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ERROR BOUNDS FOR A DIRICHLET BOUNDARY CONTROL PROBLEM BASED ON ENERGY SPACES

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ABSTRACT. In this article, an alternative energy-space based approach is proposed for the Dirichlet boundary control problem and then a finite-element based numerical method is designed and analyzed for its numerical approximation. A priori error estimates of optimal order in the energy norm and the L_2-norm are derived. Moreover, a reliable and efficient a posteriori error estimator is derived with the help of an auxiliary problem. The theoretical results are illustrated by the numerical experiments.

1. INTRODUCTION

The study of PDE constrained optimal control problems is one of the important research areas in the past few decades. The numerical analysis for this class of problems began in the 1970s [16, 22]. Subsequently, there are many important contributions to this field. It is difficult to list all the results in this introduction; we refer to some of the articles and references therein for the development of numerical methods and their error analysis. Refer to the monograph [33] for the theory of optimal control problems and for the development of numerical methods. There are mainly two types of controls proposed in the literature: one, which is said to be distributed control, is proposed through interior force, and the other is the boundary control applied through either Neumann or Robin or Dirichlet boundary conditions. For work related to the distributed control problems, see for example [13, 19, 32] and references therein. Regarding super-convergence results for distributed control problems, refer to [30] for discretized control and to [23] for undiscretized control. The Neumann boundary control problem with graded mesh refinement is analyzed in [1] and a super-convergence result is derived in [10]. For the a posteriori error analysis of conforming finite element methods for distributed control problems see [24, 25, 27] and for the a posteriori error analysis of the Neumann control problem refer to [28]. A framework for *a priori* and *a posteriori* energy norm error analysis for Neumann and distributed control problems by discontinuous Galerkin discretization can be found in [11]. Local error analysis of discontinuous Galerkin methods for the distributed control problem for the advection-diffusion equation is studied in [26]. Unlike the Neumann or distributed control problems, the Dirichlet control problem stimulates additional difficulties in formulating the problem when the control is

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sought from the space $L_2(\Gamma)$, where Γ is the boundary of the domain Ω. This is due to the fact that the trace of any $H^1(\Omega)$ function belongs to $H^{1/2}(\Gamma)$. The difficulty is resolved by reformulating the model problem into a very (ultra) weak formulation and when the domain Ω is a smooth or convex polygonal domain, the control is shown to have additional regularity and thereby the standard weak formulation is recovered; see [9, 14, 29]. When the domain Ω is not smooth (or polygonal), the state and the control have restricted regularity. The error analysis of finite element methods for the Dirichlet control problem uses the elliptic regularity results on convex domains; see for example [9, 29]. Alternatively the Dirichlet control problem is handled by perturbing the Dirichlet boundary condition into a Robin type boundary condition with singular perturbation by penalty; see [2, 8]. Another alternative is to seek the control from an $H^{1/2}(\Gamma)$ space [21, 31] or $H^1(\Gamma)$ space [20].

The finite element analysis of an $H^{1/2}(\Gamma)$-space based Dirichlet control problem was discussed recently in [31]. Therein the continuous and the discrete problems are defined by using the Steklov-Poincaré operator arising from the harmonic extension and then the optimality system is written as an $H^{1/2}$-space variational inequality on the boundary. Subsequently the numerical method and its analysis is discussed by using the same setting. The numerical method in [31] is converted into a system of equations defined in the interior of the domain using harmonic and continuous extension operators. The harmonic extension is used (as a consequence of the Steklov-Poincaré operator) for the trial functions and a continuous extension is used for the test functions of the control variable. This subsequently implies that one has to solve a Dirichlet problem for each trial function from the control space. In this article, we revisit the study of the Dirichlet boundary control problem and propose an alternative numerical algorithm with the aim of obtaining optimal order error estimates. For this, we propose a different approach from [31] for defining a Dirichlet boundary control problem which also produces a sufficiently regular control. In [31], the Steklov-Poincaré operator was used to define the cost functional with the help of a harmonic extension of the given boundary data. In our approach the control has been sought in the $H^1(\Omega)$ space and the resulting control is a harmonic function without this being explicitly imposed. This leads to the optimality conditions in a system of PDEs posed over the domain Ω. Based on this formulation, we propose a finite element numerical method and derive its corresponding optimal order error estimates in the energy norm and in the L_2-norm. The arguments in our analysis of this article are new and different from [31]. Further, using an auxiliary system of PDEs, we derive a reliable and efficient a posteriori error estimator for the development of an efficient adaptive algorithm. Numerical experiments are performed to illustrate the theoretical results on *a priori* as well as *a posteriori* error estimates.

The rest of the article is organized as follows. In section 2, we formulate the Dirichlet control problem, prove its well-posedness, derive corresponding optimality conditions and deduce the elliptic regularity. In section 3, we define the discrete control problem, derive discrete optimality conditions and prove the existence and uniqueness of the discrete solution. In section 4 and section 5, we derive *a priori* and *a posteriori* error estimates respectively. In section 6, we present numerical experiments to illustrate the theoretical results. Finally, we conclude the article in section 7.

2. Dirichlet control problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary Γ. Assume there is some $m \geq 1$ such that the boundary Γ is the union of line segments Γ_i $(1 \leq i \leq m)$ whose interior in the induced topology are pair-wise disjoint. Define the norm on the Sobolov space $H^s(\Omega)$ by $\|\cdot\|_s$ for $s \geq 0$. Let $(\cdot,\cdot)$ (resp. $\|\cdot\|$) denote the $L_2(\Omega)$-inner product (resp. norm). Let $V := H_0^1(\Omega)$ and $Q := H^1(\Omega)$. Define

$$a(w,v) = (\nabla w, \nabla v), \quad w, v \in Q,$$

where ∇ is the standard gradient operator. For given $g \in L_2(\Omega)$ and $p \in Q$, the Lax-Milgram lemma [7] implies that there exists a unique solution $w(g,p) \in V$ such that

$$a(w(g,p), v) = (g,v) - a(p,v) \quad \forall v \in V. \tag{2.1}$$

Then note that $w = w(g,p) + p$ is the weak solution of the Dirichlet problem:

$$\begin{aligned} -\Delta w &= g \quad \text{in} \quad \Omega, \\ w &= p \quad \text{on} \quad \Gamma. \end{aligned}$$

Definition 2.1. Define the solution operator $S : L_2(\Omega) \times Q \to Q$ by $S(g,p) := w$, where $w = w(g,p) + p$ and $w(g,p)$ is the solution of (2.1).

The quadratic cost functional $J : Q \times Q \to R$ is defined by

$$J(w,p) = \frac{1}{2}\|w - u_d\|^2 + \frac{\alpha}{2}\|\nabla p\|^2, \quad w \in Q,\ p \in Q, \tag{2.2}$$

where α is a given positive real number and $u_d \in L_2(\Omega)$ is a given desired function.

Model problem. The Dirichlet boundary control problem consists of finding $(u,q) \in Q \times Q$ such that

$$J(u,q) = \min_{(w,p) \in Q \times Q} J(w,p), \tag{2.3}$$

subject to the condition that $(w,p) \in Q \times Q$ satisfies $w = S(f,p)$, where $f \in L_2(\Omega)$ is a given function.

Remark 2.2. We may also view the minimization problem (2.3) as follows: find $(u_q, q) \in Q \times Q$ such that

$$J_0(u_q, q) = \min_{(w_p,p) \in Q \times Q} J_0(w_p, p),$$

where

$$J_0(w_p, p) = \frac{1}{2}\|w_p + u_f - u_d\|^2 + \frac{\alpha}{2}\|\nabla p\|^2,$$

subject to the condition that $w_p \in Q$ satisfies $a(w_p, v) = 0$ for all $v \in V$ with $w_p|_\Gamma = p|_\Gamma$, and $u_f \in V$ satisfies $a(u_f, v) = (f, v)$ for all $v \in V$. Also we note that the minimum energy in (2.3) will be realized with an equivalent $H^{1/2}(\Gamma)$-norm of the control q; see Remark 2.4.

Proposition 2.3. *There exists a unique solution* $(u, q) \in Q \times Q$ *for the optimal control problem* (2.3)*. Furthermore, there exists an adjoint state* $\phi \in V$ *satisfying the following:*

$$u = u_f + q, \quad u_f \in V,$$
$$a(u_f, v) = (f, v) - a(q, v) \quad \forall v \in V, \tag{2.4}$$
$$a(v, \phi) = (u - u_d, v) \quad \forall v \in V, \tag{2.5}$$
$$\alpha a(q, p) = a(p, \phi) + (u_d - u, p) \quad \forall p \in Q. \tag{2.6}$$

Proof. Using the solution operator S defined in Definition 2.1, we introduce the reduced functional $j : Q \to R$ by

$$j(p) = J(S(f, p), p).$$

Then by the theory of elliptic optimal control problems [33, Theorem 2.14], there exists a unique $q \in Q$ such that

$$j(q) = \min_{p \in Q} j(p).$$

Denoting the corresponding state by $u = S(f, q) \in Q$, we find that $u = u_f + q$ and $u_f \in V$ satisfies (2.4). From the first order optimality condition that $j'(q)(p) = 0$ for all $p \in Q$, we find that

$$0 = j'(q)(p) = \alpha a(q, p) + (u - u_d, S(0, p)) \quad \forall p \in Q. \tag{2.7}$$

