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WELL-BALANCED SCHEMES TO CAPTURE NON-EXPLICIT STEADY STATES: RIPA MODEL

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Abstract. The present paper concerns the derivation of numerical schemes to approximate the weak solutions of the Ripa model, which is an extension of the shallow-water model where a gradient of temperature is considered. Here, the main motivation lies in the exact capture of the steady states involved in the model. Because of the temperature gradient, the steady states at rest, of prime importance from the physical point of view, turn out to be very nonlinear and their exact capture by a numerical scheme is very challenging. We propose a relaxation technique to derive the required scheme. In fact, we exhibit an approximate Riemann solver that satisfies all the needed properties (robustness and well-balancing). We show three relaxation strategies to get a suitable interpretation of this adopted approximate Riemann solver. The resulting relaxation scheme is proved to be positive preserving, entropy satisfying and to exactly capture the nonlinear steady states at rest. Several numerical experiments illustrate the relevance of the method.

1. Introduction

The present work is devoted to simulating shallow-water flows where the temperature fluctuations are of prime importance. To model such flows, we adopt the Ripa model as introduced in [19, 36, 37] to investigate ocean currents. In one space dimension, the Ripa model is governed by the following set of partial differential equations:

$$\partial_t h + \partial_x (hu) = 0, \tag{1.1a}$$

$$\partial_t (hu) + \partial_x \left(hu^2 + g\Theta \frac{h^2}{2} \right) = -g\Theta h \partial_x z, \tag{1.1b}$$

$$\partial_t (h\Theta) + \partial_x (h\Theta u) = 0, \tag{1.1c}$$

where $h(x,t) > 0$ denotes the water height, $u(x,t) \in \mathbb{R}$ is the velocity and $\Theta(x,t) > 0$ designates a potential temperature field. In the derivation of the model, Θ is defined by the ratio $\Delta T / T_{\text{ref}}$, computed as the temperature difference ΔT from some reference value T_{ref}. Here, g is the positive gravity constant and $z(x)$ stands for a given smooth topography function.

In fact this system improves the so-called shallow-water model by inserting the horizontal temperature gradient. From now on, we emphasize that the usual

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shallow-water model, with a gravity constant given by $g\Theta$, is recovered as soon as the potential temperature Θ is assumed to be constant.

Here we have imposed a positive water height instead of nonnegative. In this work, we will stay far away from dry areas in order to focus on the difficulties coming from the steady state approximation. Actually, the resulting scheme we will obtain easily extends to dealing with dry regions by adopting some known techniques [7].

To shorten the notation, we rewrite the system (1.1) in the following condensed form:

$$\partial_t w + \partial_x f(w) = S(w), \tag{1.2}$$

with

$$w = \begin{pmatrix} h \\ hu \\ h\Theta \end{pmatrix}, \quad f(w) = \begin{pmatrix} hu \\ hu^2 + g\Theta h^2/2 \\ h\Theta u \end{pmatrix}, \quad S(w) = \begin{pmatrix} 0 \\ -g\Theta h \partial_x z \\ 0 \end{pmatrix},$$

where the unknown state vector w is assumed to belong to the set of admissible states given by

$$\Omega = \{w \in \mathbb{R};\ h > 0,\ u \in \mathbb{R},\ \Theta > 0\}.$$

When deriving numerical schemes to approximate the weak solutions of (1.1), particular attention is paid to the steady state solutions because of their physical interest. As a consequence, an important property to be satisfied by the derived numerical scheme is to accurately approximate such steady solutions. For the Ripa model (1.1), the steady solutions are defined by the following set of partial differential equations:

$$\begin{cases} \partial_x(hu) = 0, \\ \partial_x\left(hu^2 + g\Theta\dfrac{h^2}{2}\right) = -g\Theta h \partial_x z, \\ \partial_x(h\Theta u) = 0. \end{cases} \tag{1.3}$$

Providing the velocity does not vanish, we easily get the following steady states:

$$\begin{cases} hu = \text{const}, \\ \Theta = \text{const}, \\ \dfrac{u^2}{2} + g\Theta(h+z) = \text{const}. \end{cases}$$

However, considering states at rest, *i.e.* $u \equiv 0$, the system (1.3) cannot be explicitly solved, and thus the steady states at rest are now characterized as follows:

$$\begin{cases} u = 0, \\ \partial_x\left(\Theta\dfrac{h^2}{2}\right) + h\Theta\partial_x z = 0. \end{cases} \tag{1.4}$$

The main discrepancy between the well-known shallow-water model and the Ripa model lies in the definition of the steady states at rest. Indeed, the Ripa model involves steady states at rest governed by an underdetermined PDE system (1.4). Nonetheless, solutions of (1.4) can be reached by enforcing assumptions on h, Θ and z. These specific steady states at rest are physically relevant and they are given

by: the lake at rest steady states:

$$\begin{cases} u = 0, \\ \Theta = \text{const}, \\ h + z = \text{const}, \end{cases} \tag{1.5a}$$

or the isobaric steady states:

$$\begin{cases} u = 0, \\ z = \text{const}, \\ h^2\Theta = \text{const}, \end{cases} \tag{1.5b}$$

or the constant water height steady states:

$$\begin{cases} u = 0, \\ h = \text{const}, \\ z + \dfrac{h}{2}\ln\Theta = \text{const}\,. \end{cases} \tag{1.5c}$$

The aim of the present paper is to derive a finite volume numerical scheme able to accurately approximate the weak solutions of (1.1) and the steady states at rest given by (1.4), and to exactly capture the particular steady states defined by (1.5). During the last decade, numerous well-balanced schemes have been designed to accurately or exactly capture the lake at rest coming from the shallow-water steady states. For instance, the reader is referred to [1, 7, 21, 23, 24, 27, 34, 39]. Since the structures of both the Ripa model and the shallow-water model are very close, we can easily suppose that all these well-balanced schemes find extensions to approximate the weak solutions of the Ripa model. Moreover, since the shallow-water model is recovered provided a constant temperature Θ, it is clear that these extensions would preserve exactly the lake at rest with constant Θ for the Ripa model. However, there are no reasons that such extensions would be able to exactly restore the two other steady states of physical interest defined by (1.5).

Recently, Chertock et al. [17] have derived a central upwind scheme to approximate the weak solutions of (1.1). Their scheme is based on an interface tracking method as introduced in [16]. The numerical technique proposed in [17] turns out to be very relevant since the obtained scheme is accurate, positive preserving and exactly captures both steady states at rest involving a constant temperature or a constant topography (see (1.5)). However, the last lake at rest type steady state, involving constant water height, is only approximated and can never be exactly restored.

In order to derive relevant well-balanced numerical schemes exactly preserving the lake at rest steady states (1.5a) or the isobaric steady states (1.5b) or the constant water height steady states (1.5c), we suggest deriving a Godunov-type Riemann solver [26]. After, for instance, [3, 7, 10–12, 20, 25], the expected well-balanced property can be reached as soon as the associated approximate Riemann solver contains, in a sense to be prescribed, the topography source term.

Following ideas introduced in [7, 12] in order to derive approximate Riemann solvers relevant for topography source terms, we adopt a Suliciu relaxation type strategy [14, 15, 18, 28]. The resulting numerical scheme turns out to be positive preserving and well-balanced with the exact capture of the three lake at rest steady

states (1.5). Moreover, the scheme is entropy preserving as soon as the topography function satisfies regularity assumptions.

The paper is organized as follows. For the sake of completeness, the next section is devoted to the presentation of the algebraic properties satisfied by the system (1.1). Particular attention will be paid to the Riemann invariants of the model since they participate in the characterization of the steady states of interest. Moreover, we consider the entropy inequalities. We remark that the usual energy estimate does not yield an entropy. To correct such a failure, we are obliged to recast the model in an equivalent formulation to get the required entropy stability.

Next, to derive suitable numerical strategies satisfying the needed properties, we suggest the derivation of relaxation schemes (for instance, see [4,5,14,15,18,28,33]). To address such an issue, we first need to develop relaxation models. Section 3 concerns the definition of the adopted relaxation model. Here, we decide to modify the Suliciu relaxation model designed for the shallow-water equations [7] to enforce a *transport property* to be satisfied by the relaxation model. Such a modification makes the algebraic analysis of the model easier than the one coming from the usual Suliciu model [7]. Unfortunately, the suggested modification makes the Riemann problem underresolved. To correct this failure, we enforce the well-balanced property, in a sense to be prescribed, and thus we obtain the required *Riemann solution* of the suggested relaxation model. As a consequence, we get an approximate Riemann solver for the entropy reformulated Ripa model, which contains the topography source terms. In fact, the enforcement of the well-balanced property inside the *Riemann solution* can be reformulated as relaxation equations. Hence, we give a full relaxation model, which admits a unique Riemann solution. In addition, we show that both relaxation approaches give the same approximate Riemann solver for (1.1). We conclude this section with some comments concerning Cargo-LeRoux's source term reformulation [11] for shallow-water with linear topography function. Recently, extensions have been proposed by Chalons et al. [12] where a Suliciu relaxation model is derived according to the Cargo-LeRoux shallow-water reformulation. From now on, let us underline that the Cargo-LeRoux reformulation cannot be applied here because of the nonlinear topography. However, we show that a relaxation model can be developed in the spirit of the work by Chalons et al. [12], and the resulting approximate Riemann solver for the Ripa model (1.1) once again coincides with those derived below. Section 4 concerns the presentation of the Godunov-type scheme associated to the above approximate Riemann solver. In addition, we establish that the scheme is positive preserving and well-balanced. In addition, it is proved to satisfy a discrete entropy inequality under appropriate regularity conditions on the source term. In fact, the resulting numerical procedure exactly preserves the steady states at rest given by (1.5). Finally, Section 5 is devoted to illustrating the relevance of the suggested Godunov-type scheme, and several numerical experiments are performed. The paper ends with a short conclusion.

2. Basic properties of the Ripa model and entropic reformulation

For the sake of completeness of the present paper, we give here the basic algebraic properties satisfied by the Ripa model (1.1). Since the topography function z does not depend on the time, we have

$$\partial_t z = 0.$$

Then, we rewrite the system Ripa (1.1) in the following equivalent form:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\Theta\dfrac{h^2}{2}\right) = -g\Theta h\partial_x z, \\ \partial_t(h\Theta) + \partial_x(h\Theta u) = 0, \\ \partial_t z = 0. \end{cases} \tag{2.1}$$

Involving the primitive state vector $U = {}^t(h, u, \Theta, z)$, the system (2.1) can be written in the following quasi-linear form:

$$\partial_t U + A(U)\partial_x U = 0,$$

where we have set

$$A(U) = \begin{pmatrix} h & h & 0 & 0 \\ g\Theta & u & gh/2 & g\Theta \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $A(U)$ are

$$\lambda_-(U) = u - c, \qquad \lambda_u(U) = u, \qquad \lambda_0(U) = 0, \qquad \lambda_+(U) = u + c,$$

where $c = \sqrt{g\Theta h}$ denotes the sound speed. The associated eigenvectors are given by

$$r_-(U) = \begin{pmatrix} h \\ -c \\ 0 \\ 0 \end{pmatrix}, \quad r_u(U) = \begin{pmatrix} h \\ 0 \\ -2\Theta \\ 0 \end{pmatrix}, \quad r_0(U) = \begin{pmatrix} h \\ -u \\ 0 \\ \dfrac{u^2}{g\Theta} - h \end{pmatrix}, \quad r_+(U) = \begin{pmatrix} h \\ c \\ 0 \\ 0 \end{pmatrix}.$$

It is worth noticing that the eigenvectors define a basis of $\mathbb{R}^4$ provided that $u \neq \pm c$. As a consequence, the system under consideration is hyperbolic as long as $u \neq \pm c$. In the present work, we do not consider the resonant regime characterized by $u = \pm c$. In the case of the shallow-water model, the reader is referred to [31], where the resonant regime is studied. To our knowledge, resonant regimes for the Ripa model (1.1) were not analyzed.

