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MORE ON STOCHASTIC AND VARIATIONAL APPROACH TO THE LAX-FRIEDRICHS SCHEME

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ABSTRACT. A stochastic and variational aspect of the Lax-Friedrichs scheme applied to hyperbolic scalar conservation laws and Hamilton-Jacobi equations generated by space-time dependent flux functions of the Tonelli type was clarified by Soga (2015). The results for the Lax-Friedrichs scheme are extended here to show its time-global stability, the large-time behavior, and error estimates. Also provided is a weak KAM-like theorem for discrete equations that is useful in the numerical analysis and simulation of the weak KAM theory. As one application, a finite difference approximation to effective Hamiltonians and KAM tori is rigorously treated. The proofs essentially rely on the calculus of variations in the Lax-Friedrichs scheme and on the theory of viscosity solutions of Hamilton-Jacobi equations.

1. Introduction

We investigate the Lax-Friedrichs scheme applied to initial value problems of hyperbolic scalar conservation laws with a constant c,

$$u_t + H(x,t,c+u)_x = 0. \tag{1.1}$$

There is a vast literature on the stability and convergence of the scheme. The standard technique is based on the L^1-framework with a priori estimates and the compactness of functions of bounded variation, where mesh-size independent boundedness of both the difference solutions and their total variation must be verified. Since the Lax-Friedrichs scheme is very simple, details of approximation can be successfully analyzed, particularly in the case of a flux function of the simple form $H(x,t,p) = H(p)$. We refer to [6], [19], [24], and the studies cited therein. However, in the case of a general flux function depending on both x and t, the problem becomes far more difficult and often requires undesirable assumptions. The results of the general case first appeared in [18], where stability and L^1-convergence are proved with a restricted time interval that is determined by the growth of $H(x,t,p)$ with respect to p. In [17], time-global stability and L^1-convergence within arbitrary time intervals are proved for a flux function of the form $H(x,t,p) = f(p) + F(x,t)$ in the periodic setting, with many details of the large-time behavior of the Lax-Friedrichs scheme. Still, it seems very difficult to obtain results similar to those in [17] for more general flux functions by the standard approach based on the L^1-framework.

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Recently, a stochastic and variational approach to the Lax-Friedrichs scheme was announced in [22]. Stability and convergence were proved on the basis of: (i) the *law of large numbers* in the hyperbolic scaling limit of random walks, and (ii) the *calculus of variations* in the theory of viscosity solutions of the Hamilton-Jacobi equations with constants c and $h(c)$,

$$v_t + H(x,t,c+v_x) = h(c). \tag{1.2}$$

This is a finite difference version of the stochastic and variational approach to the vanishing viscosity method in [11]. Now we briefly review the stochastic and variational approach in [22]. Consider initial value problems of the inviscid hyperbolic scalar conservation law and the corresponding Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x,t,c+u)_x = 0 \ \text{ in } \mathbb{T}\times(0,T],\\ u(x,0) = u^0(x) \in L^\infty(\mathbb{T}) \ \text{ on } \mathbb{T},\\ \displaystyle\int_{\mathbb{T}} u^0(x)dx = 0, \quad \parallel u^0 \parallel_{L^\infty} \le r, \end{cases} \tag{1.3}$$

$$\begin{cases} v_t + H(x,t,c+v_x) = h(c) \ \text{ in } \mathbb{T}\times(0,T],\\ v(x,0) = v^0(x) \in \mathrm{Lip}(\mathbb{T}) \ \text{ on } \mathbb{T}, \quad \parallel v^0_x \parallel_{L^\infty} \le r, \end{cases} \tag{1.4}$$

where $c \in [c_0, c_1]$ is a varying parameter, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the standard torus, $h(c)$ is an arbitrary continuous function, and $r > 0$ is a constant. We arbitrarily fix T, r, and $[c_0, c_1]$. Note that (1.3) and (1.4) are equivalent in the sense that the entropy solution u or viscosity solution v is derived from the other if $u^0 = v^0_x$. In particular, we have $u = v_x$ (see, e.g., [1]). Hereinafter we assume that $u^0 = v^0_x$. The flux function H is assumed to satisfy the following:

(A1) $H(x,t,p) : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$, C^2,

(A2) $H_{pp} > 0$,

(A3) $\displaystyle\lim_{|p|\to+\infty} \frac{H(x,t,p)}{|p|} = +\infty$.

From (A1)–(A3) we obtain the Legendre transform $L(x,t,\xi)$ of $H(x,t,\cdot)$, which is given by

$$L(x,t,\xi) = \sup_{p\in\mathbb{R}}\{\xi p - H(x,t,p)\}$$

and satisfies

(A1)′ $L(x,t,\xi) : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$, C^2,

(A2)′ $L_{\xi\xi} > 0$,

(A3)′ $\displaystyle\lim_{|\xi|\to+\infty} \frac{L(x,t,\xi)}{|\xi|} = +\infty$.

The final assumption is the following:

(A4) There exists $\alpha > 0$ such that $|L_x| \le \alpha(|L| + 1)$.

Note that (A4) implies completeness of the Euler-Lagrange flow generated by L and Hamiltonian flow generated by H. Such a function H is common in Hamiltonian dynamics and is called a Tonelli Hamiltonian.

We discretize the equation in (1.3) by the Lax-Friedrichs scheme as follows:

$$\frac{u^{k+1}_{m+1} - \frac{(u^k_m + u^k_{m+2})}{2}}{\Delta t} + \frac{H(x_{m+2}, t_k, c + u^k_{m+2}) - H(x_m, t_k, c + u^k_m)}{2\Delta x} = 0. \tag{1.5}$$

We also discretize the equation in (1.4) by the following scheme:

$$(1.6)\qquad \frac{v_m^{k+1}-\frac{(v_{m-1}^k+v_{m+1}^k)}{2}}{\Delta t}+H(x_m,t_k,c+\frac{v_{m+1}^k-v_{m-1}^k}{2\Delta x})=h(c).$$

Note that (1.5) and (1.6) are also equivalent in the sense that u_m^k or v_{m+1}^k is derived from the other. In particular, we have

$$u_m^k=\frac{v_{m+1}^k-v_{m-1}^k}{2\Delta x},$$

which is an important relation in this paper. In the stochastic and variational approach, the stochastic feature comes from the numerical viscosity intrinsic to (1.5) and (1.6), while the variational feature comes from the variational structures of Hamilton-Jacobi equations. The stochastic and variational approach in [22] led to several results:

(1) Stochastic and variational representation formulas (value functions) for v_{m+1}^k and u_m^k were obtained.
(2) Stability of the Lax-Friedrichs scheme up to arbitrary $T>0$ was derived by variational techniques.
(3) Pointwise convergence of u_m^k to $u=v_x$ was proved almost everywhere. In particular, this yielded uniform convergence, except for neighborhoods of shocks with arbitrarily small measure.
(4) Uniform convergence of v_{m+1}^k to v with an error $O(\sqrt{\Delta x})$ was proved from a stochastic viewpoint.
(5) Random walks played a role as characteristic curves of the difference equations, which converged to the genuine characteristic curves of (1.1) and (1.2).

The purpose of this paper is to show further results for the Lax-Friedrichs scheme on the basis of (1)–(5) under the assumptions (A1)–(A4) with techniques from the theory of viscosity solutions of Hamilton-Jacobi equations. We refer only to the results for the Lax-Friedrichs scheme, but similar results for other finite difference schemes with numerical viscosity (e.g., the upwind/downwind scheme) are available as well. The main results are on the following:

- Time-global stability of the Lax-Friedrichs scheme with a fixed mesh size.
- Error estimates for entropy solutions.
- Application to the weak KAM theory.

It is proved that genuine entropy solutions at $t=1$ are uniformly bounded, regardless of the magnitudes of the initial data. Since the genuine solutions are well approximated by the difference entropy solutions for small mesh sizes, the difference entropy solutions at $t=1$ are also uniformly bounded. Due to the periodic setting, iteration of the time-1 analysis yields time-global properties. Combining these facts, we obtain time-global stability of the Lax-Friedrichs scheme. As a result, we can show that the large-time behavior of the Lax-Friedrichs scheme is such that any solutions associated with each c fall into the time periodic state uniquely determined by each c. This means that for each c we obtain the unique space-time periodic difference entropy solution and the unique (up to a constant) space-time periodic difference viscosity solution. These periodic difference solutions approximate the genuine $\mathbb{Z}^2$-periodic entropy (resp. viscosity) solutions of (1.1) (resp. (1.2)). For the periodic states, we naturally have the notion of the effective Hamiltonian for the difference Hamilton-Jacobi equation (1.6). We reveal its properties and prove that it converges to the effective Hamiltonian for the exact equation (1.2) with an error estimate of $O(\sqrt{\Delta x})$.

It is known that the optimal estimate of the L^1-error between u^k_m and u is $O(\sqrt{\Delta x})$ in the case of $H(x,t,p) = H(p)$ [19]. The upper bound $O(\sqrt{\Delta x})$ is due to properties of functions of bounded variation [14]. It is not clear whether the result in [14] is applicable to the case of our general flux functions. Through a different approach, we obtain an L^1-error estimate of $O(\Delta x^{\frac{1}{4}})$. This error estimate is based on the order $O(\sqrt{\Delta x})$ of the "error" between random walks and their space-time continuous limit (i.e., a backward characteristic curve) under hyperbolic scaling. For a technical reason, we lose the exponent $1/4$ in the case of the general flux function $H(x,t,p)$. In addition, we show that, if the genuine entropy solution is Lipschitz, a C^0-error estimate of $O(\Delta x^{\frac{1}{4}})$ is available.

Unlike the case for initial value problems, it is challenging to show convergence of full sequences and estimate the error for $\mathbb{Z}^2$-periodic entropy (resp. viscosity) solutions of (1.1) (resp. (1.2)), because the uniqueness of such genuine $\mathbb{Z}^2$-periodic solutions with respect to c is not valid in general. However, we can manage the special case in which a genuine $\mathbb{Z}^2$-periodic entropy solution $\bar{u}$ with some c is C^1 and the dynamics of its characteristic curves $C^*(s) := (q(s) \mod 1, s \mod 1)$ is C^1-conjugate to the dynamics of the linear flow on $\mathbb{T}^2$ with a Diophantine rotation vector. Such a solution $\bar{u}$ is known as a *KAM* torus in Hamiltonian dynamics (e.g., see [13], [16], [20]). We show a C^0-error estimate depending on the Diophantine nature of the rotation vector, which is a rigorous result on finite difference approximation of KAM tori. Our proof is based on the fact that one orbit of the linear flow on $\mathbb{T}^2$ with a Diophantine rotation vector is ergodic on $\mathbb{T}^2$ and hence so is each $C^*(s)$.

Finally, we note that our motivation comes not only from the viewpoint of PDEs in continuum mechanics but also from the recent theory of Lagrangian and Hamiltonian dynamics that is called the Aubry-Mather theory or the weak KAM theory [9], [8], [13]. Our periodic setting is standard, and $\mathbb{Z}^2$-periodic entropy (resp. viscosity) solutions of (1.1) (resp. (1.2)) and the effective Hamiltonian play central roles in the weak KAM theory. The results of this paper provide basic tools for numerical analysis of the weak KAM theory through finite difference approximation. We remark that from the standpoint of accuracy it is better to approximate entropy solutions and characteristic curves as well as viscosity solutions, because the central objects in the weak KAM theory, such as KAM tori, Aubry-Mather sets, effective Hamiltonians, and calibrated curves, are obtained from the derivatives of viscosity solutions or entropy solutions. The "derivatives" of approximate viscosity solutions obtained through a scheme that has no relation to a scheme for entropy solutions, are not accurate in general. Some developments in finite difference approximation methods and numerical simulations for the weak KAM theory are found in [17]. However, the results therein are mathematically restricted by the absence of the stochastic and variational approach to the Lax-Friedrichs scheme. We also point to [2] and [13] for results on smooth approximation methods for the weak KAM theory based on the vanishing viscosity method. In particular, [2] successfully applies the stochastic and variational approach to the vanishing viscosity method given in [11], where the genuine characteristic curves are approximated by solutions of stochastic ODEs with the standard Brownian motion.

The advantage of our stochastic and variational approach is that structures and properties similar to those of the exact equations (1.1) and (1.2) are available in the most common finite difference schemes, which provide much more information on the schemes. In particular, we can trace genuine characteristic curves by means of random walks. This provides further development of finite difference approximation methods for the classical and weak KAM theories.

2. Preliminary results

In this section, we state several important preliminary results.

2.1. **Entropy solution and viscosity solution.** It is well known that the viscosity solution v of (1.4) is Lipschitz and is characterized by the calculus of variations. The value of v at each point $(x,t)\in\mathbb{T}\times(0,T]$, $T\in(0,\infty)$ is given by

$$v(x,t)=\inf_{\gamma\in AC,\ \gamma(t)=x}\left\{\int_0^t L^{(c)}(\gamma(s),s,\gamma'(s))ds+v^0(\gamma(0))\right\}+h(c)t, \tag{2.1}$$

where AC is the family of absolutely continuous curves $\gamma:[0,t]\to\mathbb{T}$ and

$$L^{(c)}(x,t,\xi):=L(x,t,\xi)-c\xi$$

is the Legendre transform of $H(x,t,c+\cdot)$. We do not distinguish $\gamma:[0,t]\to\mathbb{T}$, AC from $\gamma(t)=\gamma(0)+\int_0^t\gamma'(s)ds:[0,t]\to\mathbb{R}$. This is allowed due to the periodic setting. We can find a minimizing curve γ^* of (2.1) that is a backward characteristic curve of (1.1) and (1.2) as well as a C^2-solution of the Euler-Lagrange equation generated by the Lagrangian $L^{(c)}$:

$$\frac{d}{dt}L^{(c)}_\xi(\gamma(s),s,\gamma'(s))=L^{(c)}_x(\gamma(s),s,\gamma'(s)). \tag{2.2}$$

On each minimizing curve, v is differentiable with respect to x:

$$v_x(\gamma^*(s),s)=L^{(c)}_\xi(\gamma^*(s),s,\gamma^{*\prime}(s)) \text{ for } 0<s<t. \tag{2.3}$$

We say that a point (x,t) is a regular point of v, or regular, if $v_x(x,t)$ exists. Since v is Lipschitz, almost every point is regular. In particular, if (x,t) is regular, the minimizing curve γ^* for (2.1) is unique and (2.3) holds for $s=t$. Usually, the entropy solution u of (1.3) is defined as an element of $C^0((0,T];L^1(\mathbb{T}))$. Here we always take the representative element given by v_x, which is still denoted by u. If (x,t) is regular and γ^* is the unique minimizing curve for $v(x,t)$, the value of the entropy solution $u=v_x$ at the point (x,t) is given by

$$u(x,t)=\int_0^t L^{(c)}_x(\gamma^*(s),s,\gamma^{*\prime}(s))ds+u^0(\gamma^*(0)),$$

where u^0 is assumed to be rarefaction-free,

$$\operatorname*{ess\,sup}_{x\neq y}\frac{u^0(x)-u^0(y)}{x-y}\le M \quad \text{for some } M>0 \text{ (one-sided Lipschitz condition)},$$

or, equivalently, v^0 is semiconcave:

$$v^0(x+h)+v^0(x-h)-2v^0(x)\le Mh^2 \text{ for all } x,h.$$

Otherwise, $u^0(\gamma^*(0))$ must be replaced with $L^{(c)}_\xi(\gamma^*(0),0,\gamma^{*\prime}(0))$. In particular, for any $\tau\in[0,t)$ we have

$$u(x,t)=\int_\tau^t L^{(c)}_x(\gamma^*(s),s,\gamma^{*\prime}(s))ds+L^{(c)}_\xi(\gamma^*(\tau),\tau,\gamma^{*\prime}(\tau)).$$

For more details on viscosity solutions and entropy solutions, see, e.g., [1] or [5].

We introduce the solution operators of (1.3) and (1.4) as follows:

$$\phi^t:L^\infty_{r,0}(\mathbb{T})\ni u^0\mapsto u(\cdot,t)\in L^\infty(\mathbb{T}),\quad \psi^t:\mathrm{Lip}_r(\mathbb{T})\ni v^0\mapsto v(\cdot,t)\in\mathrm{Lip}(\mathbb{T}),$$

where $L^\infty_{r,0}(\mathbb{T})$ is the set of all functions $u^0\in L^\infty(\mathbb{T})$ with $\parallel u^0\parallel_{L^\infty}\le r$ and $\int_{\mathbb{T}}u^0dx=0$, while $\mathrm{Lip}_r(\mathbb{T})$ is the set of all Lipschitz functions on $\mathbb{T}$ with a Lipschitz constant bounded by r. When we specify the value of c, we write $\phi^t(\cdot;c)$, $\psi^t(\cdot;c)$, $u^{(c)}$, $v^{(c)}$.

We would like to prove a priori boundedness of $u(x,t)=v_x(x,t)$. This is closely related to a priori compactness of minimizers for (2.1). We remark that a priori compactness of minimizers plays an important role in the Aubry-Mather theory and the weak KAM theory, and details are known for more general settings (e.g., [15], [12]). The basic assumptions for this are (A1)′–(A3)′ and completeness of the Euler-Lagrange flow. Here we adopt (A4), which is stronger than the completeness assumption. We need (A4) to obtain compactness of minimizers for our stochastic and variational problems, where these minimizers do not satisfy the Euler-Lagrange equation (2.2). In order to provide a self-contained treatment, we give brief proofs by modifying Section 4.1 of [10].