Introduce the adjoint problem of finding $\phi \in V$ such that

$$a(v, \phi) = (u - u_d, v) \quad \forall v \in V.$$

Then, we find

$$\begin{aligned} (u - u_d, S(0, p)) &= (u - u_d, S(0, p) - p) + (u - u_d, p) \\ &= a(S(0, p) - p, \phi) + (u - u_d, p) \\ &= (u - u_d, p) - a(p, \phi), \end{aligned} \tag{2.8}$$

since $S(0, p) - p \in V$ for any $p \in Q$ and $a(S(0, p), v) = 0$ for any $v \in V$. This completes the proof. ∎

For the rest of the article $(u, q, \phi) \in Q \times Q \times V$ denote the optimal state, optimal control and optimal costate. By taking $p = v \in V$ in (2.6) and using (2.5), we find that

$$a(q, v) = 0 \quad \text{for all} \quad v \in V. \tag{2.9}$$

Therefore the optimality system (2.4)–(2.6) can be written in the reduced form as follows:

$$u = u_f + q, \quad u_f \in V,$$
$$a(u_f, v) = (f, v) \quad \forall v \in V, \tag{2.10}$$
$$a(v, \phi) - (q, v) = (u_f - u_d, v) \quad \forall v \in V, \tag{2.11}$$
$$\alpha a(q, p) + (q, p) - a(p, \phi) = (u_d - u_f, p) \quad \forall p \in Q. \tag{2.12}$$

Remark 2.4. For any $p \in H^{1/2}(\Gamma)$, its $H^{1/2}(\Gamma)$ semi-norm can be equivalently defined by the Dirichlet norm:

$$|p|_{H^{1/2}(\Gamma)} := \|\nabla u_p\| = \min_{w\in Q, w=p \text{ on } \Gamma} \|\nabla w\|,$$

where the minimizer $u_p \in Q$ satisfies

$$\begin{aligned} -\Delta u_p &= 0 \quad \text{in} \quad \Omega, \\ u_p &= p \quad \text{on} \quad \Gamma. \end{aligned}$$

For the optimal control q, we have $q = u_q$ by uniqueness and hence

$$|q|_{H^{1/2}(\Gamma)} := \|\nabla q\|.$$

Therefore the minimum energy in the minimization problem (2.3) is realized with an equivalent $H^{1/2}(\Gamma)$-norm of the control q.

For the subsequent discussions, we use the following notation. For $f \in L_2(\Omega)$, $q, p \in Q$, let

$$\begin{aligned} u_f &= S(f, 0) \in V, \\ u_q &= S(0, q) \in Q, \\ \phi_f &= S(u_f - u_d, 0) \in V, \\ \phi_q &= S(u_q, 0) \in V, \\ u_p &= S(0, p) \in Q. \end{aligned}$$

Throughout the article, the letters p, r and q are used to denote the test and trial functions for the control. The state variable u with suffixes p, r and q denote harmonic extensions with Dirichlet data p, r and q, respectively. The adjoint variable ϕ with any suffix always solves a homogeneous Dirichlet boundary value problem. The suffixes f and q of ϕ indicate the forces corresponding to u_f and u_q, respectively, in the equation. Later on similar notation will be used for discrete functions.

Note from (2.9) that $u_q = q$. We write the state u and the adjoint state ϕ as $u = u_f + u_q$ and $\phi = \phi_f + \phi_q$, respectively. Using the notation and (2.7), we rewrite (2.6) as follows:

$$\alpha a(q, p) + (u_q, u_p) = (u_d - u_f, u_p) \quad \forall p \in Q. \tag{2.13}$$

Throughout the article, C denotes a generic positive constant that is independent of the solutions and subsequently independent of the mesh-size.

Lemma 2.5. *If the domain Ω is convex, then $u, \phi, q \in H^2(\Omega)$ and*

$$\|u\|_2 + \|\phi\|_2 + \|q\|_2 \le C(\|f\| + \|u_d\|).$$

Proof. Since $f \in L_2(\Omega)$, we find from the elliptic regularity on convex polygonal domains [18, Theorem 3.1.2.1] that $u_f = S(f, 0) \in H^2(\Omega)$ and $\|u_f\|_2 \le C\|f\|$. By taking $p = q$ in (2.13), we can easily derive the following stability estimate:

$$\|q\|_1 = \|u_q\|_1 \le C(\|u_d\| + \|f\|). \tag{2.14}$$

Since $u - u_d = u_q + u_f - u_d \in L_2(\Omega)$, we find $\phi = S(u - u_d, 0) \in H^2(\Omega)$ and using (2.14) that

$$\|\phi\|_2 \le C\|u - u_d\| = C\|u_q + u_f - u_d\| \le C(\|u_d\| + \|f\|).$$

Using integration by parts in (2.6), we note that q solves

$$\alpha a(q,p) = \int_\Gamma \frac{\partial \phi}{\partial n} p \, ds \quad \forall p \in Q,$$

i.e., q solves the Neumann problem

$$-\Delta q = 0 \quad \text{in} \quad \Omega,$$
$$\alpha \partial q/\partial n = \partial \phi/\partial n \quad \text{on} \quad \Gamma.$$

Take $p = 1$ in (2.6) and find that $(u_d - u, 1) = 0$. Using this in (2.5), we find that

$$\int_\Gamma \frac{\partial \phi}{\partial n} \, ds = 0,$$

which is the compatibility condition for the Neumann problem for q. Further from (2.12), q satisfies $(q,1) = (u_d - u_f, 1)$. Since $\phi \in H^2(\Omega)$, by the trace theorem [18, Theorem 1.5.2.1] its normal derivative $\partial\phi/\partial n|_{\Gamma_i} \in H^{1/2}(\Gamma_i)$ for $1 \le i \le m$. Then the elliptic regularity theory for the Neumann problem implies that $q \in H^2(\Omega)$ (see for example [17, Theorem 1.10], [18, Theorem 3.1.2.3]) and $\|q\|_2 \le C\left(\|\phi\|_2 + \|u_d - u_f\|\right)$. Finally, $\|u\|_2 = \|u_q + u_f\|_2 = \|q + u_f\|_2 \le C(\|u_d\| + \|f\|)$. $\square$

3. Discrete Dirichlet control problem

Let $\mathcal{T}_h$ be a regular triangulation of Ω; see [7, p. 108], [12, p. 124]. Denote the set of all interior edges of $\mathcal{T}_h$ by $\mathcal{E}_h^i$, the set of boundary edges by $\mathcal{E}_h^b$, and define $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$. Let h_T=diam(T) and set $h = \max\{h_T : T \in \mathcal{T}_h\}$. The set of all vertices of $\mathcal{T}_h$ is denoted by $\mathcal{V}_h$.

The finite element spaces are defined by

$$V_h = \{v \in H_0^1(\Omega) : v|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h\} \tag{3.1}$$

and

$$Q_h = \{q \in H^1(\Omega) : q|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h\}, \tag{3.2}$$

where $\mathbb{P}_1(D)$ is the space of polynomials of degree less than or equal to one restricted to the set D.

In the error analysis, we need the jump of the normal derivative of discrete functions across the inter-element boundaries. For any $e \in \mathcal{E}_h^i$, there are two elements T_+ and T_- such that $e = \partial T_+ \cap \partial T_-$. Let n_+ be the unit normal of e pointing from T_+ to T_- and let $n_- = -n_+$. Then for any $v_h \in Q_h$, define the jump of the normal derivative of v_h on e by

$$[\![\nabla v]\!] = \nabla v_+|_e \cdot n_+ + \nabla v_-|_e \cdot n_-,$$

where $v_\pm = v\big|_{T_\pm}$, the restriction of the function v to $T_\pm$.

As in the continuous case, we define the discrete solution operator S_h. To this end, for $g \in L_2(\Omega)$ and $p \in Q$, let $w_h(g,p) \in V_h$ be the unique solution of

$$a(w_h(g,p), v) = (g, v) - a(p, v) \quad \forall v \in V_h. \tag{3.3}$$

Definition 3.1. Denote the solution operator $S_h : L_2(\Omega) \times Q \to Q$ by $S_h(g,p) := w_h$, where $w_h = w_h(g,p)$ and $w_h(g,p)$ solves (3.3) with data p and g. Note that $w_h = S_h(g,p) \in Q_h$ whenever $p \in Q_h$.

The discrete problem will be defined in the following.

Discrete problem. The discrete Dirichlet boundary control problem consists of finding $(u_h, q_h) \in Q_h \times Q_h$ such that

$$J(u_h, q_h) = \min_{(w_h,p_h)\in Q_h\times Q_h} J(w_h, p_h), \tag{3.4}$$

subject to the condition that $(w_h, p_h) \in Q_h \times Q_h$ satisfies $w_h = S_h(f, p_h)$.

As in the continuous case, the first order optimality conditions lead to the following system of discrete problems: Find $(u_h, q_h, \phi_h) \in Q_h \times Q_h \times V_h$ such that

$$u_h = u_f^h + q_h, \quad u_f^h \in V_h,$$

$$a(u_f^h, v) = (f, v) - a(q_h, v) \quad \forall v \in V_h, \tag{3.5}$$

$$a(v, \phi_h) = (u_h - u_d, v) \quad \forall v \in V_h, \tag{3.6}$$

$$\alpha a(q_h, p) = a(p, \phi_h) + (u_d - u_h, p) \quad \forall p \in Q_h. \tag{3.7}$$

By taking $p = v_h \in V_h$ in (3.7) and using (3.6), note that the discrete control q_h satisfies

$$a(q_h, v_h) = 0 \quad \forall v_h \in V_h. \tag{3.8}$$

Therefore the discrete problem reduces to

$$u_h = u_f^h + q_h, \quad u_f^h \in V_h,$$

$$a(u_f^h, v) = (f, v) \quad \forall v \in V_h, \tag{3.9}$$

$$a(v, \phi_h) - (q_h, v) = (u_f^h - u_d, v) \quad \forall v \in V_h, \tag{3.10}$$

$$\alpha a(q_h, p) + (q_h, p) - a(p, \phi_h) = (u_d - u_f^h, p) \quad \forall p \in Q_h. \tag{3.11}$$

It is useful to note that (3.9) can be solved independently of the system and then the coupled system (3.6)–(3.7) can be solved successively.