Next, the characteristic fields associated with $\lambda_+(U)$ and $\lambda_-(U)$ are genuinely nonlinear, while the characteristic fields associated with $\lambda_0(U)$ and $\lambda_u(U)$ are linearly degenerated (see [7, 22, 29, 38] for further details). Indeed, straightforward calculations give

$$\nabla_U\lambda_-(U)\cdot r_-(U) = -\frac{3}{2}c, \qquad \nabla_U\lambda_+(U)\cdot r_+(U) = \frac{3}{2}c,$$
$$\nabla_U\lambda_u(U)\cdot r_u(U) = 0, \qquad \nabla_U\lambda_0(U)\cdot r_0(U) = 0.$$

We next exhibit the Riemann invariants. We denote by $(I_-^{(\ell)})_{\ell=1,2,3}$, $(I_+^{(\ell)})_{\ell=1,2,3}$, $(I_u^{(\ell)})_{\ell=1,2,3}$ and $(I_0^{(\ell)})_{\ell=1,2,3}$ the Riemann invariants associated with λ_-, λ_+, λ_u and λ_0, respectively. The Riemann invariants being defined by (see [22, 29, 38])

$$\nabla_U I^{(\ell)}\cdot r = 0, \qquad \ell = 1, 2, 3,$$

we easily obtain

$$\begin{aligned} &I_{\pm}^{(1)} = u - (\pm c), && I_{\pm}^{(2)} = \Theta, && I_{\pm}^{(3)} = z, \\ &I_{u}^{(1)} = h^2\Theta, && I_{u}^{(2)} = u, && I_{u}^{(3)} = z, \\ &I_{0}^{(1)} = hu, && I_{0}^{(2)} = \Theta, && I_{0}^{(3)} = (h+z) + \frac{u^2}{2g\Theta}. \end{aligned}$$

Since the system (1.1) is hyperbolic, discontinuous solutions may occur in a finite time. In order to rule out nonphysical solutions, the system is endowed with entropy inequalities.

Lemma 2.1. *Let w be a smooth solution of* (1.1)*. Then w satisfies the following additional conservation law:*

$$\partial_t \left(\eta(w) + g\Theta hz\right) + \partial_x \left(G(w) + g\Theta hzu\right) = 0, \tag{2.2}$$

where we have set

$$\eta(w) = h\frac{u^2}{2} + g\Theta\frac{h^2}{2}, \tag{2.3}$$

$$G(w) = \left(h\frac{u^2}{2} + g\Theta h^2\right) u. \tag{2.4}$$

Proof. From the momentum equation (1.1b), we get the relation satisfied by the kinetic energy as follows:

$$\partial_t \frac{u^2}{2} + u\partial_x \frac{u^2}{2} + \frac{u}{h}\partial_x \left(g\Theta\frac{h^2}{2}\right) + g\Theta u\partial_x z = 0. \tag{2.5}$$

Next, multiplying the water height equation (1.1a) by $g\Theta/2$, and multiplying the transport temperature equation (1.1c) by $gh/2$, we respectively get

$$\Theta\partial_t \left(g\frac{h}{2}\right) + \Theta u\partial_x \left(g\frac{h}{2}\right) + g\Theta\frac{h}{2}\partial_x u = 0, \tag{2.6}$$

$$g\frac{h}{2}\partial_t\Theta + g\frac{h}{2}u\partial_x\Theta = 0. \tag{2.7}$$

The sum of (2.5)-(2.6)-(2.7) easily gives

$$\partial_t \left(\frac{u^2}{2} + g\Theta\frac{h}{2}\right) + u\partial_x \left(\frac{u^2}{2} + g\Theta\frac{h}{2}\right) + \frac{1}{h}\partial_x \left(g\Theta\frac{h^2}{2}\right) + g\Theta u\partial_x z = 0.$$

Next, by combining the above relation with the water height equation (1.1a) multiplied by $\left(\frac{u^2}{2} + g\Theta\frac{h}{2}\right)$, we directly obtain

$$\partial_t \left(h\frac{u^2}{2} + g\Theta\frac{h^2}{2}\right) + \partial_x \left(h\frac{u^2}{2} + g\Theta h^2\right) u + gh\Theta u\partial_x z = 0. \tag{2.8}$$

Since we have $\partial_t z = 0$, we deduce from (1.1c) the following relation:

$$\partial_t(g\Theta hz) + z\partial_x(g\Theta hu) = 0. \tag{2.9}$$

The required relation (2.2) comes from the sum of (2.8) and (2.9). The proof is thus achieved. □

As a consequence of the above result, the pair

$$(\eta(w) + g\Theta hz, G(w) + g\Theta hzu)$$

stands for a good candidate to define an entropy pair. After [7], we get the required entropy pair provided that the partial entropy function $w \mapsto \eta(w)$ is a convex function.

Unfortunately, the function $w \mapsto \eta(w)$ is neither convex nor concave. In fact, involving a suitable equivalent reformulation, the system (1.1) can be endowed with a convex entropy. It is worth noticing that, because discontinuous solutions may occur, nonlinear changes of variables do not preserve, in general, the weak solutions of nonlinear hyperbolic systems. However, since the temperature Θ is governed by a transport equation (1.1c), nonlinear reformulation of Θ can be performed. As a consequence, the Ripa model (1.1) admits nonlinear equivalent reformulations as now stated.

Lemma 2.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a smooth invertible function. Let us set*

$$\Theta = \varphi(\theta). \tag{2.10}$$

Then the weak solutions of (1.1) *define weak solutions of the following system:*

$$\partial_t h + \partial_x(hu) = 0, \tag{2.11a}$$

$$\partial_t(hu) + \partial_x\left(hu^2 + g\varphi(\theta)\frac{h^2}{2}\right) = -g\varphi(\theta)h\partial_x z, \tag{2.11b}$$

$$\partial_t(h\theta) + \partial_x(h\theta u) = 0. \tag{2.11c}$$

Conversely, the weak solutions of (2.11) *are also weak solutions of* (1.1).

Proof. First, let us consider smooth solutions. Since Θ satisfies (1.1c), it also satisfies the transport equation

$$\partial_t\Theta + u\partial_x\Theta = 0.$$

By definition of θ, given by (2.10), we easily get

$$\partial_t\theta + u\partial_x\theta = 0,$$

and thus $h\theta$ satisfies the conservation law (2.11c). As a consequence, we have established the equivalence between both systems (1.1) and (2.11) for smooth solutions.

Now, we turn to considering weak solutions. In fact, the proof will be achieved as soon as the equivalence is established for discontinuous solutions. Let us consider a discontinuity connecting w_L to w_R and propagating with velocity σ. This discontinuity is governed by the well-known Rankine-Hugoniot relations (see [22,29,32,38]). Since the topography function is assumed to be smooth, it does not enter the Rankine-Hugoniot relations, and thus the triple (w_L, w_R, σ) satisfies

$$-\sigma(h_R - h_L) + (h_R u_R - h_L u_L) = 0, \tag{2.12a}$$

$$-\sigma(h_R u_R - h_L u_L) + \left(h_R u_R^2 + g\Theta_R\frac{h_R^2}{2} - h_L u_L^2 - g\Theta_L\frac{h_L^2}{2}\right) = 0, \tag{2.12b}$$

$$-\sigma(h_R\Theta_R - h_L\Theta_L) + (h_R\Theta_R u_R - h_L\Theta_L u_L) = 0. \tag{2.12c}$$

From (2.12a), we have

$$h_R(u_R - \sigma) = h_L(u_L - \sigma) = M,$$

to rewrite (2.12b) and (2.12c) as follows:

$$M(u_R - u_L) + g\left(\Theta_R \frac{h_R^2}{2} - \Theta_L \frac{h_L^2}{2}\right) = 0,$$
$$M(\Theta_R - \Theta_L) = 0.$$

Next, by involving the definition of θ given by (2.10), we have to recover the Rankine-Hugoniot relations satisfied by (2.11):

$$-\sigma(h_R - h_L) + (h_R u_R - h_L u_L) = 0, \tag{2.13a}$$

$$-\sigma(h_R u_R - h_L u_L) + \left(h_R u_R^2 + g\varphi(\theta_R)\frac{h_R^2}{2} - h_L u_L^2 - g\varphi(\theta_L)\frac{h_L^2}{2}\right) = 0, \tag{2.13b}$$

$$-\sigma(h_R\theta_R - h_L\theta_L) + (h_R\theta_R u_R - h_L\theta_L u_L) = 0. \tag{2.13c}$$

If $M = 0$, the equivalence between (2.12) and (2.13) is immediate. If $M \neq 0$, we get $\Theta_R = \Theta_L$ to write

$$\varphi(\theta_R) = \varphi(\theta_L).$$

Since φ is invertible, we obviously obtain $\theta_R = \theta_L$, and the required equivalence is established. □

To make concise the notation, let us introduce

$$\tilde{w} = \begin{pmatrix} h \\ hu \\ h\theta \end{pmatrix}, \qquad \tilde{f}(\tilde{w}) = \begin{pmatrix} hu \\ hu^2 + g\varphi(\theta)\dfrac{h^2}{2} \\ h\theta u \end{pmatrix}, \qquad \tilde{S}(\tilde{w}) = \begin{pmatrix} 0 \\ -g\varphi(\theta)h\partial_x z \\ 0 \end{pmatrix},$$

so that the equivalent system (2.11) rewrites as follows:

$$\partial_t \tilde{w} + \partial_x \tilde{f}(\tilde{w}) = \tilde{S}(\tilde{w}), \tag{2.14}$$

where the state vector $\tilde{w}$ belongs to the phase space

$$\tilde{\Omega} = \left\{\tilde{w} \in \mathbb{R}^3;\ h > 0\right\}. \tag{2.15}$$

Equipped with this new formulation of the Ripa model, we are able to exhibit the needed convex entropy.

Lemma 2.3. *The smooth solutions of* (2.11) *satisfy the following additional conservation law:*

$$\partial_t\left(\tilde{\eta}(\tilde{w}) + g\varphi(\theta)hz\right) + \partial_x\left(\tilde{G}(\tilde{w}) + g\varphi(\theta)hzu\right) = 0, \tag{2.16}$$

$$\tilde{\eta}(\tilde{w}) = h\frac{u^2}{2} + g\varphi(\theta)\frac{h^2}{2}, \tag{2.17}$$

$$\tilde{G}(\tilde{w}) = \left(h\frac{u^2}{2} + g\varphi(\theta)h^2\right)u. \tag{2.18}$$

Moreover, the partial entropy function $\tilde{w} \mapsto \tilde{\eta}(\tilde{w})$ is convex as soon as the function $\varphi : \mathbb{R} \to \mathbb{R}^+$ satisfies for all $\theta \in \mathbb{R}$:

$$\begin{aligned} &\varphi''(\theta)\varphi(\theta) - \frac{1}{2}\varphi'(\theta)^2 > 0, \\ &\varphi(\theta) - \theta\varphi'(\theta) + \frac{\theta^2}{2}\varphi''(\theta) > 0. \end{aligned} \tag{2.19}$$

Proof. The additional law (2.16) satisfied by the smooth solutions of (2.11) is a direct consequence of the equivalence between (1.1) and (2.11), and the additional law (2.2) satisfied by the smooth solutions of (1.1). Next, the condition to enforce the convexity of $\tilde{w} \mapsto \tilde{\eta}(\tilde{w})$ comes from a straightforward evaluation of the Hessian matrix of $\tilde{\eta}$. □

From now on, let us emphasize that functions φ which satisfy the assumptions (2.19) can be easily reached. For instance, we can select

$$\varphi(\theta) = \mathrm{e}^{\theta}\,.$$

To summarize this section, in order to exhibit entropy requirements, we have introduced a reformulation of the Ripa model. This reformulation, given by (2.11), is equivalent to the initial system (1.1) for the weak solutions. Moreover, the Ripa reformulated model is endowed with an entropy pair.

