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ON THE ACCURACY OF FINITE ELEMENT APPROXIMATIONS TO A CLASS OF INTERFACE PROBLEMS

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ABSTRACT. We define piecewise linear and continuous finite element methods for a class of interface problems in two dimensions. Correction terms are added to the right-hand side of the natural method to render it second-order accurate. We prove that the method is second-order accurate on general quasi-uniform meshes at the nodal points. Finally, we show that the natural method, although non-optimal near the interface, is optimal for points $\mathcal{O}(\sqrt{h\log(\frac{1}{h})})$ away from the interface.

1. Introduction

In this paper we consider finite element approximations to the following problem. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with an immersed smooth, closed interface Γ such that $\overline{\Omega} = \overline{\Omega}^- \cup \overline{\Omega}^+$ and Γ encloses Ω^-. Consider the problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1b}$$

$$[u] = \alpha \quad \text{on } \Gamma, \tag{1c}$$

$$[\nabla u \cdot \boldsymbol{n}] = \beta \quad \text{on } \Gamma. \tag{1d}$$

The jump is defined as

$$[\nabla u \cdot \boldsymbol{n}] = \nabla u^- \cdot \boldsymbol{n}^- + \nabla u^+ \cdot \boldsymbol{n}^+,$$

where $u^\pm = u|_{\Omega^\pm}$ and $\boldsymbol{n}^\pm$ is the unit outward pointing normal to $\Omega^\pm$ (see Figure 1). Also, we denote $[u] = u^+ - u^-$.

Many numerical methods have been developed for problem (1). Perhaps the most notable ones are the finite difference method of Peskin [27] (i.e., *immersed boundary method*) and the method of LeVeque and Li [20] (i.e., the *immersed interface method*; see also the method of Mayo [23–25]). The method of LeVeque and Li [20] was developed for the more general problem with discontinuous diffusion coefficients, while the method of Peskin [27] was developed for fluid flow problems with an immersed boundary. Although the method of Peskin [27] is formulated with a force function F that incorporates the elastic force of the immersed boundary Γ, it was shown in [28] that it can be reformulated as an interface problem (with $\alpha = 0$) where β encodes the elastic force.

Since the two important papers [20, 27] there have been many articles extending or improving these methods. In particular, finite element versions of these methods

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have appeared; see for example [1, 2, 4, 5, 7–12, 14, 16–19]. For the above problem ($\alpha = 0$), it is well known that the method of Peskin [27] is only first-order accurate whereas the method of LeVeque and Li [20] is second-order accurate. In fact, Beale and Layton [3] give a rigorous analysis of the LeVeque and Li [20] method and the method of Mayo [24] on rectangular grids.

One of the attractive features of the methods [5, 14, 16, 20, 24, 25, 27] is that the stiffness matrix for the problem (1) is the same as the standard piecewise linear stiffness matrix. Instead, only the load vector needs to be modified, which is important for time dependent problems where the interface is moving.

We provide a pointwise error analysis of finite element methods approximating (1). We give sufficient conditions on the finite element method that guarantee optimal estimates for the gradient error. We prove the error estimates for general quasi-uniform and shape regular meshes and assuming Ω is convex. We assume that Ω is convex to avoid unnecessary boundary complications and to single out the interface analysis issues. Our error analysis relies on standard estimates for approximate Green's functions and their finite element approximations; see [29, 31].

The main idea in the analysis will be to compare $u_h - I_h u$ where $I_h u$ is an interpolant of u and u_h is the finite element approximation. More specifically, the numerical method that we analyze satisfies

$$\int_\Omega \nabla(I_h - u_h) \cdot \nabla v dx = F_u(\nabla v) \quad \forall v \in V_h,$$

where V_h is the space of piecewise linear functions vanishing on $\partial\Omega$. Of course, different methods lead to different F_u. Roughly speaking, we will prove that $u_h - I_h u$ will be optimally convergent if $F_u(\nabla v) \leq C\, h \|\nabla v\|_{L^1(\Omega)}$ for all $v \in V_h$.

Guided by the analysis we develop a simple finite element method that satisfies these conditions. We call the method the edge-based correction finite element interface (EBC-FEI) method. We then show that the EBC-FEI method is very similar to the method of He, Lin and Lin [16], applied to problem (1) and using continuous piecewise polynomials, and this allows us to also analyze their method.

Moreover, we give an error analysis of the method considered by Boffi and Gastaldi [5]. This finite element method is in some sense the *natural* method for (1), and it can be thought of as the finite element version of the method by Peskin [27] for problem (1). Although this method is first-order accurate near the interface Γ, we show how far one has to be from the interface in order to recover optimal estimates for the gradient of the error. More specifically, we show that optimal estimates hold for points that are $O(\sqrt{\log(\frac{1}{h})h})$ away from the interface Γ. Mori [26] proves that the immersed boundary method of Peskin is optimal if one is sufficiently away from the interface, but does not quantify how far away one has to be.

The rest of the paper is organized as follows. In the next section we present our simple finite element method and give a derivation. In Section 3, we give an abstract error analysis which includes the analysis of our method. In Section 4, we present other methods in the literature. In particular, we show that our method is very similar to the method of He et al. [16], applied to problem (1) and using continuous piecewise polynomials, and hence can easily analyze their method. Also, in Section 4, we analyze the method of Boffi and Gastaldi [5]. Finally, in Section 5, we illustrate our results with some numerical examples.

2. The EBC-FEI method

In this section we present a simple finite element method for problem (1) that is second-order accurate. To do so, we assume that the data f, β and α are smooth. Furthermore, we assume that $u^\pm \in C^2(\overline{\Omega}^\pm)$.

We next develop notation. Let $\mathcal{T}_h$, $0 < h < 1$ be a sequence of triangulations of Ω, $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$, with the elements T mutually disjoint. Let h_T denote the diameter of the element T and $h = \max_T h_T$. Let V_h be the space of piecewise linear functions which are continuous on Ω and vanish on $\partial\Omega$, i.e.,

$$V_h = \{v \in H_0^1(\Omega) \,:\, v|_T \in \mathbb{P}^1(T) \quad \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}^k(T)$ is the space of polynomials of degree k on T.

We assume the mesh is shape-regular; see [6]. For our pointwise estimates, we will assume the mesh is also quasi-uniform.

Let $\mathcal{E}_h$ denote the set of all the edges of $\mathcal{T}_h$ whereas $\mathcal{E}_h^i$ denotes the interior edges. Suppose that $e \in \mathcal{E}_h^i$ with $\overline{e} = \overline{T}_1 \cap \overline{T}_2$ and $T_1, T_2 \in \mathcal{T}_h$; then we define

$$[\nabla v \cdot \boldsymbol{n}]|_e = \nabla v|_{T_1} \cdot \boldsymbol{n}_1 + \nabla v|_{T_2} \cdot \boldsymbol{n}_2,$$

where $\boldsymbol{n}_i$ is the unit normal pointing out of T_i for $i = 1, 2$. We note that we adopt the convention that edges e, elements T, subedges $e^\pm := e \cap \Omega^\pm$, subelements $T^\pm := T \cap \Omega^\pm$, and subregions $\Omega^\pm$ are open sets, and we use the overline symbol to refer to their closure.

Hypothesis 1. *Throughout the paper, we assume that the interface Γ intersects the boundary of each triangle $T \in \mathcal{T}_h$ at most at two points unless it coincides completely with an edge of T. If Γ intersects the boundary of a triangle T in exactly two points, then these two points must be on different edges $\bar{e}$ of T.*

Remark 1. We make this hypothesis to make the presentation simpler. As we will see below, the construction of $\tilde{u}_e$ in Lemma 2 can be easily generalized to cases where Γ crosses an edge $\bar{e}$ in more than one place.

We define the set of edges that intersect and do not intersect the immersed interface Γ as follows:

$$\begin{aligned}\mathcal{E}_h^{\Gamma,a} &= \{e \in \mathcal{E}_h \,:\, \overline{e} \cap \Gamma \neq \emptyset\}, \\ \mathcal{E}_h^{\Gamma^\perp} &= \mathcal{E}_h^i \setminus \mathcal{E}_h^{\Gamma,a}.\end{aligned}$$

We further separate the edges $\mathcal{E}_h^{\Gamma,a}$ depending on whether the intersection of the edge and Γ is the entire edge, an interior point of the edge (not an endpoint) or an endpoint of the edge, respectively, as

$$\begin{aligned}\mathcal{E}_h^{\Gamma,0} &= \{e \in \mathcal{E}_h^{\Gamma,a} : e \subset \Gamma\}, \\ \mathcal{E}_h^{\Gamma} &= \{e \in \mathcal{E}_h^{\Gamma,a} \setminus \mathcal{E}_h^{\Gamma,0} : e \cap \Gamma \neq \emptyset\}, \\ \mathcal{E}_h^{\Gamma,\pm} &= \{e \in \mathcal{E}_h^{\Gamma,a} : e \subset \Omega^\pm\},\end{aligned}$$

so that $\mathcal{E}_h^{\Gamma,a} = \mathcal{E}_h^{\Gamma} \cup \mathcal{E}_h^{\Gamma,+} \cup \mathcal{E}_h^{\Gamma,-} \cup \mathcal{E}_h^{\Gamma,0}$. Note that if an edge $e \in \mathcal{E}_h^{\Gamma,\pm} \subset \mathcal{E}_h^{\Gamma,a}$, then there exists one and only one endpoint of e on Γ.

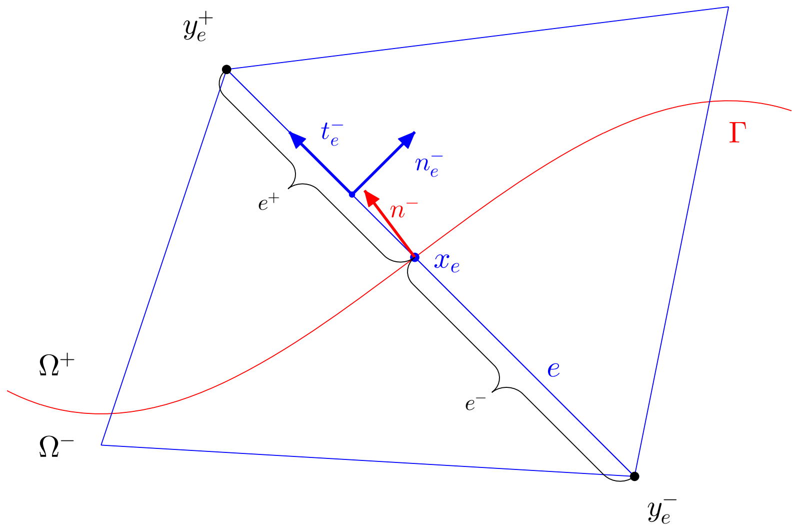

FIGURE 1. Illustration of the definitions of x_e, $y_e^{\pm}$, $\boldsymbol{t}_{e^-}$, $\boldsymbol{n}_{e^-}$, $e^{\pm}$, $\boldsymbol{n}^-$.

