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THE EULER BINARY PARTITION FUNCTION AND SUBDIVISION SCHEMES

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Abstract. For an arbitrary set D of nonnegative integers, we consider the Euler binary partition function $b(k)$ which equals the total number of binary expansions of an integer k with "digits" from D. By applying the theory of subdivision schemes and refinement equations, the asymptotic behaviour of $b(k)$ as $k \to \infty$ is characterized. For all finite D, we compute the lower and upper exponents of growth of $b(k)$, find when they coincide, and present a sharp asymptotic formula for $b(k)$ in that case, which is done in terms of the corresponding refinable function. It is shown that $b(k)$ always has a constant exponent of growth on a set of integers of density one. The sets D for which $b(k)$ has a regular power growth are classified in terms of cyclotomic polynomials.

1. Introduction

We consider the generalized Euler binary partition function and solve several problems on its asymptotic behaviour by applying the theory of refinement equations and subdivisions. Let $\mathbb{Z}_+$ be the set of nonnegative integers and let $D \subset \mathbb{Z}_+$ arbitrarily be its subset containing zero and call it *dictionary*. The *Euler binary partition function* $b_D : \mathbb{Z}_+ \to \mathbb{Z}_+$ is defined for each k as the total number of binary expansions of k with "digits" from the set D. Thus,

(1)
$$b_D(k) \quad = \quad \#\Bigl\{ (a_0, a_1, a_2, \ldots) \quad \Bigm| \quad k \ = \ \sum_{m=0}^{\infty} a_m 2^{\,m}\,, \quad a_m \in D\,,\ m \in \mathbb{Z}_+ \Bigr\}.$$

We often assume that the dictionary is fixed and use the simplified notation $b(k) = b_D(k)$. Clearly, for $D = \{0, 1\}$ we have $b(k) \equiv 1$. For other dictionaries, expansion (1) may not exist at all for some k or may not be unique. A natural question arises to characterize the asymptotic behavior of the function $b(k)$ as $k \to \infty$. For two special cases, $D = \mathbb{Z}_+$ and $D = \{0, \ldots, n\}$, this problem is well known and has a rich history.

The case $D = \mathbb{Z}_+$ was studied by L. Euler [18] in connection with the analytic function $F(z) \ = \ \prod_{j=0}^{\infty} \bigl(1 - x^{2^j}\bigr)^{-1} \ = \ \sum_{n=0}^{\infty} b(k)\, x^{\,k}$, where $b = b_{\mathbb{Z}_+}$. In 1918 Tanturri [53] introduced a recurrent formula for $b(k)$; in 1940 Mahler [33] found its asymptotic behaviour proving that $\log_2 b(k) \ \sim \ \frac{1}{2}\,(\log_2 k)^2$ as $k \to \infty$. Then de Bruijn [5], Pennington [40], Knuth [29], and Fröberg [21] improved Mahler's

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result. Lotary [32] applied discrete dynamics to this problem. Pfaltz [41] analyzed the special case of partitions of powers of two.

In this paper we consider finite dictionaries D with natural assumptions $0 \in D$ and $\gcd(D) = 1$. For the function $b = b_D$, we define its lower and upper exponents of growth:

$$(2) \qquad p_1 \;=\; \liminf_{k\to\infty} \frac{\log\, b(k)}{\log\, k}\;; \qquad p_2 \;=\; \limsup_{k\to\infty} \frac{\log\, b(k)}{\log\, k}\;.$$

In contrast to the case $D = \mathbb{Z}_+$, when, according to Mahler's result, $p_1 = p_2 = \infty$, for finite dictionaries, both those exponents are finite. Our main problem is to find or to estimate them, and also to characterize the dictionaries for which $p_1 = p_2$. The latter case, when $b(k)$ has a constant exponent of growth p, will be referred to as the *power growth*. In this case $p = \log_2(\#(D)) - 1$, and the asymptotics of $b(k)$ is characterized by the numbers

$$(3) \qquad \nu_1 \;=\; \liminf_{k\to\infty} k^{-p}\, b(k); \qquad \nu_2 \;=\; \limsup_{k\to\infty} k^{-p} b(k).$$

The next step is to find dictionaries D with $\nu_1 = \nu_2$ when $b(k)$ has the so-called *regular power growth*, i.e., $b(k) \sim c\, k^p$ as $k \to \infty$.

Only one class of finite dictionaries is well studied in the literature: the *full dictionaries*, when $D = \{0, \ldots, n\}$. They were first considered by Tanturri in 1918 (see [53] and two references therein). There are closed formulas for some small n. If $n = 1$, then, of course, $b(k) \equiv 1$. If $n = 2$, then $b(k) = s(k+1)$ [50], where $s(n+1)$ is the Stern sequence [52]. If $n = 3$, then $b(k) = \left\lfloor \frac{k}{2} \right\rfloor + 1$ [30]. Most likely, these are all "good" cases, and for other n, one should analyze the asymptotics of $b(k)$ as $k \to \infty$ rather than closed formulas. Reznick [50] proved that for every odd n, we have $p_1 = p_2 = \log_2 \frac{n+1}{2}$ and left an open problem to find ν_1 and ν_2 in this case. If $n = 2^{\,r} - 1$, where $r \in \mathbb{N}$, then $\nu_1 = \nu_1 = \frac{1}{(r-1)!}$. This was established in [50], with a remark that it can also be derived from the results of Tanturri [53]. On the other hand, Carlitz [6] showed that for $n = 5$, we have $\nu_1 \neq \nu_2$. The second open problem formulated in [50] is to classify full dictionaries with $\nu_1 = \nu_2$, i.e., with the regular power growth of $b(k)$. In case of even n, for $n = 2$, Reznick found both p_1 and p_2, and they turned out to be distinct. He made a conjecture that $p_1 < p_2$ for all even n, which was proved in 2004 [45]. The matrix approach originated in [15], and [43] made it possible to evaluate p_1 and p_2, at least, for some small n, by means of spectral characteristics of special Boolean $n \times n$ matrices. For $n = 4$, both p_1 and p_2 were computed in [15]; for $n = 6, \ldots, 12$, they were done in [43]. The technique from the latter work was developed further by Hare [22], who made computation till $n = 54$ and left a conjecture on the explicit form of p_1, p_2 (see Remark 2 in Section 7). For full dictionaries with odd n, both problems of Reznick were solved in [45]. In particular, it was shown that the power growth of $b(k)$ takes place only if $n = 2^{\,r} - 1$.

The case of arbitrary dictionary is much more complicated and richer in content. For instance, if $1 \notin D$, then it is not obvious that each big number k has at least one expansion (1). In fact, there are dictionaries for which $b(k) = 0$ for arbitrarily large k. Moreover, if D has only one even or one odd element, then $p_1 = 0$ (Proposition 1), so $b(k)$ does not grow at all. Nevertheless, apart from that special case, $b(k)$ always grows at least linearly in k (Corollary 1) and both p_1 and p_2 can be found by means of the joint spectral characteristics of certain Boolean matrices A_0, A_1 defined in Section 3 (Theorem 2). For reasonably small D, these exponents of growth are computed in Tables 1 and 2 in the Appendix.

In Section 8 we establish the existence of a subset $\mathcal{M} \subset \mathbb{N}$ of density one on which $b(k)$ has a constant exponent of growth, which is equal to the Lyapunov exponent of the matrices A_0, A_1. As for the entire $\mathbb{N}$, the problem of constant exponent of growth is addressed in Section 9. We show that $p_1 = p_2$ if and only if the characteristic polynomial $\boldsymbol{d}(z) = \sum_{m \in D} z^m$ is divisible by $1 + z^q$ for some integer q (Theorem 4), in which case $p_1 = p_2 = p = \log_2 \frac{1}{2} \boldsymbol{d}(1)$. Moreover, we derive a sharp asymptotic formula for $b(k)$ and evaluate both ν_1 and ν_2 (Theorems 6–8). Finally, we characterize the case of regular polynomial growth when $\nu_1 = \nu_2$. Theorem 11 gives the general form of dictionaries with this property. For example, if $\boldsymbol{d}(z) = \prod_{i=0}^{p} (1 + z^{n_i})$ for some natural $n_0 < n_1 < \cdots < n_p$, then $b(k) = \frac{1}{p!} k^p + o(k^p)$ as $k \to \infty$ (Theorem 10). The question whether this is the only case of regular polynomial growth is left to Problem 1.

Our approach is based on applying the theory of subdivisions and refinement equations. This idea appears to be fruitful, at least in analyzing the asymptotic behaviour of $b(k)$.

Subdivision schemes are iterative algorithms of linear approximation of functions from their values on a mesh. They originated in the late 1980s with Dyn, Levin, Gregory, De Boor, Dubuc, Deslariers, Micchelli, Dahmen, Cavaretta, etc., and are widely used now for interpolation and approximation of smooth functions and in modelling of curves and surfaces; see [7, 10, 16, 59] and references therein. Some special cases of subdivisions appeared earlier in works of De Rham (cutting angle scheme) [11, 12], Chaikin [8], etc. We consider only stationary univariate schemes defined by a sequence of real numbers $\{c_i\}_{i=0}^{n}$, $n \geq 2$. The *subdivision operator* $S : l_\infty \to l_\infty$ acts on the set of bounded sequences l_∞ as follows:

$$(4) \qquad (Sg)_k = \sum_i c_{k-2i}\, g_i$$

where $g = (g_i)_{i \in \mathbb{Z}} \in l_\infty$. We say that a *subdivision scheme converges* if for each $g \in l_\infty$, there is a function $\Phi \in C(\mathbb{R})$ such that

$$(5) \qquad \|S^j g - \Phi(2^{-j} \cdot)\|_\infty \to 0\,, \qquad j \to \infty\,,$$

where $\| \cdot \|$ is the supremum norm in l_∞. For the convergence, it suffices to check this assertion only for the sequence δ defined as $\delta_0 = 1, \delta_i = 0$, $i \in \mathbb{Z} \setminus \{0\}$. The limit function $\varphi \in C(\mathbb{R})$ is called a *refinable function.* Thus,

$$(6) \qquad \|S^j \delta - \varphi(2^{-j} \cdot)\|_\infty \to 0\,, \qquad j \to \infty\,.$$

Then for any $g \in l_\infty$, the subdivision scheme converges to the function $\Phi(t) = \sum_k g_k \varphi(t-k)$. The refinable function φ is a solution of the *refinement equation*:

$$(7) \qquad \varphi(t) = \sum_{k=0}^{n} c_k\, \varphi(2t-k)\,,$$

which is the difference functional equation with a double contraction of the argument. To every refinement equation we associate its *symbol*, the polynomial $\boldsymbol{m}(z) = \frac{1}{2} \sum_{k=0}^{n} c_k z^k$. The refinement equation can be written in a short form $T\varphi = \varphi$, where $[Tf](t) = \sum_{k=0}^{n} c_k f(2t-k)$ is the *transition operator*.

If the subdivision scheme converges, then $\varphi \in C(\mathbb{R})$ and the coefficients satisfy the following conditions ("sum rules"):

$$\sum_j c_{2j} \ = \ \sum_j c_{2j+1} \ = 1 \qquad \Leftrightarrow \qquad \boldsymbol{m}(1) \ = \ 1 \ , \ \boldsymbol{m}(-1) \ = \ 0 \, . \tag{8}$$

These conditions are necessary but, in general, not sufficient for convergence of a subdivision scheme.

The relation between binary partition functions and refinement equations is revealed in Sections 7–11. In particular, we establish the following:

– Upper and lower bounds for $b(k)$ are derived from the convergence of a special nonnegative subdivision scheme (Theorem 2 in Section 7).