Proposition 2.1. *For each $t\in(0,T]$, there exists a constant $\beta_1(t)>0$ (independent of r, $c\in[c_0,c_1]$, and the initial data v^0,u^0) for which*

$$\parallel\phi^t(u^0;c)\parallel_{L^\infty}\le\beta_1(t),\quad \parallel\psi^t(v^0;c)_x\parallel_{L^\infty}\le\beta_1(t).$$

Proof. Fix $t\in(0,T]$. If (x,t) is regular, then (2.3) holds for $s=t$. Therefore, it is sufficient to estimate $L^{(c)}_\xi(\gamma^*(t),t,\gamma^{*\prime}(t))$ for each minimizing curve γ^* of (2.1), since (x,t) is regular for almost every $x\in\mathbb{T}$ with each fixed t and $u(x,t)=v_x(x,t)=L^{(c)}_\xi(\gamma^*(t),t,\gamma^{*\prime}(t))$. We now prepare two lemmas.

Lemma 2.2. *Let γ^* be a minimizing curve for $v(x,t)$. Set $y:=\gamma^*(0)$. Then, γ^* attains*

$$\inf_{\gamma\in AC,\gamma(t)=x,\gamma(0)=y}\int_0^t L^{(c)}(\gamma(s),s,\gamma'(s))ds.$$

Proof. If not, there exists $\gamma^\sharp$ such that

$$\int_0^t L^{(c)}(\gamma^\sharp(s),s,\gamma^{\sharp\prime}(s))ds<\int_0^t L^{(c)}(\gamma^*(s),s,\gamma^{*\prime}(s))ds.$$

Since $v^0(\gamma^\sharp(0))=v^0(y)=v^0(\gamma^*(0))$, we have

$$\int_0^t L^{(c)}(\gamma^\sharp(s),s,\gamma^{\sharp\prime}(s))ds+v^0(\gamma^\sharp(0))<\int_0^t L^{(c)}(\gamma^*(s),s,\gamma^{*\prime}(s))ds+v^0(\gamma^*(0)).$$

Therefore, γ^* is not a minimizing curve for $v(x,t)$, which is a contradiction. ☐

We define the following set:

$$\Gamma(t) := \Big\{\gamma^{(c)} \mid \gamma^{(c)} \text{ attains } \inf_{\gamma(t)=x,\gamma(0)=y}\int_0^t L^{(c)}(\gamma(s),s,\gamma'(s))ds, \\ x,y\in\mathbb{T}, c\in[c_0,c_1]\Big\}.$$

By Lemma 2.2, any minimizing curve γ^* for $v(x,t)$, $x\in\mathbb{T}$, belongs to $\Gamma(t)$.

Lemma 2.3. (1) *There exists a constant $C_1(t)>0$ such that for any $x,y\in\mathbb{T}$ we have a C^1-curve γ that satisfies*

$$\gamma(t)=x,\ \ \gamma(0)=y,\ \ \int_0^t L^{(c)}(\gamma(s),s,\gamma'(s))ds\le C_1(t).$$

In particular, any $\gamma^{(c)}\in\Gamma(t)$ satisfies

$$\int_0^t L^{(c)}(\gamma^{(c)}(s),s,\gamma^{(c)\prime}(s))ds\le C_1(t).$$

(2) *There exists a constant $C_2(t)>0$ such that for any $\gamma^{(c)}\in\Gamma(t)$ we have $\tau\in(0,t)$ depending on $\gamma^{(c)}$ which satisfies*

$$|\gamma^{(c)\prime}(\tau)|\le C_2(t).$$

(3) *There exists a constant $C_3(t)>0$ such that for any $\gamma^{(c)}\in\Gamma(t)$ we have*

$$|L^{(c)}_\xi(\gamma^{(c)}(s),s,\gamma^{(c)\prime}(s))|\le C_3(t) \text{ for all } s\in[0,t].$$

Proof. (1) Consider $\gamma(s):=x+\frac{x-y}{t}(s-t)$. Since $|x-y|\le 1$, we have $|\gamma'(s)|\le t^{-1}$. Therefore, we obtain

$$\int_0^t L^{(c)}(\gamma(s),s,\gamma'(s))ds\le \sup_{x,s\in\mathbb{T},|\xi|\le t^{-1},c\in[c_0,c_1]}|L^{(c)}(x,s,\xi)|t.$$

Set $C_1(t):=\sup_{x,s\in\mathbb{T},|\xi|\le t^{-1},c\in[c_0,c_1]}|L^{(c)}(x,s,\xi)|t$ and claim (1) is proved.

(2) Note that $\gamma^{(c)}$ is a C^2-solution of the Euler-Lagrange equation (2.2). Due to claim (1), we have $\tau\in(0,t)$ which satisfies

$$C_1(t)\ge\int_0^t L^{(c)}(\gamma^{(c)}(s),s,\gamma^{(c)\prime}(s))ds=L^{(c)}(\gamma^{(c)}(\tau),\tau,\gamma^{(c)\prime}(\tau))t.$$

By (A3), $|\gamma^{(c)\prime}(\tau)|$ must be bounded by a constant $C_2(t)$ independent of $\gamma^{(c)}\in\Gamma(t)$.

(3) It follows from (A1)–(A4) that there exists α_1 for which $|L^{(c)}_x|\le\alpha_1(|L^{(c)}|+1)$ for any $c\in[c_0,c_1]$ and that $L_*:=|\min\{0,\inf_{x,s,\xi,c}L^{(c)}\}|$ is bounded. We have $\tau^*\in[0,t]$, which attains the maximum of $|L^{(c)}_\xi(\gamma^{(c)}(s),s,\gamma^{(c)\prime}(s))|$ within $0\le s\le t$.

Suppose that $\tau^* \neq \tau$, where τ is the value in claim (2). Then, using the Euler-Lagrange equation (2.2), we have

$$\begin{aligned}
&|\int_\tau^{\tau^*} \frac{d}{dt} L^{(c)}_\xi(\gamma^{(c)}(s), s, \gamma^{(c)\prime}(s))ds| \\
= \ & |L^{(c)}_\xi(\gamma^{(c)}(\tau^*), \tau^*, \gamma^{(c)\prime}(\tau^*)) - L^{(c)}_\xi(\gamma^{(c)}(\tau), \tau, \gamma^{(c)\prime}(\tau))| \\
\le \ & \int_0^t |L^{(c)}_x(\gamma^{(c)}(s), s, \gamma^{(c)\prime}(s))|ds \\
\le \ & \int_0^t \alpha_1(1 + |L^{(c)}(\gamma^{(c)}(s), s, \gamma^{(c)\prime}(s))|)ds \\
\le \ & \alpha_1 \int_0^t 1 + (L^{(c)}(\gamma^{(c)}(s), s, \gamma^{(c)\prime}(s)) + L_*) + L_* ds \\
= \ & \alpha_1(2L_* + 1)t + \alpha_1 \int_0^t L^{(c)}(\gamma^{(c)}(s), s, \gamma^{(c)\prime}(s))ds \\
\le \ & \alpha_1(2L_* + 1)t + \alpha_1 C_1(t).
\end{aligned}$$

Therefore, setting

$$C_3(t) := \alpha_1(2L_* + 1)t + \alpha_1 C_1(t) + \sup_{x,s\in\mathbb{T},|\xi|\le C_2(t),c\in[c_0,c_1]} |L^{(c)}_\xi(x, s, \xi)|,$$

we obtain for all $0 \le s \le t$,

$$|L^{(c)}_\xi(\gamma^{(c)}(s), s, \gamma^{(c)\prime}(s))| \le |L^{(c)}_\xi(\gamma^{(c)}(\tau^*), \tau^*, \gamma^{(c)\prime}(\tau^*))| \le C_3(t).$$

The case $\tau^* = \tau$ is included in the above inequality. □

We obtain Proposition 2.1 by setting $\beta_1(t) := C_3(t)$. □

We show continuity of $\phi^t(v^0_x; c)$ and $\psi^t(v_0; c)$ with respect to v^0 and c.

Proposition 2.4. *Fix $t \in (0, T]$. For each sequence $v^0_j \to v^0$ uniformly and $c^j \to c$ as $j \to \infty$ ($v^0_{j\,x}$ is not necessarily convergent), we have*

$$\psi^t(v^0_j; c^j) \to \psi^t(v^0; c) \text{ uniformly}, \quad \phi^t(v^0_{j\,x}; c^j) \to \phi^t(v^0_x; c) \text{ in } L^1(\mathbb{T}) \quad \text{as } j \to \infty.$$

Proof. By the variational representation, we have minimizing curves γ^*, γ^*_j such that

$$\begin{aligned}
\psi^t(v^0; c)(x) &= \int_0^t L(\gamma^*(s), s, \gamma^{*\prime}(s)) - c\gamma^{*\prime}(s)ds + v^0(\gamma^*(0)) + h(c)t, \\
\psi^t(v^0_j; c^j)(x) &= \int_0^t L(\gamma^*_j(s), s, \gamma^{*\prime}_j(s)) - c^j\gamma^{*\prime}_j(s)ds + v^0_j(\gamma^*_j(0)) + h(c^j)t
\end{aligned}$$

and hence

$$\begin{aligned}
\psi^t(v^0_j; c^j)(x) - \psi^t(v^0; c)(x) &\le \int_0^t -(c^j - c)\gamma^{*\prime}(s)ds + v^0_j(\gamma^*(0)) - v^0(\gamma^*(0)) \\
&\quad +(h(c^j) - h(c))t, \\
\psi^t(v^0_j; c^j)(x) - \psi^t(v^0; c)(x) &\ge \int_0^t -(c^j - c)\gamma^{*\prime}_j(s)ds + v^0_j(\gamma^*_j(0)) - v^0(\gamma^*_j(0)) \\
&\quad +(h(c^j) - h(c))t.
\end{aligned}$$

It follows from claim (3) of Lemma 2.3 that any minimizing curves for $v(x,t)$ are Lipschitz with a common Lipschitz constant for all $x\in\mathbb{T}$ and $v^0\in \mathrm{Lip}_r(\mathbb{T})$. Since h is continuous, we conclude that $\psi^t(v^0_j;c^j)\to\psi^t(v^0;c)$ uniformly as $j\to\infty$.

Let $x\in\mathbb{T}$ be a common regular point of all $\psi^t(v^0_j;c^j)$, $j=1,2,3,\dots$. Almost every point is such a point. Through a variational technique, we find that $\gamma^*_j\to\gamma^*$ uniformly and $\gamma^{*\prime}_j\to\gamma^{*\prime}$ in L^2 on $[0,t]$ as $j\to\infty$ (e.g., see Lemma 3.4 in [22]). Note that for each $\tau\in[0,t]$ we have

$$\begin{aligned}
\phi^t(v^0_x;c)(x) &= \psi^t(v^0;c)_x(x)\\
&= \int_\tau^t L_x(\gamma^*(s),s,\gamma^{*\prime}(s))ds + L_\xi(\gamma^*(\tau),\tau,\gamma^{*\prime}(\tau)) - c,\\
\phi^t(v^0_{j\,x};c^j)(x) &= \psi^t(v^0_j;c^j)_x(x)\\
&= \int_\tau^t L_x(\gamma^*_j(s),s,\gamma^{*\prime}_j(s))ds + L_\xi(\gamma^*_j(\tau),\tau,\gamma^{*\prime}_j(\tau)) - c^j.
\end{aligned}$$

For any $\varepsilon>0$, there exists an integer J such that, if $j\ge J$, we have $\|\gamma^*_j-\gamma^*\|_{C^0}\le\varepsilon$ and $\|\gamma^{*\prime}_j-\gamma^{*\prime}\|_{L^2}\le\varepsilon$. Note that we have τ (depending on $j\ge J$) such that $|\gamma^{*\prime}_j(\tau)-\gamma^{*\prime}(\tau)|\le\varepsilon$. Therefore, we conclude that $\phi^t(v^0_{j\,x};c^j)\to\phi^t(v^0_x;c)$ pointwise almost everywhere. This immediately leads to $L^1(\mathbb{T})$-convergence. □

2.2. **Stochastic and variational approach to the Lax-Friedrichs scheme.** In this subsection, we state several results of the stochastic and variational approach to the Lax-Friedrichs scheme that are shown in [22]. Let N,K be natural numbers with $N\le K$. The mesh-size $\Delta=(\Delta x,\Delta t)$ is defined by $\Delta x:=(2N)^{-1}$ and $\Delta t:=(2K)^{-1}$. We set $\lambda:=\Delta t/\Delta x$. We also set $x_m:=m\Delta x$ for $m\in\mathbb{Z}$ and $t_k:=k\Delta t$ for $k=0,1,2,\dots$. Let $(\Delta x\mathbb{Z})\times(\Delta t\mathbb{Z}_{\ge0})$ be the set of all (x_m,t_k), and let

$$\mathcal{G}_{\mathrm{even}}\subset(\Delta x\mathbb{Z})\times(\Delta t\mathbb{Z}_{\ge0}),\qquad \mathcal{G}_{\mathrm{odd}}\subset(\Delta x\mathbb{Z})\times(\Delta t\mathbb{Z}_{\ge0})$$

be the set of all (x_m,t_k) with $k=0,1,2,\dots$ and $m\in\mathbb{Z}$ such that $m+k$ is even (resp. odd), which is called the even grid (resp. odd grid). For $x\in\mathbb{R}$ and $t>0$, the notation $m(x),k(t)$ denotes the integers m,k for which $x\in[x_m,x_m+2\Delta x)$ on the even or odd grid and $t\in[t_k,t_k+\Delta t)$, respectively. Note that $m(x)$ on the even grid and $m(x)$ on the odd grid are different for the same x.

We consider the discretization of (1.3) by the Lax-Friedrichs scheme in $\mathcal{G}_{\mathrm{even}}$:

$$(2.4)\quad\begin{cases}\dfrac{u^{k+1}_{m+1}-\frac{(u^k_m+u^k_{m+2})}{2}}{\Delta t} + \dfrac{H(x_{m+2},t_k,c+u^k_{m+2})-H(x_m,t_k,c+u^k_m)}{2\Delta x}=0,\\[2ex] u^0_m=u^0_\Delta(x_m),\quad u^k_{m\pm2N}=u^k_m,\end{cases}$$

where for m even

$$(2.5)\quad u^0_\Delta(x):=\frac{1}{2\Delta x}\int_{x_m-\Delta x}^{x_m+\Delta x}u^0(y)dy\quad\text{for } x\in[x_m-\Delta x,x_m+\Delta x).$$

Note that $\sum_{\{m\,|\,0\le m<2N,\, m+k \text{ even}\}} u^k_m \cdot 2\Delta x$ is conservative with respect to k and is zero for u^0 that has zero mean. We also discretize (1.4) in $\mathcal{G}_{\rm odd}$:

$$\text{(2.6)}\quad \begin{cases} \dfrac{v^{k+1}_m - \frac{(v^k_{m-1}+v^k_{m+1})}{2}}{\Delta t} + H(x_m, t_k, c + \dfrac{v^k_{m+1} - v^k_{m-1}}{2\Delta x}) = h(c), \\ v^0_{m+1} = v^0_\Delta(x_{m+1}), \quad v^k_{m+1\pm 2N} = v^k_{m+1}, \end{cases}$$

where, in addition to $u^0 = v^0_x$, we take v^0_Δ defined as

$$\text{(2.7)}\quad v^0_\Delta(x) := v^0(-\Delta x) + \int_{-\Delta x}^{x} u^0_\Delta(y)dy \ (v^0_\Delta(x_{m+1}) = v^0(x_{m+1}) \text{ for } m \text{ even}).$$

Note that $u^0_\Delta \to u^0$ in $L^1(\mathbb{T})$ and $v^0_\Delta \to v^0$ uniformly with $\| v^0_\Delta - v^0 \|_{C^0} \le \| u^0 \|_{L^\infty} \cdot 2\Delta x$, as $\Delta \to 0$. We introduce the following difference operators:

$$D_t w^{k+1}_m := \frac{w^{k+1}_m - \frac{w^k_{m-1}+w^k_{m+1}}{2}}{\Delta t}, \quad D_x w^k_{m+1} := \frac{w^k_{m+1} - w^k_{m-1}}{2\Delta x}.$$

Throughout the paper, we follow the convention that the sum of the superscript and subscript of variables defined on the even grid (resp. odd grid) is always even (resp. odd) in the notation $u^k_m, u^k_{m(x)}, v^k_{m+1}, v^k_{m(x)}$, etc. The problems (2.4) and (2.6) are equivalent under (2.5) and (2.7). In particular, we have $D_x v^k_{m+1} = u^k_m$ [22]. Let u_Δ be the step function derived from the solution u^k_m of (2.4); namely,

$$u_\Delta(x,t) := u^k_m \text{ for } (x,t) \in [x_{m-1}, x_{m+1}) \times [t_k, t_{k+1}).$$

Let v_Δ be the linear interpolation with respect to the space variable derived from the solution v^k_{m+1} of (2.6); namely,

$$v_\Delta(x,t) := v^k_{m-1} + D_x v^k_{m+1} \cdot (x - x_{m-1}) \text{ for } (x,t) \in [x_{m-1}, x_{m+1}) \times [t_k, t_{k+1}).$$

We remark that $v_\Delta(x,\cdot)$ is a step function for each fixed x and that $(v_\Delta)_x = u_\Delta$.

