficients in $\mathbf{Z}[\zeta_{36}]$ already. Since computing (3.9) involves taking various quotients in $\mathbf{Z}[\zeta_{36}]$, it is important for the practical performance to view $T_{ij}(r,s) \in \mathbf{Z}[\zeta_{36}]((1/3)^s, (1/4)^s)$.

Putting $V = (1/3)^s, W = (1/4)^s$, we have found that T_{ij} always has denominator V^5W^3. The coefficients $c_{jw} = c_{j,(w_1,w_2)}$ are only non-zero for $(w_1, w_2) \in \{-5, \ldots, 4\} \times \{-3, \ldots, 5\}$. An excellent check for our computations is to compute the product

$$T_{ij}(r,s)T_{ij}(r,-s).$$

By the functional equation (3.1), this product is a *diagonal matrix*. Simply by computing a few coefficients of the product we can check if the matrix has been computed correctly.

4.5. Computing the integrals. To compute the sums (3.6) and (3.8), we loop over all $\mathcal{O}_S$-ideals I and for each I, we compute a generator c that is contained in V. Using the formulas from Subsections 3.1 and 3.2 we then compute the contribution from c to the respective sums. However, since the Gauss sum $g_6(1, \varepsilon, c)$ is *zero* for c that are not squarefree, we can restrict our attention to squarefree ideals I and the I that have a non-trivial gcd with (r).

To simplify the exposition, we restrict to the case that r is coprime to 6. We then have the following algorithm for approximating the Fourier coefficient $\tau(r, V) = N_{K/\mathbf{Q}}(r)^{1/12}\mathrm{Res}_{s=1/n} \sum_{\eta \in V} \Psi(r, s, \eta)$.

Algorithm 4.1.

Input. An element $r \in \mathcal{O}$ that is coprime to 6, a control parameter $x > 0$, a bound $B > 0$, a choice of representatives V, and a precision bound X.

Output. An approximation (3.6)–(3.8) to $\tau(r, V)$ coming from taking all ideals of norm up to B into account, and by performing all computations with precision X.

Step 1. Fix the embedding $\varepsilon(\zeta_6) = \exp(2\pi i/6)$. Compute and store $g_6(1, \varepsilon, \pi)$ for all prime ideals $(\pi) \subset \mathcal{O}$ with $7 \leq |\pi| \leq B$ using the method from Subsection 4.2.

Step 2. Compute and store the coefficients $\varepsilon(c_{j,w})$ of the transition matrix using formula (3.9) for all $1 \leq j \leq 216$ and $w \in \mathbf{Z}^2 \cap [-5, 4] \times [-3, 5]$.

Step 3. Compute and store the coefficients $a(m), b(m)$ for all $m \geq 1$ until both $a(m), b(m) \leq X$ using formula (3.5).

Step 4. Set $V_1 \leftarrow 0$, $V_2 \leftarrow 0$. Initialize an empty list $L \subset \mathcal{O}_S \times \mathbf{C} \times \mathbf{Z}_{>1}$. (We will add triples $(\alpha, g_6(r, \varepsilon, \alpha), N_{K/\mathbf{Q}}(\alpha))$ to L later.) Initialize an empty list $M \subset \mathbf{Z}_{>1} \times \mathbf{C}$. (We will add values $F_1(\cdot)$ to M later for all norms we encounter.)

Step 5. (*Constant term*) Determine $j \in \{1, \ldots, 216\}$ with $\eta_j = 1 \in K_S^*/K_S^{*6}\mathcal{O}_S^*$. For all $w = (w_1, w_2) \in \{-5, \ldots, 4\} \times \{-3, \ldots, 5\}$ do the following.

- *(a)* Set $f_{2,w} \leftarrow 0$.
- *(b)* For all $k \geq 1$ do the following.
- *(c)* Compute $s = F_2(y_1 x^{-1}(1/4)^{w_1}(1/3)^{w_2}k^6)b(k^6)$. If $|s| < X$, goto step 5d, else set $f_{2,w} \leftarrow f_{2,w} + s$ and repeat for the next k.
- *(d)* Set $V_2 \leftarrow V_2 + f_{2,w} \cdot c_{j,(w_1,w_2)}$.

Step 6. (*Constant term*) Set $f_1 \leftarrow 0$. For all $k \geq 1$ do the following.