– The case of power growth ($p_1 = p_2$) corresponds to refinement equations (7) with continuous solutions φ (Theorem 4 in Section 9).

– The sharp asymptotic formula for $b(k)$ as $k \to \infty$ and the bounds ν_1 and ν_2 can be found in terms of the refinable function φ (Theorems 5 – 8).

– The case of regular power growth ($\nu_1 = \nu_2$) corresponds to the spline refinable function φ (Section 11).

We begin with necessary notation and definitions in Section 2; then in Section 3 we describe the main ideas and the facts behind our analysis. In Section 4 we make the main assumptions and formulate the problems. Then in Section 5 we recall some necessary facts from the theory of subdivisions and refinement equations and prove one new result on convergence of subdivisions (Section 6). In Sections 7–11 we formulate and prove the main results on the asymptotics of the binary partition function.

Let us finally mention some related works on the binary partition functions and generalizations. The congruences of $b(k)$ modulo two was the subject of [1, 9]. The case of full dictionary $D = \{0, \ldots, n\}$ with the partition by a general sequence (the partition by the sequence $\{2^j\}_{j \in \mathbb{Z}_+}$ is replaced by an arbitrary increasing sequence $\{g_j\}_{j \in \mathbb{Z}_+}$) was studied in [19] and [15] (the case of a recurrent sequence). The papers [20, 54] analyze the relation of the binary partition functions to the Bernuolli convolutions and metric properties of the continued fractions. The matrix approach for counting of partitions of 2^k to powers of two was developed in [37].

2. Notation and definitions

We denote by $\gcd(K)$ the greatest common divisor of a set of integers K. The cardinality is denoted either by $\#(K)$ or by $|K|$. By the *support* of a sequence $\{c_k\}_{k=1}^n$ we mean the set of indices for which $c_k \neq 0$. For an $n \times n$ matrix A, we denote by A^* its adjoint matrix and by $\rho(A)$ its spectral radius, i.e., the largest modulus of eigenvalue. For two functions $f(k)$ and $g(k)$, we use the standard notation $f = O(g)$, $f(k) = o(g)$, $f \asymp g$, and $f \sim g$, meaning that $f(k)/g(k)$ is bounded, tends to zero, is bounded and away from zero, and tends to one respectively as $k \to \infty$.

In what follows, we deal with a finite dictionary $D \subset \mathbb{Z}_+$. The largest element of D will be denoted by n. For $k \in \mathbb{Z}$, we set $d_k = 1$ if $k \in D$ and $d_k = 0$ otherwise; $\boldsymbol{d}(z) = \sum_{k \in \mathbb{Z}} d_k z^k$ is the characteristic polynomial of D, $\deg \boldsymbol{d} = n$. We write $D_{\mathrm{even}} = D \cap (2\mathbb{Z})$ and $D_{\mathrm{odd}} = D \cap (2\mathbb{Z} + 1)$ for the even and odd part of D respectively. We use the notation $p = \log_2 \frac{|D|}{2}$.

For a given natural number k, we denote $j(k) = \lceil \log_2 k \rceil - 1$. If j is not specified we always assume $j = j(k)$. Thus, $2^{j-1} \le k < 2^j$.

By $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{R}^n$. $\boldsymbol{e} = (1, \dots, 1) \in \mathbb{R}^n$ is a vector of ones. We use the Schwartz space $\mathcal{S}$ of rapidly decreasing smooth functions on $\mathbb{R}$ and the corresponding space $\mathcal{S}'$ of tempered distributions; $\mathcal{S}_0$ and $\mathcal{S}_0'$ are subspaces of compactly supported functions and distributions respectively. Every nonnegative distribution is a Borel measure, which by the Lebesgue theorem can be decomposed as a sum of absolutely continuous, purely singular, and atomic parts. We use the spaces L_p of functions integrable with the pth power, $p \in [1, +\infty]$, $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i\, x\xi} dt$ is the Fourier transform. $\boldsymbol{P}\{A\}$ is a probability of an event A; $\boldsymbol{E}$ is the mathematical expectation.

A number of the form $e^{-\frac{2\pi i k}{m}}$, where $k, m \in \mathbb{N}$ are co-prime, is referred to as a primitive root of unity of degree m. The mth *cyclotomic polynomial* $\boldsymbol{C}_m(z)$ is a unique irreducible integer polynomial which is a divisor of $x^m - 1$ and is not a divisor of $x^k - 1$ for any $k < m$. Thus, $\boldsymbol{C}_m(z) = \prod_{k<m,\, \gcd(k,m)=1} \bigl(z - e^{-\frac{2\pi i k}{m}}\bigr)$. If m is a prime, then $\boldsymbol{C}_m(z) = \sum_{k=1}^{m-1} z^k = \frac{z^m - 1}{z-1}$. In case $m = 2^q$, we have

$$\boldsymbol{C}_{2^q}(z) \quad = \quad \prod_{s=1}^{2^{q-1}} \Bigl(z \;-\; e^{-\frac{2\pi i (2s-1)}{2^q}} \Bigr) \quad = \quad z^{2^{q-1}} + 1. \tag{9}$$

3. Binary partition functions versus subdivisions

We begin with observing the main facts behind the relation between binary partition functions and subdivisions. A short overview of the theory of subdivision schemes and refinable functions needed for us will be given in Section 5. Then in Section 6 we derive one new result on subdivisions, after which we prove our main theorems on the binary partition functions. We start with the following simple observation.

Lemma 1. *For every dictionary, the corresponding binary partition function $b(k)$ satisfies the following recurrent relations:*

$$b(2q) \;=\; \sum_s d_{2s}\, b(q-s); \qquad b(2q+1) \;=\; \sum_s d_{2s+1}\, b(q-s). \tag{10}$$

Proof. We establish the first equality; the second one is proved in the same way. In expansion (1) written for the number $k = 2q$, the digit $a_0 \in D$ must be even; hence $a_0 = 2s$ and $q - s = \sum_{m \in \mathbb{N}} a_m\, 2^{m-1}$, provided $2s \in D$, i.e., $d_{2s} = 1$. Thus, the total number of expansions of $2q$ is equal to the sum of numbers of expansions of $q - s$ over all s such that $d_{2s} = 1$. This completes the proof. $\Box$

Equations (10) can be written in a compact form using the following operator Γ on l_∞:

$$(\Gamma g)_k \quad = \quad \sum_{i \in \mathbb{Z}} d_{k-2i}\, g_i\,. \tag{11}$$

Applying induction we conclude from formulas (10):

$$b(k) \;=\; \bigl(\Gamma^j\, \delta\bigr)_k \qquad \text{for all} \quad k \le 2^j - 1. \tag{12}$$

Comparing formulas (4) and (11) we see that Γ is nothing else but the subdivision operator corresponding to the sequence $\{d_i\}_{i=0}^n$. However, this operator produces a nonconverging scheme, because Γ does not satisfy conditions (8). Indeed, $\sum_i d_i = |D| \neq 2$. To apply the theory of subdivisions, we use the normalized sequence $c_i = \frac{2}{|D|} d_i$, $i = 0, \ldots, n$, the corresponding symbol $\boldsymbol{m} = \boldsymbol{m}_D = \frac{1}{|D|}\boldsymbol{d}$ and the subdivision operator $S = S_D = \frac{2}{|D|}\Gamma$. Thus,

$$(13) \qquad \big(S_D\, g\big)_k \quad = \quad \frac{2}{|D|} \sum_{i \in \mathbb{Z}} d_{k-2i}\, g_i\,.$$

Among all results on subdivisions, the following three will be crucial for our analysis:

– *Under some mild assumptions, a subdivision scheme with nonnegative coefficients satisfying* (8) *converges to a positive function* φ. This was a long-standing conjecture formulated by Melkman in 1997 [34]. Partial results were obtained in [27, 35, 36]. The paper [56] of Wang was the first breakthrough, after which Zhou got the final solutions: in [58] (convergence) and in [59] (positivity of the refinable function). We shall use those results to prove sharp upper and lower bounds (27) for $b(k)$ in Theorem 2.

– *A criterion of absolute continuity of a solution of refinement equation with nonnegative coefficients.* Partial results in this direction were obtained in [13, 55]. The final statement originated in [44] in 2000. We use it to classify all dictionaries for which $b(k)$ has a power growth as $k \to \infty$, i.e., $b(k) \asymp k^p$, and to find a sharp asymptotic formula for $b(k)$ in this case (Sections 9 and 10).

– *Classification of piecewise-smooth refinable functions.* Cavaretta, Dahmen and Micchelli in [7] characterized spline solutions of refinement equations; then Lawton, Lee and Shen [31] derived the final version of that result. Later some partial results were presented independently by Berg and Plonka [2] and Hirn [23]. In 2005 we showed that every piecewise-smooth solution of a refinement equation is a spline [46]. This classifies all piecewise-smooth refinable functions. We apply that result in Section 11 to characterize dictionaries for which $b(k)$ has a regular power growth, i.e., $b(k) \sim c\,k^p$.

All details and rigorous statements on subdivisions and refinement equations are provided in Section 5. Another tool to study the binary partition function relates to the joint spectral characteristics of matrices. The main idea originated in [15, 43] is to consider the vector-function

$$(14) \qquad u(k) \;=\; \Big(b(k)\,,\;\ldots\;,\,b(k-n+1)\Big) \in \mathbb{R}^n\,.$$

Here we set $b(k) = 0$ for negative k. Thus, $u(0) = (1, 0, \ldots, 0) \in \mathbb{R}^n$. In the vector form, formulas (10) read

$$(15) \qquad u(2q+s) \quad = \quad A_s\, u(q)\,, \quad s \;=\; 0, 1\,,$$

where A_0, A_1 are two Boolean $n \times n$-matrices defined by coefficients as follows:

$$(16) \qquad (A_s)_{ij} \quad = \quad \begin{cases} d_{2j-i-1+s}\,, & \text{if} \quad 1-s \;\le\; 2j-i \;\le\; n+1-s\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$$

This leads to the explicit formula for the binary partition function:

$$(17) \qquad u(k) \quad = \quad A_{k_0} \cdots A_{k_{j-1}} u(0)\,,$$

where $k_{j-1}, \ldots, k_0$ are digits in the (standard) binary expansion of the number k:

$$k \;=\; \sum_{s=0}^{j-1} k_m 2^{\,m}, \quad k_m \in \{0, 1\}.$$

As we shall see in Section 7 (Theorem 2), the upper and lower exponents of growth of $b(k)$ are expressed by means of the following matrix characteristics:

$$\check{\rho}\,(A_0, A_1) \;=\; \lim_{j\to\infty}\ \min_{\Pi_j} \|\Pi_j\|^{1/j}\ ; \qquad \hat{\rho}\,(A_0, A_1) \;=\; \lim_{j\to\infty}\ \max_{\Pi_j} \|\Pi_j\|^{1/j}\ , \tag{18}$$

where the maxima and minima are computed over all 2^j products Π_j of length j of matrices A_0, A_1. These numbers $\check{\rho}$ and $\hat{\rho}$ are called the *lower spectral radius* and the *joint spectral radius* respectively. We often omit the matrices A_0, A_1 from that notation when it is clear what pair of matrices is considered. These joint spectral characteristics have a rich history and countless applications; see [25] for an overview. There is an algorithm [28] for the computation of the joint and lower spectral radii, which for a vast majority of matrix families of dimension ≤ 20 (for Boolean matrices the computation can be done with dimensions of several hundreds) provides their exact values.

Note that the matrix approach is also closely related to subdivisions. Indeed, the *transition matrices* T_0, T_1 of a refinement equation defined in (23) are nothing else but the normalized transpose matrices to A_0, A_1. More precisely $T_s = \frac{2}{|D|}\, A_s^*$, $s = 0, 1$.