We introduce space-time inhomogeneous random walks in $\mathcal{G}_{\rm odd}$, which correspond to characteristic curves of (1.3) and (1.4). For each point $(x_n, t_{l+1}) \in \mathcal{G}_{\rm odd}$, we define backward random walks γ that start from x_n at t_{l+1} and move by $\pm\Delta x$ in each backward time step:

$$\gamma = \{\gamma^k\}_{k=0,1,\dots,l+1}, \quad \gamma^{l+1} = x_n, \quad \gamma^{k+1} - \gamma^k = \pm\Delta x.$$

More precisely, for each $(x_n, t_{l+1}) \in \mathcal{G}_{odd}$ we introduce the following objects:

$$X^k := \{x_{m+1} \,|\, (x_{m+1}, t_k) \in \mathcal{G}_{\rm odd}, |x_{m+1} - x_n| \le (l+1-k)\Delta x\} \text{ for } k \le l+1,$$

$$G := \bigcup_{1\le k\le l+1} \left(X^k \times \{t_k\}\right) \subset \mathcal{G}_{\rm odd},$$

$$\xi : G \ni (x_{m+1}, t_k) \mapsto \xi^k_{m+1} \in [-\lambda^{-1}, \lambda^{-1}], \quad \lambda = \Delta t/\Delta x,$$

$$\bar{\bar{\rho}} : G \ni (x_{m+1}, t_k) \mapsto \bar{\bar{\rho}}^k_{m+1} := \frac{1}{2} - \frac{1}{2}\lambda \xi^k_{m+1} \in [0,1],$$

$$\bar{\rho} : G \ni (x_{m+1}, t_k) \mapsto \bar{\rho}^k_{m+1} := \frac{1}{2} + \frac{1}{2}\lambda \xi^k_{m+1} \in [0,1],$$

$$\gamma : \{0,1,2,\dots,l+1\} \ni k \mapsto \gamma^k \in X^k, \ \gamma^{l+1} = x_n, \ \gamma^{k+1} - \gamma^k = \pm\Delta x,$$

Ω : the family of the above γ.

We regard $\bar{\bar{\rho}}^k_{m+1}$ (resp. $\bar{\rho}^k_{m+1}$) as the probability of transition from (x_{m+1}, t_k) to $(x_{m+1} + \Delta x, t_k - \Delta t)$ (resp. from (x_{m+1}, t_k) to $(x_{m+1} - \Delta x, t_k - \Delta t)$). Note that ξ is a control for random walks, which plays the role of a velocity field on the grid. We define the density of each path $\gamma \in \Omega$ as

$$\mu(\gamma) := \prod_{1 \le k \le l+1} \rho(\gamma^k, \gamma^{k-1}),$$

where $\rho(\gamma^k, \gamma^{k-1}) = \bar{\bar{\rho}}^k_{m(\gamma^k)}$ (resp. $\bar{\rho}^k_{m(\gamma^k)}$) if $\gamma^k - \gamma^{k-1} = -\Delta x$ (resp. Δx). The density $\mu(\cdot) = \mu(\cdot; \xi)$ yields a probability measure for Ω; namely,

$$\mathrm{prob}(A) = \sum_{\gamma \in A} \mu(\gamma; \xi) \quad \text{for } A \subset \Omega.$$

The expectation with respect to this probability measure is denoted by $E_{\mu(\cdot;\xi)}$; namely, for a random variable $f : \Omega \to \mathbb{R}$ we have

$$E_{\mu(\cdot;\xi)}[f(\gamma)] := \sum_{\gamma \in \Omega} \mu(\gamma; \xi) f(\gamma).$$

We use γ as the symbol for random walks or a sample path. If necessary, we write $\gamma = \gamma(x_n, t_{l+1}; \xi)$ in order to specify its initial point and control.

We now state an important result on the scaling limit of inhomogeneous random walks. Let $\eta(\gamma) = \{\eta^k(\gamma)\}_{k=0,1,2,\dots,l+1}$, $\gamma \in \Omega$ be a random variable that is induced by a random walk $\gamma = \gamma(x_n, t_{l+1}; \xi)$ and is defined by

$$\eta^{l+1} := \gamma^{l+1}, \quad \eta^k(\gamma) := \gamma^{l+1} - \sum_{k < k' \le l+1} \xi(\gamma^{k'}, t_{k'}) \Delta t \quad \text{for } 0 \le k \le l.$$

Proposition 2.5 ([21]). *Set* $\tilde{\sigma}^k := E_{\mu(\cdot;\xi)}[|\gamma^k - \eta^k(\gamma)|^2]$ *and* $\tilde{d}^k := E_{\mu(\cdot;\xi)}[|\gamma^k - \eta^k(\gamma)|]$ *for* $0 \le k \le l+1$. *Then, we have*

$$(\tilde{d}^k)^2 \le \tilde{\sigma}^k \le \frac{t^{l+1} - t^k}{\lambda} \Delta x.$$

If we take the hyperbolic scaling limit, in which $\Delta = (\Delta x, \Delta t) \to 0$ under

$$0 < \lambda_0 \le \lambda = \Delta t / \Delta x < \lambda_1,$$

where $\lambda_0, \lambda_1 > 0$ are some fixed constants, then $\tilde{d}^k$ and $\sqrt{\tilde{\sigma}^k}$ always tend to zero with the order $O(\sqrt{\Delta x})$. Note that the variance does not necessarily do so for inhomogeneous random walks. We refer to [21] for more details of the hyperbolic scaling limit of inhomogeneous random walks. We always take the limit $\Delta \to 0$ under hyperbolic scaling. In the arguments below, finding appropriate $\lambda_1 > 0$ is one of the major tasks (once λ_1 is found, λ_0 can be arbitrarily chosen from $(0, \lambda_1)$).

Now we state results for the stochastic and variational approach to the Lax-Friedrichs scheme.

Theorem 2.6 ([22]). *There exists* $\lambda_1 > 0$ *(depending on* T, $[c_0, c_1]$ *and* r*) such that for any small* $\Delta = (\Delta x, \Delta t)$ *with* $\lambda = \Delta t/\Delta x < \lambda_1$ *we have the following:*

(1) *The expectation*

$$E_{\mu(\cdot;\xi)}\Big[\sum_{0 < k \le l+1} L^{(c)}(\gamma^k, t_{k-1}, \xi^k_{m(\gamma^k)}) \Delta t + v^0_\Delta(\gamma^0) \Big] + h(c) t_{l+1}$$

which is given by $\gamma=\gamma(x_n,t_{l+1};\xi)$, has an infimum with respect to $\xi: G\to[-\lambda^{-1},\lambda^{-1}]$ for each $n\in\mathbb{Z}$ and $0<l+1\le k(T)$. There exists ξ^* that attains the infimum, and this ξ^* satisfies $|\xi^*|\le\lambda_1^{-1}<\lambda^{-1}$.

(2) *For each $n\in\mathbb{Z}$ and $0<l+1\le k(T)$ the solution of (2.6) satisfies*

$$v_n^{l+1}=\inf_\xi E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le l+1}L^{(c)}(\gamma^k,t_{k-1},\xi^k_{m(\gamma^k)})\Delta t+v^0_\Delta(\gamma^0)\Big]+h(c)t_{l+1}.$$

(3) *For each v_n^{l+1} the minimizing velocity field ξ^* is unique and satisfies the equality*

$$L^{(c)}_\xi(x_m,t_k,{\xi^*}^{k+1}_m)=D_xv^k_{m+1}\ \ (\Leftrightarrow {\xi^*}^{k+1}_m=H_p(x_m,t_k,c+D_xv^k_{m+1}))\ \ \text{on } G.$$

(4) *Let ξ^* (resp. $\tilde\xi^*$) be the minimizing velocity field for v_n^{l+1} (resp. v_{n+2}^{l+1}). Let $\gamma=\gamma(x_n,t_{l+1};\xi^*)$ and $\mu(\cdot;\xi^*)$ (resp. $\tilde\gamma=\gamma(x_{n+2},t_{l+1};\tilde\xi^*)$ and $\tilde\mu(\cdot;\tilde\xi^*)$) be the minimizing random walk and its probability measure generated by ξ^* (resp. $\tilde\xi^*$). Then, $u^{l+1}_{n+1}=D_xv^{l+1}_{n+2}$ satisfies*

$$u^{l+1}_{n+1}\le E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le l+1}L^{(c)}_x(\gamma^k,t_{k-1},{\xi^*}^k_{m(\gamma^k)})\Delta t+u^0_\Delta(\gamma^0+\Delta x)\Big]+O(\Delta x),$$

$$u^{l+1}_{n+1}\ge E_{\tilde\mu(\cdot;\tilde\xi^*)}\Big[\sum_{0<k\le l+1}L^{(c)}_x(\tilde\gamma^k,t_{k-1},\tilde{\xi}^{*k}_{m(\tilde\gamma^k)})\Delta t+u^0_\Delta(\tilde\gamma^0-\Delta x)\Big]+O(\Delta x),$$

where $O(\Delta x)$ stands for a number of $(-\theta\Delta x,\theta\Delta x)$ with $\theta>0$ independent of Δx.

Now we take the hyperbolic scaling limit.

(5) *Let v be the viscosity solution of (1.4). Then, for each $t\in[0,T]$ we have*

$$v_\Delta(\cdot,t)\to v(\cdot,t)\ \text{uniformly on } \mathbb{T} \text{ as } \Delta\to0.$$

In particular, we have an error estimate. That is, there exists $\beta_2>0$ (independent of Δ, $c\in[c_0,c_1]$, and the initial data $v^0\in\mathrm{Lip}_r(\mathbb{T})$) such that

$$\sup_{t\in[0,T]}\parallel v_\Delta(\cdot,t)-v(\cdot,t)\parallel_{C^0(\mathbb{T})}\le\beta_2\sqrt{\Delta x}.$$

(6) *Let $(x,t)\in\mathbb{T}\times(0,T]$ be a regular point and let $\gamma^*:[0,t]\to\mathbb{R}$ be the minimizing curve for $v(x,t)$. Let (x_n,t_{l+1}) be a point of $[x-2\Delta x,x+2\Delta x)\times[t-\Delta t,t+\Delta t)$ and let $\gamma_\Delta:[0,t]\to\mathbb{R}$ be the linear interpolation of the random walk $\gamma=\gamma(x_n,t_{l+1};\xi^*)$ given by the minimizing velocity field ξ^* for v_n^{l+1}. Then,*

$$\gamma_\Delta\to\gamma^*\ \text{uniformly on } [0,t] \text{ in probability as } \Delta\to0.$$

In particular, the average of γ_Δ converges uniformly to γ^* as $\Delta\to0$.

(7) *Let $u=v_x$ be the entropy solution of (1.3). Then, for each regular point $(x,t)\in\mathbb{T}\times[0,T]$ we have*

$$u_\Delta(x,t)\to u(x,t)\ \text{as } \Delta\to0.$$

In particular, u_Δ converges uniformly to u on $(\mathbb{T}\times[0,T])\setminus\Theta$, where Θ is a neighborhood of the set of points of discontinuity of u with an arbitrarily small measure. (If u^0 is not rarefaction-free, $\lambda=\Delta t/\Delta x$ must satisfy (2.8) and (2.9) in addition to $\lambda\le\lambda_1$.)

Note that claims (1) and (3) give the stability condition of the Lax-Friedrichs scheme,

$$|\lambda H_p(x_m, t_k, c + u^k_m))| < 1,$$

which is called the *CFL condition*.

We further state preliminary results on the Lax-Friedrichs scheme. The solution operators of (2.4) and (2.6) are denoted by

$$\phi^t_\Delta : L^\infty_{r,0}(\mathbb{T}) \ni u^0 \mapsto u_\Delta(\cdot, t) \in L^\infty(\mathbb{T}), \quad \psi^t_\Delta : \mathrm{Lip}_r(\mathbb{T}) \ni v^0 \mapsto v_\Delta(\cdot, t) \in \mathrm{Lip}(\mathbb{T}).$$

Note that we first obtain the step function u^0_Δ from u^0 with (2.5) and then we map u^0_Δ to $u_\Delta(\cdot, t)$ with ϕ^t_Δ. Similarly, we first obtain the piecewise linear function v^0_Δ from v^0 with (2.7), in which $u^0 = v^0_x$, and then we map v^0_Δ to $v_\Delta(\cdot, t)$ with ψ^t_Δ. When we specify the value of c, we write $\phi^t_\Delta(\cdot; c), \psi^t_\Delta(\cdot; c), u^{(c)}_\Delta, u^k_m(c), v^{(c)}_\Delta, v^k_{m+1}(c)$.

Proposition 2.7. *Fix $t \in [0, T]$. For each sequence $v^0_j \to v^0$ uniformly and $c^j \to c$ as $j \to \infty$ ($v^0_{j\,x}$ is not necessarily convergent), we have*

$$\psi^t_\Delta(v^0_j; c^j) \to \psi^t_\Delta(v^0; c) \text{ uniformly,}$$
$$\phi^t_\Delta(v^0_{j\,x}; c^j) \to \phi^t_\Delta(v^0_x; c) \text{ in } L^1(\mathbb{T}) \quad \text{as } j \to \infty.$$

Proof. It is sufficient to show that $\psi^{t_{l+1}}_\Delta(v^0_j; c^j)(x_n) \to \psi^{t_{l+1}}_\Delta(v^0; c)(x_n)$ uniformly with respect to x_n as $j \to \infty$. Using the stochastic and variational representation, we have

$$\begin{aligned}
\psi^{t_{l+1}}_\Delta(v^0; c)(x_n) &= E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le l+1} L(\gamma^k, t_{k-1}, \xi^{*k}_{\ m(\gamma^k)}) \\
&\qquad -c\xi^{*k}_{\ m(\gamma^k)}\Delta t + v^0_\Delta(\gamma^0)\Big] + h(c)t_{l+1}, \\
\psi^{t_{l+1}}_\Delta(v^0_j; c^j)(x_n) &= E_{\mu(\cdot;\xi^*_j)}\Big[\sum_{0<k\le l+1} L(\gamma^k, t_{k-1}, \xi^{*k}_{j\,m(\gamma^k)}) \\
&\qquad -c^j\xi^{*k}_{j\,m(\gamma^k)}\Delta t + v^0_{j\Delta}(\gamma^0)\Big] + h(c^j)t_{l+1},
\end{aligned}$$

where ξ^*, ξ^*_j are minimizing velocity fields. Hence, by the stochastic and variational representation again, we have

$$\begin{aligned}
\psi^{t_{l+1}}_\Delta(v^0_j; c^j)(x_n) &- \psi^{t_{l+1}}_\Delta(v^0; c)(x_n) \\
&\le E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le l+1} -(c^j - c)\xi^{*k}_{\ m(\gamma^k)}\Delta t + v^0_{j\Delta}(\gamma^0) - v^0_\Delta(\gamma^0)\Big] \\
&\quad +(h(c^j) - h(c))t_{l+1}, \\
\psi^{t_{l+1}}_\Delta(v^0_j; c^j)(x_n) &- \psi^{t_{l+1}}_\Delta(v^0; c)(x_n) \\
&\ge E_{\mu(\cdot;\xi^*_j)}\Big[\sum_{0<k\le l+1} -(c^j - c)\xi^{*k}_{j\,m(\gamma^k)}\Delta t + v^0_{j\Delta}(\gamma^0) - v^0_\Delta(\gamma^0)\Big] \\
&\quad +(h(c^j) - h(c))t_{l+1}.
\end{aligned}$$

Since ξ^*, ξ^*_j are uniformly bounded and h is continuous, we obtain the assertion.

The second convergence follows from the first one and the relation

$$\phi^t_\Delta(v^0_{j\,x}; c^j)(x_m) = \frac{\psi^t_\Delta(v^0_j; c^j)(x_{m+1}) - \psi^t_\Delta(v^0_j; c^j)(x_{m-1})}{2\Delta x}. \qquad \square$$

We now show details of the one-sided Lipschitz condition on u^k_m or, equivalently, the semiconcave property of v^k_{m+1}; namely, we obtain the Δ-independent upper boundedness of

$$E^k_\Delta := \max_m \frac{u^k_{m+2} - u^k_m}{2\Delta x} = \max_m \frac{v^k_{m+3} + v^k_{m-1} - 2v^k_{m+1}}{(2\Delta x)^2}.$$

If we assume that v^0 is semiconcave, it is easy to find an upper bound for E^k_Δ through the semiconcavity of v^k_{m+1} due to its variational structure. However, we would like to avoid this assumption and we want to know about the k-dependence of the upper bound. Therefore, we use a direct method similar to that of Lemma 2 in [18]. The direct method is available for arbitrary $T > 0$, because we already know by claims (1) and (3) of Theorem 2.6 that the difference solutions are bounded up to T.

This leads to the entropy condition on $u(\cdot, t)$ and semiconcavity of $v(\cdot, t)$. The one-sided Lipschitz condition on u^k_m is essential in the standard L^1-framework of difference approximation, because this condition yields Δ-independent boundedness of the total variation of $u_\Delta(\cdot, t)$ and then L^1-convergence of the approximation follows with the aid of the compactness of functions of bounded variation. We remark that, in our framework, the one-sided Lipschitz condition is indirectly used to prove claim (7) of Theorem 2.6 when u^0 is not rarefaction-free. In the sections below, we further use the one-sided Lipschitz condition on u^k_m for different purposes.