The following stability result is useful in establishing the existence and uniqueness of the discrete problem.

Lemma 3.2. *Let* $(u_h, q_h, \phi_h) \in Q_h \times Q_h \times V_h$ *solve* (3.5)–(3.7). *Then*

$$\|\phi_h\|_1 + \|q_h\|_1 + \|u_h\|_1 \le C\left(\|f\| + \|u_d\|\right).$$

Consequently the discrete problem (3.5)–(3.7) *has a unique solution.*

Proof. From (3.9) and the Poincaré inequality, note that

$$\|u_f^h\|_1 \le C\|f\|. \tag{3.12}$$

Take $p = q_h$ in (3.11) and use (3.8) to arrive at

$$\alpha a(q_h, q_h) + (q_h, q_h) = (u_d - u_f^h, q_h).$$

The Cauchy-Schwarz inequality and (3.12) imply

$$\|q_h\|_1 \le C\left(\|f\| + \|u_d\|\right).$$

Since $u_h = {u_f}^h + q_h$, we also get $\|u_h\|_1 \le C\left(\|f\| + \|u_d\|\right)$. From (3.10) and the Poincaré inequality, it is now easy to deduce that

$$\|\phi_h\|_1 \le C\left(\|f\| + \|u_d\|\right).$$

This completes the proof of the stability estimates. Existence and uniqueness of the solution to the discrete problem is then a consequence of these estimates and finite dimensionality. □

Throughout the article u_h, q_h and ϕ_h denote the discrete optimal state, discrete optimal control and discrete optimal costate, respectively.

4. A priori error analysis

In this section, we derive *a priori* error estimates in the energy and the L_2-norms. In the analysis, we introduce an enriched discrete control in order to derive optimal order L_2-norm error estimates. Further, we derive a consistency equation which will be the key to obtaining the error estimates, particularly helping to design the dual problem for the Aubin-Nitsche duality argument.

To present the arguments systematically, we introduce the following notation: For $f \in L_2(\Omega)$ and $q_h, p_h \in Q_h$, let

$$\begin{aligned} u_f^h &= S_h(f,0) \in V_h, \\ u_{q_h}^h &= S_h(0,q_h) \in Q_h, \\ \phi_f^h &= S_h(u_f^h - u_d, 0) \in V_h, \\ \phi_{q_h}^h &= S_h(u_{q_h}^h, 0) \in V_h, \\ u_{p_h}^h &= S_h(0,p_h) \in Q_h. \end{aligned}$$

Note that (3.8) implies that $u_{q_h}^h = q_h$.

The following lemma establishes some auxiliary estimates which are useful in the subsequent *a priori* error analysis.

Lemma 4.1. *Let $u_q^h = S_h(0,q)$ and $u_p^h = S_h(0,p)$ for $q, p \in Q$. Then*

$$\|u_p - u_p^h\|_1 \leq C \inf_{v_h \in V_h} \|(u_p - p) - v_h\|_1, \tag{4.1}$$

$$\|u_q - u_p\|_1 \leq C\|q - p\|_1, \tag{4.2}$$

$$\|u_q^h - u_p^h\|_1 \leq C\|q - p\|_1. \tag{4.3}$$

Proof. Note that $u_p = S(0,p) = w_0 + p$ with $w_0 \in V$ satisfying

$$a(w_0, v) = -a(p, v) \quad \forall v \in V.$$

Similarly we can write $u_p^h = S_h(0,p) = w_0^h + p$, where $w_0^h \in V_h$ satisfies

$$a(w_0^h, v_h) = -a(p, v_h) \quad \forall v_h \in V_h.$$

Then

$$\|u_p - u_p^h\|_1 = \|w_0 - w_0^h\|_1 \leq C \inf_{v_h \in V_h} \|w_0 - v_h\|_1 = \inf_{v_h \in V_h} \|(u_p - p) - v_h\|_1.$$

This proves (4.1). Writing $u_q = S(0,q) = v_0 + q$ with v_0 satisfying

$$a(v_0, v) = -a(q, v) \quad \forall v \in V,$$

we find

$$\|u_q - u_p\|_1 \leq \|w_0 - v_0\|_1 + \|q - p\|_1 \leq C\|q - p\|_1,$$

since $a(w_0 - v_0, v) = -a(q - p, v)$ for all $v \in V$. This proves (4.2). Similarly, we can prove the estimate in (4.3). □

We prove some more auxiliary results which are very useful for the L_2-norm error estimates. For this, we construct an enriched discrete optimal control q_h which is denoted by $\tilde{q}_h$. Let $\mathcal{H}(T)$ be the cubic Hermite finite element space defined on T with degrees of freedom defined as shown in Figure 4.1; see [7, p. 75], [12, p. 66]. Further, whenever T shares an edge e with the boundary Γ, we define the enriched Hermite element $\tilde{\mathcal{H}}(T)$ on T by

$$\tilde{\mathcal{H}}(T) := \mathcal{H}(T) \oplus \operatorname{span}\{\mathbf{b}_e\},$$

where $\mathbf{b}_e = \lambda_i^2\lambda_j^2(3\lambda_i - 1)(3\lambda_j - 1)$ and λ_i, λ_j are the barycentric coordinates associated to the endpoints of the boundary edge $e = \partial T \cap \Gamma$.

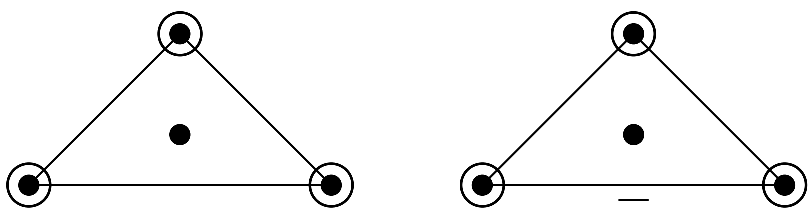

FIGURE 4.1. Degrees of freedom for the P_3-Hermite element $\mathcal{H}(T)$ (left) and the enriched P_3-Hermite element $\tilde{\mathcal{H}}(T)$ (right). Thick dots denote the values of the function, circles denote the values of its first order derivatives and the horizontal line parallel to the edge denotes the integral mean of the function over the edge.

The set of degrees of freedom defined for $\mathcal{H}(T)$ in Figure 4.1 is uni-solvent [7, p. 75], [12, p. 66] and then subsequently the choice of $\mathbf{b}_e$ implies that the set of degrees of freedom for $\tilde{\mathcal{H}}(T)$ defined in Figure 4.1 is also uni-solvent.

Define the enriched cubic Hermite finite element space by

$$\tilde{Q}_h := \{v \in C^0(\bar{\Omega}) : v|_T \in \mathcal{Q}(T) \quad \forall T \in \mathcal{T}_h\},$$

where $\mathcal{Q}(T) = \mathcal{H}(T)$ if T does not share an edge with Γ and $\mathcal{Q}(T) = \tilde{\mathcal{H}}(T)$ if T shares an edge e with Γ. In the latter case, the shape function $\mathbf{b}_e$ is associated to the boundary edge e. If a triangle has two boundary edges, then two such $\mathbf{b}_e$ functions can be added to the space $\mathcal{H}(T)$. Usually the triangles in triangulations in practice share at most one edge with boundary Γ.

We construct the enriched optimal control $\tilde{q}_h \in \tilde{Q}_h$ by averaging: Denote any vertex of the triangulation $\mathcal{T}_h$ by ν and let $\mathcal{T}_\nu$ denote the set of all triangles sharing the vertex ν with cardinality denoted by $|\mathcal{T}_\nu|$. Denote the barycenter of any T by

c_T. Then define

$$
\begin{aligned}
\tilde{q}_h(\nu) &= q_h(\nu) \quad \forall \nu \in \mathcal{V}_h,\\
\tilde{q}_h(c_T) &= q_h(c_T) \quad \forall T \in \mathcal{T}_h,\\
\nabla \tilde{q}_h(\nu) &= \frac{1}{|\mathcal{T}_\nu|} \sum_{T\in\mathcal{T}_\nu} \nabla (q_h|_T)(\nu) \quad \forall \nu \in \mathcal{V}_h,\\
\int_e \tilde{q}_h \, ds &= \int_e q_h \, ds \quad \forall e \in \mathcal{E}_h^b, \qquad (4.4)
\end{aligned}
$$

where $q_h|_T$ is the restriction of q_h to T. Note that the enriched control $\tilde{q}_h \in C^0(\bar{\Omega})$ and $\tilde{q}_h|_{\Gamma_i} \in C^1(\bar{\Gamma}_i)$ for all $1 \le i \le m$.

We now prove an approximation property of the enriched function $\tilde{q}_h$ in the following lemma.