4. Statement of the problem

Before presenting the main results we need to make several assumptions on the dictionary D. Everywhere below we assume that $0 \in D$ and $\gcd\,(D) = 1$. This is natural. The next assumption is less obvious and needs to be motivated. Observe first that both parts D_{even} and D_{odd} of the dictionary D are nonempty. Indeed, $|D_{\text{even}}| \ge 1$ because D contains zero, $|D_{\text{odd}}| \ge 1$ because $\gcd\,(D) = 1$. It turns out that the cases $|D_{\text{even}}| = 1$ and $|D_{\text{odd}}| = 1$, although possible, are exceptional and can be excluded.

Proposition 1. *Suppose either* $|D_{\text{even}}| = 1$ *or* $|D_{\text{odd}}| = 1$*; then, for each* $j \in \mathbb{N}$, *we have*

$$\min_{2^{j-1} \le k < 2^j} b(k) \quad = \quad \begin{cases} 1\,, & \text{if } 1 \in D\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Proof. If D contains both 0 and 1, then, of course, $b(k) \ge 1$ for all $k \ge 0$. On the other hand, in expansion (1) for the number $k = 2^{j-1}$, the digit a_0 must be even, and hence $a_0 = 0$ whenever $|D_{\text{even}}| = 1$. Dividing by two, we obtain $b(2^{j-1}) = b(2^{j-2})$ and, by induction, $b(2^{j-1}) = b(2^0) = 1$. Similarly we conclude that if $|D_{\text{odd}}| = 1$, then $b(2^j - 1) = 1$ for all j. Assume now $1 \notin D$. Then, in case $|D_{\text{even}}| = 1$, we have $b(2^{j-1}) = 0$ for all j. Indeed, if we take the smallest s for which $a_s \ne 0$ in expansion (1), then a_s is odd, and hence $s = j-1$ and $a_s = 1$, which contradicts our assumption. Suppose now that $|D_{\text{odd}}| = 1$, and so D contains a unique odd number $s \ge 3$. In this case, $b\big(2^m(s+1)-s\big) = 0$. For $m = 0$ this is true; for arbitrary $m \ge 1$, we have $a_0 = s$, and hence $b\big(2^m(s+1)-s\big) = b\big(\frac{2^m(s+1)-s-s}{2}\big) = b\big(2^{m-1}(s+1)-s\big)$, and the assertion follows by induction in m. □

Thus, in cases $|D_{\text{even}}| = 1$ and $|D_{\text{odd}}| = 1$, the lower exponent of growth p_1 is always zero. Hence, the case of power growth $(p_1 = p_2)$ is impossible. This allows us to make the following assumption:

Condition 1. The dictionary D has at least two even and at least two odd elements.

If the converse is not stated, we always assume this condition. Note that our main results on the upper exponent of growth (Theorem 2) remain valid in the general case, without Condition 1. One can easily check that in the proof of equality $p_2 = \log_2 \hat{\rho}$ and of the upper bound $b(k) \le c_2 \, k^{p_2}$ we do not use that condition. In contrast, the lower bound for $b(k)$, as well as all results of Sections 9–11, holds only under Condition 1; see Remark 1. As we saw, without that condition the lower exponent of growth p_1 for $b(k)$ is zero, and hence all problems addressed in Sections 9–11 do not make sense.

5. Refinement equations and subdivisions: Basic facts

We recall several known facts from the theory of subdivisions and refinement equations. They will be collected in Theorems A – D and sorted by the subject. Then, in Section 6, we establish one new result on subdivisions, which is, probably, of some independent interest. All those results will be used in Sections 7–11 to study the binary partition function.

We consider refinement equation (7). It will always be assumed that $c_0 c_n \neq 0$ and $\sum_k c_k = 2$. The whole theory can actually be reduced to this case [38]. Thus, $\boldsymbol{m}(1) = 1$. Applying the Fourier transform, we obtain the form of the transition operator in the frequency domain: $\widehat{Tf}(\xi) = \boldsymbol{m}(e^{-\pi i \xi}) \widehat{f}\big(\frac{\xi}{2}\big)$. The refinement equation becomes $\widehat{\varphi}(\xi) = \boldsymbol{m}(e^{-\pi i \xi}) \widehat{\varphi}\big(\frac{\xi}{2}\big)$.

Theorem A. *A refinement equation with $\sum_k c_k = 2$ always possesses a unique, up to multiplication by a constant, nontrivial solution $\varphi \in \mathcal{S}_0'$. This solution satisfies $c = \int_{\mathbb{R}} \varphi \, dt \neq 0$ (we assume $c = 1$), supported on $[0, n]$, and for every $f \in \mathcal{S}_0'$ such that $\int_{\mathbb{R}} f \, dt = 1$, we have*

$$T^j f \quad \xrightarrow{\mathcal{S}'} \quad \varphi \, . \tag{19}$$

If $\varphi \in L_1(\mathbb{R})$, then φ possesses the partition of unity property:

$$\sum_{k \in \mathbb{Z}} \varphi(t-k) \ \equiv \ 1 \qquad a.e. \tag{20}$$

(as usual, "a.e." means "almost everywhere").

This theorem is generally known; its proof can be found in [38]. In the sequel we denote by φ the unique solution of a refinement equation normalized by the condition $\widehat{\varphi}(0) = \int_{\mathbb{R}} \varphi \, dt = 1$. The power of the transition operator T in (19) can be expressed by means of the corresponding subdivision operator S as follows:

$$[\, T^j f \,](t) \quad = \quad \sum_{k \in \mathbb{Z}} (S^j \delta)_k \, f(2^j x - k) \ , \qquad f \in \mathcal{S}', \, j \in \mathbb{N}. \tag{21}$$

This equality holds even if the subdivision scheme does not converge [7]. In this sense, the transition and subdivision operators are dual to each other. Theorem A guarantees the existence of the solution only in distributions. The existence of a regular or continuous solution φ can be checked by criteria involving joint spectral characteristics of matrices [10, 38]. The Hölder regularity of the solution never exceeds $n-1$; in particular, $\varphi \notin C^{n-1}(\mathbb{R})$ [10]. If the solution is continuous, then it has a varying local regularity and other fractal-like properties, with the only exceptional case when φ is a spline.

Theorem B ([7,31]). *A refinable function φ is a spline of order $p \geq 0$ if and only if the symbol has the form $\boldsymbol{m}(z) = 2^{-p-1}\frac{\boldsymbol{q}(z^2)}{\boldsymbol{q}(z)}$, where $\boldsymbol{q}$ is a polynomial. Moreover,*

a) *The spline φ has integral nodes.*

b) *The polynomial $\boldsymbol{q}$ has zero of order $p+1$ at the point $z=1$. The set of roots of $\boldsymbol{q}$ is closed with respect to taking square. In particular, each root of $\boldsymbol{m}$ is a root of unity.*

c) *For each $n \geq 2$, there are finitely many refinable splines.*

This theorem was proved in [7] (the case of splines with integral nodes) and in [31] (the general case). So, for every n, there are finitely many refinement equations of degree n with spline solutions. Actually, Theorem B classifies not only refinable splines but also all piecewise smooth refinable functions, as the next theorem asserts:

Theorem C ([46]). *If the smoothness of a refinable function φ on some interval $(0, \varepsilon)$ exceeds its smoothness on $\mathbb{R}$, then φ is a spline.*

Thus, a refinable function on any interval $(0, \varepsilon)$ is as bad as on the whole real line, unless it is a spline. In particular, if φ is infinitely smooth on some interval $(0, \varepsilon)$, then it is a spline with integral nodes classified in Theorem B.

Another exceptional case of refinement equations is when all the coefficients $\{c_k\}_{k=0}^n$ are nonnegative. In particular, the existence of summable and of continuous solutions of nonnegative equations can be ensured without computing the joint spectral characteristics. Moreover, under some mild assumptions, the conditions in (8) are sufficient for the convergence of the subdivision scheme.

To formulate the results on nonnegative subdivisions we need one more notation. We consider a binary tree $\mathcal{T}$ that has the number -1 at the root, two of its children $\pm i$ on the first level, etc. Each vertex v of the tree has two children $\pm\sqrt{v}$ on the next level. Thus, the kth level contains 2^k vertices associated to the numbers $e^{-2\pi i(2s+1)\,2^{-k-1}}$, $s = 0, \ldots, 2^k - 1$. They are primitive roots of unity of degree $k+1$. A finite subset of vertices $\mathcal{B}$ is called *blocking* if each path starting at the foot (all paths are without backtracking) has exactly one common element with $\mathcal{B}$. For example, the root $\{-1\}$ is a one-element blocking set; the set $\{i, -i\}$ is also blocking.

Theorem D ([34,39,44,55,58,59]). *If all coefficients of a refinement equation are nonnegative, then the following hold:*

a) *φ is a Borel measure with* $\operatorname{supp}\varphi = [0, n]$.

b) *The measure φ is of pure type, i.e., is either purely singular or belongs to L_1. The latter holds if and only if there is a blocking set $\mathcal{B} \subset \mathcal{T}$ such that $\boldsymbol{m}(\mathcal{B}) = 0$.*

c) *If $\varphi \in L_1(\mathbb{R})$, then $\varphi \in C(\mathbb{R})$, unless all coefficients c_{2k} vanish for $2k \neq 0$ or all coefficients c_{2k+1} vanish for $2k+1 \neq n$.*

d) *A continuous φ does not vanish on the interval $(0, n)$ unless $c_{2k+1} = 0$ for all but one $k \in \mathbb{Z}$.*

e) *If φ is continuous, then the subdivision scheme converges precisely when the conditions in* (8) *are satisfied and* $\gcd\{i \,|\, c_i > 0\} = 1$.

Item a) was established in [55] and [44], item b) was proved in [44], and item e) was done in [39]. Item d) was a long-standing conjecture eventually proved in [59]. The first version of item c) was proved by Micchelli and Prautzsch [36]. They assumed that all coefficients $\{c_k\}_{k=1}^n$ are strictly positive. Then this condition was successively relaxed by Gonsor [27] and Melknan [34]. In 2001 Wang [56] proved an important special case of three positive coefficients, after which Zhou [58] obtained the final solution.

If we consider the vector-function $v : [0,1] \to \mathbb{R}^n$, $v(t) = \big(\varphi(t), \ldots, \varphi(t-n+1)\big)$, the refinement equation can be written in the following matrix form:

$$v(t) \quad = \quad \begin{cases} T_0\, v(2t)\,, & t \in [0\,,\, \frac{1}{2}]\,; \\ T_1\, v(2t-1)\,, & t \in [\frac{1}{2}\,,\, 1]\,, \end{cases} \tag{22}$$

where T_0, T_1 are the $n \times n$ *transition matrices* of the refinement equation defined entrywise as follows:

$$(T_s)_{\,ij} \;=\; c_{\,2i-j+s-1}\,, \qquad s \in \{0,1\}\,,\; i,j \;=\; 1,\ldots,n\,. \tag{23}$$

As usual, we set $c_k = 0$ for $k < 0$ and for $k > n$. We need the following proposition that ensures that a pair of transition matrices with nonnegative coefficients is always *positively irreducible*; i.e., they do not have a common nontrivial invariant coordinate subspace spanned by several (not all!) vectors of the canonical basis of $\mathbb{R}^n$.