We introduce the following notation with the λ_1 in Theorem 2.6:

$$\begin{aligned}
u^* &:= \sup_{x,t\in\mathbb{T},c\in[c_0,c_1],|\xi|\le\lambda_1^{-1}} |L^{(c)}_\xi(x,t,\xi)| \ \text{(note that } |u^k_m| \le u^*\text{)},\\
H^*_{xx} &:= \sup_{x,t\in\mathbb{T},c\in[c_0,c_1],|u|\le u^*} |H_{xx}(x,t,c+u)|,\\
H^*_{xp} &:= \sup_{x,t\in\mathbb{T},c\in[c_0,c_1],|u|\le u^*} |H_{xp}(x,t,c+u)|,\\
H^*_{pp} &:= \inf_{x,t\in\mathbb{T},c\in[c_0,c_1],|u|\le u^*} |H_{pp}(x,t,c+u)| \quad (H^*_{pp} > 0 \text{ due to (A2)}),\\
\eta &:= \max\{2H^*_{xp} + H^*_{pp}, \frac{1}{2}H^*_{pp} + H^*_{xx}\},\\
E^* &:= \frac{2H^*_{xp}}{H^*_{pp}} + \sqrt{4\Big(\frac{H^*_{xp}}{H^*_{pp}}\Big)^2 + \frac{2H^*_{xx}}{H^*_{pp}}}.
\end{aligned}$$

Before giving details, we summarize our strategy as follows: We estimate E^{k+1}_Δ from E^k_Δ by using the difference equation. We find that each $E^{k+1}_\Delta - E^k_\Delta$ is bounded from above by $P(E^k_\Delta)$, where $P(y)$ is a concave parabola whose zero point on the right-hand side is E^*; i.e., $P(y) > 0$ for $0 \le y < E^*$, $P(E^*) = 0$, and $P(y) < 0$ for $y > E^*$ (see (2.10) below). Hence, if $E^k_\Delta > E^*$ (resp. $E^k_\Delta < E^*$), then E^{k+1}_Δ decreases by at least $P(E^k_\Delta) < 0$ (resp. increases by at most $P(E^k_\Delta) > 0$) and can remain near E^* for large $k \le k(T)$. If E^0_Δ is very large, E^k_Δ decays rapidly at first in a way similar to that of solutions to $w'(s) = -(w(s))^2$, where $w(s) \sim 1/s$.

Proposition 2.8. *Let $\lambda_1>0$ be that of Theorem 2.6. Suppose that $\Delta=(\Delta x,\Delta t)$ satisfies $\lambda=\Delta t/\Delta x<\lambda_1$, $\Delta t<\min\{(2\eta)^{-1},(E^*H^*_{pp}+2H^*_{xp})^{-1}\}$,*

$$\sup_{x,t\in\mathbb{T},c\in[c_0,c_1],|u|\le u^*}\lambda(|H_p(x,t,c+u)|+H^*_{xp}\cdot 2\Delta x)<1, \tag{2.8}$$

$$\lambda\le\frac{1-2H^*_{xp}\Delta t}{rH^*_{pp}+(1+H^*_{pp})\Delta x}. \tag{2.9}$$

Then, the following hold:

(1) *For $1\le k\le k(T)$ we have*

$$E^k_\Delta=\max_m\frac{u^k_{m+2}-u^k_m}{2\Delta x}\le\frac{2e^{\eta t_k}}{H^*_{pp}}\frac{1}{t_k}\ \ (t_k=k\Delta t).$$

(2) *If $E^0_\Delta\le E^*$, we have $E^k_\Delta\le E^*$ for $1\le k\le k(T)$.*

(3) *If $k>k(\eta^{-1})$, we have $E^k_\Delta\le\dfrac{4e\eta}{H^*_{pp}}$.*

(4) *If u^k_m is extended to $k\to\infty$ with $|u^k_m|\le u^*$, we have $\limsup_{k\to\infty}E^k_\Delta\le E^*$.*

Proof. Using the difference equation and Taylor's formula, we obtain an estimate of E^{k+1}_Δ from E^k_Δ. For brevity, the remainders in Taylor's formula are denoted by H_{pp}, H_{xx}, and H_{xp}, which satisfy

$$H_{pp}\ge H^*_{pp},\ \ |H_{xx}|\le H^*_{xx},\ \ |H_{xp}|\le H^*_{xp}.$$

Set $z^k_m:=u^k_{m+2}-u^k_m$. Then, we have

$$\begin{aligned}
z^{k+1}_{m+1} &= \frac{z^k_m+z^k_{m+2}}{2}-\frac{\Delta t}{2\Delta x}\Big\{H(x_{m+4},t_k,c+u^k_{m+4})-H(x_{m+2},t_k,c+u^k_{m+4})\\
&\quad +H(x_{m+2},t_k,c+u^k_{m+4})-H(x_{m+2},t_k,c+u^k_{m+2})\\
&\quad +H(x_m,t_k,c+u^k_m)-H(x_{m+2},t_k,c+u^k_m)\\
&\quad +H(x_{m+2},t_k,c+u^k_m)-H(x_{m+2},t_k,c+u^k_{m+2})\Big\}\\
&= \Big(\frac12+\frac{\lambda}{2}H_p(x_{m+2},t_k,c+u^k_{m+2})\Big)z^k_m\\
&\quad +\Big(\frac12-\frac{\lambda}{2}H_p(x_{m+2},t_k,c+u^k_{m+2})\Big)z^k_{m+2}\\
&\quad -\frac{\Delta t}{2\Delta x}\Big\{\Big(H_x(x_{m+2},t_k,c+u^k_{m+4})-H_x(x_{m+2},t_k,c+u^k_m)\Big)(2\Delta x)\\
&\quad +\frac12 H_{pp}\cdot(z^k_{m+2})^2+\frac12 H_{pp}\cdot(z^k_m)^2+\frac12 H_{xx}\cdot(2\Delta x)^2\\
&\quad +\frac12 H_{xx}\cdot(2\Delta x)^2\Big\}\\
&= \Big\{\frac12+\frac{\lambda}{2}H_p(x_{m+2},t_k,c+u^k_{m+2})-\frac{\lambda}{2}H_{xp}\cdot 2\Delta x\Big\}z^k_m\\
&\quad +\Big\{\frac12-\frac{\lambda}{2}H_p(x_{m+2},t_k,c+u^k_{m+2})-\frac{\lambda}{2}H_{xp}\cdot 2\Delta x\Big\}z^k_{m+2}\\
&\quad -\frac{\Delta t}{2\Delta x}\Big\{\frac12 H_{pp}\cdot(z^k_{m+2})^2+\frac12 H_{pp}\cdot(z^k_m)^2+\frac12 H_{xx}\cdot(2\Delta x)^2\\
&\quad +\frac12 H_{xx}\cdot(2\Delta x)^2\Big\}.
\end{aligned}$$

It follows from (2.8) that

$$\Big\{\frac{1}{2}\pm\frac{\lambda}{2}H_p(x_{m+2},t_k,c+u^k_{m+2})-\frac{\lambda}{2}H_{xp}\cdot 2\Delta x\Big\}>0.$$

Hence, setting $\tilde z^k_m:=\max\{z^k_m,z^k_{m+2}\}$, we obtain

$$\begin{aligned} z^{k+1}_{m+1} &\le (1-2H_{xp}\Delta t)\tilde z^k_m+H^*_{xx}\cdot 2\Delta x\Delta t-\frac{H^*_{pp}}{2}\frac{\Delta t}{2\Delta x}(\tilde z^k_m)^2,\\ \frac{z^{k+1}_{m+1}}{2\Delta x} &\le (1-2H_{xp}\Delta t)\frac{\tilde z^k_m}{2\Delta x}+H^*_{xx}\Delta t-\frac{H^*_{pp}}{2}\Delta t(\frac{\tilde z^k_m}{2\Delta x})^2. \end{aligned}$$

Note that $g(y):=(1-2H_{xp}\Delta t)y+H^*_{xx}\Delta t-(\frac{H^*_{pp}}{2}\Delta t)y^2$ is monotonically increasing if

$$y\le\frac{1-2H_{xp}\Delta t}{H^*_{pp}\Delta t}.$$

It follows from (2.9) that $\lambda\le(1-2H^*_{xp}\Delta t)/(rH^*_{pp}+\Delta x)$ and hence

$$E^0_\Delta\le\frac{2r}{2\Delta x}\le\frac{1-2H^*_{xp}\Delta t}{H^*_{pp}\Delta t}\le\frac{1-2H_{xp}\Delta t}{H^*_{pp}\Delta t}$$

for all initial data in $L^\infty_{r,0}(\mathbb{T})$. Suppose that $E^k_\Delta\le(1-2H^*_{xp}\Delta t)/(H^*_{pp}\Delta t)$. Then, we obtain

$$(2.10)\quad E^{k+1}_\Delta\le E^k_\Delta+P(E^k_\Delta),\ \ P(y):=-\Delta t\Big(\frac{H^*_{pp}}{2}y^2-2H^*_{xp}y-H^*_{xx}\Big).$$

It follows from $\Delta t<(E^*H^*_{pp}+2H^*_{xp})^{-1}$ that $E^*<(1-2H^*_{xp}\Delta t)/(H^*_{pp}\Delta t)$, and from $\Delta t<(2\eta)^{-1}$ that $|y-E^*|\ge|P(y)|$ for all $0\le y\le(1-2H^*_{xp}\Delta t)/(H^*_{pp}\Delta t)$. Hence, we have two cases:

(i) if $E^k_\Delta\le E^*$, we may have $E^{k+1}_\Delta\ge E^k_\Delta$, but we certainly have $E^{k+1}_\Delta\le E^k_\Delta+P(E^k_\Delta)\le E^*$,

(ii) if $E^*<E^k_\Delta$, we have $E^{k+1}_\Delta<E^k_\Delta$.

Therefore, we have $E^{k+1}_\Delta\le(1-2H^*_{xp}\Delta t)/(H^*_{pp}\Delta t)$ and, by induction, we see that $E^k_\Delta\le(1-2H^*_{xp}\Delta t)/(H^*_{pp}\Delta t)$ for all $0\le k\le k(T)$. Thus, (2.10) holds for all $0\le k<k(T)$. It is now easy to verify that

- if $E^0_\Delta\le E^*$, then E^k_Δ may increase but never exceeds E^*,
- if $E^0_\Delta>E^*$, then the E^k_Δ are bounded from above by a monotonically decreasing sequence.

Claims (2) and (4) are clear. Now we follow Lemma 2 in [18]. Set $V^k:=E^k_\Delta+1\ge 1$. Then, by (2.10) we have

$$V^{k+1}\le(1+\eta\Delta t)V^k-\frac{H^*_{pp}}{2}\Delta t(V^k)^2.$$

Now, set $W^k:=(1-\eta\Delta t)^kV^k$ for $k\ge 0$ (the inequality $1-\eta\Delta t>0$ holds, since $\Delta t<(2\eta)^{-1}$). Then, for $k\ge 1$ we have

$$\begin{aligned} W^{k+1} &\le (1-\eta\Delta t)(1+\eta\Delta t)W^k-\frac{H^*_{pp}}{2}\Delta t(W^k)^2(1-\eta\Delta t)^{-k+1}\\ &\le W^k-\frac{H^*_{pp}}{2}\Delta t(W^k)^2. \end{aligned}$$

Consider $w'(t) = -\frac{H^*_{pp}}{2}(w(t))^2$, with $w(0) = w^0 := 2/(H^*_{pp}\Delta t)$. The solution satisfies

$$w(t) = \frac{1}{\frac{H^*_{pp}}{2}t + \frac{1}{w^0}} \le \frac{2}{H^*_{pp}t}.$$

We can show that $W^k \le w(k\Delta t)$ for $k \ge 1$ by noting that $w(\Delta t) = 1/(H^*_{pp}\Delta t)$ and $W^1 = (1-\eta\Delta t)(E^1_\Delta + 1) \le (r/\Delta x) + 1$. It follows from (2.9) that $\lambda \le 1/\{(r+\Delta x)H^*_{pp}\}$ and hence that $W^1 \le w(\Delta t)$. Suppose that $W^k \le w(k\Delta t)$ for some $k \ge 1$. Then, since $\tilde{g}(y) := y - \frac{H^*_{pp}\Delta t}{2}y^2$ is monotonically increasing for $y \le 1/(H^*_{pp}\Delta t)$, $w(k\Delta t) \le 1/(H^*_{pp}\Delta t)$ and $w'' > 0$, we have

$$\begin{aligned} W^{k+1} &\le W^k - \frac{H^*_{pp}\Delta t}{2}(W^k)^2 \le w(k\Delta t) - \frac{H^*_{pp}\Delta t}{2}(w(k\Delta t))^2 \\ &= w(k\Delta t) + \Delta t w'(k\Delta t) = w(k\Delta t + \Delta t) - \frac{1}{2}w''(k\Delta t + \theta\Delta t)\cdot(\Delta t)^2 \\ &\le w((k+1)\Delta t) \ (\theta \in (0,1)). \end{aligned}$$

Thus, we obtain

$$E^k_\Delta \le (1-\eta\Delta t)^{-k}\frac{2}{H^*_{pp}k\Delta t} \le (1-\eta\Delta t)^{-\frac{\eta k\Delta t}{\eta\Delta t}}\frac{2}{H^*_{pp}k\Delta t} \le \frac{2e^{\eta t_k}}{H^*_{pp}}\frac{1}{t_k}.$$

Now, set $f(t) := \frac{2e^{\eta t}}{H^*_{pp}}\frac{1}{t}$. The minimum of f is $f(\eta^{-1}) = \frac{2e\eta}{H^*_{pp}}$, which is greater than E^*. Therefore, due to cases (i) and (ii), the E^k_Δ are bounded from above by $f(t_k)(> E^*)$ for $k \le k(\eta^{-1})$ and never exceed $f(\eta^{-1} - \Delta t) \le \frac{4e\eta}{H^*_{pp}}$ for $k > k(\eta^{-1})$. This demonstrates claims (1) and (3). □

3. Time-global stability and large-time behavior

We prove time-global stability of the Lax-Friedrichs scheme with a fixed mesh size. Then, we show the large-time behavior of the scheme in which each difference solution falls into a time periodic state with the unit period. Each time periodic state corresponds to a space-time periodic difference solution. There arises the notion of the effective Hamiltonian of (1.6).

3.1. Time-global stability.

The main result of this section is the following theorem.

Theorem 3.1. *There exist $\lambda_1 > 0$ and $\delta > 0$ such that, if $\Delta = (\Delta x, \Delta t)$ satisfies $0 < \lambda_0 \le \lambda = \Delta t/\Delta x < \lambda_1$ and $\Delta x \le \delta$, the Lax-Friedrichs scheme starting from any $u^0 \in L^\infty_{r,0}(\mathbb{T})$ succeeds up to an arbitrary time index and satisfies the CFL condition*

$$|H_p(x_m, t_k, c + u^k_m)| \le \lambda_1^{-1} < \lambda^{-1} \quad \textit{for all } (x_m, t_k) \in \mathcal{G}_{\rm even}.$$

In order to prove this theorem, we need uniform boundedness of $\| \phi^1_\Delta(u^0; c) \|_{L^\infty}$ with respect to $(u^0; c)$, which is similar to Proposition 2.1. First, we observe the following lemma.

