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ON THE CRANK-NICOLSON ANISOTROPIC *A POSTERIORI* ERROR ANALYSIS FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

G. MURALI MOHAN REDDY AND RAJEN K. SINHA

ABSTRACT. In this exposition, we derive two anisotropic error estimators for parabolic integro-differential equations in a two-dimensional convex polygonal domain. A continuous, piecewise linear finite element space is employed for the space discretization and the time discretization is based on the Crank-Nicolson method. The *a posteriori* contributions corresponding to space discretization is derived using the anisotropic interpolation estimates together with the Zienkiewicz-Zhu error estimator to approximate the error gradient. Two different continuous, piecewise quadratic reconstructions are used to obtain the error due to time discretization. Moreover, linear approximations of the Volterra integral term are used in a crucial way to estimate the quadrature error in the approximation of the Volterra integral term.

1. Introduction

The aim of this article is to derive anisotropic *a posteriori* error bounds for the Crank-Nicolson finite element method for linear parabolic integro-differential equations (PIDE) of the form

$$(1)\qquad \begin{aligned} u_t(x,t) + \mathcal{A}u(x,t) &= \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T], \\ u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) &= u_0(x), \quad x \in \Omega. \end{aligned}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$ and $u_t(x,t) = \frac{\partial u}{\partial t}(x,t)$ with $T < \infty$. Further, $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator of the form

$$\mathcal{A}u = -\nabla \cdot (A\nabla u)$$

and the operator $\mathcal{B}(t,s)$ is of the form

$$\mathcal{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u),$$

where "∇" denotes the spatial gradient and the coefficient matrices A and $B(t,s)$, $0 \le s \le t$ are assumed to be smooth. Moreover, the initial function $u_0(x)$ and the forcing term $f(x,t)$ are assumed to be smooth for our purpose.

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The second author is the corresponding author.

Equations of type (1) arise in many applications such as heat conduction in material with memory [7], the compression of poro-viscoelasticity media [8], nuclear reactor dynamics [9] and the epidemic phenomena in biology [4]. Concerning the existence and uniqueness results of (1), we refer to [21] and the references therein.

A priori error estimates [19, 22] can give asymptotic rates of convergence as the mesh parameters goes to zero, but are not designed to give an actual error estimate for a given mesh. The question of quantifying the error brings attention to *a posteriori* error estimation technique. *A posteriori* error estimation [17, 18] is the basis for efficient adaptive meshing procedures designed to control and minimize the error. Most of adaptive algorithms are based on isotropic *a posteriori* error estimates; see [1, 3, 20] for elliptic and parabolic problems. In an isotropic finite element method the aspect ratio (ratio of the diameters of the circumscribed and inscribed circles of a finite element) is bounded by a constant. But, the recent literature survey reveals that this restriction on mesh can be relaxed and one can achieve a given level of accuracy with fewer vertices using anisotropic mesh [5, 6, 10–12, 15]. Anisotropic mesh reduces the number of degrees of freedom and computational effort leading to the reduction of memory to achieve the same convergence as compared to the isotropic mesh.

Both *a priori* and *a posteriori* error estimators for elliptic and parabolic problems on anisotropic meshes have been proposed and analyzed in [5, 6, 10–12, 15] during the last decade. In the absence of the Volterra integral term (when $\mathcal{B}(t, s) = 0$), *a posteriori* error analysis for linear parabolic problems have been investigated by several authors in recent years; see [3, 20] for isotropic error analysis and [10–12, 15] for the anisotropic error analysis. In particular, *a posteriori* error estimates concerning the Crank-Nicolson method for the parabolic problems were thoroughly studied in [2, 3, 10, 15, 20]. For a continuous, piecewise linear approximation in time, the author of [20] has derived suboptimal (with respect to time steps) error bounds for the heat equation using the standard energy techniques. Subsequently, a continuous, piecewise quadratic polynomial function in time, so-called Crank-Nicolson reconstruction, was then introduced in [2] to restore the appropriate second order of convergence for the semi-discrete scheme in time of a general parabolic problem. The result of [2] was then extended to the fully discrete Crank-Nicolson method in [10] by introducing the quadratic reconstructions based on approximations on one time level (two-point reconstruction) and two time levels (three-point reconstruction).

Since PIDE may be thought of as a perturbation of the parabolic problem it is, therefore, natural to see how the anisotropic error analysis for the parabolic problem [10] can be carried over to the present situation. Due to the presence of the Volterra integral term in (1) such an extension is not straightforward. To the best of our knowledge no article is available in the literature concerning anisotropic *a posteriori* error analysis for PIDE.

We derive two residual based *a posteriori* error estimates for PIDE in the $L^2(H^1)$-norm for the implicit Crank-Nicolson fully discrete scheme in an anisotropic framework. The error due to space discretization is obtained by using the anisotropic interpolation error estimates [5, 6] and the Z-Z estimator [23, 24]. The error due to time discretization is derived by introducing a continuous, piecewise quadratic polynomial function so-called Crank-Nicolson memory reconstruction in time which is a direct transposition of the two point reconstruction (cf. [10]). A linear approximation of the Volterra integral term is used in a crucial way to estimate the quadrature error for the approximation of the memory term. However, due to the presence of the memory term this reconstruction depends on all the previous time levels and, therefore, it is not locally defined in time. Thus, we define a local time reconstruction (based on two subintervals) by considering an analogue of the Crank-Nicolson memory reconstruction, so-called three-point reconstruction, based on a finite difference approximation of u_{tt} rather than an approximation of $f_t - Au_t + \frac{\partial}{\partial t}\{\int_0^t \mathcal{B}(t,s)u(x,s)ds\}$. Further, an extended linear approximation of the Volterra integral term is used to estimate the quadrature error while analyzing it with the later reconstruction. It is noteworthy that one can recover optimal order isotropic error estimators through our anisotropic error estimators which is a new trending topic of investigation for PIDE by the finite element method in recent years [16].

We organize the paper as follows. In Section 2, we introduce some standard notation and preliminary materials to be used in the subsequent sections. Section 3 is devoted to quadratic reconstructions for PIDE. Further, *a posteriori* error estimates in the $L^2(H^1)$-norm are derived for the Crank-Nicolson method in Section 4.

2. Preliminaries

In this section, we introduce some standard notation, function spaces, and recall some basic results to be used in our analysis.

2.1. Notation and abstract formulation.

Given a Lebesgue measurable set $\omega \subset \mathbb{R}^2$, we denote by $L^p(\omega), 1 \le p \le \infty$, the Lebesgue spaces with corresponding norms $\|\cdot\|_{L^p(\omega)}$. When $p = 2$, the space $L^2(\omega)$ is equipped with inner product $\langle\cdot,\cdot\rangle_\omega$ and the induced norm $\|\cdot\|_{L^2(\omega)}$. Whenever $\omega = \Omega$, we omit the subscripts of $\|.\|_{L^2(\omega)}$ and $\langle\cdot,\cdot\rangle_\omega$. Further, we shall use the standard notation for Sobolev spaces $W^{m,p}(\omega)$ with $1 \le p \le \infty$. The norm on $W^{m,p}(\omega)$ is defined by

$$\|v\|_{m,p,\omega} = \left(\int_\omega \sum_{|\alpha|\le m} |D^\alpha v|^p dx\right)^{1/p}, \ 1 \le p < \infty$$

with the standard modification for $p = \infty$. When $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$ and denote the associated norm by $\|\cdot\|_m$. The space $H_0^1(\Omega)$ denotes the space of functions in $H^1(\Omega)$ that vanish on the boundary of Ω (boundary values are taken in the sense of traces).

Let $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the bilinear form corresponding to the elliptic operator $\mathcal{A}$ defined by

$$a(\phi,\psi) := \langle A\nabla\phi, \nabla\psi\rangle, \quad \forall\, \phi,\psi \in H_0^1(\Omega).$$

Similarly, let $b(t,s;\cdot,\cdot)$ be the bilinear form corresponding to the operator $\mathcal{B}(t,s)$ defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$b(t,s;\phi,\psi) \ := \ \langle B(t,s)\nabla\phi, \nabla\psi\rangle, \quad \forall\, \phi,\psi \in H_0^1(\Omega).$$

Let $b_s(t,s;\cdot,\cdot)$ and $b_{ss}(t,s;\cdot,\cdot)$ be the bilinear forms obtained by differentiating the coefficients of $b(t,s;\cdot,\cdot)$ with respect to s once and twice, respectively. Further, the bilinear form $b_t(t,s;\cdot,\cdot)$ is obtained by differentiating $b(t,s;\cdot,\cdot)$ with respect to t.

We assume that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on $H_0^1(\Omega)$ i.e.,

$$a(\phi,\phi) \geq \alpha\|\nabla\phi\|^2 \quad \text{and} \quad |a(\phi,\psi)| \leq \beta\|\nabla\phi\|\|\nabla\psi\|, \quad \forall\, \phi,\psi \in H_0^1(\Omega) \tag{2}$$

with $\alpha,\beta \in \mathbb{R}^+$. Here, $\|\nabla(\cdot)\|$ defines a norm on $H_0^1(\Omega)$ in view of the Poincaré inequality. Further, we assume that the bilinear forms $b(t,s;\cdot,\cdot)$, $b_s(t,s;\cdot,\cdot)$, $b_{ss}(t,s;\cdot,\cdot)$ and $b_t(t,s;\cdot,\cdot)$ are continuous on $H_0^1(\Omega)$, i.e.,

$$|b(t,s;\phi(s),\psi)| \leq \gamma\|\nabla\phi(s)\|\|\nabla\psi\|, \quad \forall\, \phi(s),\psi \in H_0^1(\Omega), \tag{3}$$

$$|b_s(t,s;\phi(s),\psi)| \leq \gamma'\|\nabla\phi(s)\|\|\nabla\psi\|, \quad \forall\, \phi(s),\psi \in H_0^1(\Omega), \tag{4}$$

$$|b_{ss}(t,s;\phi(s),\psi)| \leq \gamma''\|\nabla\phi(s)\|\|\nabla\psi\|, \quad \forall\, \phi(s),\psi \in H_0^1(\Omega), \tag{5}$$

and

$$|b_t(t,s;\phi(s),\psi)| \leq \gamma'''\|\nabla\phi(s)\|\|\nabla\psi\|, \quad \forall\, \phi(s),\psi \in H_0^1(\Omega) \tag{6}$$

with $\gamma,\gamma',\gamma'',\gamma''' \in \mathbb{R}^+$.

The weak formulation of problem (1) may be stated as follows: Find $u:(0,T] \to H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega u_t\phi dx + a(u,\phi) = &\int_0^t b(t,s;u(s),\phi)ds \\ &+ \int_\Omega f\phi dx, \quad \forall\phi \in H_0^1(\Omega),\ t \in (0,T], \\ u(.,0) = u_0&. \end{aligned} \tag{7}$$

2.2. **Interpolation error for anisotropic finite elements.** Let $\mathcal{T}_h$ $(0 < h < 1)$ denote a conforming triangulation of $\bar{\Omega}$ into triangles K (not necessarily satisfying the minimum angle condition) with diameter $h_K \leq h$. We define by V_h the usual finite element space of continuous, piecewise linear functions on $\mathcal{T}_h$:

$$V_h = \left\{v_h \in C(\bar{\Omega}) : v_h|_K \in \mathbb{P}_1(K), \quad \forall K \in \mathcal{T}_h\right\},$$

where $\mathbb{P}_1$ is the space of polynomials of degree ≤ 1. Now, we set

$$V_h^0 = V_h \cap H_0^1(\Omega).$$

We recall some basic definitions and approximation properties concerning anisotropic interpolation estimates [5,6]. Let $T_K : \hat{K} \to K$ denote the standard invertible affine map which maps the reference triangle $\hat{K}$ into the general element K of the triangulation $\mathcal{T}_h$. Let $P_K \in \mathbb{R}^{2\times 2}$ denote the affine transformation matrix corresponding to T_K, i.e.,

$$x = T_K(\hat{x}) = P_K\hat{x} + t_K, \quad \forall\hat{x} \in \mathbb{R}^2,$$

where $t_K \in \mathbb{R}^2$ is a vector. Here, we study the spectral properties of the map T_K since it will be helpful in obtaining the anisotropic information about the size and orientation of the mesh element K. Now, invertibility of P_K ensures that it has singular value decomposition

$$P_K = R_K^T\Lambda_K S_K,$$

where R_K and S_K are both orthogonal and Λ_K is diagonal with positive entries. Let us denote

$$\Lambda_K = \begin{pmatrix} \lambda_{1,K} & 0 \\ 0 & \lambda_{2,K} \end{pmatrix} \quad \text{and} \quad R_K = \begin{pmatrix} r_{1,K}^T \\ r_{2,K}^T \end{pmatrix}$$

with the choice $\lambda_{1,K} \geq \lambda_{2,K}$. Thus, the deformation of any $K \in \mathcal{T}_h$ with respect to $\hat{K}$ can be measured in terms of the stretching factor $\lambda_{1,K}/\lambda_{2,K} (\geq 1)$. For examples of such types of transformation, we refer to [12].