Proposition 2. *If all coefficients $c_0, \ldots, c_n$ are nonnegative, then the pair of transition matrices $\{T_0, T_1\}$ is positively irreducible.*

Proof. Assume the contrary: there is a coordinate subspace $L \subset \mathbb{R}^n$ spanned by several (not all) vectors of the canonical basis such that $T_i L \subset L, i = 0, 1$. Consider a space $\mathcal{L} \subset \mathcal{S}'$ of distributions supported on the set $\bigcup_{i\in L}[i-1, i]$. Equation (22) implies that $T\mathcal{L} \subset \mathcal{L}$. Hence, taking arbitrary $f_0 \in \mathcal{L}$ with $(1, f_0) = 0$, we obtain $T^j f_0 \subset \mathcal{L}$, $j \in \mathbb{N}$, which, in view of Theorem A, means that $\varphi \in \mathcal{L}$. Hence, $\operatorname{supp}\varphi \subset \bigcup_{i\in L}[i-1, i]$, and we come to the contradiction with item a) of Theorem D. $\square$

In the next section we establish a new result on subdivisions, which will be applied in Section 10.

6. Convergence of nonconvergent subdivisions

As we know, a subdivision scheme cannot converge unless the conditions in (8) are satisfied. We will relax that condition so that the subdivision scheme still converges, although in some weaker sense (*residual convergence*). For the sake of simplicity, we restrict ourselves to the case of nonnegative coefficients, which is needed in further sections.

Definition 1. Let an integer $\ell \ge 0$ be given. For an algebraic polynomial $\boldsymbol{p}(z) = \sum_{k=0}^{n} p_k z^k$, we denote

$$\boldsymbol{p}_r(z) \;=\; \sum_{k \equiv r \,(\mathrm{mod}\; 2^\ell)} p_k\, z^k\;, \qquad r = 0, \ldots, 2^{\,\ell} - 1\,.$$

Thus, we write $\boldsymbol{p}$ as a sum of 2^ℓ polynomials, each containing powers of residual r modulo 2^ℓ. For example, if $\ell = 0$, then we have only one polynomial $\boldsymbol{p}_0 = \boldsymbol{p}$. If $\ell = 1$, then we have two polynomials $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$. The polynomial $\boldsymbol{p}_0$ is the “even part” of $\boldsymbol{p}$ collecting all its even powers, while $\boldsymbol{p}_1$ is the “odd part”. We have

$$\boldsymbol{p}_0(z) \;=\; \frac{1}{2}\Big(\boldsymbol{p}(z) \,+\, \boldsymbol{p}(-z)\Big)\,; \qquad \boldsymbol{p}_1(z) \;=\; \frac{1}{2}\Big(\boldsymbol{p}(z) \,-\, \boldsymbol{p}(-z)\Big)\,. \tag{24}$$

Generalizations of these formulas for all $\ell \geq 1$ (Lemma 3 in the Appendix) will be needed in Section 10. For a polynomial $\boldsymbol{m}$ and numbers $\ell, r \geq 0$ such that $0 \leq r \leq 2^{\ell-1}$, we denote

$$\mu_r(\boldsymbol{m}, \ell) \;=\; 2^{\ell}\, \boldsymbol{Q}_r(1)\;, \quad \text{where} \quad \boldsymbol{Q}(z) \;=\; \boldsymbol{m}(z)\boldsymbol{m}(z^2)\cdots \boldsymbol{m}(z^{2^{\ell-1}})\,. \tag{25}$$

This is a function of the polynomial $\boldsymbol{m}$, the number ℓ, and the residual r. For instance, $\mu_0(\boldsymbol{m}, 1) = 2\boldsymbol{m}_0(1)$ and $\mu_1(\boldsymbol{m}, 1) = 2\boldsymbol{m}_1(1)$.

Theorem 1. *If a refinement equation with nonnegative coefficients has a continuous solution φ, satisfies $\gcd\{c_k \mid c_k > 0\} = 1$, and its symbol $\boldsymbol{m}$ is a multiple of $z^{2^{\ell}} + 1$, then, for each $r = 0, \ldots, 2^{\ell} - 1$, we have*

$$\sup_{k \equiv r \,(\mathrm{mod}\, 2^{\ell})} \bigl|(S^j\delta)_k \,-\, \mu_r\, \varphi(2^{-j}k)\bigr| \;\to\; 0\;, \qquad j \to \infty\,, \tag{26}$$

where $\mu_r = \mu_r(\boldsymbol{m}, \ell)$.

Thus, if a refinable function is continuous, then the subdivision scheme converges under a weaker assumption than (8): the symbol $\boldsymbol{m}$ is a multiple of $z^{2^{\ell}} + 1$ for some $\ell \geq 0$. If $\ell = 0$, then this is the usual convergence; if $\ell \geq 1$, then this is a *residual convergence* or *ℓ-convergence* defined by (26) as follows: for each residual $r = 0, \ldots 2^{\ell} - 1$, the sampled sequence $(S^j\delta)_k,\ k \equiv r \,(\mathrm{mod}\, 2^{\ell})$, converges to the values of the function $\mu_r\varphi$ at the corresponding points $2^{-j}k$. If (8) is satisfied, then all μ_r are equal, and the scheme converges in the usual sense. Otherwise, those multipliers are different. The proof of Theorem 1 is in the Appendix.

7. Asymptotics of the binary partition function: The upper and lower exponents

In this section we compute the lower and upper exponents of growth p_1, p_2 for each dictionary satisfying Condition 1 and prove a tight asymptotic inequality for $b(k)$. In the next sections we analyze when $p_1 = p_2$ and what can be said more about the growth of the function $b(k)$ in that case. Besides, we shall see that in a subset of $\mathbb{N}$ of density one, $b(k)$ always has a constant exponent of growth equal to the Lyapunov exponents of the matrices A_0, A_1 defined in (16).

In view of formula (17), it is natural that the slowest and fastest growth of the norm of the vector $u(k) = \bigl(b(k), \ldots, b(k-n+1)\bigr) \in \mathbb{R}^n$ is expressed by the values $\check{\rho}$ and $\hat{\rho}$ respectively. The main difficulty is that the first component $b(k)$ of this vector may grow slower than $\|u(k)\|$. Moreover, as we see in Remark 1, this phenomenon really occurs if Condition 1 fails. That is why, to establish the asymptotic behaviour of $b(k)$ under Condition 1, we invoke powerful results from the theory of subdivisions (those listed in Theorem D).

Theorem 2. *For every dictionary D satisfying Condition 1, we have*

$$p_1 \;=\; \log_2 \check{\rho}(A_0, A_1) \quad \text{and} \quad p_2 \;=\; \log_2 \hat{\rho}(A_0, A_1)\,.$$

Moreover, there are constants $c_1, c_2 > 0$ such that

$$c_1 k^{p_1} \;\leq\; b(k) \;\leq\; c_2 k^{p_2}\,, \qquad \text{for all large } k \in \mathbb{N}\,. \tag{27}$$

Remark 1. As was noted in Section 4, all the results concerning the upper exponent of growth, the upper bound in (27) and the equality $p_2 = \log_2 \hat{\rho}$, are true for all finite dictionaries, including those not satisfying Condition 1. Indeed, in the proof below we do not use that condition when establishing the upper bound. As for the lower exponent, neither equality $p_1 = \log_2 \check\rho$ nor the lower bound in (27) holds without Condition 1. Consider, for example, the dictionary $D = \{0, 1, 3\}$. Since $D_{\rm even} = 1$, Proposition 1 yields $\min\limits_{2^{j-1} \le k < 2^j} b(k) = 1$ for all j and hence $p_1 = 0$. On the other hand, for the corresponding matrices

$$A_0 \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\;; \qquad A_0 \;=\; \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\,,$$

one can apply the algorithm from [28] and prove that $\hat\rho = \rho(A_0) = \frac{\sqrt{5}+1}{2} = 1.6180\ldots$ and $\check\rho = \sqrt{\rho(A_0A_1)} = 1.4850\ldots$. Hence $\log_2 \check\rho > 0 = p_1$, which contradicts Theorem 2. The reason is that the lower spectral radius $\check\rho$ measures the growth of the minimal norm $\|u(k)\|$, which, in this case, exceeds the growth of the minimal component $b(k)$.

For the upper bound, Theorem 2 holds: $p_2 = \log_2 \hat\rho = \log_2 \frac{\sqrt{5}+1}{2} = 0.6942\ldots$. Thus, in this case $1 \le b(k) \le c_2\, k^{\,0.6942\ldots}$.

Proof of Theorem 2. The pair of transition matrices $T_i = \frac{2}{|D|}\, A_i^*$, $i = 0, 1$, is positively irreducible by Proposition 2; hence so is the pair $\{A_0, A_1\}$. It is well known that for a positively irreducible pair of matrices A_0, A_1 and for a nonzero vector $v \in \mathbb{R}^n_+$, there is a constant $C_1 > 0$ such that

$$C_2\, \hat\rho^{\,j} \quad \le \quad \max_{\Pi_j} \, \|\Pi_j v\| \quad \le \quad C_1\, \hat\rho^{\,j}\,, \qquad j \,\in\, \mathbb{N},$$

where the maximum is over all products of length j [25]. Note that $2^{\,p_2(j-1)} \le k^{p_2} < 2^{\,p_2 j}$, where $j = j(k)$. This, in view of (17) proves equality $p_2 = \log_2 \hat\rho$ and the upper bound $b(k) \le c_2\, k^{\,p_2}$. Furthermore, (17) yields

$$\min_{2^{j-1} \le k < 2^j} b(k) \;\le\; \min_{2^{j-1} \le k < 2^j} \|u(k)\| \;\le\; \min_{\Pi_j} \|\Pi_j\|\,,$$

where the minimum is computed over all products Π_j of matrices A_0, A_1 of length j. Taking the power $1/j$ and a limit as $j \to \infty$, we obtain $p_1 \le \check\rho$. The inverse inequality will follow from the lower bound in (27), which is the crucial point of the whole proof. To establish that $b(k) \ge c_1 k^{\,p_1}$, we consider an arbitrary subdivision scheme generated by a nonnegative sequence $\{c_i\}_{i=0}^n$ that has support D. Condition 1 implies that one can choose the values of nonzero coefficients c_i so that conditions (8) are fulfilled. Then, by Theorem D, e), the corresponding scheme S converges. Its limit function φ is continuous and is strictly positive on the interval $(0, n)$ (item d) of Theorem D). Hence,

$$h \quad = \quad \inf_{\frac{1}{2} \le t \le 1} \quad \varphi(t) \quad > \quad 0\,.$$

By definition (6) of the convergent scheme, there is l_0 such that for all $l \ge l_0$, we have

$$(S^{\,l}\,\delta)_r \;>\; \frac{h}{2}\,, \qquad \text{whenever} \quad 2^{\,l-1} \;\le r \;<\; 2^{\,l}\,, \;\; \text{i.e.,} \quad \frac{r}{2^{\,l}} \;\in\; \Bigl[\frac{1}{2}\,,\, 1\Bigr)\,.$$

On the other hand, $(S^{\,l}\delta)_r > 0$ yields $(\Gamma^{\,l}\delta)_r > 0$, and hence $b(r) = (\Gamma^{\,l}\delta)_r \ge 1$. Thus, $b(r) \ge 1$ for all $r \in [2^{\,l-1}, 2^{\,l})$ whenever $l \ge l_0$. It can be assumed that $2^{\,l_0 - 1} > n$, and so $u(r) \ge \boldsymbol{e}$ for all $r \in [2^{\,l-1}, 2^{\,l})$ whenever $l \ge l_0 + 1$. We fix some of those l. Applying the shift-invariance of a subdivision operator, we have

$$u(k) \;\ge\; b(k_0)\, \boldsymbol{e} \quad \text{whenever} \quad k - k_0 \in [2^{\,l-1}, 2^{\,l})\,. \tag{28}$$