Now, for every $e \in \mathcal{E}_h^{\Gamma}$ we define (see Figure 1):

- $y_e^{\pm} \in \Omega^{\pm}$: the nodes (endpoints) of the edge e.
- x_e : the intersection of e and Γ.
- $e^{\pm}$: the subedge defined by $e^{\pm} = e \cap \Omega^{\pm}$.
- $\boldsymbol{t}_{e^{\pm}}$: the tangential unit vector for the edge e pointing out of $\Omega^{\pm}$.
- $\boldsymbol{n}_{e^{\pm}}$: the normal unit vector for the edge e, defined as a clockwise rotation of the tangential vector $\boldsymbol{t}_{e^{\pm}}$.
- $\boldsymbol{t}^{\pm}$: the tangential unit vector for the interface Γ defined as a counterclockwise rotation of the normal unit vector $\boldsymbol{n}^{\pm}$.
- $h_{e^{\pm}}$: define the length of $e^{\pm}$.
- $a_{e^{\pm}}$: defined as $a_{e^{\pm}} = \boldsymbol{n}^{\pm} \cdot \boldsymbol{t}_{e^{\pm}}$.
- $b_{e^{\pm}}$: defined as $b_{e^{\pm}} = \boldsymbol{t}^{\pm} \cdot \boldsymbol{t}_{e^{\pm}}$.

Note that $a_e^- = a_e^+$ ($b_e^- = b_e^+$) and so we denote them by a_e (b_e). For $e \in \mathcal{E}_h^{\Gamma,+}$ we let x_e be the endpoint of e that is contained in Γ.

We write a finite element method for problem (1) as follows: find $u_h \in V_h$ such that

$$\int_{\Omega} \nabla u_h \cdot \nabla v \, dx = E_h(v) \quad \forall v \in V_h. \tag{2}$$

Then, for our method (EBC-FEI), E_h is given by

$$\begin{aligned} E_h(v) = & \int_\Omega f\, v dx + \int_\Gamma (\beta v - \alpha \nabla v^+ \cdot \boldsymbol{n}^+)\, ds \\ & - \sum_{e \in \mathcal{E}_h^\Gamma} \frac{h_{e^-} h_{e^+}}{2} \left(a_e \beta(x_e) + b_e \frac{d}{ds}\alpha(x_e) + \frac{1}{2}(h_{e^+} - h_{e^-})\alpha(x_e) \right) [\nabla v \cdot \boldsymbol{n}]|_e \\ & - \frac{1}{2} \sum_{e \in \mathcal{E}_h^{\Gamma,+}} h_e \alpha(x_e) [\nabla v \cdot \boldsymbol{n}]|_e. \end{aligned} \tag{3}$$

Remark 2. We emphasize that, in order to define and analyze the method (2)-(3), we only require that $\alpha \in C^1([0, A])$, $\beta \in C^0([0, A])$ and there exists the arc-length parametrization $X : [0, A) \to \Gamma$ of Γ. Then, we denote $\frac{d}{ds}\alpha(x) = \frac{d}{ds}\alpha(X(s))$ for $x = X(s)$. Here A denotes the arc-length of Γ.

It is important to note the natural finite element method to consider for (1) will satisfy (2) with

$$E_h(v) = \int_\Omega f\, v dx + \int_\Gamma (\beta v - \alpha \nabla v^+ \cdot \boldsymbol{n}^+)\, ds. \tag{4}$$

This turns out to be the method of Boffi and Gastaldi [5] for (1) (in the case $\alpha = 0$). It is well known that this method is only first-order accurate, and hence the terms we add in (3) are correction terms that make the method second-order accurate at the nodes. This is, of course, in the spirit of the correction LeVeque and Li [20] gives for their immersed interface finite difference method.

2.1. Derivation of the EBC-FEI method.

As mentioned in the introduction, the derivation of our method is guided by trying to see the weak formulation that the interpolant of u satisfies (mod a higher order term). In order to do so, let us be precise about the interpolant.

Definition 2.1. Given $u^\pm \in C^2(\overline{\Omega}^\pm)$ define $I_h u \in V_h$ such that $I_h u(x) = u^-(x)$ for all vertices x of $\mathcal{T}_h$ with $x \in \overline{\Omega}^-$ and $I_h u(x) = u^+(x)$ for all vertices $x \in \Omega^+$.

Note that if u is continuous (i.e. $\alpha = 0$) $I_h u$ is simply the Lagrange interpolant of u. However, if $\alpha \neq 0$, then $I_h u$ interpolates values of u on vertices not intersecting Γ, and for vertices lying on Γ it takes the values of u coming from Ω^- (this is without loss of generality).

The next lemmas show the weak form $I_h u$ solves.

Lemma 1. *For any $u \in C^2(\overline{\Omega}^\pm)$ and any $v \in V_h$, it holds that*

$$\begin{aligned} \int_\Omega \nabla(I_h u) \cdot \nabla v dx = & \int_\Omega f\, v dx + \int_\Gamma (\beta v - \alpha \nabla v^+ \cdot \boldsymbol{n}^+)\, ds \\ & + \sum_{e \in \mathcal{E}_h^\Gamma \cup \mathcal{E}_h^{\Gamma,+}} \int_e (I_h u - u) [\nabla v \cdot \boldsymbol{n}]\, ds \\ & + \sum_{e \in \mathcal{E}_h^{\Gamma,0}} \int_e (I_h u - u^-) [\nabla v \cdot \boldsymbol{n}]\, ds \\ & + \sum_{e \in \mathcal{E}_h^{\Gamma^\perp} \cup \mathcal{E}_h^{\Gamma,-}} \int_e (I_h u - u) [\nabla v \cdot \boldsymbol{n}]\, ds. \end{aligned}$$

The first two terms are computable; it does not require the knowledge of u. The last term is high-order since $\|u - I_h u\|_{L^\infty(e)} = O(h^2)$ because u is smooth on edges $e \in \mathcal{E}_h^{\Gamma^\perp} \cup \mathcal{E}_h^{\Gamma,-}$ and $I_h u$ interpolates the values of u on those edges; see the definition of I_h. For edges in $e \in \mathcal{E}_h^{\Gamma,0}$, $I_h u - u^-$ is also high-order from the definition of I_h. These terms will be neglected to define an equation for u_h; see (2). The third term is neither computable nor high-order. To manage this term, we introduce a function $\tilde{u}_e$ on edges e such that $\|u - \tilde{u}_e\|_{L^\infty(e)} = O(h^2)$ in order to write the third term as a sum of computable terms and high-order terms. See also the proof of Lemma 4.

Lemma 2. *For any $u \in C^2(\overline{\Omega}^\pm)$ and any $v \in V_h$, it holds that*

$$\sum_{e\in\mathcal{E}_h^\Gamma} \int_e (I_h u - u)[\nabla v \cdot \boldsymbol{n}]\, ds$$

$$= \quad -\sum_{e\in\mathcal{E}_h^\Gamma} \left(\frac{h_{e^-} h_{e^+}}{2} (a_e \beta(x_e) + b_e \frac{d}{ds}\alpha(x_e)) + \frac{1}{2}(h_{e^+} - h_{e^-})\alpha(x_e) \right) [\nabla v \cdot \boldsymbol{n}]|_e$$

$$+ \sum_{e\in\mathcal{E}_h^\Gamma} \frac{h_{e^-} h_{e^+}}{2} \left(\nabla(u^+ - \tilde{u}_e^+)(x_e) \cdot \boldsymbol{t}_{e^+} + \nabla(u^- - \tilde{u}_e^-)(x_e) \cdot \boldsymbol{t}_{e^-} \right) [\nabla v \cdot \boldsymbol{n}]|_e$$

$$+ \sum_{e\in\mathcal{E}_h^\Gamma} \int_e (\tilde{u}_e - u)[\nabla v \cdot \boldsymbol{n}]\, ds$$

and

$$\sum_{e\in\mathcal{E}_h^{\Gamma,+}} \int_e (I_h u - u)[\nabla v \cdot \boldsymbol{n}]\, ds = -\frac{1}{2} \sum_{e\in\mathcal{E}_h^{\Gamma,+}} h_e \alpha(x_e)[\nabla v \cdot \boldsymbol{n}]|_e$$

$$+ \sum_{e\in\mathcal{E}_h^{\Gamma,+}} \int_e (u - \tilde{u}_e)[\nabla v \cdot \boldsymbol{n}]\, ds,$$

where for each $e \in \mathcal{E}_h^\Gamma$ we define $\tilde{u}_e$ so that it is linear on e^+ and on e^- and such that $\tilde{u}_e(y_e^\pm) = u(y_e^\pm)$ and $\tilde{u}_e^\pm(x_e) = u^\pm(x_e)$. For $e \in \mathcal{E}_h^{\Gamma,+}$, we define $\tilde{u}_e$ to be the unique linear function that agrees with u^+ on the endpoints of e.

We now turn to the proofs of these lemmas.

Proof. (Lemma 1) Let $v \in V_h$. Then we have

$$\begin{aligned}\int_\Omega \nabla(I_h u) \cdot \nabla v\, dx &= \int_{\Omega^-} \nabla(I_h u) \cdot \nabla v\, dx + \int_{\Omega^+} \nabla(I_h u) \cdot \nabla v\, dx \\ &= \int_{\Omega^-} \nabla(I_h u - u) \cdot \nabla v\, dx + \int_{\Omega^+} \nabla(I_h u - u) \cdot \nabla v\, dx \\ &\quad + \int_{\Omega^-} \nabla u \cdot \nabla v\, dx + \int_{\Omega^+} \nabla u \cdot \nabla v\, dx.\end{aligned}$$

Integration by parts gives

$$\begin{aligned}\int_{\Omega^-} \nabla u \cdot \nabla v\, dx + \int_{\Omega^+} \nabla u \cdot \nabla v\, dx &= \int_\Omega f\, v\, dx + \int_\Gamma (\nabla u^- \cdot \boldsymbol{n}^- + \nabla u^+ \cdot \boldsymbol{n}^+) v\, ds \\ &= \int_\Omega f\, v\, dx + \int_\Gamma \beta v\, ds.\end{aligned}$$

Hence, we have

$$\int_{\Omega} \nabla(I_h u)\cdot\nabla v\,dx = \int_{\Omega} f\,v\,dx + \int_{\Gamma} \beta v\,ds + \int_{\Omega^-} \nabla(I_h u - u)\cdot\nabla v\,dx + \int_{\Omega^+} \nabla(I_h u - u)\cdot\nabla v\,dx.$$

For every $T\in\mathcal{T}_h$ we define $T^\pm = T\cap\Omega^\pm$. Using integration by parts on each triangle and using that $\Delta v = 0$ on each triangle one gets

$$\begin{aligned}
&\int_{\Omega^-} \nabla(I_h u - u)\cdot\nabla v\,dx + \int_{\Omega^+} \nabla(I_h u - u)\cdot\nabla v\,dx\\
&\quad= \sum_{T\in\mathcal{T}_h}\left(\int_{T^-} \nabla(I_h u - u)\cdot\nabla v\,dx + \int_{T^+} \nabla(I_h u - u)\cdot\nabla v\,dx\right)\\
&\quad= \sum_{T\in\mathcal{T}_h}\left(\int_{\partial T^-} (I_h u - u)\nabla v\cdot\boldsymbol{n}\,ds + \int_{\partial T^+} (I_h u - u)\nabla v\cdot\boldsymbol{n}\,ds\right)\\
&\quad= \sum_{e\in\mathcal{E}_h^i\setminus\mathcal{E}_h^{\Gamma,0}} \int_e (I_h u - u)[\nabla v\cdot\boldsymbol{n}]\,ds\\
&\qquad+ \int_{\Gamma} (I_h u)[\nabla v\cdot\boldsymbol{n}]\,ds - \int_{\Gamma} u^-\nabla v^-\cdot\boldsymbol{n}^-\,ds - \int_{\Gamma} u^+\nabla v^+\cdot\boldsymbol{n}^+\,ds.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\int_{\Omega} \nabla(I_h u)\cdot\nabla v dx = &\int_{\Omega} f\,v dx + \int_{\Gamma} \beta v\,ds - \int_{\Gamma} \alpha\nabla v^+\cdot\boldsymbol{n}^+\,ds\\
&+ \int_{\Gamma} (I_h u - u^-)[\nabla v\cdot\boldsymbol{n}]\,ds\\
&+ \sum_{e\in\mathcal{E}_h^i\setminus\mathcal{E}_h^{\Gamma,0}} \int_e (I_h u - u)[\nabla v\cdot\boldsymbol{n}]\,ds.
\end{aligned}$$

Using that $[\nabla v\cdot\boldsymbol{n}] = 0$ on $\Gamma\setminus\bigcup_{e\in\mathcal{E}_h^{\Gamma,0}}\{e\}$ we have that

$$\int_{\Gamma} (I_h u - u^-)[\nabla v\cdot\boldsymbol{n}]\,ds = \sum_{e\in\mathcal{E}_h^{\Gamma,0}} \int_e (I_h u - u^-)[\nabla v\cdot\boldsymbol{n}]\,ds.$$

The proof is completed after observing that $\mathcal{E}_h^i\setminus\mathcal{E}_h^{\Gamma,0} = \mathcal{E}_h^{\Gamma^\perp}\cup\mathcal{E}_h^{\Gamma}\cup\mathcal{E}_h^{\Gamma,+}\cup\mathcal{E}_h^{\Gamma,-}$. ☐

Proof. (Lemma 2) Here we only prove the first identity. The second identity is in fact easier to prove.

