MATHEMATICS OF COMPUTATION
Volume 85, Number 302, November 2016, Pages 2715–2742
http://dx.doi.org/10.1090/mcom/3091
Article electronically published on March 22, 2016

SUPERCONVERGENT HDG METHODS FOR LINEAR, STATIONARY, THIRD-ORDER EQUATIONS IN ONE-SPACE DIMENSION

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ABSTRACT. We design and analyze *the first* hybridizable discontinuous Galerkin methods for stationary, third-order linear equations in one-space dimension. The methods are defined as discrete versions of characterizations of the exact solution in terms of local problems and transmission conditions. They provide approximations to the exact solution u and its derivatives $q := u'$ and $p := u''$ which are piecewise polynomials of degree k_u, k_q and k_p, respectively. We consider the methods for which the difference between these polynomial degrees is at most two. We prove that all these methods have superconvergence properties which allows us to prove that their numerical traces converge at the nodes of the partition with order at least $2\,k+1$, where k is the minimum of k_u, k_q, and k_p. This allows us to use an element-by-element postprocessing to obtain new approximations for u, q and p converging with order at least $2k+1$ uniformly. Numerical results validating our error estimates are displayed.

1. Introduction

We design, analyze and numerically test the performance of the *first* hybridizable discontinuous Galerkin (HDG) methods for the following linear, stationary, third-order model problem:

$$u''' + ru = f \qquad \text{on } \Omega := (0,\ 1), \tag{1.1a}$$

$$u = u_D \qquad \text{at } \partial\Omega := \{0,1\}, \tag{1.1b}$$

$$u' = q_N \qquad \text{at } x = 1, \tag{1.1c}$$

where $f \in L^2(\Omega)$ and r is a non-negative constant. This is a necessary stepping stone towards our long-term goal of constructing HDG methods for steady-state and time-dependent equations involving third-order derivatives in space, like the Korteweg-de Vries (KdV) [1–3, 17, 26, 27] and the Zakharov-Kuznetsov [19, 22, 28] equations.

Our contribution is part of an ongoing effort to develop discontinuous Galerkin (DG) methods for partial differential equations involving third-order derivatives. Indeed, in 2002 the first DG method, the local discontinuous Galerkin (LDG) method, for the KdV equation was introduced in [25] and further studied in [18, 23].

Received by the editor May 18, 2014 and, in revised form, February 2, 2015.
2010 *Mathematics Subject Classification.* Primary 65M60, 65N30.
The research of the second author was partially supported by the National Science Foundation (grant DMS-1115331).
The research of the third author was partially supported by the National Science Foundation (grant DMS-1419029).

In [9], a DG method for the KdV equation was devised by using repeated integration by parts. For these two methods, optimal convergence rates were observed in numerical experiments, but only sub-optimal error estimates were proved. In [24], the error estimates of the LDG method discussed in [23, 25] were improved since the solution and its derivatives were shown to have optimal convergence rates. This result was extended in [16] where a superconvergence rate of $k + 3/2$ of the LDG method is obtained for the linearized KdV equation by using a special projection of the initial condition. Finally, let us mention that in [4], a conservative DG method was developed for the KdV equation, which has the unique feature of preserving the mass and the L^2-norm of the solution. Suboptimal convergence rate was proved and shown to be sharp for odd polynomial approximations; numerical tests showed that the convergence rate is optimal for even polynomial degrees.

These methods use explicit time-marching schemes since the coefficient before the third-order derivative is very small. However, when such a coefficient is of order one, for example, implicit time-marching methods might be the methods of choice. Thus, although the methods we introduce here can also be used for explicit time-marching schemes, their use becomes significantly advantageous with implicit time-marching schemes like BDF, DIRK or even DG methods. In particular, when used with BDF or DIRK methods, a steady-state equation of the form given by our model problem (with r proportional to $1/\Delta t$) has to be solved several times per time step; see, for example, [21]. The methods proposed here can thus be efficiently implemented especially given that, in our one-dimensional setting, the sparsity structure of the global matrix to be numerically inverted is independent of the polynomial degree of the approximations, its size is only $2N + 1$, where N is the number of intervals of the mesh, and its condition number is of the order of h^{-2}, where h denotes the size of the intervals of the mesh.

To construct the HDG methods for our model problem, we follow the approach introduced in 2009 [10] for the devising of those methods for diffusion problems. Thus, given any mesh of the domain, we show that the exact solution can be obtained as follows. First, on each element, we provide data for solving the equation therein and then find the equations satisfied by the data by imposing the transmission conditions and the boundary conditions. We show six different ways of doing this for which the data are boundary conditions for the local problems. Since the HDG methods are obtained as a discrete version of these characterizations, we ensure that the only globally-coupled degrees of freedom are those associated to the data of the local problems. In particular, we can take as data of the local problems the approximation to the trace of u at both ends of the interval and the approximation to the trace of u'' at the right-most end. The resulting global matrix has the same size, sparsity structure and even condition number as the corresponding matrix of the exact solution.

On each element, the local problem is defined by using a discontinuous Galerkin method to approximate the exact solution u and its derivatives $q := u'$ and $p := u''$ by polynomials of degree k_u, k_q and k_p, respectively. We find simple conditions on the stabilization function and on (k_u, k_q, k_p) guaranteeing that a projection of the errors in u, q and p converges with order at least $k + 2$ in the L^2-norm, where k is the minimum of k_u, k_q and k_p, for $k \geq 1$. We also show that the errors of all the numerical traces at the nodes of the partition are of order at least $2\,k+1$. This allows us to use a fast, element-by-element postprocessing to obtain new approximations for u, q and p converging with order at least $2k + 1$ uniformly.

Three methods have emerged as particularly interesting. The first is the one for which $(k_u, k_q, k_p) := (k, k, k)$. It seems to be quite robust with respect to the choice of the stabilization function, and provides a converging postprocessed solution with order $2k+1$ with the least amount of degrees of freedom. It seems to be the method of choice for this one-dimensional setting. Moreover, limit cases of this method are the two local discontinuous Galerkin (LDG) methods introduced and studied in [23–25] for the time-dependent version of our steady-state model problem with *periodic* boundary conditions. In [24], it was shown that the method provides approximations converging to u, q and p with the optimal order $k+1$ in the L^2-norm. Our results give the same orders of convergence for the steady-state case, and provide, in addition, the above-mentioned superconvergence results.

The other two methods worth mentioning provide the highest superconvergence orders for the lowest number of degrees of freedom. They correspond to the choices $(k_u, k_q, k_p) := (k+1, k, k+1)$ and $(k_u, k_q, k_p) := (k+1, k, k+2)$. Indeed, these methods can provide superconvergence approximations to u, q and p with orders $(k+4, k+3, k+3)$ and $(k+5, k+4, k+3)$ for $k \geq 2$ and $k \geq 3$, respectively.

We achieve our theoretical results by using two main ingredients. The first is the introduction of a new projection tailored to the structure of the numerical traces. This technique is an extension to HDG methods of the classical employed to analyze mixed methods; see [5]; it was introduced for HDG methods in [11]. The second is the direct use of a *duality* argument; no *energy* argument is used. Thus, our approach to estimate the approximations over the whole domain is similar to the one used to study HDG methods for the fourth-order problem of Timoshenko beams in [8]. On the other hand, the approach we use to obtain the estimates of the numerical traces follows along the lines of the analysis used in [7] for studying DG and mixed methods for convection-diffusion problems in one-space dimension.

The paper is organized as follows. In Section 2, we define the HDG method and state and discuss our main results. The details of all the proofs are given in Section 3. We display the numerical results in Section 4 and end with some concluding remarks in Section 5.

2. Main results

In this section, we state and discuss our main results. We begin by describing the characterizations of the exact solution the HDG methods are a discrete version of. We then introduce the methods and state and discuss the main results of our a priori error analysis. We end this section by showing how to use the structure of the method to implement it efficiently even in the case in which some values of the stabilization function go to infinity.

2.1. **Characterizations of the exact solution.** To display the characterizations of the exact solution we are going to work with, let us first rewrite our third-order model equation as the following first-order system:

$$q - u' = 0 \quad \text{in } \Omega, \tag{2.1a}$$

$$p - q' = 0 \quad \text{in } \Omega, \tag{2.1b}$$

$$p' + ru = f \quad \text{in } \Omega, \tag{2.1c}$$

and, let us introduce a partition of the domain Ω,

$$\mathcal{T}_h = \{I_i := (x_{i-1}, x_i) : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1\},$$

and the set of the boundaries of its elements, $\partial\mathcal{T}_h := \{\partial I_i : i = 1, \dots, N\}$. We also set $\mathscr{E}_h := \{x_i\}_{i=0}^N$, $h_i = x_i - x_{i-1}$ and $h := \max_{i=1}^N h_i$.

We know that, when f is smooth enough, if we provide the values $\{\widehat{u}_i\}_{i=0}^N$ and $\{\widehat{p}_i\}_{i=1}^N$ and, for each $i = 1, \dots, N$, solve the local problem

$$Q - U' = 0, \qquad P - Q' = 0, \quad P' + rU = f \quad \text{in } I_i, \tag{2.2a}$$

$$U(x_{i-1}^+) = \widehat{u}_{i-1}, \quad U(x_i^-) = \widehat{u}_i, \quad P(x_i^-) = \widehat{p}_i, \tag{2.2b}$$

then (P, Q, U) coincides with the solution (p, q, u) of our model problem if and only if the transmission conditions

$$Q(x_i^-) = Q(x_i^+), \quad P(x_i^-) = P(x_i^+), \qquad i = 1, \dots, N-1, \tag{2.2c}$$

and the boundary conditions

$$U(0) = u_D(0), \quad U(1) = u_D(1), \quad \text{and} \quad Q(1) = q_N(1), \tag{2.2d}$$

are satisfied. This characterization is succinctly displayed in the first row of Table 1 where we denote the jump of the function φ across the interior point x_i by

$$[\![\varphi]\!](x_i) := \varphi(x_i^-) - \varphi(x_i^+) \quad \text{for} \quad j = 1, \dots, N-1,$$

where $\varphi(x^\pm) := \lim_{\varepsilon \downarrow 0} \varphi(x \pm \varepsilon)$. Five other possible characterizations of the exact solution are also displayed in Table 1. Note that for these characterizations, the data of the local problems are the unknowns of a global problem and the system of equations for the global unknowns is square. Indeed, for the first three characterizations, the number of global unknowns is $2N + 1$; there are $2(N - 1)$ transmission conditions and three boundary conditions. Similarly, for the last three characterizations, the number of global unknowns is $3N$; there are $3(N - 1)$ transmission conditions and three boundary conditions.

Table 1. Characterizations of the exact solution.

characterization	data on ∂I_i			transmission conditions
I	$\widehat{u}_{i-1}$	$\widehat{u}_i$	$\widehat{p}_i$	$[\![Q]\!] = [\![P]\!] = 0$
II	$\widehat{u}_{i-1}$	$\widehat{u}_i$	$\widehat{q}_i$	$[\![Q]\!] = [\![P]\!] = 0$
III	$\widehat{q}_{i-1}$	$\widehat{u}_i$	$\widehat{q}_i$	$[\![U]\!] = [\![P]\!] = 0$
IV	$\widehat{p}_{i-1}$	$\widehat{u}_i$	$\widehat{q}_i$	$[\![U]\!] = [\![Q]\!] = [\![P]\!] = 0$
V	$\widehat{u}_{i-1}$	$\widehat{p}_i$	$\widehat{q}_i$	$[\![U]\!] = [\![Q]\!] = [\![P]\!] = 0$
VI	$\widehat{q}_{i-1}$	$\widehat{u}_i$	$\widehat{p}_i$	$[\![U]\!] = [\![Q]\!] = [\![P]\!] = 0$

Let us illustrate the characterization I for uniform partitions and $r = 0$; in this case the computations are particularly easy. Indeed, the solution of the local problem is $(P, Q, U) = (U''_{\widehat{u}}, U'_{\widehat{u}}, U_{\widehat{u}}) + (U''_{\widehat{p}}, U'_{\widehat{p}}, U_{\widehat{p}}) + (U''_f, U'_f, U_f)$, where $U_{\widehat{u}}(x) = \frac{x - x_{i-1}}{h}\widehat{u}_i + \frac{x_i - x}{h}\widehat{u}_{i-1}$, $U_{\widehat{p}}(x) = -\frac{1}{2}(x_i - x)(x - x_{i-1})\,\widehat{p}_i$ and $U_f(x) = \int_{I_i} G_i(x, s)\, f(s)\, ds$. Here G_i is the Green's function associated to the corresponding homogeneous boundary conditions. The transmission conditions are simply

$$\begin{aligned} \frac{1}{h}(\widehat{u}_{i-1} - 2\,\widehat{u}_i + \widehat{u}_{i+1}) - \frac{h}{2}(\widehat{p}_i + \widehat{p}_{i-1}) &= [\![U'_f]\!](x_i) \\ -(\widehat{p}_i - \widehat{p}_{i+1}) &= [\![U''_f]\!](x_i), \end{aligned}$$

for $i = 1, \ldots, N-1$, and the boundary conditions are

$$\widehat{u}_0 = u_D(0), \quad \widehat{u}_N = u_D(1), \quad -\frac{1}{2}\widehat{p}_N - \frac{\widehat{u}_N - \widehat{u}_{N-1}}{h} = U_f'(1) - q_N(1).$$

The above global matrix has condition number of order h^{-2}. The HDG methods based on this characterization produce global matrices of exactly the same size, sparsity structure and condition number.

2.2. **The HDG methods associated with characterization I.** To define these methods, we begin by noting that they provide approximations $(u_h, q_h, p_h, \widehat{u}_h, \widehat{q}_h, \widehat{p}_h)$ to $(u|_\Omega, q|_\Omega, p|_\Omega, u|_{\mathscr{E}_h}, q|_{\mathscr{E}_h}, p|_{\mathscr{E}_h})$ in the space $W_h^{k_u} \times W_h^{k_q} \times W_h^{k_p} \times L^2(\mathscr{E}_h) \times L^2(\partial\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h)$ where

$$W_h^k = \{w \in L^2(\mathcal{T}_h) : \quad w|_K \in P_k(K) \quad \forall\, K \in \mathcal{T}_h\}. \tag{2.3a}$$

Here $P_k(K)$ is the space of polynomials of degree at most k on the domain K. For any ζ lying in $L^2(\partial\mathcal{T}_h)$, we denote its values on $\partial I_i : \{x_{i-1}^+, x_i^-\}$ by $\zeta(x_{i-1}^+)$ (or simply ζ_{i-1}^+) and $\zeta(x_i^-)$ (or simply ζ_i^-). Note that $\zeta(x_i^+)$ is not necessarily equal to $\zeta(x_i^-)$. In contrast, for any η in the space $L^2(\mathscr{E}_h)$, its value at x_i, $\eta(x_i)$ (or simply η_i) is uniquely defined; in this case, $\eta(x_i^-)$ or $\eta(x_i^+)$ mean nothing but $\eta(x_i)$.