Furthermore, $\Pi_l\, u(0) \ge \boldsymbol{e}$ for all products Π_l of length l of matrices A_0, A_1. Take arbitrary $k_1 \in [2^{j_1-1}, 2^{j_1})$, $j_1 > l$. We have

$$u(k_1) \;=\; A_{q_0}\cdots A_{q_{j_1-1}}u(0) \;\ge\; A_{q_0}\cdots A_{q_{j_1-1-l}}\boldsymbol{e} \;\ge\; C\,\check{\rho}^{\;j_1-l}$$

(the last inequality follows from [43, Lemma 7]). Now take arbitrary $k > 2^{l+2}$. Let $k_1 = k - 2^{\,l-1}$ and $k_0 \in \{k_1 - n + 1, \dots, k_1\}$ be such that $b(k_0)$ is the largest component of the vector $u(k_1)$. Thus,

$$b(k_0) \quad\ge\quad \frac{1}{\sqrt{n}}\,\|u(k_1)\| \quad\ge\quad \frac{C}{\sqrt{n}}\,\check{\rho}^{\;j_1-l}\,.$$

Applying (28) and taking into account that $j_1 \ge \log_2 k_1 \ge \log_2 k - 1$, we obtain $u(k) \ge C_2\,\check{\rho}^{\log_2 k} = C_2\,k^{p_1}$, which completes the proof. □

Remark 2. Thus, the upper and lower exponents of growth of $b(k)$ are expressed by values $\check{\rho}$ and $\hat{\rho}$ of the Boolean $n \times n$ matrices A_0, A_1 defined by (16). For each dictionary with not very large n, say, $n \le 200$, it is possible to find those values using the method of computation of the joint spectral characteristics from [28]. In Table 1 in the Appendix we list those values for all dictionaries with the largest element ≤ 5; in Table 2 we compute and present them for full dictionaries with $n \le 30$. As we see in Table 2, for full dictionaries, the lower and the joint spectral radii of A_0, A_1 take values $\rho(A_0)$ or $\sqrt{\rho(A_0A_1)}$. This means that the fastest growth of $b(k)$ is attained for one of the sequences $k = 2^j$, $j \in \mathbb{N}$, or $k = \frac{1}{3}\left(2^{\,2j} - 1\right)$, $j \in \mathbb{N}$, and the slowest growth is attained for the other. It is an interesting question whether this is true for all n. As for now, this conjecture formulated in 2000 [43] is open. Hare [22] made an interesting observation that, for full dictionaries, the numbers $\check{\rho}, \hat{\rho}$ are equal to $\rho(A_0)$ and $\sqrt{\rho(A_0A_1)}$ respectively, precisely when the sum of binary digits of the number n is odd; otherwise those values interchange. This was proved numerically for $n \le 54$ in [22].

In Table 1 we see that for general dictionaries, the situation is more complicated. The fastest and slowest growths can appear for longer periods.

The exponents of growth can be estimated as shown in the proposition below.

Proposition 3. *For every dictionary D, we have $\check{\rho} \le \frac{|D|}{2} \le \hat{\rho}$.*

Proof. The sum of elements in each row of the matrix $A_0 + A_1$ is equal to $|D|$; hence, $(A_0 + A_1)\boldsymbol{e} = |D|\,\boldsymbol{e}$, where $\boldsymbol{e} \in \mathbb{R}^n$ is the vector of ones. Consequently, $\sum_{\Pi_j} \Pi_j \boldsymbol{e} = |D|^j \boldsymbol{e}$, where the sum is computed over all 2^j matrix products of length j. Since for a nonnegative matrix, $\|A\| \asymp \|A\boldsymbol{e}\|$, we have $\sum_{\Pi_j} \|\Pi_j\| \asymp |D|^j$, $j \in \mathbb{N}$. Therefore, $\max_{\Pi_j} \|\Pi_j\| \ge C\left(\frac{|D|}{2}\right)^j$, and hence $\hat{\rho} \ge \frac{|D|}{2}$. Similarly, $\check{\rho} \le \frac{|D|}{2}$. □

Let us recall that $p = \log_2 \frac{|D|}{2}$. Proposition 3 yields

$$p_1 \;\le\; p \;\le\; p_2\,. \tag{29}$$

Furthermore, each row of the matrices A_0, A_1 contains either $|D_{\text{even}}|$ or $|D_{\text{odd}}|$ ones. By Condition 1, both those numbers are at least two. Therefore,

$$2 \;\le\; \check{\rho} \;\le\; \hat{\rho} \;\le\; |D| - 2\,.$$

Consequently,

$$1 \;\le\; p_1 \;\le\; p \;\le\; p_2 \;\le\; \log_2(|D| - 2)\,. \tag{30}$$

Combining with Theorem 2, we come to the following conclusion.

Corollary 1. *For an arbitrary dictionary satisfying Condition* 1*, we have*

$$c_1 k \;\le\; b(k) \;\le\; c_2 k^{\log_2(|D|-2)} \qquad \textit{for all large } k\,. \tag{31}$$

Thus, under Condition 1, $b(k)$ grows at least linearly in k as $k \to \infty$.

8. The growth of $b(k)$ on a set of density one

In view of Theorem 2, the binary partition function has lower and upper exponents of growth p_1 and p_2 and they are a priori different. In the next section we will classify dictionaries for which $p_1 = p_2$. On the other hand, on a set of density one, the exponent of growth is always constant. For full dictionaries, this phenomenon was observed in [19, 20, 54] for much more general partitions. We are going to see that for binary partitions, this is true for every dictionary, even not satisfying Condition 1.

A set $\mathcal{M} \subset \mathbb{N}$ is said to be *of density one* if $N^{-1}\,\#\{k \in \mathcal{M} \mid k \le N\} \to 1$ as $N \to \infty$.

Lemma 2. *For every dictionary, we have* $b(k) \ge 1$ *on a set of density one.*

In view of Theorem 2, this lemma obviously holds under Condition 1. Moreover, by Proposition 1 it holds if $1 \in D$. Thus, it remains to prove the lemma for the case $1 \notin D$ and $D_{\text{even}} = 1$ or $D_{\text{odd}} = 1$. This can be done in several possible ways. We choose the argument involving refinement equations. The proof is in the Appendix.

For a pair of matrices A_0, A_1, we consider a random matrix product $\xi_j = X_j \ldots X_1$, where all X_s, $s \in \mathbb{N}$, are independent matrix random variables taking values A_0 and A_1 with equal probabilities 1/2. The *Lyapunov exponent* of the pair $\{A_0, A_1\}$ is

$$\lambda\,(A_0, A_1) \;=\; \lim_{j\to\infty} j^{-1}\,\boldsymbol{E}\,\log_2 \xi_j\,. \tag{32}$$

See [24, 42, 48, 49, 57] for properties and computation of the Lyapunov exponent.

Theorem 3. *For an arbitrary dictionary D, not necessarily satisfying Condition* 1*, there is a set $\mathcal{M} \subset \mathbb{N}$ of density one for which*

$$\lim_{k\to\infty\,,\,k\in\mathcal{M}} \frac{\log b(k)}{\log k} \;=\; \lambda\,(A_0, A_1)\,. \tag{33}$$

Proof. Let $M(\xi_j)$ and $m(\xi_j)$ denote the maximal and the minimal nonzero entries of the random matrix product ξ_j. By [48, Theorem 1], we have

$$\lim_{j\to\infty} j^{-1}\,\log_2 m(\xi_j) \;=\; \lim_{j\to\infty} j^{-1}\,\log_2 M(\xi_j) \;=\; \lambda \qquad \text{a.s.}$$

By Lemma 2, $\boldsymbol{P}\,\{(\xi_j)_{11} > 0\} \;\to\; 1$ and hence

$$\boldsymbol{P}\,\{b(k) \ge m(\xi_j) \mid 2^{j-1} \le k < 2^j\} \;\to\; 1 \qquad \text{as } j \to \infty.$$

Thus, for every $\varepsilon > 0$, the probability $\boldsymbol{P}\left\{\,|\log_2\,b(k) \,-\, \lambda| \,>\, \varepsilon \quad \middle|\; 2^{j-1} \le k < 2^j\,\right\}$ tends to zero as $j \to \infty$, which completes the proof. □

In what follows, we assume again Condition 1 is satisfied.

9. The case of power growth

By Theorem 2, the binary partition function has constant exponent of growth if $\check{\rho}(A_0, A_1) = \hat{\rho}(A_0, A_1)$. For a concrete dictionary D with reasonably small n, this condition can be checked by mere computation of those values using the algorithm from [28]. Our aim is to classify all such dictionaries. This problem turns out to be solvable without computing $\check{\rho}$ and $\hat{\rho}$.

In view of inequality (29), if $p_1 = p_2$, then both those exponents are equal to $p = \log_2 |D| - 1$.

Theorem 4. *We have $p_1 = p_2$ precisely when $\boldsymbol{d}(z)$ is divisible by $z^{2^\ell} + 1$ for some integer $\ell \geq 0$; otherwise $p_1 < p < p_2$.*

Remark 3. Clearly, this is the same as saying that $\boldsymbol{d}(z)$ is divisible by $z^{q} + 1$ for some natural q.

Proof of Theorem 4. We use the subdivision operator $S = S_D$ defined by (13). Assuming $p_2 = p$ and applying the upper bound from (27) we obtain

$$\begin{aligned} \left(S^j \delta\right)_k &= \left(\frac{2}{|D|}\right)^j \left(\Gamma^j \delta\right)_k = \left(\frac{2}{|D|}\right)^j b(k) \\ &\leq 2^{-pj} c_2\, k^{p_2} \leq 2^{-pj} c_2\, 2^{j\, p_2} = c_2 \end{aligned}$$

for all $k \in [2^{j-1}, 2^j)$. On the other hand, applying (21) for $f = \chi_{[0,1)}$, we obtain a piecewise constant function $T^j f$ with dyadic nodes of order j supported on the segment $[0, n]$. We see that this function does not exceed c_2 on the interval $\left[\frac{1}{2}, 1\right)$. By Theorem A, $T^j f \to \varphi$ as $j \to \infty$ (the convergence is in the space $\mathcal{S}'$). Hence, $\varphi \leq c_2$ on $\left[\frac{1}{2}, 1\right)$, in the sense that $(\varphi, g) \leq c_2 \int_{\mathbb{R}} g\, dx$ for all test functions $g \in \mathcal{S}$ such that $\operatorname{supp} g \subset \left[\frac{1}{2}, 1\right]$. Hence, $\varphi \in L_1\left[\frac{1}{2}, 1\right]$. Thus, φ cannot be purely singular on $[0, n]$, and therefore (Theorem D, a)), it is absolutely continuous on $[0, n]$.

If $p_1 = p$, then we show as above that $\varphi \geq c_1$ on $\left[\frac{1}{2}, 1\right)$. This implies again that φ cannot be purely singular, and hence it is absolutely continuous. By the same item a) of Theorem D, this means that the symbol $\boldsymbol{m}(z)$ has a blocking set $\mathcal{B} \subset \mathcal{T}$ such that $\boldsymbol{m}(\mathcal{B}) = 0$. Each element of $\mathcal{B}$ has the form $z = e^{-\frac{2\pi i k}{2^q}}$ for some natural q and odd k. Hence, $\boldsymbol{m}$ is divisible by the corresponding cyclotomic polynomial $\boldsymbol{C}_{2^q} = z^{2^{q-1}} + 1$ (see (9)). This concludes the proof of necessity.