- *(a)* Compute $s = F_1(y_1 x k^6)a(k^6)$. If $|s| < X$, goto step 6b, else set $f_1 \leftarrow f_1 + s$ and repeat for the next k.
- *(b)* Set $V_1 \leftarrow V_1 + f_1$. Add $(1, f_1)$ to M.

Step 7. (*Constant term*) Add $(1, 1, 1)$ to L.

Step 8. For all primes ideals $(\pi) \subset \mathcal{O}_S$ with $|\pi| < B$ (ordered by norm) do the following.
Step 9. For all $(\alpha, g_6(r,\varepsilon,\alpha), N_{K/\mathbf{Q}}(\alpha)) \in L$ do the following.
Step 10. Find $k \in \{1,\dots 6\}, j \in \{1,\dots,216\}$ with $\zeta_6^k a\pi \sim \eta_j \in V$. Set $a \leftarrow \zeta_6^k a$. Set $N \leftarrow N_{K/\mathbf{Q}}(a\pi)$.
Step 11. Compute $g_6(r,\varepsilon,a\pi)$ using Lemma 3.2.
Step 12. If $N \leq B/p$, add $(a\pi, g_6(r,\varepsilon,a\pi), N)$ to L.
Step 13. For all $w = (w_1, w_2) \in \{-5,\dots,4\} \times \{-3,\dots,5\}$ do the following.
  (a) Set $f_{2,w} \leftarrow 0$.
  (b) For all $k \geq 1$ do the following.
  (c) Compute $s = F_2(y_1 x^{-1}(1/4)^{w_1}(1/3)^{w_2}k^6 N)b(k^6)$. If $|s| < X$ goto step 13d, else set $f_{2,w} \leftarrow f_{2,w} + s$ and repeat for the next k.
  (d) Set $V_2 \leftarrow V_2 + f_{2,w} \cdot c_{j,(w_1,w_2)} \frac{g_6(r,\varepsilon,a\pi)}{N}$.
Step 14. If (N, x) is present in M, set $f_1 \leftarrow x$. Else, for all $k \geq 1$ do the following.
  (a) Compute $s = F_1(y_1 x N k^6)a(k^6)$. If $|s| < X$, goto step 14b, else set $f_1 \leftarrow f_1 + s$ and repeat for the next k.
  (b) Add (N, f_1) to M.
Step 15. Set $V_1 \leftarrow V_1 + f_1 \frac{g_6(r,\varepsilon,a\pi)}{N}$. Go to step 9.
Step 16. If $|\pi| < B$, go to Step 8.
Step 17. Set $\tau \leftarrow (V_1 - 6\sqrt{3}V_2)x^{1/6}$. Set $\tau \leftarrow \tau \cdot \Gamma(1/6)\Gamma(7/6)y_1^{1/6} N_{K/\mathbf{Q}}(r)^{1/12}$. Return τ.

We make some remarks about the algorithm. First, Step 1 of the algorithm is independent of r and one should store the Gauss sums in a file once and for all if we are computing several coefficients $\tau(r,V)$. Second, the coefficients $c_{j,w}$ in Step 2 only depend on $r \in K_S^*/K_S^{*6}$. Although this quotient group has size $6^6 = 46656$, we still store the matrix in a file. This is particularly convenient if we are running the algorithm for various choices of x, X and L.

Since the values f_1, f_2 that we compute get multiplied by a Gauss sum later on, we only need to consider squarefree elements of $\mathcal{O}_S$. Our loop over all squarefree elements of norm at most B is basically a variant of the sieve of Eratosthenes. The purpose of the list M is simply to avoid some (costly) evaluations of the function F_1.

Finally, we remark that although all computations are done with precision X in the algorithm, this does not mean that the *output* is correct with the same precision. Indeed, the precision of the output depends on the combined choice of B and x as well. We refer to [5] for an analysis of the convergence properties of sums (3.6) and (3.8).

We close this section with some remarks about our implementation of the algorithm. Our implementation consists of three parts: first we compile a list of the Gauss sums $g_6(1,\varepsilon,\pi)$ for all π with $|\pi| < 10^8$. We used Magma for this part and stored the result as a large text file. The computation of the Gauss sums ran for several days.

The second stage consists of a Magma program to compute, for a given r, the coefficients $\varepsilon(c_{j,w})$ and save these coefficients in a text file. Our Magma code took roughly 20 minutes to complete on our 2.40GHz Intel Xeon processor. We stress that we did not attempt to optimize this part of the computation. The bottle neck of the computation is working with power series in 2 variables over $\mathbf{Q}(\zeta_{36})$. We believe that tailored code for power series over $\mathbf{Q}(\zeta_{36})$ would lead to a significant speed up of the computation of the coefficients $\varepsilon(c_{j,w})$.