Let $\Pi_h : H^1(\Omega) \to V_h^0$ be the standard *Clément* interpolation operator. There are two restrictions made on the patch Δ_K involved in the definition of the operator Π_h, where Δ_K denotes the set of triangles having a common vertex with K (cf. [12]). For each vertex, the number of neighboring vertices should be bounded from above uniformly with respect to the mesh size h. The second restriction is that the diameter of the reference patch $T_K^{-1}(\Delta_K)$ should be bounded above uniformly with respect to h. The later restriction on the mesh prevents the stretching directions $r_{1,K}$ and $r_{2,K}$ from changing too abruptly between the adjacent triangles of the mesh though the classical minimum angle condition is not required in this context.

Proposition 2.1. *There is a constant C independent of the mesh size and aspect ratio such that, for any $v \in H^1(\Omega)$ and any $K \in \mathcal{T}_h$, we have*

$$\|v - \Pi_h v\|_{L^2(K)} + \lambda_{2,K} \|\nabla(v - \Pi_h v)\|_{L^2(K)} + \lambda_{2,K}^{1/2} \|v - \Pi_h v\|_{L^2(\partial K)} \leq C\, \omega_K(v). \tag{8}$$

Here, $\omega_K(v)$ is defined by

$$\omega_K^2(v) = \lambda_{1,K}^2 (r_{1,K}^T G_K(v) r_{1,K}) + \lambda_{2,K}^2 (r_{2,K}^T G_K(v) r_{2,K}),$$

where $\lambda_{i,K}$ and $r_{i,K}$ are as defined above and $G_K(v)$ is the following 2×2 matrix

$$G_K(v) = \sum_{K \in \Delta_K} \begin{pmatrix} \displaystyle\int_K \left(\frac{\partial v}{\partial x_1}\right)^2 dx & \displaystyle\int_K \frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2} dx \\ \displaystyle\int_K \frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2} dx & \displaystyle\int_K \left(\frac{\partial v}{\partial x_2}\right)^2 dx \end{pmatrix}.$$

3. Quadratic reconstructions for PIDE

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0,T]$ with $I_n := [t_{n-1}, t_n]$ and $\tau_n := t_n - t_{n-1}$. Set $f^n(\cdot) = f(\cdot, t_n)$ for $t = t_n$, $n \in [0:N]$. Now, for $n = 1, 2, \ldots, N$, we define

$$\bar{\partial} v^n := \frac{v^n - v^{n-1}}{\tau_n}, \quad t_{n-1/2} = \frac{t_n + t_{n-1}}{2} \quad \text{and} \quad v^{n-1/2} := \frac{v^n + v^{n-1}}{2}.$$

We now state the Crank-Nicolson scheme as follows: Given U_h^0, where $U_h^0 = I_h u_0$, find $U_h^n \in V_h^0$, $n \in [1:N]$ such that

$$\begin{aligned} \int_\Omega \bar{\partial} U_h^n \phi_h dx + a(U_h^{n-1/2}, \phi_h) &= \sigma^n(b(t_{n-1/2}; U_h, \phi_h)) \\ &\quad + \int_\Omega f^{n-1/2} \phi_h dx, \quad \forall \phi_h \in V_h^0, \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\sigma^n(b(t_{n-1/2}; U_h, \phi_h)) \\ &\quad := \left\langle \sum_{j=0}^{n-2} \frac{\tau_{j+1}}{2} \Big[B(t_{n-1/2}, t_j) \nabla U_h^j + B(t_{n-1/2}, t_{j+1}) \nabla U_h^{j+1} \Big], \nabla \phi_h \right\rangle \\ &\qquad + \left\langle \frac{\tau_n}{4} \Big[B(t_{n-1/2}, t_{n-1}) \nabla U_h^{n-1} + B(t_{n-1/2}, t_{n-1/2}) \nabla U_h^{n-1/2} \Big], \nabla \phi_h \right\rangle. \end{aligned}$$

Here, the trapezoidal rule is used to discretize the integral term in order to be consistent with the Crank-Nicolson scheme and I_h denotes Lagrange's interpolant corresponding to V_h^0.

For $t \in I_n$, let U_h be a continuous function in time defined by linearly interpolating the nodal values U_h^n and U_h^{n-1}, i.e.,

$$\begin{aligned} U_h(t) &:= l_n(t)U_h^n + l_{n-1}(t)U_h^{n-1} \\ &:= U_h^{n-1/2} + (t - t_{n-1/2})\bar{\partial}U_h^n, \quad 1 \le n \le N. \end{aligned} \tag{10}$$

where

$$l_n(t) := \frac{(t - t_{n-1})}{\tau_n}, \qquad l_{n-1}(t) := \frac{(t_n - t)}{\tau_n}. \tag{11}$$

We shall now introduce quadratic time reconstructions which play an instrumental role in deriving *a posteriori* error bounds with second-order accuracy for the Crank-Nicolson scheme.

Crank-Nicolson memory reconstruction. Let $\check{U}_h$ be a continuous, piecewise quadratic memory dependent reconstruction of U_h in time defined for all $t \in I_n$, $1 \le n \le N$ by

$$\begin{aligned} \check{U}_h(t) &:= U_h(t) + \frac{1}{2}(t - t_{n-1})(t - t_n)\check{w}_h^n, \\ &:= U_h^{n-1/2} + (t - t_{n-1/2})\bar{\partial}U_h^n + \frac{1}{2}(t - t_{n-1})(t - t_n)\check{w}_h^n, \end{aligned} \tag{12}$$

and $\check{w}_h^n \in V_h^0$ is defined by

$$\begin{aligned} \langle \check{w}_h^n, \phi_h \rangle &:= \langle \bar{\partial} f^n, \phi_h \rangle - a(\bar{\partial}U_h^n, \phi_h) + \int_0^{t_n} \bar{\partial} b(t_n, s; U_h(s), \phi_h) ds \\ &\quad + b(t_n, t_n; U_h^n, \phi_h). \end{aligned} \tag{13}$$

Note that the quadratic time approximation $\check{U}_h$ is a generalization to the two point reconstruction introduced by the authors of [10]. For $t \in I_n$, $1 \le n \le N$, we note that

$$\frac{\partial \check{U}_h(t)}{\partial t} = \bar{\partial}U_h^n + (t - t_{n-1/2})\check{w}_h^n \tag{14}$$

and

$$\frac{\partial^2 \check{U}_h(t)}{\partial t^2} = \check{w}_h^n. \tag{15}$$

The following definition will be useful in estimating the quadrature error in approximating the Volterra integral term. Set

$$\check{\phi}(t) := \int_0^t B(t, s)\nabla \check{U}_h(s) ds. \tag{16}$$

For $t \in I_n$, $1 \leq n \leq N$, we define $\phi_1(t)$ to be the linear interpolant associated with the integral vector $\check{\phi}(t_{n-1/2})$ given by

$$\phi_1(t) \quad := \quad \check{\phi}(t_{n-1/2}) + (t - t_{n-1/2})\frac{d}{dt}\check{\phi}(t)\big|_{t=t_n}. \tag{17}$$

In order to handle the time reconstruction error, we introduce the following definition. Set

$$\check{\psi}(t) := \int_0^t B(t,s)(\nabla U_h(s) - \nabla \check{U}_h(s))ds. \tag{18}$$

For $t \in I_n$, we define $\psi_1(t)$ to be the linear interpolant associated with the integral vectors $\check{\psi}(t_n)$ and $\check{\psi}(t_{n-1})$ given by

$$\psi_1(t) := l_n(t)\check{\psi}(t_n) + l_{n-1}(t)\check{\psi}(t_{n-1}), \;\; 1 \leq n \leq N, \tag{19}$$

where $l_n(t)$ and $l_{n-1}(t)$ are given by (11).

In the quadratic reconstruction (12), the integral term $\int_0^{t_n} \bar{\partial} B(t_n, s)\nabla U_h(s)ds$ appears due to the application of the Leibniz formula on the memory term. Further, we notice that, by applying any basic quadrature approximation to the memory term which is consistent with the Crank-Nicolson scheme, the quadratic reconstruction needs to be evaluated at all the previous time levels and thus, it is not locally defined in time. Therefore, in addition to the above memory reconstruction, we define a local reconstruction by considering an analogue of it.

Three-point reconstruction. Let $\hat{U}_h$ be a three-point continuous, piecewise quadratic reconstruction of U_h in time defined for all $t \in I_n$, $1 \leq n \leq N$ by

$$\begin{aligned} \hat{U}_h(t) \quad &:= \quad U_h(t) + \frac{1}{2}(t - t_{n-1})(t - t_n)\hat{w}_h^n, \\ &:= \quad U_h^{n-1/2} + (t - t_{n-1/2})\bar{\partial} U_h^n + \frac{1}{2}(t - t_{n-1})(t - t_n)\hat{w}_h^n, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \hat{w}_h^n \quad &:= \quad \check{w}_h^1 && \text{for } \; n = 1, \\ &:= \quad \bar{\partial}^2 U_h^n := \frac{2(\bar{\partial} U_h^n - \bar{\partial} U_h^{n-1})}{(\tau_n + \tau_{n-1})} && \text{for } \; 2 \leq n \leq N. \end{aligned}$$

Here, U_h, $\check{U}_h$ and $\hat{U}_h$ coincide at $t_1, t_2, \ldots, t_N$ and later this property will play a crucial role in obtaining *a posteriori* estimate for quadrature approximation of the Volterra integral term. An argument similar to [10] (see, Remark 2) ensures that $\hat{U}_h$ vanishes on the boundary so that $\hat{U}_h \in V_h^0$.