Sufficiency. If $\boldsymbol{m}$ is divisible by $z^{2^\ell} + 1$, then (Theorem D, a) and c)) the refinable function φ is continuous and hence possesses the partition of unity property (20). Thus, for the vector-function $v = v_\varphi$, we have $\big(v(t), \boldsymbol{e}\big) \equiv 1$. Hence, $\left(\Pi\, v\left(\frac{1}{2}\right), \boldsymbol{e}\right) = 1$ for each product Π of the transition matrices $\{T_0, T_1\}$. Since $v\left(\frac{1}{2}\right) > 0$ (Theorem D, d)), it follows that norms of all those products are uniformly bounded both from above and from below. Therefore, $\check{\rho}(T_0, T_1) = \hat{\rho}(T_0, T_1) = 1$. Since $T_i = 2^{-p} A_i^*$, $i = 0, 1$, it follows that $\check{\rho}(A_0, A_1) = \hat{\rho}(A_0, A_1) = 2^p$, and consequently $p_1 = p_2 = p$. $\square$

Thus, the function $b(k)$ has a constant exponent of growth if and only if $\boldsymbol{d}$ is divisible by $z^{2^\ell} + 1$ for some $\ell \geq 0$. If $\ell = 0$, this condition becomes $\boldsymbol{d}(-1) = 0$ or, equivalently, $|D_{\text{even}}| = |D_{\text{odd}}|$.

Corollary 2. *If $|D_{\text{even}}| = |D_{\text{odd}}|$, then $b(k)$ has a constant exponent of growth.*

In particular, this condition is satisfied if $D = \{0, \dots, n\}$ and n is odd. Thus, for full dictionaries with odd n, we have $p_1 = p_2 = \log_2 \frac{n+1}{2}$. This was proved by Reznick [50]. On the other hand, if n is even, then all roots of the polynomial $\boldsymbol{d}(z) = \sum_{s=0}^{n} z^s = \frac{z^{n+1}-1}{z-1}$ are primitive roots of unity of *odd* degrees; hence $\boldsymbol{d}$ cannot be a multiple of $z^{2^\ell} + 1$. Consequently, for full dictionaries with even n, we have $p_1 < p < p_2$. For $n = 2$, this was proved in [50]; for general even n, it was done in [45].

Example 1. If $\ell = 1$, then $\boldsymbol{d}(\pm i) = 0$, which is equivalent to the following system:

$$(34) \qquad \sum_m d_{4m} = \sum_m d_{4m+2} \quad \text{and} \quad \sum_m d_{4m+1} = \sum_m d_{4m+3}.$$

By Theorem 4, if the dictionary satisfies (34), then $p_1 = p_2 = p$.

10. The sharp asymptotics for $b(k)$ and the values of ν_1 and ν_2

Theorem 4 characterizes all dictionaries with the constant exponent of growth $\lim_{k\to\infty} \frac{\log b(k)}{\log k}$. If this limit exists, then it is $p = \log_2 |D| - 1$. Of course, this does not guarantee the regular asymptotic behaviour of $b(k)$, since the sequence $b(k)k^{-p}$ may not have a limit as $k \to \infty$. Our goal is to estimate its lower and upper limits, ν_1 and ν_2, respectively. It turns out that they can be not only estimated but also found explicitly by means of the refinable function φ. Moreover, we obtain a sharp asymptotic expression for $b(k)$ as $k \to \infty$. Then, in Section 11, we characterize the case of regular asymptotic behaviour when $\nu_1 = \nu_2$.

In view of Theorem 4, we assume in the sequel that $\boldsymbol{d}(z)$ is a multiple of $z^{2^\ell}+1$ for some $\ell \geq 0$. We begin with the case $\ell = 0$ to explain the main idea and technique. Then we formulate and prove the results for all $\ell \geq 0$ when everything is essentially the same but expressed in more complicated terms.

Thus, we consider **the case $\ell = 0$**, i.e., $\sum d_{2i} = \sum d_{2i+1}$. By item e) of Theorem D, the subdivision scheme $S = S_D$ defined in (13) converges; therefore

$$(35) \qquad \delta_j = \max_{k \leq 2^j - 1} \left| \left(S^j \delta\right)_k - \varphi\left(2^{-j}k\right) \right| \to 0, \qquad j \to \infty.$$

Since $(S^j\delta)_k = \left(\frac{2}{|D|}\right)^{-j} b(k)$ for $k \leq 2^j - 1$, we have

$$(36) \qquad \delta_j = \max_{k \leq 2^j - 1} \left| \left(\frac{2}{|D|}\right)^{-j} b(k) - \varphi\left(2^{-j}k\right) \right| \to 0, \qquad j \to \infty.$$

Let us recall that φ is the continuous solution of the refinement equation

$$(37) \qquad \varphi(t) = \frac{2}{|D|} \sum_{q=0}^{n} d_q \, \varphi\,(2\,t - q)$$

normalized by the condition $\int_{\mathbb{R}} \varphi(t)\,dt = 1$. We introduce a function $\psi(t) = t^{-p}\,\varphi(t)$. Dividing (36) by $(2^{-j}k)^p$ and taking into account that $\frac{2}{|D|} = 2^{-p}$ we obtain

$$\left|k^{-p}\, b(k) - \psi\left(2^{-j}k\right)\right| \leq \left(\frac{2^j}{k}\right)^p \delta_j\,.$$

Note that $\frac{2^j}{k} < 2$, which leads to the following conclusion.

Proposition 4. *If $\boldsymbol{d}$ is divisible by $z+1$, then for every $j \geq 0$ and $k \leq 2^j - 1$, we have*

$$\big|k^{-p}\, b(k) \ - \ \psi\big(2^{-j}k\big)\big| \ \leq \ 2^p\, \delta_j, \tag{38}$$

where δ_j defined in (35) *tends to zero as $j \to \infty$.*

Remark 4. The refinable function $\varphi(t)$ and hence the function $\psi(t)$ can be found explicitly at all dyadic rational points $t = \frac{k}{2^j}$ [10]. Besides, φ can be evaluated numerically with a prescribed accuracy by the subdivision scheme S [7, 16].

Example 2. Let $D = \{0, \ldots, n\}$ with an odd n. This case satisfies assumptions of Proposition 4, because $\boldsymbol{d}(-1) = \sum_{s=0}^{n}(-1)^s = 0$. Hence, (38) holds with $\psi(t) = t^{-p}\varphi(t)$, where $p = \log_2(n+1) - 1$ and φ is a solution of the equation $\varphi(t) = \frac{2}{n+1}\sum_{s=0}^{n}\varphi(2\,t - s)$ normalized by the condition $\int_{\mathbb{R}} \varphi\, dt = 1$.

Theorem 5. *If $\boldsymbol{d}$ is divisible by $z+1$, then for an arbitrary integer $s \geq 1$, one has*

$$\nu_1 \ = \ \min_{t\in[2^{-s}, 2^{1-s}]} \psi(t); \qquad \nu_2 \ = \ \max_{t\in[2^{-s}, 2^{1-s}]} \psi(t). \tag{39}$$

Proof. Denote $K(s,j) = \big\{\frac{k}{2^j},\ \ 2^{j-s} \leq k \leq 2^{j-s+1}\big\}$. Since $\varphi \in C(\mathbb{R})$, it follows that ψ is uniformly continuous on the segment $[2^{-s}, 2^{1-s}]$. Therefore $\min\limits_{t\in[2^{-s},2^{1-s}]} \psi(t) = \lim\limits_{j\to+\infty} \min\limits_{t\in K(s,j)} \psi(t)$. By Proposition 4, the value $\min\limits_{t\in K(s,j)} \psi(t)$ is equivalent to $\min\limits_{2^{j-s}\leq k\leq 2^{j-s+1}} k^{-p}b(k)$ as $j \to \infty$. Clearly, $\lim\limits_{j\to\infty} \min\limits_{2^{j-s}\leq k\leq 2^{j-s+1}} k^{-p}\, b(k) = \liminf\limits_{k\to\infty} k^{-p}b(k)$. Thus,

$$\inf_{t\in[2^{-s},2^{1-s}]} \psi(t) \ = \ \liminf_{k\to\infty} k^{-p}b(k) \ = \ \nu_1\,.$$

The same holds for ν_2, with inf replaced by sup. □

Theorem 5 gives explicit expressions for the values ν_1, ν_2. This allows us to compute both ν_1 and ν_2 by evaluating the function φ numerically with a subdivision scheme.

Corollary 3. *If $\boldsymbol{d}$ is divisible by $z+1$, then*

$$\nu_1 \ = \ \inf_{t\in(0,1)} \psi(t)\ ; \qquad \nu_2 \ = \ \sup_{t\in(0,1)} \psi(t)\,.$$

The case of arbitrary $\ell \geq 0$. Applying Theorem 1 we treat this case by the same line of reasoning, using the residual convergence of a subdivision scheme in the sense of (26). Extending the definition of the infinitesimal sequence $\{\delta_j\}_{j\in\mathbb{N}}$ as follows:

$$\delta_j \quad = \quad \max_{r=0,\ldots,2^\ell-1} \quad \max_{k\leq 2^j-1,\ k\equiv r\,(\mathrm{mod}\,2^\ell)} \ \big|\, \big(S^j\delta\big)_k \ - \ \mu_r\, \varphi\big(2^{-j}k\big)\,\big|\ , \tag{40}$$

we, by Theorem 1, have $\delta_j \to 0$ as $j \to \infty$. Thus, δ_j measures the slowest convergence of the scheme $S = S_D$ over all residuals $r = 0, \ldots, 2^\ell - 1$. As above, φ is a solution of refinement equation (37). Arguing exactly as in the proof of Proposition 4, we obtain

Theorem 6. *If $\boldsymbol{d}$ is divisible by $z^{2^\ell} + 1$, then for every $j \geq 0$ and $k \leq 2^j - 1$, we have*

$$\big|k^{-p}\, b(k) \ - \ \mu_r\, \psi\big(2^{-j}k\big)\big| \ \leq \ 2^p\, \delta_j\,, \tag{41}$$

where r is the residual of k modulo 2^{ℓ}, μ_r is defined in (25)*, and δ_j defined in* (40) *tends to zero as $j \to \infty$.*

Taking $j = j(k)$, for which $2^{j-1} \le k < 2^j$, we obtain the following sharp asymptotic formula for the binary partition function.

Theorem 7. *If $\boldsymbol{d}$ is divisible by $z^{2^{\ell}} + 1$, then*

$$b(k) \quad = \quad \mu_r \psi\bigl(2^{-j}k\bigr)\, k^{p} \ + \ o\bigl(k^{p}\bigr) \qquad as \quad k \to \infty\,, \tag{42}$$

where $j = j(k)$ and r is the residual of k modulo 2^{ℓ}.

Thus, in case of power growth, $b(k)$ can be approximately evaluated by the sharp asymptotic formula (42). Note that the precision $o(1)$ in this formula has an exponential decay as $k \to \infty$; i.e., it does not exceed $C\gamma^k$ for some $\gamma \in (0,1)$, because the rate of convergence of a subdivision scheme is always exponential [7].

Remark 5. The constant μ_r is a rational number that can be simply evaluated as follows. Using formula (25) for the polynomial $\boldsymbol{m} = \frac{2}{|D|}\boldsymbol{d}$, we obtain $\mu_r = \bigl(\frac{4}{|D|}\bigr)^{\ell} \boldsymbol{P}_r(1)$, where $\boldsymbol{P}(z) = \boldsymbol{d}(z)\cdots \boldsymbol{d}(z^{2^{\ell-1}})$ and $\boldsymbol{P}_r$ is the r-part of the polynomial $\boldsymbol{P}$ according to Definition 1. The number $\boldsymbol{P}_r(1)$ has the following meaning. Let $b_{\ell}(k)$ denote the total number of binary expansions (1) using the first ℓ digits: $b_{\ell}(k) = \#\{(a_0, \ldots, a_{\ell-1}) \in D^{\ell} \mid k = \sum_{m=0}^{\ell-1} a_m 2^m\}$. Then $\boldsymbol{P}_r(1) = \sum_{k\in\mathbb{Z}\,,\, k\equiv r \,(\mathrm{mod}\, 2^{\ell})} b_{\ell}(k)$.