Lemma 3.2. *Let $\lambda_1 > 0$ be that of Theorem 2.6 and let $\Delta = (\Delta x, \Delta t)$ be such that $0 < \lambda_0 \le \lambda = \Delta t/\Delta x < \lambda_1$. Fix $t \in (0, T]$ arbitrarily. Then, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon; t) > 0$ such that, if $\Delta x \le \delta$, we have*

$$\sup_{u^0 \in L^\infty_{r,0}(\mathbb{T}), c \in [c_0, c_1]} \| \phi^t_\Delta(u^0; c) - \phi^t(u^0; c) \|_{L^1(\mathbb{T})} \le \varepsilon.$$

Proof. If not, then for some $\varepsilon_0 > 0$ and $\delta_j \to 0$ as $j \to \infty$, we have $\Delta x_j \le \delta_j$ such that

$$(3.1)\qquad \sup_{u^0\in L^\infty_{r,0}(\mathbb{T}),c\in[c_0,c_1]} \parallel \phi^t_{\Delta_j}(u^0;c) - \phi^t(u^0;c) \parallel_{L^1(\mathbb{T})} > 2\varepsilon_0$$

for all j, where $\Delta_j = (\Delta x_j, \lambda\Delta x_j)$. We have (u^0_j, c^j) such that

$$\parallel \phi^t_{\Delta_j}(u^0_j;c^j) - \phi^t(u^0_j;c^j) \parallel_{L^1(\mathbb{T})} > \varepsilon_0$$

for each j. Since u^0_j has the average 0 with $|u^0_j| \le r$, we have its primitive v^0_j that belongs to $\mathrm{Lip}_r(\mathbb{T})$ with $|v^0| \le r$. By the Arzela-Ascoli theorem, we have a subsequence of (v^0_j, c^j), still denoted by (v^0_j, c^j), which converges to (v^0, c). It follows from claim (7) of Theorem 2.6 that there exists $\delta_0 > 0$ such that, if $\Delta x \le \delta_0$, we have $\parallel \phi^t_\Delta(v^0_x;c) - \phi^t(v^0_x;c) \parallel_{L^1(\mathbb{T})} < \frac{\varepsilon_0}{2}$. Hence,

$$\begin{aligned}\frac{\varepsilon_0}{2} &> \parallel \phi^t_\Delta(v^0_x;c) - \phi^t(v^0_x;c) \parallel_{L^1(\mathbb{T})}\\ &\ge \parallel \phi^t_\Delta(v^0_{j\,x};c^j) - \phi^t(v^0_{j\,x};c^j) \parallel_{L^1(\mathbb{T})} - \parallel \phi^t_\Delta(v^0_x;c) - \phi^t_\Delta(v^0_{j\,x};c^j) \parallel_{L^1(\mathbb{T})}\\ &\quad - \parallel \phi^t(v^0_{j\,x};c^j) - \phi^t(v^0_x;c) \parallel_{L^1(\mathbb{T})} .\end{aligned}$$

By Propositions 2.4 and 2.7, we have

$$\parallel \phi^t_\Delta(v^0_x;c) - \phi^t_\Delta(v^0_{j\,x};c^j) \parallel_{L^1(\mathbb{T})} + \parallel \phi^t(v^0_{j\,x};c^j) - \phi^t(v^0_x;c) \parallel_{L^1(\mathbb{T})} \le \frac{\varepsilon_0}{2}$$

for large j. Therefore, we have for any $\Delta x \le \delta_0$,

$$\parallel \phi^t_\Delta(v^0_{j\,x};c^j) - \phi^t(v^0_{j\,x};c^j) \parallel_{L^1(\mathbb{T})} = \parallel \phi^t_\Delta(u^0_j;c^j) - \phi^t(u^0_j;c^j) \parallel_{L^1(\mathbb{T})} < \varepsilon_0.$$

This is a contradiction. □

Next, we see that the convergence $\parallel \phi^1_\Delta(u^0;c) - \phi^1(u^0;c) \parallel_{L^1(\mathbb{T})} \to 0$ as $\Delta \to 0$, which is uniform with respect to (u^0, c), yields uniform boundedness of $\|\phi^1_\Delta(u^0;c)\|_{L^\infty}$ with the aid of the one-sided Lipschitz condition.

Proposition 3.3. *Let $\lambda_1 > 0$ be that of Theorem 2.6 with $T = 1$. Let $\Delta = (\Delta x, \Delta t)$ be such that $0 < \lambda_0 \le \lambda = \Delta t/\Delta x < \lambda_1$, satisfying the conditions in Proposition 2.8. Then, there exists $\delta > 0$ such that, if $\Delta x \le \delta$, we have*

$$\sup_{u^0\in L^\infty_{r,0}(\mathbb{T}),c\in[c_0,c_1]} \parallel \phi^1_\Delta(u^0;c) \parallel_{L^\infty} \le \beta_1(1) + 1,$$

where $\beta_1(\cdot)$ is the one in Proposition 2.1

Proof. Let $0 < \varepsilon < 1$ be such that $\frac{1-\sqrt{\varepsilon}}{3\sqrt{\varepsilon}} > 2e^\eta/H^*_{pp} \ge E^{2K}_\Delta$, where $2K\Delta t = 1$. With this ε and $t = 1$, we have $\delta > 0$ in Lemma 3.2. We take $\Delta x \le \min\{\delta, \sqrt{\varepsilon}\}$. Consider

$$A := \{y \in \mathbb{T} \mid |\phi^1_\Delta(u^0;c)(y) - \phi^1(u^0;c)(y)| > \sqrt{\varepsilon}\}.$$

Since $\parallel \phi^1_\Delta(u^0;c) - \phi^1(u^0;c) \parallel_{L^1(\mathbb{T})} \le \varepsilon$, we have meas $[A] \le \sqrt{\varepsilon}$. Hence, for $y \in A$ there exists $x, \tilde{x} \in \mathbb{R} \setminus A$ such that $0 < y - x \le \sqrt{\varepsilon}$ and $0 < \tilde{x} - y \le \sqrt{\varepsilon}$. For $x \in \mathbb{T} \setminus A$, we have $|\phi^1_\Delta(u^0;c)(x) - \phi^1(u^0;c)(x)| \le \sqrt{\varepsilon}$ and $|\phi^1_\Delta(u^0;c)(x)| \le |\phi^1(u^0;c)(x)| + \sqrt{\varepsilon} \le \beta_1(1) + \sqrt{\varepsilon} \le \beta_1(t) + 1$. Consider

$$\tilde{A} := \{y \in A \mid |\phi^1_\Delta(u^0;c)(y)| > \beta_1(1) + 1\}.$$

Suppose that $\tilde{A}$ is not empty. Note that $\phi^1_\Delta(u^0;c)(\cdot)$ is piecewise constant derived from u^{2K}_m. Then, there exists $x_n \in \tilde{A} \cap (\Delta x\mathbb{Z})$ such that $u^{2K}_n > \beta_1(1) + 1$ (resp.

$u_n^{2K} < -\beta_1(1)-1)$. Since $x_n \in A$ and meas $[A] \le \sqrt{\varepsilon}$, there exist $x_m, x_{m'} \in (\mathbb{R}\setminus A)\cap(\Delta x\mathbb{Z})$ such that $0 < x_n - x_m \le \sqrt{\varepsilon}+\Delta x \le 3\sqrt{\varepsilon}$ and $0 < x_{m'} - x_n \le \sqrt{\varepsilon}+\Delta x \le 3\sqrt{\varepsilon}$. Therefore we have

$$\frac{u_n^{2K}-u_m^{2K}}{x_n-x_m} > \frac{\beta_1(1)+1-(\beta_1(1)+\sqrt{\varepsilon})}{3\sqrt{\varepsilon}} = \frac{1-\sqrt{\varepsilon}}{3\sqrt{\varepsilon}} > E_\Delta^{2K}, \quad \text{resp.,}$$
$$\frac{u_{m'}^{2K}-u_n^{2K}}{x_{m'}-x_n} > \frac{-(\beta_1(1)+\sqrt{\varepsilon})-(-\beta_1(1)-1)}{3\sqrt{\varepsilon}} = \frac{1-\sqrt{\varepsilon}}{3\sqrt{\varepsilon}} > E_\Delta^{2K}.$$

These two inequalities contradict the one-sided Lipschitz condition. □

Remark. Claim (7) of Theorem 2.6 states that $\phi_\Delta^1(u^0;c)$ converges to $\phi^1(u^0;c)$ uniformly on $\mathbb{T}\setminus\Theta$ as $\Delta\to 0$, where Θ is an arbitrary small neighborhood of shocks. However, we cannot use this fact for Proposition 3.3, because uniformity of the convergence with respect to $(u^0;c)$ is unverified.

Proof of Theorem 3.1. Recall that $\beta_1(\cdot)$ in Proposition 2.1 is independent of r and $c\in[c_0,c_1]$. Let $\lambda_1>0$ be that of Theorem 2.6 with $T=1$ and $r\ge\beta_1(1)+1$. Let $\delta>0$ be that of Proposition 3.3. Then, $\tilde u^0 := \phi_\Delta^1(u^0;c)$ belongs to $L^\infty_{\beta_1(1)+1,0}(\mathbb{T})$ for any $u^0\in L^\infty_{r,0}(\mathbb{T})$. Hence, by the choice of λ_1, we are guaranteed that $\phi_\Delta^1(\tilde u^0;c)=\phi_\Delta^2(u^0;c)$ is well defined and bounded by $\beta_1(1)+1$ again. In this way, $\phi_\Delta^t(u^0;c)$ can be defined for $t\to\infty$ with the CFL condition. □

3.2. **Large-time behavior.** If we take $r\ge\beta_1(1)+1$, then $\phi_\Delta^1(u^0;c)$ belongs to $L^\infty_{r,0}(\mathbb{T})$. Therefore, $\phi_\Delta^1(\cdot;c)$ maps $L^\infty_{r,0}(\mathbb{T})$ into itself. We can find the fixed points of the map for each c. In this subsection, we consider the fixed points and their stability, which makes clear the large-time behavior of the Lax-Friedrichs scheme. Note that the Lax-Friedrichs scheme has a contraction property under the CFL condition. That is, for $0\le t\le t'$ we have

$$\| \phi_\Delta^{t'}(u^0;c)-\phi_\Delta^{t'}(\tilde u^0;c)\|_{L^1(\mathbb{T})} \quad\le\quad \|\phi_\Delta^t(u^0;c)-\phi_\Delta^t(\tilde u^0;c)\|_{L^1(\mathbb{T})}\,.$$

This can be refined to become a strict contraction property. Let $\sum_{m;k}$ denote summation with respect to $\{m\,|\,0\le m<2N,\ m+k \text{ even}\}$ for each fixed k, and let $\| x\|_1 := \sum_{1\le j\le n}|x_j|$ for $x\in\mathbb{R}^n$.

Proposition 3.4. *The family of maps* $\{\phi_\Delta^t(\cdot;c)\}_{t\ge0}$ *has a strict contraction property within the unit time period. That is, for any two distinct initial data* u^0 *and* $\tilde u^0$*, we have*

$$\|\phi_\Delta^{t+1}(u^0;c)-\phi_\Delta^{t+1}(\tilde u^0;c)\|_{L^1(\mathbb{T})}<\|\phi_\Delta^t(u^0;c)-\phi_\Delta^t(\tilde u^0;c)\|_{L^1(\mathbb{T})}\,.$$

Proof. It is sufficient to show that for all $k\ge0$ any two difference solutions u^k and $\tilde u^k$ of (2.4) satisfy

$$\| u^{k+2K}-\tilde u^{k+2K}\|_1<\| u^k-\tilde u^k\|_1\,.$$

Set $z_m^k := u_m^k-\tilde u_m^k$ and $\sigma_m^k := \operatorname{sign} z_m^k = 1$ or -1 ($\operatorname{sign} 0 := 1$). Then, $\| u^k-\tilde u^k\|_1 = \sum_{m;k}|z_m^k| = \sum_{m;k}\sigma_m^k z_m^k$. By the difference equation of (2.4), we have

$$\sum_{m;k}|z_{m+1}^{k+1}| = \sum_{m;k}\sigma_{m+1}^{k+1}z_{m+1}^{k+1} = \sum_{m;k}\sigma_{m+1}^{k+1}\Big\{\frac12 z_{m+2}^k(1-\lambda\delta_{m+2}^k)+\frac12 z_m^k(1+\lambda\delta_m^k)\Big\},$$

where $\delta^k_m := H_p(x_m,t_k,c+u^k_m+\theta^k_m)$ with a constant θ^k_m derived from Taylor's formula. Switching the order of the summations above, we obtain

$$\begin{aligned}\sum_{m;k}|z^{k+1}_{m+1}| &= \sum_{m;k} z^k_m\left\{\frac{1}{2}\sigma^{k+1}_{m-1}(1-\lambda\delta^k_m)+\frac{1}{2}\sigma^{k+1}_{m+1}(1+\lambda\delta^k_m)\right\}\\ &= \sum_{m;k}|z^k_m|\\ &\quad+\sum_{m;k}|z^k_m|\left[-1+\sigma^k_m\left\{\frac{1}{2}\sigma^{k+1}_{m-1}(1-\lambda\delta^k_m)+\frac{1}{2}\sigma^{k+1}_{m+1}(1+\lambda\delta^k_m)\right\}\right].\end{aligned}$$

Let R^k denote the sum in the third line of the above equality. We find that $R^k\le 0$, since for each term of R^k the factor [ ] belongs to one of two cases:

(i) If $\sigma^{k+1}_{m-1}+\sigma^{k+1}_{m+1}=\pm 2$, then [ ] $=-1\pm\sigma^k_m=0$ or -2.

(ii) If $\sigma^{k+1}_{m-1}+\sigma^{k+1}_{m+1}=0$, then [ ] $=-1\pm\lambda\delta^k_m<0$ due to the CFL condition.

Since each of u^k and $\tilde u^k$ has the zero mean and $u^0\neq\tilde u^0$, the sign of z^k_m necessarily changes and case (ii) occurs. It seems possible that even though u^k and $\tilde u^k$ are such, we may have $R^k=0$; namely, $z^k_m=0$ for all the integers m for which case (ii) occurs. However, after further k^*-time evolution ($k^*<N<2K$), case (ii) certainly occurs and $R^{k+k^*}<0$, because such zero-points disappear as k increases due to the monotonicity of the Lax-Friedrichs scheme under the CFL condition (see also Remark 2.5 in [17]). □

We show that time periodic difference solutions exist and they are stable, which provides the large-time behavior of the Lax-Friedrichs scheme.

Theorem 3.5. *Take $r\ge\beta_1(1)+1$ and fix $\Delta=(\Delta x,\Delta t)$ so that Theorems* 2.6 *and* 3.1 *hold. Then, for each c there exists a fixed point $\bar u^0_\Delta\in L^\infty_{r,0}(\mathbb{T})$ of $\phi^1_\Delta(\cdot;c)$, which yields a time periodic difference solution $\phi^t_\Delta(\bar u^0_\Delta;c)$. Such a periodic solution is unique with respect to c. Any other solution $\phi^t_\Delta(u^0;c)$ exponentially falls into the periodic state; namely, there exist $\rho\in(0,1)$ and $a>0$ depending on Δ, but independent of u^0, such that $\|\phi^t_\Delta(u^0;c)-\phi^t_\Delta(\bar u^0_\Delta;c)\|_{L^\infty}\le a\rho^t$ for $t\in\mathbb{N}$.*

Proof. The map $\phi^1_\Delta(\cdot;c)$ is actually a map from $\mathbb{R}^N$ to $\mathbb{R}^N$, since the step functions have only N different values at most. Let B_r be the set of all $x\in\mathbb{R}^N$ with $\|x\|_\infty\le r$. If $r\ge\beta_1(1)+1$, then the map $\phi^1_\Delta(\cdot;c)$ is actually a map from B_r to B_r. Therefore, we obtain a fixed point through Brouwer's fixed point theorem. By Proposition 3.4, periodic solutions must be unique. Exponential decay can be proved in the same way as the proof of (5) of Theorem 2.1 in [17]. □

Remark. It is likely that in general ρ becomes arbitrarily close to 1 as Δ tends to zero. Numerical experiments imply such a property of ρ [17]. The uniqueness does not hold for the exact equation (1.1) in general. There may exist time periodic entropy solutions of (1.1) with the minimum period greater than 1 [1].

The following theorem for the discrete Hamilton-Jacobi equation is like the weak KAM theorem.

Theorem 3.6. *Take $r\ge\beta_1(1)+1$ and fix $\Delta=(\Delta x,\Delta t)$ so that Theorems* 2.6 *and* 3.1 *hold. Then, for each c there exists a constant $\bar h_\Delta(c)\in\mathbb{R}$ such that if $h(c)=\bar h_\Delta(c)$, we have a fixed point $\bar v^0_\Delta\in\mathrm{Lip}_r(\mathbb{T})$ of $\psi^1_\Delta(\cdot;c)$, which yields a time periodic difference solution $\psi^t_\Delta(\bar v^0_\Delta;c)$. Such a periodic solution is unique with* *respect to c up to constants. Any other solution $\psi^t_\Delta(v^0;c)$ exponentially falls into a periodic state; namely, for the $\rho\in(0,1)$ and $a>0$ in Theorem 3.5 and for $d\in\mathbb{R}$ depending on v^0 and c, we have $\| \psi^t_\Delta(v^0;c)-\psi^t_\Delta(\bar v^0_\Delta+dI_1;c)\|_{C^0}\le a\rho^t$ for $t\in\mathbb{N}$, where $I_1(x):=1$ for all x and $\psi^t_\Delta(v^0+dI_1;c)=\psi^t_\Delta(v^0;c)+dI_1$.*

Proof. We imitate the proof of the weak KAM theorem [9]. Let us write $v\sim w$ for $v,w\in C^0(\mathbb{T})$ if there exists $b\in\mathbb{R}$ such that $w=v+bI_1$. We introduce $\hat v:=\{w\in C^0(\mathbb{T})\,|\,w\sim v\}$, $\|\hat v\|:=\inf_{w\in\hat v}\|w\|_{C^0(\mathbb{T})}$, $\hat{C^0(\mathbb{T})}:=C^0(\mathbb{T})/\sim$, and $\hat{\mathrm{Lip}_r(\mathbb{T})}:=\mathrm{Lip}_r(\mathbb{T})/\sim$. From the Arzela-Ascoli theorem it follows that $\hat{\mathrm{Lip}_r(\mathbb{T})}$ is a compact convex subset of the Banach space $\hat{C^0(\mathbb{T})}$. Due to the property $\psi^t_\Delta(v^0+dI_1;c)=\psi^t_\Delta(v^0;c)+dI_1$, we have

$$\psi^1_\Delta(v^0;c)\sim\psi^1_\Delta(w^0;c)\text{ for all }v^0,w^0\in\hat v^0.$$

Hence, the map

$$\begin{aligned}&\hat\psi^1_\Delta(\cdot;c):\hat{\mathrm{Lip}_r(\mathbb{T})}\to\hat{\mathrm{Lip}_r(\mathbb{T})},\\&\hat\psi^1_\Delta(\hat v^0;c):=\{w\in\mathrm{Lip}_r(\mathbb{T})\,|\,w\sim\psi^1_\Delta(v^0;c)\}\ \ (v^0\in\hat v^0)\end{aligned}$$

is well defined and continuous. By Schauder's fixed point theorem, we have a fixed point $\hat{\bar v}^0_\Delta$ satisfying $\hat\psi^1_\Delta(\hat{\bar v}^0_\Delta;c)=\hat{\bar v}^0_\Delta$. Therefore, we have an element $\bar v^0_\Delta\in\hat{\bar v}^0_\Delta$ and constant $b(c)\in\mathbb{R}$ such that

$$\bar v^0_\Delta=\psi^1_\Delta(\bar v^0_\Delta;c)+b(c)I_1.$$

This relation means that $\bar v^0_\Delta$ yields a time periodic solution of (2.6) with $h(c)+b(c)$ instead of $h(c)$.