Lemma 4.2. *Let Ω be convex, let $\mathcal{T}_h$ be a quasi-uniform triangulation and let $\tilde{q}_h$ be the enriched q_h. Then*

$$\|q_h - \tilde{q}_h\| + h\|q_h - \tilde{q}_h\|_1 \le Ch\left(\|q - q_h\|_1 + h\|q\|_2\right).$$

Proof. The definition of $\tilde{q}_h$ and the arguments in [5, 6] imply that

$$h^{-2}\|q_h - \tilde{q}_h\| + h^{-1}\|q_h - \tilde{q}_h\|_1 \le C\Big(\sum_{e\in\mathcal{E}_h^i} \frac{1}{h}\|[\![\nabla q_h]\!]\|^2_{L_2(e)}\Big)^{1/2}.$$

Since Ω is assumed to be convex, the optimal control $q \in H^2(\Omega)$ and hence $[\![\nabla q]\!] = 0$ on all the interior edges. Therefore by using the trace inequality [12, p. 146], we obtain

$$
\begin{aligned}
\|[\![\nabla q_h]\!]\|^2_{L_2(e)} &= \|[\![\nabla (q_h - q)]\!]\|^2_{L_2(e)}\\
&\le Ch^{-1} \sum_{T\in\mathcal{T}_e} \Big(\|q - q_h\|^2_{H^1(T)} + h^2\|q\|^2_{H^2(T)}\Big).
\end{aligned}
$$

We sum over all the interior edges and rearrange the terms to find

$$\Big(\sum_{e\in\mathcal{E}_h^i} \frac{1}{h}\|[\![\nabla q_h]\!]\|^2_{L_2(e)}\Big) \le Ch^{-2}\left(\|q - q_h\|_1^2 + h^2\|q\|_2^2\right).$$

The rest of the proof now follows by taking the square root on both sides. ☐

Lemma 4.3. *Let Ω be convex, let $\mathcal{T}_h$ be a quasi-uniform triangulation and let $\tilde{q}_h$ be the enriched q_h. Let $u_{q_h} = S(0, q_h)$, $u_{\tilde{q}_h} = S(0, \tilde{q}_h)$, $u^h_{q_h} = S_h(0, q_h)$ and $u^h_{\tilde{q}_h} = S_h(0, \tilde{q}_h)$. Then*

$$\|u_{q_h} - u_{\tilde{q}_h}\| + h\|u_{q_h} - u_{\tilde{q}_h}\|_1 \le Ch\left(\|q - q_h\|_1 + h\|q\|_2\right), \qquad (4.5)$$

$$\|u_{\tilde{q}_h} - u^h_{\tilde{q}_h}\| + h\|u_{\tilde{q}_h} - u^h_{\tilde{q}_h}\|_1 \le Ch\left(\|q - q_h\|_1 + h\|q\|_2\right), \qquad (4.6)$$

$$\|u^h_{q_h} - u^h_{\tilde{q}_h}\| + h\|u^h_{q_h} - u^h_{\tilde{q}_h}\|_1 \le Ch\left(\|q - q_h\|_1 + h\|q\|_2\right). \qquad (4.7)$$

Proof. We can write $u_{q_h} = S(0, q_h) = w_0 + q_h$ and $u_{\tilde{q}_h} = S(0, \tilde{q}_h) = \tilde{w}_0 + \tilde{q}_h$ with w_0, $\tilde{w}_0 \in V$ satisfying

$$
\begin{aligned}
a(w_0, v) &= -a(q_h, v) \quad \forall v \in V,\\
a(\tilde{w}_0, v) &= -a(\tilde{q}_h, v) \quad \forall v \in V,
\end{aligned}
$$

respectively. Now it is immediate that $\|w_0 - \tilde{w}_0\|_1 \leq \|q_h - \tilde{q}_h\|_1$. This together with the result in Lemma 4.2 proves the estimate of $\|u_{q_h} - u_{\tilde{q}_h}\|_1$. In order to derive the L_2-norm estimate, we apply the Aubin-Nitsche duality argument. Let $\psi \in V \cap H^2(\Omega)$ be the solution of

$$\begin{aligned} -\Delta\psi &= w_0 - \tilde{w}_0 \quad \text{in} \quad \Omega, \\ \psi &= 0 \quad \text{on} \quad \Gamma. \end{aligned}$$

Then since $w_0 - \tilde{w}_0 \in V$, we find

$$\begin{aligned} \|w_0 - \tilde{w}_0\|^2 &= a(w_0 - \tilde{w}_0, \psi) \\ &= a(w_0 - \tilde{w}_0, \psi - \psi_h) - a(q_h - \tilde{q}_h, \psi_h - \psi) - a(q_h - \tilde{q}_h, \psi), \end{aligned}$$

where $\psi_h \in V_h$ is the Lagrange interpolation of ψ. Now integration by parts implies

$$a(q_h - \tilde{q}_h, \psi) = -\int_\Omega (q_h - \tilde{q}_h)\, \Delta\psi \, dx + \int_\Gamma \frac{\partial \psi}{\partial n}(q_h - \tilde{q}_h)\, ds.$$

By the construction of $\tilde{q}_h$, we have from (4.4) that $\int_e (q_h - \tilde{q}_h)\, ds = 0$ for all $e \in \mathcal{E}_h^b$. Hence,

$$a(q_h - \tilde{q}_h, \psi) = -\int_\Omega (q_h - \tilde{q}_h)\, \Delta\psi \, dx + \int_\Gamma \frac{\partial(\psi - \psi_h)}{\partial n}(q_h - \tilde{q}_h)\, ds$$

and

$$|a(q_h - \tilde{q}_h, \psi)| \leq \|q_h - \tilde{q}_h\| \|\psi\|_2 + Ch^{1/2}\|q_h - \tilde{q}_h\|_{L_2(\Gamma)} \|\psi\|_2 \leq C\|q_h - \tilde{q}_h\| \|\psi\|_2,$$

where we have used the approximation property of ψ_h and a trace inequality on Q_h. Further using the approximation property of ψ_h, we find that

$$|a(w_0 - \tilde{w}_0, \psi - \psi_h) - a(q_h - \tilde{q}_h, \psi_h - \psi)| \leq Ch \left(\|w_0 - \tilde{w}_0\|_1 + \|q_h - \tilde{q}_h\|_1\right) \|\psi\|_2.$$

Using the elliptic regularity of ψ, we arrive at

$$\|w_0 - \tilde{w}_0\| \leq C \left(h\|q_h - \tilde{q}_h\|_1 + \|q_h - \tilde{q}_h\|\right).$$

Now an appeal to Lemma 4.2 completes the estimate for $\|u_{q_h} - u_{\tilde{q}_h}\|$. This proves the estimates in (4.5). The estimate in (4.7) follows by the same arguments. It remains to derive the estimate in (4.6). Since $\tilde{q}_h$ is Γ_i-wise C^1 and it is continuous on $\bar{\Omega}$, we note that $u_{\tilde{q}_h} \in H^2(\Omega)$ as Ω is assumed to be convex [17, Theorem 1.8] and

$$\|u_{\tilde{q}_h}\|_2 \leq C \sum_{i=1}^m \|\tilde{q}_h\|_{H^{3/2}(\Gamma_i)}.$$

The standard arguments using Cea's lemma and the Aubin-Nitsche duality technique imply that

$$\|u_{\tilde{q}_h} - u_{\tilde{q}_h}^h\| + h\|u_{\tilde{q}_h} - u_{\tilde{q}_h}^h\|_1 \leq Ch^2 \|u_{\tilde{q}_h}\|_2 \leq Ch^2 \sum_{i=1}^m \|\tilde{q}_h\|_{H^{3/2}(\Gamma_i)}.$$

Using an inverse estimate [3, Proposition 3.2] and a suitable approximation $I_h q$ (a C^1 interpolation of Clément type) of q, we find that

$$\begin{aligned} h^2 \sum_{i=1}^m \|\tilde{q}_h\|_{H^{3/2}(\Gamma_i)} &\leq Ch^2 \sum_{i=1}^m \left(\|\tilde{q}_h - I_h q\|_{H^{3/2}(\Gamma_i)} + \|I_h q\|_{H^{3/2}(\Gamma_i)}\right) \\ &\leq Ch^{3/2} \sum_{i=1}^m \|\tilde{q}_h - I_h q\|_{H^1(\Gamma_i)} + Ch^2 \|q\|_2 \\ &\leq Ch \left(\|\tilde{q}_h - I_h q\|_1 + h\|q\|_2\right) \\ &\leq Ch \left(\|\tilde{q}_h - q\|_1 + h\|q\|_2\right). \end{aligned}$$

Now using the triangle inequality and Lemma 4.2, we complete the proof. $\square$

In the following, we rewrite the discrete problem to derive a consistency equation for the error analysis. For any $p_h \in Q_h$, we note from (3.7) that

$$\begin{aligned} a(p_h, \phi_h) - (u_h - u_d, p_h) &= a(p_h - S_h(0, p_h), \phi_h) - (u_h - u_d, p_h) \\ &= -(u_h - u_d, S_h(0, p_h)) = -(u^h_{q_h} + u^h_f - u_d, u^h_{p_h}). \end{aligned}$$

Therefore (3.7) can be written as

$$\alpha a(q_h, p_h) + (u^h_{q_h}, u^h_{p_h}) = (u_d - u^h_f, u^h_{p_h}) \quad \forall p_h \in Q_h. \tag{4.8}$$

The following result on the consistency is a key result in obtaining error estimates in the energy and the L_2-norms.