3. Relaxation models

Our objective is to derive accurate numerical schemes to approximate the weak solutions of the Ripa model (1.1). According to the results stated in the above section, we now consider the entropic reformulated Ripa model (2.11).

Following the work by Harten, Lax and van Leer [26], we suggest developing finite volume methods based on a suitable approximate Riemann solver. The main purpose of this section is to design an approximate Riemann solver by adopting relaxation techniques.

We propose to approximate the weak solutions of the system (2.11) by the weak solutions of a first-order system with singular perturbations, namely the relaxation model. After the work by Chen et al. [15] (for instance, see also [2, 14, 18, 28, 35]), we introduce relaxation models which restore the initial system within the limit of a relaxation parameter. Here, we present three relaxation strategies, but they lead to the same approximate Riemann solver.

3.1. **An incomplete Suliciu-type model.**

Involving the work by Bouchut [7] (see also [2, 12, 13, 18]) concerning the derivation of the Suliciu relaxation model for shallow-water equations, we suggest linearizing the hydrostatic pressure $g\varphi(\theta)h^2/2$. To address such an issue, we approximate the hydrostatic pressure by a new unknown π governed by the following PDE:

$$\partial_t \pi + u\partial_x \pi + \frac{a^2}{h}\partial_x u = \frac{1}{\varepsilon}\left(g\varphi(\theta)\frac{h^2}{2} - \pi\right).$$

The relaxation parameter $a > 0$ will be restricted according to robustness and stability conditions detailed later. Here, $\varepsilon > 0$ represents the relaxation time dedicated to tending to zero. Formally, in the limit of ε to zero, the relaxation source term implies π converges with the expected pressure law $g\varphi(\theta)h^2/2$.

Now, unlike the original Suliciu model proposed in [7] for the shallow-water equations, we decide to approximate the topography. In general, the exact topography equation,

$$\partial_t z = 0,$$

is considered within the relaxation model. But such a topography stationary equation may introduce some nonlinearities in the relaxation model which makes difficult the resolution of the associated Riemann problem. As a consequence, we propose

approximating the topography by a new unknown Z governed by a transport relaxation equation as follows:

$$\partial_t Z + u\partial_x Z = \frac{1}{\varepsilon}(z - Z).$$

We will see that this approximation makes easy the algebra of the adopted relaxation model, but will result in a missing relation to solve the Riemann problem. In fact, an additional relation, related to the steady states, will be considered to solve the Riemann problem.

As suggested, we approximate the entropy weak solutions of (2.11) by the weak solutions of the following relaxation model:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \pi\right) + g\varphi(\theta)h\partial_x Z = 0, \\ \partial_t(h\theta) + \partial_x(h\theta u) = 0, \\ \partial_t(h\pi) + \partial_x((h\pi + a^2)u) = \dfrac{h}{\varepsilon}\left(g\varphi(\theta)\dfrac{h^2}{2} - \pi\right), \\ \partial_t(hZ) + \partial_x(hZu) = \dfrac{h}{\varepsilon}(z - Z). \end{cases} \tag{3.1}$$

It is worth noticing that as soon as the relaxation parameter ε goes to zero, from the equations governing $(h, hu, h\theta)$ in (3.1) we formally recover the initial model (2.11).

For the sake of clarity in the notation, we set

$$W = {}^t(h, hu, h\theta, h\pi, hZ) \tag{3.2}$$

to designate the state vectors in the following phase space:

$$\mathcal{O} = \left\{W \in \mathbb{R}^5;\ h > 0\right\}.$$

In the limit of ε to zero, a relaxation equilibrium state is reached; namely, for all $\tilde{w} \in \tilde{\Omega}$:

$$W^{eq}(\tilde{w}) = {}^t\left(h, hu, h\theta, hg\varphi(\theta)\frac{h^2}{2}, hz\right). \tag{3.3}$$

In order to derive the expected relaxation scheme, we have to exhibit the solutions of the Riemann problem associated to the homogeneous system extracted from (3.1). For the sake of simplicity in the forthcoming developments, we denote by $(3.1)_{\text{homo}}$ the homogeneous system obtained by vanishing the source term in (3.1).

Concerning the algebra of the homogeneous system extracted from (3.1), with clear notation denoted by $(3.1)_{\text{homo}}$, we have the following easy statement given without proof (for instance, see [2, 5, 12]):

Lemma 3.1. *Let $a > 0$ be given. The homogeneous system extracted from* (3.1) *is hyperbolic for all $W \in \mathcal{O}$. The eigenvalues of the system are $\lambda^{\pm} = u \pm a/h$ and $\lambda^u = u$. The eigenvalue $\lambda^u = u$ is of multiplicity three. All the fields are linearly degenerated.*

Now, we are interested in solving the Riemann problem associated to $(3.1)_{\text{homo}}$. We thus consider initial data made of two constant states separated by a discontinuity located at $x = 0$ as follows:

$$W_0(x) = \begin{cases} W_L & \text{if } x < 0, \\ W_R & \text{if } x > 0. \end{cases} \tag{3.4}$$

We note that if the solution of the Riemann problem for $(3.1)_{\text{homo}}$ exists, then it is made of four constant states separated by three contact discontinuities. Hence, we consider solutions in the form

$$W_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right) = \begin{cases} W_L & \text{if } x/t < \lambda^-, \\ W_L^\star & \text{if } \lambda^- < x/t < \lambda^u, \\ W_R^\star & \text{if } \lambda^u < x/t < \lambda^+, \\ W_R & \text{if } x/t > \lambda^+. \end{cases} \tag{3.5}$$

Since a contact discontinuity is defined by the continuity of the Riemann invariants associated with the characteristic field under consideration, we now exhibit the Riemann invariants.

For the characteristic fields defined by $\lambda^\pm$, we easily get the following Riemann invariants:

$$I_1^\pm(W) = u \pm \frac{a}{h}, \qquad I_2^\pm(W) = \pi - (\pm a u), \qquad I_3^\pm(W) = \theta, \qquad I_4^\pm(W) = Z. \tag{3.6}$$

Concerning the characteristic field with eigenvalue $\lambda^u = u$, since the multiplicity is three, we are waiting for two independent Riemann invariants.

Lemma 3.2. *The characteristic field of* $(3.1)_{\text{homo}}$ *defined by the eigenvalue* $\lambda^u = u$ *admits only one Riemann invariant given by*

$$I_1^u(W) = u.$$

Proof. First, since the derivation of the Riemann invariants is free from any changes of variables, it is here convenient to rewrite $(3.1)_{\text{homo}}$ in the following quasi-linear form:

$$\partial_t V + B(V)\partial_x V = 0 \qquad \text{with } V = {}^t(h, u, \theta, \pi, Z),$$

where the matrix $B(V)$ finds an easy explicit form. After straightforward computations, the eigenspace associated with $\lambda^u = u$ is defined by the following three eigenvectors:

$$r_1^u(V) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad r_2^u(V) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad r_3^u(V) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -g\varphi(\theta)h \\ 1 \end{pmatrix}.$$

By definition of the Riemann invariants, we look for functions $I^u(V)$ such that $\nabla_V I^u(V) \cdot r_i^u(V) = 0$ with $i = 1$, 2 and 3. We directly see that the function $V_1^u(V) = u$ is the only function that satisfies the Riemann invariant definition. $\square$

As a consequence of the above result, it is not possible to solve the Riemann problem associated with $(3.1)_{\text{homo}}$. Indeed, the intermediate states $W_L^\star$ and $W_R^\star$

are solutions of the following system:

$$\begin{aligned} &I_i^-(W_L^\star) = I_i^-(W_L), \qquad i = 1,2,3,4,\\ &I_i^+(W_R^\star) = I_i^+(W_R), \qquad i = 1,2,3,4,\\ &I_1^u(W_L^\star) = I_1^u(W_R^\star), \end{aligned} \tag{3.7}$$

and thus one relation is missing. To close the system (3.7), we arbitrarily impose the following relation:

$$\pi_R^\star - \pi_L^\star = -g\varphi(\bar\theta(W_L,W_R))\bar h(W_L,W_R)(Z_R^\star - Z_L^\star), \tag{3.8}$$

where the functions $\bar h : \mathcal{O}\times\mathcal{O} \to \mathbb{R}^+$ and $\bar\theta : \mathcal{O}\times\mathcal{O} \to \mathbb{R}$ denote h-average and θ-average functions, respectively. At this stage, we only assume that these two functions satisfy the consistency property

$$h_L = h_R = h \;\Rightarrow\; \bar h(W_L,W_R) = h \quad\text{and}\quad \theta_L = \theta_R = \theta \;\Rightarrow\; \bar\theta(W_L,W_R) = \theta$$

and the symmetry property

$$\bar h(W_R,W_L) = \bar h(W_L,W_R) \quad\text{and}\quad \bar\theta(W_R,W_L) = \bar\theta(W_L,W_R).$$

A precise definition of $\bar h$ and $\bar\theta$ will be given later in order to get the required well-balanced property.

From now on, let us underline that the additional relation (3.8) has been fixed according to the definition of the steady states at rest given by (1.4)-(2.10):

$$u = 0 \qquad\text{and}\qquad \partial_x\left(g\varphi(\theta)\frac{h^2}{2}\right) = -g\varphi(\theta)\partial_x z. \tag{3.9}$$

Indeed, since both relaxation unknowns π and Z will have the behavior of $g\varphi(\theta)h^2/2$ and z, we notice that the identity (3.8) can be understood, in a sense to be prescribed, as an approximation of (3.9).

We are now able to give the full characterization of the Riemann solution of $(3.1)_{\text{homo}}$-(3.4) supplemented by the relation (3.8).

Lemma 3.3. *The Riemann solution of the system* $(3.1)_{\text{homo}}$-(3.4) *completed by the relation* (3.8) *is given by* (3.5), *where the intermediate states* $W_L^\star$ *and* $W_R^\star$ *are given by*

$$\theta_L^\star = \theta_L, \qquad \theta_R^\star = \theta_R, \tag{3.10a}$$

$$Z_L^\star = Z_L, \qquad Z_R^\star = Z_R, \tag{3.10b}$$

$$\begin{aligned} u^\star &= u_L^\star = u_R^\star\\ &= \frac{1}{2}(u_L+u_R) - \frac{1}{2a}(\pi_R - \pi_L)\\ &\quad - \frac{g}{2a}\varphi(\bar\theta(W_L,W_R))\bar h(W_L,W_R)(Z_R - Z_L), \end{aligned} \tag{3.10c}$$

$$\pi_L^\star = \pi_L + a(u_L - u^\star), \qquad \pi_R^\star = \pi_R + a(u^\star - u_R), \tag{3.10d}$$

$$\frac{1}{h_L^\star} = \frac{1}{h_L} + \frac{1}{a}(u^\star - u_L), \qquad \frac{1}{h_R^\star} = \frac{1}{h_R} + \frac{1}{a}(u_R - u^\star). \tag{3.10e}$$

Proof. The intermediate states are solutions of the system (3.7)-(3.8). A straightforward calculation gives the expected result. □

In fact, when deriving the full discrete scheme, the main numerical properties will be implied by the Riemann solution (3.5)-(3.10) with initial data given by (3.4) satisfying the relaxation equilibrium condition (3.3). Put in other words, the initial data for the relation unknowns π and Z coincide with the relaxed values respectively given by $g\varphi(\theta)h^2/2$ and z. For the sake of clarity in the notation, we denote by

$$\tilde{w}^{eq}\left(\frac{x}{t};\tilde{w}_L,\tilde{w}_R\right) = W_{\mathcal{R}}^{(h,hu,h\theta)}\left(\frac{x}{t};W^{eq}(\tilde{w}_L),W^{eq}(\tilde{w}_R)\right) \tag{3.11}$$

such a specific approximate solution. From now on, let us underscore that $\tilde{w}^{eq}$ is nothing but an approximate Riemann solver for (2.11).

