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SUPERCONVERGENCE BY M-DECOMPOSITIONS. PART I: GENERAL THEORY FOR HDG METHODS FOR DIFFUSION

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Abstract. We introduce the concept of an M-decomposition and show how to use it to systematically construct hybridizable discontinuous Galerkin and mixed methods for steady-state diffusion methods with superconvergence properties on unstructured meshes.

1. Introduction

This is the first of a series of papers in which we develop the concept of an M-decomposition as an effective tool for devising hybridizable discontinuous Galerkin (HDG) methods that superconverge on unstructured meshes of shape-regular polyhedral elements. We introduce our approach in the framework of the following diffusion problem:

$$\begin{aligned} \mathrm{c}\boldsymbol{q} + \nabla u &= 0 && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{q} &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded polyhedral domain, c is a uniformly bounded, uniformly positive definite symmetric matrix-valued function, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$.

Let us place our results within the ongoing effort of devising superconvergent methods defined on *unstructured* meshes. In 1985, it was shown, for the Raviart-Thomas [1,26] and the Brezzi-Douglas-Marini [3] mixed methods, that the approximation to the scalar variable u *superconverged* in unstructured meshes; see also [2,6]. As consequence, by using a local postprocessing, a new approximation to u could be obtained which converges faster than the original approximation. Since the computational effort needed to compute the postprocessing is negligible (and can be done in parallel) with respect to the computational effort needed to compute the original approximation, this superconvergence property has highly practical value.

Since there is a strong relation between the HDG and the mixed methods (see [11]), it is reasonable to ask if similar superconvergence properties could be obtained for the HDG methods. The first positive answer was given in 2008, when the first

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error analysis of an HDG method, namely, the so-called single-face hybridizable (SFH) discontinuous Galerkin method (defined on unstructured simplicial meshes), was obtained in [10]. By exploiting the above-mentioned relation between HDG and mixed methods, the SFH method was proven to have superconvergence properties similar to those of the Raviart-Thomas [1,26] and Brezzi-Douglas-Marini [3] mixed methods. In 2009, the first family of DG methods, not necessarily hybridizable, proven to be superconvergent was identified in [13], improving on the previous error analysis carried out in [4]. In 2010, a wide family of HDG methods, also defined in simplicial meshes, was proven to be superconvergent in [12]. Finally, in 2012, general sufficient conditions for HDG methods to be superconvergent were found in [14] for various polyhedral elements; the extension to curved elements was carried out in [15]. Our results are a refinement of the work on superconvergent HDG methods done in [14].

The HDG methods. To better describe them, let us begin by defining the HDG methods; we follow [11]. Let $\mathcal{T}_h := \{K\}$ denote a conforming triangulation of Ω, where K is a polyhedral element. Let $\mathcal{E}_h$ denote the set of faces F of the elements $K \in \mathcal{T}_h$, and let $\mathcal{F}(K)$ denote the set of faces F of the element K. The HDG methods seek an approximation to $(u, \boldsymbol{q}, u|_{\mathcal{E}_h})$, $(u_h, \boldsymbol{q}_h, \widehat{u}_h)$, in the finite dimensional space $W_h \times \boldsymbol{V}_h \times M_h$, where

$$\begin{aligned}
\boldsymbol{V}_h &:= \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{T}_h) : \boldsymbol{v}|_K \in \boldsymbol{V}(K),\ K \in \mathcal{T}_h\},\\
W_h &:= \{w \in L^2(\mathcal{T}_h) : w|_K \in W(K), K \in \mathcal{T}_h\},\\
M_h &:= \{\mu \in L^2(\mathcal{E}_h) : \mu|_F \in M(F),\ F \in \mathcal{E}_h\},
\end{aligned}$$

and determine it as the only solution of the following weak formulation:

$$(\mathrm{c}\,\boldsymbol{q}_h\,,\,\boldsymbol{v})_{\mathcal{T}_h} - (u_h\,,\,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle \widehat{u}_h\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{1.1a}$$

$$-(\boldsymbol{q}_h\,,\,\nabla w)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\,,\,w\rangle_{\partial\mathcal{T}_h} = (f\,,\,w)_{\mathcal{T}_h}, \tag{1.1b}$$

$$\langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{1.1c}$$

$$\langle \widehat{u}_h, \mu\rangle_{\partial\Omega} = \langle g, \mu\rangle_{\partial\Omega}, \tag{1.1d}$$

for all $(w, \boldsymbol{v}, \mu) \in W_h \times \boldsymbol{V}_h \times M_h$, where

$$\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \alpha(u_h - \widehat{u}_h) \quad \text{on} \quad \partial\mathcal{T}_h. \tag{1.1e}$$

Here we write $(\eta\,,\,\zeta)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}(\eta,\zeta)_K$, where $(\eta,\zeta)_D$ denotes the integral of $\eta\zeta$ over the domain $D \subset \mathbb{R}^n$. We also write $\langle\eta\,,\,\zeta\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\langle\eta\,,\,\zeta\rangle_{\partial K}$, where $\langle\eta\,,\,\zeta\rangle_D$ denotes the integral of $\eta\zeta$ over the domain $D \subset \mathbb{R}^{n-1}$ and where $\partial\mathcal{T}_h := \{\partial K : K \subset \mathcal{T}_h\}$. When vector-valued functions are involved, we use a similar notation.

As pointed out in [11], by taking particular choices of the local spaces $\boldsymbol{V}(K)$, $W(K)$ and

$$M(\partial K) := \{\mu \in L^2(\partial K) :\ \mu|_F \in M(F) \text{ for all } F \in \mathcal{F}(K)\},$$

and of the *linear local stabilization* function α, the different HDG methods are obtained; when we can set $\alpha = 0$, we obtain the so-called mixed methods.

Superconvergent HDG methods. In [14], sufficient conditions on the local spaces $\boldsymbol{V}(K), W(K)$ and $M(\partial K)$, and on the stabilization function α were found under which there is an auxiliary projection $(\boldsymbol{\Pi}_V \boldsymbol{q}, \Pi_W u)$ such that

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{\mathcal{T}_h} \le C\,\|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_{\mathcal{T}_h},$$
$$\|\Pi_W u-u_h\|_{\mathcal{T}_h} \le C\,h\|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_{\mathcal{T}_h},$$

where $\|\cdot\|_{\mathcal{T}_h}$ denotes the $L^2(\mathcal{T}_h)$-norm. It was also proven that, if the error $\Pi_W u - u_h$ converges to zero *faster* than the error $u-u_h$, in which case we say that the HDG method is *superconvergent*, this property can be advantageously exploited as done before for mixed methods in [1, 3]. Indeed, following [23, 27, 28], we can compute elementwise a new approximation to u, u_h^*, in a space W_h^* such that

$$\|u-u_h^*\|_{\mathcal{T}_h} \le C\,h\,(\|\boldsymbol{q}-\Pi_V\boldsymbol{q}\|_{\mathcal{T}_h} + \inf_{\omega\in W_h^*}\|\nabla(u-\omega)\|_{\mathcal{T}_h}),$$

which means that it is possible to define u_h^* converging to u as fast as $\Pi_W u$ converges to u_h.

Let us re-emphasize that superconvergence of finite element methods can be typically obtained whenever the mesh is invariant under translations, or when it presents some special structure; see [29]. In our setting, we say a method is superconvergent only whenever we have unstructured meshes of shape-regular elements and when $u-u_h^*$ converges to zero *faster* than $u-u_h$.

The concept of an M-decomposition. Here, we further refine the work started in [14] and show that if, for all the elements $K\in\mathcal{T}_h$, the local space $\boldsymbol{V}(K)\times W(K)$ admits an $M(\partial K)$-decomposition, then it is possible to find a stabilization function α such that the above-mentioned conditions hold. In other words, we reduce the problem of finding superconvergent HDG methods on unstructured meshes to the problem of finding $\boldsymbol{V}(K), W(K)$ admitting $M(\partial K)$-decompositions.

Moreover, by using the concept of M-decompositions we obtain new stability estimates (of the piecewise divergence $\nabla\cdot\boldsymbol{q}_h$, the piecewise gradient ∇u_h, and the jumps $u_h-\widehat{u}_h$ in terms of the approximate flux and the source data f) and the corresponding error estimates. To obtain the new error estimates, we extend the techniques used in [7] for the staggered DG method on simplexes and in [22] for the Raviart-Thomas method also defined on simplexes.

The concept of an M-decomposition came about by extending to our setting what could be considered to be the *first* example of what we now call an M-decomposition in previous work [17, Lemma 4.1] on HDG methods for the incompressible Stokes flow.

The construction of spaces admitting M-decompositions. We also propose a systematic way of *constructing* local spaces admitting M-decompositions for a given space $M(\partial K)$. We proceed as follows. We pick *any* given space $\boldsymbol{V}_g\times W_g$ satisfying the inclusion properties:

(I.1) $\gamma\boldsymbol{V}_g+\gamma W_g\subset M$,
(I.2) $\nabla W_g\times\nabla\cdot\boldsymbol{V}_g\subset\boldsymbol{V}_g\times W_g$,

where $\gamma\boldsymbol{V}_g := \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v}\in\boldsymbol{V}_g\}$ and $\gamma W_g := \{w|_{\partial K} : w\in W_g\}$. Our main result, Theorem 2.4, states that $\boldsymbol{V}_g\times W_g$ admits an M-decomposition if and only if the natural number

$$I_M(\boldsymbol{V}_g \times W_g) := \dim M - \dim\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\} \\ - \dim\{w|_{\partial K} : w \in W_g, \nabla w = 0\}$$

is zero. In this case, we have that

$$M = \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W_g, \nabla w = 0\}.$$

Thanks to this result, if $I_M(\boldsymbol{V}_g \times W_g)$ is not zero, we see that we have to add to $\boldsymbol{V}_g$ a space $\delta\boldsymbol{V}_{\text{fillM}}$ of *solenoidal* functions of dimension $I_M(\boldsymbol{V}_g \times W_g)$ (this number is positive thanks to the inclusion properties (I)) to obtain a new space admitting an M-decomposition. See the space displayed in the middle row on Table 1; it gives rise to an HDG method.

The two other spaces also admit M-decompositions and give rise to mixed methods since $\nabla\cdot\boldsymbol{V} = W$; they are called the *sandwiching* mixed method spaces. Thanks to our main result, Theorem 2.4, we see that, for the first space to verify this, we must add a space $\delta\boldsymbol{V}_{\text{fillW}}$ of *non-solenoidal* functions (such that $\nabla\cdot\delta\boldsymbol{V}_{\text{fillW}} \subset W_g$) of dimension

$$I_S(\boldsymbol{V}_g \times W_g) := \dim W_g - \dim \nabla\cdot\boldsymbol{V}_g.$$

In Table 2, we list succintly the properties of the spaces $\delta\boldsymbol{V}_{\text{fillM}}$ and $\delta\boldsymbol{V}_{\text{fillW}}$.

TABLE 1. Construction of spaces $\boldsymbol{V} \times W$ admitting an M-decomposition, where the space of traces $M(\partial K)$ includes the constants. The given space $\boldsymbol{V}_g \times W_g$ satisfies the inclusion properties (I).

$\boldsymbol{V}$	W	$\nabla\cdot\boldsymbol{V}$
$\boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}} \oplus \delta\boldsymbol{V}_{\text{fillW}}$	W_g (if $\supset \mathcal{P}_0(K)$)	$= W_g$
$\boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}}$	W_g (if $\supset \mathcal{P}_0(K)$)	$\subset W_g$
$\boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}}$	$\nabla\cdot\boldsymbol{V}_g$ (if $\supset \mathcal{P}_0(K)$)	$= \nabla\cdot\boldsymbol{V}_g$

TABLE 2. The properties of the spaces $\delta\boldsymbol{V}$. Here, $\boldsymbol{V}_{g_s}$ denotes the subspace of solenoidal functions in $\boldsymbol{V}_g$.

$\delta\boldsymbol{V}$	$\nabla\cdot\delta\boldsymbol{V}$	$\gamma\delta\boldsymbol{V}$	$\dim\delta\boldsymbol{V}$
$\delta\boldsymbol{V}_{\text{fillM}}$	$\{\boldsymbol{0}\}$	$\subset M, \cap\gamma\boldsymbol{V}_{g_s} = \{0\}$	$I_M(\boldsymbol{V}_g \times W_g)$
$\delta\boldsymbol{V}_{\text{fillW}}$	$\subset W_g, \cap\nabla\cdot\boldsymbol{V}_g = \{0\}$	$\subset M$	$I_S(\boldsymbol{V}_g \times W_g)$

Let us note that, once we find spaces $\boldsymbol{V}\times W$ spaces admitting M-decompositions, we still have to check the conditions

(J.1) $\mathcal{P}_0(K) \subset \nabla\cdot\boldsymbol{V}$,
(J.2) $\mathcal{P}_1(K) \subset W$.

They are not necessary for achieving M-decompositions, but guarantee that the corresponding methods are actually superconvergent. We illustrate this general approach with several examples of old and new methods in two-space dimensions. The actual practical construction of those and other spaces is carried out in Part II (two-dimensional case) and Part III (three-dimensional case).

Applications. We end by providing two applications of the above construction which take advantage of the relation between the superconvergent HDG method and its two sandwiching mixed methods. In particular, we show that the stiffness matrix for the degrees of freedom of the unknown $\widehat{u}_h$ of each of these three methods can be made to be *the same* if the stabilization function is properly chosen. This extends a similar result obtained in [10] for an HDG_k method, which is sandwiched between the RT_k and the BDM_k mixed methods, all defined on simplicial meshes. We also provide two local, $H(\mathrm{div})$-conforming flux postprocessings of the HDG approximation that use the spaces of its two sandwiching mixed methods. This extends similar results for the HDG_k method [12] defined on simplicial meshes.

Another approach to superconvergence. The hybrid high-order (HHO) methods introduced in [20, 21] superconverge (in the sense we understand it here) on unstructured meshes of general polyhedral elements. Although the relation between the HHO and HDG methods is far from being apparent, in [9], it was shown that the HHO method is also a superconvergent HDG method (see [8] for the different ways of rewriting HDG methods) even though their spaces do *not* admit M-decompositions. The technique of achieving superconvergence for the HHO methods was thus incorporated into the family of HDG methods. Roughly speaking, it consists in the incorporation of a postprocessing u_h^* into the very definition of the method through a suitable definition of the stabilization function α. This new stabilization function uses the so-called Lehrenfeld-Schöberl projection suggested back in 2010 in [24, Remark 1.2.4] and used later to analyze HDG methods in [25]; other related superconvergent HDG methods were also proposed in [9].

Organization of the paper. In Section 2 we introduce the notion of an M-decomposition and state and then prove our main result, namely, the characterization of M-decompositions of Theorem 2.4. Then by using the definition of the M-decompositions, we obtain, in Section 3, the error analysis previously obtained in [14], and, in Section 4, new stability and superconvergence results. Next, we use the characterization of M-decompositions to introduce, in Section 5, a systematic way of actually constructing M-decompositions, and in Section 6, to get the two above-mentioned results on sandwiching mixed methods. We end in Section 7 with some concluding remarks.

2. The M-decompositions

In this section, we introduce and motivate the notion of an M-decomposition. Then we state and discuss our main result, namely, a characterization of the M-decompositions; see Theorem 2.4. The remainder of the section is devoted to proving this result.

To simplify the notation, when there is no possible confusion, we do not indicate the domain on which the functions of a given space are defined. For example, instead of $\boldsymbol{V}(K)$, we simply write $\boldsymbol{V}$.

2.1. Definition.

The notion of an M-decomposition relates the trace of the normal component of the space of approximate fluxes $\boldsymbol{V} \subset \boldsymbol{H}(\mathrm{div}, K)$ and the trace of the space of approximate scalars $W \subset H^1(K)$ with the space of approximate traces

$M \subset L^2(\partial K)$. To define it, we need to consider the combined trace operator

$$\begin{aligned} \mathrm{tr} : \boldsymbol{V} \times W &\longrightarrow L^2(\partial K) \\ (\boldsymbol{v}, w) &\longmapsto (\boldsymbol{v} \cdot \boldsymbol{n} + w)|_{\partial K}, \end{aligned}$$

where $\boldsymbol{n} : \partial K \to \mathbb{R}^d$ is the unit outward pointing normal field on ∂K. We note that, properly speaking, we are considering vector fields $\boldsymbol{V}$ with divergence in $L^2(K)$ and such that $\boldsymbol{v} \cdot \boldsymbol{n} \in L^2(\partial K)$.