For $t \in I_n$, we observe that

$$\frac{\partial \hat{U}_h(t)}{\partial t} = \bar{\partial} U_h^n + (t - t_{n-1/2})\hat{w}_h^n \tag{21}$$

and

$$\frac{\partial^2 \hat{U}_h(t)}{\partial t^2} = \hat{w}_h^n. \tag{22}$$

In order to account for the time discretization error due to quadrature approximation of the Volterra integral term in the context of three-point reconstruction, set

$$\hat{\phi}(t) := \int_0^t B(t,s)\nabla \hat{U}_h(s)ds. \tag{23}$$

For $t \in I_n$, $2 \le n \le N$, define $\phi_2(t)$ to be the extended piecewise linear interpolant associated to the integral vectors $\hat{\phi}(t_{n-1/2})$ and $\hat{\phi}(t_{n-3/2})$ given by

$$\phi_2(t) = l_{n-1/2}(t)\hat{\phi}(t_{n-1/2}) + l_{n-3/2}(t)\hat{\phi}(t_{n-3/2}), \tag{24}$$

where

$$l_{n-1/2}(t) = 1 - \frac{2(t_{n-1/2}-t)}{(\tau_{n-1}+\tau_n)} \quad \text{and} \quad l_{n-3/2}(t) = \frac{2(t_{n-1/2}-t)}{(\tau_{n-1}+\tau_n)}. \tag{25}$$

Set

$$\hat{\psi}(t) := \int_0^t B(t,s)(\nabla U_h(s) - \nabla \hat{U}_h(s))ds. \tag{26}$$

For $t \in I_n$, we define $\psi_2(t)$ to be the extended linear interpolant associated with the integral vectors $\hat{\psi}(t_{n-1/2})$ and $\hat{\psi}(t_{n-3/2})$ and is given by

$$\psi_2(t) := l_{n-1/2}(t)\hat{\psi}(t_{n-1/2}) + l_{n-3/2}(t)\hat{\psi}(t_{n-3/2}), \quad 2 \le n \le N, \tag{27}$$

where $l_{n-1/2}(t)$ and $l_{n-3/2}(t)$ are given by (25).

4. Error analysis

In this section, we derive optimal order *a posteriori* upper bounds for the error $e := u - U_h$ in the $L^2(H^1)$-norm, where u and U_h satisfy (1) and (9), respectively. The standard energy argument is used in our analysis. First, we relate the error to the equation residual and introduce Clément interpolant. Then by localizing the residual term over each of the elements and the edges of the triangulation, we use the anisotropic interpolation error estimates. In the first estimator (see, Theorem 4.1), a linear approximation of the Volterra integral term is used in a crucial way to estimate the quadrature error for the approximation of the memory term. Moreover, an extended linear approximation of the Volterra integral term is used to estimate the error due to the quadrature approximation of the memory term in the the later estimator (see, Theorem 4.2).

Setting $\check{e} := u - \check{U}_h$ and $\hat{e} := u - \hat{U}_h$, we state the main results of this section in the following theorems.

Theorem 4.1. *Suppose that the mesh is such that there exists a constant c independent of the time step, mesh size and aspect ratio such that*

$$\lambda_{1,K}^2 (r_{1,K}^T G_K(\check{e}) r_{1,K}) \le c\lambda_{2,K}^2 (r_{2,K}^T G_K(\check{e}) r_{2,K}), \quad \forall K \in \mathcal{T}_h. \tag{28}$$

For any $1 \le n \le N$, $t \in I_n$, let $\check{\phi}(t)$, $\phi_1(t)$, $\check{\psi}(t)$ and $\psi_1(t)$ be given by (16), (17), (18) *and* (19), *respectively. Then there exists a constant C depending on the interpolation constants of Proposition* 2.1 *(hence independent of the time step, mesh size and aspect ratio) and the final time T such that the following a posteriori error bound holds:*

$$\|e(.,T)\|^2 + \alpha \int_0^T \|\nabla e\|^2 dt \le \|e(.,0)\|^2$$
$$+ C \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h} \Bigg\{ \int_{t_{n-1}}^{t_n} \Bigg[\Big\| f - \bar\partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds \Big\|_{L^2(K)}$$
$$+ \frac{1}{\lambda_{2,K}^{1/2}} \|[A\nabla U_h \cdot \boldsymbol{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}} \Big\| \Big[\int_0^t B(t,s)\nabla U_h(s)\cdot \boldsymbol{n} ds \Big] \Big\|_{L^2(\partial K)} \Bigg] \omega_K(\check e) dt$$
$$+ \tau_n^3 \lambda_{2,K}^2 \|\breve w_h^n\|_{L^2(K)}^2 + \int_{t_{n-1}}^{t_n} \|f - \check f\|_{L^2(K)}^2 dt + \tau_n \breve\theta_{n,K}^2 + \tau_n^3 \breve\zeta_{n,K}^2$$
$$+ \tau_n^5 \|\nabla \breve w_h^n\|_{L^2(K)}^2 + \tau_n \Big[\breve\zeta_{n,K}^2 + \breve\zeta_{n-1,K}^2 + \|\check\phi(t) - \phi_1(t)\|_{L^2(K)}^2$$
$$+ \|\check\psi(t) - \psi_1(t)\|_{L^2(K)}^2 \Big] \Bigg\},$$

where $\breve\theta_{n,K}$ *and* $\breve\zeta_{n,K}$ *are defined by*

$$\begin{aligned} \breve\theta_{n,K} \quad = \quad & \sum_{j=1}^{n} \tau_j^3 \left[\|\nabla U_h^j\|_{L^2(K)} + \|\nabla U_h^{j-1}\|_{L^2(K)} + \tau_j^2 \|\nabla \breve w_h^j\|_{L^2(K)} \right] \\ & + \sum_{j=1}^{n} \tau_j^3 \left[\|\nabla \bar\partial U_h^j\|_{L^2(K)} + \tau_j \|\nabla \breve w_h^j\|_{L^2(K)} \right] + \sum_{j=1}^{n} \tau_j^3 \|\nabla \breve w_h^j\|_{L^2(K)} \end{aligned} \tag{29}$$

and

$$\breve\zeta_{n,K} := \sum_{j=1}^{n} \tau_j^3 \|\nabla \breve w_h^j\|_{L^2(K)}, \tag{30}$$

$\breve w_h^n$ *is given by* (13) *and* $\check f$ *is defined by*

$$\check f(\cdot,t) = l_{n-1}(t) f^{n-1} + l_n(t) f^n. \tag{31}$$

Here, $[\cdot]$ *denotes the jump of the bracketed quantity across an internal edge,* $[\cdot] = 0$ *for an edge on the boundary* $\partial\Omega$ *and* $\boldsymbol{n}$ *is the unit edge normal.*

Theorem 4.2. *Suppose that the mesh is such that there exists a constant c independent of the time step, mesh size and aspect ratio such that*

$$\lambda_{1,K}^2 (r_{1,K}^T G_K(\hat e) r_{1,K}) \le c \lambda_{2,K}^2 (r_{2,K}^T G_K(\hat e) r_{2,K}), \quad \forall K \in \mathcal{T}_h. \tag{32}$$

For any $2 \leq n \leq N$ and $t \in I_n$, let $\hat{\phi}(t)$, $\phi_2(t)$, $\hat{\psi}(t)$ and $\psi_2(t)$ be given by (23), (24), (26) and (27), respectively. Then there exists a constant C depending on the interpolation constants of Proposition 2.1 *(hence independent of the time step, mesh size and aspect ratio) and the final time T such that the following a posteriori error bound holds:*

$$
\begin{aligned}
&\|e(.,T)\|^2 + \alpha \int_{t_1}^{T} \|\nabla e\|^2 dt \leq \|e(.,t_1)\|^2 \\
&+ C \sum_{n=2}^{N} \sum_{K \in \mathcal{T}_h} \Bigg\{ \int_{t_{n-1}}^{t_n} \Bigg[\Bigg\| f - \bar{\partial} U_h^n + \nabla \cdot (A \nabla U_h) - \int_0^t \nabla \cdot (B(t,s) \nabla U_h(s)) ds \Bigg\|_{L^2(K)} \\
&+ \frac{1}{\lambda_{2,K}^{1/2}} \|[A \nabla U_h \cdot \boldsymbol{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}} \Bigg\| \Bigg[\int_0^t B(t,s) \nabla U_h(s) \cdot \boldsymbol{n} ds \Bigg] \Bigg\|_{L^2(\partial K)} \Bigg] \omega_K(\hat{e}) dt \\
&+ \tau_n^3 \lambda_{2,K}^2 \|\hat{w}_h^n\|_{L^2(K)}^2 + \int_{t_{n-1}}^{t_n} \|f - \hat{f}\|_{L^2(K)}^2 dt \\
&+ \tau_n \left[\theta_{n,K}^2 + \theta_{n-1,K}^2 + \|\hat{\phi}(t) - \phi_2(t)\|_{L^2(K)}^2 \right] + \left[\tau_{n-1}^2 \tau_n^3 + \tau_n^5 \right] \|\nabla \hat{w}_h^n\|_{L^2(K)}^2 \\
&+ \tau_n \left[\zeta_{n,K}^2 + \zeta_{n-1,K}^2 + \|\hat{\psi}(t) - \psi_2(t)\|_{L^2(K)}^2 \right] \Bigg\},
\end{aligned}
$$

where $\theta_{n,K}$, $\zeta_{n,K}$ are defined by

$$
\begin{aligned}
\theta_{n,K} \;=\; & \sum_{j=1}^{n} \tau_j^3 \left[\|\nabla U_h^j\|_{L^2(K)} + \|\nabla U_h^{j-1}\|_{L^2(K)} + \tau_j^2 \|\nabla \hat{w}_h^j\|_{L^2(K)} \right] \\
& + \sum_{j=1}^{n} \tau_j^3 \left[\|\nabla \bar{\partial} U_h^j\|_{L^2(K)} + \tau_j \|\nabla \hat{w}_h^j\|_{L^2(K)} \right] + \sum_{j=1}^{n} \tau_j^3 \|\nabla \hat{w}_h^j\|_{L^2(K)}, \qquad (33)
\end{aligned}
$$

$$
\zeta_{n,K} := \sum_{j=1}^{n} \tau_j^3 \|\nabla \hat{w}_h^j\|_{L^2(K)} \qquad (34)
$$

and $\hat{f}$ is given by

$$
\hat{f}(\cdot,t) = f^{n-1/2} + (t - t_{n-1/2}) \left(\frac{f^n - f^{n-2}}{\tau_{n-1} + \tau_n} \right). \qquad (35)
$$

Here, $[\cdot]$ denotes the jump of the bracketed quantity across an internal edge, $[\cdot] = 0$ for an edge on the boundary $\partial\Omega$ and $\boldsymbol{n}$ is the unit edge normal.

The proofs of Theorem 4.1 and Theorem 4.2 require some preparation. We shall first proceed to prove Theorem 4.2. For this purpose, we need to prove a sequence of lemmas.

Lemma 4.1. *Let U_h and $\hat{U}_h$ be as defined by* (10) *and* (20)*, respectively. Then, for any $2 \le n \le N$, $t \in I_n$ and for all $\phi_h \in V_h^0$, we have*

$$\int_\Omega \frac{\partial \hat{U}_h}{\partial t}\phi_h dx + a(U_h,\phi_h) = \int_\Omega \hat{f}\phi_h dx + \frac{\tau_{n-1}(t-t_{n-1/2})}{2}a(\hat{w}_h^n,\phi_h)$$
$$+\; l_{n-1/2}(t)\sigma^n(b(t_{n-1/2};U_h,\phi_h)) + l_{n-3/2}(t)\sigma^{n-1}(b(t_{n-3/2};U_h,\phi_h)),$$

where $\hat{f}$ is given by (35)*, $l_{n-1/2}(t)$ and $l_{n-3/2}(t)$ are given by* (25)*.*

Proof. Let $t \in I_n$ with $2 \le n \le N$. For all $\phi_h \in V_h^0$, we use (10) and (9) to get

$$\begin{aligned} \int_\Omega \bar{\partial} U_h^n \phi_h dx + a(U_h,\phi_h) &= \sigma^n(b(t_{n-1/2};U_h,\phi_h)) + \int_\Omega f^{n-1/2}\phi_h dx \\ &\quad + (t-t_{n-1/2})a(\bar{\partial}U_h^n,\phi_h). \end{aligned} \tag{36}$$

Using (21), we rewrite (36) as

$$\begin{aligned} \int_\Omega \frac{\partial \hat{U}_h}{\partial t}\phi_h dx + a(U_h,\phi_h) &= \sigma^n(b(t_{n-1/2};U_h,\phi_h)) + \int_\Omega f^{n-1/2}\phi_h dx \\ &\quad + (t-t_{n-1/2})\left(a(\bar{\partial}U_h^n,\phi_h) + \int_\Omega \hat{w}_h^n\phi_h dx\right). \end{aligned} \tag{37}$$