Now, we establish the main properties satisfied by $\tilde{w}^{eq}$. First, we prove the robustness of this approach.

Lemma 3.4. *Let $\tilde{w}_L$ and $\tilde{w}_R$ be two constant states in $\tilde{\Omega}$, given by* (2.15)*. Assume that the relaxation parameter a is large enough to endure the following estimations:*

$$u_L - \frac{a}{h_L} < u^\star < u_R + \frac{a}{h_R}, \tag{3.12}$$

where $u^\star$ is defined by (3.10c)*. Then $\tilde{w}^{eq}(x/t;\tilde{w}_L,\tilde{w}_R)$ belongs to $\tilde{\Omega}$.*

Proof. First, because of the definition of $u^\star$ given by (3.10c), we immediately remark that (3.12) is satisfied provided a is large enough.

Next, by definition of the Riemann invariants (3.6), we have

$$u_L - \frac{a}{h_L} = u^* - \frac{a}{h_L^*} \quad \text{and} \quad u_R + \frac{a}{h_R} = u^* + \frac{a}{h_R^*},$$

so that (3.12) can be rewritten as $-h_L^* < 0 < h_R^*$. As a consequence, we obtain the positiveness of the intermediate water heights, and the proof is thus completed. □

Next, we exhibit a specific behavior of the approximate Riemann solver $\tilde{w}^{eq}$, as soon as $\tilde{w}_L$ and $\tilde{w}_R$ satisfy steady state at rest type conditions.

Lemma 3.5. (i) *Let $\tilde{w}_L$ and $\tilde{w}_R$ be given in $\tilde{\Omega}$ such that*

$$u_L = u_R = 0, \tag{3.13}$$

$$\varphi(\theta_R)\frac{h_R^2}{2} - \varphi(\theta_L)\frac{h_L^2}{2} + \varphi(\bar{\theta}^{eq}(\tilde{w}_L,\tilde{w}_R))\bar{h}^{eq}(\tilde{w}_L,\tilde{w}_R)(z_R - z_L) = 0, \tag{3.14}$$

where we have set

$$\bar{h}^{eq}(\tilde{w}_L,\tilde{w}_R) = \bar{h}(W^{eq}(\tilde{w}_L),W^{eq}(\tilde{w}_R)),$$
$$\bar{\theta}^{eq}(\tilde{w}_L,\tilde{w}_R) = \bar{\theta}(W^{eq}(\tilde{w}_L),W^{eq}(\tilde{w}_R)).$$

Then we get $\tilde{w}^{eq}$ at rest:

$$\tilde{w}^{eq}(x/t;\tilde{w}_L,\tilde{w}_R) = \begin{cases} \tilde{w}_L & \text{if } x < 0, \\ \tilde{w}_R & \text{if } x > 0. \end{cases} \tag{3.15}$$

(ii) *Let $\tilde{w}_L$ and $\tilde{w}_R$ be in $\tilde{\Omega}$ such that the lake at rest steady state condition* (1.5a) *is satisfied:*

$$u_L = u_R = 0,$$
$$\theta_L = \theta_R,$$
$$h_L + z_L = h_R + z_R.$$

In addition, assume

$$\bar{h}^{eq}(\tilde{w}_L, \tilde{w}_R) = \frac{1}{2}(h_L + h_R). \tag{3.16}$$

Then we get $\tilde{w}^{eq}$ *at rest given by* (3.15).

(iii) *Let* $\tilde{w}_L$ *and* $\tilde{w}_R$ *in* $\tilde{\Omega}$ *be such that the isobaric steady state condition* (1.5b) *is satisfied:*

$$\begin{aligned} u_L &= u_R = 0, \\ z_L &= z_R, \\ \varphi(\theta_L)h_L^2 &= \varphi(\theta_R)h_R^2. \end{aligned}$$

In addition, assume that $\bar{h}^{eq}$ *is given by* (3.16). *Then we get* $\tilde{w}^{eq}$ *at rest given by* (3.15).

(iv) *Let* $\tilde{w}_L$ *and* $\tilde{w}_R$ *in* $\tilde{\Omega}$ *be such that the constant water height steady state condition* (1.5c) *is satisfied:*

$$\begin{aligned} u_L &= u_R = 0, \\ h_L &= h_R, \\ z_L + \frac{h_L}{2}\ln\varphi(\theta_L) &= z_R + \frac{h_R}{2}\ln\varphi(\theta_R). \end{aligned} \tag{3.17}$$

In addition, assume that $\bar{\theta}^{eq}$ *is defined as follows:*

$$\bar{\theta}^{eq}(\tilde{w}_L, \tilde{w}_R) = \begin{cases} \varphi^{-1}\left(\dfrac{\varphi(\theta_R) - \varphi(\theta_L)}{\ln(\varphi(\theta_R)) - \ln(\varphi(\theta_L))}\right), & \text{if } \theta_L \neq \theta_R, \\ \theta_L & \text{if } \theta_L = \theta_R. \end{cases} \tag{3.18}$$

Then we get $\tilde{w}^{eq}$ *at rest given by* (3.15).

Proof. To establish that $\tilde{w}^{eq}$ stays at rest, since $u_L = u_R = 0$ and because of the definition of the intermediate states given by (3.10), it suffices to prove that $u^\star = 0$. By involving equilibrium left and right nonmoving states, $u^\star$ is now defined by

$$u^\star = -\frac{g}{2a}\left(\varphi(\theta_R)\frac{h_R^2}{2} - \varphi(\theta_L)\frac{h_L^2}{2} + \varphi(\bar{\theta}^{eq}(\tilde{w}_L, \tilde{w}_R))\bar{h}(\tilde{w}_L, \tilde{w}_R)(z_R - z_L)\right).$$

The statement (i) immediately comes from (3.14).

Concerning the statement (ii), since $\theta_L = \theta_R$, we have $\bar{\theta}(\tilde{w}_L, \tilde{w}_R) = \theta_L = \theta_R$. Then, we rewrite $u^\star$ as follows:

$$u^\star = -\frac{g}{2a}\theta_R\left((h_R - h_L)\frac{h_L + h_R}{2} + \bar{h}^{eq}(\tilde{w}_L, \tilde{w}_R)(z_R - z_L)\right),$$

to deduce $u^\star = 0$ from (3.16) and $h_L + z_L = h_R + z_R$.

Next, statement (iii) is obtained in a similar way.

To establish the last statement (iv), since $h_L = h_R$, we get $\bar{h}(\tilde{w}_L, \tilde{w}_R) = h_L = h_R$. Then $u^\star$ reads

$$u^\star = -\frac{g}{2a}h_R\left(\frac{h_R}{2}(\varphi(\theta_R) - \varphi(\theta_L)) + \varphi(\bar{\theta}^{eq}(\tilde{w}_L, \tilde{w}_R))(z_R - z_L)\right).$$

But from (3.17), we have

$$z_R - z_L = -\frac{h_R}{2}\left(\ln(\varphi(\theta_R)) - \ln(\varphi(\theta_L))\right).$$

Then we immediately deduce $u^\star = 0$ from the definition of the θ-average (3.18). The proof is thus achieved. $\square$

The derivation of an approximate Riemann solver based on an incomplete relaxation model may appear a bit cavalier. We now exhibit additional interpretation of $\tilde{w}^{eq}$ coming from a complete relaxation model.

3.2. **A complete Suliciu relaxation model.** We suggest modifying the relaxation model (3.1) in order to obtain a well-posed relaxation system. In fact, the ill-posed failure of (3.1) comes from the topography source term $g\varphi(\theta)h\partial_x z$. As a consequence, we propose to introduce new relaxation variables to enforce a stronger linearization. We thus adopt the following equation to govern the relaxation momentum:

$$\partial_t(hu) + \partial_x(hu^2 + \pi) + g\varphi(\bar{\theta}(X^-, X^+))\bar{h}(X^-, X^+)\partial_x Z = 0,$$

where $\bar{h} : \mathcal{O} \times \mathcal{O} \to \mathbb{R}^+$ and $\bar{\theta} : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ designate the average functions introduced in (3.8). The new variables $X^\pm$ must represent W_L and W_R according to (3.2). To address such an issue, we adopt the following law to govern $X^\pm$:

$$\begin{aligned}
\partial_t X^+ + (u+\delta)\partial_x X^+ &= \frac{1}{\varepsilon}(W - X^+),\\
\partial_t X^- + (u-\delta)\partial_x X^- &= \frac{1}{\varepsilon}(W - X^-),
\end{aligned}$$

where $\delta > 0$ is a small parameter to be fixed later.

We are now able to give the full relaxation model:

$$\left\{\begin{aligned}
&\partial_t h + \partial_x(hu) = 0,\\
&\partial_t(hu) + \partial_x\left(hu^2 + \pi\right) + g\varphi(\bar{\theta}(X^-, X^+))\bar{h}(X^-, X^+)\partial_x Z = 0,\\
&\partial_t(h\theta) + \partial_x(h\theta u) = 0,\\
&\partial_t(h\pi) + \partial_x((h\pi + a^2)u) = \frac{h}{\varepsilon}\left(g\varphi(\theta)\frac{h^2}{2} - \pi\right),\\
&\partial_t(hZ) + \partial_x(hZu) = \frac{h}{\varepsilon}(z - Z),\\
&\partial_t(hX^+) + \partial_x(hX^+u) + \delta h\partial_x X^+ = \frac{h}{\varepsilon}(W - X^+),\\
&\partial_t(hX^-) + \partial_x(hX^-u) - \delta h\partial_x X^- = \frac{h}{\varepsilon}(W - X^-).
\end{aligned}\right. \tag{3.19}$$

To simplify the notation, let us set

$$\mathcal{W} = {}^t(W, hX^+, hX^-).$$

In order to exhibit the Riemann solution associated to the system (3.19), we first give the algebra of the first-order system extracted from (3.19), which we denote $(3.19)_{\text{homo}}$.

From now on, let us emphasize that $(3.19)_{\text{homo}}$ involves more unknowns than $(3.1)_{\text{homo}}$. In fact, the additional equations in $(3.19)_{\text{homo}}$ can be understood as a reformulation of the complementary relation (3.8). As a consequence and as established in this section, the resulting scheme derived from the relaxation model (3.19) will exactly coincide with the relaxation scheme derived from the relaxation model (3.1) supplemented by the additional relation (3.8).

Lemma 3.6. *Let $a > 0$ be given. Then the homogeneous system* $(3.19)_{\text{homo}}$ *is hyperbolic for all $\mathcal{W} \in \mathcal{O} \times \mathbb{R}^5 \times \mathbb{R}^5$. The eigenvalues of the system are $\lambda^{\pm} = u \pm \frac{a}{h}$ (single), $\lambda^u = u$ (triple) and $\lambda^{\pm\delta} = u \pm \delta$ (multiplicity five). All the fields are linearly degenerated.*

To solve the Riemann problem made of $(3.19)_{\text{homo}}$ with initial data given by

$$\mathcal{W}_0(x) = \begin{cases} \mathcal{W}_L, & \text{if } x < 0, \\ \mathcal{W}_R, & \text{if } x > 0, \end{cases} \tag{3.20}$$

we need the Riemann invariants for each field.