We found that the overhead of using a computer algebra package like Magma or Sage was too high for the third stage, the actual computation of $\tau(r, V)$. We implemented this part of the algorithm in C. The evaluations of the function F_1 requires multi-precision arithmetic, and we used GMP and MPFR for this part. This is the only part of the code where multi-precision is required. Although the C-code is parallelizable in, e.g., Step 5 of the algorithm, we have not done this. Our single-thread code has been reasonably optimized for the rest. With a bound of $B = 10^8$ for the Gauss sums and a precision bound $X = 10^{-20}$, the code approximates $\tau(r, V)$ in roughly 45 minutes on the same 2.40GHz Intel Xeon processor.

Looking beyond $n = 6$, the next case is $n = 5$. Since $\mathbf{Q}(\zeta_5)/\mathbf{Q}$ has degree 4, the convergence of our algorithm will be a lot slower. As a practical consequence, this means we will need a higher bound than 10^8 on the norms of the π's we consider. We are in the process of developing fast code to approximate coefficients of the quintic theta series using a bound of 10^{10}.

5. The conjecture

The first thing we need to decide for actual computations is which set of representatives V to use. We want to pick a set V such that the relation

$$\tau(\pi^4) = \tau(\pi^4, V) = \overline{G_1(1, \pi)} = \frac{\overline{g_6(1, \varepsilon, \pi)}}{|\pi|^{1/2}}$$

from the introduction is correct in the more precise setup of Section 2. The relation above is derived in the introduction by disregarding the primes dividing 6, and if we take those primes into account, then the relation for $\tau(\pi^4, V)$ is more subtle; see below. However, since one goal of this paper is to check the conjecture from [3], we pick a set V that mimics the equality $\tau(\pi^4, V) = \overline{G_1(1, \pi)}$ as closely as possible.

After trying several possibilities, we have found that the set

$$V = \left\{ \eta \in K_S^*/K_S^{*6} \mid \alpha_1 = \beta_1 = \beta_2 = 0 \text{ and } \begin{cases} \alpha_3 = 0, & \text{if } \alpha_2 = 0 \\ \alpha_3 = 1, & \text{if } \alpha_2 = 1 \end{cases} \right\}$$

gives the cleanest results. We note that this is the set V_2 considered by Wellhausen in his thesis [18]. The set V can be characterized as follows:

$$\{x \in \mathcal{O} \mid \gcd(x, 6) = 1, x \sim \eta \in V\} = \{x \in \mathcal{O} \mid x \equiv y \bmod 12\}$$

with $y \in$

(5.1)

$$\{1, 5, 4+3\zeta_6, 8+3\zeta_6, 1+6\zeta_6, 5+6\zeta_6, 1+9\zeta_6, 2+9\zeta_6, 5+9\zeta_6, 7+9\zeta_6, 10+9\zeta_6, 11+9\zeta_6\}.$$

Our choice of V has the following property:

$$\forall v, w \in V : (v, w)_6 = 1 \qquad \text{or} \qquad (v, -w)_6 = 1,$$

which can be proved easily. This property does not hold for the perhaps easier choice of V' characterized by

$$\{x \in \mathcal{O} \mid \gcd(x, 6) = 1, x \sim \eta \in V'\} = \{x \in \mathcal{O} \mid x \equiv 1 \bmod 3\}.$$

The fact that the Hilbert symbol is particularly easy on V has consequences for the coefficient $\tau(\pi^4, V)$. The theory of *Hecke operators* is used in the proof of the following lemma.

Lemma 5.1. *Let* $\pi \in \mathcal{O}_S$ *be prime with* $|\pi| \equiv 1 \bmod 4$ *and* $\pi \equiv y \bmod 12$ *for some* y *in set* (5.1), *and let* V *be as above. Then we have*

$$\frac{\tau(\pi^4, V)}{\tau(1, V)} = |\pi|^{-1/2}\overline{g_6(1, \varepsilon, \pi)}.$$