Definition 2.1 (The M-decomposition)**.** We say that $\boldsymbol{V} \times W$ admits an M-decomposition when

(a) $\mathrm{tr}(\boldsymbol{V} \times W) \subset M$,

and there exists a subspace $\widetilde{\boldsymbol{V}} \times \widetilde{W}$ of $\boldsymbol{V} \times W$ satisfying

(b) $\nabla W \times \nabla \cdot \boldsymbol{V} \subset \widetilde{\boldsymbol{V}} \times \widetilde{W}$,

(c) $\mathrm{tr} : \widetilde{\boldsymbol{V}}^\perp \times \widetilde{W}^\perp \to M$ is an isomorphism.

Here $\widetilde{\boldsymbol{V}}^\perp$ and $\widetilde{W}^\perp$ are the $L^2(K)$-orthogonal complements of $\widetilde{\boldsymbol{V}}$ in $\boldsymbol{V}$, and of $\widetilde{W}$ in W, respectively.

As pointed out in the Introduction, this definition can be considered to be a further refinement of the work on sufficient conditions for superconvergence of HDG methods for diffusion done in [14]. It is *essential* to defining the auxiliary projection used to obtain the a priori error analysis obtained in [14] (see Section 3), as well as to obtaining the new stability estimates in Section 4.

Note that, by properties (a) and (c), if $\boldsymbol{V} \times W$ admits an M-decomposition, we have that $M = \mathrm{tr}(\boldsymbol{V} \times W)$. Less obvious is the fact that there are, not two, but *three* orthogonality decompositions associated with an M-decomposition, as we see in the next result.

Proposition 2.2 (The triple orthogonality property)**.** *If $\boldsymbol{V} \times W$ admits an M-decomposition, we have the following three orthogonal decompositions:*

$$\boldsymbol{V} = \widetilde{\boldsymbol{V}} \oplus \widetilde{\boldsymbol{V}}^\perp, \quad W = \widetilde{W} \oplus \widetilde{W}^\perp, \quad M = \gamma \widetilde{\boldsymbol{V}}^\perp \oplus \gamma \widetilde{W}^\perp,$$

where $\gamma \widetilde{\boldsymbol{V}}^\perp := \{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} \, : \, \boldsymbol{v} \in \widetilde{\boldsymbol{V}}^\perp\}$ *and* $\gamma \widetilde{W}^\perp := \{w|_{\partial K} \, : \, w \in \widetilde{W}^\perp\}$.

Proof. We only need to show that the third direct sum is $L^2(\partial K)$-orthogonal. To see this, note that if $(\boldsymbol{v}^\perp, w^\perp) \in \widetilde{\boldsymbol{V}}^\perp \times \widetilde{W}^\perp$, then

$$\langle \boldsymbol{v}^\perp \cdot \boldsymbol{n}, w^\perp \rangle_{\partial K} = (\nabla \cdot \boldsymbol{v}^\perp, w^\perp)_K + (\boldsymbol{v}^\perp, \nabla w^\perp)_K = 0,$$

since $\nabla \cdot \boldsymbol{v}^\perp \in \nabla \cdot \boldsymbol{V} \subset \widetilde{W}$ and $\nabla w^\perp \in \nabla W \subset \widetilde{\boldsymbol{V}}$. This completes the proof. □

The computation of the auxiliary spaces $\widetilde{\boldsymbol{V}}$ and $\widetilde{W}$ is not specified by the definition. We show how to find them in Proposition 2.7.

Finally, let us illustrate the definition of an M-decomposition with the simplest examples. In Table 3, we display four examples of spaces $\boldsymbol{V} \times W$ admitting M-decompositions with $M(\partial K) := \mathcal{P}_0(\partial K)$. For the first two, K is the unit triangle, that is, the triangle with vertices $(0,0), (1,0)$ and $(0,1)$; for the other two, K is the unit square $(0,1) \times (0,1)$. The properties (a) and (b) can be easily verified. To facilitate the verification of property (c), we provide a pictorial representation of the traces of the basis functions: Adjacent to each edge, we write the value of the corresponding trace.

Table 3. Examples of spaces $\boldsymbol{V} \times W$ admitting an M-decomposition for $M(\partial K) := \mathcal{P}_0(\partial K)$. To verify that the isomorphism property (c) holds, we display, in the case of the standard unit triangle and the unit square, the normal trace of the elements of the basis of $\widetilde{\boldsymbol{V}}^{\perp}$ and the trace of the basis of $\widetilde{W}^{\perp}$ (when nonzero). Note that they do form a basis for $M(\partial K)$.

K	basis of $\boldsymbol{V}$	basis of W	$\widetilde{\boldsymbol{V}}$	$\widetilde{W}$	$\widetilde{\boldsymbol{V}}^{\perp}$	$\widetilde{W}^{\perp}$
triangle	$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$	1	$\{\boldsymbol{0}\}$	$\{0\}$	$\boldsymbol{V}$	W
triangle	$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}x\\y\end{bmatrix}$	1	$\{\boldsymbol{0}\}$	W	$\boldsymbol{V}$	$\{0\}$
square	$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}x\\-y\end{bmatrix}$	1	$\{\boldsymbol{0}\}$	$\{0\}$	$\boldsymbol{V}$	W
square	$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}x\\-y\end{bmatrix}, \begin{bmatrix}x\\y\end{bmatrix}$	1	$\{\boldsymbol{0}\}$	W	$\boldsymbol{V}$	$\{0\}$

K	normal trace of the basis of $\widetilde{\boldsymbol{V}}^{\perp}$	trace of the basis of $\widetilde{W}^{\perp}(\neq \{0\})$
triangle	◺ (left -1, top $\frac{1}{\sqrt{2}}$, bottom 0), ◺ (left 0, top $\frac{1}{\sqrt{2}}$, bottom -1)	◺ (left 1, top 1, bottom 1)
triangle	◺ (left -1, top $\frac{1}{\sqrt{2}}$, bottom 0), ◺ (left 0, top $\frac{1}{\sqrt{2}}$, bottom -1), ◺ (left 0, top $\sqrt{2}$, bottom 0)	
square	□ (left -1, top 0, right 1, bottom 0), □ (left 0, top 1, right 0, bottom -1), □ (left 0, top -1, right 1, bottom 0)	□ (left 1, top 1, right 1, bottom 1),
square	□ (left -1, top 0, right 1, bottom 0), □ (left 0, top 1, right 0, bottom -1), □ (left 0, top -1, right 1, bottom 0), □ (left 0, top 1, right 1, bottom 0)	

2.2. **A characterization of M-decompositions.** Next, we give the main result of this section, namely, a characterization of the M-decompositions given solely in terms of the spaces $\boldsymbol{V} \times W$. Roughly speaking, it states that $\boldsymbol{V} \times W$ admits an M-decomposition if and only if the space M is the orthogonal sum of the traces of the kernels of $\nabla\cdot$ in $\boldsymbol{V}$ and of ∇ in W. It is expressed in terms of a special integer that we define next.

Definition 2.3 (The M-index)**.** The M-index of the space $\boldsymbol{V} \times W$ is the number

$$I_M(\boldsymbol{V} \times W) := \dim M - \dim\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} - \dim\{w|_{\partial K} : w \in W, \nabla w = 0\}.$$

Theorem 2.4 (A characterization of M-decompositions)**.** *For a given space of traces M, the space $\boldsymbol{V} \times W$ admits an M-decomposition if and only if*

(a) $\mathrm{tr}(\boldsymbol{V} \times W) \subset M$,
(b) $\nabla W \times \nabla \cdot \boldsymbol{V} \subset \boldsymbol{V} \times W$,
(c) $I_M(\boldsymbol{V} \times W) = 0$.

In this case, we have

$$M = \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v}\in\boldsymbol{V}, \nabla\cdot\boldsymbol{v}=0\} \oplus \{w|_{\partial K} : w\in W, \nabla w = 0\}, \tag{2.1}$$

where the sum is orthogonal.

The importance of this result resides in that it allows us to know if any given space $\boldsymbol{V}\times W$ admits an M-decomposition by just verifying some inclusion properties and by computing a single number, namely, $I_M(\boldsymbol{V}\times W)$—a natural number, by property (a). There is no need of finding the auxiliary subspaces $\widetilde{\boldsymbol{V}}$ and $\widetilde{W}$, something that was *essential* for defining the auxiliary projections for the a priori error analysis of the method. Moreover, this result shows explicitly how M can be expressed in terms of traces of the kernels of the divergence in $\boldsymbol{V}$ and the trace of the kernel of the gradient in W; we call the identity *the kernels' trace decomposition.* This identity is the guiding principle for the systematic construction of M-decompositions we develop in Section 5.

In Table 4, we illustrate this result by verifying the kernels' trace decomposition for the four examples considered in Table 3.

TABLE 4. Verification of the kernels' trace decomposition. For the four examples of Table 3, we display the normal trace of the elements of the basis of $\{\boldsymbol{v}\in\boldsymbol{V} : \nabla\cdot\boldsymbol{v}=0\}$ and the trace of the element of the basis of $\{w\in W : \nabla w = 0\}$. Note that they do form a basis for $M(\partial K)$ and that, as a consequence, $I_M(\boldsymbol{V}\times W)=0$.

K	trace of the basis of $\{\boldsymbol{v}\in\boldsymbol{V} : \nabla\cdot\boldsymbol{v}=0\}$	trace of the basis of $\{w\in W : \nabla w = 0\}$
triangle (both cases)	$-1\,\overset{\frac{1}{\sqrt{2}}}{\underset{0}{\triangle}}$, $\;0\,\overset{\frac{1}{\sqrt{2}}}{\underset{-1}{\triangle}}$	$1\,\overset{1}{\underset{1}{\triangle}}$
square (both cases)	$-1\,\overset{0}{\underset{0}{\square}}\,1$, $\;0\,\overset{1}{\underset{-1}{\square}}\,0$, $\;0\,\overset{-1}{\underset{0}{\square}}\,1$	$1\,\overset{1}{\underset{1}{\square}}\,1$,

Note that the characterization Theorem 2.4 implies that, if M has at least one element with non-zero average on ∂K, then the space W, and hence M, *must* contain the constants. This is the case for the four cases considered in Table 4. However, if in Table 3, we replace $M(\partial K) := \mathcal{P}_0(\partial K)$ and $W := \mathcal{P}_0(K)$ by $M(\partial K) := \{\mu\in\mathcal{P}_0(\partial K) : \langle\mu,1\rangle_{\partial K}=0\}$ and $W := \{0\}$, respectively, then we would still have an M-decomposition in all four cases. This is why it is not necessary that W contain the constants to have an M-decomposition.

2.3. **Properties of the M-decompositions.** Here, we provide several properties of M-decompositions which we are going to use to prove the main result of this section, Theorem 2.4. First, we show that the subspace $\widetilde{W}$ is necessarily unique. Next, we show that this is not true for the subspace $\widetilde{\boldsymbol{V}}$ but that its orthogonal complement $\widetilde{\boldsymbol{V}}^{\perp}$ is defined up to the so-called *solenoidal bubble* functions. Using these results, we introduce what could be called the *canonical* M-decomposition (see Proposition 2.7), since it provides a simple way of computing $\widetilde{\boldsymbol{V}}$ and $\widetilde{W}$ in terms of $\boldsymbol{V}$ and W. We are then going to be ready to prove Theorem 2.4 in the next subsection.

2.3.1. *Uniqueness of* $\widetilde{W}$. We begin by showing that the subspace $\widetilde{W}$ is actually unique.

Proposition 2.5 (Uniqueness of $\widetilde{W}$). *If $\boldsymbol{V} \times W$ admits an M-decomposition, then* $\widetilde{W} = \nabla \cdot \boldsymbol{V}$.

Proof. Since $\nabla \cdot \boldsymbol{V} \subset \widetilde{W}$, we only need to prove that $\widetilde{W} \cap (\nabla \cdot \boldsymbol{V})^{\perp} = \{0\}$. So, if we take $\widetilde{w} \in \widetilde{W}$ satisfying

$$(\widetilde{w}, \nabla \cdot \boldsymbol{v})_K = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V},$$

we have to show that $\widetilde{w} = 0$. To do that, we integrate by parts to get that

$$-(\nabla \widetilde{w}, \boldsymbol{v})_K + \langle \widetilde{w}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V},$$

and, in particular, that

$$\langle \widetilde{w}, \boldsymbol{v}^{\perp} \cdot \boldsymbol{n} \rangle_{\partial K} = 0 \; \forall \boldsymbol{v}^{\perp} \in \widetilde{\boldsymbol{V}}^{\perp}$$

since $\nabla W \subset \widetilde{\boldsymbol{V}}$. This implies that there exists $\widetilde{w}^{\perp} \in \widetilde{W}^{\perp}$ such that $\widetilde{w} + \widetilde{w}^{\perp} = 0$ on ∂K. As a consequence, we have that, for any $\boldsymbol{v} \in \boldsymbol{V}$,

$$(\nabla \widetilde{w}, \boldsymbol{v})_K = -\langle \widetilde{w}^{\perp}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = -(\nabla \widetilde{w}^{\perp}, \boldsymbol{v})_K - (\widetilde{w}^{\perp}, \nabla \cdot \boldsymbol{v})_K = -(\nabla \widetilde{w}^{\perp}, \boldsymbol{v})_K,$$

since $\nabla \cdot \boldsymbol{V} \subset \widetilde{W}$. This is equivalent to $(\nabla(\widetilde{w} + \widetilde{w}^{\perp}), \boldsymbol{v})_K = 0 \; \forall \boldsymbol{v} \in \boldsymbol{V}$, and therefore to $\nabla(\widetilde{w} + \widetilde{w}^{\perp}) = 0$, since $\nabla W \subset \boldsymbol{V}$ by hypothesis. Given the fact that $\widetilde{w} + \widetilde{w}^{\perp} = 0$ on ∂K, this implies that $\widetilde{w} = -\widetilde{w}^{\perp} \in \widetilde{W} \cap \widetilde{W}^{\perp} = \{0\}$ and the proof is complete. ☐

2.3.2. *Relation between the spaces* $\widetilde{\boldsymbol{V}}^{\perp}$. Unlike the subspace $\widetilde{W}^{\perp}$, the subspace $\widetilde{\boldsymbol{V}}^{\perp}$ is not necessarily unique. It is defined up to functions in the space

$$\boldsymbol{V}_{\text{sbb}} := \{\boldsymbol{v} \in \boldsymbol{V} \; : \; \nabla \cdot \boldsymbol{v} = 0, \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0\}.$$

We call this the space of *solenoidal bubbles* in $\boldsymbol{V}$ since the extension by zero of $\boldsymbol{V}_{\text{sbb}}$ lies in $H(\text{div}, \Omega)$. Next, we establish two properties relating the different spaces $\widetilde{\boldsymbol{V}}^{\perp}$.

Proposition 2.6 (Relation between the spaces $\widetilde{\boldsymbol{V}}^{\perp}$). *If $\boldsymbol{V} \times W$ admits two M-decompositions with associated subspaces $\widetilde{\boldsymbol{V}}_1$ and $\widetilde{\boldsymbol{V}}_2$, then*

$$\dim \widetilde{\boldsymbol{V}}_1^{\perp} = \dim \widetilde{\boldsymbol{V}}_2^{\perp} = \dim M - \dim W + \dim \nabla \cdot \boldsymbol{V},$$

and $\widetilde{\boldsymbol{V}}_1^{\perp} + \boldsymbol{V}_{\text{sbb}} = \widetilde{\boldsymbol{V}}_2^{\perp} + \boldsymbol{V}_{\text{sbb}}$.

Proof. Let us calculate the dimension of $\widetilde{\boldsymbol{V}}_i$ for $i = 1, 2$. By property (c) of an M-decomposition, we have that

$$\begin{aligned}
\dim \widetilde{\boldsymbol{V}}_i^{\perp} &= \dim M - \dim \widetilde{W}_i^{\perp} \\
&= \dim M - (\dim W - \dim \widetilde{W}_i) \\
&= \dim M - (\dim W - \dim \nabla \cdot \boldsymbol{V}),
\end{aligned}$$

by Proposition 2.5.