Note that $\psi^t_\Delta(v^0;c)\le\psi^t_\Delta(\tilde v^0;c)$ if $v^0\le\tilde v^0$, due to the variational representation. Let v^0 be an arbitrary function in $\mathrm{Lip}_r(\mathbb{T})$ and a^0,b^0 be constants such that for all $x\in\mathbb{T}$ we have

$$\bar v^0_\Delta(x)+b^0\le v^0(x)\le\bar v^0_\Delta(x)+a^0,$$

with at least one point attaining the equality in each inequality. Then, we have $\bar v^0_\Delta(x)+b^0\le\psi^1_\Delta(v^0;c)(x)\le\bar v^0_\Delta(x)+a^0$ for all $x\in\mathbb{T}$. Let a^1,b^1 be constants such that for all $x\in\mathbb{T}$ we have

$$\bar v^0_\Delta(x)+b^1\le\psi^1_\Delta(v^0;c)(x)\le\bar v^0_\Delta(x)+a^1,$$

with at least one point attaining the equality in each inequality. Note that $a^1\le a^0$ and $b^1\ge b^0$. Then, we have $\bar v^0_\Delta(x)+b^1\le\psi^2_\Delta(v^0;c)(x)\le\bar v^0_\Delta(x)+a^1$ for all $x\in\mathbb{T}$. In this way, we obtain the bounded monotone sequences a^j and b^j. Take d such that $\lim_{j\to\infty}b^j\le d\le\lim_{j\to\infty}a^j$. Then, for each $t\in\mathbb{N}$, $\bar v^0_\Delta+dI_1$ and $\psi^t_\Delta(v^0;c)$ coincide for at least one point. Let $x_0\in\mathbb{T}$ be such that $\bar v^0_\Delta(x_0)+d=\psi^t_\Delta(v^0;c)(x_0)$. Then, we obtain for all $x\in\mathbb{T}$,

$$|\psi^t_\Delta(v^0;c)(x)-\psi^t_\Delta(\bar v^0_\Delta+dI_1;c)(x)|\le\left|\int_{x_0}^x|\phi^t_\Delta(v^0_x;c)-\phi^t_\Delta(\bar u^0_\Delta;c)|dy\right|\le a\rho^t.$$

∎

We introduce the map $\bar h_\Delta(c):c\mapsto h(c)+b(c)$, which is the effective Hamiltonian of the difference Hamilton-Jacobi equation (1.6). Note that $\bar h_\Delta(c)$ plays an important role in the numerical analysis of the weak KAM theory. We investigate its properties.

3.3. **Effective Hamiltonian.** Below is the characterization of $\bar{h}_\Delta(c)$, which is very similar to that of the effective Hamiltonian $\bar{h}(c)$ of the exact Hamilton-Jacobi equations (1.2). We refer to [1] for the characterization of $\bar{h}(c)$.

Theorem 3.7. (1) *$h(c)=\bar{h}_\Delta(c)$ is the unique value for which* (1.6) *admits a space-time periodic difference solution.*

(2) *$\bar{h}_\Delta(c)$ is the averaged Hamiltonian. That is, for the space-time periodic difference solution $\bar{u}^k_m$ of* (1.5) *we have*

$$\bar{h}_\Delta(c)=\sum_{0\le k<2K}\sum_{m;k}H(x_m,t_k,c+\bar{u}^k_m(c))\cdot 2\Delta x\Delta t.$$

(3) *Let $v^{l+1}_n(c)$ be a time-global solution of the difference equation*

$$D_t v^{k+1}_m+H(x_m,t_k,c+D_x v^k_{m+1})=0. \tag{3.2}$$

Then, for all n we have

$$\lim_{l\to\infty}\frac{v^{l+1}_n(c)}{t_{l+1}}=-\bar{h}_\Delta(c).$$

(4) *$\bar{h}_\Delta(c)$ is a convex C^1-function.*

(5) *$\bar{h}_\Delta(c)$ uniformly converges to the exact effective Hamiltonian $\bar{h}(c)$ of* (1.2) *as $\Delta\to 0$:*

$$\sup_{c\in[c_0,c_1]}|\bar{h}_\Delta(c)-\bar{h}(c)|\le\beta_3\sqrt{\Delta x}.$$

Proof. (1) Let $\tilde{\bar{v}}^k_{m+1}$ be another space-time periodic solution of (1.6) with $h(c)=\tilde{\bar{h}}_\Delta(c)$. Extending the periodic solutions to the entire odd grid, we have the following stochastic and variational representation formulas up to any negative time index l_0:

$$\begin{aligned}
\bar{v}^{l+1}_n &= E_{\mu(\cdot;\xi^*)}\Big[\sum_{l_0<k\le l+1}L^{(c)}(\gamma^k,t_{k-1},{\xi^*}^k_{m(\gamma^k)})\Delta t+\bar{v}^{l_0}_{m(\gamma^{l_0})}\Big]\\
&\quad+\bar{h}_\Delta(c)(t_{l+1}-t_{l_0}),\\
\tilde{\bar{v}}^{l+1}_n &= E_{\mu(\cdot;\tilde{\xi}^*)}\Big[\sum_{l_0<k\le l+1}L^{(c)}(\gamma^k,t_{k-1},{\tilde{\xi}^*}{}^k_{m(\gamma^k)})\Delta t+\tilde{\bar{v}}^{l_0}_{m(\gamma^{l_0})}\Big]\\
&\quad+\tilde{\bar{h}}_\Delta(c)(t_{l+1}-t_{l_0}).
\end{aligned}$$

By the variational property, we have

$$\tilde{\bar{v}}^{l+1}_n-\bar{v}^{l+1}_n\le E_{\mu(\cdot;\xi^*)}\Big[\tilde{\bar{v}}^{l_0}_{m(\gamma^{l_0})}-\bar{v}^{l_0}_{m(\gamma^{l_0})}\Big]+(\tilde{\bar{h}}_\Delta(c)-\bar{h}_\Delta(c))(t_{l+1}-t_{l_0}). \tag{3.3}$$

Note that $\bar{v}^k_{m+1},\tilde{\bar{v}}^k_{m+1}$ are periodic and hence bounded. Dividing (3.3) by $t_{l+1}-t_{l_0}$ and letting $l_0\to-\infty$, we obtain

$$0\le\tilde{\bar{h}}_\Delta(c)-\bar{h}_\Delta(c).$$

Similar reasoning yields the converse inequality.

(2) Since $\bar{v}^k_{m+1}$ satisfies $D_t\bar{v}^{k+1}_m+H(x_m,t_k,c+D_x\bar{v}^k_{m+1})=\bar{h}_\Delta(c)$, we have

$$\begin{aligned}
\bar{h}_\Delta(c) &= \sum_{0\le k<2K}\sum_{m;k}D_t\bar{v}^{k+1}_m\cdot 2\Delta x\Delta t\\
&\quad+\sum_{0\le k<2K}\sum_{m;k}H(x_m,t_k,c+D_x\bar{v}^k_{m+1})\cdot 2\Delta x\Delta t.
\end{aligned}$$

The first term on the right-hand side is equal to zero due to the periodicity of $\bar v^k_{m+1}$.

(3) Let $\tilde v^{l+1}_n(c)$ be the solution of $D_t\tilde v^{k+1}_m + H(x_m, t_k, c + D_x\tilde v^k_{m+1}) = \bar h_\Delta(c)$ with the same mesh size as (3.2) and with $\tilde v^0_{m+1} = v^0_{m+1}$. From Theorem 3.6 it follows that we have $|\tilde v^{l+1}_n(c) - \bar v^{l+1}_n(c)| \to 0$ as $l \to \infty$, adding a constant if necessary. Therefore, $\tilde v^{l+1}_n$ is bounded for $l \to \infty$. Since

$$
\begin{aligned}
v^{l+1}_n(c) &= \inf_\xi E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le l+1} L^{(c)}(\gamma^k, t_{k-1}, \xi^k_{m(\gamma^k)})\Delta t + v^0_{m(\gamma^0)}\Big],\\
\tilde v^{l+1}_n(c) &= \inf_\xi E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le l+1} L^{(c)}(\gamma^k, t_{k-1}, \xi^k_{m(\gamma^k)})\Delta t + v^0_{m(\gamma^0)}\Big] + \bar h_\Delta(c)t_{l+1},
\end{aligned}
$$

and the minimizing velocity fields of these are the same, we obtain $v^{l+1}_n(c) - \tilde v^{l+1}_n(c) = -\bar h_\Delta(c)t_{l+1}$.

(4) Following the proof of (6) of Theorem 2.1 in [17], we can prove that $c+\bar u^k_m(c)$ is a C^1-function of c for each m, k. Therefore, claim (2) yields C^1-regularity of $\bar h_\Delta$. Let $v^{l+1}_n(c)$ be a solution of (3.2) and fix n. We show that the map $c \mapsto v^{l+1}_n(c)$ is a concave function for each $l+1 \ge 1$. Let ξ^* be the minimizing velocity field for $v^{l+1}_n(c^*)$ with $c^* := \theta c + (1-\theta)\tilde c$, $\theta \in [0,1]$. Then, we have

$$
\begin{aligned}
v^{l+1}_n(c^*) &- \{\theta v^{l+1}_n(c) + (1-\theta)v^{l+1}_n(\tilde c)\}\\
&\ge \theta E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le l+1} -(c^*-c){\xi^*}^k_{m(\gamma^k)}\Delta t\Big]\\
&\quad +(1-\theta)E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le l+1} -(c^*-\tilde c){\xi^*}^k_{m(\gamma^k)}\Delta t\Big]\\
&= 0.
\end{aligned}
$$

Therefore, the map $c \mapsto v^{l+1}_n(c)/t_{l+1}$ is also a concave function and

$$
\bar h_\Delta(c) = -\lim_{l\to\infty} \frac{v^{l+1}_n(c)}{t_{l+1}}
$$

is a convex function.

(5) From here on $b_1, b_2, \dots$ are positive constants independent of Δ and c. For each $x \in \mathbb{T}$, we have m such that $x \in [x_{m+1}, x_{m+3})$. Note that $\bar v^{2K}_{n_*+1} \le \bar v_\Delta(x,1) \le \bar v^{2K}_{n^*+1}$ with $(n_*, n^*) = (m, m+2)$ or $(n_*, n^*) = (m+2, m)$, and

$$
\begin{aligned}
\bar v^{2K}_{n_*+1} - \bar v(x_{n_*+1},1) - 2r\Delta x &\le \bar v_\Delta(x,1) - \bar v(x,1)\\
&\le \bar v^{2K}_{n^*+1} - \bar v(x_{n^*+1},1) + 2r\Delta x,
\end{aligned}
\tag{3.4}
$$

since $\bar u(\cdot,1) = \bar v_x(\cdot,1) \in L^\infty_{r,0}$.

Let $x \in \mathbb{T}$ attain $\max_{y\in\mathbb{T}}(\bar v_\Delta(y,1) - \bar v(y,1))$ and let n^* be defined in the above manner with this x. Let γ^* be a minimizing curve for $\bar v(x_{n^*+1}, t)$. Define ξ as $\xi^k_m := {\gamma^*}'(t_k)$. Note that, for all $\gamma \in \Omega$, the $\eta^k(\gamma)$ defined by this ξ satisfies $|\eta^k(\gamma) - \gamma^*(t_k)| \le b_1\Delta x$ for any $0 \le k \le 2K$. By the representation formulas and Proposition 2.5, we have

$$\bar v(x_{n^*+1},1)=\int_0^1 L^{(c)}(\gamma^*(s),s,\gamma^{*\prime}(s))ds+\bar v(\gamma^*(0),0)+\bar h(c),$$

(3.5)
$$\begin{aligned}\bar v^{2K}_{n^*+1}&\le E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le 2K}L^{(c)}(\gamma^k,t_{k-1},\xi^k_{m(\gamma^k)})\Delta t+\bar v_\Delta(\gamma^0,0)\Big]+\bar h_\Delta(c)\\ &\le E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le 2K}L^{(c)}(\eta^k(\gamma),t_{k-1},\xi^k_{m(\gamma^k)})\Delta t+\bar v_\Delta(\eta^0(\gamma),0)\Big]\\ &+\bar h_\Delta(c)+b_2\sqrt{\Delta x}\\ &\le E_{\mu(\cdot;\xi)}\Big[\sum_{0<k\le 2K}L^{(c)}(\gamma^*(t_k),t_{k-1},\gamma^{*\prime}(t_k))\Delta t+\bar v_\Delta(\gamma^*(0),0)\Big]\\ &+\bar h_\Delta(c)+b_3\sqrt{\Delta x}\\ &\le \int_0^1 L^{(c)}(\gamma^*(s),s,\gamma^{*\prime}(s))ds+\bar v_\Delta(\gamma^*(0),0)+\bar h_\Delta(c)+b_4\sqrt{\Delta x}.\end{aligned}$$

Therefore, noting (3.4), we have

$$\bar v_\Delta(x,1)-\bar v(x,1)\le \bar v_\Delta(\gamma^*(0),0)-\bar v(\gamma^*(0),0)+\bar h_\Delta(c)-\bar h(c)+b_5\sqrt{\Delta x}.$$

From the time-periodicity of $\bar v_\Delta,\bar v$ and the above choice of x, it follows that

$$(\bar v_\Delta(x,1)-\bar v(x,1))-(\bar v_\Delta(\gamma^*(0),0)-\bar v(\gamma^*(0),0))\ge 0.$$

Therefore we obtain

$$-b_5\sqrt{\Delta x}\le \bar h_\Delta(c)-\bar h(c).$$

Let $x\in\mathbb{T}$ attain $\min_{y\in\mathbb{T}}(\bar v_\Delta(y,1)-\bar v(y,1))$ and let n_* be defined in the above manner with this x. Let ξ^* be the minimizing velocity field for $\bar v^{2K}_{n_*+1}$. Then we have

$$\begin{aligned}\bar v^{2K}_{n_*+1}&=&E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le 2K}L^{(c)}(\gamma^k,t_{k-1},{\xi^*}^k_{m(\gamma^k)})\Delta t+\bar v_\Delta(\gamma^0,0)\Big]+\bar h_\Delta(c),\\ &\ge& E_{\mu(\cdot;\xi^*)}\Big[\sum_{0<k\le 2K}L^{(c)}(\eta^k(\gamma),t_{k-1},{\xi^*}^k_{m(\gamma^k)})\Delta t+\bar v_\Delta(\eta^0(\gamma),0)\Big]\\ &&+\bar h_\Delta(c)-b_6\sqrt{\Delta x}.\end{aligned}$$

Let $\eta_\Delta(\gamma)$ be the linear interpolation of $\eta^k(\gamma)$. Note that $\eta_\Delta(\gamma)'(t)={\xi^*}^k_{m(\gamma^k)}$ for $t\in(t_{k-1},t_k)$. For each γ we have

(3.6)
$$\begin{aligned}\bar v(x_{n_*+1},1)\le&\int_0^1 L^{(c)}(\eta_\Delta(\gamma)(s),s,\eta_\Delta(\gamma)'(s))ds+\bar v(\eta_\Delta(\gamma)(0),0)+\bar h(c)\\ \le&\sum_{0<k\le 2K}L^{(c)}(\eta^k(\gamma),t_{k-1},{\xi^*}^k_{m(\gamma^k)})\Delta t\\ &+\bar v(\eta^0(\gamma),0)+\bar h(c)+b_7\Delta x.\end{aligned}$$

Therefore, noting (3.4), we have

$$\begin{aligned}\bar v_\Delta(x,1)-\bar v(x,1)&\ge& E_{\mu(\cdot;\xi^*)}\Big[\bar v_\Delta(\eta^0(\gamma),0)-\bar v(\eta^0(\gamma),0)\Big]\\ &&+\bar h_\Delta(c)-\bar h(c)-b_8\sqrt{\Delta x}.\end{aligned}$$

From the time-periodicity of $\bar{v}_\Delta, \bar{v}$ and the above choice of x it follows that

$$(\bar{v}_\Delta(x,1) - \bar{v}(x,1)) - (\bar{v}_\Delta(\eta^0(\gamma),0) - \bar{v}(\eta^0(\gamma),0)) \le 0$$

for all γ. Thus we obtain

$$\bar{h}_\Delta(c) - \bar{h}(c) \le b_8\sqrt{\Delta x}. \qquad \square$$

3.4. **Convergence of periodic solutions.** We prove that for space-time periodic solutions the difference solutions converge to the exact ones up to a subsequence. Note that viscosity solutions and entropy solutions with space-time periodicity are not unique with respect to c in general. The selection problem in finite difference approximation remains open. It is also challenging to investigate details of the convergence even in the case where the uniqueness holds. We will make some progress with this issue in the next section.