We have

$$\int_e (I_h u - u)[\nabla v\cdot\boldsymbol{n}]\,ds = [\nabla v\cdot\boldsymbol{n}]|_e \int_e w_e\,ds + [\nabla v\cdot\boldsymbol{n}]|_e \int_e (\tilde{u}_e - u)\,ds,$$

where we set $w_e = I_h u - \tilde{u}_e$. Note that $w_e(y_e^{\pm}) = 0$. Since w_e is piecewise linear, we can easily show that

$$\begin{aligned}
\int_e w_e\,ds &= \frac{1}{2}(h_{e^-} w_e^-(x_e) + h_{e^+} w_e^+(x_e)) \\
&= \frac{1}{2}(h_{e^-} w_e^+(x_e) + h_{e^+} w_e^-(x_e)) \\
&\quad + \frac{1}{2}h_{e^-}(w_e^-(x_e) - w_e^+(x_e)) + \frac{1}{2}h_{e^+}(w_e^+(x_e) - w_e^-(x_e)) \\
&= \frac{1}{2}(h_{e^-} w_e^+(x_e) + h_{e^+} w_e^-(x_e)) \\
&\quad + \frac{1}{2}(h_{e^+} - h_{e^-})[w_e(x_e)] \\
&= \frac{h_{e^-} h_{e^+}}{2}(\nabla w_e^+ \cdot \boldsymbol{t}_{e^+} + \nabla w_e^- \cdot \boldsymbol{t}_{e^-}) \\
&\quad + \frac{1}{2}(h_{e^+} - h_{e^-})[w_e(x_e)].
\end{aligned}$$

In the last step we used $w_e^{\pm}(x_e) = h_{e^{\pm}} \nabla w_e \cdot \boldsymbol{t}_{e^{\pm}}$ $(w_e(y_e^{\pm}) = 0)$. Since $I_h u$ is continuous on e, we have

$$\begin{aligned}
\int_e w_e\,ds &= -\frac{h_{e^-} h_{e^+}}{2}(\nabla \tilde{u}_e^+ \cdot \boldsymbol{t}_{e^+} + \nabla \tilde{u}_e^- \cdot \boldsymbol{t}_{e^-}) \\
&\quad - \frac{1}{2}(h_{e^+} - h_{e^-})[\tilde{u}_e(x_e)] \\
&= -\frac{h_{e^-} h_{e^+}}{2}(\nabla u^+(x_e) \cdot \boldsymbol{t}_{e^+} + \nabla u^-(x_e) \cdot \boldsymbol{t}_{e^-}) \\
&\quad - \frac{1}{2}(h_{e^+} - h_{e^-})[u(x_e)] \\
&\quad + \frac{h_{e^-} h_{e^+}}{2}(\nabla(u^+ - \tilde{u}_e^+)(x_e) \cdot \boldsymbol{t}_{e^+} + \nabla(u^- - \tilde{u}_e^-)(x_e) \cdot \boldsymbol{t}_{e^-}) \\
&= -\frac{h_{e^-} h_{e^+}}{2}\left(a_e \beta(x_e) + b_e \frac{d}{ds}\alpha(x_e)\right) - \frac{1}{2}(h_{e^+} - h_{e^-})\alpha(x_e) \\
&\quad + \frac{h_{e^-} h_{e^+}}{2}(\nabla(u^+ - \tilde{u}_e^+) \cdot \boldsymbol{t}_{e^+} + \nabla(u^- - \tilde{u}_e^-) \cdot \boldsymbol{t}_{e^-}). \qquad \square
\end{aligned}$$

Of course, we defined our method (2)-(3) precisely using Lemmas 1 and 2. The right-hand side $F_u(\nabla v)$ collects all the high-order (also non-computable) terms, while the computable terms are used to define the proposed discrete method (2)-(3), and the following result follows straightforwardly.

Lemma 3. *Let $u_h \in V_h$ solve* (2) *with E_h given by* (3)*. Then it holds that*

$$\int_\Omega \nabla(I_h u - u_h) \cdot \nabla v\,dx = F_u(\nabla v) \quad \textit{for all } v \in V_h,$$

where

$$\begin{aligned} F_u(\boldsymbol{\phi}) = & \sum_{e\in\mathcal{E}_h^{\Gamma\perp}\cup\mathcal{E}_h^{\Gamma,-}} \int_e (I_h u - u)[\boldsymbol{\phi}\cdot\boldsymbol{n}]\,ds + \sum_{e\in\mathcal{E}_h^{\Gamma,+}} \int_e (u-\tilde{u}_e)[\boldsymbol{\phi}\cdot\boldsymbol{n}]\,ds \\ & + \sum_{e\in\mathcal{E}_h^{\Gamma}} [\boldsymbol{\phi}\cdot\boldsymbol{n}]|_e \frac{h_{e^-} h_{e^+}}{2} (\nabla(u^+ - \tilde{u}_e^+)(x_e)\cdot\boldsymbol{t}_{e^+} + \nabla(u^- - \tilde{u}_e^-)(x_e)\cdot\boldsymbol{t}_{e^-}) \\ & + \sum_{e\in\mathcal{E}_h^{\Gamma,0}} \int_e (I_h u - u^-)[\boldsymbol{\phi}\cdot\boldsymbol{n}]\,ds, \quad \textit{for all } \boldsymbol{\phi} = \nabla v,\ v\in V_h. \end{aligned}$$

In order to establish a priori error estimates in $W^{i,\infty}$, for $i = 0, 1$, we extend the domain of definition of $F_u(\boldsymbol{\phi})$ to the space Φ_h of non-conforming Raviart-Thomas elements

$$\Phi_h = \{\boldsymbol{\phi} \in [L^2(\Omega)]^2 : \boldsymbol{\phi}|_T \in RT_0(T) \text{ for all } T \in \mathcal{T}_h\},$$

where $RT_0(T) = [\mathbb{P}^0(T)]^2 \oplus x\mathbb{P}^0(T)$. Note that the space of ∇v for all $v \in V_h$ is contained in Φ_h. Moreover, clearly we have

$$F_u(\boldsymbol{\phi}) = 0 \quad \text{for all } \boldsymbol{\phi} \in \Phi_h^D, \tag{5}$$

where

$$\Phi_h^D = \Phi_h \cap H(\mathrm{div};\Omega)$$

is the conforming Raviart-Thomas space (see Raviart-Thomas [30]), since $[\boldsymbol{\phi}\cdot\boldsymbol{n}]|_e = 0$ for all $\boldsymbol{\phi} \in \Phi_h^D$. Below we will use $F_u(\Pi\nabla g) = 0$ where Π is the Raviart-Thomas projection and g is an approximate Green's function. This is the reason we introduce Raviart-Thomas elements. Note that $F_u(\nabla(I_h g))$ is not necessarily zero.

We also have the following lemma.

Lemma 4. *For any $u \in C^2(\overline{\Omega}^{\pm})$ and any $\boldsymbol{\phi} \in \Phi_h$, it holds that*

$$|F_u(\boldsymbol{\phi})| \le hC_F \|\boldsymbol{\phi}\|_{L^1(\Omega)} \quad \textit{for all } \boldsymbol{\phi} \in \Phi_h, \tag{6}$$

where

$$C_F \le C(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}),$$

where C depends only on the shape regularity and quasi-uniformity of the elements of $\mathcal{T}_h$.

Proof. First note that by inverse estimates we have

$$|e|[\boldsymbol{\phi}\cdot\boldsymbol{n}]|_e = \|[\boldsymbol{\phi}\cdot\boldsymbol{n}]|\|_{L^1(e)} \le Ch^{-1}(\|\boldsymbol{\phi}\|_{L^1(T_e^1)} + \|\boldsymbol{\phi}\|_{L^1(T_e^2)}),$$

where the elements T_e^1 and T_e^2 share the edge e. The first term of $F_u(\boldsymbol{\phi})$ is easy to bound since u is smooth in each entire edge $e \in \mathcal{E}_h^{\Gamma\perp} \cup \mathcal{E}_h^{\Gamma,-}$, and so

$$\|u - I_h u\|_{L^\infty(e)} \le Ch^2\|u\|_{C^2(e)} \le Ch^2(\|u^+\|_{C^2(\Omega^+)} + \|u^-\|_{C^2(\Omega^-)}).$$

It is also easy to obtain bounds for the fourth term since u^- is $C^2(e)$ on the entire edge e.

The second and third terms can be bounded easily using that $\tilde{u}_e^{\pm}$ is the linear interpolation of $u^{\pm}$ on $e^{\pm}$ and $u^{\pm}$ is $C^2(e^{\pm})$, so that

$$\|u^{\pm} - \tilde{u}_e^{\pm}\|_{L^\infty(e^{\pm})} + |e|\|\nabla(u^{\pm} - \tilde{u}_e^{\pm})\cdot\boldsymbol{t}_{e^{\pm}}\|_{L^\infty(e^{\pm})} \le C\,|e|^2\|u\|_{C^2(e^{\pm})}. \qquad \square$$

The properties (6) and (5) will be important to prove optimal estimates, which we do in the next section.

3. Abstract error analysis

In this section we give an abstract error analysis of finite element methods. Estimates for the method we have defined in the previous section follow from these abstract estimates.

The finite element methods we consider in this paper read as follows: find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = E_h(v) \quad \forall v \in V_h, \tag{7}$$

where E_h is a linear functional.

Now we can state a positive result. The proof turns out to be a simple consequence of approximate Green's functions estimates derived by Rannacher and Scott [29].

Theorem 1. *Suppose that Ω is a convex polygon and suppose the family of meshes $\{\mathcal{T}_h\}_{h>0}$ are shape regular and quasi-uniform. Suppose that $u^\pm \in C^2(\overline{\Omega}^\pm)$ and $u_h \in V_h$ are the solutions of (1) and (7), respectively. Suppose that*

$$\int_\Omega \nabla(I_h u - u_h) \cdot \nabla v \, dx = F_u(\nabla v) \quad \textit{for all } v \in V_h$$

and F_u satisfies the following:

$$F_u(\boldsymbol{\phi}) = 0 \textit{ for any } \boldsymbol{\phi} \in \Phi_h^D, \tag{8}$$

$$|F_u(\boldsymbol{\phi})| \le C_F h \|\boldsymbol{\phi}\|_{L^1(\Omega)} \textit{ for all } \boldsymbol{\phi} \in \Phi_h^D, \tag{9}$$

for a constant C_F. Then, there exists a constant C such that

$$\|\nabla(I_h u - u_h)\|_{L^\infty(\Omega)} \le C C_F \, h, \tag{10}$$

where C depends only on the quasi-uniformity and shape regularity of the mesh.