With $t = t_{n-1}$, (9) becomes

$$\int_\Omega \bar{\partial}U_h^{n-1}\phi_h dx + a(U_h^{n-3/2},\phi_h) = \sigma^{n-1}(b(t_{n-3/2};U_h,\phi_h)) + \int_\Omega f^{n-3/2}\phi_h dx. \tag{38}$$

A little simplification after subtracting (38) from (9) and then multiplying both sides by the term $2/(\tau_n + \tau_{n-1})$ we obtain

$$\begin{aligned} \int_\Omega \hat{w}_h^n\phi_h dx &+ a\left(\frac{U_h^n - U_h^{n-2}}{\tau_{n-1}+\tau_n},\phi_h\right) = \int_\Omega \frac{(f^n - f^{n-2})}{(\tau_{n-1}+\tau_n)}\phi_h dx \\ &+ \frac{2}{(\tau_{n-1}+\tau_n)}\Big[\sigma^n(b(t_{n-1/2};U_h,\phi_h)) - \sigma^{n-1}(b(t_{n-3/2};U_h,\phi_h))\Big]. \end{aligned} \tag{39}$$

Noting the fact that

$$\frac{U_h^n - U_h^{n-2}}{\tau_n + \tau_{n-1}} = \bar{\partial}U_h^n - \frac{\tau_{n-1}}{2}\hat{w}_h^n,$$

we have from (39),

$$\begin{aligned} \int_\Omega \hat{w}_h^n\phi_h dx &+ a(\bar{\partial}U_h^n,\phi_h) = \int_\Omega \frac{(f^n - f^{n-2})}{(\tau_{n-1}+\tau_n)}\phi_h dx + \frac{\tau_{n-1}}{2}a(\hat{w}_h^n,\phi_h) \\ &+ \frac{2}{(\tau_{n-1}+\tau_n)}\Big[\sigma^n(b(t_{n-1/2};U_h,\phi_h)) - \sigma^{n-1}(b(t_{n-3/2};U_h,\phi_h))\Big]. \end{aligned} \tag{40}$$

Substituting (40) on the right of (37), we complete the rest of the proof. □

The following lemma yields a bound on the quadrature error.

Lemma 4.2 (Quadrature error estimate)**.** *Let $\hat{U}_h$ and $\theta_{n,K}$ be as defined by* (20) *and* (33)*, respectively. Moreover, let $\hat{\phi}(t)$ and $\phi_2(t)$ be given by* (23) *and* (24)*. Then, for any $\phi_h \in V_h^0$ and $t \in I_n$ with $2 \leq n \leq N$, the following bound holds:*

$$\Big| \int_0^t b(t,s;\hat{U}_h(s),\phi_h)ds - l_{n-1/2}(t)\sigma^n(b(t_{n-1/2};\hat{U}_h,\phi_h)) - l_{n-3/2}(t)\sigma^{n-1}(b(t_{n-3/2};\hat{U}_h,\phi_h)) \Big|$$
$$\leq \bar{\gamma} \sum_{K\in\mathcal{T}_h} \Big[|l_{n-1/2}(t)|\theta_{n,K} + |l_{n-3/2}(t)|\theta_{n-1,K} + \|\hat{\phi}(t)-\phi_2(t)\|_{L^2(K)} \Big] \|\nabla\phi_h\|_{L^2(K)},$$

where $\bar{\gamma} = \max\left\{\frac{\gamma''}{4}, \gamma', \frac{\gamma}{2}, 1\right\}$.

Proof. We choose any $2 \leq n \leq N$ and $t \in I_n$. Taking the L^2 inner product with $\nabla\phi_h$ on both sides of (24), we have

$$\begin{aligned} \langle \phi_2(t), \nabla\phi_h \rangle &= l_{n-1/2}(t) \int_0^{t_{n-1/2}} b(t_{n-1/2},s;\hat{U}_h(s),\phi_h)ds \\ &\quad + l_{n-3/2}(t) \int_0^{t_{n-3/2}} b(t_{n-3/2},s;\hat{U}_h(s),\phi_h)ds, \quad \forall \phi_h \in V_h^0. \end{aligned} \tag{41}$$

Since $l_{n-1/2}(t) + l_{n-3/2}(t) \equiv 1,\ t \in I_n$, it follows that

(42)
$$\begin{aligned} &\int_0^t b(t,s;\hat{U}_h(s),\phi_h)ds - l_{n-1/2}(t)\sigma^n(b(t_{n-1/2};\hat{U}_h,\phi_h)) \\ &\quad - l_{n-3/2}(t)\sigma^{n-1}(b(t_{n-3/2};\hat{U}_h,\phi_h)) \\ &= l_{n-1/2}(t)\Big[\int_0^{t_{n-1/2}} b(t_{n-1/2},s;\hat{U}_h(s),\phi_h)ds - \sigma^n(b(t_{n-1/2};\hat{U}_h,\phi_h))\Big] \\ &\quad + l_{n-3/2}(t)\Big[\int_0^{t_{n-3/2}} b(t_{n-3/2},s;\hat{U}_h(s),\phi_h)ds - \sigma^{n-1}(b(t_{n-3/2};\hat{U}_h,\phi_h))\Big] \\ &\quad + \langle \hat{\phi}(t) - \phi_2(t), \nabla\phi_h \rangle \\ &:= l_{n-1/2}(t)\mathcal{I}_1 + l_{n-3/2}(t)\mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

To estimate the quadrature error for approximating the Volterra integral term, we first need to estimate $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$. A standard Trapezoidal rule argument for a sufficiently smooth function $g(s)$ yields

$$\int_a^b g(s)ds - \frac{(b-a)}{2}(g(a)+g(b)) = \frac{1}{2}\int_a^b (s-a)(s-b)g''(s)ds.$$

If we define

$$\psi_{2j}(s) := \begin{cases} (s-t_{j-1})(s-t_j) & \text{for } s \in [t_{j-1}, t_j] \text{ and } 1 \le j \le n-1, \\ (s-t_{j-1})(s-t_{j-1/2}) & \text{for } s \in [t_{j-1}, t_{j-1/2}] \text{ and } j = n, \end{cases}$$

then

$$\int_{t_{j-1}}^{t_j} g(s)ds - \frac{\tau_j}{2}[g(t_j) + g(t_{j-1})] = \frac{1}{2}\int_{t_{j-1}}^{t_j} \psi_{2j}(s) g''(s)ds, \tag{43}$$

and

$$\int_{t_{n-1}}^{t_{n-1/2}} g(s)ds - \frac{\tau_n}{4}[g(t_{n-1}) + g(t_{n-1/2})] = \frac{1}{2}\int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s) g''(s)ds. \tag{44}$$

Using (43) and (44), we obtain

$$\begin{aligned}
\mathcal{I}_1 &:= \int_0^{t_{n-1/2}} b(t_{n-1/2}, s; \hat{U}_h(s), \phi_h)ds - \sigma^n(b(t_{n-1/2}; \hat{U}_h, \phi_h)) \\
&= \Big\langle \int_0^{t_{n-1/2}} B(t_{n-1/2}, s)\nabla \hat{U}_h(s)ds, \nabla\phi_h \Big\rangle \\
&\quad -\Big\langle \sum_{j=0}^{n-2} \frac{\tau_{j+1}}{2}\Big[B(t_{n-1/2}, t_j)\nabla U_h^j + B(t_{n-1/2}, t_{j+1})\nabla U_h^{j+1}\Big], \nabla\phi_h \Big\rangle \\
&\quad -\Big\langle \frac{\tau_n}{4}\Big[B(t_{n-1/2}, t_{n-1})\nabla U_h^{n-1} + B(t_{n-1/2}, t_{n-1/2})\nabla U_h^{n-1/2}\Big], \nabla\phi_h \Big\rangle \\
&= \frac{1}{2}\Big\langle \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} \psi_{2j}(s)\frac{d^2}{ds^2}\big\{B(t_{n-1/2}, s)\nabla\hat{U}_h(s)\big\}ds, \nabla\phi_h \Big\rangle \\
&\quad +\frac{1}{2}\Big\langle \int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s)\frac{d^2}{ds^2}\big\{B(t_{n-1/2}, s)\nabla\hat{U}_h(s)\big\}ds, \nabla\phi_h \Big\rangle.
\end{aligned}$$

Thus, over each $K \in \mathcal{T}_h$, we have

$$\begin{aligned}
\mathcal{I}_1 &:= \int_0^{t_{n-1/2}} b(t_{n-1/2}, s; \hat{U}_h(s), \phi_h)ds - \sigma^n(b(t_{n-1/2}; \hat{U}_h, \phi_h)) \\
&= \frac{1}{2}\sum_{K\in\mathcal{T}_h}\Bigg\{\Big\langle \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} \psi_{2j}(s)\Bigg[\frac{d^2(B(t_{n-1/2}, s))}{ds^2}\nabla\hat{U}_h(s) \\
&\quad +2\frac{d(B(t_{n-1/2}, s))}{ds}\frac{d(\nabla\hat{U}_h(s))}{ds} + B(t_{n-1/2}, s)\frac{d^2(\nabla\hat{U}_h(s))}{ds^2}\Bigg]ds, \nabla\phi_h\Big\rangle_K \\
&\quad +\Big\langle \int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s)\Bigg[\frac{d^2(B(t_{n-1/2}, s))}{ds^2}\nabla\hat{U}_h(s) + 2\frac{d(B(t_{n-1/2}, s))}{ds}\frac{d(\nabla\hat{U}_h(s))}{ds} \\
&\quad +B(t_{n-1/2}, s)\frac{d^2(\nabla\hat{U}_h(s))}{ds^2}\Bigg]ds, \nabla\phi_h\Big\rangle_K\Bigg\}.
\end{aligned}$$

Using (20) together with (3), (4) and (5), we obtain

$$\begin{aligned}
|\mathcal{I}_1| &:= \left| \int_0^{t_{n-1/2}} b(t_{n-1/2}, s; \hat{U}_h(s), \phi_h) ds - \sigma^n(b(t_{n-1/2}; \hat{U}_h, \phi_h)) \right| \\
&\leq \frac{1}{2} \sum_{K \in \mathcal{T}_h} \Bigg\{ \Bigg(\sum_{j=1}^{n-1} \tau_j^2 \gamma'' \Bigg[\frac{\tau_j}{2} \|\nabla U_h^j\|_{L^2(K)} + \frac{\tau_j}{2} \|\nabla U_h^{j-1}\|_{L^2(K)} \\
&\quad + \frac{\tau_j^3}{12} \|\nabla \hat{w}_h^j\|_{L^2(K)} \Bigg] + \sum_{j=1}^{n-1} \tau_j^2 \gamma' \Bigg[2\tau_j \|\nabla \bar{\partial} U_h^j\|_{L^2(K)} + \frac{\tau_j^2}{2} \|\nabla \hat{w}_h^j\|_{L^2(K)} \Bigg] \\
&\quad + \sum_{j=1}^{n-1} \tau_j^3 \gamma \|\nabla \hat{w}_h^j\|_{L^2(K)} + \tau_n^2 \gamma'' \Bigg[\frac{3\tau_n}{8} \|\nabla U_h^n\|_{L^2(K)} + \frac{\tau_n}{8} \|\nabla U_h^{n-1}\|_{L^2(K)} \\
&\quad + \frac{\tau_n^3}{24} \|\nabla \hat{w}_h^n\|_{L^2(K)} \Bigg] + \tau_n^2 \gamma' \Bigg[\tau_n \|\nabla \bar{\partial} U_h^n\|_{L^2(K)} + \frac{\tau_n^2}{4} \|\nabla \hat{w}_h^n\|_{L^2(K)} \Bigg] \\
&\quad + \frac{\tau_n^3}{2} \gamma \|\nabla \hat{w}_h^n\|_{L^2(K)} \Bigg) \|\nabla \phi_h\|_{L^2(K)} \Bigg\} \\
&\leq \bar{\gamma_1} \sum_{K \in \mathcal{T}_h} \theta_{n,K} \|\nabla \phi_h\|_{L^2(K)}, \qquad (45)
\end{aligned}$$

where $\bar{\gamma_1} = \max\left\{ \frac{\gamma''}{4}, \gamma', \frac{\gamma}{2} \right\}$ and $\theta_{n,K}$ is given by (33). Similarly,

$$\begin{aligned}
|\mathcal{I}_2| &:= \left| \int_0^{t_{n-3/2}} b(t_{n-3/2}, s; \hat{U}_h(s), \phi_h) ds - \sigma^n(b(t_{n-3/2}; \hat{U}_h, \phi_h)) \right| \\
&\leq \bar{\gamma_1} \sum_{K \in \mathcal{T}_h} \theta_{n-1,K} \|\nabla \phi_h\|_{L^2(K)}. \qquad (46)
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\mathcal{I}_3| &:= |\langle \hat{\phi}(t) - \phi_2(t), \nabla \phi_h \rangle| \\
&\leq \sum_{K \in \mathcal{T}_h} \|\hat{\phi}(t) - \phi_2(t)\|_{L^2(K)} \|\nabla \phi_h\|_{L^2(K)}. \qquad (47)
\end{aligned}$$

Substituting the estimates (45)–(47) in (42), we obtain the desired result and this completes the proof. □

The following lemma reveals the contributions for the time reconstruction error.