To determine the approximations, we simply use a discrete version of the characterization I of the exact solution. Thus, we assume that we are given the values $\{\widehat{u}_{hi}\}_{i=0}^N$ and $\{\widehat{p}_{hi}^-\}_{i=1}^N$ and, for each I_i, $i = 1, \ldots, N$, solve a local problem by using a Galerkin method. To describe it, we are going to use the following notation. By $(\varphi, v)_{I_i}$, we denote the integral of φ times v on the interval I_i, and by $\langle \varphi, vn\rangle_{\partial I_i}$ we simply mean the expression $\varphi(x_i^-)v(x_i^-)n(x_i^-) + \varphi(x_{i-1}^+)v(x_{i-1}^+)n(x_{i-1}^+)$. Here n denotes the outward unit normal to I_i: $n(x_{i-1}^+) := -1$ and $n(x_i^-) := 1$.

On the element I_i, we give f and the boundary data $\widehat{u}_{h\,i-1}, \widehat{u}_{h\,i}$ and $\widehat{p}_{h\,i}^-$ and take $(p_h, q_h, u_h) \in P_{k_p}(I_i) \times P_{k_q}(I_i) \times P_{k_u}(I_i)$ to be the solution of the equations

$$\begin{aligned}
(q_h, v)_{I_i} + (u_h, v')_{I_i} - \langle \widehat{u}_h, vn\rangle_{\partial I_i} &= 0,\\
(p_h, z)_{I_i} + (q_h, z')_{I_i} - \langle \widehat{q}_h, zn\rangle_{\partial I_i} &= 0,\\
(ru_h, w)_{I_i} - (p_h, w')_{I_i} + \langle \widehat{p}_h, wn\rangle_{\partial I_i} &= (f, w)_{I_i},
\end{aligned}$$

for all $(v, z, w) \in P_{k_q}(I_i) \times P_{k_u}(I_i) \times P_{k_p}(I_i)$, where the remaining undefined numerical traces are given by

$$\begin{cases}
\widehat{p}_h = p_h + \tau_{pu}\,(\widehat{u}_{h\,i-1} - u_h)\,n & \text{at } x_{i-1}^+,\\
\widehat{q}_h = q_h + \tau_{qu}\,(\widehat{u}_{h\,i-1} - u_h)\,n & \text{at } x_{i-1}^+,\\
\widehat{q}_h = q_h + \tau_{qu}\,(\widehat{u}_{h\,i} - u_h)\,n + \tau_{qp}\,(\widehat{p}_{h\,i}^- - p_h)\,n & \text{at } x_i^-.
\end{cases}$$

The functions τ_{qu}, τ_{qp} and τ_{pu} are defined on $\partial\mathcal{T}_h$ and are called the components of the *stabilization function*; they have to be properly chosen to ensure that the above problem has a unique solution.

It remains to impose the transmission conditions

$$[\![\widehat{q}_h]\!](x_i) = 0, \quad \text{and} \quad [\![\widehat{p}_h]\!](x_i) = 0 \qquad \text{for all} \quad i = 1, \ldots, N-1,$$

and the boundary conditions

$$\widehat{u}_h(0) = u_D(0), \quad \widehat{u}_h(1) = u_D(1), \quad \text{and} \quad \widehat{q}_h(1) = q_N(1).$$

This completes the definition of the HDG methods using the characterization I. Let us emphasize the fact that this way of introducing the HDG methods immediately provides a way to implement them.

On the other hand, the above presentation of the HDG methods is not very well suited for their analysis, or for comparison with HDG methods associated to other characterizations of the exact solution. This is why we now rewrite it in a more compact form. The approximation provided by the HDG methods, $(u_h, q_h, p_h, \widehat{u}_h, \widehat{q}_h, \widehat{p}_h)$, is the element of $W_h^{k_u} \times W_h^{k_q} \times W_h^{k_p} \times L^2(\mathscr{E}_h) \times L^2(\partial\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h)$ which solves the equations

$$(q_h, v)_{\mathcal{T}_h} + (u_h, v')_{\mathcal{T}_h} - \langle \widehat{u}_h, vn\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.4a}$$

$$(p_h, z)_{\mathcal{T}_h} + (q_h, z')_{\mathcal{T}_h} - \langle \widehat{q}_h, zn\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.4b}$$

$$(ru_h, w)_{\mathcal{T}_h} - (p_h, w')_{\mathcal{T}_h} + \langle \widehat{p}_h, wn\rangle_{\partial\mathcal{T}_h} = (f, w)_{\mathcal{T}_h}, \tag{2.4c}$$

for all $(v, z, w) \in W_h^{k_q} \times W_h^{k_u} \times W_h^{k_p}$, where, on $\partial\mathcal{T}_h$, we have

$$\begin{cases} \widehat{p}_h^{\,+} = p_h^+ + \tau_{pu}^+ \,(\widehat{u}_h - u_h^+)\, n^+, \\ \widehat{q}_h^{\,+} = q_h^+ + \tau_{qu}^+ \,(\widehat{u}_h - u_h^+)\, n^+, \\ \widehat{q}_h^{\,-} = q_h^- + \tau_{qu}^- \,(\widehat{u}_h - u_h^-)\, n^- + \tau_{qp}^- \,(\widehat{p}_h^{\,-} - p_h^-)\, n^-, \end{cases} \tag{2.4d}$$

and

$$[\![\widehat{q}_h]\!](x_i) = 0, \quad [\![\widehat{p}_h]\!](x_i) = 0 \qquad \text{for} \quad i = 1, \dots, N-1, \tag{2.4e}$$

$$\widehat{u}_h(0) = u_D(0), \quad \widehat{u}_h(1) = u_D(1), \quad \widehat{q}_h(1) = q_N(1). \tag{2.4f}$$

2.3. **The HDG methods associated to the other characterizations.** It is not difficult to find the HDG methods associated to the other characterizations of the exact solution. For example, the HDG methods associated to characterization II, have the same compact form (2.4) except that now the global unkowns are $\{\widehat{u}_{hi}\}_{i=0}^N$ and $\{\widehat{q}_{hi}^{\,-}\}_{i=1}^N$, and the remaining numerical traces are given by

$$\begin{cases} \widehat{q}_h^{\,+} = q_h^+ + \tau_{qu}^+ \,(\widehat{u}_h - u_h^+)\, n^+, \\ \widehat{p}_h^{\,+} = p_h^+ + \tau_{pu}^+ \,(\widehat{u}_h - u_h^+)\, n^+, \\ \widehat{p}_h^{\,-} = p_h^- + \tau_{pu}^-(\widehat{u}_h - u_h^-)\, n^- + \tau_{pq}^- \,(\widehat{q}_h^{\,-} - q_h^-)\, n^-. \end{cases}$$

This means that we recover the HDG method associated with characterization I if we can take $\tau_{pu}^- := -\tau_{qu}^-\, n^-/\tau_{qp}^-$ and $\tau_{pq}^- := 1/\tau_{qp}^-$, that is, if $\tau_{qp}^- \neq 0$. It is now not difficult to see that the very *same* HDG methods are obtained by using any of the six characterizations of the exact solution, provided that the corresponding stabilization function allows for the transition from one characterization to the other; see also [12], where four characterizations were used to define HDG methods for the Stokes equations of incompressible fluid flow. This is not surprising, as the exact solution can itself be characterized in those ways. Thus, the particular characterization we use is more relevant for the actual implementation of the method, rather than for its actual definition. We compare the six characterizations giving rise to the HDG methods in Table 2, where we display the corresponding global unknowns and the transmission conditions, and in Table 3, where we give the nonzero components of the stabilization function for each characterization in terms of the stabilization function of the first one.

TABLE 2. Characterizations of the HDG methods.

characterization	data on ∂I_i			transmission conditions
I	$\widehat{u}_{hi-1}$	$\widehat{u}_{hi}$	$\widehat{p}_{hi}^-$	$[\![\widehat{q}_h]\!] = [\![\widehat{p}_h]\!] = 0$
II	$\widehat{u}_{hi-1}$	$\widehat{u}_{hi}$	$\widehat{q}_{hi}^-$	$[\![\widehat{q}_h]\!] = [\![\widehat{p}_h]\!] = 0$
III	$\widehat{q}_{hi-1}$	$\widehat{u}_{hi}^-$	$\widehat{q}_{hi}$	$[\![\widehat{u}_h]\!] = [\![\widehat{p}_h]\!] = 0$
IV	$\widehat{p}_{hi-1}^+$	$\widehat{u}_{hi}^-$	$\widehat{q}_{hi}^-$	$[\![\widehat{u}_h]\!] = [\![\widehat{q}_h]\!] = [\![\widehat{p}_h]\!] = 0$
V	$\widehat{u}_{hi-1}^+$	$\widehat{p}_{hi}^-$	$\widehat{q}_{hi}^-$	$[\![\widehat{u}_h]\!] = [\![\widehat{q}_h]\!] = [\![\widehat{p}_h]\!] = 0$
VI	$\widehat{q}_{hi-1}^+$	$\widehat{u}_{hi}^-$	$\widehat{p}_{hi}^-$	$[\![\widehat{u}_h]\!] = [\![\widehat{q}_h]\!] = [\![\widehat{p}_h]\!] = 0$

TABLE 3. Components of the stabilization function for each of the characterizations of the HDG methods expressed in terms of the stabilization function of characterization I.

char.	components of the stabilization function τ			
I	τ_{pu}^+	τ_{qu}^+	τ_{qu}^-	τ_{qp}^-
II	τ_{pu}^+	τ_{qu}^+	$\tau_{pu}^- = -\tau_{qu}^- \, n^-/\tau_{qp}^-$	$\tau_{pq}^- = 1/\tau_{qp}^-$
III	$\tau_{pq}^+ = \tau_{pu}^+ \, n^+/\tau_{qu}^+$	$\tau_{uq}^+ = 1/\tau_{qu}^+$	$\tau_{pu}^- = -\tau_{qu}^- \, n^-/\tau_{qp}^-$	$\tau_{pq}^- = 1/\tau_{qp}^-$
IV	$\tau_{up}^+ = 1/\tau_{pu}^+$	$\tau_{qp}^+ = \tau_{qu}^+ \, n^+/\tau_{pu}^+$	$\tau_{pu}^- = -\tau_{qu}^- \, n^-/\tau_{qp}^-$	$\tau_{pq}^- = 1/\tau_{qp}^-$
V	τ_{pu}^+	τ_{qu}^+	$\tau_{uq}^- = 1/\tau_{qu}^-$	$\tau_{up}^- = -\tau_{qp}^- \, n^-/\tau_{qu}^-$
VI	$\tau_{pq}^+ = \tau_{pu}^+ \, n^+/\tau_{qu}^+$	$\tau_{uq}^+ = 1/\tau_{qu}^+$	τ_{qu}^-	τ_{qp}^-

TABLE 4. Numerical traces of the HDG methods for which the stabilization function is zero in terms of the characterization of the exact solution they were derived from.

characterization	τ_{pu}^+	τ_{qu}^+	τ_{qu}^-	τ_{qp}^-	$\widehat{u}_h$	$\widehat{q}_h$	$\widehat{p}_h$
I	0	0	0	0	unknown	$q_h^+ = q_h^-$	p_h^+
II	0	0	0	∞	unknown	q_h^+	$p_h^+ = p_h^-$
III	0	∞	0	∞	u_h^+	unknown	$p_h^- = p_h^+$
IV	∞	0	0	∞	u_h^+	q_h^+	p_h^-
V	0	0	∞	0	u_h^-	q_h^+	p_h^+
VI	0	∞	0	0	u_h^+	q_h^-	p_h^+

In Table 3, we can see that these characterizations are *not* equivalent only when some of the components of the stabilization function are zero. The six limiting cases in which all the components of the stabilization function are zero are displayed in Table 4. The limit cases associated to characterizations IV and V correspond to nothing but the LDG methods considered in [23–25].

2.4. **The a priori error estimates.** Next, we present our a priori error analysis of the HDG methods under very mild conditions on the polynomial degrees k_u, k_q and k_p, and on the stabilization function τ. We assume that the difference between these polynomial degrees is never larger than two; see Table 5; our results seem to indicate that a wider difference does not result in a better method. Also, as suggested by the structure of the numerical traces given by (2.4d), we take the stabilization function of the following form:

$$\tau^+_{pu} = c^+_{pu}\, h^{k_p-k_u},\ \tau^+_{qu} = c^+_{qu}\, h^{k_q-k_u},\ \tau^-_{qu} = c^-_{qu}\, h^{k_q-k_u},\ \tau^-_{qp} = c^-_{qp}\, h^{k_q-k_p}, \tag{2.5}$$

where $c^+_{pu}, c^+_{qu}, c^-_{qu}$ and c^-_{qp} are fixed constants. Although this is not the most general assumption on the stabilization function, and is actually not essential for our analysis, it will allow us to treat all the choices of polynomial degrees k_u, k_q, k_p displayed in Table 5 in a very concise manner.

TABLE 5. The thirteen HDG methods under consideration. Here k denotes the minimum of the polynomial degrees k_u, k_q and k_p.

	$k_u = k_q + 1$				$k_u = k_q$					$k_u = k_q - 1$			
method	#1	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	#12	#13
$k_u - k$	1	1	1	2	0	0	0	1	2	0	0	0	1
$k_q - k$	0	0	0	1	0	0	0	1	2	1	1	1	2
$k_p - k$	0	1	2	0	0	1	2	0	0	0	1	2	0

For all these methods, we obtain upper bounds on L^2-norms of a projection of the errors in all three variables u_h, q_h, and p_h by using a duality argument. We also get superconvergence results in $L^\infty(\mathscr{E}_h)$-norm for the numerical traces $\widehat{u}_h$, $\widehat{q}_h$ and $\widehat{p}_h$, and in $L^\infty(\mathfrak{T}_h)$-norm of new approximations to u_h, q_h, and p_h obtained by using a local postprocessing. We also obtain similar results for some of the limit cases just discussed.