Proof. Let $z \in \Omega \subset \mathbb{R}^2$. To prove estimate (10), we need to bound $|\nabla(I_h u - u_h)(z)|$ for any z. It is enough to get a bound of $|\nabla(I_h u - u_h)(z)|$ for every z interior to a triangle of the mesh since we can bound the error at points on the boundary of triangles by continuity. To this end, let $z \in T_z$ for some $T_z \in \mathcal{T}_h$.

Consider now the regularized Dirac delta function $\delta_h^z = \delta_h \in C_0^1(T_z)$ (see [6]), which satisfies

$$r(z) = (r, \delta_h)_{T_z} \qquad \forall r \in P^1(T_z) \tag{11}$$

and has the property

$$\|\delta_h\|_{W^{k,q}(T_z)} \le C h^{-k-2(1-1/q)}, \qquad 1 \le q \le \infty, \ k = 0, 1, \tag{12}$$

where the constant C does not depend on z. For a more detailed construction of δ_h see the appendix in [31].

For each $i = 1, 2$, define the approximate Green's function $g \in H_0^1(\Omega)$, which solves the following equations:

$$-\Delta g = \partial_{x_i} \delta_h \qquad \text{in } \Omega, \tag{13a}$$

$$g = 0 \qquad \text{on } \partial\Omega. \tag{13b}$$

The finite element approximation of g, $g_h \in V_h$ solves

$$\int_\Omega \nabla g_h \cdot \nabla v \, dx = \int_\Omega v \partial_{x_i} \delta_h \, dx \quad \text{for all } v \in V_h. \tag{14}$$

Then, using the definition of δ_h and equation (14), we have

$$\begin{aligned}\partial_{x_i}(I_h u - u_h)(z) &= \int_\Omega \delta_h \partial_{x_i}(I_h u - u_h)dx \\ &= -\int_\Omega (\partial_{x_i}\delta_h)(I_h u - u_h)dx \\ &= -\int_\Omega \nabla g_h \cdot \nabla(I_h u - u_h)dx \\ &= -F_u(\nabla g_h).\end{aligned}$$

We will use the Raviart-Thomas projection $\Pi : H^1(\Omega) \to \Phi_h^D$ (see [30]), defined locally, for any $T \in \mathcal{T}_h$, by $\Pi|_T : H^1(T) \to RT_0(T)$, such that

$$\int_e (\boldsymbol{q} - \Pi|_T \boldsymbol{q}) \cdot \boldsymbol{n}_e ds = 0 \quad \text{for each edge } e \subset \partial T.$$

By (8) we have $F_u(\Pi(\nabla g)) = 0$ and so

$$\partial_{x_i}(I_h u - u_h)(z) = F_u(\Pi(\nabla g) - \nabla g_h).$$

Hence, by (9) we have

$$|\partial_{x_i}(I_h u - u_h)(z)| = |F_u(\Pi(\nabla g) - \nabla g_h)| \le C_F h \|\Pi(\nabla g) - \nabla g_h\|_{L^1(\Omega)}.$$

The proof will be completed once we prove that

$$\|\nabla g - \nabla g_h\|_{L^1(\Omega)} \le C, \tag{15}$$

$$\|\Pi(\nabla g) - \nabla g\|_{L^1(\Omega)} \le C. \tag{16}$$

Estimate (15) is a known result (see [29]) where the constant C depends on the quasi-uniformity and shape regularity of the mesh and it is assumed that Ω is convex. The proof of estimate (16) is much easier, and we give a sketch of the proof in the Appendix. □

It turns out that we can remove (8) from the above theorem and still obtain a good result as the next theorem states.

Theorem 2. *Suppose all the hypotheses of the previous theorem except* (8). *Then, there exists a constant C such that*

$$\|\nabla(I_h u - u_h)\|_{L^\infty(\Omega)} \le CC_F\, h \log(1/h), \tag{17}$$

where C depends only on quasi-uniformity and shape regularity of the mesh.

Proof. Following the proof of the previous theorem we have

$$|\partial_{x_i}(I_h u - u_h)(z)| = |F_u(\nabla g_h)| \le C_F h \|\nabla g_h\|_{L^1(\Omega)},$$

where we used (9). Using the triangle inequality and since we have (15), it is enough to prove

$$\|\nabla g\|_{L^1(\Omega)} \le C \log(1/h).$$

This is a well-known result, and the proof is very similar to the proof of (16). We give a proof in the Appendix to make the paper more self-contained. □

Now we turn our attention to an estimate for $\|I_h u - u_h\|_{L^\infty(\Omega)}$. First we prove an estimate in L^p norm, for any $2 \le p < \infty$, by a standard duality argument (see [6]).

Theorem 3. *Assume the hypotheses of Theorem* 1. *Then for any* $2 \le p < \infty$ *there exists a constant* C *such that*

$$\|I_h u - u_h\|_{L^p(\Omega)} \le Chp\,(\|\nabla(I_h u - u_h)\|_{L^p(\Omega)} + hC_F),$$

and in particular

$$\|I_h u - u_h\|_{L^p(\Omega)} \le CC_F h^2 p,$$

where C *depends only on quasi-uniformity and shape regularity of the mesh.*

Proof. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we know that

$$\|I_h u - u_h\|_{L^p(\Omega)} = \sup_{\phi \in C_c(\Omega)} \frac{\int_\Omega (I_h u - u_h)\phi}{\|\phi\|_{L^q(\Omega)}}.$$

Given ϕ as above define ψ as the solution to the problem

$$\begin{aligned} -\Delta\psi &= \phi \qquad \text{in } \Omega, \\ \psi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

We know that the following regularity holds for $2 \ge q > 1$ for smooth domain Ω (see for example [13]):

$$\|\psi\|_{W^{2,q}(\Omega)} \le C\,p\,\|\phi\|_{L^q(\Omega)}, \tag{18}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Note that the constant $C\,p$ blows up as q approaches 1. The estimate (18) for convex domains also holds, although an explicit formula for the constant does not seem to be in the literature (see for example [6]).

Then, we see that

$$\begin{aligned} \int_\Omega (I_h u - u_h)\phi\,dx &= \int_\Omega \nabla(I_h u - u_h)\cdot\nabla\psi\,dx \\ &= \int_\Omega \nabla(I_h u - u_h)\cdot\nabla(\psi - I_h\psi)\,dx + \int_\Omega \nabla(I_h u - u_h)\cdot\nabla I_h\psi\,dx \\ &= \int_\Omega \nabla(I_h u - u_h)\cdot\nabla(\psi - I_h\psi)\,dx + F_u(\nabla(I_h\psi)) \\ &= \int_\Omega \nabla(I_h u - u_h)\cdot\nabla(\psi - I_h\psi)\,dx + F_u(\nabla(I_h\psi) - \Pi\nabla\psi), \end{aligned}$$

where we used (8) in the last step. Hence, using (9) we have

$$\int_\Omega (I_h u - u_h)\phi \le \|\nabla(I_h u - u_h)\|_{L^p(\Omega)}\|\nabla(\psi - I_h\psi)\|_{L^q(\Omega)} + C_F h\|\nabla(I_h\psi) - \Pi\nabla\psi\|_{L^1(\Omega)}. \tag{19}$$

Using the properties of I_h and Π we can easily show that

$$\|\nabla(\psi - I_h\psi)\|_{L^q(\Omega)} \le C\,h\|\psi\|_{W^{2,q}(\Omega)}$$

and

$$\|\nabla(I_h\psi) - \Pi\nabla\psi\|_{L^1(\Omega)} \le Ch\|\psi\|_{W^{2,1}(\Omega)}.$$

The proof will be completed if we use (18) and take the supremum over ϕ. ☐

Corollary 1. *Assume the hypotheses of Theorem* 1. *Then, we have*

$$\|I_h u - u_h\|_{L^\infty(\Omega)} \le CC_F h^2 \log(1/h),$$

where C *depends only on quasi-uniformity and shape regularity of the mesh.*

Proof. Using the inverse inequality we have

$$\|I_h u - u_h\|_{L^\infty(\Omega)} \leq Ch^{-2/p}\|I_h u - u_h\|_{L^p(\Omega)}.$$

The result follows after applying the previous theorem and setting $\frac{p}{2} = \log(1/h)$. $\square$

We conclude this section by stating the estimates for the method we derived in the previous section. Of course, the estimates are simple consequences of Theorem 1, Corollary 1 and Lemma 4.

Corollary 2. *Suppose that Ω is convex. Let $u_h \in V_h$ be the solution to* (2) *with E_h given by* (3)*. Then we have the estimates*

$$\|\nabla(I_h u - u_h)\|_{L^\infty(\Omega)} \leq Ch(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)})$$

and

$$\|I_h u - u_h\|_{L^\infty(\Omega)} \leq Ch^2 \log(1/h)(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}),$$

where C depends only on the quasi-uniformity and shape regularity of the mesh.

Remark 3. We have established a priori error estimates between u_h and u at the nodes of the mesh. There are many ways to use u_h to obtain an approximation of $O(h^2)$ at points near Γ (e.g. using extrapolation from nearby nodes). In the following we sketch one such way. For an edge $e \in \mathcal{E}_h^\Gamma$, we construct the piecewise linear function $\tilde{u}_h$ on $e^\pm$ by imposing appropriate jump conditions of the function and tangential derivative at x_e and $\tilde{u}_h(y_e^\pm) = u_h(y_e^\pm)$. If $e \in \mathcal{E}_h^{\Gamma,+}$, $\tilde{u}_h$ is defined as the linear function satisfying $\tilde{u}_h(y_e^-) = u_h(y_e^-) + \alpha(y_e^-)$ and $\tilde{u}_h(y_e^+) = u_h(y_e^+)$. For the remaining edges, define $\tilde{u}_h = u_h$. It is possible to show that if $u^\pm \in C^2(\Omega^\pm)$, the error $\|\tilde{u}_h - u\|_{L^\infty(e)}$ is of the same order as the error of $|u_h - u|$ at the endpoints of e plus $O(h^2)$. To extend $\tilde{u}_h$ to inside an element T, we can interpolate the values of $\tilde{u}_h$ on each $\partial T \cap \overline{\Omega}^\pm$ to $T \cap \overline{\Omega}^\pm$, respectively. Let us denote this interpolation again by $\tilde{u}_h$. For instance, if Γ crosses ∂T in two points, for the region $T \cap \bar{\Omega}^\pm$ which has a triangular shape we use linear interpolation, while for the region with a quadrangular shape we use isoparametric bilinear interpolation.

4. Other methods

4.1. **The method of He et al.** [16]**.** It turns out that our method is very similar to a method introduced by He et al. [16] applied to problem (1) and using continuous piecewise polynomials. It should be mentioned that the methodology used to derive the method in [16] is quite different from the methodology that we used to derive the method in the previous section. We should also mention that He et al. [16] considered the much more difficult problem of discontinuous diffusion coefficients. Although for their method F_u does not satisfy (8) and so we cannot prove the $h^2 \log(1/h)$ estimate for the pointwise error as in Corollary 1, we develop a more complicated analysis to prove an optimal result.