Proof. The following holds for $\rho(\pi^4, \eta) = \mathrm{Res}_{s=1/6}\psi(\pi^4, s, \eta)$:

$$\rho(\pi^4, \eta) = |\pi|^{-5/6} g_6(1, \varepsilon^5, \pi)\varepsilon((-\eta, \pi^5)_S)\rho(1, \eta\pi^{-5});$$

see [9]. We sum over all $\eta \in V$ and renormalize to τ to obtain

$$\tau(\pi^4, V) = |\pi|^{-1/2} g_6(1, \varepsilon^5, \pi) \sum_{\eta=\alpha\pi^5 \in V} \varepsilon((\alpha, \pi^5)_S)\rho(1, \alpha),$$

where we have used the equality $(-\pi^5, \pi^5)_6 = 1$ in the sum. By replacing π by $-\pi$ if necessary, we may assume that $\alpha \in V$. The lemma follows from checking that for $|\pi| \equiv 1 \bmod 4$ and $\alpha\pi \in V$, the equalities $(\alpha, \pi^5)_6$ hold for all $\eta \in V$. □

We remark that the proof hinges on the special property of V. If we change V, then the lemma need not be true. Lemma 5.1 is as close to the relation $\tau(\pi^4, V) = \overline{G(1, \pi)}$ as we can get. We note though that for $|\pi| \equiv 3 \bmod 4$, not all of the Hilbert symbols $(\alpha, \pi^5)_S$ are trivial. The lemma is false in this case.

To see what happens for $|\pi| \equiv 3 \bmod 4$, we use Algorithm 4.1. We have found that the choices

$$B = 10^8, \qquad x = 1/300, \qquad X = 10^{-20},$$

work very well. We compute and store all Gauss sums for primes up to norm 10^8. This computation is highly parallelizable, and it is of great help here to have a cluster of CPU's available. We used the method from Subsection 3.4 to compute the individual Gauss sums, noting that the equality

$$g_6(1, \varepsilon, \overline{\pi}) = (-1, \pi)_6 g_6(1, \varepsilon, \pi)$$

saves us half the computations.

Lemma 5.1 is a very good test for the implementation, since a small mistake in the implementation will cause the equality in Lemma 5.1 to be false. Furthermore, the output of the algorithm should be roughly independent of x. By letting x vary over $1/500, 1/400, 1/300, 1/200, 1/100$, we can check that the algorithm is performing correctly. The first quantity to compute is $\tau(1, V)$. In agreement with [18], we find that

$$\tau(1, V) \approx 0.1358547858696091.$$

By letting x vary and checking the independence of x in the computations, we are confident that the expression above is correct up to 16 decimal digits. We remark that for other choices of V, the 'constant term' $\tau(1, V)$ need not be real.

Conjecture 5.2. *Let* $\pi \in \mathcal{O}_S$ *be prime with* $|\pi| \equiv 3 \bmod 4$ *and* $\pi \equiv y \bmod 12$ *for some* y *in set* (5.1). *Then we have*

$$\frac{\tau(\pi^4, V)}{\tau(1, V)} = |\pi|^{-1/2}\frac{(-1, \pi)_6\overline{g_6(1, \varepsilon, \pi)}}{\sqrt{3}}.$$

Evidence. This conjecture is purely based on computational evidence. In fact, since the norm of π^4 grows rather quickly, we only computed a few cases. The conjecture is correct for $|\pi| = 7, 19, 31$ for several decimal digits. To *prove* this conjecture, one should examine the relations between the $\rho(\pi^4, \eta)$ for varying η more closely. □

We conclude that for this choice of V, the coefficients $\tau(\pi^4, V)$ are almost in agreement with the general philosophy explained in the introduction. We now move on to the coefficients $\tau(\pi^2, V)$. In this case, the Hecke operators relate $\rho(\pi^2, \eta)$ to itself. However, we can still derive the following.

Lemma 5.3. *Let $\pi \in \mathcal{O}_S$ be prime with $|\pi| \equiv 1 \bmod 12$, and $\pi \equiv y \bmod 12$ for some y in set* (5.1)*. If we have $\tau(\pi^2, V) \neq 0$, then $\left(\frac{\overline{\pi}}{\pi}\right)_6 = 1$.*

Proof. The Hecke operators now give

$$\rho(\pi^2, \eta) = |\pi|^{-3/6} g_2(1, \varepsilon^3, \pi)\varepsilon(-\eta, \pi^3)_S \rho(\pi^2, \eta\pi^{-3});$$

see [9]. Analagous to the proof of Lemma 5.1 we derive that

$$\tau(\pi^2, V) = |\pi|^{-1/2} g_2(1, \varepsilon^3, \pi)\tau(\pi^2, V)$$

holds for $|\pi| \equiv 1 \bmod 12$. Furthermore, we have $g_2(1, \varepsilon^3, \pi) = \left(\frac{\overline{\pi}}{\pi}\right)_6 \sqrt{N_{K/\mathbf{Q}}(\pi)}$ in this case. The lemma follows. □