2.4.1. *The auxiliary projection.* Next, we define a key auxiliary projection tailored to the numerical traces. The projection of the function $(u, q, p) \in H^1(\mathfrak{T}_h)\times H^1(\mathfrak{T}_h)\times H^1(\mathfrak{T}_h)$, $\mathit{\Pi}(u, q, p) := (\mathit{\Pi} u, \mathit{\Pi} q, \mathit{\Pi} p)$, is defined as follows. On an element I_i, the projection is the element of $P_{k_u}(I_i) \times P_{k_q}(I_i) \times P_{k_p}(I_i)$ which solves the following equations:

$$(\delta_u, v)_{I_i} = 0 \quad \forall\, v \in P_{k_q-1}(I_i), \tag{2.6a}$$
$$(\delta_q, z)_{I_i} = 0 \quad \forall\, z \in P_{k_u-1}(I_i), \tag{2.6b}$$
$$(\delta_p, w)_{I_i} = 0 \quad \forall\, w \in P_{k_p-1}(I_i), \tag{2.6c}$$
$$\delta_p - \tau_{pu}\, \delta_u\, n = 0 \quad \text{on } x^+_{i-1}, \tag{2.6d}$$
$$\delta_q - \tau_{qu}\, \delta_u\, n = 0 \quad \text{on } x^+_{i-1}, \tag{2.6e}$$
$$\delta_q - \tau_{qu}\, \delta_u\, n - \tau_{qp}\, \delta_p\, n = 0 \quad \text{on } x^-_i, \tag{2.6f}$$

where we use the notation $\delta_\omega := \omega - \mathit{\Pi}\omega$ for $\omega = u, q$, and p. Note that the last three equations have exactly the same structure of the numerical traces of the HDG method (2.4d).

The following result contains the approximation properties of the projection Π. To state it, we use the following notation. The $H^s(D)$-norm is denoted by $\|\cdot\|_{s,D}$. We drop the first subindex if $s=0$, and the second one if $D=\Omega$ or $D=\Omega_h$.

Theorem 2.1. *Suppose that any of the following conditions holds:*

$$\begin{array}{lll} (i) & k_u = k_q + 1 & \text{and} \quad \tau^+_{qu}\tau^-_{qu} \neq 0, \\ (ii) & k_u = k_q & \text{and} \quad \tau^+_{qu} + \tau^-_{qu} - (-1)^{k_u+k_p}\tau^+_{pu}\tau^-_{qp} \neq 0, \\ (iii) & k_u = k_q - 1. & \end{array}$$

Then the projection Π is well defined on any interval I_i. In addition, if the stabilization function has the form given in (2.5)*, we have that, for $\omega = u, q$ and p, there is a constant C such that*

$$\|\omega - \Pi\omega\|_{I_i} \leq C\, C_\omega\, h^{\ell_\omega+1} \quad \text{for} \quad \ell_\omega \in [1, k_\omega],$$

provided $\omega \in H^{\ell_\omega+1}(I_i)$. In the case (i), we have

$$\begin{aligned} C_u &:= H_{u,u} + \left(|1/c^-_{qu}| + |1/c^+_{qu}| + |(c^-_{qp}c^+_{pu})/(c^-_{qu}c^+_{qu})|\right) H_{u,q} + \left|c^-_{qp}/c^-_{qu}\right| H_{u,p}, \\ C_q &:= H_{q,q}, \\ C_p &:= |c^+_{pu}/c^+_{qu}|\, H_{p,q} + H_{p,p}. \end{aligned}$$

In the case (ii),

$$\begin{aligned} C_u &:= \left(1 + \left|c^-_{qu}/c\right|\right) H_{u,u} + |1/c|\ H_{u,q} + \left|c^-_{qp}/c\right|\ H_{u,p}, \\ C_q &:= \left|c^+_{qu}\, c^-_{qu}/c\right|\ H_{q,u} + \left(1 + \left|c^+_{qu}/c\right|\right) H_{q,q} + \left|c^+_{qu}\, c^-_{qp}/c\right|\ H_{q,p}, \\ C_p &:= \left|c^+_{pu}\, c^-_{qu}/c\right|\ H_{p,u} + \left|c^+_{pu}/c\right|\ H_{p,q} + \left(1 + \left|c^+_{pu}c^-_{qp}/c\right|\right) H_{p,p}, \end{aligned}$$

where $c := c^+_{qu} + c^-_{qu} - (-1)^{k_u+k_p}c^+_{pu}c^-_{qp} \neq 0$. Finally, in the case (iii),

$$\begin{aligned} C_u &= H_{u,u}, \\ C_q &:= \left(|c^-_{qu}| + |c^+_{qu}| + |c^-_{qp}c^+_{pu}|\right) H_{q,u} + H_{q,q} + |c^-_{qp}|\, H_{q,p}, \\ C_p &:= |c^+_{pu}|\, H_{p,u} + H_{p,p}. \end{aligned}$$

Here $H_{\nu,\omega} := h^{(\ell_\omega-\ell_\nu)-(k_\omega-k_\nu)}\|\omega\|_{\ell_\omega+1,I_i}$ for ν, ω equal to u, q and p.

The proof is given in the Appendix. Note that, if $\omega \in H^{k_\omega}(\mathcal{T}_h)$ for $\omega = u, q$ and p, we can take $\ell_\omega := k_\omega$ and conclude that the projection has optimal approximation properties with respect to h, that is, that $\|\omega - \Pi\omega\| \leq C\, h^{k_\omega+1}$ for $\omega = u, q$ and p.

2.4.2. *Estimates of the L^2-norm of the errors.* Here we provide estimates for the L^2-norm of the projection of the errors

$$\epsilon_u := \Pi u - u_h, \quad \epsilon_q := \Pi q - q_h, \quad \epsilon_p := \Pi p - p_h,$$

and deduce from them the estimates for the L^2-norm of the errors

$$e_u := u - u_h, \quad e_q := q - q_h, \quad e_p := p - p_h.$$

Theorem 2.2. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied. Moreover, assume that $(u, q, p) \in H^{k_u+1}(\mathcal{T}_h) \times H^{k_q+1}(\mathcal{T}_h) \times H^{k_p+1}(\mathcal{T}_h)$. Then, for h small enough, we have that $\|\epsilon_u\|$, $\|\epsilon_q\|$ and $\|\epsilon_p\|$ converge with the orders displayed in Table* 6*, and that $\|e_u\|$, $\|e_q\|$ and $\|e_p\|$ converge with the orders displayed in Table* 7*. The analysis is inconclusive when $c^+_{pu} \neq 0$ for method* 9*, and when $k = 0$ for methods* 8 *and* 13*. This is indicated with the symbols $*$ and $\dagger$, respectively.*

TABLE 6. Orders of convergence of $(\epsilon_u, \epsilon_q, \epsilon_p)$ in terms of the polynomial degrees of the approximation (k_u, k_q, k_p) and the constants c^+_{pu} and c^-_{qp}. Below, $(a, b, c) \wedge k$ means $(a \wedge k, b \wedge k, c \wedge k)$ where $a \wedge k$ denotes the minimum between a and k.

	m.	$c^+_{pu} = 0$, $c^-_{qp} = 0$	$c^+_{pu} \neq 0$, $c^-_{qp} = 0$	$c^+_{pu} = 0$, $c^-_{qp} \neq 0$	$c^+_{pu} \neq 0$, $c^-_{qp} \neq 0$
$k_u = k_q + 1$	#1	$k+1+(2,1,2)\wedge k$	$k+1+(2,1,1)\wedge k$	$k+1+(2,1,2)\wedge k$	$k+1+(2,1,1)\wedge k$
	#2	$k+2+(2,1,1)\wedge k$	$k+2+(2,1,1)\wedge k$	$k+2+(1,1,1)\wedge k$	$k+2+(1,1,1)\wedge k$
	#3	$k+2+(3,2,1)\wedge k$	$k+2+(3,2,1)\wedge k$	$k+2+(0,2,1)\wedge k$	$k+2+(0,2,1)\wedge k$
	#4	$k+1+(2,1,3)\wedge k$	$k+1+(2,1,0)\wedge k$	$k+1+(2,1,3)\wedge k$	$k+1+(2,1,0)\wedge k$
$k_u = k_q$	#5	$k+1+(1,1,1)\wedge k$	$k+1+(1,1,1)\wedge k$	$k+1+(1,1,1)\wedge k$	$k+1+(1,1,1)\wedge k$
	#6	$k+1+(1,1,1)\wedge k$	$k+1+(1,1,1)\wedge k$	$k+1+(0,0,1)\wedge k$	$k+1+(0,0,1)\wedge k$
	#7	$k+1+(1,1,1)\wedge k$	$k+1+(1,1,1)\wedge k$	$k+0+(0,0,1)\wedge k$	$k+0+(0,0,1)\wedge k$
	#8	$k+1+(1,1,2)\wedge k$	$k+1+(1,1,0)\wedge k\,\dagger$	$k+1+(1,1,2)\wedge k$	$k+1+(1,1,0)\wedge k\,\dagger$
	#9	$k+1+(1,1,2)\wedge k$	$*$	$k+1+(1,1,2)\wedge k$	$*$
$k_u = k_q - 1$	#10	$k+1+(2,1,3)\wedge k$	$k+1+(2,1,2)\wedge k$	$k+1+(2,1,3)\wedge k$	$k+1+(2,1,2)\wedge k$
	#11	$k+2+(1,1,0)\wedge k$	$k+2+(1,1,0)\wedge k$	$k+2+(1,0,0)\wedge k$	$k+2+(1,0,0)\wedge k$
	#12	$k+2+(1,1,0)\wedge k$	$k+2+(1,1,0)\wedge k$	$k+1+(1,0,1)\wedge k$	$k+1+(1,0,1)\wedge k$
	#13	$k+1+(2,1,3)\wedge k$	$k+1+(2,1,1)\wedge k\,\dagger$	$k+1+(2,1,3)\wedge k$	$k+1+(2,1,1)\wedge k\,\dagger$

TABLE 7. Orders of convergence of (e_u, e_q, e_p) in terms of the polynomial degrees of the approximation (k_u, k_q, k_p) and the constants c^+_{pu} and c^-_{qp}. Below, $(a, b, c) \wedge k$ means $(a \wedge k, b \wedge k, c \wedge k)$ where $a \wedge k$ denotes the minimum between a and k.

	method	$c^-_{qp} = 0$	$c^-_{qp} \neq 0$
$k_u = k_q + 1$	#1	$k+1+(1,0,0)\wedge k$	$k+1+(1,0,0)\wedge k$
	#2	$k+1+(1,0,1)$	$k+1+(1,0,1)$
	#3	$k+2+(0,-1,1)\wedge k$	$k+2+(0,-1,1)\wedge k$
	#4	$k+1+(2,1,0)\wedge k$	$k+1+(2,1,0)\wedge k$
$k_u = k_q$	#5	$k+1+(0,0,0)\wedge k$	$k+1+(0,0,0)\wedge k$
	#6	$k+1+(0,0,1)\wedge k$	$k+1+(0,0,1)\wedge k$
	#7	$k+1+(0,0,1)\wedge k$	$k+0+(0,0,1)\wedge k$
	#8	$k+1+(1,1,0)\wedge k\,\dagger$	$k+1+(1,1,0)\wedge k\dagger$
	#9	$k+1+(1,1,0)\wedge k\,*$	$k+0+(1,1,0)\wedge k\,*$
$k_u = k_q - 1$	#10	$k+1+(0,1,0)\wedge k$	$k+1+(0,1,0)\wedge k$
	#11	$k+1+(0,1,1)$	$k+1+(0,1,1)$
	#12	$k+1+(0,1,1)$	$k+1+(0,0,1)\wedge k$
	#13	$k+1+(1,1,0)\wedge k\dagger$	$k+1+(1,1,0)\wedge k\dagger$

Let us briefly discuss these results. First, we are interested in finding out the effect of the stabilization function τ on the performance of the HDG methods. In Tables 6 and 7, we can see that

- the best choice of stabilization function occurs when $\tau^+_{pu} = \tau^-_{qp} = 0$,
- the *only* HDG method whose orders of convergence are *unaffected* by the choice of the stabilization function is the method #5, that is, the method for which $(k_u, k_q, k_p) := (k, k, k)$.

Second, it would be helpful to identify schemes that can achieve the highest orders of convergence with the lowest number of degrees of freedom. For that purpose, we list in Table 8 all the "most efficient" methods with $k_u + k_q + k_p = 0, \ldots, 12$.

TABLE 8. Methods achieving the highest orders of convergence of $(\epsilon_u, \epsilon_q, \epsilon_p)$ for the smallest number of degrees of freedom for $c^+_{pu} = c^-_{qp} = 0$ in terms of the polynomial degrees $k_u + k_q + k_p \leq 12$. Here $k := \min\{k_u, k_q, k_p\}$.

$k_u + k_q + k_p$	method	orders
0	#5 ($k = 0$)	(1,1,1)
1	#1, #6, #10 ($k = 0$)	(1,1,1)
2	#2, #11 ($k = 0$)	(2,2,2)
3	#5 ($k = 1$)	(3,3,3)
4	#1, #6, #10 ($k = 1$)	(3,3,3)
5	#2 ($k = 1$)	(4,4,4)
6	#3 ($k = 1$),#5 ($k = 2$)	(4,4,4)
7	#1, #10 ($k = 2$)	(5,4,5)
8	#2 ($k = 2$)	(6,5,5)
9	#3 ($k = 2$)	(6,6,5)
10	#10 ($k = 3$)	(6,5,7)
11	#2 ($k = 3$)	(7,6,6)
12	#3 ($k = 3$)	(8,7,6)

Finally, let us explore what is the effect of the polynomial degrees (k_u, k_q, k_p) on the superconvergence properties of the methods. Let us denote the order of convergence of the projection of the error in ω by s_ω for $\omega = u, q$ and p. Ideally, we would like to have s_ω larger than or equal to the optimal order of convergence for ω, $k_\omega + 1$. Thus, the larger the difference $s_\omega - (k_\omega + 1)$ the better the superconvergence of the method. For this reason, the quantity

$$\text{gain} := \sum_{\omega=u,q,p} s_\omega - (k_\omega + 1),$$

which we call, for lack of a better term, the "gain", seems to be a reasonable measure of the quality of the superconvergence properties of the method.

In Table 9, we classify the HDG methods for the case $c^+_{pu} = c^-_{qp} = 0$ according to the "gain" versus the relative number of degrees of freedom on each interval, which we define as

$$\text{rndof} := (k_u - k, k_q - k, k_p - k).$$

Here we take $k := \min\{k_u, k_q, k_p\}$, in agreement with the description of the methods in Table 5. In this manner, the methods with higher "gain" are higher in the table, and those with relatively more local degrees of freedom are more to the right. We also indicate if the number $s_\omega - (k_\omega + 1)$ is positive, zero, or negative by the symbols $\uparrow, -$ and $\downarrow$, respectively. It is interesting to see that:

- The method #5 with $k = 0$ converges optimally in the three variables and in the three numerical traces.
- The two methods with highest "gain" are method #2, $(k + 1, k, k + 1)$, which superconverges with orders $(k+4, k+3, k+3)$ for $k \geq 2$, and method #3, $(k + 1, k, k + 2)$, which superconverges with orders $(k + 5, k + 4, k + 3)$ for $k \geq 3$. Thus, for $k \geq 3$, method #3 seems to be the method of choice if we are looking for superconvergence properties.
- The methods #7, #9, #12, and #13 have suboptimal convergence properties which render them not particularly attractive for any k.