Remark 6. For the full dictionary $D = \{0, 1, \ldots, 5\}$, it was announced in [50] (without proof) that there exists a continuous function ψ such that $b(k) = \psi(2^{-j}k)k^p + o(k^p)$ as $k \to \infty$. In this case, $p = \log_2 3$. Theorem 7 implies that such a function does exist and this is $\psi(t) = t^{-\log_2 3}\varphi(t)$, where φ is a continuous solution of the refinement equation $\varphi(t) = \frac{1}{3}\sum_{q=0}^{5}\varphi(2t - q)$. Indeed, the polynomial $\boldsymbol{d}(z) = \frac{1}{6}\sum_{q=0}^{5} z^q$ is a multiple of $z + 1$; hence we have the case $\ell = 0$, and asymptotic equality (42) holds with $\mu_r = 1$. By Theorem 7, such a function $\psi(t)$ exists for each dictionary D whose polynomial $\boldsymbol{d}(z)$ is divisible by $1 + z$. If it is divisible by $z^{2^{\ell}} + 1$, this function is replaced by 2^{ℓ} functions $\mu_r\, \psi(t)$, $r = 0, \ldots, 2^{\ell} - 1$, depending on the residual of k modulo 2^{ℓ}.

As for the values ν_1 and ν_2, Theorem 7 immediately implies:

Theorem 8. *If $\boldsymbol{d}$ is divisible by $z^{2^{\ell}} + 1$, then for an arbitrary integer $s \ge 1$, one has*

$$\nu_1 \ = \ \Bigl[\min_{r\in\{1,\ldots,2^{\ell}-1\}} \mu_r\Bigr] \min_{t\in[2^{-s},2^{1-s}]} \psi(t); \quad \nu_2 \ = \ \Bigl[\max_{r\in\{1,\ldots,2^{\ell}-1\}} \mu_r\Bigr] \max_{t\in[2^{-s},2^{1-s}]} \psi(t). \tag{43}$$

11. The regular power growth

Our last problem is to characterize dictionaries for which $\nu_1 = \nu_2$, i.e., when the sequence $k^{-p}\, b(k)$ tends to some limit $\nu > 0$ as $k \to \infty$. This "most regular" case offers a surprising resistance. Although we present a complete classification of such dictionaries in Theorem 11, the question if that classification can be simplified remains open (Problem 1 below). We start with several observations.

If $b(k) \sim k^{p}$, then $b(s+1)/b(s) \to 1$ as $s \to \infty$. The recurrence (10) yields that all terms in the right hand sides are asymptotically equivalent. Dividing the first equality by the second one, we obtain $\frac{b(2k+1)}{b(2k)} \to \frac{|D_{\rm even}|}{|D_{\rm odd}|}$ as $k \to \infty$. Hence $|D_{\rm even}| = |D_{\rm odd}|$, and therefore $\boldsymbol{d}$ is a multiple of $z+1$. In particular, Corollary 3 is applicable for this case. We obtain $\psi(t) \equiv {\rm const}\,,\ t \in (0,1)$, and hence $\varphi(t) \equiv t^p$ on the interval $(0,1)$. Invoking now Theorem C we conclude that φ must be a spline. By Theorem B, $p \in \mathbb{N}$ and so $|D_{\rm even}| = |D_{\rm odd}| = 2^p$. The order of the spline is p; hence $\psi(t) \equiv c$, and therefore $\nu_1 = \nu_2 = c$. Finally, by Theorem B, $\boldsymbol{d} = 2^{p+1}\boldsymbol{m}$ is a polynomial of the form $\frac{\boldsymbol{q}(z^2)}{\boldsymbol{q}(z)}$, whose roots are complex roots of unity. We arrive at the following theorem.

Theorem 9. *We have $\nu_1 = \nu_2$ if and only if φ is a refinable spline. In this case $p \in \mathbb{N}\,,\ |D_{\rm even}| = |D_{\rm odd}| = 2^p$, the spline has degree p, and*

$$b(k) \quad \sim \quad ck^{\,p} \qquad as \quad k \to \infty\,.$$

This holds precisely when $\boldsymbol{d}(z) = \frac{\boldsymbol{q}(z^2)}{\boldsymbol{q}(z)}$, where $\boldsymbol{q}$ is an integer polynomial that has zero of order $p+1$ at the point $z = 1$. All roots of $\boldsymbol{d}$ are roots of unity.

Example 3. For a full dictionary $D = \{0, \dots, n\}$, Theorem 9 immediately implies that $|D| = 2^{p+1}$; i.e., $n+1$ is a power of two. In this case $b(k)$ indeed has a regular power growth. This was first shown in [50]; then in [45] the necessity of the condition that $n+1$ is a power of two was proved. A generalization of this fact follows from Theorem 10 below.

There is at least one class of dictionaries satisfying assumptions of Theorem 9:

Theorem 10. *We have $b(k) \sim \frac{1}{p\,!}\,k^{\,p}$ as $k \to \infty$ whenever $\boldsymbol{d}$ has the form*

$$\boldsymbol{d}(z) \;=\; \bigl(z^{\,n_0}+1\bigr)\cdots\bigl(z^{\,n_p}+1\bigr)\,, \tag{44}$$

where $n_0, \dots, n_p$ are different natural numbers.

Proof. Note that $z^{n_r}+1 = \frac{(z^2)^{n_r}-1}{z^{n_r}-1}$; hence $\boldsymbol{d}(z) = \frac{\boldsymbol{q}(z^2)}{\boldsymbol{q}(z)}$ with $\boldsymbol{q}(z) = z^{n_r} - 1$. Applying Theorem 9 concludes the proof. ☐

Example 4. For $D = \{0,1,3,4\}$, we have $\boldsymbol{d}(z) = (z+1)(z^3+1)$. Hence, $b(k) \sim ck$ as $k \to \infty$.

For $D = \{0,2,3,4,5,6,7,9\}$, we have $\boldsymbol{d}(z) = (z^2+1)(z^3+1)(z^4+1)$. Therefore, $b(k) \sim ck^2$ as $k \to \infty$.

For $D = \{0,1,2,5\}$, the function $b(k)$ has a power growth, but not regular. Since $\boldsymbol{d}(-1) = 0$, we have $p_1 = p_2 = \log_2|D| - 1 = 1$. Thus, $\nu_1 k \le b(k) \le \nu_2 k$. On the other hand, not all roots of $\boldsymbol{d}(z) = z^5 + z^2 + z + 1$ lie on the unit circle, hence $\nu_1 < \nu_2$.

The case of full dictionary with $2^{\,p+1}$ elements (Example 3) corresponds to polynomial (44) with $n_r = 2^{\,r}$, $r = 0, \dots, p$.

Not every polynomial of the form (44) is a Newman polynomial, i.e., has $0,1$ coefficients, only when all sums of numbers from the set $\{n_r\}_{r=0}^{p}$ are different. Under this assumption, polynomial (44) indeed defines a dictionary with the regular behaviour of $b(k)$. We are not aware of other examples.

Problem 1. Are there dictionaries with the regular power growth of the binary partition function different from those given by (44)?

As for now, we have only partial results in this direction that give a more detailed description of dictionaries corresponding to the regular power growth. One of them is by cyclotomic polynomials given below.

Theorem 11. *We have* $\nu_1 = \nu_2$ *if and only if*

$$\boldsymbol{d}(z) \;=\; \prod_{s=1}^{m} \boldsymbol{C}_{q_s}\,, \tag{45}$$

where $\boldsymbol{C}_q$ *is the* q*th cyclotomic polynomial,* $q_1, \ldots, q_m$ *are even natural numbers (may be coinciding), precisely* $p+1$ *of which are powers of two and* $q_s = 2$ *for some of them.*

Proof. Since $\boldsymbol{d}$ is an integer polynomial and all its roots are roots of unity (Theorem 9), it follows that it is a product of cyclotomic polynomials. Assume some q_s is odd. The number $w = e^{-\frac{2\pi i s_j}{q_j}}$ is a root of $\boldsymbol{d}$ of some multiplicity h. Hence it is a root of $\boldsymbol{q}(z^2)$ of the same multiplicity. Since the set of roots of $\boldsymbol{q}$ is closed with respect to squaring, it follows that this set contains all powers $w^{2^k}, k \geq 0$. Therefore, $w^{2^k} = w$ for some k, and consequently, $\frac{\boldsymbol{q}(z^2)}{\boldsymbol{q}(z)}$ is not divisible by $z - w$, which contradicts the assumption. Thus, all q_s are even. Furthermore, since $\boldsymbol{d}(1) = 2^{p+1}$, it follows that $\boldsymbol{q}$ has a root $z = 1$ of multiplicity $p+1$. Hence, $p+1$ multipliers in (45) are powers of two. Moreover, since $\boldsymbol{d}(-1) = 0$ (because $D_{\mathrm{even}} = D_{\mathrm{odd}}$ by Theorem 9), at least one of those multipliers is $z + 1 = \boldsymbol{C}_2$.

Conversely, if $q = 2q'$ is even, then $\boldsymbol{C}_{2q'}$ has the form $\frac{\boldsymbol{q}(z^2)}{\boldsymbol{q}(z)}$ (with $\boldsymbol{q} = \boldsymbol{C}_{q'}$), and hence any product of such polynomials has this form. Hence, φ is a spline. By Theorem B, its degree is equal to the multiplicity of the root $z = 1$ of the polynomial $\boldsymbol{q}$, which coincides with the total number of powers of two among the numbers $\{q_j\}_{j=1}^{m}$. □

Remark 7. Every polynomial of the form (45) produces a refinable spline of order p, but there is no guarantee that this polynomial has $0, 1$ coefficients. Cyclotomic polynomials with specified coefficients have been studied in a number of papers (see [4, 14] and references therein). By (9) we have $\boldsymbol{C}_{2^q}(z) = z^{2^{q-1}} + 1$; hence $p+1$ multipliers in the product (45) have this simple form, including at least one multiplier $z+1$. The question is what other cyclotomic polynomials can be taken to obtain a Newman polynomial in the product?

12. Appendix

12.1. The values of p_1 and p_2 for some dictionaries. We begin with dictionaries D with the largest element $n \leq 5$. As usual, we denote by ρ the spectral radius and by A_0, A_1 the $n \times n$ Boolean matrices defined in (16) for the corresponding dictionaries. The computation was done by evaluating the joint spectral characteristics of matrices with the invariant polytope algorithm from [28]. The numerical values are rounded to the sixth digit after the decimal point. We take all dictionaries containing zero and with $\gcd(D) = 1$. There are in total 25 dictionaries. Six of them have power growth, i.e, $p_1 = p_2$.

Table 1 presents the values of p_1 and p_2 for full dictionaries $D = \{0, \ldots, n\}$. As we mentioned, if n is odd, then $p_1 = p_2 = \log_2 \frac{n+1}{2}$. Therefore, we focus on even n. Table 2 presents those values for $n \leq 30$. As we see, the distance between p_1 and p_2 decreases fast and already for $n = 30$ becomes smaller than 10^{-6}.