We caution that the converse of the lemma does not hold. In our computations we have found several cases where $\tau(\pi^2, V) \approx 0$ even though $\left(\frac{\overline{\pi}}{\pi}\right)_6 = 1$. Specifically, we conjecture that $\tau(\pi^2, V) = 0$ for

$$|\pi| = 37, 313, 373, 661, 769.$$

These five norms are the only norms less than 1300 for which $\left(\frac{\overline{\pi}}{\pi}\right)_6 = 1$ and $\tau(\pi^2, V) \approx 0$. We have not been able to determine a pattern in this small set of primes.

Conjecture 5.4. *Let $\pi \in \mathcal{O}_S$ be prime with $|\pi| \equiv 1 \bmod 12$, and $\pi \equiv y \bmod 12$ for some y in set* (5.1)*. If $\tau(\pi^2, V) \neq 0$, then we have*

$$\frac{\tau(\pi^2, V)}{\tau(1, V)} = \zeta_6^{g(\pi)} \frac{2 g_3(1, \varepsilon^2, \overline{\pi})}{|\pi|^{1/2}}$$

for some value $g(\pi) \in \{0, \ldots, 6\}$ satisfying $g(\pi) + g(\overline{\pi}) \equiv 0 \bmod 6$.

Evidence. The support for this conjecture is numerical. We have approximated $\tau(\pi^2, V)$ for all $|\pi| < 1300$. We list the values for $g(\pi)$ below for all $|\pi| < 10^3$.

$N(\pi)$	π	$g(\pi)$	$N(\pi)$	π	$g(\pi)$	$N(\pi)$	π	$g(\pi)$
61	$-9\zeta_6 + 4$	5	433	$-24\zeta_6 + 13$	0	853	$-27\zeta_6 + 31$	1
157	$-12\zeta_6 + 13$	2	577	$27\zeta_6 - 8$	5	877	$3\zeta_6 + 28$	4
193	$-9\zeta_6 + 16$	1	601	$-24\zeta_6 + 25$	3	937	$3\zeta_6 - 32$	2
349	$-3\zeta_6 - 17$	1	613	$9\zeta_6 + 19$	2	977	$36\zeta_6 - 23$	3
397	$12\zeta_6 - 23$	1	673	$21\zeta_6 - 29$	4			

We have not been able to find a pattern in the exponents $g(\pi)$.

For the π^2-case, it remains to consider the inert primes and the primes of norm congruent to $7 \bmod 12$. For an inert prime π, we remark that although $|\pi| \equiv 1 \bmod 4$, the residue symbol in Lemma 5.3 is undefined and the proof therefore does not follow through. We have the following conjecture.

Conjecture 5.5. *Let $\pi \in \mathcal{O}_S$ be an inert prime with $\tau(\pi^2, V) \neq 0$, and $\pi \equiv y \bmod 12$ for some y in set* (5.1)*. Then we have*

$$\frac{\tau(\pi^2, V)}{\tau(1, V)} = -\frac{2\sqrt{|\pi|}}{|\pi|^{1/2}}.$$

Evidence. We have computed the coefficients $\tau(\pi^2, V)$ for $\pi = 5, 11, \ldots, 89$. We have $\tau(\pi^2, V) \approx 0$ for

$$\pi = 5, 17, 29, 41, 53, 59, 89$$

and the conjecture is true for the other cases $\pi = 11, 23, 47, 71, 83$ with several decimal digits precision. From this data one can furthermore conjecture that $\tau(\pi^2, V) = 0$ for $|\pi| \equiv 5 \bmod 12$. Note that the prime 59 contradicts the converse statement. □

We remark that the $\sqrt{|\pi|}$ in Conjecture 5.5 equals the cubic Gauss sum for π, just like in Conjecture 5.4.

Conjecture 5.6. *Let $\pi \in \mathcal{O}_S$ be prime with $|\pi| \equiv 7 \bmod 12$, and $\pi \equiv y \bmod 12$ for some y in set* (5.1)*. Then we have*

$$\frac{\tau(\pi^2, V)}{\tau(1, V)} = \zeta_{12}^{h(\pi)} \frac{2g_3(1, \varepsilon^2, \overline{\pi})}{\sqrt{3}|\pi|^{1/2}}$$

with $h(\pi) \in \{1, 3, 5, 7, 9, 11\}$ satisfying $h(\pi) + h(\overline{\pi}) \equiv 0 \bmod 12$.