Lemma 4.3 (Time reconstruction error estimate)**.** *Let U_h and $\hat{U}_h$ be as defined by (10) and (20), respectively. Let $\zeta_{n,K}$ be given by (34). Further, let $\hat{\psi}(t)$ and $\psi_2(t)$ be given by (26) and (27), respectively. Then, for any $\phi_h \in V_h^0$ and $t \in I_n$ with $2 \leq n \leq N$, we have*

$$\begin{aligned}
&\left| \int_0^t b(t, s; U_h(s) - \hat{U}_h(s), \phi_h) ds \right| \\
&\quad \leq \sum_{K \in \mathcal{T}_h} \Bigg\{ \frac{\gamma}{12} \Big[|l_{n-1/2}(t)| \zeta_{n,K} \|\nabla \phi_h\|_{L^2(K)} + |l_{n-3/2}(t)| \zeta_{n-1,K} \|\nabla \phi_h\|_{L^2(K)} \Big] \\
&\qquad + \|\hat{\psi}(t) - \psi_2(t)\|_{L^2(K)} \|\nabla \phi_h\|_{L^2(K)} \Bigg\}.
\end{aligned}$$

Proof. We choose any $2 \le n \le N$ and $t \in I_n$. Taking the L^2 inner product with $\nabla\phi_h$ on both sides of (26) and using the identity (27), we obtain for all $\phi_h \in V_h^0$,

$$
\begin{aligned}
&|\int_0^t b(t,s;U_h(s)-\hat{U}_h(s),\phi_h)ds| \\
&\quad \le |l_{n-1/2}(t)| \int_0^{t_{n-1/2}} |b(t_{n-1/2},s;U_h(s)-\hat{U}_h(s),\phi_h)|ds \\
&\qquad + |l_{n-3/2}(t)| \int_0^{t_{n-3/2}} |b(t_{n-3/2},s;U_h(s)-\hat{U}_h(s),\phi_h)|ds \\
&\qquad + |\langle\hat{\psi}(t)-\psi_2(t),\nabla\phi_h\rangle| \\
&\quad \le \frac{1}{2}|l_{n-1/2}(t)| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} |b(t_{n-1/2},s;(s-t_{j-1})(s-t_j)\hat{w}_h^j,\phi_h)|ds \\
&\qquad + \frac{1}{2}|l_{n-1/2}(t)| \int_{t_{n-1}}^{t_{n-1/2}} |b(t_{n-1/2},s;(s-t_{n-1})(s-t_n)\hat{w}_h^n,\phi_h)|ds \\
&\qquad + \frac{1}{2}|l_{n-3/2}(t)| \sum_{j=1}^{n-2} \int_{t_{j-1}}^{t_j} |b(t_{n-3/2},s;(s-t_{j-1})(s-t_j)\hat{w}_h^j,\phi_h)|ds \\
&\qquad + \frac{1}{2}|l_{n-3/2}(t)| \int_{t_{n-2}}^{t_{n-3/2}} |b(t_{n-3/2},s;(s-t_{n-2})(s-t_{n-1})\hat{w}_h^{n-1},\phi_h)|ds \\
&\qquad + |\langle\hat{\psi}(t)-\psi_2(t),\nabla\phi_h\rangle| \\
&\quad := |\mathcal{J}_1|+|\mathcal{J}_2|+|\mathcal{J}_3|+|\mathcal{J}_4|+|\mathcal{J}_5|. \qquad (48)
\end{aligned}
$$

To estimate $|\mathcal{J}_1|$, we use (3) to obtain over each element $K \in \mathcal{T}_h$,

$$
\begin{aligned}
|\mathcal{J}_1| &:= \frac{1}{2}|l_{n-1/2}(t)| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} |b(t_{n-1/2},s;(s-t_j)(s-t_{j+1})\hat{w}_h^j,\phi_h)|ds \\
&\le \frac{\gamma|l_{n-1/2}(t)|}{12} \sum_{K\in\mathcal{T}_h} \sum_{j=1}^{n-1} \tau_j^3 \|\nabla\hat{w}_h^j\|_{L^2(K)} \|\nabla\phi_h\|_{L^2(K)}.
\end{aligned}
$$

Similarly,

$$|\mathcal{J}_2| \le \frac{\gamma|l_{n-1/2}(t)|}{24} \sum_{K\in\mathcal{T}_h} \tau_n^3 \|\nabla\hat{w}_h^n\|_{L^2(K)} \|\nabla\phi_h\|_{L^2(K)},$$

$$|\mathcal{J}_3| \le \frac{\gamma|l_{n-3/2}(t)|}{12} \sum_{K\in\mathcal{T}_h} \sum_{j=1}^{n-2} \tau_j^3 \|\nabla\hat{w}_h^j\|_{L^2(K)} \|\nabla\phi_h\|_{L^2(K)},$$

$$|\mathcal{J}_4| \le \frac{\gamma|l_{n-3/2}(t)|}{24} \sum_{K\in\mathcal{T}_h} \tau_{n-1}^3 \|\nabla\hat{w}_h^{n-1}\|_{L^2(K)} \|\nabla\phi_h\|_{L^2(K)}$$

and

$$|\mathcal{J}_5| \le \sum_{K\in\mathcal{T}_h} \|\hat{\psi}(t)-\psi_2(t)\|_{L^2(K)} \|\nabla\phi_h\|_{L^2(K)}.$$

We now combine these estimates with (48) to complete the proof. ◻

Proof of Theorem 4.2. Using the weak formulation (7), we have for any $2 \le n \le N$ and $t \in I_n$,

$$\int_\Omega \frac{\partial \hat{e}}{\partial t}\phi dx + a(e,\phi) - \int_0^t b(t,s;e(s),\phi)ds$$
$$= \int_\Omega f\phi dx - \int_\Omega \frac{\partial \hat{U}_h}{\partial t}\phi dx - a(U_h,\phi) + \int_0^t b(t,s;U_h(s),\phi)ds, \ \ \forall \phi \in H_0^1(\Omega).$$

Choose $\phi = \hat{e}$ in the above error equation. Then, using (21) with some rearrangement of terms, it follows that

$$\int_\Omega \frac{\partial \hat{e}}{\partial t}\hat{e} dx + a(e,\hat{e}) - \int_0^t b(t,s;e(s),\hat{e})ds$$
$$= \int_\Omega (f - \bar{\partial}U_h^n)(\hat{e} - \Pi_h\hat{e})dx - a(U_h,\hat{e}-\Pi_h\hat{e}) + \int_0^t b(t,s;U_h(s),\hat{e}-\Pi_h\hat{e})ds$$
$$- (t - t_{n-1/2})\int_\Omega \hat{w}_h^n(\hat{e}-\Pi_h\hat{e})dx + \int_\Omega f\Pi_h\hat{e}dx + \int_0^t b(t,s;\hat{U}_h(s),\Pi_h\hat{e})ds$$
$$- a(U_h,\Pi_h\hat{e}) - \int_\Omega \frac{\partial \hat{U}_h}{\partial t}\Pi_h\hat{e}dx + \int_0^t b(t,s;U_h(s)-\hat{U}_h(s),\Pi_h\hat{e})ds.$$

By using Lemma 4.1, we obtain

(49)
$$\int_\Omega \frac{\partial \hat{e}}{\partial t}\hat{e} dx + a(e,\hat{e}) - \int_0^t b(t,s;e(s),\hat{e})ds$$
$$= \int_\Omega (f - \bar{\partial}U_h^n)(\hat{e} - \Pi_h\hat{e})dx - a(U_h,\hat{e}-\Pi_h\hat{e}) + \int_0^t b(t,s;U_h(s),\hat{e}-\Pi_h\hat{e})ds$$
$$- (t - t_{n-1/2})\int_\Omega \hat{w}_h^n(\hat{e}-\Pi_h\hat{e})dx + \int_\Omega (f-\hat{f})\Pi_h\hat{e}dx$$
$$- \frac{\tau_{n-1}(t-t_{n-1/2})}{2}a(\hat{w}_h^n,\Pi_h\hat{e}) + \Bigg[\int_0^t b(t,s;\hat{U}_h(s),\Pi_h\hat{e})ds$$
$$- l_{n-1/2}(t)\sigma^n(b(t_{n-1/2};U_h,\Pi_h\hat{e})) - l_{n-3/2}(t)\sigma^{n-1}(b(t_{n-3/2};U_h,\Pi_h\hat{e}))\Bigg]$$
$$+ \int_0^t b(t,s;U_h(s)-\hat{U}_h(s),\Pi_h\hat{e})ds.$$