To prove the second equality, it is clear that we only need to show that $\widetilde{\boldsymbol{V}}_1^\perp \subset \widetilde{\boldsymbol{V}}_2^\perp + \boldsymbol{V}_{\rm sbb}$. Let $\widetilde{\boldsymbol{v}}_1^\perp \in \widetilde{\boldsymbol{V}}_1^\perp$. By hypothesis (c) in the definition of an M-decomposition, there exists $(\widetilde{\boldsymbol{v}}_2^\perp, \widetilde{w}_2^\perp) \in \widetilde{\boldsymbol{V}}_2^\perp \times \widetilde{W}_2^\perp$ such that $\widetilde{\boldsymbol{v}}_1^\perp \cdot \boldsymbol{n} = \widetilde{\boldsymbol{v}}_2^\perp \cdot \boldsymbol{n} + \widetilde{w}_2^\perp$ on ∂K. However,

$$\begin{aligned}\langle \widetilde{w}_2^\perp, \widetilde{w}_2^\perp \rangle_{\partial K} &= \langle (\widetilde{\boldsymbol{v}}_1^\perp - \widetilde{\boldsymbol{v}}_2^\perp) \cdot \boldsymbol{n}, \widetilde{w}_2^\perp \rangle_{\partial K} \\ &= (\widetilde{\boldsymbol{v}}_1^\perp - \widetilde{\boldsymbol{v}}_2^\perp, \nabla \widetilde{w}_2^\perp)_K + (\nabla \cdot (\widetilde{\boldsymbol{v}}_1^\perp - \widetilde{\boldsymbol{v}}_2^\perp), \widetilde{w}_2^\perp)_K = 0,\end{aligned}$$

since $\widetilde{\boldsymbol{v}}_1^\perp - \widetilde{\boldsymbol{v}}_2^\perp$ is orthogonal to $\widetilde{\boldsymbol{V}}_1 \cap \widetilde{\boldsymbol{V}}_2 \supset \nabla W$ and $\widetilde{w}_2^\perp$ is orthogonal to $\widetilde{W}_2 \supset \nabla \cdot \boldsymbol{V}$. This proves that $\widetilde{\boldsymbol{v}}_1^\perp \cdot \boldsymbol{n} = \widetilde{\boldsymbol{v}}_2^\perp \cdot \boldsymbol{n}$ on ∂K. Finally,

$$(\nabla \cdot (\widetilde{\boldsymbol{v}}_1^\perp - \widetilde{\boldsymbol{v}}_2^\perp), w)_K = -(\widetilde{\boldsymbol{v}}_1^\perp - \widetilde{\boldsymbol{v}}_2^\perp, \nabla w) = 0 \qquad \forall w \in W,$$

and therefore $\nabla \cdot (\widetilde{\boldsymbol{v}}_1^\perp - \widetilde{\boldsymbol{v}}_2^\perp) = 0$, since W contains all divergences of elements of $\boldsymbol{V}$. This proves that $\widetilde{\boldsymbol{v}}_1^\perp - \widetilde{\boldsymbol{v}}_2^\perp \in \boldsymbol{V}_{\rm sbb}$. $\square$

2.3.3. *The canonical M-decomposition.* The two results above suggest the introduction of what could be called the canonical M-decomposition.

Proposition 2.7 (The canonical M-decomposition)**.** *If the space $\boldsymbol{V} \times W$ admits an M-decomposition, then it admits an M-decomposition based on the subspaces*

$$\widetilde{\boldsymbol{V}} = \nabla W \oplus \boldsymbol{V}_{\rm sbb} \qquad (\textit{orthogonal sum}), \qquad \widetilde{W} = \nabla \cdot \boldsymbol{V}.$$

To prove this proposition, we are going to use the following auxiliary result.

Lemma 2.8. *Let $\boldsymbol{V} \times W$ admit an M-decomposition with associated subspace $\widetilde{\boldsymbol{V}}_1 \times \widetilde{W}$. Then it admits an M-decomposition with associated subspace $\widetilde{\boldsymbol{V}}_2 \times \widetilde{W}$ if $\widetilde{\boldsymbol{V}}_2 \supset \nabla W$ is such that the restricted trace map*

$$\widetilde{\boldsymbol{V}}_2^\perp \ni \boldsymbol{v} \longmapsto \boldsymbol{v} \cdot \boldsymbol{n} \in M_{\boldsymbol{n}} := \{\mu \in M \,:\, \langle \mu, w \rangle_{\partial K} = 0 \quad \forall w \in (\nabla \cdot \boldsymbol{V})^\perp\}$$

is an isomorphism.

Proof. This result follows from the fact that $\widetilde{W} = \nabla \cdot \boldsymbol{V}$, by Proposition 2.5, and by the definition of an M-decomposition. $\square$

We are now ready to prove Proposition 2.7.

Proof. Let us begin by noting that

$$(\nabla w, \boldsymbol{v}_{\rm sbb})_K = -(w, \nabla \cdot \boldsymbol{v}_{\rm sbb})_K + \langle w, \boldsymbol{v}_{\rm sbb} \cdot \boldsymbol{n} \rangle_{\partial K} = 0 \quad \forall w \in W, \ \forall \boldsymbol{v}_{\rm sbb} \in \boldsymbol{V}_{\rm sbb}.$$

This shows that the sum $\nabla W \oplus \boldsymbol{V}_{\rm sbb}$ is orthogonal.

We construct the space $\widetilde{\boldsymbol{V}}$ in two steps. In the first step, we introduce an auxiliary space V as follows. Let $\widetilde{\boldsymbol{V}}_1$ be the space associated to the existing M-decomposition, let $\Pi_{\rm sbb} : \boldsymbol{V} \to \boldsymbol{V}_{\rm sbb}$ be the $\boldsymbol{L}^2(K)$-orthogonal projector onto $\boldsymbol{V}_{\rm sbb}$, and let us set

$$\mathsf{V} := \{\widetilde{\boldsymbol{v}}^\perp - \Pi_{\rm sbb} \widetilde{\boldsymbol{v}}^\perp : \widetilde{\boldsymbol{v}}^\perp \in \widetilde{\boldsymbol{V}}_1^\perp\}.$$

In the second step, we prove that $\widetilde{\boldsymbol{V}} = \mathsf{V}^\perp$.

Let us prove this identity. First, we show that $\widetilde{\boldsymbol{V}} \subset \mathsf{V}^\perp$, that is, that $\nabla W \subset \mathsf{V}^\perp$ and $\boldsymbol{V}_{\rm sbb} \subset \mathsf{V}^\perp$. The second inclusion follows by the definition of V. Let us prove the first step. Take any $w \in W$ and any $\mathsf{v} \in \mathsf{V}$. By construction, there is $\widetilde{\boldsymbol{v}}^\perp \in \widetilde{\boldsymbol{V}}_1^\perp$ such that $\mathsf{v} = \widetilde{\boldsymbol{v}}^\perp - \Pi_{\rm sbb} \widetilde{\boldsymbol{v}}^\perp$ and so,

$$(\nabla w, \mathsf{v})_K = (\nabla w, \widetilde{\boldsymbol{v}}^\perp - \Pi_{\rm sbb} \widetilde{\boldsymbol{v}}^\perp)_K = (\nabla w, \widetilde{\boldsymbol{v}}^\perp)_K = 0,$$

because $\widetilde{\boldsymbol{V}}_1 \supset \nabla W$. This implies that $\nabla W \subset \mathsf{V}^\perp$ and hence that $\widetilde{\boldsymbol{V}} \subset \mathsf{V}^\perp$.

Next, let us show that $\boldsymbol{V} \times W$ admits an M-decomposition with the associated subspace $\mathbf{V}^{\perp} \times \widetilde{W}$. By definition of $\mathbf{V}$, it is clear that $\dim \mathbf{V} \leq \dim \widetilde{\boldsymbol{V}}_1^{\perp}$. Noting that $(\Pi_{\rm sbb}\boldsymbol{v}) \cdot \boldsymbol{n} = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}$, it follows that the range of the normal trace operators $\boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}$ from $\mathbf{V}$ and from $\widetilde{\boldsymbol{V}}_1^{\perp}$ is the same. From Lemma 2.8, it follows that $\widetilde{\boldsymbol{V}}_1^{\perp} \ni \boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n} \in M_{\boldsymbol{n}}$ is an isomorphism of finite dimensional spaces, and therefore the normal trace mapping from $\mathbf{V}$ to $M_{\boldsymbol{n}}$ is also an isomorphism. Since we have that $\nabla W \subset \mathbf{V}^{\perp}$, this implies that $\boldsymbol{V} \times W$ admits an M-decomposition with the associated subspace $\mathbf{V}^{\perp} \times \widetilde{W}$.

It remains to show that $\widetilde{\boldsymbol{V}}^{\perp} \cap \mathbf{V}^{\perp} = \{\mathbf{0}\}$, which proves the reverse inclusion. Then let $\mathbf{v}^{\perp} \in \mathbf{V}^{\perp}$ satisfy

$$(\mathbf{v}^{\perp}, \nabla w + \boldsymbol{v}_{\rm sbb})_K = 0 \qquad \forall w \in W,\ \forall \boldsymbol{v}_{\rm sbb} \in \boldsymbol{V}_{\rm sbb}. \tag{2.2}$$

Then, for all $w^{\perp} \in \widetilde{W}^{\perp} = (\nabla \cdot \boldsymbol{V})^{\perp}$, we have that

$$0 = (\mathbf{v}^{\perp}, \nabla w^{\perp})_K = -(\nabla \cdot \mathbf{v}^{\perp}, w^{\perp})_K + \langle \mathbf{v}^{\perp} \cdot \boldsymbol{n}, w^{\perp}\rangle_{\partial K} = \langle \mathbf{v}^{\perp} \cdot \boldsymbol{n}, w^{\perp}\rangle_{\partial K}.$$

Since $\boldsymbol{V} \times W$ admits an M-decomposition with the associated subspace $\mathbf{V}^{\perp} \times \widetilde{W}$, we have that $M = \gamma\mathbf{V} \oplus \gamma\widetilde{W}^{\perp}$ with orthogonal sum. Hence, there exists $\mathbf{v} \in \mathbf{V}$ such that $(\mathbf{v} + \mathbf{v}^{\perp}) \cdot \boldsymbol{n} = 0$ on ∂K. Using (2.2) it follows that, for all $w \in W$,

$$\begin{aligned} 0 &= (\mathbf{v}^{\perp}, \nabla w)_K \\ &= -(\nabla \cdot \mathbf{v}^{\perp}, w) - \langle \mathbf{v} \cdot \boldsymbol{n}, w\rangle_{\partial K} \\ &= -(\nabla \cdot (\mathbf{v}^{\perp} + \mathbf{v}), w)_K - (\mathbf{v}, \nabla w)_K \\ &= -(\nabla \cdot (\mathbf{v}^{\perp} + \mathbf{v}), w)_K, \end{aligned}$$

because $\nabla W \subset \mathbf{V}^{\perp}$. Therefore $\nabla \cdot (\mathbf{v}^{\perp} + \mathbf{v}) = 0$, and thus $\mathbf{v}^{\perp} + \mathbf{v} \in \boldsymbol{V}_{\rm sbb}$. Finally, by the identity (2.2) with $\boldsymbol{v}_{\rm sbb} := \mathbf{v}^{\perp} + \mathbf{v}$ and $w := 0$, we get that

$$0 = (\mathbf{v}^{\perp}, \mathbf{v}^{\perp} + \mathbf{v})_K = (\mathbf{v}^{\perp}, \mathbf{v}^{\perp})_K,$$

and therefore $\mathbf{v}^{\perp} = \mathbf{0}$. This completes the proof. □

2.4. **Proof of the characterization of M-decompositions theorem.** Armed with the concept of the canonical M-decomposition, we are now ready to prove Theorem 2.4.

Since properties (a) and (b) are part of the definition of an M-decomposition, we can always assume they hold. Since, by Proposition 2.7, $\boldsymbol{V} \times W$ admits an M-decomposition if and only if it admits the canonical decomposition, we can always take the choice $\widetilde{\boldsymbol{V}} := \nabla W \oplus \boldsymbol{V}_{\rm sbb}$ and $\widetilde{W} := \nabla \cdot \boldsymbol{V}$. Next, we prove that properties (a) and (b) imply

$$\dim \gamma\widetilde{\boldsymbol{V}}^{\perp} = \dim \widetilde{\boldsymbol{V}}^{\perp}, \quad \dim \gamma\widetilde{W}^{\perp} = \dim \widetilde{W}^{\perp}. \tag{2.3}$$

Let us show that the last equality always holds. Indeed, if $\widetilde{w}^{\perp} \in \widetilde{W}^{\perp}$ and is zero on ∂K, then, for any $\boldsymbol{v} \in \boldsymbol{V}$, we have

$$0 = \langle \widetilde{w}^{\perp}, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial K} = (\nabla \widetilde{w}^{\perp}, \boldsymbol{v})_K + (\widetilde{w}^{\perp}, \nabla \cdot \boldsymbol{v})_K = (\nabla \widetilde{w}^{\perp}, \boldsymbol{v})_K$$

since $\widetilde{W} = \nabla \cdot \boldsymbol{V}$. Since $\boldsymbol{V} \supset \nabla W$, we can now take $\boldsymbol{v} := \nabla \widetilde{w}^{\perp}$ and conclude that $\widetilde{w}^{\perp}$ is a constant on K. As a consequence $\widetilde{w}^{\perp} = 0$, and the last equality follows.

Let us now show that the first equality in (2.3) also holds. Indeed, if $\widetilde{\boldsymbol{v}}^\perp \in \widetilde{\boldsymbol{V}}^\perp$ and its normal trace is zero on ∂K, then, for any $w \in W$, we have

$$0 = \langle w, \widetilde{\boldsymbol{v}}^\perp \cdot \boldsymbol{n}\rangle_{\partial K} = (\nabla w, \widetilde{\boldsymbol{v}}^\perp)_K + (w, \nabla \cdot \widetilde{\boldsymbol{v}}^\perp)_K = (w, \nabla \cdot \widetilde{\boldsymbol{v}}^\perp)_K$$

since $\widetilde{\boldsymbol{V}} \supset \nabla W$. Since $W \supset \nabla \cdot \boldsymbol{V}$, we can now take $w := \nabla \cdot \widetilde{\boldsymbol{v}}^\perp$ and conclude that $\nabla \cdot \widetilde{\boldsymbol{v}}^\perp$ is zero on K. As a consequence, $\widetilde{\boldsymbol{v}}^\perp \in \boldsymbol{V}_{\text{sbb}} \subset \widetilde{\boldsymbol{V}}$, we have that $\widetilde{\boldsymbol{v}}^\perp = \boldsymbol{0}$, and the result follows.

On the other hand, we have that the trace operator $\text{tr} : \widetilde{\boldsymbol{V}}^\perp \times \widetilde{W}^\perp \to M$ is an isomorphism if and only if

$$\dim M = \dim \gamma \widetilde{\boldsymbol{V}}^\perp + \dim \gamma \widetilde{W}^\perp.$$

Thus, we have to show that this equality holds if and only if $I_M(\boldsymbol{V} \times W) = 0$, assuming properties (a) and (b). However, we have that

$$\begin{aligned} I &:= \dim M - \dim \widetilde{\boldsymbol{V}}^\perp - \dim \widetilde{W}^\perp \\ &= \dim M - (\dim \boldsymbol{V} - \dim \widetilde{\boldsymbol{V}}) - (\dim W - \dim \widetilde{W}) \\ &= \dim M - (\dim \boldsymbol{V} - \dim \nabla W - \dim \boldsymbol{V}_{\text{sbb}}) - (\dim W - \dim \nabla \cdot \boldsymbol{V}) \end{aligned}$$

by the definition of $\widetilde{\boldsymbol{V}}$ and $\widetilde{W}$. After rearranging terms, we get that

$$\begin{aligned} I &= \dim M - (\dim \boldsymbol{V} - \dim \nabla \cdot \boldsymbol{V} - \dim \boldsymbol{V}_{\text{sbb}}) - (\dim W - \dim \nabla W) \\ &= \dim M - (\dim\{\boldsymbol{v} \in \boldsymbol{V} : \nabla \cdot \boldsymbol{v} = 0\} - \dim\{\boldsymbol{v} \in \boldsymbol{V} : \nabla \cdot \boldsymbol{v} = 0, \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} = 0\}) \\ &\quad - \dim\{w \in W : \nabla w = 0\} \\ &= \dim M - \dim\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} - \dim\{w|_{\partial K} : w \in W, \nabla w = 0\} \\ &= I_M(\boldsymbol{V} \times W), \end{aligned}$$

and the result follows.

Now, by the inclusion property (a), we have that

$$\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W, \nabla w = 0\} \subset M,$$

where the sum is $L^2(\partial K)$-orthogonal since

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}, w\rangle_{\partial K} = (\nabla \cdot \boldsymbol{v}, w)_K + (\boldsymbol{v}, \nabla w)_K = 0$$

if $\nabla \cdot \boldsymbol{v} = 0$ and $\nabla w = 0$. Finally, since the M-index is zero by property (c), the equality holds. This completes the proof of the characterization Theorem 2.4.