Evidence. We have approximated the coefficients $\tau(\pi^2, V)$ for all $|\pi| < 8000$. We list the first few values for $h(\pi)$ below.

$N(\pi)$	π	$h(\pi)$	$N(\pi)$	π	$h(\pi)$	$N(\pi)$	π	$h(\pi)$
7	$3\zeta_6 - 2$	9	79	$3\zeta_6 - 10$	7	163	$3\zeta_6 + 11$	3
19	$3\zeta_6 + 2$	11	103	$-9\zeta_6 - 2$	1	199	$15\zeta_6 - 13$	11
31	$6\zeta_6 - 1$	3	127	$6\zeta_6 - 13$	7	211	$15\zeta_6 - 1$	7
43	$-6\zeta_6 + 7$	1	139	$3\zeta_6 - 13$	7	223	$6\zeta_6 + 11$	3
67	$-9\zeta_6 + 7$	9	151	$-9\zeta_6 + 14$	7	271	$-9\zeta_6 + 19$	1

We have not been able to find a pattern in the exponents $h(\pi)$. Since $h(\pi)$ appears to always be odd, we can replace the $\sqrt{3}$ in the denominator of the conjecture by $\zeta_{12}^{11}(1+\zeta_6)$ to force $h(\pi)$ to be even and $\zeta_{12}^{h(\pi)}$ is then a *sixth* root of unity. However, complex conjugation does not act nicely in this case. This is the reason we have stated the conjecture with a $\sqrt{3}$ instead of $1 + \zeta_6$. □

We proceed with the investigation of $\tau(\pi, V)$. We believe that the quantity

$$\left(\frac{\tau(\pi, V)}{\tau(1, V)}\right)^2$$

has interesting algebraic properties.

Conjecture 5.7. *Let $\pi \in \mathcal{O}_S$ be prime with $\pi \equiv y \bmod 12$ for some y in set* (5.1). *If $|\pi| \equiv 1 \bmod 4$, then we have*

$$\left(\frac{\tau(\pi, V)}{\tau(1, V)}\right)^2 = \zeta_6^{k(\pi)} \frac{g_3(1, \varepsilon^2, \overline{\pi})}{|\pi|^{1/2}} 3^{l(\pi)} (1 - 3\zeta_6)^{m(\pi)} (-2 + 3\zeta_6)^{n(\pi)}$$

with $k(\pi) \in \{1, \ldots, 6\}$ satisfying $k(\pi) + k(\overline{\pi}) = 0 \bmod 6$. We have $l(\pi) \in \{-1, 0\}$. The elements $1 - 3\zeta_6, -2 + 3\zeta_6$ have norm 7, and we have $m(\pi), n(\pi) \in \{0, 2\}$ with the restriction that they cannot both be equal to 2. If $|\pi| \equiv 3 \bmod 4$, then we have

$$\left(\frac{\tau(\pi, V)}{\tau(1, V)}\right)^2 = \zeta_6^{k(\pi)} \frac{g_3(1, \varepsilon^2, \overline{\pi})}{|\pi|^{1/2}} 3^{l(\pi)} (1 + 3\zeta_6)^{m(\pi)} (4 - 3\zeta_6)^{n(\pi)}$$

with the same restrictions on k, l, m, n. The elements $1 + 3\zeta_6, 4 - 3\zeta_6$ both have norm 13.

Evidence. In the case $|\pi| \equiv 1 \bmod 4$, we have computed $\tau(\pi, V)$ for all $|\pi| < 900$. The norms where an element of norm 7 appears in τ are

$$73, 193, 241, 349, 373, 421, 613, 661, 709, 757, 829.$$

For these norms, we have $l(\pi) = -1$. The norms with $l(\pi) = 0$ are

$$97, 229, 313, 457, 577, 877.$$

The conjecture is on thinner ice for $|\pi| \equiv 1 \bmod 4$. In this case, our implementation is not entirely independent of the parameter $x > 0$, which has the practical impact that we can only rely on very few digits. In his thesis, Wellhausen computed $\tau(\pi, V)$ for all $|\pi| < 100$ and the only $|\pi|$ where the element of norm 13 appears is 79. The norms with $l(\pi) = 0$ are

$$19, 31.$$

We have not found a pattern in the exponents $k(\pi)$, nor a condition when the elements of norm 7, 13 appear. □

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