TABLE 9. Orders of convergence of $(\epsilon_u, \epsilon_q, \epsilon_p)$ minus (k, k, k) when $c_{pu}^+ = c_{qp}^- = 0$ in terms of the polynomial degrees (k_u, k_q, k_p) for $k := \min\{k_u, k_q, k_p\} \geq 3$.

k	gain	rndof= 0	rndof= 1	rndof= 2	rndof= 3	rndof= 4
	6				#3 $(5,4,3)$ $(\uparrow\uparrow,-)$	
	5		#10 $(3,2,4)$ $(\uparrow,-,\uparrow)$	#2 $(4,3,3)$ $(\uparrow\uparrow,\uparrow)$		
	4		#1 $(3,2,3)$ $(\uparrow,\uparrow,\uparrow)$			
≥ 3	3	#5 $(2,2,2)$ $(\uparrow,\uparrow,\uparrow)$		#11 $(3,3,2)$ $(\uparrow,\uparrow,-)$	#4 $(3,2,4)$ $(-,-,\uparrow)$ and #13 $(3,2,4)$ $(\uparrow,\downarrow,\uparrow)$	
	2		#6 $(2,2,2)$ $(\uparrow,\uparrow,-)$	#8 $(2,2,3)$ $(-,-,\uparrow)$	#12 $(3,3,2)$ $(\uparrow,\uparrow,\downarrow)$	
	1			#7 $(2,2,2)$ $(\uparrow,\uparrow,\downarrow)$		
	0					#9 $(2,2,3)$ $(\downarrow,\downarrow,\uparrow)$

2.4.3. *Superconvergence of numerical traces.* Next, we discuss the superconvergence properties of the error in the numerical traces

$$\widehat{e}_\omega := \omega - \widehat{\omega}_h$$

for $\omega = u, q$ and p. We have the following result.

Theorem 2.3. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied. Moreover, assume that* $(u, q, p) \in H^{k_u+1}(\mathcal{T}_h) \times H^{k_q+1}(\mathcal{T}_h) \times H^{k_p+1}(\mathcal{T}_h)$. *Then, for* h *sufficiently small, we have that*

$$\|\widehat{e}_\omega\|_\infty := \max_{i=0,\ldots,N} |\widehat{e}_\omega(x_i)|,$$

converges with the orders given in Table 10 *for* $\omega = u, q$ *and* p. *The analysis is inconclusive when* $c_{pu}^+ \neq 0$ *for method* 9, *and when* $k = 0$ *for methods* 8 *and* 13. *This is indicated with the symbols* $*$ *and* $\dagger$, *respectively.*

2.4.4. *Postprocessing for better accuracy.* Since the numerical traces superconverge to the solution with an order $s+1$ typically larger than the order of the approximations inside the elements, we can postprocess the approximate solution (u_h, q_h, p_h) to get new appproximations (u_h^*, q_h^*, p_h^*) which supercoverge with order $s+1$ in the L^∞-norm. The postprocessing we use here is similar to the one in [6]. On the element $I_i = (x_{i-1}, x_i)$, $1 \leq j \leq N$, we define (u_h^*, q_h^*, p_h^*) as the element in the space $[P_s(I_i)]^3$ such that

$$(ru_h^*, w)_{I_i} - (p_h^*, w')_{I_i} + p_h^*(x_i^-)w(x_i^-) = (f, w)_{I_i} + \widehat{p}_h(x_{i-1})w(x_{i-1}), \tag{2.7a}$$

$$(p_h^*, z)_{I_i} + (q_h^*, z')_{I_i} - q_h^*(x_i^-)z(x_i^-) = -\widehat{q}_h(x_{i-1})z(x_{i-1}), \tag{2.7b}$$

$$(q_h^*, v)_{I_i} + (u_h^*, v')_{I_i} - u_h^*(x_i^-)v(x_i^-) = -\widehat{u}_h(x_{i-1})v(x_{i-1}), \tag{2.7c}$$

TABLE 10. Orders of convergence of $\max_{\omega=u,q,p} \|\widehat{e}_\omega\|_\infty$, in terms of the polynomial degrees of the approximation (k_u, k_q, k_p) and the constants c_{pu}^+ and c_{qp}^-.

	method	$c_{pu}^+ = 0$, $c_{qp}^- = 0$	$c_{pu}^+ \neq 0$, $c_{qp}^- = 0$	$c_{pu}^+ = 0$, $c_{qp}^- \neq 0$	$c_{pu}^+ \neq 0$, $c_{qp}^- \neq 0$
$k_u{=}k_q+1$	#1	$2k+1$	$2k+1$	$2k+1$	$2k+1$
	#2	$2k+2$	$2k+2$	$2k+2$	$2k+2$
	#3	$2k+2$	$2k+2$	$2k+2$	$2k+2$
	#4	$2k+1$	$2k+1$	$2k+1$	$2k+1$
$k_u{=}k_q$	#5	$2k+1$	$2k+1$	$2k+1$	$2k+1$
	#6	$2k+1$	$2k+1$	$2k+1$	$2k+1$
	#7	$2k+1$	$2k+1$	$2k-1$	$2k-1$
	#8	$2k+1$	$2k+1\,\dagger$	$2k+1$	$2k+1\,\dagger$
	#9	$2k+1$	$*$	$2k+1$	$*$
$k_u{=}k_q-1$	#10	$2k+1$	$2k+1$	$2k+1$	$2k+1$
	#11	$2k+2$	$2k+2$	$2k+2$	$2k+2$
	#12	$2k+2$	$2k+2$	$2k+1$	$2k+1$
	#13	$2k+1$	$2k+1\,\dagger$	$2k+1$	$2k+1\,\dagger$

for all $v, z, w \in P_s(I_i)$ for all $I_i \in \mathcal{T}_h$. It is easy to see that the above equations are the standard DG descretization for the following simple initial value problem:

$$\begin{aligned} ru^* + (p^*)' &= f \text{ in } I_i, & p^*(x_{i-1}) &= \widehat{p}_h(x_{i-1}),\\ p^* - (q^*)' &= 0 \text{ in } I_i, & q^*(x_{i-1}) &= \widehat{q}_h(x_{i-1}),\\ q^* - (u^*)' &= 0 \text{ in } I_i, & u^*(x_{i-1}) &= \widehat{u}_h(x_{i-1}). \end{aligned}$$

Note that when $r = 0$, the three equations in (2.7) can be solved separately. We can first obtain p_h^* by solving (2.7a), then insert it into (2.7b) to solve for q_h^*, and then solve (2.7c) for u_h^*. When r is non-zero, we solve the ODE system element by element.

To state the convergence property of the postprocessed solutions, we use the following norm:

$$\|\varphi\|_{\infty,h} = \max_{1\le i\le N} \sup_{x\in[x_{i-1},x_i]} |\varphi(x)|.$$

Theorem 2.4. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied. Moreover, assume that* $(u,q,p) \in H^{k_u+1}(\mathcal{T}_h) \times H^{k_q+1}(\mathcal{T}_h) \times H^{k_p+1}(\mathcal{T}_h)$*. Suppose also that* $(f,u,q,p) \in [H^s(\Omega)]^4$*. Then for h sufficiently small, we have that*

$$\max\{\|u - u_h^*\|_{\infty,h}, \|q - q_h^*\|_{\infty,h}, \|p - p_h^*\|_{\infty,h}\}$$

converges with the order $s+1$ given in Table 10*. The analysis is inconclusive when $c_{pu}^+ \neq 0$ for method* 9*, and when $k = 0$ for methods* 8 *and* 13*. This is indicated with the symbols $*$ and $\dagger$, respectively.*

Thanks to this result, we can take the point of view that, in the one-dimensional case, the only relevant computation is that of the numerical traces since, the orders of convergence of the postprocessed solutions are superior (or equal) to those of the projections of the errors for all the methods in Table 8 when $k \geq 3$.

Note that the post-processed approximations of all HDG methods converge with order $2k+1$, except for those of methods #2, $(k+1, k, k+1)$, #3 $(k+1, k, k+2)$, #11, $(k, k+1, k+1)$, and #12, $(k, k+1, k+2)$, which converge with order $2k+2$. Note also that the method #5 with polynomial degrees $(k+1, k+1, k+1)$ has the same number of degrees of freedom as methods #3 and #12 and converges with order $2k+3$, that is, with an additional order. For this reason, method #5 seems to be the method of choice in this one-dimensional setting.

2.5. **The limit cases.** For some of the limit cases displayed in Table 4, we have the following result. The limit cases that do not appear there are either not well defined or our analysis is inconclusive.

Theorem 2.5. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied. Moreover, assume that* $(u, q, p) \in H^{k_u+1}(\mathcal{T}_h) \times H^{k_q+1}(\mathcal{T}_h) \times H^{k_p+1}(\mathcal{T}_h)$. *Then, for some of the limit cases, we have that* $\|\epsilon_u\|$, $\|\epsilon_q\|$ *and* $\|\epsilon_p\|$, *and* $\|e_u\|$, $\|e_q\|$ *and* $\|e_p\|$ *converge with the orders displayed in Table* 11.

If, in addition, $(f, u, q, p) \in [H^s(\Omega)]^4$, *then*

$$\max_{\omega=u,q,p} \|\widehat{e}_\omega\|_\infty \quad \text{and} \quad \max\{\|u - u_h^*\|_{\infty,h}, \|q - q_h^*\|_{\infty,h}, \|p - p_h^*\|_{\infty,h}\}$$

converge with the order $s+1$ *given in Table* 11.

TABLE 11. Orders of convergence of $(\epsilon_u, \epsilon_q, \epsilon_p)$, (e_u, e_q, e_p), and $\max_{\omega=u,q,p} \|\widehat{e}_\omega\|_\infty$ in the limit cases. Below, $(a, b, c) \wedge k$ means $(a \wedge k, b \wedge k, c \wedge k)$ where $a \wedge k$ denotes the minimum between a and k.

limit case	method	$(\epsilon_u, \epsilon_q, \epsilon_p)$	(e_u, e_q, e_p)	$\max_{\omega=u,q,p} \|\widehat{e}_\omega\|_\infty$
	#10	$k+1+(2,1,3) \wedge k$	$k+1+(0,1,0) \wedge k$	$2k+1$
I	#11	$k+2+(1,1,0) \wedge k$	$k+1+(0,1,1)$	$2k+2$
$k_u = k_q - 1$	#12	$k+2+(1,1,0) \wedge k$	$k+1+(0,1,1)$	$2k+2$
	#13	$k+1+(2,1,3) \wedge k$	$k+1+(1,1,0) \wedge k$	$2k+1$
	#5	$k+1+(1,1,1) \wedge k$	$k+1+(0,0,0) \wedge k$	$2k+1$
	#6	$k+1+(1,2,1) \wedge k$	$k+1+(0,0,1) \wedge k$	$2k+1$
IV, V, VI	#7	$k+1+(1,2,1) \wedge k$	$k+1+(0,0,1) \wedge k$	$2k+1$
$k_u = k_q$	#8	$k+1+(2,1,3) \wedge k$	$k+1+(1,1,0) \wedge k$	$2k+1$
	#9	$k+1+(2,1,3) \wedge k$	$k+1+(2,1,0) \wedge k$	$2k+1$

Note that for the limit case I, the approximation for q turns out to be continuous. The convergence orders of methods #10-13 are the same as those in the first column of Table 6, and the method #11 seems to be the best choice. For the limit cases IV, V and VI, the method #5 has the same order of convergence as in the first column of Table 6, and it also seems to be the best choice for those limit cases. Note that in the limit cases IV and V, the method #5 is nothing but the LDG method considered in [23, 24]. Finally, note that the methods #6-9 have better convergence orders in the limit cases; compare with the orders in the first column of Table 6.

3. Proofs

In this section, we provide detailed proofs of our main results, Theorems 2.2, 2.3 and 2.4.

3.1. **The dual problem and the dual projection.** Since our analysis is based on a duality argument, we begin by introducing the so-called dual problem we are going to use. For any given $\eta_p, \eta_q, \eta_u \in L^2(\Omega)$, we define (Ψ_u, Ψ_q, Ψ_p) to be the solution of

$$\Psi_u' - \Psi_q = \eta_p \quad \text{in} \quad \Omega, \tag{3.1a}$$

$$\Psi_q' - \Psi_p = -\eta_q \quad \text{in} \quad \Omega, \tag{3.1b}$$

$$\Psi_p' - r\Psi_u = \eta_u \quad \text{in} \quad \Omega, \tag{3.1c}$$

$$\Psi_u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.1d}$$

$$\Psi_q(0) = 0. \tag{3.1e}$$

The main property we are going to use is the following regularity result; its proof is provided in the Appendix.

Lemma 3.1. *For any $r \geq 0$, we have*

$$\|\Psi_u\|_1 + \|\Psi_q\|_1 + \|\Psi_p\|_1 \leq C_{\mathrm{R}}(\|\eta_p\|^2 + \|\eta_q\|^2 + \|\eta_u\|^2)^{1/2},$$

where C_{R} depends only on r.

We are also going to need a variation of the projection $\mathit{\Pi}$ defined by (2.6), $\mathit{\Pi}^*$. The projection of the function $(u, q, p) \in H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h)$, $\mathit{\Pi}^*(u, q, p) := (\mathit{\Pi}^* u, \mathit{\Pi}^* q, \mathit{\Pi}^* p)$, is defined as follows. On an element I_i, the projection is the element of $P_{k_p}(I_i) \times P_{k_u}(I_i) \times P_{k_q}(I_i)$ which solves the following equations:

$$(\delta_q^*, z)_{I_i} = 0 \quad \forall\, z \in P_{k_q-1}(I_i), \tag{3.2a}$$

$$(\delta_p^*, w)_{I_i} = 0 \quad \forall\, w \in P_{k_u-1}(I_i), \tag{3.2b}$$

$$(\delta_u^*, v)_{I_i} = 0 \quad \forall\, v \in P_{k_p-1}(I_i), \tag{3.2c}$$

$$\delta_u^* - \tau_{qp}\, \delta_q^*\, n = 0 \quad \text{on } x_i^-, \tag{3.2d}$$

$$\delta_p^* - \tau_{qu}\, \delta_q^*\, n = 0 \quad \text{on } x_i^-, \tag{3.2e}$$

$$\delta_p^* - \tau_{qu}\, \delta_q^*\, n + \tau_{pu}\, \delta_u^*\, n = 0 \quad \text{on } x_{i-1}^+, \tag{3.2f}$$

where we use the notation $\delta_w^* := w - \mathit{\Pi}^* w$ for $w = u, q$, and p. Note that, if the first projection is denoted by $\mathit{\Pi}(u, q, p; \tau_{pu}^+, \tau_{qu}^+, \tau_{qu}^-, \tau_{qp}^-)$, we have, for $x \in I_i$, that

$$\mathit{\Pi}^*(u, q, p)(x) := \mathit{\Pi}(q, p, u; -\tau_{qp}^-, -\tau_{qu}^-, -\tau_{qu}^+, \tau_{pu}^+)(x_i + x_{i-1} - x). \tag{3.3}$$

The following result gives the approximation properties of this projection. Thanks to the previous identity, it follows directly from Theorem 2.1.