<u>The triple orthogonal decomposition</u>:

$$\boldsymbol{V} = \widetilde{\boldsymbol{V}} \oplus \widetilde{\boldsymbol{V}}^\perp, \qquad W = \widetilde{W} \oplus \widetilde{W}^\perp, \qquad M = \gamma \widetilde{\boldsymbol{V}}^\perp \oplus \gamma \widetilde{W}^\perp.$$

<u>The canonical (orthogonal) decomposition</u>:

$$\boldsymbol{V} = \underbrace{\nabla W \oplus \boldsymbol{V}_{\text{sbb}}}_{\widetilde{\boldsymbol{V}}} \oplus \widetilde{\boldsymbol{V}}^\perp, \qquad W = \underbrace{(\nabla \cdot \boldsymbol{V})}_{\widetilde{W}} \oplus \widetilde{W}^\perp.$$

<u>The kernels' trace (orthogonal) decomposition</u>:

$$M = \{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W, \nabla w = 0\}.$$

2.5. **Summary.** We end this section by succinctly putting together the main results about M-decompositions, namely, the triple orthogonal decomposition, Proposition 2.2, the canonical decomposition, Proposition 2.7, and the characterization of M-decompositions, Theorem 2.4.

3. Recovering previous results

Here we briefly show that, when the local spaces $\boldsymbol{V}\times W$ admit M-decompositions, we do recover the estimates of the projection of the errors obtained in [14]. In what follows, Id denotes the identity operator and the operators P_W, $P_{\boldsymbol{V}}$, and P_M are the L^2 projections onto W, $\boldsymbol{V}$, and M, respectively. Moreover, we set

$$e_{\boldsymbol{q}} := \boldsymbol{q}-\boldsymbol{q}_h,\quad \varepsilon_{\boldsymbol{q}} := \boldsymbol{\Pi}_V\boldsymbol{q}-\boldsymbol{q}_h,\quad \varepsilon_u := \Pi_W u-u_h \quad \text{and} \quad \varepsilon_{\widehat{u}} := P_M u-\widehat{u}_h.$$

3.1. **Existence and uniqueness of the auxiliary projection.** Let us begin by recalling the definition of the auxiliary projection.

Definition 3.1 ([14] The auxiliary HDG-projection)**.** Let $(\boldsymbol{q},u)$ be smooth enough so that their boundary traces are in $L^2(\partial K)$. Let $\boldsymbol{V}\times W$ admit an M-decomposition. Then, the pair $\Pi_h(\boldsymbol{q},u)=(\boldsymbol{\Pi}_V\boldsymbol{q},\Pi_W u)\in\boldsymbol{V}\times W$ defined by the equations

$$\begin{aligned}
(\Pi_W u-u,w)_K &= 0 && \forall w\in\widetilde{W},\\
(\boldsymbol{\Pi}_V\boldsymbol{q}-\boldsymbol{q},\boldsymbol{v})_K &= 0 && \forall \boldsymbol{v}\in\widetilde{\boldsymbol{V}},\\
\langle\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}+\alpha(\Pi_W u),\mu\rangle_{\partial K} &= \langle\boldsymbol{q}\cdot\boldsymbol{n}+\alpha(P_M u),\mu\rangle_{\partial K} && \forall\mu\in M,
\end{aligned}$$

is the auxiliary HDG-projection associated to the M-decomposition.

The HDG-projection depends on a linear stabilization operator $\alpha : L^2(\partial K)\to L^2(\partial K)$. Sufficient hypotheses on α ensuring that Π_h is actually well defined are given next. The following proposition is a small variation of a similar result obtained in [14].

Proposition 3.2 (The HDG-projection)**.** *Let $\boldsymbol{V}\times W$ admit an M-decomposition. Then the auxiliary HDG-projection Π_h is well defined if we take the linear stabilization operator $\alpha : L^2(\partial K)\to L^2(\partial K)$ such that*

$$w\in\widetilde{W}^{\perp}:\quad \langle\alpha(w),w\rangle_{\partial K}=0 \quad\Longrightarrow\quad w=0. \tag{3.1}$$

Note that we can take the stabilization function α equal to zero if $\widetilde{W}^{\perp}=\{0\}$, that is, if $W=\nabla\cdot\boldsymbol{V}$. This suggests the introduction of the following number.

Definition 3.3 (The S-index)**.** The S-index ("S" for stabilization) of the space $\boldsymbol{V}\times W$ is the number

$$I_S(\boldsymbol{V}\times W) := \dim W-\dim\nabla\cdot\boldsymbol{V}.$$

Note also that, for any $w\in W$, we have

$$\begin{aligned}
(P_W\nabla\cdot\boldsymbol{q},w)_K &= -(\boldsymbol{q},\nabla w)_K+\langle\boldsymbol{q}\cdot\boldsymbol{n},w\rangle_{\partial K}\\
&= -(\boldsymbol{\Pi}_V\boldsymbol{q},\nabla w)_K+\langle\boldsymbol{\Pi}_V\boldsymbol{q}\cdot\boldsymbol{n}+\alpha(\Pi_W u-P_M u),w\rangle_{\partial K}\\
&= (\nabla\cdot\boldsymbol{\Pi}_V\boldsymbol{q},w)_K+\langle\alpha(\Pi_W u-P_M u),w\rangle_{\partial K},
\end{aligned}$$

and if we define $L_W(m)$ as the element of W such that

$$(L_W(m),w)_K=\langle m,w\rangle_{\partial K}\quad\forall\, w\in W,$$

then we can write

$$P_W \nabla \cdot \boldsymbol{q} = \nabla \cdot \boldsymbol{\Pi}_V \boldsymbol{q} + L_W(\alpha(\Pi_W u - P_M u)).$$

This extends to our framework the commutativity properties of the projections P_W and $\boldsymbol{\Pi}_V$ for the mixed methods; that is, for the case in which we can take $\alpha = 0$.

Let us now prove Proposition 3.2.

Proof. Let us start by noting that, if $\boldsymbol{V} \times W$ admits an M-decomposition with associated spaces $\widetilde{\boldsymbol{V}}$ and $\widetilde{W}$, then, by the third condition defining an M-decomposition,

$$\dim \boldsymbol{V} + \dim W = \dim \widetilde{\boldsymbol{V}} + \dim \widetilde{W} + \dim M. \tag{3.2}$$

This means that the system defining the projection is square and we only have to prove uniqueness.

Thus, if we set $(\boldsymbol{q}, u) = (\boldsymbol{0}, 0)$, the first two equations defining the projection show that $\boldsymbol{\Pi}_V \boldsymbol{q}$ belongs to $\widetilde{\boldsymbol{V}}^\perp$, and that $\Pi_W u$ belongs to $\widetilde{W}^\perp$. Since $\gamma W \subset M$, we can take $\mu := \Pi_W u$ in the third equation defining the projection to get

$$\langle \alpha(\Pi_W u), \Pi_W u \rangle_{\partial K} = -\langle \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n}, \Pi_W u \rangle_{\partial K} = 0,$$

since the sum $\gamma \widetilde{\boldsymbol{V}}^\perp \oplus \gamma \widetilde{W}^\perp$ is orthogonal. Therefore, by the assumption on the stabilization function α, it follows that $\Pi_W u = 0$. Finally, since $\gamma \boldsymbol{V} \subset M$, we can take $\mu := \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n}$ in the third equation defining the projection to get

$$\langle \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n}, \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n} \rangle_{\partial K} = 0$$

which implies, by the third condition defining an M-decomposition, that $\boldsymbol{\Pi}_V \boldsymbol{q} = \boldsymbol{0}$. This completes the proof. ☐

3.2. **Approximation properties of the auxiliary projection.** Note that, in view of the second equation defining the auxiliary HDG-projection, one might think that its approximation properties depend on the choice of the subspace $\widetilde{\boldsymbol{V}}$. This would be rather unpleasant given that, unlike the subspace $\widetilde{W}$, the subspace $\widetilde{\boldsymbol{V}}$ of an M-decomposition is *not* uniquely defined. Fortunately, this is not so as we see in the next result which is a small variation of a similar result in [14]; for the sake of completeness, we include a proof in the Appendix. To state it, we need to introduce the quantities

$$a_{\widetilde{W}^\perp} := \begin{cases} \inf_{\mu \in \gamma \widetilde{W}^\perp \setminus \{0\}} \langle \alpha(\mu), \mu \rangle_{\partial K} / \|\mu\|^2_{\partial K} & \text{if } \widetilde{W}^\perp \neq \{0\}, \\ \infty & \text{if } \widetilde{W}^\perp = \{0\}, \end{cases}$$

and

$$\|\alpha\| := \sup_{\lambda, \mu \in M \setminus \{0\}} \langle \alpha(\lambda), \mu \rangle_{\partial K} / (\|\lambda\|_{\partial K} \|\mu\|_{\partial K}).$$

When $\widetilde{W}^\perp = \{0\}$, i.e., when $\widetilde{W} = W$, we will assume that $\alpha = 0$.

Proposition 3.4 (Approximation properties of the HDG-projection)**.** *Let $\boldsymbol{V} \times W$ admit an M-decomposition and let the stabilization function α satisfy the condition*

$$a_{\widetilde{W}^\perp} > 0.$$

Then, we have

$$\begin{aligned}
\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_K \le{}& \|(Id - P_{\boldsymbol{V}})\, \boldsymbol{q}\|_K + \mathsf{C}_1\, h_K^{1/2}\, \|((Id - P_{\boldsymbol{V}})\boldsymbol{q}) \cdot \boldsymbol{n}\|_{\partial K}\\
&+ \mathsf{C}_2\, h_K\, \|(Id - P_{\widetilde{W}})\nabla \cdot \boldsymbol{q}\|_K + \mathsf{C}_3\, h_K^{1/2}\, \|(Id - P_W)u\|_{\partial K},\\
\|u - \Pi_W u\|_K \le{}& \|(Id - P_W)u\|_K + \mathsf{C}_4\, h_K^{1/2}\, \|(Id - P_W)u\|_{\partial K}\\
&+ \mathsf{C}_5\, h_K\, \|(Id - P_{\widetilde{W}})\nabla \cdot \boldsymbol{q}\|_K,
\end{aligned}$$

where $\mathsf{C}_1 := C_{\widetilde{\boldsymbol{V}}^\perp}$ *and*

$$\begin{aligned}
\mathsf{C}_2 &:= \frac{C_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}}\, C_{\widetilde{\boldsymbol{V}}^\perp}\, \|\alpha\|, \quad \mathsf{C}_3 := \left(1 + \frac{\|\alpha\|}{a_{\widetilde{W}^\perp}}\right) C_{\widetilde{\boldsymbol{V}}^\perp}\, \|\alpha\|,\\
\mathsf{C}_4 &:= \frac{C_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}}\, \|\alpha\|, \qquad\qquad \mathsf{C}_5 := \frac{C^2_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}}.
\end{aligned}$$

Here

$$\begin{aligned}
C_{\widetilde{\boldsymbol{V}}^\perp} &:= \sup_{\boldsymbol{v} \in \widetilde{\boldsymbol{V}}^\perp \setminus \{\boldsymbol{0}\}} h_K^{-1/2}\, \|\boldsymbol{v}\|_K / \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{\partial K},\\
C_{\widetilde{W}^\perp} &:= \sup_{w \in \widetilde{W}^\perp \setminus \{0\}} h_K^{-1/2}\, \|w\|_K / \|w\|_{\partial K}.
\end{aligned}$$

Note that the fact that the coercivity constant $a_{\widetilde{W}^\perp}$ is positive implies the property (3.1) of the stabilization function α used in Proposition 3.2; this is due to the third condition in the definition of M-decomposition. Note also that, if $W = \widetilde{W} = \nabla \cdot\ \boldsymbol{V}$, then $\mathsf{C}_i = 0$ for $i = 2, 3, 4, 5$ since in this case we are taking $a_{\widetilde{W}^\perp} = \infty$.

Finally, note that the above error estimates depend on the choice of the space $\widetilde{\boldsymbol{V}}$ only through the stability constant $C_{\widetilde{\boldsymbol{V}}^\perp}$. The constants $C_{\widetilde{\boldsymbol{V}}^\perp}$ and $C_{\widetilde{W}^\perp}$ are optimal bounds for inverse inequalities bounding the $L^2(K)$-norm of $\boldsymbol{v} \in \widetilde{\boldsymbol{V}}^\perp$ and $w \in \widetilde{W}^\perp$ by the $L^2(\partial K)$-norm of their respective traces.

3.3. **Estimates of the projection of the errors.** In Section 2, we have been working on general choices $\boldsymbol{V}, W$ and M on a fixed element. We now revert to the situation of the introduction, where Ω is partitioned into polyhedral elements and where $M(\partial K) = \prod_{F \in \mathcal{F}(K)} M(F)$.

Next, we obtain the error estimates of [14]. We introduce a (slightly different) postprocessing of the scalar variable u_h^* which we take in the space

(3.3a) $$W_h^* := \{w \in L^2(\mathcal{T}_h) :\ w|_K \in W^*(K),\ K \in \mathcal{T}_h\},$$

and define as follows. On each element $K \in \mathcal{T}_h$, the function u_h^* is the element of $W^*(K)$ such that

(3.3b) $$(\nabla u_h^*, \nabla w)_K = -\,(c\, \boldsymbol{q}_h, \nabla w)_K \quad \forall\, w \in \widetilde{W}^*(K)^\perp,$$

(3.3c) $$(u_h^*, w)_K = (u_h, w)_K \qquad \forall\, w \in \widetilde{W}^*(K),$$

where $W^*(K) = \widetilde{W}^*(K) \oplus \widetilde{W}^*(K)^\perp$ and $\widetilde{W}^*(K)$ is any non-trivial subspace of $\widetilde{W}(K)$ containing constant functions. We have the following result which follows directly from the analysis carried out in [14].

Theorem 3.5 (A priori error estimates)**.** *Suppose that for every $K \in \mathcal{T}_h$, the space $\boldsymbol{V}(K)\times W(K)$ admits an $M(\partial K)$-decomposition and that the stabilization function α satisfies the following properties:*

(i) $w \in \widetilde{W}^\perp(K), \quad \langle \alpha(w), w\rangle_{\partial K} = 0 \quad \Longrightarrow \qquad w = 0,$
(ii) $\langle \alpha(\mu), \mu\rangle_{\partial K} \geq 0$ *for all* $\mu \in M(\partial K)$,
(iii) $\langle \alpha(\lambda), \mu\rangle_{\partial K} = \langle \lambda, \alpha(\mu)\rangle_{\partial K}$, *for all* $\lambda, \mu \in M(\partial K)$.

Then, for the solution of (1.1)*, we have*

$$\|\boldsymbol{\Pi}_V \boldsymbol{q} - \boldsymbol{q}_h\|_{\mathcal{T}_h} \leq \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\mathcal{T}_h},$$
$$\|\Pi_W u - u_h\|_{\mathcal{T}_h} \leq C\,H\,\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\mathcal{T}_h},$$

where $H = 1$ for general polyhedral domains. For convex polyhedral domains, we have that $H = h$ provided $\boldsymbol{\mathcal{P}}_0(K) \subset \nabla W(K)$ for all elements $K \in \mathcal{T}_h$. Moreover, if $\mathcal{P}_0(K) \subset \nabla\cdot \boldsymbol{V}(K)$ for all elements $K \in \mathcal{T}_h$, then

$$\|u - u_h^*\|_{\mathcal{T}_h} \leq \|\Pi_W u - u_h\|_{\mathcal{T}_h} + C\,h\,(\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\mathcal{T}_h} + \inf_{\omega\in W_h^*} \|\nabla(u-\omega)\|_{\mathcal{T}_h}).$$

Note that this result guarantees the superconvergence of the HDG (or mixed, if $\alpha = 0$) method *provided* that $\boldsymbol{\mathcal{P}}_0(K) \subset \nabla W(K)$ and that $\mathcal{P}_0(K) \subset \nabla\cdot\boldsymbol{V}(K)$. No similar condition is required for $\boldsymbol{V}(K)\times W(K)$ to admit an $M(\partial K)$-decomposition though.

4. New stability and superconvergence estimates

In this section, we obtain new stability and superconvergence results.

4.1. **Existence and uniqueness of the auxiliary adjoint HDG projection.**

We begin by introducing an auxiliary adjoint HDG projection onto $\boldsymbol{V}\times W$ for functions in the finite dimensional space $\boldsymbol{V}\times W\times M$.

Definition 4.1 (The auxiliary adjoint HDG projection)**.** Let $\boldsymbol{V}\times W$ admit an M-decomposition. Let $d := (\mathbf{d}_v, \mathrm{d}_w, \mathrm{d}_\mu) \in \boldsymbol{V}\times W\times M$. Then, the pair $\Pi_h^* d := (\boldsymbol{\Pi}_V^* d,\, \Pi_W^* d) \in \boldsymbol{V}\times W$ defined by the equations

$$\begin{aligned}
(\Pi_W^* d, w)_K &= (\mathrm{d}_w, w)_K && \forall w \in \widetilde{W},\\
(\boldsymbol{\Pi}_V^* d, \boldsymbol{v})_K &= (\mathbf{d}_v, \boldsymbol{v})_K && \forall \boldsymbol{v} \in \widetilde{\boldsymbol{V}},\\
\langle \boldsymbol{\Pi}_V^* d\cdot\boldsymbol{n} - \alpha(\Pi_W^* d), \mu\rangle_{\partial K} &= \langle \mathrm{d}_\mu, \mu\rangle_{\partial K} && \forall \mu \in M,
\end{aligned}$$

is the auxiliary adjoint HDG projection associated to the M-decomposition.