For the characteristic fields associated with the eigenvalues $\lambda^{\pm}$, the Riemann invariants are $I_j^{\pm}$, $1 \le j \le 4$, defined by (3.6) completed by

$$I_5^{\pm} = X^-, \qquad I_6^{\pm} = X^+.$$

Concerning the Riemann invariants coming from the field with eigenvalue $\lambda^u = u$, we have

$$I_1^u = u, \quad I_2^u = X^+, \quad I_3^u = X^-, \quad I_4^u = \pi + g\varphi(\bar{\theta}(X^-, X^+))\bar{h}(X^-, X^+)Z.$$

Finally, the Riemann invariants associated to $\lambda^{\pm\delta}$ are

$$I_1^{\pm\delta} = h, \quad I_2^{\pm\delta} = u, \quad I_3^{\pm\delta} = \theta, \quad I_4^{\pm\delta} = \pi, \quad I_5^{\pm\delta} = Z, \quad I_6^{\pm\delta} = X^{\mp}.$$

Lemma 3.7. *Assume $a > 0$ large enough such that the robustness condition* (3.12) *is satisfied. Moreover, assume $\delta > 0$ small enough to satisfy*

$$u_L - \frac{a}{h_L} < u^\star - \delta < u^\star < u^\star + \delta < u_R + \frac{a}{h_R},$$

where $u^\star$ is defined by (3.10c)*. Then the Riemann problem* $(3.19)_{\text{homo}}$*-*(3.20) *admits a unique solution denoted by $\mathcal{W}_{\mathcal{R}}(x/t; \mathcal{W}_L, \mathcal{W}_R)$.*

In addition, both Riemann solutions of $(3.19)_{\text{homo}}$*-*(3.20) *and* $(3.1)_{\text{homo}}$*-*(3.4) *coincide in the following sense:*

$$\mathcal{W}_{\mathcal{R}}^{(h,hu,h\theta,h\pi,hZ)}\left(\frac{x}{t}; \mathcal{W}_L, \mathcal{W}_R\right) = W_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right), \tag{3.21}$$

provided we have

$$\mathcal{W}_{L,R} = {}^t(W_{L,R}, h_{L,R}W_{L,R}, h_{L,R}W_{L,R}).$$

As a consequence, we obtain the same approximate Riemann solver (3.11)*:*

$$\begin{aligned} \mathcal{W}_L &= {}^t\left(W^{eq}(\tilde{w}_L), h_L W^{eq}(\tilde{w}_L), h_L W^{eq}(\tilde{w}_L)\right), \\ \mathcal{W}_R &= {}^t\left(W^{eq}(\tilde{w}_R), h_R W^{eq}(\tilde{w}_R), h_R W^{eq}(\tilde{w}_R)\right), \\ \mathcal{W}_{\mathcal{R}}^{(h,hu,h\theta)}\left(\frac{x}{t}; \mathcal{W}_L, \mathcal{W}_R\right) &= \tilde{w}^{eq}\left(\frac{x}{t}; \tilde{w}_L, \tilde{w}_R\right). \end{aligned}$$

Proof. Since $(3.19)_{\text{homo}}$ is hyperbolic with only linearly degenerated fields, the Riemann solution is made of six constant states separated by contact discontinuities. By involving the trivial Riemann invariants, we easily obtain a Riemann solution

in the following form:

$$\mathcal{W}_{\mathcal{R}}\left(\frac{x}{t};\mathcal{W}_L,\mathcal{W}_R\right)=\begin{cases}\mathcal{W}_L & \text{if } \dfrac{x}{t}<\lambda^-,\\ {}^t(W_L^\star,h_L^\star W_L,h_L^\star W_L) & \text{if } \lambda^-<\dfrac{x}{t}<\lambda^{-\delta},\\ {}^t(W_L^\star,h_L^\star W_R,h_L^\star W_L) & \text{if } \lambda^{-\delta}<\dfrac{x}{t}<\lambda^{u},\\ {}^t(W_R^\star,h_R^\star W_R,h_R^\star W_L) & \text{if } \lambda^{u}<\dfrac{x}{t}<\lambda^{+\delta},\\ {}^t(W_R^\star,h_R^\star W_R,h_R^\star W_R) & \text{if } \lambda^{+\delta}<\dfrac{x}{t}<\lambda^{-},\\ \mathcal{W}_R & \text{if } \dfrac{x}{t}>\lambda^{+},\end{cases}$$

where $W_{L,R}^\star$ are defined according to (3.5). We here skip the details of computations, but the continuity of the Riemann invariants across each contact wave gives a system made of (3.7) and (3.8) to solve the intermediate states $W_L^\star$ and $W_R^\star$. As a consequence, we exactly recover the same definition of $W_L^\star$ and $W_R^\star$ as stated in Lemma 3.3. The equivalence relation (3.21) is thus established and the proof is completed. □

3.3. **A Cargo-LeRoux formulation** [11]**.** We now propose a last reformulation of the approximate Riemann solver (3.11). It is based on an interpretation of the source term introduced by Cargo and LeRoux [11] and recently revisited by Chalons et al. [12]. In fact, Cargo-LeRoux's reformulation concerns linear topography functions. Hence, such an approach is not directly available here since we deal with general topography functions.

However, for the sake of simplicity in the forthcoming developments, let us momentarily assume

$$z(x)=x.$$

By introducing a potential function $q(x,t)$ as follows:

$$\partial_x q = g\varphi(\theta)h, \tag{3.22}$$

it is possible to establish the following law satisfied by q:

$$\partial_t(hq)+\partial_x(hqu)=0. \tag{3.23}$$

Then, by extension of the work by Cargo and LeRoux [11], the weak solutions of the Ripa model (2.11) are solutions of the Ripa model with potential:

$$\begin{cases}\partial_t h+\partial_x(hu)=0,\\ \partial_t(hu)+\partial_x\left(hu^2+g\varphi(\theta)\dfrac{h^2}{2}\right)+\partial_x q=0,\\ \partial_t(h\theta)+\partial_x(h\theta u)=0,\\ \partial_t(hq)+\partial_x(hqu)=0.\end{cases}$$

By adopting this equivalent formulation, Chalons et al. [12] have derived (for the Euler with friction equations) a Suliciu relaxation model as follows:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \pi\right) + \partial_x q = 0, \\ \partial_t(h\theta) + \partial_x(h\theta u) = 0, \\ \partial_t(hq) + \partial_x(hqu) = 0, \\ \partial_t(h\pi) + \partial_x((h\pi + a^2)u) = \dfrac{h}{\varepsilon}\left(g\varphi(\theta)\dfrac{h^2}{2} - \pi\right). \end{cases} \tag{3.24}$$

Now, since the present work is devoted to the Ripa model with general topography function, we have to consider a potential in the form

$$\partial_x q = g\varphi(\theta)h\partial_x z.$$

Because $\partial_x z \neq 1$, the relation (3.23) is no longer satisfied by the potential function q. Hence, the natural derivation of the Suliciu-type model (3.24) cannot be performed. However, we suggest considering the model (3.24) but for a suitable relaxation of q. The relaxation model under consideration thus reads:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \pi\right) + \partial_x q = 0, \\ \partial_t(h\theta) + \partial_x(h\theta u) = 0, \\ \partial_t(h\pi) + \partial_x((h\pi + a^2)u) = \dfrac{1}{\varepsilon}\left(g\varphi(\theta)\dfrac{h^2}{2} - \pi\right), \\ \partial_t(hq) + \partial_x(hqu) = \dfrac{h}{\varepsilon}\left(\displaystyle\int^x g\varphi(\theta)h\partial_x z\, dx - q\right). \end{cases} \tag{3.25}$$

We set

$$\tilde{\mathcal{W}} = {}^t(h, hu, h\theta, h\pi, hq)$$

in the open set

$$\tilde{\mathcal{O}} = \left\{\tilde{\mathcal{W}} \in \mathbb{R}^5;\ h > 0\right\}.$$

It is worth noticing the very specific form of the relaxation equilibrium reached as soon as ε goes to zero. Indeed, we get the following equilibrium state:

$$\tilde{\mathcal{W}}^{eq} = {}^t\left(h, hu, h\theta, h\int^x g\varphi(\theta)h\partial_x z\, dx, g\varphi(\theta)\frac{h^2}{2}\right),$$

where the primitive function $\int^x g\varphi(\theta)h\partial_x z\, dx$ will be approximated later.

Once again, let us denote by $(3.25)_{\text{homo}}$ the homogeneous first-order system extracted from (3.25). We now solve the Riemann problem with initial data given by

$$\tilde{\mathcal{W}}_0(x) = \begin{cases} \tilde{\mathcal{W}}_L & \text{if } x < 0, \\ \tilde{\mathcal{W}}_R & \text{if } x > 0. \end{cases} \tag{3.26}$$

Lemma 3.8. *The system* $(3.25)_{\text{homo}}$ *is hyperbolic. The eigenvalues are* $\lambda^- = u_L - a/h_L$, $\lambda^u = u^\star$ *and* $\lambda^+ = u_R + a/h_R$, *where* $u^\star$ *is defined by*

$$u^\star = \frac{1}{2}(u_L + u_R) - \frac{1}{2a}(\pi_R + q_R - \pi_L - q_L). \tag{3.27}$$

The eigenvalue $\lambda^u = u^\star$ is of multiplicity three. Moreover, assume that the relaxation parameter satisfies (3.12)*. Then the Riemann problem* $(3.25)_{\text{homo}}$-(3.26) *admits a unique solution*

$$\tilde{\mathcal{W}}_{\mathcal{R}}\left(\frac{x}{t};\tilde{\mathcal{W}}_L,\tilde{\mathcal{W}}_R\right)=\begin{cases}\tilde{\mathcal{W}}_L & \text{if } \dfrac{x}{t}<\lambda^-,\\ \tilde{\mathcal{W}}_L^\star & \text{if } \lambda^-<\dfrac{x}{t}<\lambda^u,\\ \tilde{\mathcal{W}}_R^\star & \text{if } \lambda^u<\dfrac{x}{t}<\lambda^+,\\ \tilde{\mathcal{W}}_R & \text{if } \dfrac{x}{t}>\lambda^+,\end{cases}$$

where the intermediate states are given by the relations (3.10a)*,* (3.10b)*,* (3.10d)*,* (3.10e) *and* (3.27)*.*

We skip the proof of this result, which turns out to be fully similar to the establishment of Lemma 3.3 or Lemma 3.7. The reader is also referred to [2, 5, 6, 8, 9, 13].

Next, we remark that the Riemann solution solely involves the quantity $q_R - q_L$ where q is defined as a primitive function. To approximate the relaxation unknown q, we adopt the following formula:

$$q_R - q_L = g\varphi(\bar{\theta}(W_L, W_R))\bar{h}(W_L, W_R)(z_R - z_L), \tag{3.28}$$

where $\bar{h}$ and $\bar{\theta}$ are average functions once again defined according to (3.8). As a consequence of this approximation, we immediately recover the introduced approximate Riemann solver

$$\tilde{\mathcal{W}}_{\mathcal{R}}^{(h,hu,h\theta)}\left(\frac{x}{t};{}^t(W^{eq}(\tilde{w}_L),h_Lq_L),{}^t(W^{eq}(\tilde{w}_R),h_Rq_R)\right)=\tilde{w}^{eq}\left(\frac{x}{t};\tilde{w}_L,\tilde{w}_R\right),$$

where q_L and q_R satisfy (3.28).