Lemma 3.2. *For $(\Psi_p, \Psi_q, \Psi_u) \in H^{s_p^*+1}(\mathcal{T}_h) \times H^{s_q^*+1}(\mathcal{T}_h) \times H^{s_u^*+1}(\mathcal{T}_h)$, we have*

$$\|\delta_{\Psi_p}^*\| \leq C\, C_p^*\, h^{\ell_p^*+1},$$

$$\|\delta_{\Psi_q}^*\| \leq C\, C_q^*\, h^{\ell_q^*+1},$$

$$\|\delta_{\Psi_u}^*\| \leq C\, C_u^*\, h^{\ell_u^*+1}.$$

In the case (i), we have

$$\begin{aligned}
C^*_q &:= H^*_{q,q} + \left(|1/c^-_{qu}| + |1/c^+_{qu}| + |(c^-_{qp}c^+_{pu})/(c^-_{qu}c^+_{qu})|\right) H^*_{q,p} + \left|c^+_{pu}/c^+_{qu}\right| h^{dk} H^*_{q,u},\\
C^*_p &:= H^*_{p,p},\\
C^*_u &:= |c^-_{qp}/c^-_{qu}|\, h^{-dk} H^*_{u,q} + H^*_{u,u}.
\end{aligned}$$

In the case (ii),

$$\begin{aligned}
C^*_q &:= \left(1 + \left|c^+_{qu}/c\right|\right) H^*_{q,q} + |1/c|\; H^*_{q,p} + \left|c^+_{pu}/c\right| h^{dk}\; H^*_{q,u},\\
C^*_p &:= \left|c^+_{qu}\, c^-_{qu}/c\right| H^*_{p,q} + \left(1 + \left|c^-_{qu}/c\right|\right) H^*_{p,p} + \left|c^-_{qu}\, c^+_{pu}/c\right| h^{dk} H^*_{p,u},\\
C^*_u &:= \left|c^-_{qp}\, c^+_{qu}/c\right| h^{-dk} H^*_{u,q} + \left|c^-_{qp}/c\right| h^{-dk} H^*_{u,p} + \left(1 + \left|c^+_{pu}c^-_{qp}/c\right|\right) H^*_{u,u},
\end{aligned}$$

and in the case (iii),

$$\begin{aligned}
C^*_q &= H^*_{q,q},\\
C^*_p &:= \left(|c^-_{qu}| + |c^+_{qu}| + |c^-_{qp}c^+_{pu}|\right) H^*_{p,q} + H^*_{p,p} + |c^+_{pu}|\, h^{dk} H^*_{p,u},\\
C^*_u &:= |c^-_{qp}|\, h^{-dk} H^*_{u,q} + H^*_{u,u}.
\end{aligned}$$

Here $dk := 2k_p - k_q - k_u$, $H^*_{\nu,\omega} := h^{(\ell^*_\omega - \ell^*_\nu)-(k^*_\omega - k^*_\nu)} \|\Psi_\omega\|_{\ell^*_\omega+1}$ *with* $\ell^*_\omega := \min\{k^*_\omega, s^*_\omega\}$, *for* ν, ω *equal to* u, q *and* p, *where* $(k^*_u, k^*_q, k^*_p) := (k_p, k_u, k_q)$.

3.2. **The error equations.** Next, we obtain the equations for the projection of the errors. Then we obtain, from the equations defining the HDG methods, see (2.4), and the fact that the exact solution also satisfies these equations, that

$$\begin{aligned}
(e_q, v)_{\mathcal{T}_h} + (e_u, v')_{\mathcal{T}_h} &= \langle \widehat{e}_u, vn\rangle_{\partial\mathcal{T}_h},\\
(e_p, z)_{\mathcal{T}_h} + (e_q, z')_{\mathcal{T}_h} &= \langle \widehat{e}_q, zn\rangle_{\partial\mathcal{T}_h},\\
(re_u, w)_{\mathcal{T}_h} - (e_p, w')_{\mathcal{T}_h} &= -\langle \widehat{e}_p, wn\rangle_{\partial\mathcal{T}_h},
\end{aligned}$$

for all $v, z, w \in W_h^{k_q} \times W_h^{k_u} \times W_h^{k_p}$, where

$$\begin{cases}
\widehat{e}^{\,+}_p = e^+_p - \tau^+_{pu}\,(\widehat{u}_h - u^+_h)\, n^+,\\
\widehat{e}^{\,+}_q = e^+_q - \tau^+_{qu}\,(\widehat{u}_h - u^+_h)\, n^+,\\
\widehat{e}^{\,-}_q = e^-_q - \tau^-_{qu}\,(\widehat{u}_h - u^-_h)\, n^- - \tau^-_{qp}\,(\widehat{p}^{\,-}_h - p^-_h)\, n^-,
\end{cases} \tag{3.4}$$

and

$$\begin{aligned}
&[\![\widehat{e}_q]\!](x_i) = 0, \quad [\![\widehat{e}_p]\!](x_i) = 0 \qquad \text{for} \quad i = 1, \dots, N-1,\\
&\widehat{e}_u(0) = 0, \quad \widehat{e}_u(1) = 0, \quad \widehat{e}_q(1) = 0.
\end{aligned}$$

Now we set $\widehat{\epsilon}_u = \widehat{e}_u$ and $\widehat{\epsilon}^{\,-}_p = \widehat{e}^{\,-}_p$, and let

$$\begin{cases}
\widehat{\epsilon}^{\,+}_p = \epsilon^+_p + \tau^+_{pu}\,(\widehat{\epsilon}_u - \epsilon^+_u)\, n^+,\\
\widehat{\epsilon}^{\,+}_q = \epsilon^+_q + \tau^+_{qu}\,(\widehat{\epsilon}_u - \epsilon^+_h)\, n^+,\\
\widehat{\epsilon}^{\,-}_q = \epsilon^-_q + \tau^-_{qu}\,(\widehat{\epsilon}_u - \epsilon^-_u)\, n^- + \tau^-_{qp}\,(\widehat{\epsilon}^{\,-}_p - \epsilon^-_p)\, n^-.
\end{cases}$$

From the definition of the projection $\mathit{\Pi}$, (2.6d)-(2.6f), we can see that $\widehat{\epsilon}^{\,+}_p = \widehat{e}^{\,+}_p$ and $\widehat{\epsilon}_q = \widehat{e}_q$. Then using the definition of the projection $\mathit{\Pi}$, (2.6a)-(2.6c), we easily get that

$$(e_q, v)_{\mathcal{T}_h} + (\epsilon_u, v')_{\mathcal{T}_h} = \langle \widehat{\epsilon}_u, vn\rangle_{\partial\mathcal{T}_h}, \tag{3.5a}$$

$$(e_p, z)_{\mathcal{T}_h} + (\epsilon_q, z')_{\mathcal{T}_h} = \langle \widehat{\epsilon}_q, zn\rangle_{\partial\mathcal{T}_h}, \tag{3.5b}$$

$$(re_u, w)_{\mathcal{T}_h} - (\epsilon_p, w')_{\mathcal{T}_h} = -\langle \widehat{\epsilon}_p, wn\rangle_{\partial\mathcal{T}_h}, \tag{3.5c}$$

for all $v, z, w \in W_h^{k_q} \times W_h^{k_u} \times W_h^{k_p}$, and

$$[\![\widehat{\epsilon}_q]\!](x_i) = 0, \quad [\![\widehat{\epsilon}_p]\!](x_i) = 0 \qquad \text{for} \quad i = 1, \dots, N-1, \tag{3.5d}$$

$$\widehat{\epsilon}_u(0) = 0, \quad \widehat{\epsilon}_u(1) = 0, \quad \widehat{\epsilon}_q(1) = 0. \tag{3.5e}$$

3.3. **Proof of the L^2-error estimates.** We are now ready to prove the L^2-error estimates of Theorem 2.2. By default, $(\varphi, v) = (\varphi, v)_{\mathcal{T}_h}$ and $\langle\varphi, vn\rangle = \langle\varphi, vn\rangle_{\partial\mathcal{T}_h}$. We begin by establishing a key identity involving the quantity

$$\epsilon := (\epsilon_u, \eta_u) + (\epsilon_q, \eta_q) + (\epsilon_p, \eta_p)$$

with which we will be able to relate the projections of the errors with the dual problem. From now on, $(\cdot)_\ell$ denotes the L^2-projection into the space of piecewise polynomials of degree $\ell \geq 0$. The operator $(\cdot)_{-1}$ denotes the multiplication by zero operator. Finally, we set $\delta_\eta^\ell := \eta - (\eta)_\ell$.

Lemma 3.3. *We have that*

$$\begin{aligned}\epsilon =&\chi_{k_u-1,k_q}\,(\delta_q, \delta^*_{\Psi_p} - \delta^{k_u-1}_{\Psi_p}) + \chi_{k_u-1,k_q}\,(\epsilon_q, \delta^*_{\Psi_p})\\ &-\chi_{k_p-1,k_u}\,(\delta_p, \delta^*_{\Psi_q} - \delta^{k_p-1}_{\Psi_q}) - \chi_{k_q-1,k_p}\,(\epsilon_p, \delta^*_{\Psi_q})\\ &-r\,\chi_{k_q-1,k_p}\,(\delta_u, \delta^*_{\Psi_u} - \delta^{k_q-1}_{\Psi_u}) - r\,\chi_{k_p-1,k_u}\,(\epsilon_u, \delta^*_{\Psi_u}),\end{aligned}$$

where $\chi_{\ell,m} = 1$ *when* $\ell < m$ *and* $\chi_{\ell,m} = 0$ *otherwise, and* $(\delta^*_{\Psi_u}, \delta^*_{\Psi_q}, \delta^*_{\Psi_p}) := (\Psi_u, \Psi_q, \Psi_p) - \mathit{\Pi}^*(\Psi_u, \Psi_q, \Psi_p)$.

Proof. Using the first three equations defining the dual problem (3.1) in the definition of ϵ gives

$$\begin{aligned}\epsilon =&(\epsilon_u, \Psi_p' - r\,\Psi_u) - (\epsilon_q, \Psi_q' - \Psi_p) + (\epsilon_p, \Psi_u' - \Psi_q)\\ =&(\epsilon_u, \Psi_p') - (\epsilon_q, \Psi_q') + (\epsilon_p, \Psi_u') - r\,(\epsilon_u, \Psi_u) + (\epsilon_q, \Psi_p) - (\epsilon_p, \Psi_q).\end{aligned}$$

Since

$$(\epsilon_\omega, \Psi_v') = (\epsilon_\omega, (\mathit{\Pi}^*\Psi_v)') + (\epsilon_\omega, (\delta^*_{\Psi_v})') = (\epsilon_\omega, (\mathit{\Pi}^*\Psi_v)') + \langle\epsilon_\omega, \delta^*_{\Psi_v} n\rangle,$$

for the pairs $(\omega, v) = (u, p), (q, q)$ and (p, u), by the first three orthogonality properties defining the projection $\mathit{\Pi}^*$, equations (3.2), we get

$$\begin{aligned}\epsilon =&(\epsilon_u, (\mathit{\Pi}^*\Psi_p)') - (\epsilon_q, (\mathit{\Pi}^*\Psi_q)') + (\epsilon_p, (\mathit{\Pi}^*\Psi_u)')\\ &- r(\epsilon_u, \Psi_u) + (\epsilon_q, \Psi_p) - (\epsilon_p, \Psi_q)\\ &+ \langle\epsilon_u, \delta^*_{\Psi_p} n\rangle - \langle\epsilon_q, \delta^*_{\Psi_q} n\rangle + \langle\epsilon_p, \delta^*_{\Psi_u} n\rangle.\end{aligned}$$

Taking $v = \Pi^* \Psi_p$, $z = \Pi^* \Psi_q$, and $w = \Pi^* \Psi_u$ in the equations satisfied by the projections of the errors, (3.5), gives

$$\begin{aligned}\epsilon =& - (e_q, \Pi^* \Psi_p) + (e_p, \Pi^* \Psi_q) + r(e_u, \Pi^* \Psi_u)\\ &+ \langle \widehat{\epsilon}_u, \Pi^* \Psi_p n\rangle - \langle \widehat{\epsilon}_q, \Pi^* \Psi_q n\rangle + \langle \widehat{\epsilon}_p, \Pi^* \Psi_u n\rangle\\ &+ (\epsilon_q, \Psi_p) - (\epsilon_p, \Psi_q) - r(\epsilon_u, \Psi_u)\\ &+ \langle \epsilon_u, \delta^*_{\Psi_p} n\rangle - \langle \epsilon_q, \delta^*_{\Psi_q} n\rangle + \langle \epsilon_p, \delta^*_{\Psi_u} n\rangle\\ =& - (\delta_q, \Pi^* \Psi_p) + (\delta_p, \Pi^* \Psi_q) + r(\delta_u, \Pi^* \Psi_u)\\ &+ (\epsilon_q, \delta^*_{\Psi_p}) - (\epsilon_p, \delta^*_{\Psi_q}) - r(\epsilon_u, \delta^*_{\Psi_u}) + \Theta,\end{aligned}$$

where

$$\begin{aligned}\Theta =& \langle \widehat{\epsilon}_u, \Pi^* \Psi_p n\rangle - \langle \widehat{\epsilon}_q, \Pi^* \Psi_q n\rangle + \langle \widehat{\epsilon}_p, \Pi^* \Psi_u n\rangle\\ &+ \langle \epsilon_u, \delta^*_{\Psi_p} n\rangle - \langle \epsilon_q, \delta^*_{\Psi_q} n\rangle + \langle \epsilon_p, \delta^*_{\Psi_u} n\rangle.\end{aligned}$$