It is easy to see that this adjoint is defined whenever the HDG projection is. In fact, a glance to the definition of the auxiliary HDG projection, (3.1), allows us to see that $(\boldsymbol{\Pi}_V\boldsymbol{q}, \Pi_W u) = (-\boldsymbol{\Pi}_V^* d,\, \Pi_W^* d)$ for $d = (-\boldsymbol{q}, u, -\boldsymbol{q}\cdot\boldsymbol{n} - \alpha(P_M u))$.

4.2. **Approximation properties of the auxiliary adjoint projection.** We have the following bounds for the adjoint projection.

Proposition 4.2 (Stability of the adjoint HDG projection)**.** *Let $\boldsymbol{V} \times W$ admit an M-decompositon and let the stabilization function α satisfy the condition*

$$a_{\widetilde{W}^\perp} > 0.$$

Then, we have, for data $d := (\mathbf{d}_v, \mathrm{d}_w, \mathrm{d}_\mu) \in \nabla W \times \nabla \cdot \boldsymbol{V} \times M$,

$$\|\boldsymbol{\Pi}_V^* d\|_K \le C_1 \|\mathrm{d}_w\|_K + C_3 \|\mathbf{d}_v\|_K + C_5\, h_K^{1/2}\, \|d_\mu\|_{\partial K},$$
$$\|\Pi_W^* d\|_K \le C_2 \|\mathrm{d}_w\|_K + C_4 \|\mathbf{d}_v\|_K + C_6\, h_K^{1/2}\, \|d_\mu\|_{\partial K},$$

where

$$C_1 := C_{\widetilde{\boldsymbol{V}}^\perp}\, C_{\nabla\cdot\boldsymbol{V}}\, \|\alpha\| \left(1 + \frac{\|\alpha\|}{a_{\widetilde{W}^\perp}}\right), \quad C_2 := 1 + C_{\nabla\cdot\boldsymbol{V}}\, \|\alpha\|\, \frac{C_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}},$$
$$C_3 := 1 + C_{\widetilde{\boldsymbol{V}}^\perp}\, C_{\nabla W} \left(1 + \frac{\|\alpha\|}{a_{\widetilde{W}^\perp}}\right), \quad C_4 := C_{\nabla W}\, \frac{C_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}},$$
$$C_5 := C_{\widetilde{\boldsymbol{V}}^\perp} \left(1 + \frac{\|\alpha\|}{a_{\widetilde{W}^\perp}}\right), \qquad C_6 := \frac{C_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}}.$$

Here

$$C_{\nabla W} := \sup_{\mathbf{0} \neq \boldsymbol{v} \in \nabla W} h_K^{1/2}\, \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{\partial K} / \|\boldsymbol{v}\|_K,$$
$$C_{\nabla\cdot\boldsymbol{V}} := \sup_{\boldsymbol{v} \in \boldsymbol{V} \,:\, \nabla\cdot\boldsymbol{v} \neq 0} h_K^{1/2}\, \|\nabla\cdot\boldsymbol{v}\|_{\partial K} / \|\nabla\cdot\boldsymbol{v}\|_K.$$

A proof is sketched in the Appendix. It is very similar to the proof of the approximation properties of the HDG projection in Proposition 3.4.

4.3. **New stability and superconvergence estimates.** We are now ready to obtain the new estimates. Similar results were obtained in an effort of obtaining inf-sup conditions in [7, Theorems 3.1, 3.2], for the so-called staggered DG method on simplexes, and then in [22, Proposition 3], for the Raviart-Thomas method also in simplexes.

Theorem 4.3. *Let $\boldsymbol{V}(K) \times W(K)$ admit an $M(\partial K)$-decomposition and let the stabilization function α satisfy the following properties:*

(i) $a_{\widetilde{W}^\perp} > 0$,
(ii) $\langle \alpha(\lambda), \mu\rangle_{\partial K} = \langle \lambda, \alpha(\mu)\rangle_{\partial K}$ *for all* $\lambda, \mu \in M$.

Then, for the solution of (1.1)*, we have the following local stability estimates:*

$$\|\nabla\cdot\boldsymbol{q}_h\|_K \le C_1\, \|\mathrm{c}\|_{L^\infty(K)}\, \|\boldsymbol{q}_h\|_K + C_2\, \|P_W f\|_K,$$
$$\|\nabla u_h\|_K \le C_3\, \|\mathrm{c}\|_{L^\infty(K)}\, \|\boldsymbol{q}_h\|_K + C_4\, \|P_{\widetilde{W}^\perp} f\|_K,$$
$$h_K^{-1/2}\, \|u_h - \widehat{u}_h\|_{\partial K} \le C_5\, \|\mathrm{c}\|_{L^\infty(K)}\, \|\boldsymbol{q}_h\|_K + C_6\, \|P_{\widetilde{W}^\perp} f\|_K.$$

Moreover, we also have the following superconvergence results:

$$\|\nabla\cdot\varepsilon_{\boldsymbol{q}}\|_K \le C_1\, \|\mathrm{c}\|_{L^\infty(K)}\, \|e_{\boldsymbol{q}}\|_K,$$
$$\|\nabla\varepsilon_u\|_K \le C_3\, \|\mathrm{c}\|_{L^\infty(K)}\, \|e_{\boldsymbol{q}}\|_K,$$
$$h_K^{-1/2}\, \|\varepsilon_u - \varepsilon_{\widehat{u}}\|_{\partial K} \le C_5\, \|\mathrm{c}\|_{L^\infty(K)}\, \|e_{\boldsymbol{q}}\|_K.$$

Proof. Let us prove the stability estimates. To do that, we begin by noting that we can rewrite the first two equations defining the HDG methods on each element K as

$$\begin{aligned}(c\,\boldsymbol{q}_h, \boldsymbol{v})_K + (\nabla u_h, \boldsymbol{v})_K - \langle u_h - \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0,\\ (\nabla\cdot\boldsymbol{q}_h, w)_K + \langle u_h - \widehat{u}_h, \alpha(w)\rangle_{\partial K} &= (f, w)_K,\end{aligned}$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K)\times W(K)$. Note that in the last term of the left-hand side of the second equation, we have used the fact that the stabilization operator α is self-adjoint; see hypothesis (ii) above. Adding these equations, we get

$$\begin{aligned}(c\,\boldsymbol{q}_h, \boldsymbol{v})_K + (\nabla u_h, \boldsymbol{v})_K + (\nabla\cdot\boldsymbol{q}_h, w)_K&\\ -\langle u_h - \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n} - \alpha(w)\rangle_{\partial K} &= (f, w)_K,\end{aligned}\tag{4.1}$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K)\times W(K)$. Therefore, testing with $(\boldsymbol{v}, w) := (\boldsymbol{\Pi}_V^* d, \Pi_W^* d)$ and using the equations that define the adjoing HDG projection, it follows that

$$(\nabla u_h, \mathbf{d}_v)_K + (\nabla\cdot\boldsymbol{q}_h, \mathrm{d}_w)_K - \langle u_h - \widehat{u}_h, \mathrm{d}_\mu\rangle_{\partial K} = (f, \Pi_W^* d)_K - (c\boldsymbol{q}_h, \boldsymbol{\Pi}_V^* d)_K \tag{4.2}$$

for an arbitrary $d = (\mathbf{d}_v, \mathrm{d}_w, \mathrm{d}_\mu)$.

To prove the first stability estimate, we take $d := (\mathbf{0}, \nabla\cdot\boldsymbol{q}_h, 0)$ in (4.2) and use that $\Pi_W^* d \in W$, so that

$$\begin{aligned}\|\nabla\cdot\boldsymbol{q}_h\|_K^2 &= (P_W f, \Pi_W^* d)_K - (c\,\boldsymbol{q}_h, \boldsymbol{\Pi}_V^* d)_K\\ &\le \|c\|_{L^\infty(K)}\|\boldsymbol{q}_h\|_K\|\boldsymbol{\Pi}_V^* d\|_K + \|P_W f\|_K\|\Pi_W^* d\|_K\\ &\le \left(C_1\,\|c\|_{L^\infty(K)}\|\boldsymbol{q}_h\|_K + C_2\,\|P_W f\|_K\right)\,\|\nabla\cdot\boldsymbol{q}_h\|_K,\end{aligned}$$

by the stability properties of the adjoint projection in Proposition 4.2. To prove the second estimate, we take $d := (\nabla u_h, 0, 0)$ in (4.2) and note that $\Pi_W^* d \in \widetilde{W}^\perp$ because $\mathrm{d}_w = 0$. Then

$$\begin{aligned}\|\nabla u_h\|_K^2 &= (P_{\widetilde{W}^\perp} f, \Pi_W^* d)_K - (c\,\boldsymbol{q}_h, \boldsymbol{\Pi}_V^* d)_K\\ &\le \|c\|_{L^\infty(K)}\|\boldsymbol{q}_h\|_K\|\boldsymbol{\Pi}_V^* d\|_K + \|P_{\widetilde{W}^\perp} f\|_K\|\Pi_W^* d\|_K\\ &\le \left(C_3\,\|c\|_{L^\infty(K)}\|\boldsymbol{q}_h\|_K + C_4\,\|P_{\widetilde{W}^\perp} f\|_K\right)\|\nabla u_h\|_K,\end{aligned}$$

by Proposition 4.2. To prove the third estimate, we take $d := -(\mathbf{0}, 0, u_h - \widehat{u}_h)$ in (4.2)

$$\begin{aligned}\|u_h - \widehat{u}_h\|_{\partial K}^2 &= (P_{\widetilde{W}^\perp} f, \Pi_W^* d)_K - (c\,\boldsymbol{q}_h, \boldsymbol{\Pi}_V^* d)_K\\ &\le \|c\|_{L^\infty(K)}\|\boldsymbol{q}_h\|_K\|\boldsymbol{\Pi}_V^* d\|_K + \|P_{\widetilde{W}^\perp} f\|_K\|\Pi_W^* d\|_K\\ &\le \left(C_5\,\|c\|_{L^\infty(K)}\|\boldsymbol{q}_h\|_K + C_6\,\|P_{\widetilde{W}^\perp} f\|_K\right) h_K^{1/2}\|u_h - \widehat{u}_h\|_{\partial K},\end{aligned}$$

by Proposition 4.2.

The remaining estimates can be proven in exactly the same way given that we have

$$(\nabla\varepsilon_u, \mathbf{d}_v)_K + (\nabla\cdot\varepsilon_{\boldsymbol{q}}, \mathrm{d}_w)_K - \langle\varepsilon_u - \varepsilon_{\widehat{u}}, \mathrm{d}_\mu\rangle_{\partial K} = -(c\,e_{\boldsymbol{q}}, \boldsymbol{\Pi}_V^* d)_K$$

for an arbitrary $d = (\mathbf{d}_v, \mathrm{d}_w, \mathrm{d}_\mu)$. This completes the proof. ☐

5. The construction of M-decompositions

In this section, we propose a systematic way of obtaining M-decompositions. We do this in two steps. In the first step, we show how to modify a given space $\boldsymbol{V}_g \times W_g$ to produce a new space $\boldsymbol{V} \times W$ admitting an M-decomposition. Since $\boldsymbol{V}_g \times W_g$ is assumed to satisfy the first two inclusion properties of an M-decomposition, the indexes $I_M(\boldsymbol{V}_g \times W_g)$ and $I_S(\boldsymbol{V}_g \times W_g)$ are non-negative. We propose five different ways of doing this according to whether the indexes are zero or not. In the second step, we apply these five ways one after the other starting from a given initial space $\boldsymbol{V}_g \times W_g$.

5.1. **An illustration of the construction.** Let us begin by illustrating the (main part of the) above-mentioned construction in the simple case in which $M(\partial K) := \mathcal{P}_0(\partial K)$, K is the unit square and the given space is $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_0(K) \times \mathcal{P}_0(K)$. This will give a concrete idea of the abstract construction. We proceed in three steps.

Step 1. We easily see that the given space $\boldsymbol{V}_1 \times W_1 := \boldsymbol{V}_g \times W_g$ (see the upper part of Table 5) does satisfy the inclusion properties (I). Moreover, a simple computation gives us that $\dim M(\partial K) = 4$, that $\dim\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\} = 2$ and that $\{w|_{\partial K} : w \in W_g, \nabla w = 0\} = 1$. Hence $I_M(\boldsymbol{V}_g \times W_g) = 4-2-1 = 1 > 0$, a reflection that $M(\partial K)$ is strictly bigger than

$$\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W_g, \nabla w = 0\}.$$

As a consequence, $\boldsymbol{V}_1 \times W_1$ does not admit an M-decomposition. See the lower part of Table 5.

Step 2. Since $I_M(\boldsymbol{V}_g \times W_g) = 1$, the space $\boldsymbol{V}_{\text{fillM}}$ is one-dimensional. It must also be solenoidal, its normal trace must lie in $M(\partial K)$ and must not be an element of the space $\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\}$. A possible choice is $\boldsymbol{V}_{\text{fillM}} := \text{span}\{(x, -y)\}$. We then set $\boldsymbol{V}_2 \times W_2 := (\boldsymbol{V}_g \oplus \boldsymbol{V}_{\text{fillM}}) \times W_g$; see the upper part of Table 5. Now, we get that $I_M(\boldsymbol{V}_2 \times W_2) = I_M(\boldsymbol{V}_1 \times W_1) - \dim \boldsymbol{V}_{\text{fillM}} = 1-1 = 0$ and so the space $\boldsymbol{V}_2 \times W_2$ does admit an M- decomposition. However, $I_S(\boldsymbol{V}_2 \times W_2) = I_S(\boldsymbol{V}_g \times W_g) = 1$ since $\dim W_g = 1$ and $\dim \nabla\cdot\boldsymbol{V}_g = 0$, a reflection of the fact that $W_2 = W$ is strictly bigger that $\nabla\cdot\boldsymbol{V}_2 = \nabla\cdot\boldsymbol{V}_g = \{0\}$; see the lower part of Table 5.

Step 3. Since $I_S(\boldsymbol{V}_g \times W_g) = 1$, the space $\boldsymbol{V}_{\text{fillS}}$ is one-dimensional. Its normal component must lie on $M(\partial K)$ and must be such that $\nabla\cdot\boldsymbol{V}_{\text{fillS}} \subset W$. A suitable choice is $\boldsymbol{V}_{\text{fillW}} = \text{span}\{(x, y)\}$; see the upper part of Table 5. Setting $\boldsymbol{V}_3 \times W_3 := (\boldsymbol{V}_g \oplus \boldsymbol{V}_{\text{fillM}} \oplus \boldsymbol{V}_{\text{fillW}}) \times W_g$, a simple computation gives us that $I_M(\boldsymbol{V}_3 \times W_3 = I_(\boldsymbol{V}_2 \times W_2) = 0$, and so the space $\boldsymbol{V}_3 \times W_3$ admits an M-decomposition. Moreover, $I_S(\boldsymbol{V}_3 \times W_3) = I_S(\boldsymbol{V}_2 \times W_2) - \dim \boldsymbol{V}_{\text{fillM}} = 1-1 = 0$, a reflection of that fact that $W = \nabla\cdot\boldsymbol{V}_3$; see the lower part of Table 5.

Let us now start the construction in the general case.