Lemma 4.4. *The equality*

$$\alpha a(q, p_h) + (u_q, u^h_{p_h}) = (u_f - u_d, u^h_{p_h}) + a(p_h, \phi - P_h\phi) \quad \forall p_h \in Q_h$$

holds, where $P_h : V \to V_h$ is the elliptic projection, i.e., $a(w_h, v - P_h v) = 0$ for all $w_h \in V_h$ and $v \in V$.

Proof. For any $p_h \in Q_h$, we find

$$\begin{aligned} (u_q, u^h_{p_h}) &= (u_q, u^h_{p_h} - p_h) + (u_q, p_h) = a(u^h_{p_h} - p_h, \phi_q) + (u_q, p_h) \\ &= a(u^h_{p_h} - p_h, P_h\phi_q) + (u_q, p_h) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} (u_f - u_d, u^h_{p_h}) &= (u_f - u_d, u^h_{p_h} - p_h) + (u_f - u_d, p_h) \\ &= a(u^h_{p_h} - p_h, \phi_f) + (u_f - u_d, p_h) \\ &= a(u^h_{p_h} - p_h, P_h\phi_f) + (u_f - u_d, p_h). \end{aligned} \tag{4.10}$$

Therefore adding (4.9) and (4.10) and using the fact that $u = u_q + u_f$, $\phi = \phi_q + \phi_f$ and $P_h\phi = P_h\phi_q + P_h\phi_f$, we find

$$\begin{aligned} (u - u_d, u^h_{p_h}) &= a(u^h_{p_h} - p_h, P_h\phi) + (u - u_d, p_h) \\ &= (u - u_d, p_h) - a(p_h, P_h\phi), \end{aligned}$$

where it has been used that $a(P_h\phi, u^h_{p_h}) = 0$. Then

$$(u - u_d, u^h_{p_h}) = (u - u_d, p_h) - a(p_h, \phi) + a(p_h, \phi - P_h\phi).$$

Using (2.6), we find $(u - u_d, p_h) - a(p_h, \phi) = -\alpha a(q, p_h)$. This completes the proof. $\square$

Using Lemma 4.4 and (2.13), we note that the following Galerkin orthogonality (perturbed) holds:

(4.11)
$$(u_q - u^h_{q_h}, u^h_{p_h}) + \alpha a(q - q_h, p_h) = (u^h_f - u_f, u^h_{p_h}) - a(p_h, \phi - P_h\phi) \quad \forall p_h \in Q_h.$$

We now turn to derive *a priori* error estimates. Firstly, we derive the energy norm error estimates in the following lemma which indeed proves the best approximation result.

Theorem 4.5. *The following holds:*

$$\|q - q_h\|_1 + \|u - u_h\|_1 + \|\phi - \phi_h\|_1 \le C \left(\inf_{p_h \in Q_h} \|q - p_h\|_1 + \inf_{v_h \in V_h} \|\phi - v_h\|_1 \right) + C \left(\inf_{v_h \in V_h} \|u_f - v_h\|_1 \right).$$

If the domain Ω is convex, then we have

$$\|q - q_h\|_1 + \|u - u_h\|_1 + \|\phi - \phi_h\|_1 \le Ch \left(\|f\| + \|u_d\| \right).$$

Proof. Let $r_h \in Q_h$, $p_h = r_h - q_h$ and $u^h_{r_h} = S_h(0, r_h)$. By the linearity, we note that $u^h_{p_h} = u^h_{r_h} - u^h_{q_h}$. From (4.11), we find

$$\begin{aligned} \|u^h_{p_h}\|^2 + \alpha \|\nabla p_h\|^2 = {} & (u^h_q - u_q, u^h_{p_h}) - (u^h_q - u^h_{\theta_h}, u^h_{p_h}) + (u^h_f - u_f, u^h_{p_h}) \\ & + \alpha a(r_h - q, p_h) - a(p_h, \phi - P_h\phi), \end{aligned}$$

where $u^h_q = S_h(0, q)$ and $u^h_f = S_h(f, 0)$. Then using (4.1) and (4.3), we find

$$\begin{aligned} |(u^h_q - u_q, u^h_{p_h})| &\le C \inf_{v_h \in V_h} \|(u_q - q) - v_h\|_1 \, \|u^h_{p_h}\| = 0, \\ |(u^h_q - u^h_{r_h}, u^h_{p_h})| &\le \|q - r_h\|_1 \, \|u^h_{p_h}\|, \end{aligned}$$

respectively, where we have used that $u_q = q$. By Cea's lemma [7],

$$|(u^h_f - u_f, u^h_{p_h})| \le C \inf_{v_h \in V_h} \|u_f - v_h\|_1 \, \|u^h_{p_h}\|.$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} |a(q - r_h, p_h)| &\le \|q - r_h\|_1 \, \|\nabla p_h\|, \\ |a(p_h, \phi - P_h\phi)| &\le C \inf_{v_h \in V_h} \|\phi - v_h\|_1 \, \|\nabla p_h\|. \end{aligned}$$

Since

$$q - q_h = u_q - u^h_{q_h} = (u_q - u^h_q) + (u^h_q - u^h_{r_h}) + u^h_{p_h}$$

and $r_h \in Q_h$ is arbitrary, the proof of the best approximation result follows from Lemma 4.1. If the domain Ω is convex, then the error estimate follows from the finite element interpolation theory ([7, Theorem 4.4.20], [12, Theorem 3.1.6]) and the elliptic regularity in Lemma 2.5. ☐

In the next theorem, we derive L_2-norm error estimates by employing the Aubin-Nitsche duality argument using an auxiliary optimal control problem.

Theorem 4.6. *Let Ω be a convex polygonal domain and assume that the mesh $\mathcal{T}_h$ is quasi-uniform. Then*

$$\|q - q_h\| \le Ch^2 \left(\|f\| + \|u_d\| \right).$$

Proof. Consider the following auxiliary problem of finding $r \in Q$ such that

$$j_a(r) = \min_{p \in Q} j_a(p),$$

where

$$j_a(p) = \frac{1}{2}\|S(0,p) - (q - q_h)\|^2 + \frac{\alpha}{2}\|\nabla p\|^2.$$

By the theory of optimal control problems [33, Theorem 2.14], the minimization problem (4.12) admits a unique solution $r \in Q$ and the solution satisfies the optimality condition

$$\alpha a(p,r) + (u_r, u_p) = (q - q_h, u_p) \quad \forall p \in Q, \tag{4.12}$$

where $u_r = S(0,r)$ and $u_p = S(0,p)$. Note that (4.12) can also be written as

$$\alpha a(p,r) + (u_r, p) - a(\zeta, p) = (q - q_h, p) \quad \forall p \in Q, \tag{4.13}$$

where $\zeta = S(u_r - (q - q_h), 0)$. Further note by taking $p = 1$ in (4.13) that

$$(u_r - (q - q_h), 1) = 0. \tag{4.14}$$

Since $\zeta = S(u_r - (q - q_h), 0)$, find using (4.14) that

$$\int_\Gamma \frac{\partial \zeta}{\partial n}\, ds = 0,$$

which is the compatibility condition for the following Neumann problem solved by ψ:

$$\begin{aligned} -\Delta r &= 0 \quad \text{in} \quad \Omega, \\ \alpha \frac{\partial r}{\partial n} &= \frac{\partial \zeta}{\partial n} \quad \text{on} \quad \Gamma. \end{aligned} \tag{4.15}$$

By taking $p = v \in V$ in (4.13) and using (4.15), we note that $u_r = S(0,r) = r$. The following stability estimate is now immediate by taking $p = r$ in (4.12):

$$\|u_r\|_1 = \|r\|_1 \leq C\|q - q_h\|. \tag{4.16}$$

Since the domain Ω is assumed to be convex, $\zeta = S(u_r - (q - q_h), 0) \in H^2(\Omega)$ and by (4.16), it follows that $\|\zeta\|_2 \leq C\|u_r - (q - q_h)\| \leq C\|q - q_h\|$. Moreover by the trace theorem [18], we have $\partial\zeta/\partial n|_{\Gamma_i} \in H^{1/2}(\Gamma_i)$ for $1 \leq i \leq m$. This implies $r \in H^2(\Omega)$ (see [17, Theorem 1.10], [18]) and $\|r\|_2 \leq C\|q - q_h\|$. Let $r_h \in Q_h$ be some approximation of r such that $\|r_h\|_1 \leq C\|r\|_2$, for example r_h can be the Lagrange interpolation of r; see [7, Lemma 4.4.1]. Denote $u_r^h = S_h(0,r)$, $u_{r_h}^h = S_h(0, r_h)$ and $u_{q_h} = S(0, q_h)$. Then by taking $p = q - q_h$ in (4.12), using