Let us now show that $\Theta = 0$. From the fact of the last two of the equations defining the projection of the errors, (3.5), and the boundary conditions of the dual problem, (3.1), we have that

$$\langle \widehat{\epsilon}_u, \Psi_p n\rangle = \langle \widehat{\epsilon}_q, \Psi_q n\rangle = \langle \widehat{\epsilon}_p, \Psi_u n\rangle = 0,$$

which means

$$\Theta = -\langle \widehat{\epsilon}_u - \epsilon_u, \delta^*_{\Psi_p} n\rangle + \langle \widehat{\epsilon}_q - \epsilon_q, \delta^*_{\Psi_q} n\rangle - \langle \widehat{\epsilon}_p - \epsilon_p, \delta^*_{\Psi_u} n\rangle.$$

Using the fourth set of equations for the projection of the errors, equations (3.5), we get (with the obvious notation) that

$$\begin{aligned}\Theta =& - \langle \widehat{\epsilon}_u - \epsilon_u, \delta^*_{\Psi_p} n\rangle\\ &+ \langle \tau_{qu}(\widehat{\epsilon}_u - \epsilon_u)n + \tau_{qp}(\widehat{\epsilon}_p - \epsilon_p)n, \delta^*_{\Psi_q} n\rangle_{\partial\mathcal{T}_h^-} + \langle \tau_{qu}(\widehat{\epsilon}_u - \epsilon_u)n, \delta^*_{\Psi_q} n\rangle_{\partial\mathcal{T}_h^+}\\ &- \langle \widehat{\epsilon}_p - \epsilon_p, \delta^*_{\Psi_u} n\rangle_{\partial\mathcal{T}_h^-} - \langle \tau_{pu}(\widehat{\epsilon}_u - \epsilon_u)n, \delta^*_{\Psi_u} n\rangle_{\partial\mathcal{T}_h^+}\\ =& - \langle (\widehat{\epsilon}_u - \epsilon_u)n, \delta^*_{\Psi_p} - \tau_{qu}\delta^*_{\Psi_q} n + \tau_{pu}\delta^*_{\Psi_u} n\rangle_{\partial\mathcal{T}_h^+}\\ &- \langle (\widehat{\epsilon}_u - \epsilon_u)n, \delta^*_{\Psi_p} - \tau_{qu}\delta^*_{\Psi_q} n\rangle_{\partial\mathcal{T}_h^-} - \langle (\widehat{\epsilon}_p - \epsilon_p)n, \delta^*_{\Psi_u} - \tau_{qp}\delta^*_{\Psi_q} n\rangle_{\partial\mathcal{T}_h^-}\\ =& \, 0,\end{aligned}$$

by the last three equations defining the projection Π^*, (3.2).

Now, by taking into account the first three orthogonality properties defining the projections Π and Π^*, see (2.6) and (3.2), respectively, we get that

$$\begin{aligned}\epsilon =& - \chi_{k_u-1,k_q}\,(\delta_q, \Pi^* \Psi_p) + \chi_{k_u-1,k_q}\,(\epsilon_q, \delta^*_{\Psi_p})\\ &+ \chi_{k_p-1,k_u}\,(\delta_p, \Pi^* \Psi_q) - \chi_{k_q-1,k_p}\,(\epsilon_p, \delta^*_{\Psi_q})\\ &+ r\,\chi_{k_q-1,k_p}\,(\delta_u, \Pi^* \Psi_u) - r\,\chi_{k_p-1,k_u}\,(\epsilon_u, \delta^*_{\Psi_u}).\end{aligned}$$

The result follows by writing $\Pi^* \Psi_\omega = -\delta^*_{\Psi_p} + \delta^{\ell(\omega)}_{\Psi_\omega} + (\Psi_\omega)_{\ell(\omega)}$, with $\ell(u) = k_q - 1, \ell(q) = k_p - 1$ and $\ell(p) = k_u - 1$, and applying the first three orthogonality properties of the projection Π, (2.6). This completes the proof. $\square$

The following is an immediate consequence of this result.

Corollary 3.4. *We have*

$$\begin{aligned}|\epsilon| \leq\ & \chi_{k_u-1,k_q}\,\|\delta_q\|\,(\|\delta^{k_u-1}_{\Psi_p}\| + \|\delta^*_{\Psi_p}\|) \quad + \chi_{k_u-1,k_q}\,\|\epsilon_q\|\,\|\delta^*_{\Psi_p}\|\\ &+\chi_{k_p-1,k_u}\,\|\delta_p\|\,(\|\delta^{k_p-1}_{\Psi_q}\| + \|\delta^*_{\Psi_q}\|) \quad + \chi_{k_q-1,k_p}\,\|\epsilon_p\|\,\|\delta^*_{\Psi_q}\|\\ &+r\,\chi_{k_q-1,k_p}\,\|\delta_u\|\,(\|\delta^{k_q-1}_{\Psi_u}\| + \|\delta^*_{\Psi_u}\|) + r\,\chi_{k_p-1,k_u}\,\|\epsilon_u\|\,\|\delta^*_{\Psi_u}\|.\end{aligned}$$

It is now clear that we need to obtain estimates of $\|\delta_\omega\|$, $\|\delta^*_{\Psi_\omega}\|$ and of $\|\delta^{\ell(\omega)}_{\Psi_\omega}\|$ for $\omega = u, q$, and p. The estimates of $\|\delta_\omega\|$ are displayed next and are a direct consequence of Theorem 2.1.

Lemma 3.5. *Assume that* $(u, q, p) \in H^{k_u+1}(\mathcal{T}_h) \times H^{k_q+1}(\mathcal{T}_h) \times H^{k_p+1}(\mathcal{T}_h)$. *Then we have*

$$\begin{aligned}\|\delta_u\| &\leq C\,h^{k_u+1}\,(\|u\|_{k_u+1} + \|q\|_{k_q+1} + \|p\|_{k_p+1}),\\ \|\delta_q\| &\leq C\,h^{k_q+1}\,(\|u\|_{k_u+1} + \|q\|_{k_q+1} + \|p\|_{k_p+1}),\\ \|\delta_p\| &\leq C\,h^{k_p+1}\,(\|u\|_{k_u+1} + \|q\|_{k_q+1} + \|p\|_{k_p+1}).\end{aligned}$$

The estimates of $\|\delta^*_{\Psi_\omega}\|$ are contained in Lemma 3.2. Finally, the estimates of $\|\delta^{\ell(\omega)}_{\Psi_\omega}\|$ are a direct consequence of the approximation properties of the L^2-projection. They are gathered in the next result.

Lemma 3.6. *For* $(\Psi_p, \Psi_q, \Psi_u) \in H^{s^*_p+1}(\mathcal{T}_h) \times H^{s^*_q+1}(\mathcal{T}_h) \times H^{s^*_u+1}(\mathcal{T}_h)$, *we have*

$$\begin{aligned}\|\delta^{k_u-1}_{\Psi_p}\| &\leq C\,h^{\min\{s^*_p+1,k_u\}}\,\|\Psi_p\|_{s^*_p+1},\\ \|\delta^{k_p-1}_{\Psi_q}\| &\leq C\,h^{\min\{s^*_q+1,k_p\}}\,\|\Psi_q\|_{s^*_q+1},\\ \|\delta^{k_q-1}_{\Psi_u}\| &\leq C\,h^{\min\{s^*_u+1,k_q\}}\,\|\Psi_u\|_{s^*_u+1}.\end{aligned}$$

To obtain the orders of convergence for the projection of the errors, see Table 6, it is just a matter of inserting the above-mentioned estimates into Corollary 3.4, applying the regularity estimates of Lemma 3.1, and carrying out simple, but long and tedious, algebraic manipulations.

Let us briefly show the proof for the method #1; the proofs for the other methods are similar. First, to estimate $\|\epsilon_u\|$, we take $\eta_u = \epsilon_u$ and $\eta_q = \eta_p = 0$ in Corollary 3.4 and get

$$\begin{aligned}\|\epsilon_u\|^2 \leq\ & \chi_{k_p-1,k_u}\,\|\delta_p\|\,(\|\delta^{k_p-1}_{\Psi_q}\| + \|\delta^*_{\Psi_q}\|) \quad + \chi_{k_q-1,k_p}\,\|\epsilon_p\|\,\|\delta^*_{\Psi_q}\|\\ &+r\,\chi_{k_q-1,k_p}\,\|\delta_u\|\,(\|\delta^{k_q-1}_{\Psi_u}\| + \|\delta^*_{\Psi_u}\|) + r\,\chi_{k_p-1,k_u}\,\|\epsilon_u\|\,\|\delta^*_{\Psi_u}\|,\end{aligned} \tag{3.6}$$

using the fact that $\chi_{k_u-1,k_q} = 0$ for the method #1. Note that when $\eta_q = \eta_p = 0$, Lemma 3.1 implies that

$$\|\Psi_u\|_{s^*_u+1} + \|\Psi_q\|_{s^*_q+1} + \|\Psi_p\|_{s^*_p+1} \leq C_R\|\epsilon_u\|,$$

where $(s^*_u, s^*_q, s^*_p) := (2, 1, 0)$. Applying Lemma 3.5, Lemma 3.6, and Lemma 3.2 to (3.6), we have that

$$\|\epsilon_u\| \leq CC_R h^{k+1+k\wedge 2} + Ch\|\epsilon_p\|.$$

Similarly, taking $\eta_q = \epsilon_q$ and $\eta_u = \eta_p = 0$ in Corollary 3.4 and Lemma 3.1 with $(s^*_u, s^*_q, s^*_p) := (1, 0, 2)$, we get

$$\|\epsilon_q\| \leq CC_R h^{k+1+1\wedge k} + CC_R h\|\epsilon_p\| + CC_R h\|\epsilon_u\|.$$

Finally, taking $\eta_p = \epsilon_p$ and $\eta_u = \eta_q = 0$ in Corollary 3.4 and Lemma 3.1 with $(s_u^*, s_q^*, s_p^*) := (0, 2, 1)$, we get

$$\|\epsilon_p\| \le C C_R h^{k+1+2\wedge k} + C\, C_R\, |c_{pu}^+|\, h^{k+2} + C C_R h \|\epsilon_u\|.$$

The orders of convergence of $(\epsilon_u, \epsilon_q, \epsilon_p)$ for the method #1 are obtained by combining the three inequalities above.

Finally, the orders of convergence for the errors, see Table 7, can be obtained by using the estimates just proven, the triangle inequality and the approximation estimates of the projection Π given in Theorem 2.1. This completes the proof of Theorem 2.2.

The proof to obtain the orders of convergence for the projection of the errors and the errors for the limit cases of Theorem 2.5 is similar. Note that our analysis can only go through provided the limit of the auxiliary projection and its dual are well defined. The cases displayed in Table 11 are the only cases for which this happens.

3.4. **Proof of the error estimates of the numerical traces.** Now we prove the error estimates of the numerical traces of Theorem 2.3. We start by defining the following Green's functions. For any $\omega = p, q, u$ and any $y \in (0, 1)$, $(\Phi_{p,y}^\omega, \Phi_{q,y}^\omega, \Phi_{u,y}^\omega)$ are the solutions of the following system:

$$\frac{d}{dx}\Phi_{u,y}^\omega = \Phi_{q,y}^\omega \qquad \text{in} \quad (0, y) \cup (y, 1), \tag{3.7a}$$

$$\frac{d}{dx}\Phi_{q,y}^\omega = \Phi_{p,y}^\omega \qquad \text{in} \quad (0, y) \cup (y, 1), \tag{3.7b}$$

$$\frac{d}{dx}\Phi_{p,y}^\omega - r\Phi_{u,y}^\omega = 0 \qquad \text{in} \quad (0, y) \cup (y, 1), \tag{3.7c}$$

with boundary condition

$$\Phi_{u,y}^\omega(x_0) = \Phi_{u,y}^\omega(x_N) = \Phi_{q,y}^\omega(x_N) = 0, \tag{3.7d}$$

and jump condition

$$[\![\Phi_{u,y}^\omega]\!](y) = \delta_{\omega p}, \;\; [\![\Phi_{q,y}^\omega]\!](y) = -\delta_{\omega q}, \;\; [\![\Phi_{p,y}^\omega]\!](y) = \delta_{\omega u}. \tag{3.7e}$$

We first obtain representation formulas for the errors in the numerical traces, and then approximate them.

Lemma 3.7. *Set* $\theta_i^\omega := \langle \widehat{e}_u, \Phi_{p,x_i}^\omega n\rangle - \langle \widehat{e}_q, \Phi_{q,x_i}^\omega n\rangle + \langle \widehat{e}_p, \Phi_{u,x_i}^\omega n\rangle$ *for* $\omega = u, q, p$. *Then we have*

$$\theta_i^\omega = \theta_{i,1}^\omega + \theta_{i,2}^\omega + \theta_{i,3}^\omega$$

where

$$\begin{aligned}
\theta_{i,1}^\omega &:= \langle \widehat{e}_u - e_u, (\Phi_{p,x_i}^\omega - v_1)n\rangle - \langle \widehat{e}_q - e_q, (\Phi_{q,x_i}^\omega - v_2)n\rangle + \langle \widehat{e}_p - e_p, (\Phi_{u,x_i}^\omega - v_3)n\rangle,\\
\theta_{i,2}^\omega &:= (e_u', (\Phi_{p,x_i}^\omega - v_1)) - (e_q', (\Phi_{q,x_i}^\omega - v_2)) + (e_p', (\Phi_{u,x_i}^\omega - v_3)),\\
\theta_{i,3}^\omega &:= (re_u, \Phi_{u,x_i}^\omega - v_3) - (e_q, \Phi_{p,x_i}^\omega - v_1) + (e_p, \Phi_{q,x_i}^\omega - v_2),
\end{aligned}$$

for any $(v_1, v_2, v_3) \in W_h^{k_q} \times W_h^{k_u} \times W_h^{k_p}$.