5.2. **The case $I_M(\boldsymbol{V}_g \times W_g) > 0$.** In this case, the space $\boldsymbol{V}_g \times W_g$ does not admit an M-decomposition and, by Theorem 2.4, we have that

$$\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W_g, \nabla w = 0\} \underset{\neq}{\subset} M.$$

To simplify the notation, we let $\boldsymbol{V}_{g_s} := \{\boldsymbol{v} \in \boldsymbol{V}_g : \nabla\cdot\boldsymbol{v} = 0\}$ be the divergence-free subspace of $\boldsymbol{V}_g$ (s stands for solenoidal), and $W_{g_{cst}} := \{w \in W_g : \nabla w = \boldsymbol{0}\}$ the gradient-free subspace of W_g (*cst* stands for constants). We see that, in order to achieve equality, we have to, roughly speaking, *fill* the remaining part of M by

TABLE 5. An illustration of the construction of M-decompositions for $M(\partial K) := \mathcal{P}_0(\partial K)$ and the unit square K. The given space $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_0(K) \times \mathcal{P}_0(K)$ satisfies the inclusion properties (I).

step	$\boldsymbol{V}_i$	basis of $\boldsymbol{V}_g$	of $\boldsymbol{V}_{\text{fillM}}$	of $\boldsymbol{V}_{\text{fillW}}$	of W_g
1	$\boldsymbol{V}_g$	$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$	-	-	1
2	$\boldsymbol{V}_g \oplus \boldsymbol{V}_{\text{fillM}}$	$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}x\\-y\end{bmatrix}$	-	1
3	$\boldsymbol{V}_g \oplus \boldsymbol{V}_{\text{fillM}} \oplus \boldsymbol{V}_{\text{fillW}}$	$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}x\\-y\end{bmatrix}$	$\begin{bmatrix}x\\y\end{bmatrix}$	1

step	trace of the basis of $\{\boldsymbol{v} \in \boldsymbol{V}_i : \nabla \cdot \boldsymbol{v} = 0\}$	trace of the basis of $\{w \in W_g : \nabla w = 0\}$	$I_M(\boldsymbol{V} \times W)$	$I_S(\boldsymbol{V} \times W)$
1	$\begin{smallmatrix} & 0 & \\ -1 & \square & 1 \\ & 0 & \end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ 0 & \square & 0 \\ & -1 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & \\ 1 & \square & 1 \\ & 1 & \end{smallmatrix}$	1	1
2	$\begin{smallmatrix} & 0 & \\ -1 & \square & 1 \\ & 0 & \end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ 0 & \square & 0 \\ & -1 & \end{smallmatrix}$, $\begin{smallmatrix} & -1 & \\ 0 & \square & 1 \\ & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & \\ 1 & \square & 1 \\ & 1 & \end{smallmatrix}$	0	1
3	$\begin{smallmatrix} & 0 & \\ -1 & \square & 1 \\ & 0 & \end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ 0 & \square & 0 \\ & -1 & \end{smallmatrix}$, $\begin{smallmatrix} & -1 & \\ 0 & \square & 1 \\ & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & \\ 1 & \square & 1 \\ & 1 & \end{smallmatrix}$	0	0

adding a space of solenoidal functions $\delta\boldsymbol{V}_{\text{fillM}}$ of dimension $I_M(\boldsymbol{V}_g \times W_g)$. The precise description of this subspace is in the following result.

Proposition 5.1 (Filling the space of traces M)**.** *Let $\boldsymbol{V}_g \times W_g$ satisfy properties* (a) *and* (b) *of Theorem* 2.4*. Assume that $\delta\boldsymbol{V}_{\text{fillM}}$ satisfies the following hypotheses:*

(a) $\gamma\delta\boldsymbol{V}_{\text{fillM}} \subset M$,
(b) $\nabla \cdot \delta\boldsymbol{V}_{\text{fillM}} = \{0\}$,
(c) $\gamma\boldsymbol{V}_{g_s} \cap \gamma\delta\boldsymbol{V}_{\text{fillM}} = \{0\}$,
(d) $\dim \delta\boldsymbol{V}_{\text{fillM}} = \dim \gamma\delta\boldsymbol{V}_{\text{fillM}} = I_M(\boldsymbol{V}_g \times W_g)$.

Then $(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillM}}) \times W_g$ admits an M-decomposition. Moreover, at least one space $\delta\boldsymbol{V}_{\text{fillM}}$ can be constructed to satisfy all the hypotheses when $W_{g_{cst}} = \mathcal{P}_0$.

We remark that the first equality in hypothesis (d), namely, $\dim \delta\boldsymbol{V}_{\text{fillM}} = \dim \gamma\delta\boldsymbol{V}_{\text{fillM}}$ is *not* used in the proof. Nevertheless, we include it in the hypotheses because it gives the *smallest* dimension of a space $\delta\boldsymbol{V}_{\text{fillM}}$ such that $\boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}} \times W_g$ admits an M-decomposition.

Proof. First note that $I_M(\boldsymbol{V}_g \times W_g) \geq 0$ since $\text{tr}(\boldsymbol{V}_g \times W_g) \subset M$, by hypothesis. This means that hypothesis (d) makes sense.

The new space $(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillM}}) \times W_g$ satisfies properties (a) and (b) of Theorem 2.4 thanks to the hypotheses on $\boldsymbol{V}_g \times W_g$ and the inclusion hypothesis (a). It remains to verify property (c) of Theorem 2.4, that is, that $I_M((\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillM}}) \times W_g) = 0$. Note that hypotheses (b) and (c) imply

$$\{\boldsymbol{v}\cdot\boldsymbol{n} : \boldsymbol{v} \in \boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillM}}, \nabla\cdot\boldsymbol{v} = 0\} = \{\boldsymbol{v}\cdot\boldsymbol{n} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\} \oplus \{\boldsymbol{v}\cdot\boldsymbol{n} : \boldsymbol{v} \in \delta\boldsymbol{V}_{\text{fillM}}\},$$

and therefore

$$\begin{aligned} I_M((\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillM}}) \times W_g) = {} & \dim M - \dim\{\boldsymbol{v}\cdot\boldsymbol{n} :\, \boldsymbol{v} \in \boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillM}}, \nabla\cdot\boldsymbol{v} = 0\} \\ & - \dim\{w|_{\partial K} :\, w \in W_g, \nabla w = 0\} \\ = {} & I_M(\boldsymbol{V}_g \times W_g) - \dim\{\boldsymbol{v}\cdot\boldsymbol{n} : \boldsymbol{v} \in \delta\boldsymbol{V}_{\text{fillM}}\} = 0, \end{aligned}$$

by hypothesis (d).

Let us now show that it is possible to construct a space $\delta\boldsymbol{V}_{\text{fillM}}$ with the required properties. Let $\mathcal{B}$ be a basis for $(\text{tr}(\boldsymbol{V}_{g_s} \times W_{g_{cst}}))^\perp$. For each $\mu \in \mathcal{B}$, we define ϕ_μ as the solution of

$$\Delta\phi_\mu = 0 \text{ in } K, \qquad \boldsymbol{n}\cdot\nabla\phi_\mu = \mu \text{ on } \partial K.$$

We claim that we can take $\delta\boldsymbol{V}_{\text{fillM}}$ to be the span of $\{\nabla\phi_\mu\}_{\mu\in\mathcal{B}}$. Since $W_{g_{cst}} = \mathcal{P}_0$, the average of μ is zero and $\nabla\phi_\mu$ is well defined. The boundary condition ensures the satisfaction of hypotheses (a) and (c), and hypothesis (b) holds by construction. Finally, condition (d) is also satisfied given that the set $\{\nabla\phi_\mu\}_{\mu\in\mathcal{B}}$ is linearly independent, and

$$\dim(\text{tr}(\boldsymbol{V}_{g_s} \times W_{g_{cst}}))^\perp = \dim M - \dim \text{tr}(\boldsymbol{V}_{g_s} \times W_{g_{cst}}) = I_M(\boldsymbol{V}_g \times W_g).$$

This completes the proof. □

Note that, although we have shown the possibility of constructing a space $\delta\boldsymbol{V}_{\text{fillM}}$ with the required properties by using an auxiliary Laplace equation, the practical construction of such spaces (done in Parts II and III) does not use such an approach.

5.3. **The case $I_M(\boldsymbol{V}_g \times W_g) = 0$ but $I_S(\boldsymbol{V}_g \times W_g) > 0$.** In this case, the space $\boldsymbol{V}_g \times W_g$ admits an M-decomposition but $\nabla\cdot\boldsymbol{V}_g$ is a proper subspace of W_g. By the kernels' trace decomposition of Theorem 2.4, we have (2.1) and we then see that, if we seek a modification of $\boldsymbol{V}_g \times W_g$ of the form $\boldsymbol{V}_g \times W$, it must be such that

$$\{w|_{\partial K} :\, w \in W, \nabla w = 0\} = \{w|_{\partial K} :\, w \in W_g, \nabla w = 0\}.$$

The following result gives a hypothesis under which we are allowed to reduce W_g to $W := \nabla\cdot\boldsymbol{V}_g$. Its proof is a direct application of Theorem 2.4.

Proposition 5.2 (Reducing the space W_g). *Assume that $\boldsymbol{V}_g \times W_g$ admits an M-decomposition. Then $\boldsymbol{V}_g \times \nabla\cdot\boldsymbol{V}_g$ admits an M-decomposition provided that*

$$\{w|_{\partial K} :\, w \in W_g, \nabla w = 0\} = \{\nabla\cdot\boldsymbol{v}|_{\partial K} :\, \boldsymbol{v} \in \boldsymbol{V}_g, \nabla(\nabla\cdot\boldsymbol{v}) = 0\}. \tag{5.1}$$

Note that the hypothesis (5.1) holds when W_g contains constant functions and there exists $\boldsymbol{v} \in \boldsymbol{V}_g$ such that $\nabla\cdot\boldsymbol{v} = 1$.

Now, let us seek a modification of $\boldsymbol{V}_g \times W_g$ of the form $\boldsymbol{V} \times W_g$, where $\boldsymbol{V}_g \subset \boldsymbol{V}$. Since

$$\nabla\cdot\boldsymbol{V}_g \underset{\neq}{\subset} W_g,$$

we see that in order to achieve the equality, we have to, roughly speaking, *fill* the remaining part of W_g by adding a space of *non*-solenoidal functions $\delta\boldsymbol{V}_{\text{fillW}}$ of dimension $I_S(\boldsymbol{V}_g \times W_g)$. In this case, we would immediately have that

$$\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} :\, \boldsymbol{v} \in \boldsymbol{V}, \nabla\cdot\boldsymbol{v} = 0\} = \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} :\, \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\},$$

and, by Theorem 2.4, the resulting space would admit an M-decomposition. The precise way of choosing $\delta\boldsymbol{V}_{\text{fillW}}$ is described in the following result.

Proposition 5.3 (Increasing the space $\boldsymbol{V}_g$)**.** *Let the space $\boldsymbol{V}_g \times W_g$ admit an M-decomposition and assume that $\nabla\cdot\boldsymbol{V}_g$ is a proper subspace of W_g. Let $\delta\boldsymbol{V}_{\text{fillW}}$ satisfy the following hypotheses:*

(a) $\gamma\delta\boldsymbol{V}_{\text{fillW}} \subset M$,
(b) $\nabla\cdot\delta\boldsymbol{V}_{\text{fillW}} \subset W_g$,
(c) $\nabla\cdot\boldsymbol{V}_g \cap \nabla\cdot\delta\boldsymbol{V}_{\text{fillW}} = \{0\}$,
(d) $\dim\delta\boldsymbol{V}_{\text{fillW}} = \dim\nabla\cdot\delta\boldsymbol{V}_{\text{fillW}} = I_S(\boldsymbol{V}_g \times W_g)$.

Then $(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillW}}) \times W_g$ admits an M-decomposition with $W_g = \nabla\cdot(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillW}})$. Moreover, at least one space $\delta\boldsymbol{V}_{\text{fillW}}$ can be constructed to satisfy all the hypotheses when M contains the constant functions.

Just as for Proposition 5.1, the first equality of hypothesis (d) is not used in the proof. Nevertheless, we include it in the hypotheses because it gives the *smallest* dimension of the space $\delta\boldsymbol{V}_{\text{fillW}}$ such that $\boldsymbol{V}_g\oplus\delta\boldsymbol{V}_{\text{fillW}}\times W_g$ admits an M-decomposition with $W_g = \nabla\cdot(\boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillW}})$.

Proof. It is easy to see that, if $\boldsymbol{V}_g \times W_g$ admits and M-decomposition, so does the space $(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillW}}) \times W_g$, due to hypotheses (a) and (b). For the verification of $W_g = \nabla\cdot(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillW}})$, we only need to show that the dimensions of these two space are the same. But

$$\begin{aligned}\dim\nabla\cdot(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{fillW}}) &= \dim\nabla\cdot\boldsymbol{V}_g + \dim\nabla\cdot\delta\boldsymbol{V}_{\text{fillW}}\\ &= \dim\nabla\cdot\boldsymbol{V}_g + I_S(\boldsymbol{V}_g \times W_g)\\ &= \dim W_g,\end{aligned}$$

where the first equation comes from hypothesis (c) and the second one comes from the second equality of hypothesis (d).

Let us now show that it is possible to construct a space $\delta\boldsymbol{V}_{\text{fillW}}$ with the required properties. Let $\mathcal{B}$ be a basis of $\widetilde{W}^{\perp}$. For $w \in \mathcal{B}$, we define ϕ_w to be the solution of

$$\Delta\phi_w = w \text{ in } K \text{ and } \boldsymbol{n}\cdot\nabla\phi_w = \frac{(w,1)_K}{|\partial K|} \text{ on } \partial K.$$

Then, we can take $\delta\boldsymbol{V}_{\text{fillW}}$ as the span of $\{\nabla\phi_w\}_{w\in\mathcal{B}}$. Note that since M contains the constants, hypothesis (a) is actually satisfied. Finally, it is not difficult to see that hypotheses (b), (c), and (d) are also satisfied by the choice of $\mathcal{B}$. This completes the proof. □

Note again that, although we have shown the possibility of constructing a space $\delta\boldsymbol{V}_{\text{fillW}}$ with the required properties by using an auxiliary Poisson equation, the practical construction of such spaces (done in Parts II and III) does not use such an approach.

5.4. **The case $I_M(\boldsymbol{V}_g \times W_g) = 0$ and $I_S(\boldsymbol{V}_g \times W_g) = 0$.** In this case, the space $\boldsymbol{V}_g \times W_g$ admits an M-decomposition and by Theorem 2.4, we have (2.1). Moreover, $\nabla\cdot\boldsymbol{V}_g = W_g$. Since $\boldsymbol{V}_g = \nabla W_g \oplus \boldsymbol{V}_{\text{sbb}} \oplus \widetilde{\boldsymbol{V}_g}^{\perp}$, we have that

$$\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\} = \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \nabla W_g \oplus \widetilde{\boldsymbol{V}_g}^{\perp}, \nabla\cdot\boldsymbol{v} = 0\},$$

and that $\nabla\cdot(\nabla W_g \oplus \widetilde{\boldsymbol{V}_g}^{\perp}) = \nabla\cdot\boldsymbol{V}_g = W_g$. We have thus proved the following result.

Proposition 5.4 (Removing all the solenoidal bubbles from the space $\boldsymbol{V}_g$)**.** *Assume that the space $\boldsymbol{V}_g \times W_g$ admits an M-decomposition with $\nabla\cdot\boldsymbol{V}_g = W_g$. Then the space $(\nabla W_g \oplus \widetilde{\boldsymbol{V}_g}^{\perp})\times W_g$ admits an M-decomposition with $\nabla\cdot(\nabla W_g \oplus \widetilde{\boldsymbol{V}_g}^{\perp}) = W_g$.*

Similarly, since for any solenoidal bubble space $\delta\boldsymbol{V}_{\text{sbb}}$,

$$\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v}\in\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{sbb}}, \nabla\cdot\boldsymbol{v} = 0\} = \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v}\in\boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\},$$

the space $\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{sbb}} \times W_g$ admits an M-decomposition. Moreover, we have that $\nabla\cdot(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{sbb}}) = \nabla\cdot\boldsymbol{V}_g = W_g$. We have thus proved the following result.

Proposition 5.5 (Adding solenoidal bubbles to the space $\boldsymbol{V}_g$)**.** *Assume that the space $\boldsymbol{V}_g \times W_g$ admits an M-decomposition with $\nabla\cdot\boldsymbol{V}_g = W_g$. Let $\delta\boldsymbol{V}_{\text{sbb}}$ satisfy*

(a) $\gamma\delta\boldsymbol{V}_{\text{sbb}} = \{0\}$,
(b) $\nabla\cdot\delta\boldsymbol{V}_{\text{sbb}} = \{0\}$,
(c) $\boldsymbol{V}_g \cap \delta\boldsymbol{V}_{\text{sbb}} = \{\boldsymbol{0}\}$.

Then $(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{sbb}})\times W_g$ admits an M-decomposition with $\nabla\cdot(\boldsymbol{V}_g + \delta\boldsymbol{V}_{\text{sbb}}) = W_g$.