Now, we integrate (49) by parts on each of the elements K of $\mathcal{T}_h$ and plug back in the estimates of Lemmas 4.2 and 4.3. Then, using the continuity of the bilinear forms $b(t,s;\cdot,\cdot)$ and $a(\cdot,\cdot)$ together with the Cauchy-Schwarz inequality and the Poincaré inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + a(e,\hat{e}) \le \gamma\|\nabla\hat{e}(t)\|\int_0^t \|\nabla e(s)\|ds$$
$$+ \sum_{K\in\mathcal{T}_h}\Bigg\{\Bigg\| f - \bar{\partial}U_h^n + \nabla\cdot(A\nabla U_h)$$

$$
\begin{aligned}
&- \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\Big\|_{L^2(K)} \|\hat{e}-\Pi_h\hat{e}\|_{L^2(K)} \\
&+ \frac{1}{2}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}\|\hat{e}-\Pi_h\hat{e}\|_{L^2(\partial K)} \\
&+ \frac{1}{2}\Big\|\Big[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds\Big]\Big\|_{L^2(\partial K)}\|\hat{e}-\Pi_h\hat{e}\|_{L^2(\partial K)} \\
&+ |t-t_{n-1/2}|\|\hat{w}_h^n\|_{L^2(K)}\|\hat{e}-\Pi_h\hat{e}\|_{L^2(K)} + C_2\|f-\hat{f}\|_{L^2(K)}\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \\
&+ \frac{|\tau_{n-1}||t-t_{n-1/2}|}{2}\beta\|\nabla\hat{w}_h^n\|_{L^2(K)}\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \\
&+ \bar{\gamma}\Big[|l_{n-1/2}(t)|\theta_{n,K} + |l_{n-3/2}(t)|\theta_{n-1,K} + \|\hat{\phi}(t)-\phi_2(t)\|_{L^2(K)}\Big]\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \\
&+ \frac{\gamma}{12}\Big[|l_{n-1/2}(t)|\zeta_{n,K} + |l_{n-3/2}(t)|\zeta_{n-1,K}\Big]\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \\
&+ \|\hat{\psi}(t)-\psi_2(t)\|_{L^2(K)}\|\nabla\Pi_h\hat{e}\|_{L^2(K)}\Bigg\},
\end{aligned}
$$

where C_2 is a constant in the Poincaré inequality. Now, we use the fact that $a(e,\hat{e}) = \frac{1}{2}\Big[a(e,e) + a(\hat{e},\hat{e}) - a(e-\hat{e}, e-\hat{e})\Big]$ together with coercivity and continuity of the bilinear form $a(\cdot,\cdot)$. Then, an application of Young's inequality and Proposition 2.1 together with $\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \le C_3\|\nabla\hat{e}\|_{L^2(K)}$ leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + \frac{\alpha}{2}\|\nabla e\|^2 + \frac{\alpha}{2}\|\nabla\hat{e}\|^2 \le \frac{\gamma}{2\nu}\|\nabla\hat{e}\|^2 + \frac{\gamma C_Y(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds \\
&+ \sum_{K\in\mathcal{T}_h}\Bigg\{C_1\Bigg[\Big\|f - \bar{\partial}U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\Big\|_{L^2(K)} \\
&+ \frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\Big\|\Big[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds\Big]\Big\|_{L^2(\partial K)}\Bigg]\omega_K(\hat{e}) \\
&+ C_1|t-t_{n-1/2}|\|\hat{w}_h^n\|_{L^2(K)}\omega_K(\hat{e}) + C_2C_3\|f-\hat{f}\|_{L^2(K)}\|\nabla\hat{e}\|_{L^2(K)} \\
&+ \frac{C_3\beta}{2}|\tau_{n-1}||t-t_{n-1/2}|\|\nabla\hat{w}_h^n\|_{L^2(K)}\|\nabla\hat{e}\|_{L^2(K)} \\
&+ \frac{\beta}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla\hat{w}_h^n\|^2_{L^2(K)} \\
&+ C_3\bar{\gamma}\Big[|l_{n-1/2}(t)|\theta_{n,K} + |l_{n-3/2}(t)|\theta_{n-1,K} + \|\hat{\phi}(t)-\phi_2(t)\|_{L^2(K)}\Big]\|\nabla\hat{e}\|_{L^2(K)} \\
&+ \frac{\gamma C_3}{12}\Big[|l_{n-1/2}(t)|\zeta_{n,K} + |l_{n-3/2}(t)|\zeta_{n-1,K}\Big]\|\nabla\hat{e}\|_{L^2(K)} \\
&+ C_3\|\hat{\psi}(t)-\psi_2(t)\|_{L^2(K)}\|\nabla\hat{e}\|_{L^2(K)}\Bigg\},
\end{aligned}
$$

where C_1 is a constant in Proposition 2.1, $C_Y(T)$ is a constant which appeared due to the application of Young's inequality and the Cauchy-Schwarz inequality.

Further, we have used the relation

$$\|\nabla(e-\hat{e})\|^2 = \|\nabla(\hat{U}_h - U_h)\|^2 = \frac{1}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla \hat{w}_h^n\|^2.$$

Now, an application of Young's inequality along with $\omega_K(\hat{e}) \le C_4\lambda_{2,K}\|\nabla\hat{e}\|_{L^2(K)}$, which follows trivially from Proposition 2.1 under the error equidistribution assumption (32), yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + \frac{\alpha}{2}\|\nabla e\|^2 + \frac{\alpha}{2}\|\nabla\hat{e}\|^2 \le \frac{\gamma}{2\nu}\|\nabla\hat{e}\|^2 + \frac{\gamma C_Y(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds \\
&+\sum_{K\in\mathcal{T}_h}\Bigg\{C_1\Bigg[\Big\|f - \bar{\partial}U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\Big\|_{L^2(K)} \\
&+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\Big\|\Big[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds\Big]\Big\|_{L^2(\partial K)}\Bigg]\omega_K(\hat{e}) \\
&+\frac{\nu}{2}(t-t_{n-1/2})^2\lambda_{2,K}^2\|\hat{w}_h^n\|_{L^2(K)}^2 + \frac{\nu}{2}\|f-\hat{f}\|_{L^2(K)}^2 \\
&+\frac{1}{2\nu}(C_1^2C_4^2 + C_2^2C_3^2 + \beta^2C_3^2 + 3C_3^2\bar{\gamma}^2 + 2C_3^2\gamma^2 + C_3^2)\|\nabla\hat{e}\|_{L^2(K)}^2 \\
&+\frac{\nu}{2}\Big[l_{n-1/2}^2(t)\theta_{n,K}^2 + l_{n-3/2}^2(t)\theta_{n-1,K}^2 + \|\hat{\phi}(t)-\phi_2(t)\|_{L^2(K)}^2\Big] \\
&+\frac{\nu}{8}\tau_{n-1}^2(t-t_{n-1/2})^2\|\nabla\hat{w}_h^n\|_{L^2(K)}^2 + \frac{\beta}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla\hat{w}_h^n\|_{L^2(K)}^2 \\
&+\frac{\nu}{288}\Big[l_{n-1/2}^2(t)\zeta_{n,K}^2 + l_{n-3/2}^2(t)\zeta_{n-1,K}^2\Big] + \frac{\nu}{2}\|\hat{\psi}(t)-\psi_2(t)\|_{L^2(K)}^2\Bigg\}.
\end{aligned}$$

We now choose $\nu = \frac{1}{\alpha}(C_1^2C_4^2 + C_2^2C_3^2 + \beta^2C_3^2 + 3C_3^2\bar{\gamma}^2 + 2C_3^2\gamma^2 + C_3^2 + \gamma)$ and apply Gronwall's lemma. Then, integrate from t_{n-1} to t_n to obtain

$$\begin{aligned}
&\|\hat{e}(t_n)\|^2 + \alpha\int_{t_{n-1}}^{t_n}\|\nabla e\|^2 dt \le \|\hat{e}(t_{n-1})\|^2 \\
&+C\sum_{K\in\mathcal{T}_h}\Bigg\{\int_{t_{n-1}}^{t_n}\Bigg[\Big\|f - \bar{\partial}U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\Big\|_{L^2(K)} \\
&+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}}\Big\|\Big[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds\Big]\Big\|_{L^2(\partial K)}\Bigg]\omega_K(\hat{e})dt \\
&+\tau_n^3\lambda_{2,K}^2\|\hat{w}_h^n\|_{L^2(K)}^2 + \int_{t_{n-1}}^{t_n}\|f-\hat{f}\|_{L^2(K)}^2 dt \\
&+\tau_n\Big[\theta_{n,K}^2 + \theta_{n-1,K}^2 + \|\hat{\phi}(t)-\phi_2(t)\|_{L^2(K)}^2\Big] + \tau_{n-1}^2\tau_n^3\|\nabla\hat{w}_h^n\|_{L^2(K)}^2 \\
&+\tau_n^5\|\nabla\hat{w}_h^n\|_{L^2(K)}^2 + \tau_n\Big[\zeta_{n,K}^2 + \zeta_{n-1,K}^2 + \|\hat{\psi}(t)-\psi_2(t)\|_{L^2(K)}^2\Big]\Bigg\},
\end{aligned}$$

where $C = \max\Big\{2C_1C(T), 4C(T)\nu, \beta C(T)\Big\}$ with $C(T)$ denoting Gronwall's constant. Noting the fact that $e(t_n) = \hat{e}(t_n)\ \forall n$ and summing over $n = 2$ to N we complete the rest of the proof. $\square$

The proof of Theorem 4.1 will follow the arguments of the proof of Theorem 4.2. However, for clarity of presentation, we present the proof which again relies on a sequence of lemmas.

Lemma 4.4. *Let U_h and $\check{U}_h$ be as defined by* (10) *and* (12)*, respectively. Then, for any $t \in I_n$ with $1 \le n \le N$, and for all $\phi_h \in V_h^0$, we have*

$$\int_\Omega \frac{\partial \check{U}_h}{\partial t}\phi_h dx + a(U_h,\phi_h) = \int_\Omega \check{f}\phi_h dx + \sigma^n(b(t_{n-1/2};U_h,\phi_h)) \\ + (t-t_{n-1/2})\int_0^{t_n} \bar{\partial} b(t_n,s;U_h(s),\phi_h)ds + (t-t_{n-1/2})b(t_n,t_n;U_h^n,\phi_h),$$

where $\check{f}$ is given by (31).

Proof. For $\phi_h \in V_h^0$, we use (10) and (9) to have

$$\begin{aligned}\int_\Omega \bar{\partial} U_h^n \phi_h dx + a(U_h,\phi_h) &= \sigma^n(b(t_{n-1/2};U_h,\phi_h)) + \int_\Omega f^{n-1/2}\phi_h dx \\ &\quad +(t-t_{n-1/2})a(\bar{\partial}U_h^n,\phi_h).\end{aligned} \tag{50}$$

In view of (14), the equation (50) becomes

$$\begin{aligned}\int_\Omega \frac{\partial \check{U}_h}{\partial t}\phi_h dx + a(U_h,\phi_h) &= \sigma^n(b(t_{n-1/2};U_h,\phi_h)) + \int_\Omega f^{n-1/2}\phi_h dx \\ &\quad +(t-t_{n-1/2})\left(a(\bar{\partial}U_h^n,\phi_h) + \int_\Omega \check{w}_h^n\phi_h dx\right).\end{aligned} \tag{51}$$

Now, an application of (13) gives the desired result. □

The following lemma shows the contributions for the time reconstruction error.

Lemma 4.5 (Time reconstruction error estimate)**.** *Let U_h and $\check{U}_h$ be defined by* (10) *and* (12)*, respectively, and let $\check{\zeta}_{n,K}$ be given by* (30)*. Then, for any $\phi_h \in V_h^0$ and $t \in I_n$, $1 \le n \le N$, the following bound holds:*

$$\left|\int_0^t b(t,s;U_h(s)-\check{U}_h(s),\phi_h)ds\right| \\ \le \sum_{K\in\mathcal{T}_h}\left\{\frac{\gamma}{12}\Big[l_n(t)\check{\zeta}_{n,K} + l_{n-1}(t)\check{\zeta}_{n-1,K}\Big] + \|\check{\psi}(t)-\psi_1(t)\|_{L^2(K)}\right\}\|\nabla\phi_h\|_{L^2(K)}.$$

Proof. The result follows analogously to the proof of Lemma 4.3 by using the definition (19) instead of (27). □

Proof of Theorem 4.1. Let $t \in I_n$ and $1 \le n \le N$. We use the weak formulation (7) to have

$$\int_\Omega \frac{\partial \check{e}}{\partial t}\phi dx + a(e,\phi) - \int_0^t b(t,s;e(s),\phi)ds \\ = \int_\Omega f\phi dx - \int_\Omega \frac{\partial \check{U}_h}{\partial t}\phi dx - a(U_h,\phi) + \int_0^t b(t,s;U_h(s),\phi)ds, \;\; \forall \phi \in H_0^1(\Omega).$$