Proof. By simple addition and subtraction, we have

$$\begin{aligned}\theta_i^\omega &= \langle \widehat{e}_u, (\Phi^\omega_{p,x_i} - v_1)n\rangle - \langle \widehat{e}_q, (\Phi^\omega_{q,x_i} - v_2)n\rangle + \langle \widehat{e}_p, (\Phi^\omega_{u,x_i} - v_3)n\rangle \\ &\quad + \langle \widehat{e}_u, v_1 n\rangle - \langle \widehat{e}_q, v_2 n\rangle + \langle \widehat{e}_p, v_3 n\rangle \\ &= \langle \widehat{e}_u, (\Phi^\omega_{p,x_i} - v_1)n\rangle - \langle \widehat{e}_q, (\Phi^\omega_{q,x_i} - v_2)n\rangle + \langle \widehat{e}_p, (\Phi^\omega_{u,x_i} - v_3)n\rangle \\ &\quad + (e_u, v_1') - (e_q, v_2') + (e_p, v_3') + (e_q, v_1) - (e_p, v_2) - (re_u, v_3),\end{aligned}$$

by the first of error equations (3.5). Now, we integrate by parts to obtain

$$\begin{aligned}\theta_i^\omega &= \langle \widehat{e}_u, (\Phi^\omega_{p,x_i} - v_1)n\rangle - \langle \widehat{e}_q, (\Phi^\omega_{q,x_i} - v_2)n\rangle + \langle \widehat{e}_p, (\Phi^\omega_{u,x_i} - v_3)n\rangle \\ &\quad - (e_u', v_1) + (e_q', v_2) - (e_p', v_3) + (e_q, v_1) - (e_p, v_2) - (r\, e_u, v_3) \\ &\quad + \langle e_u, v_1\, n\rangle - \langle e_q, v_2\, n\rangle + \langle e_p, v_3\, n\rangle \\ &= \theta^\omega_{i,1} - (e_u', v_1) + (e_q', v_2) - (e_p', v_3) + (e_q, v_1) - (e_p, v_2) - (r\, e_u, v_3) \\ &\quad + \langle e_u, \Phi^\omega_{p,x_i}\, n\rangle - \langle e_q, \Phi^\omega_{q,x_i}\, n\rangle + \langle e_p, \Phi^\omega_{u,x_i}\, n\rangle \\ &= \theta^\omega_{i,1} + \theta^\omega_{i,2} + \theta^\omega_{i,3} \\ &\quad - (e_u', \Phi^\omega_{p,x_i}) + (e_q', \Phi^\omega_{q,x_i}) - (e_p', \Phi^\omega_{u,x_i}) \\ &\quad + (e_q, \Phi^\omega_{p,x_i}) - (e_p, \Phi^\omega_{q,x_i}) - (r\, e_u, \Phi^\omega_{u,x_i}) \\ &\quad + \langle e_u, \Phi^\omega_{p,x_i}\, n\rangle - \langle e_q, \Phi^\omega_{q,x_i}\, n\rangle + \langle e_p, \Phi^\omega_{u,x_i}\, n\rangle \\ &= \theta^\omega_{i,1} + \theta^\omega_{i,2} + \theta^\omega_{i,3} \\ &\quad + (e_q, -\frac{d}{dx}\Phi^\omega_{q,x_i} + \Phi^\omega_{p,x_i}) + (e_p, \frac{d}{dx}\Phi^\omega_{u,x_i} - \Phi^\omega_{q,x_i}) + (e_u, \frac{d}{dx}\Phi^\omega_{p,x_i} - r\,\Phi^\omega_{u,x_i})\end{aligned}$$

and the identity follows by using the first three equations defining the Green's functions (3.7). This completes the proof. □

We are now ready to obtain a representation of the errors in the numerical traces.

Lemma 3.8. *For $\omega = u, q, p$, we have*

$$\widehat{e}_\omega(x_i) = \Gamma^\omega_{i,1} + \Gamma^\omega_{i,2},$$

where

$$\begin{aligned}\Gamma^\omega_{i,1} &= (\delta^{k_u-1}_{u'}, \delta^*\Phi^\omega_{p,x_i}) - (\delta^{k_q-1}_{q'}, \delta^*\Phi^\omega_{q,x_i}) + (\delta^{k_p-1}_{p'}, \delta^*\Phi^\omega_{u,x_i}), \\ \Gamma^\omega_{i,2} &= (re_u, \delta^*\Phi^\omega_{u,x_i}) - (e_q, \delta^*\Phi^\omega_{p,x_i}) + (e_p, \delta^*\Phi^\omega_{q,x_i}),\end{aligned}$$

where $(\delta^*\Phi^\omega_{u,x_i}, \delta^*\Phi^\omega_{q,x_i}, \delta^*\Phi^\omega_{p,x_i}) := (\Phi^\omega_{u,x_i}, \Phi^\omega_{q,x_i}, \Phi^\omega_{p,x_i}) - \mathit{\Pi}^*(\Phi^\omega_{u,x_i}, \Phi^\omega_{q,x_i}, \Phi^\omega_{p,x_i})$.

Proof. By the boundary and transmission conditions defining the Green's functions, (3.7), we have that

$$\theta_i^\omega = \widehat{e}_\omega(x_i).$$

Taking $(v_1, v_2, v_3) := (\mathit{\Pi}^*\Phi^\omega_{p,x_i}, \mathit{\Pi}^*\Phi^\omega_{q,x_i}, \mathit{\Pi}^*\Phi^\omega_{u,x_i})$ in the identity of the previous lemma, we get that

$$\widehat{e}_\omega(x_i) = \theta^\omega_{i,1} + \theta^\omega_{i,2} + \theta^\omega_{i,3},$$

where

$$\begin{aligned}\theta^\omega_{i,1} &= \langle \widehat{e}_u - e_u,\ \delta^*\Phi^\omega_{p,x_i} n\rangle - \langle \widehat{e}_q - e_q,\ \delta^*\Phi^\omega_{q,x_i} n\rangle + \langle \widehat{e}_p - e_p,\ \delta^*\Phi^\omega_{u,x_i} n\rangle, \\ \theta^\omega_{i,2} &= (e_u',\ \delta^*\Phi^\omega_{p,x_i}) - (e_q',\ \delta^*\Phi^\omega_{q,x_i}) + (e_p',\ \delta^*\Phi^\omega_{u,x_i}), \\ \theta^\omega_{i,3} &= (re_u,\ \delta^*\Phi^\omega_{u,x_i}) - (e_q,\ \delta^*\Phi^\omega_{p,x_i}) + (e_p,\ \delta^*\Phi^\omega_{q,x_i}).\end{aligned}$$

First, we show that $\theta^{\omega}_{i,1} = 0$. By the definition of $\widehat{e}_p^{+}$ and $\widehat{e}_q$ in the error equations, (3.4), we get that

$$\begin{aligned}\theta^{\omega}_{i,1} &= \langle(\widehat{e}_u - e_u)n,\, \delta^*\Phi^{\omega}_{p,x_i} - \tau^{-}_{qu}\,\delta^*\Phi^{\omega}_{q,x_i}\rangle_{\partial\mathcal{T}_h^-}\\ &\quad + \langle(\widehat{e}_u - e_u)n,\, \delta^*\Phi^{\omega}_{p,x_i} - \tau^{+}_{qu}\,\delta^*\Phi^{\omega}_{q,x_i} + \tau^{+}_{pu}\,\delta^*\Phi^{\omega}_{u,x_i}\rangle_{\partial\mathcal{T}_h^+}\\ &\quad + \langle(\widehat{e}_p - e_p)n,\, \delta^*\Phi^{\omega}_{u,x_i} - \tau^{-}_{qp}\,\delta^*\Phi^{\omega}_{q,x_i}\rangle_{\partial\mathcal{T}_h^-}\\ &= 0,\end{aligned}$$

by the definition of $\mathit{\Pi}^*$.

Since obviously $\theta^{\omega}_{i,3} = \Gamma^{\omega}_{i,2}$, we only need to show that $\theta^{\omega}_{i,2} = \Gamma^{\omega}_{i,1}$. But the equality follows from the first orthogonality properties of the projection $\mathit{\Pi}^*$, (3.2). This completes the proof. □

A straightforward corollary of the previous result is the following estimate of the errors of the numerical traces.

Corollary 3.9. *For $\omega = u, q$ and p, we have*

$$\begin{aligned}|\widehat{e}_\omega(x_i)| \le{}& \|\delta^{k_u-1}_{u'}\|\,\|\delta^*\Phi^{\omega}_{p,x_i}\| + r\,\|e_u\|\,\|\delta^*\Phi^{\omega}_{u,x_i}\|\\ &+ \|\delta^{k_q-1}_{q'}\|\,\|\delta^*\Phi^{\omega}_{q,x_i}\| + \|e_q\|\,\|\delta^*\Phi^{\omega}_{p,x_i}\|\\ &+ \|\delta^{k_p-1}_{p'}\|\,\|\delta^*\Phi^{\omega}_{u,x_i}\| + \|e_p\|\,\|\delta^*\Phi^{\omega}_{q,x_i}\|.\end{aligned}$$

It is now clear that to prove the error estimates for the numerical traces of Theorem 2.3, we simply have to use the estimates of $\|\delta^{\ell(\omega)}_{\omega'}\|$ given in Lemma 3.6, the estimates of $\|e_\omega\|$ of Theorem 2.2, and those of $\|\delta^*_{\Psi_\omega}\|$ given by Lemma 3.2 with $\ell^*_\omega := k^*_\omega$. Again, this is a simple, but long and tedious manipulation, which we do not display here. This completes the proof of Theorem 2.3.

3.5. **Proof of error estimates of the postprocessed solution.** Now we prove the error estimates for the postprocessed solution in Theorem 2.4. Here we need to use the following lemma, which is proved in [6] in the proof of Theorem 3.1.

Lemma 3.10. *Let $I_i = (x_{i-1}, x_i)$. Suppose φ is the solution to the initial value problem*

$$\varphi'(x) = f(x),\ \ x \in I_i \quad \textit{and} \quad \varphi(x_{i-1}) = \varphi_0$$

and $\varphi^*_h \in P_s(I_i)$ is an approximate solution given by

$$-(\varphi^*_h, v')_{I_i} + \varphi^*_h(x_i)v(x_i) = (f^*, v)_{I_i} + \varphi^*_0\,v(x_{i-1})$$

for all $v \in P_s(I_i)$. Then

$$\|\varphi - \varphi^*_h\|_{L^\infty(I_i)} \le Ch^{s+1}\|f^{(s)}\|_{L^\infty(I_i)} + |\varphi_0 - \varphi^*_0| + Ch\|f - f^*\|_{L^\infty(I_i)}.$$

Now, let us prove Theorem 2.4. Applying Lemma 3.10 to the exact solutions in (2.1) and the postprocessed solutions in (2.7), we get

$$\begin{aligned}\|u - u^*_h\|_{L^\infty(I_i)} \le{}& Ch^{s+1}\|q^{(s)}\|_{L^\infty(I_i)} + |(u - \widehat{u}_h)(x_{i-1})| + Ch\|q - q^*_h\|_{L^\infty(I_i)},\\ \|q - q^*_h\|_{L^\infty(I_i)} \le{}& Ch^{s+1}\|p^{(s)}\|_{L^\infty(I_i)} + |(q - \widehat{q}_h)(x_{i-1})| + Ch\|p - p^*_h\|_{L^\infty(I_i)},\\ \|p - p^*_h\|_{L^\infty(I_i)} \le{}& Ch^{s+1}\Big(\|f^{(s)}\|_{L^\infty(I_i)} + \|ru^{(s)}\|_{L^\infty(I_i)}\Big) + |(p - \widehat{p}_h)(x_{i-1})|\\ &+ Ch\|r(u - u^*_h)\|_{L^\infty(I_i)}.\end{aligned}$$

Adding these three inequalities, we have, for h sufficiently small,

$$\begin{aligned}&\|u-u_h^*\|_{L^\infty(I_i)}+\|q-q_h^*\|_{L^\infty(I_i)}+\|p-p_h^*\|_{L^\infty(I_i)}\\ \le& Ch^{s+1}(\|ru^{(s)}\|_{L^\infty(I_i)}+\|q^{(s)}\|_{L^\infty(I_i)}+\|p^{(s)}\|_{L^\infty(I_i)}+\|f^{(s)}\|_{L^\infty(I_i)})\\ &+(|(u-\widehat{u}_h)(x_{i-1})|+|(q-\widehat{q}_h)(x_{i-1})|+|(p-\widehat{p}_h)(x_{i-1})|).\end{aligned}$$

Now it is only a matter of taking the maximum over all the intervals $I_i \in \Omega_h$ and using the estimates of the error in the numerical traces of Theorem 2.3. This completes the proof of Theorem 2.4.

The proof for the limit cases of Theorem 2.5 is similar.

4. Numerical results

In this section, we carry out several numerical experiments to explore the sharpness of our theoretical orders of convergence. To do that, we set $r = 1$ and take f such that the exact solutions are $u = \sin(\pi x)$ and $u = e^x$ on the domain $(0, 1)$.

Our numerical results indicate that the error estimates of $(\epsilon_u, \epsilon_q, \epsilon_p)$ and, as a consequence, those of (e_u, e_q, e_p), in Theorem 2.2 are sharp except for the method #8 with $\tau_{pu}^+ = 0$ and the methods #7, #9, and #12 with $\tau_{qp}^- \neq 0$. The orders of convergence we observe in the numerical experiments for these cases are displayed in Table 12.

Table 12. Observed orders of convergence of $(\epsilon_u, \epsilon_q, \epsilon_p)$. Only the methods with possibly non-sharp orders are considered. Below, $(a, b, c) \wedge k$ means $(a \wedge k, b \wedge k, c \wedge k)$ where $a \wedge k$ denotes the minimum between a and k.

method	$c_{pu}^+ = 0$, $c_{qp}^- = 0$	$c_{pu}^+ \neq 0$, $c_{qp}^- = 0$	$c_{pu}^+ = 0$, $c_{qp}^- \neq 0$	$c_{pu}^+ \neq 0$, $c_{qp}^- \neq 0$
#7	sharp	sharp	$k+1+(0,0,1)\wedge k$	$k+1+(0,0,1)\wedge k$
#8	$k+1+(1,1,3)\wedge k$	sharp	$k+1+(1,1,3)\wedge k$	sharp
#9	$k+1+(2,1,3)\wedge k$	$k+1+(2,1,0)\wedge k$	$k+1+(2,1,3)\wedge k$	$k+1+(2,1,0)\wedge k$
#12	sharp	sharp	$k+2+(1,0,0)\wedge k$	$k+2+(1,0,0)\wedge k$

Although our analysis is inconclusive for the method #9 with $\tau_{pu}^+ \neq 0$ and for the methods #8 and #13 with $\tau_{pu}^+ \neq 0$ and $k = 0$, our numerical results show that these methods are well defined.

We also observed that the orders of convergence for the numerical traces in Theorem 2.3 are *all* sharp except for the methods #7 and #12 with $\tau_{qp}^- \neq 0$. It seems that having a non-zero τ_{qp}^- does *not* affect the convergence orders of numerical traces at all. This is also true for the postprocessed solutions.

Finally, we observed that the orders of convergence for the limit cases listed in Table 11 of Theorem 2.5 are *all* sharp. For methods that are not listed in Table 11, our numerical results (not reported here) show that they are either not well defined or have suboptimal convergence rates for at least one of the variables.

5. Concluding remarks

There are many more characterizations of the exact solution we could have used to define HDG methods. For example, we could have taken as data for the local problem on the interval I_i, two boundary values at the point x_{i-1} and one at the point x_i. We could also have used some data which is not necessarily a boundary value, like the average of p over the interval I_i instead of the boundary value $\widehat{p}_i$. The equivalence of the resulting HDG methods with the ones presented here does not have an obvious answer and remains an open problem.