$q = u_q$, $q_h = u^h_{q_h}$ and Lemma 4.4, we find

$$
\begin{aligned}
\|q-q_h\|^2 &= (q-q_h, u_q - u^h_{q_h}) = (q-q_h, u_q - u_{q_h} + u_{q_h} - u^h_{q_h}) \\
&= (q-q_h, u_{q-q_h}) + (q-q_h, u_{q_h} - u^h_{q_h}) \\
&= (u_q - u_{q_h}, u_r) + \alpha a(q-q_h, r) + (q-q_h, u_{q_h} - u^h_{q_h}) \\
&= (u_q - u_{q_h}, u^h_{r_h}) + \alpha a(q-q_h, r_h) + (u_q - u_{q_h}, u_r - u^h_{r_h}) \\
&\quad + \alpha a(q-q_h, r-r_h) + (q-q_h, u_{q_h} - u^h_{q_h}) \\
&= (u_q - u^h_{q_h}, u^h_{r_h}) + \alpha a(q-q_h, r_h) + (u_q - u_{q_h}, u_r - u^h_{r_h}) \\
&\quad + \alpha a(q-q_h, r-r_h) + (u^h_{q_h} - u_{q_h}, u^h_{r_h} + q_h - q) \\
&= (u_f - u^h_f, u^h_{r_h}) - a(r_h, \phi - P_h\phi) + (u_q - u_{q_h}, u_r - u^h_r) \\
&\quad + (u_q - u_{q_h}, u^h_r - u^h_{r_h}) + (u^h_{q_h} - u_{q_h}, u^h_{r_h} + q_h - q) + \alpha a(q-q_h, r-r_h). \qquad (4.17)
\end{aligned}
$$

The arguments in Lemma 4.1 imply that

$$\|u^h_{r_h}\|_1 \le C\|r\|_1 \le C\|q-q_h\|.$$

In the following, we estimate each term on the right-hand side of (4.17): The Aubin-Nitsche duality argument implies that

$$|(u_f - u^h_f, u^h_{r_h})| \le Ch^2 \|u_f\|_2 \, \|r\|_1.$$

Since $\phi - P_h\phi \in V$, we find by integration by parts that

$$
\begin{aligned}
a(r_h, \phi - P_h\phi) &= a(r, \phi - P_h\phi) + a(r_h - r, \phi - P_h\phi) \\
&= -(\phi - P_h\phi, \Delta r) + a(r_h - r, \phi - P_h\phi) = a(r_h - r, \phi - P_h\phi),
\end{aligned}
$$

and hence by the Cauchy-Schwarz inequality

$$|a(r_h, \phi - P_h\phi)| \le Ch^2 \|\phi\|_2 \, \|r\|_2.$$

Using (4.1)–(4.3), we find

$$
\begin{aligned}
|(u_q - u_{q_h}, u_r - u^h_r)| &\le Ch^2 \left(\|f\| + \|u_d\|\right) \|r\|_2, \\
|(u_q - u_{q_h}, u^h_r - u^h_{r_h})| &\le Ch^2 \left(\|f\| + \|u_d\|\right) \|r\|_2.
\end{aligned}
$$

It is obvious that

$$|a(q-q_h, r-r_h)| \le Ch^2 \left(\|f\| + \|u_d\|\right) \|r\|_2.$$

Using (4.5)–(4.7) and the energy norm error estimate in Theorem 4.5, we find that

$$
\begin{aligned}
|(u^h_{q_h} - u_{q_h}, u^h_{r_h} + q_h - q)| &= |(u^h_{q_h} - u^h_{\tilde q_h} + u^h_{\tilde q_h} - u_{\tilde q_h} + u_{\tilde q_h} - u_{q_h}, u^h_{r_h} + q_h - q)| \\
&\le Ch \left(\|q-q_h\|_1 + h\|q\|_2\right) \left(\|r\|_1 + \|q-q_h\|\right) \\
&\le Ch^2 \left(\|f\| + \|u_d\|\right) \left(\|r\|_1 + \|q-q_h\|\right).
\end{aligned}
$$

Finally, the elliptic regularity of r completes the proof. □

Remark 4.7. Since $q_h|_{\Gamma_i} \in C^0(\Gamma_i)$ and it is piece-wise linear, we have $q_h \in H^{3/2-\epsilon}(\Gamma_i)$ for any $\epsilon > 0$; see [4]. Therefore u_{q_h} cannot be an $H^2(\Omega)$ function in general. This is the reason to introduce the enriched $\tilde q_h$ of q_h.

Next, we derive optimal order L_2-norm error estimates for the state and adjoint state in the following corollary.

Corollary 4.8. *Let Ω be a convex polygon and assume that the mesh $\mathcal{T}_h$ is quasi-uniform. Then*

$$\begin{aligned}\|u-u_h\| &\le Ch^2(\|f\|+\|u_d\|),\\ \|\phi-\phi_h\| &\le Ch^2(\|f\|+\|u_d\|).\end{aligned}$$

Proof. Since $u-u_h = q-q_h+u_f-u_f^h$, we have from Theorem 4.6 and the triangle inequality that

$$\|u-u_h\| \le C\|q-q_h\| + \|u_f-u_f^h\| \le Ch^2\left(\|f\|+\|u_d\|\right).$$

The estimate for $\phi-\phi_h$ follows by the Aubin-Nitsche duality and the estimate for $u-u_h$. □

5. A posteriori error analysis

In this section, we derive a reliable and efficient residual based *a posteriori* error estimator for the development of an adaptive algorithm. The a posteriori error analysis of optimal control problems can be effectively derived by using some auxiliary problems; see [11, 25]. For this purpose, we introduce the following system of auxiliary problems: Find $(\tilde{u},\tilde{\phi},\tilde{q}) \in Q\times V\times Q$ such that

$$\tilde{u} = \tilde{u_f} + q_h,$$

$$a(\tilde{u_f}, v) = (f, v) - a(q_h, v) \quad \forall v \in V, \tag{5.1}$$

$$a(v, \tilde{\phi}) = (u_h - u_d, v) \quad \forall v \in V, \tag{5.2}$$

$$\alpha a(\tilde{q}, p) = a(p, \phi_h) - (u_h - u_d, p) \quad \forall p \in Q. \tag{5.3}$$

The following lemma is the key result in establishing *a posteriori* error estimates.

Lemma 5.1. *The following holds:*

$$\begin{aligned}&\|\nabla(q-\tilde{q})\| + \|u-u_h\|_1 + \|\phi-\phi_h\|_1\\ &\quad\le C\left(\|\nabla(q_h-\tilde{q})\| + \|u_h-\tilde{u}\|_1 + \|\phi_h-\tilde{\phi}\|_1\right).\end{aligned}$$

Proof. From (2.4)–(2.6) and (5.1)–(5.3), we find the following error equations:

$$u-\tilde{u} = u_f - \tilde{u_f} + q - q_h,$$

$$a(u_f - \tilde{u_f}, v) = -a(q-q_h, v) \quad \forall v \in V, \tag{5.4}$$

$$a(v, \phi-\tilde{\phi}) = (u-u_h, v) \quad \forall v \in V, \tag{5.5}$$

$$\alpha a(q-\tilde{q}, p) = a(p, \phi-\phi_h) - (u-u_h, p) \quad \forall p \in Q. \tag{5.6}$$

Take $v = u_f - \tilde{u_f}$ in (5.5), $v = \phi - \tilde{\phi}$ in (5.4) and then subtract the resulting equations to obtain

$$-a(q-q_h, \phi-\tilde{\phi}) - (u-u_h, u_f-\tilde{u_f}) = 0. \tag{5.7}$$

Take $p = q-\tilde{q}$ in (5.6) and find

$$\begin{aligned}\alpha a(q-\tilde{q}, q-\tilde{q}) &= a(q-\tilde{q}, \phi-\phi_h) - (u-u_h, q-\tilde{q})\\ &= a(q-q_h, \phi-\tilde{\phi}) + a(q_h-\tilde{q}, \phi-\tilde{\phi})\\ &\quad + a(q-\tilde{q}, \tilde{\phi}-\phi_h) - (u-u_h, q-\tilde{q}).\end{aligned} \tag{5.8}$$

Using (5.7) in (5.8), we arrive at

$$\begin{aligned}\alpha a(q-\tilde{q},q-\tilde{q}) &= a(q_h-\tilde{q},\phi-\tilde{\phi})+a(q-\tilde{q},\tilde{\phi}-\phi_h)-(u-u_h,q-\tilde{q}+u_f-\tilde{u_f})\\ &= a(q_h-\tilde{q},\phi-\tilde{\phi})+a(q-\tilde{q},\tilde{\phi}-\phi_h)-(u-u_h,u-\tilde{u}-(\tilde{q}-q_h)).\end{aligned}$$

Therefore

$$\begin{aligned}\alpha\|\nabla(q-\tilde{q})\|^2+\|u-\tilde{u}\|^2 &= a(q_h-\tilde{q},\phi-\tilde{\phi})+a(q-\tilde{q},\tilde{\phi}-\phi_h)\\ &\quad -(\tilde{u}-u_h,u-\tilde{u})-(u-u_h,\tilde{q}-q_h).\end{aligned}$$

From (5.5), we note that $\|\phi-\tilde{\phi}\|_1\le C\|u-u_h\|\le C\left(\|u-\tilde{u}\|+\|\tilde{u}-u_h\|\right)$. Now the Cauchy-Schwarz inequality and Young's inequality imply

$$\|\nabla(q-\tilde{q})\|+\|u-\tilde{u}\|_1+\|\phi-\tilde{\phi}\|_1\le C\left(\|\nabla(q_h-\tilde{q})\|+\|u_h-\tilde{u}\|_1+\|\phi_h-\tilde{\phi}\|_1\right).$$