Choosing $\phi = \check{e}$ in the above error equation and using (14) with some rearrangement of terms, we obtain

$$\begin{aligned}
&\int_\Omega \frac{\partial \check{e}}{\partial t}\check{e}dx + a(e,\check{e}) - \int_0^t b(t,s;e(s),\check{e})ds \\
&\quad = \int_\Omega (f - \bar{\partial}U_h^n)(\check{e} - \Pi_h\check{e})dx - a(U_h, \check{e} - \Pi_h\check{e}) + \int_0^t b(t,s;U_h(s), \check{e} - \Pi_h\check{e})ds \\
&\qquad - (t - t_{n-1/2})\int_\Omega \check{w}_h^n(\check{e} - \Pi_h\check{e})dx + \int_\Omega f\Pi_h\check{e}dx + \int_0^t b(t,s;\check{U}_h(s), \Pi_h\check{e})ds \\
&\qquad - a(U_h, \Pi_h\check{e}) - \int_\Omega \frac{\partial \check{U}_h}{\partial t}\Pi_h\check{e}dx + \int_0^t b(t,s;U_h(s) - \check{U}_h(s), \Pi_h\check{e})ds.
\end{aligned}$$

Using (17) and Lemma 4.4, the above equation leads to

(52)

$$\begin{aligned}
&\int_\Omega \frac{\partial \check{e}}{\partial t}\check{e}dx + a(e,\check{e}) - \int_0^t b(t,s;e(s),\check{e})ds \\
&\quad = \int_\Omega (f - \bar{\partial}U_h^n)(\check{e} - \Pi_h\check{e})dx - a(U_h, \check{e} - \Pi_h\check{e}) + \int_0^t b(t,s;U_h(s), \check{e} - \Pi_h\check{e})ds \\
&\qquad - (t - t_{n-1/2})\int_\Omega \check{w}_h^n(\check{e} - \Pi_h\check{e})dx + \int_\Omega (f - \check{f})\Pi_h\check{e}dx \\
&\qquad + \left[\int_0^{t_{n-1/2}} b(t_{n-1/2}, s;\check{U}_h(s), \Pi_h\check{e})ds - \sigma^n(b(t_{n-1/2};U_h, \Pi_h\check{e}))\right] \\
&\qquad + \int_0^t b(t,s;U_h(s) - \check{U}_h(s), \Pi_h\check{e})ds + \langle \check{\phi}(t) - \phi_1(t), \nabla\Pi_h\check{e}\rangle \\
&\qquad + (t - t_{n-1/2})\int_0^{t_n} \bar{\partial}b(t_n, s;\check{U}_h(s) - U_h(s), \Pi_h\check{e})ds.
\end{aligned}$$

Now, we integrate (52) by parts on each of the elements K of $\mathcal{T}_h$ and substitute the estimate (45) with $\check{U}$ replacing $\hat{U}$. Then, using Lemma 4.5, (3), (6) and (2) together with the Cauchy-Schwarz inequality and the Poincaré inequality, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\check{e}(t)\|^2 + a(e,\check{e}) \le \gamma\|\nabla\check{e}(t)\|\int_0^t \|\nabla e(s)\|ds \\
&\quad + \sum_{K\in\mathcal{T}_h}\Bigg\{\Bigg\| f - \bar{\partial}U_h^n + \nabla\cdot(A\nabla U_h) \\
&\qquad\qquad - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\Bigg\|_{L^2(K)} \|\check{e} - \Pi_h\check{e}\|_{L^2(K)} \\
&\quad + \frac{1}{2}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}\|\check{e} - \Pi_h\check{e}\|_{L^2(\partial K)} \\
&\quad + \frac{1}{2}\left\|\left[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds\right]\right\|_{L^2(\partial K)}\|\check{e} - \Pi_h\check{e}\|_{L^2(\partial K)} \\
&\quad + |t - t_{n-1/2}|\|\check{w}_h^n\|_{L^2(K)}\|\check{e} - \Pi_h\check{e}\|_{L^2(K)} + C_2\|f - \check{f}\|_{L^2(K)}\|\nabla\Pi_h\check{e}\|_{L^2(K)}
\end{aligned}$$

$$+ \bar{\gamma}_1 \breve{\theta}_{n,K} \|\nabla \Pi_h \breve{e}\|_{L^2(K)} + \frac{\gamma'''}{12}|t - t_{n-1/2}|\breve{\zeta}_{n,K} \|\nabla \Pi_h \breve{e}\|_{L^2(K)}$$
$$+ \frac{\gamma}{12}\Big[|l_{n-1}(t)|\breve{\zeta}_{n-1,K} + |l_n(t)|\breve{\zeta}_{n,K}\Big] \|\nabla \Pi_h \breve{e}\|_{L^2(K)}$$
$$+ \|\breve{\psi}(t) - \psi_1(t)\|_{L^2(K)} \|\nabla \Pi_h \breve{e}\|_{L^2(K)} + \|\breve{\phi}(t) - \phi_1(t)\|_{L^2(K)} \|\nabla \Pi_h \breve{e}\|_{L^2(K)} \Bigg\},$$

where C_2 is a constant in the Poincaré inequality and $\breve{\theta}_{n,K}$ is given by (29). Now, we use the fact $a(e, \breve{e}) = \frac{1}{2}\Big[a(e,e) + a(\breve{e},\breve{e}) - a(e-\breve{e}, e-\breve{e})\Big]$ together with the coercivity and continuity of the bilinear form $a(\cdot,\cdot)$. Then, an application of Young's inequality and Proposition 2.1 together with $\|\nabla \Pi_h \breve{e}\|_{L^2(K)} \le C_3 \|\nabla \breve{e}\|_{L^2(K)}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\breve{e}(t)\|^2 + \frac{\alpha}{2}\|\nabla e\|^2 + \frac{\alpha}{2}\|\nabla \breve{e}\|^2 \le \frac{\gamma}{2\nu}\|\nabla \breve{e}\|^2 + \frac{\gamma C_Y(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds$$
$$+ \sum_{K\in\mathcal{T}_h} \Bigg\{ C_1 \Bigg[\Big\| f - \bar{\partial} U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds \Big\|_{L^2(K)}$$
$$+ \frac{1}{2\lambda_{2,K}^{1/2}} \|[A\nabla U_h \cdot \mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}} \Big\| \Big[\int_0^t B(t,s)\nabla U_h(s)\cdot \mathbf{n} ds \Big] \Big\|_{L^2(\partial K)} \Bigg] \omega_K(\breve{e})$$
$$+ C_1|t - t_{n-1/2}| \|\breve{w}_h^n\|_{L^2(K)} \omega_K(\breve{e}) + C_2 C_3 \|f - \breve{f}\|_{L^2(K)} \|\nabla \breve{e}\|_{L^2(K)}$$
$$+ \frac{\beta}{4}(t - t_{n-1})^2 (t - t_n)^2 \|\nabla \breve{w}_h^n\|_{L^2(K)}^2 + C_3 \bar{\gamma}_1 \breve{\theta}_{n,K} \|\nabla \breve{e}\|_{L^2(K)}$$
$$+ \frac{C_3\gamma'''}{12}|t - t_{n-1/2}|\breve{\zeta}_{n,K} \|\nabla \breve{e}\|_{L^2(K)}$$
$$+ \frac{\gamma C_3}{12}\Big[|l_{n-1}(t)|\breve{\zeta}_{n-1,K} + |l_n(t)|\breve{\zeta}_{n,K}\Big] \|\nabla \breve{e}\|_{L^2(K)}$$
$$+ C_3\|\breve{\psi}(t) - \psi_1(t)\|_{L^2(K)} \|\nabla \breve{e}\|_{L^2(K)} + C_3\|\breve{\phi}(t) - \phi_1(t)\|_{L^2(K)} \|\nabla \breve{e}\|_{L^2(K)} \Bigg\},$$

where C_1 is a constant in Proposition 2.1, $C_Y(T)$ is the constant which appeared due to the application of Young's inequality and the Cauchy-Schwarz inequality. Here, we have used the following relation:

$$\|\nabla(e - \breve{e})\|^2 = \|\nabla(\breve{U}_h - U_h)\|^2 = \frac{1}{4}(t - t_{n-1})^2 (t - t_n)^2 \|\nabla \breve{w}_h^n\|^2.$$

Now, an application of Young's inequality along with $\omega_K(\breve{e}) \le C_4 \lambda_{2,K} \|\nabla \breve{e}\|_{L^2(K)}$ (which follows trivially from Proposition 2.1 under the error equidistribution assumption (32)) yields

$$\frac{1}{2}\frac{d}{dt}\|\breve{e}(t)\|^2 + \frac{\alpha}{2}\|\nabla e\|^2 + \frac{\alpha}{2}\|\nabla \breve{e}\|^2 \le \frac{\gamma}{2\nu}\|\nabla \breve{e}\|^2 + \frac{\gamma C_Y(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds$$
$$+ \sum_{K\in\mathcal{T}_h} \Bigg\{ C_1 \Bigg[\Big\| f - \bar{\partial} U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds \Big\|_{L^2(K)}$$
$$+ \frac{1}{2\lambda_{2,K}^{1/2}} \|[A\nabla U_h \cdot \mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}} \Big\| \Big[\int_0^t B(t,s)\nabla U_h(s)\cdot \mathbf{n} ds \Big] \Big\|_{L^2(\partial K)} \Bigg] \omega_K(\breve{e})$$

$$
\begin{aligned}
&+ \frac{\nu}{2}(t-t_{n-1/2})^2\lambda_{2,K}^2\|\check{w}_h^n\|_{L^2(K)}^2 + \frac{\nu}{2}\|f-\check{f}\|_{L^2(K)}^2 \\
&+ \frac{1}{2\nu}(C_1^2C_4^2 + C_2^2C_3^2 + C_3^2(\gamma''')^2 + C_3^2\bar{\gamma}_1^2 + 2C_3^2\gamma^2 + 2C_3^2)\|\nabla\check{e}\|_{L^2(K)}^2 + \frac{\nu}{2}\check{\theta}_{n,K}^2 \\
&+ \frac{\nu}{288}(t-t_{n-1/2})^2\check{\zeta}_{n,K}^2 + \frac{\beta}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla\check{w}_h^n|_{L^2(K)}^2 \\
&+ \frac{\nu}{288}\Big[l_n^2(t)\check{\zeta}_{n,K}^2 + l_{n-1}^2(t)\check{\zeta}_{n-1,K}^2\Big] + \frac{\nu}{2}\|\check{\psi}(t)-\psi_1(t)\|_{L^2(K)}^2 \\
&\qquad\qquad\qquad\qquad + \frac{\nu}{2}\|\check{\phi}(t)-\phi_1(t)\|_{L^2(K)}^2\bigg\}.
\end{aligned}
$$

Choosing $\nu = \frac{1}{\alpha}(C_1^2C_4^2 + C_2^2C_3^2 + C_3^2(\gamma''')^2 + C_3^2\bar{\gamma}_1^2 + 2C_3^2\gamma^2 + 2C_3^2 + \gamma)$, we apply Gronwall's lemma and then integrate from t_{n-1} to t_n to obtain

$$
\begin{aligned}
&\|\check{e}(t_n)\|^2 + \alpha\int_{t_{n-1}}^{t_n}\|\nabla e\|^2dt \le \|\check{e}(t_{n-1})\|^2 \\
&+ C\sum_{K\in\mathcal{T}_h}\Bigg\{\int_{t_{n-1}}^{t_n}\Bigg[\bigg\|f - \bar{\partial}U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\bigg\|_{L^2(K)} \\
&+ \frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}}\bigg\|\Big[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds\Big]\bigg\|_{L^2(\partial K)}\Bigg]\omega_K(\check{e})dt \\
&+ \tau_n^3\lambda_{2,K}^2\|\check{w}_h^n\|_{L^2(K)}^2 + \int_{t_{n-1}}^{t_n}\|f-\check{f}\|_{L^2(K)}^2dt + \tau_n\check{\theta}_{n,K}^2 + \tau_n^3\check{\zeta}_{n,K}^2 + \tau_n^5\|\nabla\check{w}_h^n\|_{L^2(K)}^2 \\
&+ \tau_n\Big[\check{\zeta}_{n,K}^2 + \check{\zeta}_{n-1,K}^2 + \|\check{\phi}(t)-\phi_1(t)\|_{L^2(K)}^2 + \|\check{\psi}(t)-\psi_1(t)\|_{L^2(K)}^2\Big]\Bigg\},
\end{aligned}
$$

where $C = \max\Big\{2C_1C(T), C(T)\nu, \beta C(T)\Big\}$ with $C(T)$ denoting Gronwall's constant. Since $e(t_n) = \check{e}(t_n)\ \forall n$, we sum over $n = 1$ to N to complete the proof. □