TABLE 1. The values of p_1 and p_2 for dictionaries with the largest element ≤ 5.

the set D			p_1			p_2
$\{0,1\}$			0			0
$\{0,1,2\}$			0	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	0.694241
$\{0,1,3\}$			0	$\log_2\rho(A_0)$	$=$	0.694241
$\{0,2,3\}$			0	$\log_2\rho(A_1)$	$=$	0.694241
$\{0,1,2,3\}$			1			1
$\{0,1,4\}$			0	$\frac{1}{3}\log_2\rho(A_0^2A_1)$	$=$	0.652578
$\{0,3,4\}$			0	$\frac{1}{3}\log_2\rho(A_1^2A_0)$	$=$	0.652578
$\{0,1,2,4\}$			0	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	1.084079
$\{0,1,3,4\}$			1			1
$\{0,2,3,4\}$			0	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	1.084079
$\{0,1,2,3,4\}$	$\log_2\rho(A_0)$	$=$	1.271552	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	1.335632
$\{0,1,5\}$			0	$\log_2\rho(A_0)$	$=$	0.694241
$\{0,2,5\}$			0	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	0.622201
$\{0,3,5\}$			0	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	0.622201
$\{0,4,5\}$			0	$\log_2\rho(A_1)$	$=$	0.694241
$\{0,1,2,5\}$			1			1
$\{0,1,3,5\}$			0	$\log_2\rho(A_1)$	$=$	1.075135
$\{0,1,4,5\}$			1			1
$\{0,2,3,5\}$			1			1
$\{0,2,4,5\}$			0	$\log_2\rho(A_0)$	$=$	1.075135
$\{0,1,2,3,5\}$	$\frac{1}{7}\log_2\rho\left(A_0A_1^2A_0A_1^3\right)$	$=$	1.306774	$\frac{1}{10}\log_2\rho\left((A_0^2A_1)^2A_0^3A_1\right)$	$=$	1.336073
$\{0,1,2,4,5\}$	$\frac{1}{4}\log_2\rho(A_0^3A_1)$	$=$	1.301875	$\frac{1}{3}\log_2\rho(A_0A_1^2)$	$=$	1.338064
$\{0,1,3,4,5\}$	$\frac{1}{4}\log_2\rho(A_1^3A_0)$	$=$	1.301875	$\frac{1}{3}\log_2\rho(A_1A_0^2)$	$=$	1.338064
$\{0,2,3,4,5\}$	$\frac{1}{7}\log_2\rho\left(A_1A_0^2A_1A_0^3\right)$	$=$	1.306774	$\frac{1}{10}\log_2\rho\left((A_1^2A_0)^2A_1^3A_0\right)$	$=$	1.336073
$\{0,1,2,3,4,5\}$	$\log_2 3$	$=$	1.584962	$\log_2 3$	$=$	1.584962

12.2. Proofs of auxiliary results.

Proof of Lemma 2. Assume to the contrary that there is a number $\gamma > 0$ such that there are arbitrarily large j with the following property: $b(k) = 0$ for at least $\gamma 2^{j-1}$ numbers k from $2^{j-1}, \ldots, 2^j - 1$. Take an arbitrary nonnegative sequence $\{c_k\}_{k=0}^n$ with the support D satisfying (8). This is possible because D contains both even and odd numbers. Then $T^j\chi_{[0,1)}$ converges to φ in the space $L_1[0,n]$ (see [47]). On the other hand, there are arbitrarily large j such that $(S^j\delta)_k = 0$ for at least $\gamma 2^{j-1}$ numbers $k \in \{2^{j-1}, \ldots, 2^j - 1\}$. Equality (21) for $f = \chi_{[0,1)}$ yields that the function $T^j\chi_{[0,1)}$ vanishes on the subset of the segment $[0,1]$ of the Lebesgue measure at least γ. Hence, so does the limit function φ. Thus, $\varphi(t) = 0$ on a set $K \subset [0,n]$ of positive Lebesgue measure. This is well known to be impossible. Indeed, consider the mapping $\Phi(K) = \frac{1}{2}K \cup \left(\frac{1}{2} + \frac{n}{2}\right)$. This is the ergodic *baker transform* on the segment $[0,n]$. Since $\Phi(K) = K$, it follows from egrodicity that the Lebesgue measure of K is either zero or n. This contradicts our assumption, which completes the proof. □

TABLE 2. The values of p_1 and p_2 for full dictionaries with even $n \le 30$.

n	p_1			p_2		
2		0		$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	0.694241
4	$\log_2\rho(A_0)$	$=$	1.271552	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	1.335632
6	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	1.804008	$\log_2\rho(A_0)$	$=$	1.812106
8	$\log_2\rho(A_0)$	$=$	2.168158	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	2.170918
10	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	2.458655	$\log_2\rho(A_0)$	$=$	2.460976
12	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	2.700205	$\log_2\rho(A_0)$	$=$	2.700920
14	$\log_2\rho(A_0)$	$=$	2.906863	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	2.906910
16	$\log_2\rho(A_0)$	$=$	3.087446	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	3.087472
18	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	3.247895	$\log_2\rho(A_0)$	$=$	3.247991
20	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	3.392291	$\log_2\rho(A_0)$	$=$	3.392368
22	$\log_2\rho(A_0)$	$=$	3.523548	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	3.523568
24	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	3.643852	$\log_2\rho(A_0)$	$=$	3.643862
26	$\log_2\rho(A_0)$	$=$	3.754890	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	3.754880
28	$\log_2\rho(A_0)$	$=$	3.857979	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	3.857981
30	$\frac{1}{2}\log_2\rho(A_0A_1)$	$=$	3.954195	$\log_2\rho(A_0)$	$=$	3.954196

In the proof of Theorem 1 we need three auxiliary facts. The first one gives a formula for the polynomial $\boldsymbol{p}_r$ for each $r = 0, \dots, 2^{\ell} - 1$. This is a generalization of formulas (24) to arbitrary $\ell \ge 1$.

Lemma 3. *If $\boldsymbol{p}$ is a polynomial and $\ell \ge 0$, then, for each residual $r = 0, \dots, 2^{\ell} - 1$, we have*

$$\boldsymbol{p}_r(z) \quad = \quad 2^{-\ell} \sum_{s=1}^{2^{\ell}-1} e^{2\pi i \cdot 2^{-\ell} sr}\, \boldsymbol{p}\left(e^{-2\pi i \cdot 2^{-\ell} s}\, z\right). \tag{46}$$

Proof. For a monomial $\boldsymbol{p}(z) = z^q$, the right hand side of (46) is equal to z^q if $q \equiv r \,(\mathrm{mod}\, 2^{\ell})$, and vanishes otherwise. Then the formula follows by linearity. ☐

Lemma 4. *Let a polynomial $\boldsymbol{p}$ and numbers $N > \ell \ge 0$ be given. If $\boldsymbol{p}$ vanishes at all roots of unity of degree 2^N, then so does the polynomial $\boldsymbol{p}_r$ for each $r = 0, \dots, 2^{\ell} - 1$.*

Proof. If w is a root of unity of degree 2^N, then so is the number $e^{-2\pi i \cdot 2^{-\ell} s}\, w$. Hence, by the assumption, the right hand side of (46) is zero at the point $z = w$. Thus, $\boldsymbol{p}_r(w) = 0$. ☐

For an integer $\ell \ge 0$, we define the following space of functions:

$$\mathcal{V}_{\ell} \quad = \quad \left\{ f \in C_0(\mathbb{R}) \;\middle|\; \widehat{f}(2^{-q}s) = \delta(s)\,,\ s \in \mathbb{Z}\,,\ \ q = 0, \dots, \ell \right\}.$$

The following fact is known; its proof can be found, for instance, in [39, 47].

Lemma 5. *Under the assumptions of Theorem* 1*, for every* $f \in \mathcal{V}_\ell$*, we have* $\|T^j f - \varphi\|_{C(\mathbb{R})} \to 0$ *as* $j \to \infty$*.*

Proof of Theorem 1. We begin by introducing the following notation: $f = \chi_{[0,1)}$, $g(t) = f\bigl(\frac{t+r}{2^\ell}\bigr) = \chi_{[-r\,,\,2^\ell - r)}$, and

$$f_{\ell,r}(t) \quad = \sum_{k \equiv r \,(\mathrm{mod}\, 2^\ell)} (S^\ell \delta)_k \, f(t - 2^{-\ell} k)\,.$$

For arbitrary $j \ge \ell$ consider two functions:

$$\sum_{k \equiv r \,(\mathrm{mod}\, 2^\ell)} \varphi(2^{-j}k)\; g(2^j t - k) \qquad \text{and} \qquad \mu_r^{-1} \sum_{k \equiv r \,(\mathrm{mod}\, 2^\ell)} (S^j \delta)_k \, g(2^j t - k)\,.$$

The first function is a piecewise constant interpolation of the function φ, which converges to φ uniformly as $j \to \infty$, since φ is continuous. The second function is also piecewise constant with the same nodes. If we prove that it converges to φ uniformly on $\mathbb{R}$, it would mean that the difference of sequences $(S^j\delta)_k$ and $\{\mu_r\, \varphi(2^{-j}k)\}_{k\in\mathbb{Z}}$ converges to zero uniformly on the set $\{k \in \mathbb{Z} |\ k \equiv r \,(\mathrm{mod}\, 2^\ell)\}$ as $j \to \infty$, which completes the proof. Writing $k = 2^\ell k' + r$, we have

$$\begin{aligned}
\sum_{k \equiv r \,(\mathrm{mod}\, 2^\ell)} (S^j \delta)_k \, g(2^j t - k) \quad &= \sum_{k \equiv r \,(\mathrm{mod}\, 2^\ell)} (S^j \delta)_k \, f\left(\frac{2^j t - (k - r)}{2^\ell}\right) \\
= \sum_{k' \in \mathbb{Z}} (S^j \delta)_{2^\ell k' + r} \, f\left(2^{j-\ell} t - k'\right) \ &= \sum_{k' \in \mathbb{Z}} (S^{j-\ell}\delta)_{k'} \, f_{\ell,r}\left(2^{j-\ell} t - k' - 2^{-\ell} r\right) \\
= \ \bigl[\, T^{j-l} f_{\ell,r}(\cdot - 2^{-\ell} r) \,\bigr]\,(t) &
\end{aligned}$$

(the latter equality follows from equality (21)). Thus, one needs to show that $\mu_r^{-1}\, T^{j-l} f_{\ell,r}(\cdot - 2^{-\ell} r)$ converges uniformly to φ as $j \to \infty$. This, in view of Lemma 5, follows from the assertion $\mu_r^{-1}\, f_{\ell,r}(\cdot - 2^{-\ell} r) \in \mathcal{V}_\ell$, which is equivalent to $\mu_r^{-1}\, f_{\ell,r} \in \mathcal{V}_\ell$. To check this, we note that

$$\widehat{f_{\ell,r}}(\xi) \quad = \quad \boldsymbol{Q}_r\bigl(e^{-2\pi i \cdot 2^{-\ell}\xi}\bigr)\, \widehat{f}(\xi)\,,$$

where $\boldsymbol{Q}(z) = \prod_{j=0}^{\ell-1} \boldsymbol{m}(z^{2^j})$. Since $\boldsymbol{m}$ is divisible by $z^{2^\ell} + 1$, it vanishes at all roots of unity of degree $2^{\ell+1}$. Hence, $\boldsymbol{Q}$ vanishes at all roots of unity of degrees $2^{\ell+1}, \ldots, 2^{2\ell}$. By Lemma 4, so does the polynomial $\boldsymbol{Q}_r$. Thus, the function $\boldsymbol{Q}_r\bigl(e^{-2\pi i \cdot 2^{-\ell}\xi}\bigr)$ vanishes at all points $2^{-q}s$ with $q = 1, \ldots, \ell$ and odd $s \in \mathbb{Z}$. Besides, $\widehat{f}$ vanishes at all integers except for zero. Thus, $\widehat{f_{\ell,r}}(2^{-q}s) = 0$ for all $q = 0, \ldots, \ell$ and $s \neq 0$. Finally, $\widehat{f}(0) = 1$ and hence $\mu_r^{-1}\widehat{f_{\ell,r}}(0) = 1$, which is required. □

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