The rest of the proof is completed by the triangle inequality. □

Define the volume residuals by

$$\eta_{f,T}=h_T\|f\|_{L_2(T)},\quad \eta_f=\left(\sum_{T\in\mathcal{T}_h}\eta_{f,T}^2\right)^{1/2},$$

$$\eta_{u,T}=h_T\|u_h-u_d\|_{L_2(T)},\quad \eta_{u,1}=\left(\sum_{T\in\mathcal{T}_h}\eta_{u,T}^2\right)^{1/2},$$

the jump residuals by

$$\eta_{\phi,e}=h_e^{1/2}\|[\![\nabla\phi_h]\!]\|_{L_2(e)},\quad \eta_\phi=\left(\sum_{e\in\mathcal{E}_h^i}\eta_{\phi,e}^2\right)^{1/2},$$

$$\eta_{u,e}=h_e^{1/2}\|[\![\nabla u_h]\!]\|_{L_2(e)},\quad \eta_{u,2}=\left(\sum_{e\in\mathcal{E}_h^i}\eta_{u,e}^2\right)^{1/2},$$

$$\eta_{q,e}=h_e^{1/2}\|[\![\nabla q_h]\!]\|_{L_2(e)},\quad \eta_{q,1}=\left(\sum_{e\in\mathcal{E}_h^i}\eta_{q,e}^2\right)^{1/2},$$

and the boundary residual by

$$\eta_{q,e,b}=h_e^{1/2}\|\alpha\partial q_h/\partial n-\partial\phi_h/\partial n\|_{L_2(e)},\quad \eta_{q,2}=\left(\sum_{e\in\mathcal{E}_h^b}\eta_{q,e,b}^2\right)^{1/2}.$$

Define the total estimator by

$$\eta_h=\left(\eta_f^2+\eta_{u,1}^2+\eta_{u,2}^2+\eta_\phi^2+\eta_{q,1}^2+\eta_{q,2}^2\right)^{1/2}.$$

The result on the residual based error estimator is proved in the following theorem.

Theorem 5.2. *The following holds:*

$$\|\nabla(q-q_h)\|+\|u-u_h\|_1+\|\phi-\phi_h\|_1\le C\eta_h.$$

Proof. Note from Lemma 5.1 that it is enough to estimate $\|\nabla(\tilde{q}-q_h)\|+\|\tilde{u}-u_h\|_1+\|\tilde{\phi}-\phi_h\|_1$. To this end, we find using (5.1)–(5.3) and (3.5)–(3.7) that

$$\tilde{u}-u_h = \tilde{u_f}-u_f^h,$$

$$a(\tilde{u}-u_h, v_h) = a(\tilde{u_f}-u_f^h, v_h) = 0 \quad \forall v_h \in V_h, \tag{5.9}$$

$$a(v_h, \tilde{\phi}-\phi_h) = 0 \quad \forall v_h \in V_h, \tag{5.10}$$

$$\alpha a(\tilde{q}-q_h, p_h) = 0 \quad \forall p_h \in Q_h, \tag{5.11}$$

and further find that

$$\begin{aligned} a(\tilde{u}-u_h, v) &= (f,v) - a(u_h, v) \quad \forall v \in V,\\ a(v, \tilde{\phi}-\phi_h) &= (u_h-u_d, v) - a(v,\phi_h) \quad \forall v \in V,\\ \alpha a(\tilde{q}-q_h, p) &= a(p,\phi_h) - (u_h-u_d,p) - \alpha a(q_h,p) \quad \forall p \in Q. \end{aligned}$$

Using the orthogonality (5.9)–(5.11), standard arguments with quasi-interpolation and integration by parts, we complete the proof. $\square$

Finally, we sketch the proof of the local efficiency estimates in the next theorem. For this, define the data oscillations of any $g \in L_2(D)$, where $D \subset \Omega$ is a subdomain, by

$$\mathrm{Osc}(g,D) = \left(\sum_{T\subset \bar{D}} h_T^2 \inf_{g_h\in P_0(T)} \|g-g_h\|^2_{L_2(T)}\right)^{1/2}.$$

The proof of local efficiency estimates is sketched in the following theorem.

Theorem 5.3. *The following hold:*

$$\begin{aligned}
\eta_{f,T} &\le C\left(\|\nabla(u-u_h)\|_{L_2(T)} + \mathrm{Osc}(f,T)\right),\\
\eta_{u,e} &\le C\left(\|\nabla(u-u_h)\|_{L_2(T_e)} + \mathrm{Osc}(f,T_e)\right),\\
\eta_{u,T} &\le C\left(\|\nabla(\phi-\phi_h)\|_{L_2(T)} + \|u-u_h\|_{L_2(T)} + \mathrm{Osc}(u_d,T)\right),\\
\eta_{\phi,e} &\le C\left(\|\nabla(\phi-\phi_h)\|_{L_2(T_e)} + \|u-u_h\|_{L_2(T_e)} + \mathrm{Osc}(u_d,T_e)\right),\\
\eta_{q,e} &\le C\left(\|u-u_h\|_{L_2(T)} + \|\nabla(\phi-\phi_h)\|_{L_2(T_e)} + \|\nabla(q-q_h)\|_{L_2(T_e)} + \mathrm{Osc}(u_d,T_e)\right),\\
\eta_{q,e,b} &\le C\left(\|u-u_h\|_{L_2(T)} + \|\nabla(\phi-\phi_h)\|_{L_2(T_b)} + \|\nabla(q-q_h)\|_{L_2(T_b)} + \mathrm{Osc}(u_d,T_b)\right),
\end{aligned}$$

where T_e is the union of triangles sharing the edge e and T_b is the triangle having the edge $e \subset \partial T_b \cap \Gamma$, ∂T_b is the boundary of T_b.

Proof. Let $(\cdot,\cdot)_T$ denote the $L_2(T)$ inner-product. Let $b_T \in P_3(T)\cap H_0^1(T)$ be the bubble function with $\|b_T\|_{L_\infty(T)} = 1$ and let $\theta = b_T f_h$ for $f_h \in P_0(T)$. Then using norm equivalence on finite dimensional space and using the fact that $(\nabla u_h, \nabla\theta)_T = 0$, we find

$$\begin{aligned} C\|f_h\|^2_{L_2(T)} &\le (f,\theta)_T + (f_h - f,\theta)_T = (\nabla(u-u_h),\nabla\theta)_T + (f_h-f,\theta)_T\\ &\le \|\nabla(u-u_h)\|_{L_2(T)}\|\nabla\theta\|_{L_2(T)} + \|f-f_h\|_{L_2(T)}\|\theta\|_{L_2(T)}. \end{aligned}$$

Note we have used from (2.4) that $(\nabla u, \nabla\theta)_T = a(u,\tilde{\theta}) = (f,\tilde{\theta}) = (f,\theta)_T$, where $\tilde{\theta} \in H_0^1(\Omega)$ is the extension of θ by zero outside of T. Now using an inverse inequality [7, Lemma 4.5.3], we find

$$h_T\|f_h\|_{L_2(T)} \le C\|\nabla(u-u_h)\|_{L_2(T)} + Ch_T\|f-f_h\|_{L_2(T)}.$$

This estimate and the triangle inequality complete the proof of the first inequality. Similarly using the edge bubble function techniques [34], we can prove the second inequality of the theorem. We will sketch the proof of the third inequality. Let $f_h = (u_h - g_h)$ and $\theta = b_T f_h$ for $g_h \in P_0(T)$. Then since $(\nabla\phi_h, \nabla\theta)_T = 0$ and $(\nabla\tilde{\phi}, \nabla\theta)_T = (u_h - u_d, \theta)_T$ from (5.2),

$$\begin{aligned}
C\|f_h\|^2_{L_2(T)} &\le (u_h - g_h, \theta)_T = (u_h - u_d, \theta)_T + (u_d - g_h, \theta)_T \\
&= (\nabla(\tilde{\phi} - \phi_h), \nabla\theta)_T + (u_d - g_h, \theta)_T \\
&= (\nabla(\tilde{\phi} - \phi), \nabla\theta)_T + (\nabla(\phi - \phi_h), \nabla\theta)_T + (u_d - g_h, \theta)_T \\
&= (u - u_h, \theta)_T + (\nabla(\phi - \phi_h), \nabla\theta)_T + (u_d - g_h, \theta)_T,
\end{aligned}$$

where we have used for $\theta \in H^1_0(T)$ that

$$(\nabla(\tilde{\phi} - \phi), \nabla\theta)_T = a(\tilde{\theta}, \tilde{\phi} - \phi) = (u_h - u, \tilde{\theta}) = (u_h - u, \theta)_T,$$

where $\tilde{\theta} \in H^1_0(\Omega)$ is the extension of θ by zero outside of T. The rest of the proof follows by the standard arguments. The proofs of the remaining inequalities follow by similar bubble function techniques. □

6. Numerical experiments

In this section, we illustrate the theoretical results by performing some numerical experiments. We conduct two experiments with two model problems. In the first experiment, we test the validity of the *a priori* error estimates derived in Theorem 4.5, Theorem 4.6 and Corollary 4.8. In the second experiment, we test the performance of the *a posteriori* error estimator derived in Theorem 5.2 and its efficiency in Theorem 5.3. To this end, we construct the model problems with known solutions. For this we modify the model problem as follows: Modify the cost functional J, denoted by $\tilde{J}$, by

$$\tilde{J}(w, p) = \frac{1}{2}\|w - u_d\|^2 + \frac{\alpha}{2}\|p - q_d\|^2_1, \quad w \in Q,\ p \in Q,$$

where q_d is a given function. Then the minimization problem reads: Find $(u, q) \in Q \times Q$ such that

$$\tilde{J}(u, q) = \min_{(w,p)\in Q\times Q} \tilde{J}(w, p),$$

subject to the condition that $(w, p) \in Q \times Q$ satisfies $w = S(f, p)$. Then it is easy to check that the optimality conditions take the form

$$\begin{aligned}
u &= u_0 + q, \\
a(u_0, v) &= (f, v) - a(q, v) \quad \forall v \in V, \\
a(v, \phi) &= (u - u_d, v) \quad \forall v \in V, \\
\alpha a(q, p) &= a(p, \phi) - (u - u_d, p) + \alpha a(q_d, p) \quad \forall p \in Q.
\end{aligned}$$