TABLE 6. Five ways of constructing spaces $\boldsymbol{V}\times W$ admitting an M-decomposition. The spaces are obtained by modifying the space $\boldsymbol{V}_g\times W_g$ according to whether it already admits an M-decomposition or not, and according to whether the space $\nabla\cdot\boldsymbol{V}_g$ is a proper subspace of W_g or not. The space $\boldsymbol{V}_g\times W_g$ is assumed to satisfy the first two inclusion properties of an M-decomposition, namely, $\text{tr}(\boldsymbol{V}_g\times W_g)\subset M$ and $\nabla W_g\times\nabla\cdot\boldsymbol{V}_g\subset\boldsymbol{V}_g\times W_g$.

way #	properties of $\boldsymbol{V}_g\times W_g$	$\boldsymbol{V}$	W	properties of $\boldsymbol{V}\times W$
I (Prop.5.1)	$I_M(\boldsymbol{V}_g\times W_g) > 0$	$\boldsymbol{V}_g\oplus\delta\boldsymbol{V}_{\text{fillM}}$	W_g	$I_M(\boldsymbol{V}\times W)=0$ $I_S(\boldsymbol{V}\times W) = I_S(\boldsymbol{V}_g\times W_g)$
II (Prop.5.2)	$I_M(\boldsymbol{V}_g\times W_g)=0$ $I_S(\boldsymbol{V}_g\times W_g)>0$	$\boldsymbol{V}_g$	$\nabla\cdot\boldsymbol{V}_g$	$I_M(\boldsymbol{V}\times W)=0$ $I_S(\boldsymbol{V}\times W)=0$ if (5.1) holds
III (Prop.5.3)	$I_M(\boldsymbol{V}_g\times W_g)=0$ $I_S(\boldsymbol{V}_g\times W_g)>0$	$\boldsymbol{V}_g\oplus\delta\boldsymbol{V}_{\text{fillW}}$	W_g	$I_M(\boldsymbol{V}\times W)=0$ $I_S(\boldsymbol{V}\times W)=0$
IV (Prop.5.4)	$I_M(\boldsymbol{V}_g\times W_g)=0$ $I_S(\boldsymbol{V}_g\times W_g)=0$	$\nabla W_g\oplus\widetilde{\boldsymbol{V}_g}^{\perp}$	W_g	$I_M(\boldsymbol{V}\times W)=0$ $I_S(\boldsymbol{V}\times W)=0$
V (Prop.5.5)	$I_M(\boldsymbol{V}_g\times W_g)=0$ $I_S(\boldsymbol{V}_g\times W_g)=0$	$\boldsymbol{V}_g\oplus\delta\boldsymbol{V}_{\text{sbb}}$	W_g	$I_M(\boldsymbol{V}\times W)=0$ $I_S(\boldsymbol{V}\times W)=0$

5.5. **A systematic procedure for obtaining M-decompositions.** We can now use these five ways of obtaining M-decompositions (see Table 6) to propose a systematic way for constructing spaces admitting M-decompositions starting from a single, given space $\boldsymbol{V}_g\times W_g$ assumed to satisfy the first two inclusion properties of an M-decomposition. Typically, the space $\boldsymbol{V}_g\times W_g$ is chosen because it either has desirable approximation properties, e.g., $\boldsymbol{\mathcal{P}}_k(K)\times\mathcal{P}_k(K)$ for general polyhedral elements, or because it has a convenient structure, e.g., $\boldsymbol{\mathcal{Q}}_k(K)\times\mathcal{Q}_k(K)$ for

TABLE 7. Spaces $\boldsymbol{V} \times W$ admitting an M-decomposition. They are constructed from the single space $\boldsymbol{V}_g \times W_g$ which is assumed to satisfy the first two inclusion properties of an M-decomposition, namely, $\mathrm{tr}(\boldsymbol{V}_g \times W_g) \subset M$ and $\nabla W_g \times \nabla \cdot \boldsymbol{V}_g \subset \boldsymbol{V}_g \times W_g$. The construction II holds whenever condition (5.1) does.

way #	$\boldsymbol{V}$	W
V	$\boldsymbol{V}^{\mathrm{MIX}} := \boldsymbol{V}^{\mathrm{mix}} \oplus \delta\boldsymbol{V}_{\mathrm{sbb}}$	$W^{\mathrm{MIX}} := W_g$
III	$\boldsymbol{V}^{\mathrm{mix}} := \boldsymbol{V}^{\mathrm{hdg}} \oplus \delta\boldsymbol{V}_{\mathrm{fillW}}$	$W^{\mathrm{mix}} := W_g$
I	$\boldsymbol{V}^{\mathrm{hdg}} := \boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\mathrm{fillM}}$	$W^{\mathrm{hdg}} := W_g$
II	$\boldsymbol{V}_{\mathrm{mix}} := \boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\mathrm{fillM}}$	$W_{\mathrm{mix}} := \nabla \cdot \boldsymbol{V}_g$
IV	$\boldsymbol{V}_{\mathrm{MIX}} := ((\boldsymbol{V}_{\mathrm{mix}})_{\mathrm{sbb}})^{\perp}$	$W_{\mathrm{MIX}} := \nabla \cdot \boldsymbol{V}_g$

TABLE 8. The main properties of the spaces $\delta\boldsymbol{V}_g$.

$\delta\boldsymbol{V}_g$	$\nabla \cdot \delta\boldsymbol{V}_g$	$\delta\boldsymbol{V}_g \cdot \boldsymbol{n}$	$\dim \delta\boldsymbol{V}_g$
$\delta\boldsymbol{V}_{\mathrm{sbb}}$	$\{\boldsymbol{0}\}$	0	> 0
$\delta\boldsymbol{V}_{\mathrm{fillW}}$	$\subset W_g, \cap \nabla \cdot \boldsymbol{V}_g = \{0\}$	$\subset M$	$I_S(\boldsymbol{V}_g \times W_g)$
$\delta\boldsymbol{V}_{\mathrm{fillM}}$	$\{\boldsymbol{0}\}$	$\subset M, \cap \boldsymbol{V}_g \cdot \boldsymbol{n} = \{0\}$	$I_M(\boldsymbol{V}_g \times W_g)$

TABLE 9. The spaces $\widetilde{\boldsymbol{V}} \times \widetilde{W}$ defining the canonical decomposition of each space $\boldsymbol{V} \times W$ in terms of the space $\boldsymbol{V}_g \times W_g$.

way #	$\widetilde{\boldsymbol{V}}$	$\widetilde{W}$
V	$\nabla W_g \oplus \boldsymbol{V}_{\mathrm{sbb}} \oplus \delta\boldsymbol{V}_{\mathrm{sbb}}$	W_g
III	$\nabla W_g \oplus \boldsymbol{V}_{\mathrm{sbb}}$	W_g
I	$\nabla W_g \oplus \boldsymbol{V}_{\mathrm{sbb}}$	$\nabla \cdot \boldsymbol{V}_g$
II	$\nabla(\nabla \cdot \boldsymbol{V}_g) \oplus \boldsymbol{V}_{\mathrm{sbb}}$	$\nabla \cdot \boldsymbol{V}_g$
IV	$\nabla(\nabla \cdot \boldsymbol{V}_g)$	$\nabla \cdot \boldsymbol{V}_g$

cubes. Here $\mathcal{Q}_k(K) = \mathcal{P}_k(x) \otimes \mathcal{P}_k(y) \otimes \mathcal{P}_k(z)$ is the scalar tensor product space on cubes, and $\boldsymbol{\mathcal{Q}}_k(K)$ is the vectorial tensor product space. The systematic construction is described in Tables 7 and 8, whereas the spaces defining their corresponding canonical decompositions are displayed in Table 9.

Note that the spaces obtained by the ways I, II and III satisfy

$$\boldsymbol{V}^{\mathrm{mix}} \times W^{\mathrm{mix}} \supset \boldsymbol{V}^{\mathrm{hdg}} \times W^{\mathrm{hdg}} \supset \boldsymbol{V}_{\mathrm{mix}} \times W_{\mathrm{mix}},$$

and so the HDG method is always *sandwiched* by two mixed methods. These two mixed methods are special in the following sense. The space $\boldsymbol{V}^{\mathrm{mix}} \times W^{\mathrm{mix}}$ has the smallest dimension among all spaces defining a mixed method that includes $\boldsymbol{V}^{\mathrm{hdg}} \times W^{\mathrm{hdg}}$. Similarly, $\boldsymbol{V}_{\mathrm{mix}} \times W_{\mathrm{mix}}$ has the biggest dimension among all spaces defining a mixed method that is included in $\boldsymbol{V}^{\mathrm{hdg}} \times W^{\mathrm{hdg}}$.

5.6. **Some examples.** We end this section with some examples obtained with the above, general construction. We present our construction in a table which is a slight modification of Table 7. Indeed, we discard the constructions associated to ways 4 and 5; see also Table 1 in the Introduction.

In Table 10, we show known and new spaces that come out of our construction. We only present the spaces that can be concisely described and so we restrict ourselves to the two-dimensional case. These examples are part of our results in Part II. Therein, we discuss several variations of these results as well as additional cases. In particular, we discuss the case in which $M = \mathcal{P}_k(\partial K)$, K is any polygon and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$. In Part III, we carry out the construction for general, flat-faced polyhedral elements K.

Let us emphasize that the space in the middle row is the space associated to an HDG method, $\boldsymbol{V}^{\text{hdg}} \times W^{\text{hdg}}$. The spaces in the top and bottom row are the spaces $\boldsymbol{V}^{\text{mix}} \times W^{\text{mix}}$ and $\boldsymbol{V}_{\text{mix}} \times W_{\text{mix}}$ associated to the corresponding sandwiching mixed methods. All these methods are superconvergent provided the conditions (J) are satisfied.

TABLE 10. Spaces $\boldsymbol{V} \times W$ admitting an $M(\partial K)$-decomposition.

$\boldsymbol{V}$	W	method
$M = \mathcal{P}_k(\partial K)$, K is a square and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{Q}}_k \times \mathcal{Q}_k$.		
$\boldsymbol{\mathcal{Q}}_k \oplus \mathbf{curl}\,\text{span}\{x^{k+1}y, x\,y^{k+1}\} \oplus \text{span}\{\boldsymbol{x}\,x^k y^k\}$	$\mathcal{Q}_k$	$\mathbf{TNT}_{[k]}$ [14]
$\boldsymbol{\mathcal{Q}}_k \oplus \mathbf{curl}\,\text{span}\{x^{k+1}y, x\,y^{k+1}\}$	$\mathcal{Q}_k$	$\mathbf{HDG}^{Q}_{[k]}$[14]
$\boldsymbol{\mathcal{Q}}_k \oplus \mathbf{curl}\,\text{span}\{x^{k+1}y, x\,y^{k+1}\}$	$\mathcal{Q}_k \setminus \{x^k\,y^k\}$	$\mathbf{BDM}_{[k]}$
$M = \mathcal{P}_k(\partial K)$, K is a triangle and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$.		
$\boldsymbol{\mathcal{P}}_k \oplus \boldsymbol{x}\,\widetilde{\mathcal{P}}_k$	$\mathcal{P}_k$	$\mathbf{RT}_k$ [26]
$\boldsymbol{\mathcal{P}}_k$	$\mathcal{P}_k$	$\mathbf{HDG}_k$[14]
$\boldsymbol{\mathcal{P}}_k$	$\mathcal{P}_{k-1}$	$\mathbf{BDM}_k$ [3]
$M = \mathcal{P}_k(\partial K)$, K is a square and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$.		
$\boldsymbol{\mathcal{P}}_k \oplus \mathbf{curl}\,\text{span}\{x^{k+1}y, x\,y^{k+1}\} \oplus \boldsymbol{x}\,\widetilde{\mathcal{P}}_k$	$\mathcal{P}_k$	(new)
$\boldsymbol{\mathcal{P}}_k \oplus \mathbf{curl}\,\text{span}\{x^{k+1}y, x\,y^{k+1}\}$	$\mathcal{P}_k$	(new)
$\boldsymbol{\mathcal{P}}_k \oplus \mathbf{curl}\,\text{span}\{x^{k+1}y, x\,y^{k+1}\}$	$\mathcal{P}_{k-1}$	$\mathbf{BDM}_{[k]}$ [3]
$M = \mathcal{P}_k(\partial K)$, K is a quadrilateral and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$.		
$\boldsymbol{\mathcal{P}}_k \oplus_{i=1}^{ne} \mathbf{curl}\,\text{span}\{\xi_4\,\lambda_3^k, \xi_4\,\lambda_4^k\} \oplus \boldsymbol{x}\,\widetilde{\mathcal{P}}_k$	$\mathcal{P}_k$	(new)
$\boldsymbol{\mathcal{P}}_k \oplus_{i=1}^{ne} \mathbf{curl}\,\text{span}\{\xi_4\,\lambda_3^k, \xi_4\,\lambda_4^k\}$	$\mathcal{P}_k$	(new)
$\boldsymbol{\mathcal{P}}_k \oplus_{i=1}^{ne} \mathbf{curl}\,\text{span}\{\xi_4\,\lambda_3^k, \xi_4\,\lambda_4^k\}$	$\mathcal{P}_{k-1}$	(new)

In Table 10, we use the notation $\mathbf{curl}\,p := (-p_y, p_x)$. We also need to define the linear function λ_i and the rational function ξ_i associated to the definition of the spaces for quadrilaterals. Let $\{\mathbf{v}_i\}_{i=1}^4$ be the set of vertices of the quadrilateral K which we take to be counterclockwise ordered. Let $\{\mathbf{e}_i\}_{i=1}^4$ be the set of edges of K where the edge $\mathbf{e}_i$ connects the vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, where we set $\mathbf{v}_5 = \mathbf{v}_1$. Then, for $1 \le i \le 4$, we define λ_i to be the linear function that vanishes on edge $\mathbf{e}_i$ and reaches maximum value 1 in the closure of K, and ξ_i to be a rational function such that $\xi_i|_{\mathbf{e}_i} \in \mathcal{P}_1(\mathbf{e}_i)$ and $\xi_i(\mathbf{v}_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. A particular choice of ξ_i is given as follows:

$$\xi_i := \eta_{i-1}\frac{\lambda_{i-2}}{\lambda_{i-2}(\mathbf{v}_i)} + \eta_i\frac{\lambda_{i+1}}{\lambda_{i+1}(\mathbf{v}_i)}, \qquad \text{where} \qquad \eta_i := \Pi_{\substack{j=1\\ j\neq i}}^4 \frac{\lambda_j}{\lambda_j+\lambda_i}.$$

The rational function η_i is constructed in such a way that its trace on ∂K is zero except on the edge $\mathbf{e}_i$, where it is equal to one.

6. Applications

Here, we provide two applications of the construction of M-decompositions just proposed. The first is a *sandwiching* theorem stating that, under proper conditions on the stabilization function, a significant part of the approximate solutions and the associated matrix equations provided by the HDG method and its two sandwiching mixed methods do coincide. The second is two elementwise $H(\text{div})$-conforming postprocessings of the approximate flux for the HDG method defined by using the two *sandwiching* mixed methods.

6.1. Preliminaries: HDG methods with a special stabilization α.

As a stepping stone towards our sandwiching result, we present a combination of a particular case of the characterization of the HDG methods [11, Theorem 2.1] with the generalization of the first property of [10, Theorem 2.4] for the so-called HDG_k method on simplexes.

To do that, we introduce the *local solvers* associated with the method. The first local solver is defined on the element $K \in \mathcal{T}_h$ as the mapping $\mathsf{m} \in L^2(\partial K) \to (\mathfrak{Q}\mathsf{m}, \mathcal{U}\mathsf{m}) \in \boldsymbol{V}(K)\times W(K)$ where

$$\begin{aligned}
(\mathrm{c}\,\mathfrak{Q}\mathsf{m}, \boldsymbol{v})_K - (\mathcal{U}\mathsf{m}, \nabla\cdot\boldsymbol{v})_K &= -\langle \mathsf{m}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K},\\
-(\nabla w, \mathfrak{Q}\mathsf{m})_K + \langle w, \widehat{\mathfrak{Q}}\mathsf{m}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0,\\
\widehat{\mathfrak{Q}}\mathsf{m}\cdot\boldsymbol{n} &= \mathfrak{Q}\mathsf{m}\cdot\boldsymbol{n} + \alpha(\mathcal{U}\mathsf{m}-\mathsf{m}),
\end{aligned}$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K)\times W(K)$.

The other local solver is defined on the element $K \in \mathcal{T}_h$ as the mapping $f \in L^2(K) \to (\mathfrak{Q}f, \mathcal{U}f) \in \boldsymbol{V}(K)\times W(K)$ where

$$\begin{aligned}
(\mathrm{c}\,\mathfrak{Q}f, \boldsymbol{v})_K - (\mathcal{U}f, \nabla\cdot\boldsymbol{v})_K &= 0,\\
-(\nabla w, \mathfrak{Q}f)_K + \langle w, \widehat{\mathfrak{Q}}f\cdot\boldsymbol{n}\rangle_{\partial K} &= (f, w)_K,\\
\widehat{\mathfrak{Q}}f\cdot\boldsymbol{n} &= \mathfrak{Q}f\cdot\boldsymbol{n} + \alpha(\mathcal{U}f),
\end{aligned}$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K)\times W(K)$.