Remark 1. Note that the error estimators derived in Theorems 4.1 and 4.2 are not traditional *a posteriori* error estimates because they contain the terms $\omega_K(\check{e})$ and $\omega_K(\hat{e})$ which involves the gradient of the exact solution u. It is known that Z-Z like error estimators (cf. [23, 24]) are asymptotically exact for smooth solutions; see [10–12, 15] for elliptic and parabolic problems. For the purpose of approximating the terms $\omega_K(\check{e})$ and $\omega_K(\hat{e})$, we now recall from [23, 24] the following Z-Z error estimator:

$$
\zeta^{ZZ}(U_h) = \begin{pmatrix} \zeta_1^{ZZ}(U_h) \\ \zeta_2^{ZZ}(U_h) \end{pmatrix} = \begin{pmatrix} (I - I_h^A)\Big(\frac{\partial U_h}{\partial x_1}\Big) \\ (I - I_h^A)\Big(\frac{\partial U_h}{\partial x_2}\Big) \end{pmatrix},
$$

where I_h^A is an approximate L^2 projection operator onto V_h and is defined by its values at each vertex P as

$$
I_h^A(\nabla U_h)(P) = \begin{pmatrix} I_h^A\Big(\frac{\partial U_h}{\partial x_1}\Big)(P) \\ I_h^A\Big(\frac{\partial U_h}{\partial x_2}\Big)(P) \end{pmatrix} = \frac{1}{\sum\limits_{P\in K, K\in\mathcal{T}_h}|K|}\begin{pmatrix} \sum\limits_{P\in K,K\in\mathcal{T}_h}|K|\Big(\dfrac{\partial U_h}{\partial x_1}\Big)|_K \\ \sum\limits_{P\in K,K\in\mathcal{T}_h}|K|\Big(\dfrac{\partial U_h}{\partial x_2}\Big)|_K \end{pmatrix}.
$$

Therefore, in Theorems 4.1 and 4.2, replacing the matrices $G_K(\check{e})$ and $G_K(\hat{e})$ in $\omega_K(\check{e})$ and $\omega_K(\hat{e})$, respectively, by the matrix $\breve{G}_K(U_h)$ we recover the usual *a posteriori* error estimators. The matrix $\breve{G}_K(v_h)$, for any $v_h \in V_h$, is defined by

$$\breve{G}_K(v_h) = \begin{pmatrix} \displaystyle\int_K \left(\zeta_1^{ZZ}(v_h)\right)^2 dx & \displaystyle\int_K \zeta_1^{ZZ}(v_h)\zeta_2^{ZZ}(v_h)dx \\ \displaystyle\int_K \zeta_1^{ZZ}(v_h)\zeta_2^{ZZ}(v_h)dx & \displaystyle\int_K \left(\zeta_2^{ZZ}(v_h)\right)^2 dx \end{pmatrix}.$$

Remark 2. One can recover the isotropic *a posteriori* error estimators from the anisotropic *a posteriori* error estimators of Theorems 4.1 and 4.2. In the case of isotropic mesh ($\lambda_{1,K} \simeq \lambda_{2,K} \simeq h_K$), Theorems 4.1 and 4.2 take the form

$$\begin{aligned}
&\|e(.,T)\|^2 + \alpha \int_0^T \|\nabla e\|^2 dt \le \|e(.,0)\|^2 \\
&\quad + C \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h} \Bigg\{ \int_{t_{n-1}}^{t_n} \Bigg[h_K^2 \Big\| f - \bar{\partial} U_h^n + \nabla\cdot(A\nabla U_h) \\
&\qquad\qquad - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds \Big\|_{L^2(K)} \\
&\quad + h_K \|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + h_K \Big\| \Big[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds \Big] \Big\|_{L^2(\partial K)} \Bigg] dt \\
&\quad + \tau_n^3 h_K^2 \|\check{w}_h^n\|_{L^2(K)}^2 + \int_{t_{n-1}}^{t_n} \|f - \check{f}\|_{L^2(K)}^2 dt + \tau_n \check{\theta}_{n,K}^2 + \tau_n^3 \check{\zeta}_{n,K}^2 + \tau_n^5 \|\nabla \check{w}_h^n\|_{L^2(K)}^2 \\
&\quad + \tau_n \Big[\check{\zeta}_{n,K}^2 + \check{\zeta}_{n-1,K}^2 + \|\check{\phi}(t) - \phi_1(t)\|_{L^2(K)}^2 + \|\check{\psi}(t) - \psi_1(t)\|_{L^2(K)}^2 \Big] \Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
&\|e(.,T)\|^2 + \alpha \int_{t_1}^T \|\nabla e\|^2 dt \le \|e(.,t_1)\|^2 \\
&\quad + C \sum_{n=2}^{N} \sum_{K\in\mathcal{T}_h} \Bigg\{ \int_{t_{n-1}}^{t_n} \Bigg[h_K^2 \Big\| f - \bar{\partial} U_h^n + \nabla\cdot(A\nabla U_h) \\
&\qquad\qquad - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds \Big\|_{L^2(K)} \\
&\quad + h_K \|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + h_K \Big\| \Big[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds \Big] \Big\|_{L^2(\partial K)} \Bigg] dt \\
&\quad + \tau_n^3 h_K^2 \|\hat{w}_h^n\|_{L^2(K)}^2 + \int_{t_{n-1}}^{t_n} \|f - \hat{f}\|_{L^2(K)}^2 dt \\
&\quad + \tau_n \Big[\theta_{n,K}^2 + \theta_{n-1,K}^2 + \|\hat{\phi}(t) - \phi_2(t)\|_{L^2(K)}^2 \Big] + \Big\{ \tau_{n-1}^2 \tau_n^3 + \tau_n^5 \Big\} \|\nabla \hat{w}_h^n\|_{L^2(K)}^2 \\
&\quad + \tau_n \Big[\zeta_{n,K}^2 + \zeta_{n-1,K}^2 + \|\hat{\psi}(t) - \psi_2(t)\|_{L^2(K)}^2 \Big] \Bigg\},
\end{aligned}$$

where the contributions $\breve{\theta}_{n,K}$, $\breve{\zeta}_{n,K}$, $\theta_{n,K}$ and $\zeta_{n,K}$ are given by (29), (30), (33) and (34), respectively.

Remark 3. When u is smooth enough, the error e in the $L^2(H^1)$-norm is $O(h+\tau^2)$ for the Crank-Nicolson scheme. Thus, the terms related to the data oscillation, interior element residual and the jump residual in Theorems 4.1 and 4.2 are of the optimal order. The rest of the terms along with the data oscillation term estimate the error due to the time discretization in both the error estimators. In particular, the error indicators $\breve{\theta}_{n,K}$ and $\theta_{n,K}$ in the estimators account for the contributions coming from each of the element K due to quadrature approximation of the Volterra integral term whereas the terms $\breve{\zeta}_{n,K}$ and $\zeta_{n,K}$ convey the contributions for the reconstruction errors coming from each of the elements K. Since the error indicators $\breve{\theta}_{n,K}$, $\theta_{n,K}$, $\breve{\zeta}_{n,K}$ and $\zeta_{n,K}$ are formally of optimal order, the terms associated with these indicators give the desired rate of convergence. The term $\left\{\tau_{n-1}^2\tau_n^3+\tau_n^5\right\}\|\nabla\hat{w}_h^n\|_{L^2(K)}^2$ in the second estimator is of optimal order provided $\sum_{n=1}^{N}\tau_n\|\nabla\hat{w}_h^n\|_{L^2(K)}^2$ is bounded with respect to h and τ. A similar argument holds for the term $\tau_n^5\|\nabla\breve{w}_h^n\|_{L^2(K)}^2$ which appears in Theorem 4.1. Also, in view of the error behaviour which is of $O(h+\tau^2)$, if we take h proportional to τ^2, the similar terms $\tau_n^3h_K^2\|\breve{w}_h^n\|_{L^2(K)}^2$, $\tau_n^3h_K^2\|\hat{w}_h^n\|_{L^2(K)}^2$ which appears in Theorem 4.1 and Theorem 4.2, respectively, are of higher order. Moreover, the terms $\|\breve{\phi}(t)-\phi_1(t)\|_{L^2(K)}^2$, $\|\hat{\phi}(t)-\phi_2(t)\|_{L^2(K)}^2$, $\|\breve{\psi}(t)-\psi_1(t)\|_{L^2(K)}^2$ and $\|\hat{\psi}(t)-\psi(t)\|_{L^2(K)}^2$ are *a posteriori* quantities and are eventually of optimal order arises due to linear interpolations.

Remark 4. The introduction of the Z-Z error estimator in Theorem 4.2 is not sufficient to provide a meaningful *a posteriori* error estimator due to the presence of the term $\|e(.,t_1)\|^2$ on right-hand side of Theorem 4.2. Since the error bound in Theorem 4.1 is of optimal order in the $L^2(H^1)$-norm, the optimality of the *a posteriori* error estimate in Theorem 4.2 is thus justified by using the estimate of $\|e(.,t_1)\|^2$ from Theorem 4.1.

Remark 5. When $\mathcal{B}(t,s)=0$, the error estimators in Theorems 4.1 and 4.2 are similar to that of the purely parabolic problems [10]. Since PIDE (1) may be thought of as a perturbation to the parabolic problem, it is natural to expect that our *a posteriori* error estimators should reflect back the quadrature error coming from the approximation of the memory term. The terms $\breve{\theta}_{n,K}$ and $\theta_{n,K}$ are indicators for the quadrature errors which are of optimal order. Therefore, the results obtained in Theorems 4.1 and 4.2 generalize the results of the parabolic problems [10] to PIDE.

Remark 6. The assumption (28) or (32) implies that the error gradient in the direction of maximum stretching is less than the error gradient in the direction of minimum stretching. For instance, the condition

$$\lambda_{1,K}^2(r_{1,K}^TG_K(e)r_{1,K})\leq\lambda_{2,K}^2(r_{2,K}^TG_K(e)r_{2,K})\quad\forall K\in\mathcal{T}_h$$

means that the error in both the directions of minimum and maximum stretching is equidistributed. Moreover, in [12–14] the authors have shown through the numerical experiments that the effectivity index (estimated error/true error) blows up when the mesh is refined in one direction only. Thus, the assumptions (28) or (32) are going to play a crucial role for the adaptive algorithms to be developed for PIDE on anisotropic meshes.

Concluding remarks

Despite the importance of the PIDE and their variants in the modeling of several physical phenomena, the topic of *a posteriori* error analysis for such types of equations remains unexplored. In this paper, we derived optimal order residual based anisotropic *a posteriori* error estimates for the PIDE (1) in the $L^2(H^1)$-norm for the fully discrete Crank-Nicolson scheme. However, we note that one can obtain optimal isotropic *a posteriori* error estimates in the $L^2(H^1)$-norm for the PIDE (1) through the anisotropic estimates presented in this paper. We believe the work presented here could be a first step towards the development of various space-time adaptive algorithms for PIDE on different meshes. Moreover, this work unlocks several new research directions. It is challenging to study the problem of obtaining *a posteriori* error estimates with the constants in the bounds being independent of the final time T and hence, they can serve as long-time estimates. Further, *a posteriori* error analysis for hyperbolic integro-differential equations is an interesting research problem which will be reported elsewhere.

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Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati - 781039, India

E-mail address: gmuralireddy1984@gmail.com

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati - 781039, India

E-mail address: rajen@iitg.ernet.in