Theorem 3.8. *There exists a sequence $\Delta = (\Delta x, \Delta t) \to 0$ for which $\{\bar{v}^{(c)}_\Delta\}$ and $\{\bar{u}^{(c)}_\Delta\}$ converge to a $\mathbb{Z}^2$-periodic viscosity solution $\bar{v}$ of* (1.2) *with $h(c) = \bar{h}(c)$ and to a $\mathbb{Z}^2$-periodic entropy solution $\bar{u} = \bar{v}_x$ of* (1.1)*, respectively:*

$$\sup_{t\in\mathbb{T}} \| \bar{v}^{(c)}_\Delta(\cdot,t) - \bar{v}(\cdot,t) \|_{C^0} \to 0, \qquad \sup_{t\in\mathbb{T}} \| \bar{u}^{(c)}_\Delta(\cdot,t) - \bar{u}(\cdot,t) \|_{L^1(\mathbb{T})} \to 0.$$

Proof. If necessary, we add a constant so that $\bar{v}^{(c)}_\Delta(\cdot,0)$ is bounded by r. Then, $\{\bar{v}^{(c)}_\Delta(\cdot,0)\}$ is a family of functions that are uniformly bounded and equicontinuous. We have a convergent subsequence, still denoted by $\bar{v}^{(c)}_\Delta(\cdot,0)$, with $\bar{v}^{(c)}_\Delta(\cdot,0) \to \bar{v}^0$ as $\Delta \to 0$. Let $\bar{v}$ be the viscosity solution of (1.4) with $v^0 = \bar{v}^0$ and $h(c) = \bar{h}(c)$. Then, we have a minimizing curve such that

$$\bar{v}(x_n, t_{l+1}) = \int_0^{t_{l+1}} L^{(c)}(\gamma^*(s), s, \gamma^{*\prime}(s))ds + \bar{v}^0(\gamma^*(0)) + \bar{h}(c)t_{l+1}.$$

By an estimate similar to (3.5), we have

$$\begin{aligned}\bar{v}_\Delta(x_n, t_{l+1}) \;\; &\le \;\; \int_0^{t_{l+1}} L^{(c)}(\gamma^*(s), s, \gamma^{*\prime}(s))ds + \bar{v}_\Delta(\gamma^*(0), 0) \\ &\quad + \bar{h}_\Delta(c)t_{l+1} + b_1\sqrt{\Delta x}.\end{aligned}$$

Since $\bar{h}_\Delta(c) \to \bar{h}(c)$, we obtain

$$\limsup_{\Delta\to 0}\{\bar{v}^{(c)}_\Delta(x_n, t_{l+1}) - \bar{v}(x_n, t_{l+1})\} \le 0,$$

which is uniform with respect to $(x_n, t_{l+1}) \in \mathbb{T}^2$. By an estimate similar to (3.6), we obtain

$$\liminf_{\Delta\to 0}\{\bar{v}^{(c)}_\Delta(x_n, t_{l+1}) - \bar{v}(x_n, t_{l+1})\} \ge 0,$$

which is uniform with respect to $(x_n, t_{l+1}) \in \mathbb{T}^2$. Therefore, we conclude that $\bar{v}^{(c)}_\Delta \to \bar{v}$ uniformly on $\mathbb{T}^2$ as $\Delta \to 0$ and $\bar{v}$ is $\mathbb{Z}^2$-periodic due to the periodicity of $\bar{v}_\Delta$.

Through reasoning similar to the proof of Theorem 2.8 in [22], it follows that $\bar{u}^{(c)}_\Delta := (\bar{v}^{(c)}_\Delta)_x$ converges to $\bar{u} = \bar{v}_x$ pointwise almost everywhere in $\mathbb{T}^2$, where $\{\bar{v}^{(c)}_\Delta\}$ is the convergent subsequence above. Hence, we have $\| \bar{u}^{(c)}_\Delta(\cdot,t) - \bar{u}(\cdot,t) \|_{L^1(\mathbb{T})} \to 0$

as $\Delta \to 0$ for each t. Through reasoning similar to the proof of Proposition 2.14 in [17], it follows that $\bar{u}^{(c)}_{\Delta}$ satisfies

$$\| \bar{u}^{(c)}_{\Delta}(\cdot, t_k) - \bar{u}^{(c)}_{\Delta}(\cdot, t_{k'}) \|_{L^1(\mathbb{T})} \leq b_2 |t_k - t_{k'}|$$

with a constant b_2 independent of k, k', c, and Δ. Therefore, $\bar{u} \in \mathrm{Lip}(\mathbb{T}; L^1(\mathbb{T}))$ with the Lipschitz constant b_2. Thus, we have demonstrated the theorem. □

4. Error estimates

We show error estimates for entropy solutions of initial value problems and for $\mathbb{Z}^2$-periodic entropy solutions in the special case where they are associated with KAM tori. The latter is a rigorous result on finite difference approximation of KAM tori. We refer to [2] for an error estimate for $\mathbb{Z}^2$-periodic entropy solutions associated with KAM tori in the limit of the vanishing viscosity method.

4.1. **Error estimates for initial value problem.** The following theorem provides error estimates for the initial value problem.

Theorem 4.1. *Let $T > 0$ be an arbitrary number. Let $\Delta = (\Delta x, \Delta t)$ satisfy the conditions in Theorem 2.6 and Proposition 2.8. Let u be the entropy solution of (1.3) and u_Δ be given by the difference solution of (2.4). Then the following hold:*

(1) *For each $t \in (0, T]$, there exists a constant $\beta_4(t) > 0$ independent of the initial data for which we have*

$$\| u_\Delta(\cdot, t) - u(\cdot, t) \|_{L^1(\mathbb{T})} \leq \beta_4(t) \Delta x^{\frac{1}{4}}.$$

In particular, if u^0 is rarefaction-free, then there exists a constant $\beta_5 > 0$ for which we have

$$\sup_{0 \leq t \leq T} \| u_\Delta(\cdot, t) - u(\cdot, t) \|_{L^1(\mathbb{T})} \leq \beta_5 \Delta x^{\frac{1}{4}}.$$

(2) *If u is Lipschitz in $\mathbb{T} \times [0, T]$, then there exists a constant $\beta_6 > 0$ for which we have*

$$\sup_{(x,t) \in \mathbb{T} \times [0,T]} |u_\Delta(x, t) - u(x, t)| \leq \beta_6 \Delta x^{\frac{1}{4}}.$$

Proof. (1) Let v_Δ and v correspond to u_Δ and u, respectively. By Theorem 2.6, for all $t \in [0, T]$ and all initial data, we have

$$\| v_\Delta(\cdot, t) - v(\cdot, t) \|_{C^0} \leq \beta_2 \sqrt{\Delta x}. \tag{4.1}$$

By Proposition 2.8, we have for each $t \in [\Delta t, T]$ and all initial data,

$$\frac{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)}{2\Delta x} \leq E^{k(t)}_{\Delta}. \tag{4.2}$$

Since $u_\Delta(\cdot, t)$ has the zero mean, we have

$$\begin{aligned} \sum_{m;k(t)} \{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)\} &= \sum_{m:+} \{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)\} \\ &\quad + \sum_{m:-} \{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)\} \\ &= 0, \end{aligned}$$

where $\sum_{m:+}$ (resp. $\sum_{m:-}$) stands for the summation with respect to m for which $u_\Delta(x_{m+2},t)-u_\Delta(x_m,t)\geq 0$ (resp. $u_\Delta(x_{m+2},t)-u_\Delta(x_m,t)<0$). Hence, it follows from (4.2) that the total variation of $u_\Delta(\cdot,t)$ on $\mathbb{T}$ is bounded:

$$(4.3)\quad \sum_{m;k(t)} |u_\Delta(x_{m+2},t)-u_\Delta(x_m,t)| = 2\sum_{m:+}\{u_\Delta(x_{m+2},t)-u_\Delta(x_m,t)\} \leq 2E_\Delta^{k(t)}.$$

For any $\varepsilon>0$, there exists $\tilde\Delta=(\tilde{\Delta x},\tilde{\Delta t})$ such that

$$\| u_{\tilde\Delta}(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbb{T})}\leq\varepsilon.$$

In particular, we take such a mesh parameter $\tilde\Delta=(\tilde{\Delta x},\tilde{\Delta t})$ that satisfies $\tilde{\Delta t}/\tilde{\Delta x}=\Delta t/\Delta x$, $\tilde{\Delta x}\leq(\beta_2^{-1}\varepsilon)^4$, and $\Delta x/\tilde{\Delta x}=3^p$ for some $p\in\mathbb{N}$. The last relation guarantees that the points of discontinuity of u_Δ are also those of $u_{\tilde\Delta}$. Then we have

$$\begin{aligned} \| u_\Delta(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbb{T})} &\leq \| u_{\tilde\Delta}(\cdot,t)-u_\Delta(\cdot,t)\|_{L^1(\mathbb{T})}+\varepsilon,\\ (4.4)\qquad \| v_{\tilde\Delta}(\cdot,t)-v_\Delta(\cdot,t)\|_{C^0} &\leq \beta_2\sqrt{\Delta x}+\beta_2\sqrt{\tilde{\Delta x}}\leq 2\beta_2\sqrt{\Delta x}.\end{aligned}$$

Now we estimate $\| u_{\tilde\Delta}(\cdot,t)-u_\Delta(\cdot,t)\|_{L^1(\mathbb{T})}$. We introduce $w_\Delta:=u_{\tilde\Delta}(\cdot,t)-u_\Delta(\cdot,t)$, $\tilde w_\Delta:=v_{\tilde\Delta}(\cdot,t)-v_\Delta(\cdot,t)$ and $\tilde k(t):=3^pk(t)$. Let $x_m\in\tilde{\Delta x}\mathbb{Z}$ and set $x_{m_0}:=0$ for $\tilde k(t)$ even or $x_{m_0}:=\tilde{\Delta x}$ for $\tilde k(t)$ odd. We divide $\tilde{\Delta x}\mathbb{Z}$ according to the sign of w_Δ. That is, $I_1,I_2,\dots,I_{n+1}$ are defined as

$$\begin{aligned} I_1 &:= \{x_{m_0},x_{m_0+2},\dots,x_{m_1}\} \text{ on which } w_\Delta(x)\geq 0 \text{ (or } <0),\\ I_2 &:= \{x_{m_1+2},x_{m_1+4},\dots,x_{m_2}\} \text{ on which } w_\Delta(x)< 0 \text{ (or } \geq 0),\\ I_3 &:= \{x_{m_2+2},x_{m_2+4},\dots,x_{m_3}\} \text{ on which } w_\Delta(x)\geq 0 \text{ (or } <0),\dots,\\ I_n &:= \{x_{m_{n-1}+2},x_{m_{n-1}+4},\dots,x_{m_n}\} \text{ on which } w_\Delta(x)< 0 \text{ (or } \geq 0),\\ I_{n+1} &:= \{x_{m_n+2},x_{m_n+4},\dots,x_{m_0}+1\} \text{ on which } w_\Delta(x)\geq 0 \text{ (or } <0),\end{aligned}$$

where n is even and $x_{m_n}\leq x_{m_0}+1-2\tilde{\Delta x}$. We then redefine I_1 as

$$I_1:=\{x_{m_0},x_{m_0+2},\dots,x_{m_1}\} \text{ with } x_{m_0}:=x_{m_n+2}-1.$$

Note that $w_\Delta(x)\geq 0$ (or <0) on I_1. Setting $|I_1|:=x_{m_1}-x_{m_0}+2\tilde{\Delta x}$ and $|I_j|:=x_{m_j}-x_{m_{j-1}+2}+2\Delta x$ for $j>1$, we have $\sum_{j=1}^n|I_j|=1$. For each I_j on which $w_\Delta(x)\geq 0$ (resp. <0), we have $y^j\in I_j$ for which $w_\Delta(x)$ takes the maximum (resp. minimum) within I_j. Suppose that $w_\Delta(x)\geq 0$ on I_1. In the other case, the argument is parallel. Note that

$$\| w_\Delta(x)\|_{L^1(\mathbb{T})}=\sum_{j=1}^{n/2}\left\{\sum_{x\in I_{2j-1}} w_\Delta(x)\cdot 2\tilde{\Delta x}-\sum_{x\in I_{2j}} w_\Delta(x)\cdot 2\tilde{\Delta x}\right\}.$$

Define

$$\begin{aligned} J &:= \{j\,|\,0\leq j\leq n/2,\ \max\{|I_{2j-1}|,|I_{2j}|\}<\Delta x^{1/4}\},\\ \tilde J &:= \{j\,|\,0\leq j\leq n/2,\ \max\{|I_{2j-1}|,|I_{2j}|\}\geq\Delta x^{1/4}\}.\end{aligned}$$

Then we have $\sharp\tilde{J}\cdot\Delta x^{1/4}\le 1$ and $\sharp\tilde{J}\le\Delta x^{-1/4}$. Therefore, noting (4.3) and (4.4) as well as $w_\Delta=(\tilde{w}_\Delta)_x$, we obtain

$$\begin{aligned}\| w_\Delta(x)\|_{L^1(\mathbb{T})} &= \sum_{j\in J}\left\{\sum_{x\in I_{2j-1}} w_\Delta(x)\cdot 2\tilde{\Delta x}-\sum_{x\in I_{2j}} w_\Delta(x)\cdot 2\tilde{\Delta x}\right\}\\ &\quad +\sum_{j\in\tilde{J}}\left\{\sum_{x\in I_{2j-1}} w_\Delta(x)\cdot 2\tilde{\Delta x}-\sum_{x\in I_{2j}} w_\Delta(x)\cdot 2\tilde{\Delta x}\right\}\\ &\le \sum_{j\in J}|w_\Delta(y^{2j-1})-w_\Delta(y^{2j})|\Delta x^{\frac14}\\ &\quad +\sum_{j\in\tilde{J}}\Big[\{\tilde{w}_\Delta(x_{m_{2j-1}}+\tilde{\Delta x})-\tilde{w}_\Delta(x_{m_{2j-2}+2}-\tilde{\Delta x})\}\\ &\quad -\{\tilde{w}_\Delta(x_{m_{2j}}+\tilde{\Delta x})-\tilde{w}_\Delta(x_{m_{2j-1}+2}-\tilde{\Delta x})\}\Big]\\ &\le (2E^{\tilde{k}(t)}_{\tilde{\Delta}}+2E^{k(t)}_\Delta)\Delta x^{\frac14}+\sharp\tilde{J}\cdot 4\cdot 2\beta_2\sqrt{\Delta x}\\ &\le 4E^{k(t)}_\Delta\Delta x^{\frac14}+8\beta_2\Delta x^{\frac14},\end{aligned}$$

Since ε is arbitrary, we conclude that

$$\| u_\Delta(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbb{T})}\le (4E^{k(t)}_\Delta+8\beta_2)\Delta x^{\frac14}.$$

If u^0 is rarefaction-free, we have $M>0$ such that $E^{k(t)}_\Delta\le\max\{M,E^*\}$ for all $0\le t\le T$.