The finite element method of [16] for (1) (with $\alpha = 0$) solves (2) with

$$E_h(v) = \int_\Omega f\, v dx + \int_\Gamma \beta v ds - \sum_{T \in \mathcal{T}_h^\Gamma} q_T \int_T \nabla u_b \cdot \nabla v\, dx, \tag{20}$$

where $\mathcal{T}_h^\Gamma$ are all the triangle in $\mathcal{T}_h$ that intersect Γ. In order to define q_T and u_b we need to first introduce some notation. Suppose that $T \in \mathcal{T}_h^\Gamma$ and Γ intersects T at two points, x_e and x_r, where e and r are edges of T. Consider the line $L_T = \overline{x_e x_r}$ that passes through x_e and x_r (i.e., the line that interpolates Γ) and let $\boldsymbol{n}_T^\pm$ be the unit normal vector of that line pointing out of $\Omega^\pm$. Then, $q_T = \frac{1}{|L_T|}\int_{\Gamma\cap T}\beta(s)ds$. Moreover the function u_b on T is piecewise linear such that it vanishes on all the three nodes of T and such that the jump of the normal derivative of u_b along L_T is 1:

$$\nabla u_b^- \cdot \boldsymbol{n}_T^- + \nabla u_b^+ \cdot \boldsymbol{n}_T^+ = 1.$$

In order to make the method of He et al. look more like our method (3) we integrate by parts and get

$$-q_T\int_T \nabla u_b\cdot\nabla v\,dx = -q_T\int_{\partial T} u_b\nabla v\cdot\boldsymbol{n}\,ds = -q_T\Big(\int_e u_b\nabla v\cdot\boldsymbol{n}\,ds + \int_r u_b\nabla v\cdot\boldsymbol{n}\,ds\Big).$$

It is not difficult to see that

$$-q_T\int_e u_b\nabla v\cdot\boldsymbol{n}\,ds = -\frac{h_{e^-}h_{e^+}}{2}q_T\tilde{a}_e^T\nabla v\cdot\boldsymbol{n},$$

where $\tilde{a}_e^T = \boldsymbol{t}_{e^-}\cdot\boldsymbol{n}_T^-$. Note that $\tilde{a}_e^T \neq a_e$ in general and more crucially that $\tilde{a}_e^T$ is also different from $\tilde{a}_e^K$ when K is the other triangle that has e as an edge. Of course, they do coincide when Γ is a line, and in fact our method will coincide with the method of He et al.

To be more precise, let $e\in\mathcal{E}_h^\Gamma$ with $\bar{e} = \overline{T}\cap\overline{K}$ and $T, K\in\mathcal{T}_h^\Gamma$; then set

$$c_e = \frac{q_T\tilde{a}_e^T + q_K\tilde{a}_e^K}{2}, \quad m_e = \frac{q_T\tilde{a}_e^T - q_K\tilde{a}_e^K}{2}.$$

Then we see that

$$\begin{aligned} (21)\qquad & -\sum_{T\in\mathcal{T}_h^\Gamma} q_T\int_T\nabla u_b\cdot\nabla v\,dx \\ &= -\sum_{e\in\mathcal{E}_h^\Gamma}\left(\frac{h_{e^-}h_{e^+}}{2}c_e[\nabla v\cdot\boldsymbol{n}]|_e + \frac{h_{e^-}h_{e^+}}{2}m_e\{\nabla v\cdot\boldsymbol{n}\}|_e\right), \end{aligned}$$

where $\{\nabla v\cdot\boldsymbol{n}\}|_e = \frac{1}{2}(\nabla v|_T + \nabla v|_K)\cdot\boldsymbol{n}_T$ where again $\bar{e} = \overline{T}\cap\overline{K}$ and $\boldsymbol{n}_T$ is unit normal pointing out of T.

As we can see our method and the method of He et al. are very similar. In fact, let $\mathbf{u}_h$ denote the solution of our method and let u_h be the solution of the method of He et al. and let $w_h = \mathbf{u}_h - u_h$. Then we see that

$$\int_\Omega \nabla w_h\cdot\nabla v\,dx = R_u(\nabla v) \quad \text{for all } v\in V_h,$$

where

$$R_u(\boldsymbol{\phi}) = \sum_{e\in\mathcal{E}_h^\Gamma}\left(\frac{h_{e^-}h_{e^+}}{2}(c_e - a_e\beta(x_e))[\boldsymbol{\phi}\cdot\boldsymbol{n}]|_e + \frac{h_{e^-}h_{e^+}}{2}m_e\{\boldsymbol{\phi}\cdot\boldsymbol{n}\}|_e\right).$$

It is easy to see that

$$\begin{aligned} |c_e - a_e\beta(x_e)| + |m_e| &\le C\,h\max_s\Big(|\beta(X(s))| + \Big|\frac{\partial\beta(X(s))}{\partial s}\Big|\Big) \\ &\le C\,h(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}). \end{aligned}$$

Hence, we can show that

$$(22)\qquad R_u(\boldsymbol{\phi}) \le C_R h\|\boldsymbol{\phi}\|_{L^1(S_\Gamma)} \quad \text{for all } \boldsymbol{\phi}\in\Phi_h,$$

where $S_\Gamma = \bigcup_{T\in\mathcal{T}_h, T\cap\Gamma\neq\emptyset} T$ and

$$C_R \leq C\,(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}).$$

However, it does not necessarily hold that $R_h(\boldsymbol{\phi}) = 0$, for all $\boldsymbol{\phi} \in \Phi_h^D$.

Using Theorem 2 we do have, however,

$$\|\nabla(I_h u - u_h)\|_{L^\infty(\Omega)} \;\leq\; C\,(C_R + C_L)\,\log(1/h)h$$

for the He et al. [16] method. We can remove the logarithm by using a more delicate analysis.

Theorem 4. *Let Ω be convex and let u_h be the solution of* (2) *with* (20)*. Then we have*

$$\|\nabla(I_h u - u_h)\|_{L^\infty(\Omega)} \;\leq\; C\,h(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}),$$

where C depends only on the quasi-uniformity and shape regularity of the mesh.

Proof. Following the proof of Theorem 2 we can show for each $i = 1, 2$ that

$$|\partial_{x_i} w_h(z)| = |R_u(\nabla g_h)| \leq C_R h \|\nabla g_h\|_{L^1(S_\Gamma)}$$

by using (22). Using the triangle inequality we have

$$|\partial_{x_i} w_h(z)| \leq C_R h(\|\nabla(g_h - g)\|_{L^1(S_\Gamma)} + \|\nabla g\|_{L^1(S_\Gamma)}).$$

Using (15) we have $\|\nabla(g_h - g)\|_{L^1(S_\Gamma)} \leq \|\nabla(g_h - g)\|_{L^1(\Omega)} \leq C$. Although it holds that

$$\|\nabla g\|_{L^1(\Omega)} \leq C\log(1/h),$$

one has the better estimate

$$\|\nabla g\|_{L^1(S_\Gamma)} \leq C.$$

The proof of this inequality follows the ideas of the proof of (16). We leave the details to the reader. This will show that

$$\|\nabla w_h\|_{L^\infty(\Omega)} \leq C\,C_R\,h.$$

The proof is completed after we apply Corollary 2. □

Now let us turn to the analysis of the pointwise error, which is more delicate.

Theorem 5. *Let Ω be a convex set and let u_h be the solution of* (2) *with E_h defined by* (20)*. Then we have*

$$\|I_h u - u_h\|_{L^\infty(\Omega)} \;\leq\; C\,h^2\log(1/h)(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}),$$

where C depends only on the quasi-uniformity and shape regularity of the mesh.

Proof. Let $z \in \Omega$ be arbitrary and let $\delta_h = \delta_h^z$ satisfy (11) and (12). Define the approximate Green's function $\tilde{g} \in H_0^1(\Omega)$, which solves the following problem:

$$\begin{aligned} -\Delta\tilde{g} &= \delta_h \qquad \text{in } \Omega,\\ \tilde{g} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Let $P_h : H_0^1(\Omega) \to V_h$ be the Scott-Zhang interpolant (see [32]). Then we have

$$\begin{aligned} w_h(z) &= \int_\Omega w_h \delta_h \, dx = \int_\Omega \nabla w_h \cdot \nabla \tilde{g} \, dx \\ &= \int_\Omega \nabla w_h \cdot \nabla P_h \tilde{g} \, dx + \int_\Omega \nabla w_h \cdot \nabla (\tilde{g} - P_h \tilde{g}) \, dx \\ &= R_u(\nabla (P_h \tilde{g})) + \int_\Omega \nabla w_h \cdot \nabla (\tilde{g} - P_h \tilde{g}) \, dx. \end{aligned}$$

Hence

$$|w_h(z)| \leq C_R h \, \|\nabla (P_h \tilde{g})\|_{L^1(S_\Gamma)} + \|\nabla w_h\|_{L^\infty(\Omega)} \|\nabla (\tilde{g} - P_h \tilde{g})\|_{L^1(\Omega)}. \tag{23}$$

The property of P_h, elliptic regularity (18) (for any $2 \geq q > 1$), and using the notation $\frac{1}{p} + \frac{1}{q} = 1$ imply that

$$\|\nabla (\tilde{g} - P_h \tilde{g})\|_{L^1(\Omega)} \leq C \, h \|\tilde{g}\|_{W^{2,1}(\Omega)} \leq C \, p \|\delta_h\|_{L^q(\Omega)}.$$

Using property (12) we get

$$\|\nabla (\tilde{g} - P_h \tilde{g})\|_{L^1(\Omega)} \leq C p h^{-2/p} h = C h \log(1/h), \tag{24}$$

where we choose $\frac{p}{2} = \log(1/h)$.

Next, we estimate $\|\nabla (P_h \tilde{g})\|_{L^1(S_\Gamma)}$. Using the stability of the Scott-Zhang interpolant we have $\|\nabla (P_h \tilde{g})\|_{L^1(S_\Gamma)} \leq C \, \|\nabla \tilde{g}\|_{L^1(\tilde{S}_\Gamma)}$, where $\tilde{S}_\Gamma = \{x : dist(x, S_\Gamma) \leq h\}$.

So far, we have the estimate

$$|w_h(z)| \leq C \, C_R h^2 \log(1/h) + C C_R h \|\nabla \tilde{g}\|_{L^1(\tilde{S}_\Gamma)}, \tag{25}$$

where we used (20).