How to extend these results to the time-dependent case (both linear and nonlinear), on the one hand, and to the multi-dimensional case, on the other, constitute the subject of ongoing work.

Appendix A. Proof of the approximation properties of the projection

Here, we prove the approximation property of the projection, Theorem 2.1. To do that, we use the following notation. For any $\omega \in L^2(I_i)$, $(\omega)_{k_\omega}$ denotes the L^2-projection of ω onto the polynomial space $P_{k_\omega}(I_i)$. Then we have $\delta_\omega = g_\omega + d_\omega$, where $g_\omega = \omega - (\omega)_{k_\omega}$ and $d_\omega = (\omega)_{k_\omega} - \Pi\omega$ for $\omega = u, q$ and p. We only need to estimate d_ω.

(i) The case $k_u = k_q + 1$. In this case, $\mathit{\Pi} q$ is nothing but the L^2-projection into $P_{k_q}(I_i)$ by the second equation defining the projection $\mathit{\Pi}$, (2.6b). Thus, we have that $d_q = 0$. From the first and third equations defining the projection $\mathit{\Pi}$, (2.6a) and (2.6c), respectively, we see that we have

$$d_p = C_p\, L_{k_p} \quad \text{and} \quad d_u = C_u^+\, \frac{(-1)^{k_u}}{2}(L_{k_u} - L_{k_u-1}) + C_u^-\, \frac{1}{2}(L_{k_u} + L_{k_u-1})$$

where L_ℓ denotes the scaled Legendre polynomial of degree ℓ. We can then rewrite the remaining equations defining the projection Π, (2.6d), (2.6e) and (2.6f), as follows:

$$\begin{aligned} d_p + \tau_{pu}^+ d_u &= -(g_p + \tau_{pu}^+ g_u) && \text{on } x_{i-1}^+,\\ d_q + \tau_{qu}^+ d_u &= -(g_q + \tau_{qu}^+ g_u) && \text{on } x_{i-1}^+,\\ d_q - \tau_{qu}^- d_u - \tau_{qp}^- d_p &= -(g_q - \tau_{qu}^- g_u - \tau_{qp}^- g_p) && \text{on } x_i^- . \end{aligned}$$

Using the form of d_u, d_q and d_p, we can recast these equations in matrix form as follows:

$$\begin{pmatrix} (-1)^{k_p} & 0 & \tau_{pu}^+ \\ 0 & 0 & \tau_{qu}^+ \\ -\tau_{qp}^- & -\tau_{qu}^- & 0 \end{pmatrix} \begin{pmatrix} C_p \\ C_u^- \\ C_u^+ \end{pmatrix} = - \begin{pmatrix} g_p^+ + \tau_{pu}^+ g_u^+ \\ g_q^+ + \tau_{qu}^+ g_u^+ \\ g_q^- - \tau_{qu}^- g_u^- - \tau_{qp}^- g_p^- \end{pmatrix}.$$

This linear system has a unique solution since $\tau_{qu}^-\, \tau_{qu}^+ \neq 0$. Inverting it, we obtain, after taking absolute values, that

$$\begin{aligned} &|C_p| \le |\tau_{pu}^+/\tau_{qu}^+|\, \|g_q\|_{\partial I_i} + \|g_p\|_{\partial I_i},\\ &|C_u^-| \le \|g_u\|_{\partial I_i} + \big(|1/\tau_{qu}^-| + |(\tau_{qp}^-\tau_{pu}^+)/(\tau_{qu}^-\tau_{qu}^+)|\big)\, \|g_q\|_{\partial I_i} + \big|\tau_{qp}^-/\tau_{qu}^-\big|\, \|g_p\|_{\partial I_i},\\ &|C_u^+| \le \|g_u\|_{\partial I_i} + |1/\tau_{qu}^+|\, \|g_q\|_{\partial I_i}. \end{aligned}$$

The estimates now follow from the approximation properties of the L^2-projection and the assumption on the stabilization function.

(ii) The case $k_q = k_u$. From the first three equations defining the projection Π, (2.6a), (2.6b) and (2.6c), we see that we can write

$$d_\omega = C_\omega L_{k_\omega} \quad \text{for} \quad \omega = u, q, \quad \text{and} \quad p.$$

Proceeding as in the previous case, we rewrite the remaining equations defining the projection Π, (2.6d), (2.6e) and (2.6f), as follows:

$$\begin{pmatrix} (-1)^{k_p} & 0 & (-1)^{k_u}\tau^+_{pu} \\ 0 & (-1)^{k_q} & (-1)^{k_u}\tau^+_{qu} \\ -\tau^-_{qp} & 1 & -\tau^-_{qu} \end{pmatrix} \begin{pmatrix} C_p \\ C_q \\ C_u \end{pmatrix} = - \begin{pmatrix} g^+_p + \tau^+_{pu} g^+_u \\ g^+_q + \tau^+_{qu} g^+_u \\ g^-_q - \tau^-_{qu} g^-_u - \tau^-_{qp} g^-_p \end{pmatrix}.$$

This linear system has a unique solution since $\theta := \tau^-_{qu} + \tau^+_{qu} - (-1)^{k_u+k_p}\tau^-_{qp}\tau^+_{pu} \neq 0$. Inverting the system, we obtain, after simple but tedious algebraic manipulations, that

$$\begin{aligned}
|C_p| &\le \left|\tau^+_{pu}\,\tau^-_{qu}/\theta\right| \, \|g_u\|_{\partial I_i} + \left|\tau^+_{pu}/\theta\right| \, \|g_q\|_{\partial I_i} + \left(1 + \left|\tau^+_{pu}\tau^-_{qp}/\theta\right|\right) \, \|g_p\|_{\partial I_i} \\
|C_q| &\le \left|\tau^+_{qu}\,\tau^-_{qu}/\theta\right| \, \|g_u\|_{\partial I_i} + \left(1 + \left|\tau^+_{qu}/\theta\right|\right) \, \|g_q\|_{\partial I_i} + \left|\tau^+_{qu}\tau^-_{qp}/\theta\right| \, \|g_p\|_{\partial I_i} \\
|C_u| &\le \left(1 + \left|\tau^-_{qu}/\theta\right|\right) \, \|g_u\|_{\partial I_i} + |1/\theta| \, \|g_q\|_{\partial I_i} + \left|\tau^-_{qp}/\theta\right| \, \|g_p\|_{\partial I_i}.
\end{aligned}$$

The estimates now follow from the approximation properties of the L^2-projection and the assumption on the stabilization function.

(iii) The case $k_u = k_q - 1$. In this case, $\mathit{\Pi} u$ is nothing but the L^2-projection into $P_{k_u}(I_i)$ by the first equation defining the projection $\mathit{\Pi}$, (2.6a). Thus, we have that $d_u = 0$. From the second and third equations defining the projection $\mathit{\Pi}$, (2.6b) and (2.6c), respectively, we see that we have

$$d_p = C_p\, L_{k_p} \quad \text{and} \quad d_q = C^+_q\, \frac{(-1)^{k_q}}{2}(L_{k_q} - L_{k_q-1}) + C^-_q\, \frac{1}{2}(L_{k_q} + L_{k_q-1}).$$

We can now rewrite the remaining equations defining the projection Π, (2.6d), (2.6e) and (2.6f), as follows:

$$\begin{pmatrix} (-1)^{k_p} & 0 & 0 \\ 0 & 0 & 1 \\ -\tau^-_{qp} & 1 & 0 \end{pmatrix} \begin{pmatrix} C_p \\ C^-_q \\ C^+_q \end{pmatrix} = - \begin{pmatrix} g^+_p + \tau^+_{pu} g^+_u \\ g^+_q + \tau^+_{qu} g^+_u \\ g^-_q - \tau^-_{qu} g^-_u - \tau^-_{qp} g^-_p \end{pmatrix}.$$

And we get

$$\begin{aligned}
|C_p| &\le |\tau^+_{pu}| \, \|g_u\|_{\partial I_i} + \|g_p\|_{\partial I_i}, \\
|C^-_q| &\le \left(|\tau^-_{qu}| + |\tau^-_{qp}\tau^+_{pu}|\right) \, \|g_u\|_{\partial I_i} + \|g_q\|_{\partial I_i} + |\tau^-_{qp}| \, \|g_p\|_{\partial I_i}, \\
|C^+_q| &\le |\tau^+_{qu}| \, \|g_u\|_{\partial I_i} + \|g_q\|_{\partial I_i}.
\end{aligned}$$

The estimates now follow from the approximation properties of the L^2-projection and the assumption on the stabilization function. This completes the proof of Theorem 2.1.

Appendix B. Proof of the regularity estimate

In this section, we prove the regularity result, Lemma 3.1. Integrating the equations (3.1a)-(3.1c) in the dual problem and using the boundary conditions

$\Psi_u(0) = \Psi_q(0) = 0$, we get

$$\Psi_u = \mathcal{I}\Psi_q + \mathcal{I}\eta_p \quad \text{in} \quad \Omega, \tag{B.1a}$$

$$\Psi_q = \mathcal{I}\Psi_p - \mathcal{I}\eta_q \quad \text{in} \quad \Omega, \tag{B.1b}$$

$$\Psi_p = r\mathcal{I}\Psi_u + \mathcal{I}\eta_u + \Psi_p(0) \quad \text{in} \quad \Omega, \tag{B.1c}$$

where we have used the notation $\mathcal{I}f(x) := \int_0^x f(t)dt$. From Lemma A.1 in [8], we have $\|\mathcal{I}^n f\| \le \|f\|$ for any $f \in L^2(\Omega)$ and $n \ge 1$.

First, we estimate the term $\Psi_p(0)$. Integrating the equation (3.1a) on $[0, 1]$ and using the boundary conditions in (3.1d),

$$0 = \Psi_u(1) - \Psi_u(0) = \int_0^1 \Psi_q(x)dx + \int_0^1 \eta_p(x)dx.$$

Using (B.1b) and (B.1c), we have

$$\begin{aligned} 0 &= \int_0^1 (\mathcal{I}\Psi_p - \mathcal{I}\eta_q)dx + \int_0^1 \eta_p(x)dx \\ &= \int_0^1 (r\mathcal{I}^2\Psi_u + \mathcal{I}^2\eta_u + \mathcal{I}\Psi_p(0))dx - \int_0^1 \mathcal{I}\eta_q dx + \int_0^1 \eta_p(x)dx \\ &= \int_0^1 (r\mathcal{I}^2\Psi_u + \mathcal{I}^2\eta_u)dx + \frac{1}{2}\Psi_p(0) - \int_0^1 \mathcal{I}\eta_q dx + \int_0^1 \eta_p(x)dx. \end{aligned}$$

By the Cauchy-Schwarz inequality, we get

$$|\Psi_p(0)| \le 2(r\|\mathcal{I}^2\Psi_u\| + \|\mathcal{I}^2\eta_u\| + \|\mathcal{I}\eta_q\| + \|\eta_p\|) \le 2(r\|\Psi_u\| + \|\eta_u\| + \|\eta_q\| + \|\eta_p\|).$$

Therefore, we have

$$|\Psi_p(0)|^2 \le 16(r^2\|\Psi_u\|^2 + \|\eta\|^2). \tag{B.2}$$

Next, we estimate the term $r\|\Psi_u\|$. Multiplying the equations (3.1a)-(3.1c) by $-\Psi_p, \Psi_q$ and $-\Psi_u$, respectively, integrating over [0,1], and adding them together, we get

$$\Psi_q^2(1) + r\int_0^1 \Psi_u^2 dx = -\int_0^1 \eta_p\Psi_p\, dx - \int_0^1 \eta_q\Psi_q\, dx - \int_0^1 \eta_u\Psi_u\, dx,$$

which implies that

$$r\|\Psi_u\|^2 \le \frac{\varepsilon}{2}(\|\Psi_u\|^2 + \|\Psi_q\|^2 + \|\Psi_p\|^2) + \frac{\varepsilon^{-1}}{2}\|\eta\|^2 \tag{B.3}$$

for any $\varepsilon > 0$.

Now let us estimate Ψ_p, Ψ_q, and Ψ_u in L^2-norm. From (B.1c), we get

$$\|\Psi_p\|^2 \le 2(r^2\|\mathcal{I}\Psi_u\|^2 + \|\mathcal{I}\eta_u\|^2 + |\Psi_p(0)|^2) \le 2(r^2\|\Psi_u\|^2 + \|\eta_u\|^2 + |\Psi_p(0)|^2).$$

Using (B.2) and (B.3),

$$\begin{aligned} \|\Psi_p\|^2 &\le 34r^2\|\Psi_u\|^2 + 34\|\eta\|^2 \\ &\le 17\varepsilon r(\|\Psi_u\|^2 + \|\Psi_q\|^2 + \|\Psi_p\|^2) + (17\varepsilon^{-1}r + 34)\|\eta\|^2. \end{aligned}$$

From (B.1b), we get

$$\begin{aligned} \|\Psi_q\|^2 &\le 2(\|\mathcal{I}\Psi\|^2 + \|\mathcal{I}\eta_q\|^2) \\ &\le 2(\|\Psi_p\|^2 + \|\eta_q\|^2) \le 34\,\varepsilon\, r\,(\|\Psi_u\|^2 + \|\Psi_q\|^2 + \|\Psi_p\|^2) + (34\varepsilon^{-1}r + 70)\|\eta\|^2. \end{aligned}$$

Similarly, from (B.1a), we get

$$\|\Psi_u\|^2 \leq 2(\|\Psi_q\|^2 + \|\eta_p\|^2) \leq 68\,\varepsilon\, r(\|\Psi_u\|^2 + \|\Psi_q\|^2 + \|\Psi_p\|^2) + (68\varepsilon^{-1}r + 142)\|\eta\|^2.$$

Hence

$$\|\Psi_u\|^2 + \|\Psi_q\|^2 + \|\Psi_p\|^2 \leq 119\varepsilon r(\|\Psi_u\|^2 + \|\Psi_q\|^2 + \|\Psi_p\|^2) + (119\varepsilon^{-1}r + 246)\|\eta\|^2.$$

Choosing $\varepsilon < \frac{1}{119r}$, we have

$$\|\Psi_u\|^2 + \|\Psi_q\|^2 + \|\Psi_p\|^2 \leq C\|\eta\|^2. \tag{B.4}$$

Now we only need to show that the H^1 seminorm of Ψ_p, Ψ_q and Ψ_u are bounded by η in L^2-norm. This follows from the dual problem (3.1a)-(3.1c), the L^2-estimates of (Ψ_p, Ψ_q, Ψ_u) in (B.4) and the estimate of $r\Psi_u$ in (B.3).

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