(2) Fix $t\in[0,T]$ arbitrarily. By (4.1), for any $x,x'\in\mathbb{T}$ we have

$$\left|\int_{x'}^{x} u_\Delta(y,t)-u(y,t)dy\right|\le 2\beta_2\sqrt{\Delta x}.$$

Let $\tilde{u}_\Delta(x)$ denote the linear interpolation of $u^{k(t)}_m$ with respect to the space variable. From (4.3) it follows that $\|\tilde{u}_\Delta-u_\Delta(\cdot,t)\|_{L^1(\mathbb{T})}\le b_1\Delta x$. Hence, setting $w_\Delta:=\tilde{u}_\Delta-u(\cdot,t)$, for all $x,x'\in\mathbb{T}$ we have

$$\left|\int_{x'}^{x} w_\Delta(y)dy\right|\le b_2\sqrt{\Delta x}. \tag{4.5}$$

Since u is Lipschitz, w_Δ still satisfies the one-sided Lipschitz condition

$$\frac{w_\Delta(x_1)-w_\Delta(x_2)}{x_1-x_2}\le b_3.$$

Note that w_Δ does not necessarily satisfy any Lipschitz condition, because $\tilde{u}_\Delta$ does not necessarily satisfy any Lipschitz condition. Suppose that $|w_\Delta(\bar{x})|>b_4\Delta x^{\frac14}$ with $(b_4)^2/(4b_2)>b_3$ for some $\bar{x}$. Let $I\ni\bar{x}$ be a connected interval on whose boundary we have $|w_\Delta(x)|=\frac{b_4}{2}\Delta x^{\frac14}$. By (4.5), we find that

$$|I|\le\frac{2b_2}{b_4}\Delta x^{\frac14}.$$

If $w_\Delta(\bar{x})>0$ (resp. <0), and with the left (resp. right) boundary of I denoted by x, we have

$$\frac{w_\Delta(\bar{x})-w_\Delta(x)}{\bar{x}-x}\ge\frac{(b_4)^2}{4b_2}>b_3\quad\left(\text{resp. }\frac{w_\Delta(x)-w_\Delta(\bar{x})}{x-\bar{x}}\ge\frac{(b_4)^2}{4b_2}>b_3\right),$$

which is a contradiction. Therefore, we obtain

$$\parallel w_\Delta \parallel_{C^0} \le b_4 \Delta x^{\frac{1}{4}}.$$

Since $|u_\Delta(x,t) - u(x,t)| = |u_m^{k(t)} - u(x,t)| \le |u_m^{k(t)} - u(x_m,t)| + b_5\Delta x = |w_\Delta(x_m)| + b_5\Delta x$, we have demonstrated the theorem. ☐

4.2. **Error estimate for KAM tori.** Let $\bar{u}^{(c)} = \bar{v}_x^{(c)}$ be a $\mathbb{Z}^2$-periodic entropy solution of the C^1-class. We recall the relationship between such a $\bar{u}^{(c)}$ and Hamiltonian dynamics. Consider the time-1 map $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ of the Hamiltonian flow generated by the flux function $H(x,t,p)$ with the initial time equal to zero. Then, $\{(x, c+\bar{u}^{(c)}(x,0)) \,|\, x \in \mathbb{T}\} \cong \mathbb{T}$ is a smooth invariant torus of f. According to the classical result of Poincaré, there exists a rotation number ω_1. Let us regard the nonautonomous Hamiltonian dynamics generated by $H(x,t,p)$ as the autonomous dynamics generated by $\mathcal{H}(q_1,q_2,p_1,p_2) := p_2 + H(q_1,q_2,p_1)$ in the extended phase space $\mathbb{T}^2 \times \mathbb{R}^2$. We define

$$\mathcal{I}(\bar{u}^{(c)}) := \{(q, g(q)) \,|\, q = (q_1,q_2) \in \mathbb{T}^2\} \cong \mathbb{T}^2,$$

where $g(q) := (c + \bar{u}^{(c)}(q_1,q_2), \bar{h}(c) - H(q_1,q_2,c+\bar{u}^{(c)}(q_1,q_2))$. Then, $\mathcal{I}(\bar{u}^{(c)})$ is a smooth invariant torus of the Hamiltonian flow $\varphi^s_{\mathcal{H}}$ generated by $\mathcal{H}$. Let $C(s) := (\gamma^*(s), s)$ be the characteristic curves of $\bar{u}^{(c)}$, which satisfy $\gamma^{*\prime}(s) = H_p(\gamma^*(s), s, c + \bar{u}^{(c)}(\gamma^*(s),s))$ for $s \in \mathbb{R}$. The dynamics of the reduced characteristic curves $C^*(s) := C(s) \mod 1 = (\gamma^*(s) \mod 1, s \mod 1)$ and that of the trajectories on $\mathcal{I}(\bar{u}^{(c)})$ are identical; namely, for all $s \in \mathbb{R}$ we have

$$\varphi^s_{\mathcal{H}}(C^*(0), g(C^*(0))) = (C^*(s), g(C^*(s))).$$

According to the classical result of Poincaré, $C(s)/s$ converges to $\omega = (\omega_1, 1) \in \mathbb{R}^2$ independently of $C(0)$ as $|s| \to \infty$. This ω is called a rotation vector of $\mathcal{I}(\bar{u}^{(c)})$. If the rotation vector is irrational, each trajectory starting from a point of $\mathcal{I}(\bar{u}^{(c)})$ is dense on $\mathcal{I}(\bar{u}^{(c)})$. Therefore, we can obtain information on $\bar{u}^{(c)}$ from merely one characteristic curve, which is the crucial fact in the subsequent argument. Approximation of $\bar{u}^{(c)}$ leads to that of the invariant torus $\mathcal{I}(\bar{u}^{(c)})$.

Here we consider a special case where $\mathcal{I}(\bar{u}^{(c)})$ is a KAM torus. $\mathcal{I}(\bar{u}^{(c)})$ is called a KAM torus if it is smooth and the dynamics of $C^*(s)$ is C^1-conjugate to the dynamics of the linear flow on $\mathbb{T}^2$ with a Diophantine rotation vector (see [13], [16], [20]).

We say that c is associated with a KAM torus if $\bar{u}^{(c)}$ is C^1 and the dynamics of $C^*(s)$ is C^1-conjugate to that of a linear flow on $\mathbb{T}^2$ with a Diophantine rotation vector; namely, there exists a diffeomorphism $F : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$C^*(s) = F(\omega s + \theta),$$

where $\theta \in \mathbb{R}$ depends on $C^*(0)$ and $\omega \in \mathbb{R}^2$ satisfies the ν, τ-Diophantine condition

$$|\omega_1 z_1 + \omega_2 z_2| \ge \nu \parallel z \parallel_1^{-\tau} \quad \text{for all } z \in \mathbb{Z}^2 \setminus \{0\}.$$

If c is associated with a KAM torus, then $\bar{u}^{(c)}$ is the unique $\mathbb{Z}^2$-periodic entropy solution of (1.1) with that c. Regarding the existence of a value c associated with a KAM torus, we refer to the classical KAM theory for the autonomous Hamiltonian systems with two degrees of freedom generated by the above $\mathcal{H}(q_1,q_2,p_1,p_2)$. We remark that with additional assumptions the classical KAM theory under Rüssmann's nondegenerate condition (e.g., see [20]) works for such a degenerate $\mathcal{H}$ in (p_1,p_2). The following theorem provides error estimates for KAM tori.

Theorem 4.2. *Let $\Delta=(\Delta x,\Delta t)$ satisfy the conditions in Theorem 2.6, Proposition 2.8, and Theorem 3.1. Suppose that c is associated with a KAM torus. Let $\bar v^{(c)}$ be a $\mathbb{Z}^2$-periodic viscosity solution such that $\bar v^{(c)}_x=\bar u^{(c)}$. Then, the space-time periodic difference solutions $\bar v^{(c)}_\Delta$ and $\bar u^{(c)}_\Delta$ satisfy*

$$\sup_{(x,t)\in\mathbb{T}^2}|\bar v^{(c)}_\Delta(x,t)-\bar v^{(c)}(x,t)|\le\beta_7\Delta x^{\frac{1}{2(1+\tau)}},$$
$$\sup_{(x,t)\in\mathbb{T}^2}|\bar u^{(c)}_\Delta(x,t)-\bar u^{(c)}(x,t)|\le\beta_8\Delta x^{\frac{1}{4(1+\tau)}},$$

where β_7 and β_8 are independent of Δ.

Proof. Let $\bar v^{(c)}_\Delta$ be a periodic difference solution. In what follows, we omit the superscript (c) in $\bar v^{(c)}$, $\bar u^{(c)}$, etc. Fix $t\in\mathbb{T}$ arbitrarily. By adding a constant to $\bar v_\Delta$ if necessary, we have $\bar v_\Delta(x,t)-\bar v(x,t)\le 0$ for all $x\in\mathbb{T}$ and $\bar v_\Delta(x^*,t)-\bar v(x^*,t)=0$ for some $x^*\in\mathbb{T}$. Then, we have n^* and l such that

$$0=\bar v_\Delta(x^*,t)-\bar v(x^*,t)\le\bar v^l_{n^*+1}-\bar v(x_{n^*+1},t_l)+b_1\Delta x,$$

where $|x^*-x_{n^*+1}|\le 2\Delta x$ and $t\in[t_l,t_{l+1})$. For any $j\in\mathbb{N}$, we have a minimizing curve γ^* such that

$$\begin{aligned}\bar v(x_{n^*+1},t_l)&=\int_{-j+t_l}^{t_l}L^{(c)}(\gamma^*(s),s,\gamma^{*\prime}(s))ds+\bar v(\gamma^*(-j+t_l),-j+t_l)+\bar h(c)j,\\ \bar v^l_{n^*+1}&\le\int_{-j+t_l}^{t_l}L^{(c)}(\gamma^*(s),s,\gamma^{*\prime}(s))ds+\bar v_\Delta(\gamma^*(-j+t_l),-j+t_l)\\ &\quad+\bar h_\Delta(c)j+b_2\sqrt{\Delta x}j,\end{aligned}$$

where we use an estimate similar to (3.5). Hence, with claim 5 of Theorem 3.7 we obtain

$$\begin{aligned}0&\le\bar v^l_{n^*+1}-\bar v(x_{n^*+1},t_l)+b_1\Delta x\\ &\le\bar v_\Delta(\gamma^*(-j+t_l),-j+t_l)-\bar v(\gamma^*(-j+t_l),-j+t_l)\\ &\quad+(\bar h_\Delta(c)-\bar h(c))j+b_2\sqrt{\Delta x}j+b_1\Delta x\\ &\le\bar v_\Delta(\gamma^*(-j+t),t)-\bar v(\gamma^*(-j+t),t)+b_3\sqrt{\Delta x}j.\end{aligned}$$

Since $\bar v_\Delta(x,t)-\bar v(x,t)\le 0$ for all $x\in\mathbb{T}$, we then obtain

$$|\bar v_\Delta(\gamma^*(-j+t),t)-\bar v(\gamma^*(-j+t),t)|\le b_3\sqrt{\Delta x}j.$$

Since $C^*(-s+t):=(\gamma^*(-s+t),-s+t)\mod 1$ is a reduced characteristic curve, we have $C^*(-s+t)=F(\omega(-s+t)+\theta)$. From [7] and [3] it follows that the set

$$\mathcal{N}_\varepsilon:=\{\theta+\omega(-s+t)\mod 1\,|\,0\le s\le\frac{b_4}{\varepsilon^\tau}\}$$

is ε-dense on $\mathbb{T}^2$; namely,

$$\bigcup_{\zeta\in\mathcal{N}_\varepsilon}B_\varepsilon(\zeta)=\mathbb{T}^2,$$

where $B_\varepsilon(\zeta)=\{\tilde\zeta\in\mathbb{T}^2\,|\,\parallel\tilde\zeta-\zeta\parallel_1\le\varepsilon\}$. We define $\mathcal{T}:=F^{-1}(\mathbb{T}\times\{t\})$ and $X:=(x,t)$ for $x\in\mathbb{T}$. For each X, we have $\zeta\in\mathcal{N}_\varepsilon\cap\mathcal{T}$ such that $\parallel\tilde X-\zeta\parallel_1\le\varepsilon$ with $\tilde X:=F^{-1}(X)$ and such that $\zeta=\omega(-s^*+t)+\theta\mod 1$ with some $0\le s^*\le\frac{b_4}{\varepsilon^\tau}$.

Note that s^* must be an integer, because $F(\zeta) = C^*(-s^*+t) \in \mathbb{T}\times\{t\}$ and $-s^*+t$ mod $1 = t$. Hence, setting $s^* = j$, we have

$$\| X - C^*(-j+t) \|_1 = \| F(\tilde{X}) - F(\zeta) \|_1 \leq \| DF \|_{op}\, \varepsilon.$$

Therefore, for all $x \in \mathbb{T}$ we obtain

$$\begin{aligned} |\bar{v}_\Delta(x,t) - \bar{v}(x,t)| &\leq |\bar{v}_\Delta(F(\tilde{X})) - \bar{v}_\Delta(F(\zeta))| + |\bar{v}_\Delta(F(\zeta)) - \bar{v}(F(\zeta))| \\ &\quad + |\bar{v}(F(\zeta)) - \bar{v}(F(\tilde{X}))| \\ &\leq b_5\varepsilon + b_3\sqrt{\Delta x}\, j + b_5\varepsilon \\ &\leq b_6(\frac{\sqrt{\Delta x}}{\varepsilon^\tau} + \varepsilon). \end{aligned}$$

Taking $\varepsilon = \Delta x^{\frac{1}{2(1+\tau)}}$, for all $x \in \mathbb{T}$ we have

$$|\bar{v}_\Delta(x,t) - \bar{v}(x,t)| \leq 2b_6 \Delta x^{\frac{1}{2(1+\tau)}}.$$

Note that b_6 is independent of the choice of t. For $\bar{u}_\Delta = (\bar{v}_\Delta)_x$, $\bar{u} = \bar{v}_x$, and all $x, x' \in \mathbb{T}$, we have

$$|\int_{x'}^{x} \bar{u}_\Delta(y,t) - \bar{u}(y,t)dy| \leq 4b_6 \Delta x^{\frac{1}{2(1+\tau)}}.$$

Since $\bar{u}_\Delta$ satisfies the one-sided Lipschitz condition, we have $\| \tilde{\bar{u}}_\Delta - \bar{u}_\Delta(\cdot,t) \|_{L^1(\mathbb{T})} \leq b_7\Delta x$, where $\tilde{\bar{u}}_\Delta(x)$ denotes the linear interpolation of $\bar{u}^l_m$ with respect to the space variable. Setting $w_\Delta := \tilde{\bar{u}}_\Delta - \bar{u}(\cdot,t)$, for all $x, x' \in \mathbb{T}$ we have

$$|\int_{x'}^{x} w_\Delta(y)dy| \leq b_8 \Delta x^{\frac{1}{2(1+\tau)}}. \tag{4.6}$$

Since $\bar{u}$ is C^1, we know that w_Δ still satisfies the one-sided Lipschitz condition

$$\frac{w_\Delta(x_1) - w_\Delta(x_2)}{x_1 - x_2} \leq b_9.$$

Suppose that $|w_\Delta(\bar{x})| > b_{10}\Delta x^{\frac{1}{4(1+\tau)}}$ with $(b_{10})^2/(4b_8) > b_9$ for some $\bar{x}$. Let $I \ni \bar{x}$ be a connected interval on whose boundary we have $|w_\Delta(x)| = \frac{b_{10}}{2}\Delta x^{\frac{1}{4(1+\tau)}}$. By (4.6), we find that

$$|I| \leq \frac{2b_8}{b_{10}} \Delta x^{\frac{1}{4(1+\tau)}}.$$

If $w_\Delta(\bar{x}) > 0$ (resp. < 0), and with the left (resp. right) boundary of I denoted by x, we have

$$\frac{w_\Delta(\bar{x}) - w_\Delta(x)}{\bar{x} - x} \geq \frac{(b_{10})^2}{4b_8} > b_9 \quad \left(\text{resp. } \frac{w_\Delta(x) - w_\Delta(\bar{x})}{x - \bar{x}} \geq \frac{(b_{10})^2}{4b_8} > b_9\right),$$

which is a contradiction. Therefore, we obtain

$$\| w_\Delta \|_{C^0} \leq b_{10}\Delta x^{\frac{1}{4(1+\tau)}}.$$

Since $|\bar{u}_\Delta(x,t) - \bar{u}(x,t)| = |\bar{u}^l_m - \bar{u}(x,t)| \leq |\bar{u}^l_m - \bar{u}(x_m,t)| + b_{11}\Delta x = |w_\Delta(x_m)| + b_{11}\Delta x$, we have demonstrated the theorem. ☐

The point of our numerical approximation of KAM tori is that the embedding of each KAM torus is connected to a certain classical solution of the PDEs (1.1) and (1.2), which are then solved numerically. Note that the existence of such a classical solution is assumed in our argument. The regularity criterion of solutions to (1.1) and (1.2) under (A1)–(A4) remains an important open problem. An estimate of the error between $\bar{u}^{(c)}_{\Delta}$ and $\bar{u}^{(c)}$ without the Diophantine condition or without the condition $\bar{u}^{(c)} \in C^1$ also remains open. The latter is particularly interesting in the context of a rigorous treatment of numerical approximations of Aubry-Mather sets.

Finally, we describe in brief the idea of another numerical approach to KAM tori, which is based on the so-called a posteriori KAM theorem. Let f be the time-1 map given at the beginning of this subsection. If there exists a smooth embedding $U^* : \mathbb{T} \to \mathbb{T} \times \mathbb{R}$ which satisfies the functional equation

$$f \circ U(q) = U \circ T_{\omega_1}(q) \quad \text{for all } q \in \mathbb{T}, \tag{4.7}$$

where $\omega_1 \in \mathbb{R}$ and $T_{\omega_1}(q) := q + \omega_1$, then $U^*(\mathbb{T})$ is a smooth invariant torus of f on which the dynamics is C^1-conjugate to that of T_{ω_1} on $\mathbb{T}$. The standard classical KAM theory leads to the fact that, if ω_1 is a Diophantine number, a unique such U^* exists under certain conditions on f. The idea of the a posteriori KAM theorem is stated below. We regard (4.7) as $\mathcal{F}(U) := f \circ U - U \circ T_{\omega_1} = 0$ in a certain family W of smooth mappings: $\mathbb{T} \to \mathbb{T} \times \mathbb{R}$, where $\mathcal{F} : W \to W$.

Idea of a posteriori KAM Theorem. *Suppose that there exists $U^0 \in W$ such that $\mathcal{F}(U^0)$ is close to 0 in the norm of W. Then, there exists unique U^* such that $\mathcal{F}(U^*) = 0$ and $\| U^0 - U^* \|_W \leq C \| \mathcal{F}(U^0) \|_W$.*

In fact, this idea has been justified with the Diophantine condition of ω_1 in many studies. The a posteriori KAM theorem describes both the existence of KAM tori and their numerical approximation, since a suitable U^0 can be numerically constructed through Newton's method. Moreover, the a posteriori KAM theorem can be successfully applied to find the magnitude of perturbation at which the classical KAM theory breaks down. We point to [4] for a nice presentation and survey with plenty of references for the a posteriori KAM theorem and its applications. The a posteriori KAM theorem provides no information on the situation after the classical KAM theory breaks down. On the other hand, the weak KAM theory still guarantees the existence of Aubry-Mather sets with arbitrary rotation numbers. It will be an important contribution to recast the results of the classical KAM theory or the weak KAM theory in terms of the other, including a more detailed comparison of our result with those based on the a posteriori KAM theorem.

Recently the author announced a partial result on selection problems of $\mathbb{Z}^2$-periodic entropy solutions and viscosity solutions in finite difference approximation [23].

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