In order to estimate $\|\nabla \tilde{g}\|_{L^1(\tilde{S}_\Gamma)}$ we write

$$S_i = \{x \in \tilde{S}_\Gamma : ih \leq |x - z| \leq (i+1)h\}.$$

Using a natural assumption on the shape of Γ, one can see that $|S_i| \leq C h^2$ for all i. Hence,

$$h \|\nabla \tilde{g}\|_{L^1(\tilde{S}_\Gamma)} = h \|\nabla \tilde{g}\|_{L^1(S_0 \cup S_1)} + h \sum_{i=2}^{M} \|\nabla \tilde{g}\|_{L^1(S_i)},$$

where $M = \mathcal{O}(1/h)$. We bound the first term:

$$\|\nabla \tilde{g}\|_{L^1(S_0 \cup S_1)} \leq h^{2-2/p} \|\nabla \tilde{g}\|_{L^p(S_0 \cup S_1)}.$$

Using the Sobolev embedding inequality we have

$$\|\nabla \tilde{g}\|_{L^p(S_0 \cup S_1)} \leq p \|\tilde{g}\|_{H^2(S_0 \cup S_1)} \leq p \|\tilde{g}\|_{H^2(\Omega)} \leq p \|\delta_h\|_{L^2(\Omega)},$$

where we used elliptic regularity (18). Hence, using (12) we have

$$h \|\nabla \tilde{g}\|_{L^1(S_0 \cup S_1)} \leq p \, h^2 h^{-1/p}.$$

Again, choosing $p = \log(1/h)$ we get

$$h \|\nabla \tilde{g}\|_{L^1(S_0 \cup S_1)} \leq C \, h^2 \log(1/h).$$

For the remaining terms we get

$$\sum_{i=2}^{M} \|\nabla \tilde{g}\|_{L^1(S_i)} \leq h^2 \sum_{i=2}^{M} \|\nabla \tilde{g}\|_{L^\infty(S_i)}.$$

Then we obtain

$$h\|\nabla\tilde{g}\|_{L^1(\tilde{S}_\Gamma)} = h^2\log(1/h) + h^3\sum_{i=2}^{M}\|\nabla\tilde{g}\|_{L^\infty(S_i)}. \tag{26}$$

Using the Green's function representation

$$\tilde{g}(x) = \int G_x(y)\delta_h(y)dy,$$

where G_x is the Green's function centered at x, we have

$$\partial_{x_i}\tilde{g}(x) = \int \partial_{x_i}G_x(y)\delta_h(y)dy.$$

It is well known that

$$|\partial_{x_i}G_x(y)| \le \frac{C}{|x-y|}.$$

If $x \in S_i$, then we know that $\|x-y\| \ge (i-1)h$ for any $y \in T_z$. Hence, we have

$$\|\nabla\tilde{g}\|_{L^\infty(S_i)} \le \frac{C}{(i-1)h}\|\delta_h\|_{L^1(T_z)} = \frac{C}{(i-1)h}.$$

Therefore,

$$h^3\sum_{i=2}^{M}\|\nabla\tilde{g}\|_{L^\infty(S_i)} \le C\,h^2\sum_{i=1}^{M-1}\frac{1}{i} \le C\,h^2\log(1/h).$$

Combining the last inequality and (26), we get

$$h\|\nabla\tilde{g}\|_{L^1(\tilde{S}_\Gamma)} \le Ch^2\log(1/h). \tag{27}$$

Taking the supremum over $z \in \Omega$ in (23) and using estimates (24) and (27) we obtain

$$\|w_h\|_{L^\infty(\Omega)} \le C\,h\log(1/h)\left(\|\nabla w_h\|_{L^\infty(\Omega)}\right).$$

The proof is completed if we apply the triangle inequality, Corollary 2 and the previous theorem. □

4.2. **The natural method.** As mentioned earlier, the natural method (for $\alpha = 0$) is given by (2) with

$$E_h(v) = \int_\Omega fv\,dx + \int_\Gamma \beta v\,ds. \tag{28}$$

It is well known that this method is suboptimal near the interface Γ. For completeness we prove error estimates for this method.

To this end, let $\mathbf{u}_h$ be the solution using our method (3) (with $\alpha = 0$) and let u_h be the method using (28) and call $w_h = u_h - \mathbf{u}_h$. Then we see that w_h satisfies

$$\int_\Omega \nabla w_h \cdot \nabla v\,dx = L_u(\nabla v) \quad \text{for all } v \in V_h, \tag{29}$$

where

$$L_u(\boldsymbol{\phi}) = \sum_{e\in\mathcal{E}_h^\Gamma}\frac{h_{e^-}h_{e^+}}{2}a_e\beta(x_e)\,[\boldsymbol{\phi}\cdot n]|_e.$$

We can easily show the following lemma.

Lemma 5. *Let $u^\pm \in C^2(\Omega^\pm)$. Then it holds that*

$$L_u(\boldsymbol{\phi}) = 0 \text{ for any } \boldsymbol{\phi} \in \Phi_h^D, \tag{30}$$

$$|L_u(\boldsymbol{\phi})| \le C_L \|\boldsymbol{\phi}\|_{L^1(S_\Gamma)} \text{ for all } \boldsymbol{\phi} \in \Phi_h, \tag{31}$$

where the $S_\Gamma = \bigcup_{T \in \mathcal{T}_h, T\cap\Gamma \neq \emptyset} T$ *and*

$$C_L \le C(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}),$$

where C depends only on the quasi-uniformity and shape regularity of the mesh.

Theorem 6. *Let Ω be a convex domain and let u_h solve* (2) *with* (28)*. Then we have that*

$$\|u_h - I_h u\|_{L^\infty(\Omega)} \le C\, h\, \log(1/h)(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}),$$

where C depends only on the quasi-uniformity and shape regularity of the mesh.

Proof. Following the argument as in the proof of Theorem 1, we can easily show that $\|\nabla w_h\|_{L^\infty(\Omega)} \le C$ where we use (30) and (31) and also the estimates (15) and (16).

Then, using a duality argument as in the proof of Theorem 3, we can easily show for $2 \le p < \infty$:

$$\|w_h\|_{L^p(\Omega)} \le Chp(\|\nabla w_h\|_{L^p(\Omega)} + C_L) \le Chp(1 + C_L).$$

Then, as we did before, we use an inverse estimate $\|w_h\|_{L^\infty(\Omega)} \le C\, h^{-2/p} \|w_h\|_{L^p(\Omega)}$ and set $\frac{p}{2} = \log(1/h)$ to get

$$\|w_h\|_{L^\infty(\Omega)} \le Ch \log(1/h)(1 + C_L).$$

We obtain the result if we apply Corollary 2. ☐

The above result is far from optimal, and this is in fact observed in numerical experiments near the interface Γ. In particular, the gradient of the error will be $O(1)$ near the interface. However, numerical experiments also show that if one is far enough away from the interface, then one obtains optimal estimates. In fact, Mori [26] showed that this was the case for the immersed boundary method [26] (see also [22]). We note, however, he did not quantify exactly how far away from the interface one has to be.

We will quantify how far from the interface one has to be to obtain optimal estimates for the gradient error. In order to this we will need Green's function estimates of the third derivatives. This holds on smooth domains Ω; however not on any convex polygonal domain. Therefore, we assume that Ω is a rectangle and we replace the Dirichlet boundary conditions with periodic boundary conditions. In this case, we will have the following estimate for the corresponding Green's function $G_x(y)$ centered at x:

$$|\partial^2_{x_i x_j} \partial_{y_j} G_x(y)| \le \frac{C}{|x-y|^3}, \tag{32}$$

for any $1 \le i, j \le 2$.

Theorem 7. *Suppose that Ω is a rectangle and assume that u solves* (1) *with the Dirichlet boundary conditions replaced with periodic boundary conditions. Let u_h*

solve (2) using (28). Let $z \in \Omega$ and let $d = dist(z, \Gamma) \geq \kappa h$ for a sufficiently large fixed constant κ. Furthermore, suppose $dist(\Gamma, \partial\Omega) > d$. Then, we have

$$|\nabla(I_h u - u_h)(z)| \leq Ch(\log(1/h)\frac{h}{d^2} + 1)(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}),$$

where C depends only on the quasi-uniformity and shape regularity of the mesh.

Proof. Let $\delta_h = \delta_h^z$ satisfy (11) and (12). Furthermore, for each $i = 1, 2$, let g satisfy (13) with Dirichlet boundary conditions replaced with periodic boundary conditions and let g_h be its finite element approximation. Then, we have

$$|\partial_{x_i} w_h(z)| = |L_u(\nabla g_h)| = |L_u(\nabla g_h - \Pi \nabla g)|,$$

where we used (30). Using (31) and the triangle inequality we have

$$|\partial_{x_i} w_h(z)| \leq CC_L(\|\nabla g_h - \nabla g\|_{L^1(S_\Gamma)} + \|\nabla g - \Pi\nabla g\|_{L^1(S_\Gamma)}). \tag{33}$$

We proceed to bound $\|\nabla g_h - \nabla g\|_{L^1(S_\Gamma)}$. The second term is easier to bound. Define the sets

$$S_i = \{x \in S_\Gamma : d_i \leq |x - z| \leq d_{i+1}\},$$

where $d_i = \sqrt{d^2 + (ih)^2}$. As one can see, by using a natural assumption on the shape of Γ, the measure of S_i is less than $O(h^2)$. Also define

$$B_r(S_i) = \{x : dist(x, S_i) \leq r\}.$$

We can then write

$$\|\nabla g_h - \nabla g\|_{L^1(S_\Gamma)} = \sum_{i=0}^{M} \|\nabla g_h - \nabla g\|_{L^1(S_i)},$$

where $M = O(1/h)$. We have

$$\|\nabla g_h - \nabla g\|_{L^1(S_i)} \leq Ch^2\|\nabla g_h - \nabla g\|_{L^\infty(S_i)}.$$

We will show the bound

$$\|\nabla g_h - \nabla g\|_{L^\infty(S_i)} \leq \frac{h}{d_i^3}\log(1/h), \tag{34}$$

and hence

$$\begin{aligned}\|\nabla g_h - \nabla g\|_{L^1(S_\Gamma)} &\leq C\,h^3 \log(1/h) \sum_{i=0}^{M} \frac{1}{d_i^3} \\ &= Ch^3 \log(1/h) \sum_{i=0}^{M} (\frac{1}{(d^2 + (ih)^2})^{3/2} \\ &\leq \frac{Ch^2}{d^2}\log(1/h).\end{aligned}$$

In the last step we bound the sum by

$$\begin{aligned} h^3 \int_1^{M+1} \left(\frac{1}{d^2 + h^2x^2}\right)^{3/2} dx &= \frac{h^3}{d^2}\left[\left(\frac{x^2}{d^2 + h^2x^2}\right)^{1/2}\right]_1^{M+1} \\ &\leq \frac{h^3}{d^2}\left(\frac{(M+1)^2}{d^2 + h^2(M+1)^2}\right)^{1/2},\end{aligned}$$

and we used the fact that $M = O(1/h)$.

Of course, we can also prove $\|\nabla g - \Pi \nabla g\|_{L^1(S_\Gamma)} \leq \frac{Ch^2}{d^2}\log(1/h)$, and therefore, in view of (33) we have

$$|\partial_{x_i} w_h(z)| \leq \frac{CC_L h^2}{d^2}\log(1/h).$$

Hence, the proof is completed if we combine this result with Corollary 2. What remains is the proof of (34). To do this, we will use a result by Schatz and Wahlbin. Note that $g - g_h$ solves the following:

$$\int_\Omega \nabla(g - g_h)\cdot\nabla v dx = 0 \quad \text{ for all } v \in V_h \cap H_0^1(B_{\frac{d_i}{2}}(S_i)).$$

Therefore, by a result of Schatz and Wahlbin [31], we have

$$\begin{aligned}\|\nabla(g-g_h)\|_{L^\infty(S_i)} \leq & C(\|\nabla(g - P_h g)\|_{L^\infty(B_{\frac{d_i}{2}}(S_i))} + \frac{1}{d_i}\|g - P_h g\|_{L^\infty(B_{\frac{d_i}{2}}(S_i))}) \\ & + \frac{1}{d^3}\|g - g_h\|_{L^1(B_{\frac{d_i}{2}}(S_i))}.\end{aligned}$$

We bound the first two terms:

$$\|\nabla(g - P_h g)\|_{L^\infty(B_{\frac{d_i}{2}}(S_i))} + \frac{1}{d_i}\|g - P_h g\|_{L^\infty(B_{\frac{d_i}{2}}(S_i))} \leq Ch\|g\|_{W^{2,\infty}(B_{\frac{3d_i}{4}}(S_i))}.$$

Let $x \in B_{\frac{3d_i}{4}}(S_i)$. Then $|x - y| \geq d_i$ for any $y \in T_z$, and hence

$$\begin{aligned}\partial^2_{x_i x_j} g(x) = & \int_\Omega \partial^2_{x_i x_j} G_x(y)\partial_{y_i}\delta_h(y)dy \; = \; -\int_\Omega \partial^2_{x_i x_j}\partial_{y_i} G_x(y)\delta_h(y)dy \\ \leq & \; \frac{C}{d_i^3}\|\delta_h\|_{L^1(T_z)} \; = \; \frac{C}{d_i^3},\end{aligned} \tag{35}$$

where we used (32).