Accordingly, we modify the discrete problem. A posteriori error estimators $\eta_{q,T}$ and $\eta_{q,e,b}$ have to be modified as follows:

$$\begin{aligned}
\eta_{q,T} &= h_T\|u_h - u_d + \alpha\Delta q_d\|_{L_2(T)}, \\
\eta_{q,e,b} &= h_e^{1/2}\|\alpha\partial(q_h - q_d)/\partial n - \partial\phi_h/\partial n\|_{L_2(e)}.
\end{aligned}$$

Example 1. For this example, the domain Ω is taken to be the unit square $(0,1)\times(0,1)$ and set $\alpha = 1$. We choose the data of the problem as follows. Choose the state to be $u(x,y) = e^{(x+y)}$, the adjoint state to be $\phi(x,y) = x^2(1-x^2)^2y^2(1-y^2)^2$ and the control to be $q(x,y) = e^{(x+y)}$. Then compute $f = -\Delta u$ and $u_d = u + \Delta\phi$. The choice of ϕ leads to $q_d = q$. We generate a sequence of meshes with mesh size h as shown in Table 6.1 by uniformly refining successively each triangulation. We have used our in-house MATLAB code for the computations. The computed errors and orders of convergence in H^1- and L_2-norms are shown in Table 6.1 and Table 6.2, respectively. The experiment clearly illustrates the expected rates of convergence.

TABLE 6.1. H^1-norm errors and orders of convergence for Example 1

h	$\|u-u_h\|_1$	order	$\|q-q_h\|_1$	order	$\|\phi-\phi_h\|_1$	order
1/4	6.2317e-001	–	6.3984e-001	—	2.8268e-002	–
1/8	3.2177e-001	0.9536	3.2435e-001	0.9802	1.5167e-002	0.8982
1/16	1.6242e-001	0.9863	1.6280e-001	0.9944	7.7765e-003	0.9637
1/32	8.1426e-002	0.9961	8.1483e-002	0.9986	3.9214e-003	0.9877
1/64	4.0743e-002	0.9989	4.0752e-002	0.9996	1.9659e-003	0.9962
1/128	2.0376e-002	0.9997	2.0377e-002	0.9999	9.8368e-004	0.9989
1/256	1.0189e-002	0.9999	1.0189e-002	1.0000	4.9195e-004	0.9997

TABLE 6.2. L_2-norm errors and orders of convergence for Example 1

h	$\|u-u_h\|$	order	$\|q-q_h\|$	order	$\|\phi-\phi_h\|$	order
1/4	5.4578e-002	–	4.9697e-002	–	2.3716e-003	–
1/8	1.3749e-002	1.9890	1.2240e-002	2.0215	7.8326e-004	1.5983
1/16	3.4659e-003	1.9880	3.0522e-003	2.0037	2.1374e-004	1.8736
1/32	8.6984e-004	1.9944	7.6263e-004	2.0008	5.5087e-005	1.9561
1/64	2.1779e-004	1.9978	1.9063e-004	2.0002	1.3903e-005	1.9863
1/128	5.4477e-005	1.9992	4.7657e-005	2.0000	3.4858e-006	1.9959
1/256	1.3622e-005	1.9997	1.1914e-005	2.0000	8.7218e-007	1.9988

Experiment 2. In this experiment, we consider the domain Ω to be L-shaped as shown in Figure 6.3 and we let $\alpha = 1$. We take the state variable to be $u = r^{2/3}\sin(2\theta/3)$, the adjoint state to be $\phi(x,y) = x^2(1-x^2)^2y^2(1-y^2)^2$ and the control to be $q = r^{2/3}\sin(2\theta/3)$. As in Example 1, we have computed $f = -\Delta u$, $u_d = u + \Delta\phi$ and $q_d = q$. The following successive iteration of the mesh refinement algorithm has been used in the experiment:

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.$$

In the step MARK, we have used the Dörfler bulk marking [15] with parameter 0.4. The marked elements are refined by using the newest vertex bisection algorithm. Figure 6.1 shows the behavior of the error estimator and the total error $\|u-u_h\|_1 + \|\phi-\phi_h\|_1 + \|q-q_h\|_1$ with the increasing number of degrees of freedom N (the total number of unknowns for state, adjoint state and control variable). The figure shows that the estimator is reliable and the total error converges at the optimal rate of $N^{-1/2}$. The efficiency of the error estimator is depicted through the efficiency indices (estimator/$(\|u-u_h\|_1 + \|\phi-\phi_h\|_1 + \|q-q_h\|_1)$) in Figure 6.2. Finally, Figure 6.3 shows the adaptive mesh refinement near the reentrant corner as expected.

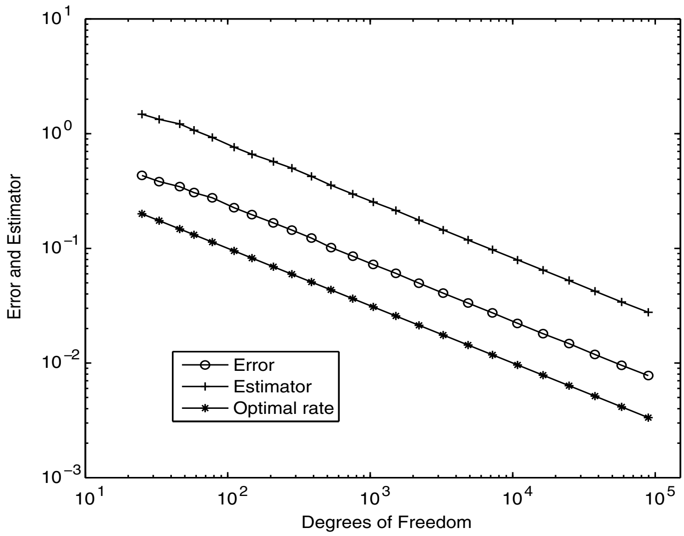

FIGURE 6.1. Error and estimator for Example 2

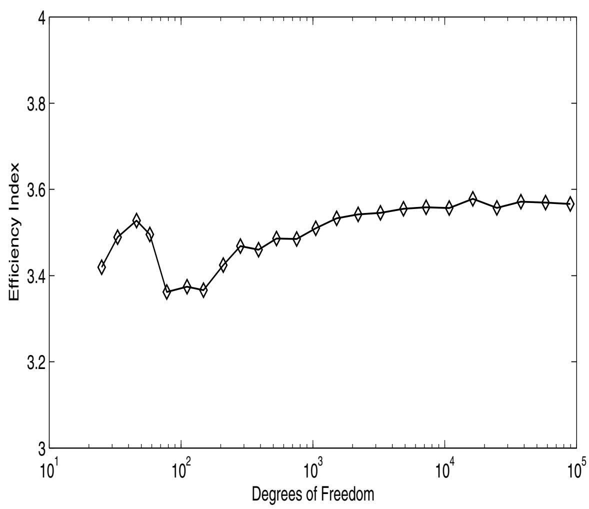

FIGURE 6.2. Efficiency index for Example 2

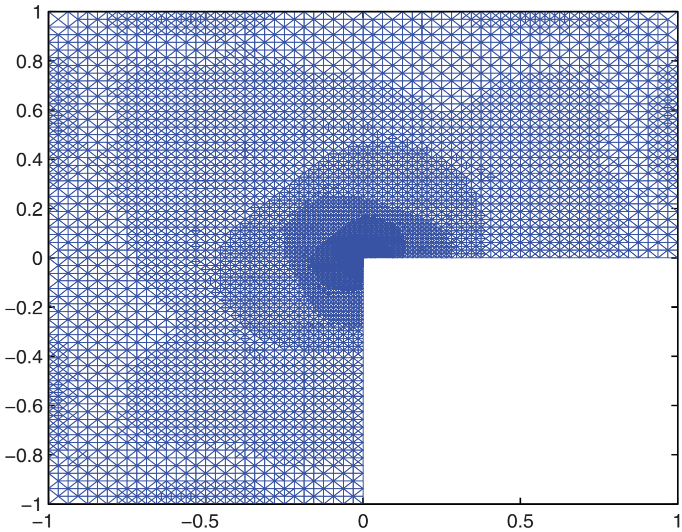

Figure 6.3. Adaptive mesh refinement for Example 2

7. Conclusions

In this article, we have proposed an energy-space based approach for defining the Dirichlet control problem. The model problem is shown to be well-posed and the regularity of the solution is deduced on convex polygonal domains. A finite-element based numerical method is proposed and its corresponding error estimates are derived. *A priori* error estimates are optimal in the H^1- and the L_2-norms. An efficient and reliable *a posteriori* error estimator is derived for the development of the adaptive mesh refinement algorithm. Numerical experiments have been performed to illustrate the theoretical results. The results in this article are derived for a two-dimensional domain. This assumption is made to make use of the elliptic regularity results for Dirichlet and Neumann problems. The results can be extended to three-dimensional domains with the help of elliptic regularity theory. However the best approximation result in the energy norm holds on any polygonal (or polyhedral) domain without being convex. The convexity of the domain is assumed in the L_2-norm error estimates for the sake of simplicity. Suboptimal rates of convergence may be worked out without assuming that the domain is convex.

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