4. The relaxation scheme

We adopt $\tilde{w}^{eq}(x/t;\tilde{w}_L,\tilde{w}_R)$ as the approximate Riemann solver to derive the relaxation numerical scheme to approximate the weak solutions of (2.11). To discretize space, we consider a uniform mesh made of cells $(x_{i-1/2}, x_{i+1/2})$ with a constant size Δx. Similarly, the time is discretized by considering constant step Δt so that $t^{n+1} = t^n + \Delta t$ with $n \in \mathbb{N}$. The time step will be restricted according to a CFL condition to be defined. The topography is discretized as follows:

$$z_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} z(x)dx. \tag{4.1}$$

At time t^n , we assume known a piecewise constant approximation of the solution of (2.11) given by

$$\tilde{w}^n(x,t^n) = \tilde{w}_i^n, \qquad x \in (x_{i-1/2}, x_{i+1/2}).$$

To evolve in time this approximation, we consider the juxtaposition of the approximate Riemann solver at each interface $x_{i+1/2}$:

$$\tilde{w}^n(x,t^n+t) = \tilde{w}^{eq}\left(\frac{x}{t};\tilde{w}_i^n,\tilde{w}_{i+1}^n\right), \qquad x \in (x_i, x_{i+1}) \quad t \in (0, \Delta t).$$

Moreover, the relaxation parameter a is fixed locally interface by interface. At interface $x_{i+1/2}$, where the approximate Riemann solver is considered to connect

$\tilde{w}_i^n$ and $\tilde{w}_{i+1}^n$, the local relaxation parameter is denoted by $a_{i+1/2}$ and it is fixed according to the robustness condition (3.12).

From now on, we underscore that the present work concerns wet region simulations and dry domains are not considered. As a consequence, the robustness condition (3.12) is allways well-defined. In order to deal with wet and dry transition, the here derived relaxation scheme can be modified according to the vacuum treatment proposed by Bouchut [7]. Bouchut's technique enriches the relaxation model by adopting a relaxation parameter a now governed by an advection equation. Here, in order to focus on a well-balanced strategy, we deliberately omit the dry areas, but for an easy extension of our approach to introduce Bouchut's method is to simulate dry regions.

The time step $\Delta t > 0$ is chosen small enough to avoid some interaction between the approximate Riemann solvers. As a consequence, we impose a CFL like condition given by

$$\frac{\Delta t}{\Delta x} \max_{i\in\mathbb{Z}} \left(u_i^n - \frac{a_{i+1/2}}{h_i^n}, u_{i+1}^n + \frac{a_{i+1/2}}{h_{i+1}^n} \right) \leq \frac{1}{2}. \tag{4.2}$$

Next, the updated state at time t^{n+1} is defined as the following average:

$$\tilde{w}_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{w}^n(x, t^n + \Delta t) dx. \tag{4.3}$$

After a straightforward computation (for instance, see [5, 7, 8, 28]), we obtain the following explicit three-point scheme:

$$\tilde{w}_i^{n+1} = \tilde{w}_i^n - \frac{\Delta t}{\Delta x} \left(\tilde{f}_{i+1/2} - \tilde{f}_{i-1/2} \right) + \frac{\Delta t}{2} \left(\tilde{S}_{i+1/2} + \tilde{S}_{i-1/2} \right), \tag{4.4}$$

where the numerical source term reads:

$$\tilde{S}_{i+1/2} = {}^t \left(0, -g\varphi(\bar{\theta}(\tilde{w}_i^n, \tilde{w}_{i+1}^n))\bar{h}(\tilde{w}_i^n, \tilde{w}_{i+1}^n) \frac{z_{i+1} - z_i}{\Delta x}, 0 \right). \tag{4.5}$$

Concerning the numerical flux function $\tilde{f}_{i+1/2} := \tilde{f}_\Delta(\tilde{w}_i^n, z_i, \tilde{w}_{i+1}^n, z_{i+1})$, we have set

$$\begin{aligned} &\tilde{f}_\Delta(\tilde{w}_L, z_L, \tilde{w}_R, z_R) \\ &= \begin{cases} {}^t\Big(h_L u_L, h_L u_L^2 + g\varphi(\theta_L)\dfrac{h_L^2}{2} + s_{LR}, h_L\theta_L u_L \Big) & \text{if } u_L - \dfrac{a}{h_L} > 0, \\ {}^t\left(h_L^\star u^\star, h_L^\star (u^\star)^2 + \pi_L^\star + s_{LR}, h_L^\star \theta_L u^\star \right) & \text{if } u_L - \dfrac{a}{h_L} < 0 < u^\star, \\ {}^t\left(h_R^\star u^\star, h_R^\star (u^\star)^2 + \pi_R^\star - s_{LR}, h_R^\star \theta_R u^\star \right) & \text{if } u^\star < 0 < u_R + \dfrac{a}{h_R}, \\ {}^t\Big(h_R u_R, h_R u_R^2 + g\varphi(\theta_R)\dfrac{h_R^2}{2} - s_{LR}, h_R\theta_R u_R \Big) & \text{if } u_R + \dfrac{a}{h_R} < 0, \end{cases} \end{aligned} \tag{4.6}$$

where we have introduced

$$s_{LR} = -\frac{g}{2}\varphi(\bar{\theta}(w_L, w_R))\bar{h}(w_L, w_R)(z_R - z_L),$$

and the intermediate states are defined according to (3.10).

Now, applying the main properties satisfied by the approximate Riemann solver $\tilde{w}^{eq}$, stated in Lemma 3.4 and Lemma 3.5, we prove the robustness and the well-balanced properties satisfied by the scheme (4.4).

Theorem 4.1. *For all i in $\mathbb{Z}$, assume that the local relaxation parameter $a_{i+1/2}$ satisfies*

$$u_i^n - \frac{a_{i+1/2}}{h_i^n} < u_{i+1/2}^\star < u_{i+1}^n + \frac{a_{i+1/2}}{h_{i+1}^n}, \tag{4.7}$$

where $u_{i+1/2}^\star$ is defined by (3.10c) *with clear notation. Let $\tilde{\Omega}$ be defined by* (2.15)*. Assume $\tilde{w}_i^n$ belongs to $\tilde{\Omega}$ for all $i \in \mathbb{Z}$. Then, under the CFL condition* (4.2)*, we get $\tilde{w}_i^{n+1}$ in $\tilde{\Omega}$ for all i in Z.*

Proof. In fact, we have just to prove that the updated water height h_i^{n+1} stays positive. Because of the updated formula (4.3), the positiveness requirement is obtained as soon as the intermediate water heights, involved within the approximate Riemann solver, are positive. Since we have imposed (4.7), Lemma 3.4 can be applied, and the proof is achieved. □

Concerning the well-balanced property, let us first establish the exact preservation of steady states given by (1.5)-(2.10).

Theorem 4.2. *Assume the average functions $\bar{h}$ and $\bar{\theta}$ are respectively given by* (3.16) *and* (3.18)*. Let us consider initial data $\tilde{w}_i^0$ given by one of the steady states at rest* (1.5)-(2.10)*:*

$$\begin{cases} u_i^0 = 0, \\ \theta_i^0 = \theta, \\ h_i^0 + z_i = H, \end{cases} \quad or \quad \begin{cases} u_i^0 = 0, \\ z_i = Z, \\ (h_i^0)^2 \varphi(\theta_i^0) = P, \end{cases} \quad or \quad \begin{cases} u_i^0 = 0, \\ h_i^0 = H, \\ z_i + \dfrac{h_i^0}{2} \ln(\varphi(\theta_i^0)) = P, \end{cases}$$

where θ, H, Z and P denote constants. Then the updated state $\tilde{w}_i^{n+1}$ stays at rest:

$$\tilde{w}_i^{n+1} = \tilde{w}_i^n, \qquad \forall i \in \mathbb{Z},\ n \in \mathbb{N}.$$

Proof. Since $\bar{h}$ and $\bar{\theta}$ are respectively defined by (3.16) and (3.18), we can apply Lemma 3.5 at each interface. By adopting an immediate induction, let us assume that, at time t^n, $\tilde{w}_i^n$ satisfies for all $i \in \mathbb{Z}$ one of the three following rest properties:

$$\begin{cases} u_i^n = 0, \\ \theta_i^n = \theta, \\ h_i^n + z_i = H, \end{cases} \quad \begin{cases} u_i^n = 0, \\ z_i = Z, \\ (h_i^n)^2 \varphi(\theta_i^n) = P, \end{cases} \quad \begin{cases} u_i^n = 0, \\ h_i^n = H, \\ z_i + \dfrac{h_i^n}{2} \ln(\varphi(\theta_i^n)) = P. \end{cases}$$

The proof is established as soon as we show $\tilde{w}_i^{n+1} = \tilde{w}_i^n$. From Lemma 3.5, we deduce that each approximate Riemann solver stays at rest, given by (3.15). With the updated state $\tilde{w}_i^{n+1}$ defined by (4.3), the proof is easily achieved. □

Theorem 4.2 states the preservation of particular steady states given by (1.5). In fact, this result can be extended to more general steady states at rest according to (1.4)-(2.10). Concerning such steady states, the main question is: What approximation of (1.4)-(2.10) is preserved by the scheme?

Actually, suppose initial data $\tilde{w}_i^0$ which approximate (1.4)-(2.10) as follows for all $i \in \mathbb{Z}$:

$$\frac{1}{\Delta x}\left(\varphi(\theta_{i+1}^0)\frac{(h_{i+1}^0)^2}{2} - \varphi(\theta_i^0)\frac{(h_i^0)^2}{2}\right) + \varphi(\bar{\theta}(\tilde{w}_i^0, \tilde{w}_{i+1}^0))\bar{h}(\tilde{w}_i^0, \tilde{w}_{i+1}^0)\frac{1}{\Delta x}(z_{i+1} - z_i) = 0. \tag{4.8}$$

We easily see that $(\tilde{w}_i^0)_{i\in\mathbb{Z}}$ coincides with, at least, a first-order approximation of the solution of (1.4). We now establish that initial data at rest such that (4.8) holds true are exactly preserved by the scheme (4.4).

Theorem 4.3. *At time $t = 0$, assume $(\tilde{w}_i^0)_{i\in\mathbb{Z}}$ satisfies $u_i^0 = 0$ and (4.8) for all i in $\mathbb{Z}$. Then, the approximate solution by (4.4) stays at rest:*

$$\tilde{w}_i^{n+1} = \tilde{w}_i^n, \qquad \forall i \in \mathbb{Z},\ n \in \mathbb{N}.$$

Proof. At each interface, the initial data satisfies (3.13)-(3.14), and Lemma 3.5 can be applied. As a consequence, at each interface the approximate Riemann solver stays at rest. Since the updated state $\tilde{w}_i^1$, at time $t = \Delta t$, is defined by (4.3), we immediately deduce the equality

$$\tilde{w}_i^1 = \tilde{w}_i^0 \qquad \forall i \in \mathbb{Z}.$$

Arguing an induction procedure, the proof is then completed. □

For the sake of completeness, we conclude this section by proving that the derived numerical scheme (4.4) is entropy preserving. From now on, let us mention that the result stated below is probably not optimal and the conditions to obtain the required stability must be improved. However, the entropy statement which is now established emphasizes the relevance of the derived numerical scheme.

In order to shorten the notation, we introduce

$$\overline{\eta}(\tilde{w}, z) = \tilde{\eta}(\tilde{w}) + g\varphi(\theta)hz, \tag{4.9}$$

$$\overline{G}(\tilde{w}, z) = \tilde{G}(\tilde{w}) + g\varphi(\theta)hzu, \tag{4.10}$$

where the partial entropy $\tilde{\eta}$ and the partial entropy flux $\tilde{G}$ are defined by (2.17) and (2.18), respectively. Here, we directly apply the entropy stability condition given by Harten, Lax and van Leer [26]. We begin by giving a local entropy estimation satisfied by the Riemann solver $\tilde{w}^{eq}$ in case the left and right velocities are different.