Finally, using a duality argument (see Appendix C), we can show that

$$\|g - g_h\|_{L^1(\Omega)} \leq C\, h\, p\, h^{-(2-2/q)}, \tag{36}$$

where $1 < q < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Combining the last inequality with $\frac{p}{2} = \log(1/h)$ and (35) proves (34). This completes the proof. □

5. Numerical examples

In this section we illustrate our results with two examples. We consider the square domain $\Omega = [-1, 1]^2$ with non-uniform triangular meshes and we tabulate the L^2 error, H^1 semi-norm error, L^∞ error and $W^{1,\infty}$ semi-norm error with their respective order of convergence for our examples. Plots of approximate solutions by our method are also provided. The interpolant I_h is introduced in Definition 2.1.

Let u be the exact solution of problem (1), u_h be the solution of our method (2)-(3) and u_h^N the solution of the first-order method (2)-(28). Define the errors with respect to the interpolant I_h as

$$e_h := u_h - I_h u, \qquad e_h^N := u_h^N - I_h u,$$

and define the respective order of convergence (associated to the error and the norm) as

$$\mathrm{r}(e, \|\cdot\|) := \frac{\log(\|e_{h_{l+1}}\|/\|e_{h_l}\|)}{\log(h_{l+1}/h_l)}.$$

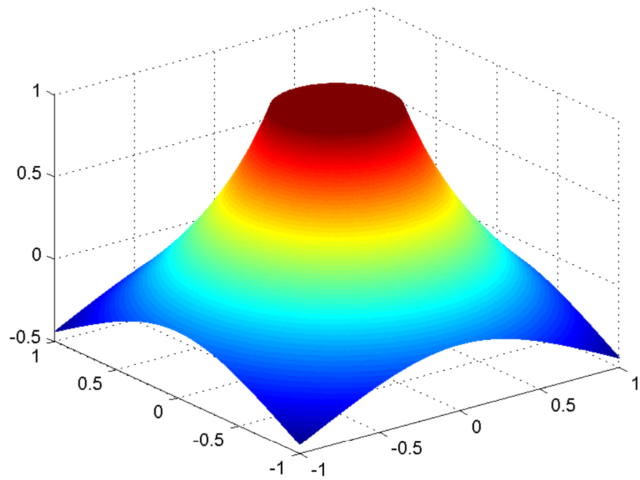

FIGURE 2. Plot of the approximate solution, example 1, by our method (EBC-FEI) on a non-uniform grid.

These examples are taken from [21].

1. Consider an exact solution of problem (1):

$$u(x) \quad = \quad \begin{cases} 1, & \text{if } r \leq R \\ 1 - \log(\frac{r}{R}), & \text{if } r > R \end{cases} \qquad x \in [-1,1]^2,$$

where $r = \|x\|_2$ and $R = 1/3$. Then, the data is given by $f^{\pm} = 0$, $\alpha = 0$ and $\beta = \frac{1}{R}$. We summarize the errors and order of convergence in the following tables.

TABLE 1. L^2 and L^∞ errors of the approximate solution, example 1, of our method (EBC-FEI) on a non-uniform grid.

h	$\|e_h\|_{L^2}$	r	$\|\nabla e_h\|_{L^2}$	r	$\|e_h\|_{L^\infty}$	r	$\|\nabla e_h\|_{L^\infty}$	r
1.8e-1	1.39e-1		4.44e-1		2.53e-1		5.20e-1	
8.8e-2	3.09e-2	2.17	1.72e-1	1.37	6.40e-2	1.98	3.84e-1	0.44
4.4e-2	7.32e-3	2.08	5.75e-2	1.58	1.58e-2	2.02	1.79e-1	1.10
2.2e-2	1.81e-3	2.02	2.18e-2	1.40	4.19e-3	1.91	1.20e-1	0.58
1.1e-2	4.50e-4	2.01	8.57e-3	1.35	8.92e-4	2.23	6.45e-2	0.89
5.5e-3	1.12e-4	2.01	3.57e-3	1.26	2.37e-4	1.91	3.17e-2	1.02
2.8e-3	2.68e-5	2.06	1.55e-3	1.21	6.23e-5	1.93	1.71e-2	0.90
1.4e-3	6.89e-6	1.96	7.68e-4	1.01	1.68e-5	1.90	8.33e-3	1.03

The results presented in Table 1 confirm the estimates obtained in Theorem 2 and Corollary 1. In the same way, Table 2 exemplifies the estimate obtained in Theorem 6.

TABLE 2. L^2 and L^∞ errors of the approximate solution, example 1, of the natural method on a non-uniform grid.

h	$\|e_h^N\|_{L^2}$	r	$\|\nabla e_h^N\|_{L^2}$	r	$\|e_h^N\|_{L^\infty}$	r	$\|\nabla e_h^N\|_{L^\infty}$	r
1.8e-1	1.02e-1		4.71e-1		1.63e-1		7.01e-1	
8.8e-2	1.57e-2	2.70	1.38e-1	1.78	4.09e-2	2.00	3.26e-1	1.10
4.4e-2	6.72e-3	1.22	1.30e-1	0.09	2.85e-2	0.52	5.48e-1	-0.75
2.2e-2	2.02e-3	1.74	7.88e-2	0.72	1.07e-2	1.42	5.87e-1	-0.10
1.1e-2	7.65e-4	1.40	6.16e-2	0.36	7.24e-3	0.56	6.24e-1	-0.09
5.5e-3	2.71e-4	1.50	4.27e-2	0.53	4.39e-3	0.72	6.24e-1	0.00
2.8e-3	9.09e-5	1.58	2.83e-2	0.59	2.04e-3	1.11	7.80e-1	-0.32
1.4e-3	3.53e-5	1.36	2.24e-2	0.34	1.38e-3	0.57	8.78e-1	-0.17

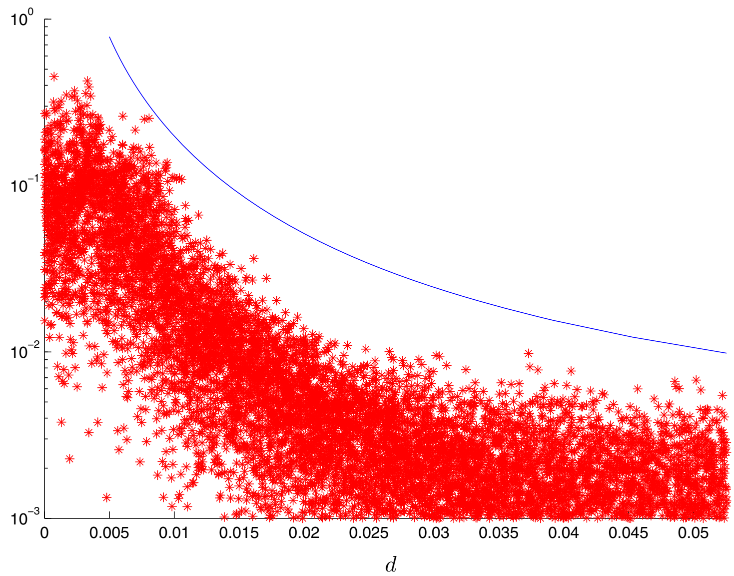

FIGURE 3. Semi-log plot of gradient error for the natural method with $h = .0028$. $|\nabla e_h^N(d_T)|$ (red) for every triangle T and curve $2h + \log(1/h)(h/d)^2$ (blue). The distance d in the x-axis varies from 0 to $\sqrt{h}$.

It is difficult to check the sharpness of Theorem 7. In an attempt to do this, we plot for each triangle T the error $|\nabla e_h^N(d_T)|$ where d_T is the distance between its centroid and the interface Γ. We compare this to the graph of the bound of the error given by Theorem 7, namely, $C(h + \frac{h^2\log(1/h)}{d^2})$; see Figure 3. We observe that the curve roughly describes the behavior of the error when the distance d is less than $\sqrt{h}$.

TABLE 3. L^2 and L^∞ errors of the approximate solution, example 2, of our method (EBC-FEI) on a non-uniform grid.

h	$\|e_h\|_{L^2}$	r	$\|\nabla e_h\|_{L^2}$	r	$\|e_h\|_{L^\infty}$	r	$\|\nabla e_h\|_{L^\infty}$	r
1.8e-1	9.28e-3		3.27e-2		1.42e-2		4.23e-2	
8.8e-2	5.41e-3	0.78	3.50e-2	-0.10	8.23e-3	0.79	6.61e-2	-0.64
4.4e-2	1.19e-3	2.18	1.18e-2	1.56	2.19e-3	1.91	3.18e-2	1.06
2.2e-2	2.89e-4	2.05	5.06e-3	1.23	7.41e-4	1.56	2.25e-2	0.50
1.1e-2	7.51e-5	1.94	2.42e-3	1.06	1.64e-4	2.17	1.15e-2	0.97
5.5e-3	1.89e-5	1.99	1.18e-3	1.04	4.45e-5	1.88	5.57e-3	1.04
2.8e-3	4.71e-6	2.00	5.74e-4	1.03	1.20e-5	1.89	2.68e-3	1.06
1.4e-3	1.18e-6	2.00	2.86e-4	1.01	3.03e-6	1.98	1.35e-3	0.98

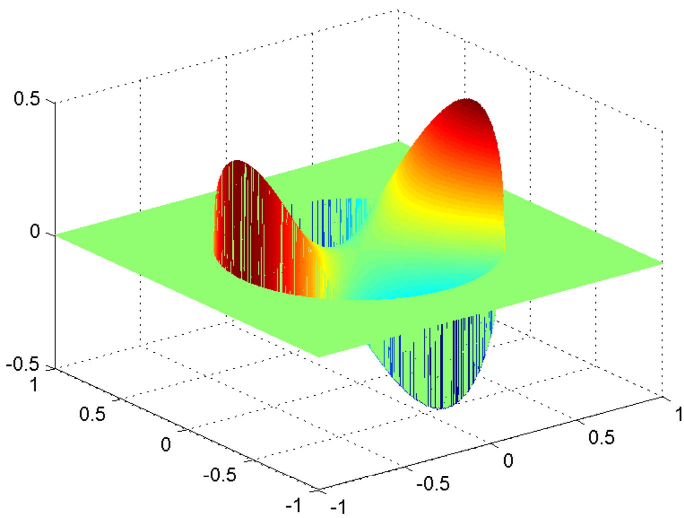

FIGURE 4. Plot of the approximate solution, example 2, by our method (EBC-FEI) on a non-uniform grid.