Lemma 4.4. *Assume the left and right velocities satisfy $u_L \neq u_R$. Then we can choose the relaxation parameter a large enough so that the following estimation holds:*

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \overline{\eta}\left(\tilde{w}^{eq}\left(\frac{x}{\Delta t}; \tilde{w}_L, \tilde{w}_R\right)\right) dx - \frac{\overline{\eta}(\tilde{w}_L) + \overline{\eta}(\tilde{w}_R)}{2} + \frac{\Delta t}{\Delta x}\left(\overline{G}(\tilde{w_R}) - \overline{G}(\tilde{w_L})\right) \leq 0. \tag{4.11}$$

Proof. A straightforward first-order expansion of the left-hand side of (4.11) with respect to a leads to

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\overline{\eta}\left(\tilde{w}^{eq}\left(\frac{x}{\Delta t};\tilde{w}_L,\tilde{w}_R\right)\right)dx-\frac{\overline{\eta}(\tilde{w}_L)+\overline{\eta}(\tilde{w}_R)}{2}$$
$$+\frac{\Delta t}{\Delta x}\left(\overline{G}(\tilde{w_R})-\overline{G}(\tilde{w_L})\right)=-a\frac{\Delta t}{\Delta x}\frac{(u_R-u_L)^2}{4}+O(1).$$

Since $u_L\neq u_R$, we can easily find a large enough such that (4.11) holds. □

Let us notice that (4.11) is exactly the entropy consistency relation introduced in [26], which is sufficient to ensure the required entropy stability of the scheme. Unfortunately, Lemma 4.4 does not hold true as soon as the velocities u_L and u_R are equal. In this case we can only obtain the following weaker result.

Lemma 4.5. *Assume the left and right velocities satisfy $u_L=u_R$. Then we can choose the relaxation parameter a large enough so that the following estimation holds:*

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\overline{\eta}\left(\tilde{w}^{eq}\left(\frac{x}{\Delta t};\tilde{w}_L,\tilde{w}_R\right)\right)dx-\frac{\overline{\eta}(\tilde{w}_L)+\overline{\eta}(\tilde{w}_R)}{2}$$
$$+\frac{\Delta t}{\Delta x}\left(\overline{G}(\tilde{w_R})-\overline{G}(\tilde{w_L})\right)\leq O(z_R-z_L). \tag{4.12}$$

Proof. Let us compute the expansion of the left-hand side of (4.12) with respect to z_R-z_L to get

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\overline{\eta}\left(\tilde{w}^{eq}\left(\frac{x}{\Delta t};\tilde{w}_L,\tilde{w}_R\right)\right)dx-\frac{\overline{\eta}(\tilde{w}_L)+\overline{\eta}(\tilde{w}_R)}{2}+\frac{\Delta t}{\Delta x}\left(\overline{G}(\tilde{w_R})-\overline{G}(\tilde{w_L})\right)$$
$$=-\frac{1}{4a}(p_R-p_L)^2+\sum_{k=2}^{\infty}\frac{(p_R-p_L)^k}{2^k a^{2k-1}}\left(p_L h_L^{k-1}+(-1)^k p_R h_R^{k-1}\right)+O(z_R-z_L).$$

We can obviously find a such that the constant term will be nonpositive, and the result is proven. □

In order to prove that a converged solution is entropy satisfying, the well-known Lax-Wendroff theorem (see [22,30,32]) usually requires the scheme to satisfy a discrete entropy inequality like (4.11). However, we can easily check that the following weaker inequality is actually sufficient to apply the Lax-Wendroff theorem:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\overline{\eta}\left(\tilde{w}^{eq}\left(\frac{x}{\Delta t};\tilde{w}_L,\tilde{w}_R\right)\right)dx-\frac{\overline{\eta}(\tilde{w}_L)+\overline{\eta}(\tilde{w}_R)}{2}$$
$$+\frac{\Delta t}{\Delta x}\left(\overline{G}(\tilde{w_R})-\overline{G}(\tilde{w_L})\right)\leq O(\Delta x^{1+\varepsilon}), \tag{4.13}$$

where ε is a positive number. As a consequence, we immediately deduce from Lemmas 4.4 and 4.5 the following stability result.

Theorem 4.6. *Assume there exists a positive number ε such that the topography function z is Hölder continuous with exponent $1+\varepsilon$:*

$$\exists C>0 \text{ such that } |z(y)-z(x)|\leq C|y-x|^{1+\varepsilon},\quad \forall x,y\in\mathbb{R}. \tag{4.14}$$

Then the scheme (4.4) *is entropy preserving in the sense of the Lax-Wendroff theorem: If the approximations given by the scheme* (4.4) *are uniformly bounded and converge in* $L_{1,loc}$ *to a function* w*, then* w *satisfies the entropy inequality* (2.16).

Classically, the topography function is only required to be continuous, which corresponds to 1-Hölder continuity. Here we need a little more regularity, which is unusual. But as stated before this condition can certainly be improved.

5. Numerical experiments

We now turn to illustrating the relevance of the derived numerical scheme (4.4)-(4.5)-(4.6). First, we present three experiments introduced in [17]: a dam break over a flat bottom, a dam break over a nonflat bottom and a perturbation of a lake at rest solution. In order to underscore the fact that the relaxation scheme (4.4) is able to preserve more complex steady states, we conclude by studying a perturbation of a nonlinear steady state. In all the numerical experiments, the gravity constant is fixed at $g = 1$ and the function φ is defined by $\varphi(\theta) = e^{\theta}$.

We point out that Lemmas 4.4 and 4.5 ensure the existence of a constant a large enough so that the entropy property is satisfied. However, they do not give a constructive procedure to obtain such a parameter. As a consequence, for the numerical experiments that follow, we had to make a choice. We decided to use a parameter $a_{i+1/2}$ at interface $x_{i+1/2}$ which satisfies the natural Whitham condition

$$a_{i+1/2} > \max\left(h_i\sqrt{g\theta_i h_i}, h_{i+1}\sqrt{g\theta_{i+1}h_{i+1}}\right),$$

supplemented by the robustness condition (4.7). Once the parameters $a_{i+1/2}$ are set for all the interfaces, the time step is limited according to the CFL restriction (4.2).

5.1. **Smooth solution.** The aim of this experiment is to show the convergence of the presented numerical scheme in case of a smooth solution. In order to do this, we consider a topography on the computational domain $[-1, 1]$ given by a smooth bump (see Figure 1):

$$z(x) = \begin{cases} 2(\cos(10\pi x)) + 1) & \text{if } -0.1 \le x \le 0.1, \\ 0 & \text{otherwise,} \end{cases}$$

with the initial condition

$$(h, u, \Theta)(x, 0) = (3 + \exp(0.1x), \exp(0.1x), 2\exp(0.1x)).$$

A reference solution, computed at time $t = 0.1$ with $25,600$ cells, is displayed in Figure 1.

For increasing values of the number of cells N, we compute the relative L^1 and L^∞ errors in water height with respect to the reference solution. The results as well as the convergence rates are displayed in Table 1. As expected, a first-order convergence is achieved either in L^1 norm or in L^∞ norm.

5.2. **Dam break over a flat bottom.** We consider a dam break problem over a flat topography ($z \equiv 0$) with the following initial condition:

$$(h, u, \Theta)(x, 0) = \begin{cases} (5, 0, 3) & \text{if } x < 0, \\ (1, 0, 5) & \text{if } x > 0. \end{cases}$$

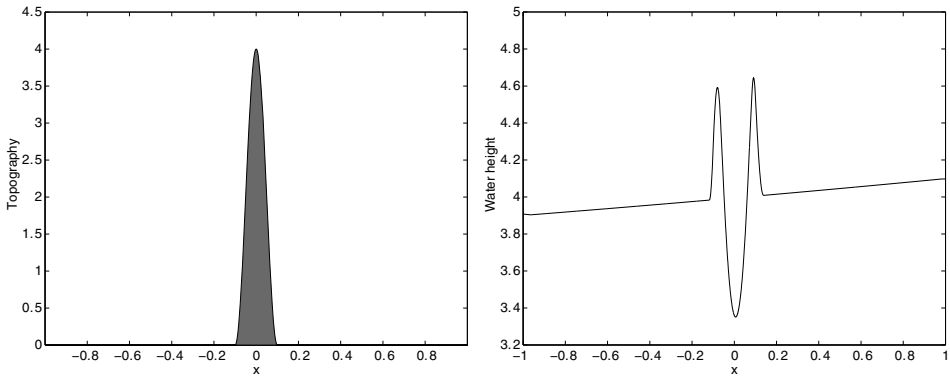

FIGURE 1. Smooth solution. Left: topography. Right: water height h at time $t = 0.1$ computed with $25,600$ cells.

TABLE 1. Relative L^1 and L^∞ errors in density for the smooth solution with respect to the reference solution.

N	L^1 error		L^∞ error	
100	6.78E-03		7.32E-02	
200	3.44E-03	0.98	3.97E-02	0.88
400	1.75E-03	0.98	2.09E-02	0.93
800	8.77E-04	1.00	1.07E-02	0.97
1600	4.34E-04	1.01	5.41E-03	0.98
3200	2.11E-04	1.04	2.65E-03	1.03

We use 200 cells to discretize the computational domain $[-1, 1]$. In Figure 2, we show the solution obtained at time $t = 0.2$ for the water height h, the temperature Θ and the pressure $p = g\Theta h^2/2$. The results are compared to a reference solution computed with $20,000$ cells.

The general shape of the solution is well captured, although there is quite a lot of diffusion, which is expected for a first-order scheme.

5.3. **Dam break over a nonflat bottom.** We focus again on a dam break problem, but this time with a nonconstant topography term, given by (see top left of Figure 3)

$$z(x) = \begin{cases} 2(\cos(10\pi(x+0.3))+1) & \text{if } -0.4 \le x \le -0.2, \\ 0.5(\cos(10\pi(x-0.3))+1) & \text{if } 0.2 \le x \le 0.4, \\ 0 & \text{otherwise.} \end{cases}$$

We use the following initial data:

$$(h, u, \Theta)(x, 0) = \begin{cases} (5 - z(x), 0, 1) & \text{if } x < 0, \\ (1 - z(x), 0, 5) & \text{if } x > 0. \end{cases}$$

The solution is computed over the computational domain $[-1, 1]$ with 200 cells. In Figure 3, we display the results at time $t = 0.3$ for the free surface $h + z$, the

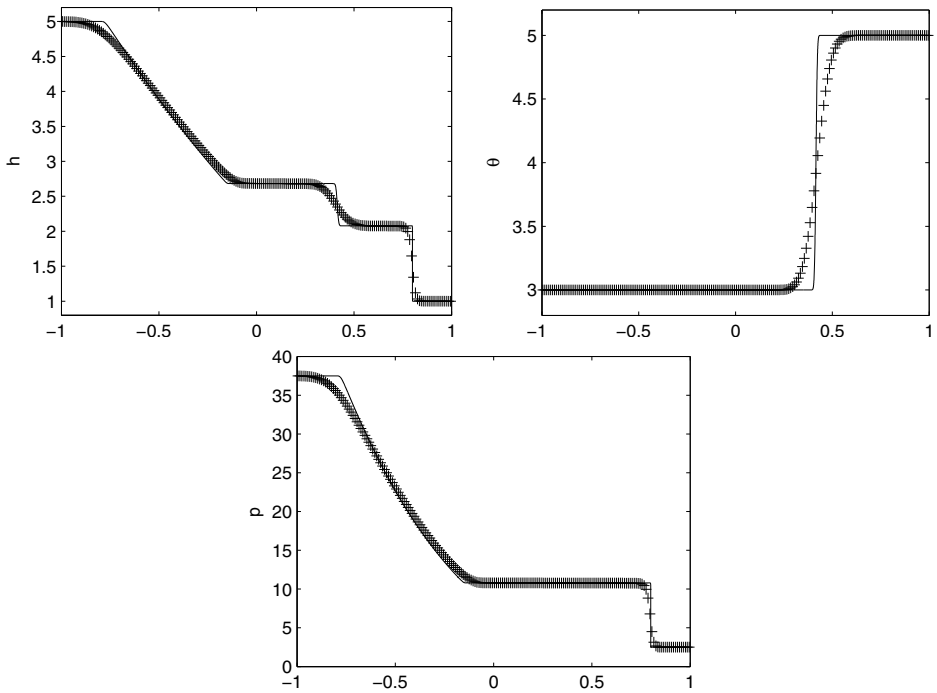

FIGURE 2. Dam break problem over a flat bottom. Results in height h (top left), temperature Θ (top right) and pressure p (bottom) at time $t = 0.2$. Solid line: reference solution.

temperature Θ and the pressure $p = g\Theta h^2/2$. We also present a reference solution computed with $20,000$ cells.

First, let us notice that initially the area near $x = 0.3$ is almost dry. As expected, the relaxation scheme preserves the positivity of the water height h. The results show that the scheme behaves well in the presence of a topography term. Although the solution is not very accurate in the vicinity of the discontinuities due to the numerical viscosity of the scheme, the shape of the solution is in good agreement with [17].

5.4. **Perturbation of a lake at rest solution.** We consider here a topography with two isolated bumps:

$$z(x) = \begin{cases} 0.85(\cos(10\pi(x+0.9))+1) & \text{if } -1 \le x \le -0.8, \\ 1.25(\cos(10\pi(x-0.4))+1) & \text{if } 0.3 \le x \le 0.5, \\ 0 & \text{otherwise.} \end{cases}$$

We can easily check that the solution

$$(h_s, u_s, \Theta_s)(x) = \begin{cases} (6 - z(x), 0, 4) & \text{if } x < 0, \\ (4 - z(x), 0, 9) & \text{if } x > 0 \end{cases}$$

is a nonmoving steady state, made of two lakes at rest connected by a stationary contact discontinuity. Of course, after Theorem 4.2, the relaxation scheme exactly preserves this solution. As a consequence there is not much interest in simulating

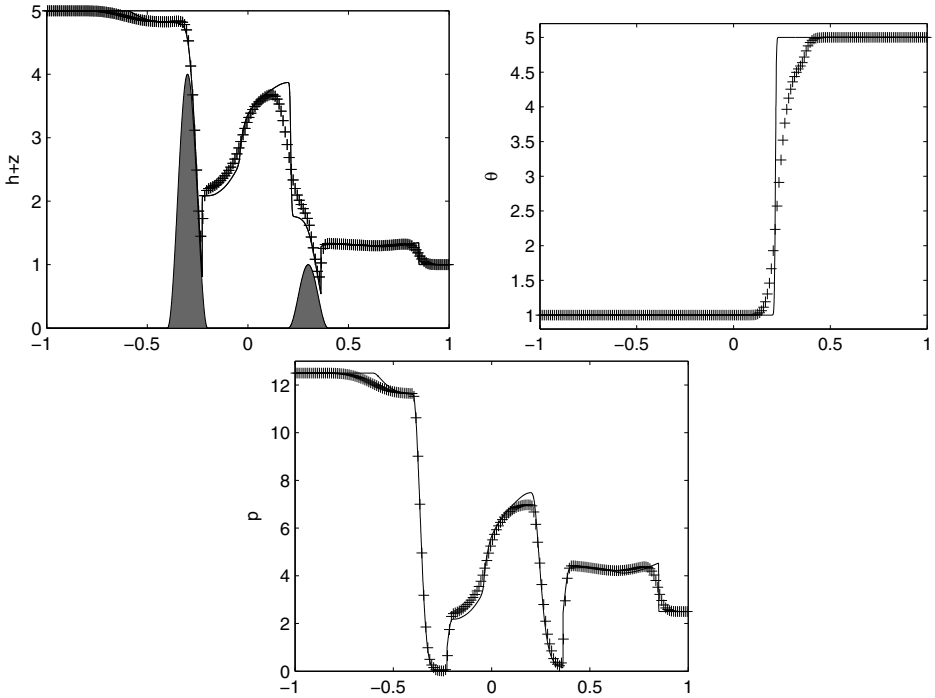

FIGURE 3. Dam break problem over a nonflat bottom. Results in free surface $h+z$ (top left), temperature Θ (top right) and pressure p (bottom) at time $t = 0.3$. Solid line: reference solution.

this solution. Instead, we investigate the behavior of a perturbation of this steady state by considering the initial data

$$(h, u, \Theta)(x, 0) = (h_s, u_s, \Theta_s)(x) + (0.1, 0, 0)\chi_{[-1.5,-1.4]}(x),$$

where $\chi_{[-1.5,-1.4]}$ is the indicator function of the set $[-1.5, -1.4]$. This perturbation splits into two waves moving in opposite directions. The one propagating towards the right successively crosses the first bump, the stationary contact discontinuity and the second bump.

The computational domain $[-2, 2]$ is discretized using 100 cells. The results in free surface $h+z$ and in pressure $p = g\Theta h^2/2$ at time $t = 0.1$ are shown in Figure 4. First, we notice that the relaxation scheme captures perfectly the stationary contact discontinuity despite the perturbation. All numerical schemes are not able to do this (see [17]). Moreover the scheme does not create spurious oscillations near the bumps. Additionally the size of the waves created by the perturbation decrease with time. This shows the stability of the relaxation scheme with respect to the lake at rest solutions.

5.5. **Perturbation of a nonlinear steady state.** We are now interested in a more complex steady state than the lake at rest described before. On the computational domain $[-1, 1]$, we consider a topography given by

$$z(x) = 6 - 2\exp(x).$$

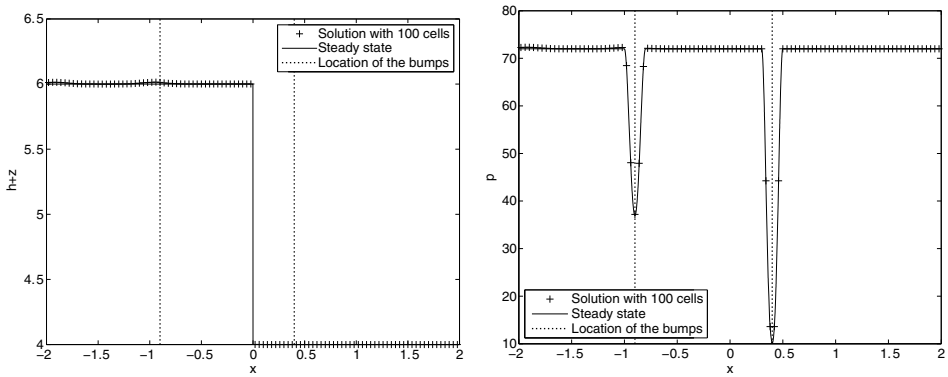

FIGURE 4. Perturbation of a lake at rest solution. Results in free surface $h + z$ (left) and pressure p (right) at time $t = 0.1$.

We can easily check that the solution

$$(h_s, u_s, \Theta_s)(x) = (\exp(x), 0, \exp(2x))$$

is a steady state at rest. We introduce a perturbation by defining the initial data as follows:

$$(h, u, \Theta)(x, 0) = (h_s, u_s, \Theta_s)(x) + (0.1, 0, 0)\chi_{[-0.1,0]}(x).$$

Figure 5 shows the evolution of the perturbation in water height $h - h_s$ and in velocity $u - u_s$. The results are computed at time $t = 0.2$ and $t = 0.4$ using 200 cells. We can see the initial perturbation splitting into two waves propagating in opposite directions. Once again, there is no spurious oscillations and the perturbation decreases with time. This result underlines the importance of the well-balancing property satisfied by the relaxation scheme (4.4).

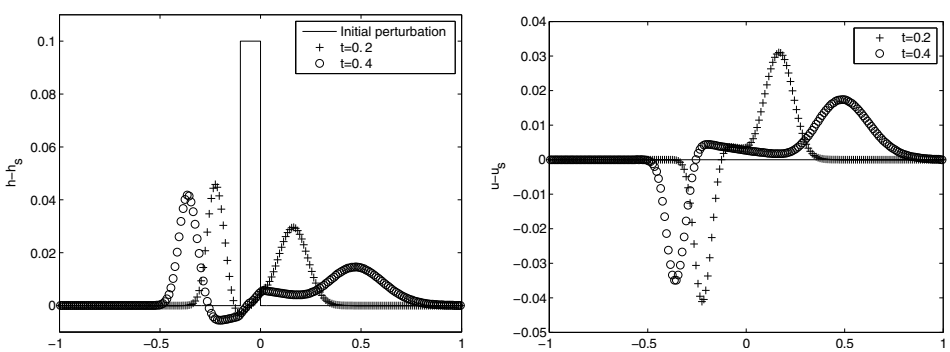

FIGURE 5. Perturbation of a nonlinear steady state. Results in perturbation of the free surface $h - h_s$ (left) and perturbation of the velocity $u - u_s$ (right). Solid line: initial perturbation.

6. Conclusion

In this work, we have derived a numerical scheme to approximate the weak solutions of the Ripa model, which is an extension of the well-known shallow-water model but with a gradient of temperature. Exhibiting the basic algebraic properties, we have shown that the initial formulation of the model does not satisfy the required entropy estimations. In fact, involving a suitable change of variables, the entropy inequalities can be restored. Moreover, the considered change of variables is admissible and it preserves the weak solutions of the Ripa model. Based on this new interpretation of the set of partial differential equations under consideration, we have designed a numerical scheme able to accurately/exactly evaluate the steady states. Indeed, the Ripa model comes with very specific steady states governed by a nonsolvable partial differential equation.

Enforcing additional conditions to be satisfied by the steady states (constant free surface for instance), some explicit classes of steady states have been exhibited. One of the objectives is to get a numerical scheme able to exactly capture these classes of lake at rest type. To address such an issue, we have considered a Suliciu relaxation type model. In order to simplify the algebra coming with such a relaxation model, we have imposed a *transport property* to be satisfied by the topography. This model modification makes the associated Riemann problem underresolved, and a closure law must be added. This additional closure law is fixed according to the required steady states. Here, we have adopted the relation (3.8) which mimics the PDE governing the steady states. The choice of the closure relation (3.8) is certainly not unique. However, in order to satisfy the required well-balanced property, this relation must enforce the steady states to be preserved by the designed approximate Riemann solver.

In order to give a more convenient interpretation of this Suliciu relaxation model supplemented by a closure law, we have proposed two interpretations of the relaxation model. The first one exactly enters the Suliciu formalism but for a more complex model in order to take into account the additional closure law. The second one coincides with a relaxation model derived in the framework of Cargo-LeRoux's source term reformulation. Of course, the three presented models lead to the same approximate Riemann solver considered next to develop the numerical scheme. Nonetheless, from our point of view, the approach based on an underresolved relaxation model supplemented by a closure law may offer more opportunities. Indeed, we have here decided to fix the closure law (3.8) according to the needed steady states preservation. But other closure law derivations can be argued to get distinct properties other than well-balancing.

Finally, from the relaxation approximate Riemann solver, we derive a fully discrete numerical scheme. This method is proved to be positive preserving, entropy satisfying and well-balanced. More precisely, we prove that the three classes of steady states—namely the lake at rest (1.5a), the isobaric steady state (1.5b) and the constant water height steady state (1.5c)—are exactly preserved. Moreover, by adopting additional regularity assumptions on the topography function, we establish a stability result by exhibiting discrete entropy inequalities. Several numerical tests are performed to attest the relevance of the derived numerical scheme.

The relaxation framework developed here in the case of the Ripa model can be extended to other systems with source terms. In particular the derivation of well-balanced relaxation schemes for the Euler equations with gravitational effects

will be the purpose of a forthcoming sequel to this paper. This derivation will be completed by a second-order MUSCL extension achieved by a new gradient reconstruction able to exactly preserve the required steady states.

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