2. Consider the exact solution

$$u(x_1, x_2) = \begin{cases} x_1^2 - x_2^2, & \text{if } r \leq R, \\ 0, & \text{if } r > R. \end{cases}$$

Therefore, the data is given by $f^{\pm} = 0$, $\alpha(\theta) = -R^2(\cos^2(\theta/R) - \sin^2(\theta/R))$ and $\beta(\theta) = 2R(\cos^2(\theta/R) - 2R\sin^2(\theta/R))$, for $\theta \in [0, 2\pi R]$, and $R = 2/3$. In Table 3 we list the rates of convergence.

6. FUTURE WORK

As one can imagine several extensions are possible. In a future work, we first plan to extend our method and analyze it for fluid flow problems. Three-dimensional problems will also be considered. Finally, in the future we will consider discontinuous diffusion coefficients.

Appendix A. Proof of estimate (16)

Here we shall prove that there exists a constant $C > 0$, independent of h, such that

$$\|\Pi(\nabla g) - \nabla g\|_{L^1(\Omega)} \quad \leq \quad C, \tag{37}$$

where Π is the lowest order Raviart-Thomas interpolant.

We proceed by a dyadic decomposition argument (see [15]). We assume without loss of generality that $|\Omega| \leq 1$. Define $d_j = 2^{-j}$ and let J be the integer such that $2^{-(J+1)} \leq Kh \leq 2^{-J}$, where K is a fixed constant that is large enough. Then, consider the decomposition of Ω,

$$\Omega \;=\; \Omega^* \cup \bigcup_{j=0}^{J} \Omega_j, \tag{38}$$

where $\Omega^* = \{x \in \Omega : |x - z| \leq Kh\}$, $\Omega_j = \{\boldsymbol{x} \in \Omega : d_{j+1} \leq |x - z| \leq d_j\}$. Henceforth, we will denote by C the generic constants not depending on K or h.

We break (37) using the dyadic decomposition (38):

$$\|\Pi(\nabla g) - \nabla g\|_{L^1(\Omega)} = \|\Pi(\nabla g) - \nabla g\|_{L^1(\Omega^*)} + \sum_{j=0}^{J} \|\Pi(\nabla g) - \nabla g\|_{L^1(\Omega_j)}.$$

First, we estimate the term involving the set Ω^*:

$$\begin{aligned}
\|\Pi(\nabla g) - \nabla g\|_{L^1(\Omega^*)} &\leq Kh\|\Pi(\nabla g) - \nabla g\|_{L^2(\Omega^*)} \\
&\leq Kh^2\|\nabla g\|_{H^1(\Omega)} \\
&\leq Kh^2\|\partial_{x_i}\delta_h\|_{L^2(\Omega)} \\
&\leq CK.
\end{aligned}$$

In the inequality we used estimate (12) with $q = 2$ and $k = 0$. For the second term we have

$$\begin{aligned}
\sum_{j=0}^{J} \|\Pi(\nabla g) - \nabla g\|_{L^1(\Omega_j)} &= C\sum_{j=0}^{J} d_j^2\|\Pi(\nabla g) - \nabla g\|_{L^\infty(\Omega_j)} \\
&\leq C\sum_{j=0}^{J} d_j^2 h^\sigma\|\nabla g\|_{C^\sigma(\Omega_j')},
\end{aligned}$$

where $\Omega_j' = \{x \in \Omega : d_{j+2} \leq |x - z| \leq d_{j-1}\}$. The bound for $\|\nabla g\|_{C^\sigma(\Omega_j')}$ is proved, for example, in [15] in the three-dimensional case. In the two-dimensional case one will have the bound

$$\|\nabla g\|_{C^\sigma(\Omega_j')} \;\leq\; Cd_j^{-2-\sigma}. \tag{39}$$

Then,

$$\sum_{j=0}^{J} \|\Pi(\nabla g) - \nabla g\|_{L^1(\Omega_j)} \;\leq\; C\sum_{j=0}^{J} d_j^2 h^\sigma d_j^{-2-\sigma} \;\leq\; Ch^\sigma d_J^\sigma \;\leq\; CK^{-\sigma}.$$

This completes the proof. $\square$

Appendix B. Proof of estimate $\|\nabla g\|_{L^1(\Omega)}$

We proceed by dyadic decomposition as in the previous proof:

$$\|\nabla g\|_{L^1(\Omega)} \;=\; \|\nabla g\|_{L^1(\Omega^*)} \,+\, \sum_{j=0}^{J} \|\nabla g\|_{L^1(\Omega_j)}.$$

We estimate the first term:

$$\|\nabla g\|_{L^1(\Omega^*)} \;\le\; Kh\|\nabla g\|_{L^2(\Omega^*)} \;\le\; Kh^2\|g\|_{H^2(\Omega)} \;\le\; Kh^2\|\partial_{x_i}\delta_h\|_{L^2(\Omega)} \;\le\; K.$$

For the remaining term we have, using estimate (39),

$$\|\nabla g\|_{L^1(\Omega_j)} \;\le\; d_j^2\|\nabla g\|_{L^\infty} \;\le\; d_j^2 C d_j^{-2} \;=\; C.$$

Since we choose J such that $2^{-(J+1)} \;\le\; Kh \;\le\; 2^{-J}$ implies that $J \approx \log(1/h)$, then

$$\|\nabla g\|_{L^1(\Omega)} \;\le\; C\log(1/h).$$

□

Appendix C. Proof of estimate (36)

We will prove the following estimate:

$$\|g - g_h\|_{L^1(\Omega)} \;\le\; Chh^{-(2-2/q)}.$$

Let P_h be the Scott-Zhang interpolant. Then for $1 < q < 2$ we have

$$\|g - g_h\|_{L^1(\Omega)} \;\le\; \|g - g_h\|_{L^q(\Omega)} \;\le\; \|g - P_h g\|_{L^q(\Omega)} + \|P_h g - g_h\|_{L^q(\Omega)}.$$

Consider the dual problem

$$\begin{aligned} -\Delta\psi = \phi &\quad \text{in } \Omega,\\ \phi = 0 &\quad \text{on } \partial\Omega, \end{aligned}$$

with the regularity result for $p > 2$,

$$\|\psi\|_{W^{2,p}(\Omega)} \;\le\; Cp\|\phi\|_{L^p(\Omega)}.$$

Using this, we have

$$\begin{aligned} \|P_h g - g_h\|_{L^q(\Omega)} &= \sup_{\substack{\phi\in C_c^\infty(\Omega)\\ \|\phi\|_{L^p(\Omega)}\le 1}} (P_h g - g_h, \phi)\\ &\le (P_h g - g_h, -\Delta\psi)\\ &= (\nabla(P_h g - g_h), \nabla\psi)\\ &= (\nabla(P_h g - g_h), \nabla(\psi - P_h\psi)) + (\nabla(P_h g - g_h), \nabla P_h\psi)\\ &= (\nabla(P_h g - g_h), \nabla(\psi - P_h\psi)) + (\nabla(P_h g - g), \nabla P_h\psi). \end{aligned}$$

For the first term we have, with $1/p = 1 - 1/q$, and applying the inverse estimate:

$$\begin{aligned} (\nabla(P_h g - g_h), \nabla(\psi - P_h\psi)) &\le \|\nabla(P_h g - g_h)\|_{L^q(\Omega)}\|\nabla(\psi - P_h\psi)\|_{L^p(\Omega)}\\ &\le Chp\|\nabla(P_h g - g_h)\|_{L^q(\Omega)}\|\phi\|_{L^p(\Omega)}\\ &\le Chph^{-(2-2/q)}\|\nabla(P_h g - g_h)\|_{L^1(\Omega)}, \end{aligned}$$

and for the second term we have

$$\begin{aligned} (\nabla(P_h g - g), \nabla P_h\psi) &= (\nabla(P_h g - g), \nabla(P_h\psi - \psi)) + (\nabla(P_h g - g), \nabla\psi)\\ &= (\nabla(P_h g - g), \nabla(P_h\psi - \psi)) + (P_h g - g, \phi). \end{aligned}$$

We estimate them by

$$\begin{aligned}(\nabla(P_h g - g), \nabla(P_h\psi - \psi)) &\le \|\nabla(P_h g - g)\|_{L^q(\Omega)}\|\nabla(P_h\psi - \psi)\|_{L^p(\Omega)}\\ &\le Ch\|\nabla(P_h g - g)\|_{L^q(\Omega)}\|\psi\|_{W^{2,p}(\Omega)}\\ &\le Chp\|\nabla(P_h g - g)\|_{L^q(\Omega)},\\ (P_h g - g, \phi) &\le \|P_h g - g\|_{L^q(\Omega)}\|\phi\|_{L^p(\Omega)} \le \|P_h g - g\|_{L^q(\Omega)}.\end{aligned}$$

Assuming the following inequalities, for $1 \le q < 2$,

$$\|P_h g - g\|_{L^q(\Omega)} \le Chh^{-(2-2/q)}, \tag{40}$$

$$\|\nabla(P_h g - g)\|_{L^q(\Omega)} \le Ch^{-(2-2/q)}, \tag{41}$$

we have

$$\|P_h g - g_h\|_{L^q(\Omega)} \le Chph^{-(2-2/q)} + Chph^{-(2-2/q)} + Chh^{-(2-2/q)},$$

and therefore

$$\|g - g_h\|_{L^q(\Omega)} \le Chph^{-(2-2/q)} + Chh^{-(2-2/q)}.$$

Proof of (40). We proceed by a dyadic decomposition argument as before. We break (40) using the dyadic decomposition (38)

$$\|P_h g - g\|_{L^q(\Omega)} = \|P_h g - g\|_{L^q(\Omega^*)} + \sum_{j=0}^{J}\|P_h g - g\|_{L^q(\Omega_j)}.$$

First, we estimate the term involving the set Ω^*:

$$\begin{aligned}\|P_h g - g\|_{L^q(\Omega^*)} &\le C(Kh)^{2(1/q-1/2)}\|P_h g - g\|_{L^2(\Omega^*)}\\ &\le C(Kh)^{2(1/q-1/2)}h^2\|\partial_{x_i}\delta_h\|_{L^2(\Omega)}\\ &\le C(Kh)^{2(1/q-1/2)}\\ &= Ch(K)^{2(1/q-1/2)}h^{-(2-2/q)}.\end{aligned}$$

In the inequality we used $\|\partial_{x_i}\delta_h\|_{L^2(\Omega)} \le Ch^{-2}$. For the second term we have

$$\begin{aligned}\sum_{j=0}^{J}\|P_h g - g\|_{L^q(\Omega_j)} &= C\sum_{j=0}^{J} d_j^{2/q}\|P_h g - g\|_{L^\infty(\Omega_j)}\\ &\le C\sum_{j=0}^{J} d_j^{2/q}h^{1+\sigma}\|g\|_{C^{1+\sigma}(\Omega_j')},\end{aligned}$$

where $\Omega_j' = \{x \in \Omega : d_{j+2} \le |x - z| \le d_{j-1}\}$. The bound for $\|\nabla g\|_{C^{1+\sigma}(\Omega_j')}$ is proved, for example, in [15] in the three-dimensional case. In the two-dimensional case one will have the bound

$$\|\nabla g\|_{C^\sigma(\Omega_j')} \le Cd_j^{-2-\sigma}.$$

Then,

$$\begin{aligned}\sum_{j=0}^{J}\|P_h g - g\|_{L^q(\Omega_j)} &\le C\sum_{j=0}^{J} d_j^{2/q}h^{1+\sigma}d_j^{-2-\sigma} \le Ch^{1+\sigma}Cd_J^{\sigma}d_J^{2/q-2}\\ &\le ChK^{-\sigma}(Kh^{-(2-2/q)}).\end{aligned}$$

This completes the proof. ∎

Proof of (41) follows by the same arguments.

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