We can now state our characterization result.

Theorem 6.1. *Suppose that, for every $K \in \mathcal{T}_h$, the space $\boldsymbol{V}(K)\times W(K)$ admits an $M(\partial K)$-decomposition and that the stabilization function α is of the form $\rho\,\alpha_0$ where $\rho > 0$ and α_0 satisfies the following properties:*

(i) $w \in \widetilde{W}^{\perp}(K), \quad \langle\alpha_0(w), w\rangle_{\partial K} = 0 \quad \Longrightarrow \quad w = 0,$
(ii) $\langle\alpha_0(\mu), \mu\rangle_{\partial K} \geq 0$ *for all* $\mu \in M(\partial K)$,
(iii) $\langle\alpha_0(\lambda), \mu\rangle_{\partial K} = \langle\lambda, \alpha_0(\mu)\rangle_{\partial K}$, *for all* $\lambda, \mu \in M(\partial K)$,
(iv) $\langle\alpha_0(\lambda), \mu\rangle_{\partial K} = 0\ \forall\mu \in \gamma\widetilde{W}(K)^{\perp}$ *implies* $\alpha_0(\lambda) = 0$ *on* ∂K.

Then the approximate solution $(\boldsymbol{q}_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h$ *is well defined and*

$$(\boldsymbol{q}_h, u_h) = (\mathfrak{Q}\widehat{u}_h, \mathcal{U}\widehat{u}_h) + (\mathfrak{Q}f, \mathcal{U}f),$$

where $\widehat{u}_h$ *is the element of* $M_h(g) := \{\mu \in M_h : \mu = P_M g \text{ on } \partial\Omega\}$ *satisfying*

$$(\mathrm{c}\,\mathfrak{Q}\widehat{u}_h, \mathfrak{Q}\mu)_{\mathcal{T}_h} = (f, \mathcal{U}\mu)_{\mathcal{T}_h} \quad \forall\, \mu \in M_h(0).$$

Moreover, the functions $(\boldsymbol{q}_h, u_h)$, $(\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \widehat{u}_h)$ *and* u_h^* *are independent of* ρ *except for* $P_{\widetilde{W}^\perp}\mathcal{U}f$ *which depends on an affine manner of* $\frac{1}{\rho}P_{\widetilde{W}^\perp}f$.

To prove Theorem 6.1 (and others in the next subsection), we are going to use the following simple result.

Lemma 6.2. *Assume that* α *satisfies the hypotheses of Theorem* 6.1. *Then the approximate solution* $(\boldsymbol{q}_h, u_h)$ *of the HDG method* (1.1) *can be obtained in terms of* f *and* $\widehat{u}_h$ *by successively solving for the expression on the left-hand side of the following weak formulations:*

$$\begin{aligned}
\langle \alpha(u_h - \widehat{u}_h), w\rangle_{\partial K} &= (P_{\widetilde{W}^\perp}f, w)_K && \forall\, w \in \widetilde{W}^\perp,\\
(\nabla\cdot\boldsymbol{q}_h, w)_K &= -\langle \alpha(u_h - \widehat{u}_h), w\rangle_{\partial K} + (f, w)_K && \forall\, w \in W,\\
(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_K &= -\langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} && \forall\, \boldsymbol{v} \in \boldsymbol{V} : \nabla\cdot\boldsymbol{v} = 0,\\
(P_{\widetilde{W}}u_h, \nabla\cdot\boldsymbol{v})_K &= \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} + (\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_K && \forall\, \boldsymbol{v} \in \boldsymbol{V},\\
\langle \alpha_0(P_{\widetilde{W}^\perp}u_h), w\rangle_{\partial K} &= (\frac{1}{\rho}P_{\widetilde{W}^\perp}f, w)_K - \langle \alpha_0(P_{\widetilde{W}}u_h - \widehat{u}_h), w\rangle_{\partial K} && \forall\, w \in \widetilde{W}^\perp.
\end{aligned}$$

Proof. We begin by noting that the first two equations defining the HDG method (1.1a)-(1.1b) can be rewritten as

$$(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_K - (P_{\widetilde{W}}u_h, \nabla\cdot\boldsymbol{v})_K = -\langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{6.1a}$$

$$(\nabla\cdot\boldsymbol{q}_h, w)_K + \langle \alpha(u_h - \widehat{u}_h), w\rangle_{\partial K} = (f, w)_K \quad \forall\, w \in W. \tag{6.1b}$$

By hypothesis (iv), for any $\lambda \in L^2(\partial K)$, the functional $\langle \alpha(\lambda), \cdot\rangle_{\partial K} : M \to \mathbb{R}$ can be determined by its action on $\gamma\widetilde{W}^\perp$. Therefore (6.1b) shows that the first equation in the statement of the lemma holds and $P_{\widetilde{W}^\perp}f$ determines $\langle \alpha(u_h - \widehat{u}_h), \cdot\rangle_{\partial K}$ as a functional on M.

The third equation follows from (6.1a). Since we already determined $\nabla\cdot\boldsymbol{q}_h$, this equation can be used to determine the remaining part of $\boldsymbol{q}_h$. We see that it can be expressed solely in terms on f and $\widehat{u}_h$.

We can now use the fourth equation to determine $P_{\widetilde{W}}u_h$. To determine $P_{\widetilde{W}^\perp}u_h$, we rewrite the first equation as the fifth equation and solve. This is possible by hypotheses (i), (ii) and (iii). This completes the proof. □

We are now ready to prove the characterization Theorem 6.1.

Proof. We begin by pointing out that this characterization of the approximate solution is a particular case of [11, Theorem 2.1], provided $\alpha(\mathcal{U}(\mathsf{m}) - \mathsf{m}) = 0$. If this is the case, we only need to prove that the only part of the approximate solution which depends on ρ is $P_{\widetilde{W}^\perp}\mathcal{U}f$ and that it depends in an affine manner of $\frac{1}{\rho}P_{\widetilde{W}^\perp}f$.

If we now take $f = 0$ and replace $\widehat{u}_h$ by m in Lemma 6.2, we have that $(\boldsymbol{q}_h, u_h) = (\mathfrak{Q}\mathsf{m}, \mathcal{U}\mathsf{m})$ and that the first equation therein and hypothesis (iv) imply that $\alpha(\mathcal{U}\mathsf{m} - \mathsf{m}) = 0$, as wanted. The remaining equations of Lemma 6.2 imply that $(\mathfrak{Q}\mathsf{m}, \mathcal{U}\mathsf{m})$ is independent of ρ.

Similarly, if we now set $\widehat{u}_h = 0$, we have that $(\boldsymbol{q}_h, u_h) = (\mathfrak{Q}f, \mathfrak{U}f)$ and by the equations of Lemma 6.2, we see that the only component of the solution that depends on ρ is $P_{\widetilde{W}^\perp}\,\mathfrak{U}f$ and that it depends in an affine manner of $\frac{1}{\rho}P_{\widetilde{W}^\perp}f$.

It remains to show that u_h^* is independent of ρ. But this is actually the case since u_h^* is defined (see (3.3)) solely in terms of $\boldsymbol{q}_h$ and $P_{\widetilde{W}}u_h$. This completes the proof. □

6.2. **A sandwiching theorem.** The following result is a generalization of the second and third properties [10, Theorem 2.4]. There the HDG method is the HDG_k method, and the sandwiching mixed methods are the RT_k and BDM_k methods, all of them defined on simplexes.

Theorem 6.3. *Suppose that for every $K \in \mathcal{T}_h$, the space $\boldsymbol{V}_g(K) \times W_g(K)$ is the one used in our construction of the HDG method and its two sandwiching mixed methods in the previous section. Let us also assume that the stabilization function α is of the form $\rho\alpha_0$ where $\rho > 0$ and α_0 satisfies the following properties:*

(i) $w \in (\nabla \cdot \boldsymbol{V}_g(K))^\perp, \quad \langle \alpha_0(w), w\rangle_{\partial K} = 0 \quad \Longrightarrow \quad w = 0,$

(ii) $\langle \alpha_0(\mu), \mu\rangle_{\partial K} \geq 0$ *for all* $\mu \in M(\partial K)$,

(iii) $\langle \alpha_0(\lambda), \mu\rangle_{\partial K} = \langle \lambda, \alpha_0(\mu)\rangle_{\partial K}$, *for all* $\lambda, \mu \in M(\partial K)$,

(iv) $\langle \alpha_0(\lambda), \mu\rangle_{\partial K} = 0\ \forall \mu \in \gamma(\nabla \cdot \boldsymbol{V}_g(K))^\perp$ *implies* $\alpha_0(\lambda) = 0$ *on* ∂K.

Then we have that:

(1) *If $f|_K = P_{\nabla\cdot \boldsymbol{V}_g} f|_K$ for all elements $K \in \mathcal{T}_h$, then the functions*

$$(\boldsymbol{q}_h, P_{\nabla\cdot\boldsymbol{V}_g}u_h), \quad (\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \widehat{u}_h) \quad \text{and} \quad u_h^*$$

are the same for the HDG and its two sandwiching mixed methods.

(2) *The bilinear form associated to the stiffness matrix of $\widehat{u}_h$, $(\lambda, \mu) \in M_h \times M_h \mapsto (\mathrm{c}\,\mathfrak{Q}\lambda, \mathfrak{Q}\mu)_{\mathcal{T}_h}$, is the same for the HDG and its two sandwiching mixed methods.*

Proof. By Lemma 6.2, when $P_{(\nabla\cdot\boldsymbol{V}_g)^\perp} f = 0$, for each of the three methods under consideration, the component of the approximate solution $(\boldsymbol{q}_h, P_{\nabla\cdot\boldsymbol{V}_g}u_h)$ can be expressed in terms of $\widehat{u}_h$ and f exactly in the same manner, that is, by solving

$$\begin{aligned}
\nabla\cdot\boldsymbol{q}_h &= P_{\nabla\cdot\boldsymbol{V}_g} f, && \\
(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_K &= -\langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} && \forall\, \boldsymbol{v} \in \boldsymbol{V}^{\mathrm{hdg}} : \nabla\cdot\boldsymbol{v} = 0,\\
(P_{\nabla\cdot\boldsymbol{V}_g}u_h, \nabla\cdot\boldsymbol{v})_K &= -\langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} + (\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_K && \forall\, \boldsymbol{v} \in \boldsymbol{V}_g.
\end{aligned}$$

This is a straightforward consequence of the definition of the local spaces

$$\begin{aligned}
\boldsymbol{V}^{\mathrm{mix}} &:= \boldsymbol{V}^{\mathrm{hdg}} \oplus \delta\boldsymbol{V}^{\mathrm{hdg}}_{\mathrm{mix}} && W^{\mathrm{mix}} := W_g,\\
\boldsymbol{V}^{\mathrm{hdg}} &:= \boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\mathrm{fillM}} && W^{\mathrm{hdg}} := W_g,\\
\boldsymbol{V}_{\mathrm{mix}} &:= \boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\mathrm{fillM}} && W_{\mathrm{mix}} := \nabla\cdot\boldsymbol{V}_g,
\end{aligned}$$

and of the following properties:

$$\{\boldsymbol{v} \in \delta\boldsymbol{V}^{\mathrm{hdg}}_{\mathrm{mix}} : \nabla\cdot\boldsymbol{v} = 0\} = \{\boldsymbol{0}\} \quad \text{and} \quad \nabla\cdot\delta\boldsymbol{V}_{\mathrm{fillM}} = \{0\}.$$

This immediately implies that the functions $(\mathfrak{Q}\mathsf{m}, P_{\nabla\cdot\boldsymbol{V}_g}\,\mathfrak{U}\mathsf{m})$ are also the same for the three methods. By Theorem 6.1, we then have that $\widehat{u}_h$ is also the same for the three methods. Since $\alpha(u_h - \widehat{u}_h) = 0$ for the three methods, this proves the first property.

By a similar argument, the second property follows from the fact that $\mathfrak{Q}\mathsf{m}$ is the same for the three methods. This completes the proof. $\square$

6.3. **Postprocessings of the approximate flux.** Here we show how to locally postprocess the approximation $(\boldsymbol{q}_h^{\text{hdg}}, \widehat{\boldsymbol{q}}_h^{\text{hdg}} \cdot \boldsymbol{n})$ provided by the HDG method with local spaces $\boldsymbol{V} \times W := \boldsymbol{V}^{\text{hdg}} \times W^{\text{hdg}}$ to obtain two different $H(\text{div}, \Omega)$-conforming, new approximations to the flux. Each is associated to one of the mixed methods obtained in our systematic construction.

We define them as follows. On the element K, we take $\boldsymbol{q}_h^{\text{mix},*}$ as the element of $\boldsymbol{V}^{\text{mix}}(K)$ such that

$$(\boldsymbol{q}_h^{\text{mix},*}, \boldsymbol{v})_K = (\boldsymbol{q}_h^{\text{hdg}}, \boldsymbol{v})_K \qquad \forall \boldsymbol{v} \in \widetilde{\boldsymbol{V}}^{\text{mix}}(K), \tag{6.2a}$$

$$\langle \boldsymbol{q}_h^{\text{mix},*} \cdot \boldsymbol{n}, \mu \rangle_{\partial K} = \langle \widehat{\boldsymbol{q}}_h^{\text{hdg}} \cdot \boldsymbol{n}, \mu \rangle_{\partial K} \qquad \forall \mu \in M(\partial K). \tag{6.2b}$$

On the element K, we take $\boldsymbol{q}_{\text{mix},h}^*$ as the element of $\boldsymbol{V}_{\text{mix}}(K)$ such that

$$(\boldsymbol{q}_{\text{mix},h}^*, \boldsymbol{v})_K = (\boldsymbol{q}_h^{\text{hdg}}, \boldsymbol{v})_K \qquad \forall \boldsymbol{v} \in \widetilde{\boldsymbol{V}}_{\text{mix}}(K), \tag{6.3a}$$

$$\langle \boldsymbol{q}_{\text{mix},h}^* \cdot \boldsymbol{n}, \mu \rangle_{\partial K} = \langle \widehat{\boldsymbol{q}}_h^{\text{hdg}} \cdot \boldsymbol{n}, \mu \rangle_{\partial K} \qquad \forall \mu \in M(\partial K). \tag{6.3b}$$

Proposition 6.4. *We have that*

(i) $\boldsymbol{q}_h^{\text{mix},*} \in H(\text{div}, \Omega)$ *and* $\nabla \cdot \boldsymbol{q}_h^{\text{mix},*} = P_W f$,
(ii) $\boldsymbol{q}_{\text{mix},h}^* \in H(\text{div}, \Omega)$ *and* $\nabla \cdot \boldsymbol{q}_{\text{mix},h}^* = P_{\nabla\cdot\boldsymbol{V}} f$.

Proof. Let us first prove property (i). Since $\boldsymbol{V}^{\text{mix}}(K) = \widetilde{\boldsymbol{V}}^{\text{mix}}(K) \oplus \widetilde{\boldsymbol{V}}^{\text{mix}}(K)^{\perp}$ and $\widetilde{\boldsymbol{V}}^{\text{mix}}(K)^{\perp} \ni \boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n} \in M(\partial K)$ is an isomorphism, it follows that equations (6.2) are uniquely solvable. Moreover, for each $F \in \mathcal{F}(K)$, $\boldsymbol{q}_{\text{mix},h}^* \cdot \boldsymbol{n} \in M(F)$ is uniquely determined by the value $\widehat{\boldsymbol{q}}_h^{\text{hdg}} \cdot \boldsymbol{n}$ on F. Since $\widehat{\boldsymbol{q}}_h^{\text{hdg}}$ is single valued (by the third equation defining the HDG solution), this proves that $\boldsymbol{q}_{\text{mix},h}^* \in H(\text{div}, \Omega)$. Finally, by the second equation defining the HDG method, and since $W^{\text{hdg}} = W^{\text{mix}} := W$, we have that, for all $w \in W^{\text{mix}}$,

$$\begin{aligned}
(f, w)_K &= -(\boldsymbol{q}_h, \nabla w)_K + \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial K} \\
&= -(\boldsymbol{q}_h^{\text{mix},*}, \nabla w)_K + \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial K} \qquad \text{since } \widetilde{\boldsymbol{V}}^{\text{mix}} \supset \nabla W^{\text{mix}}, \\
&= -(\boldsymbol{q}_h^{\text{mix},*}, \nabla w)_K + \langle \boldsymbol{q}_h^{\text{mix},*} \cdot \boldsymbol{n}, w \rangle_{\partial K} \qquad \text{since } \gamma W^{\text{mix}} \subset M, \\
&= (\nabla \cdot \boldsymbol{q}_h^{\text{mix},*}, w)_K.
\end{aligned}$$

Since this is another way of writing that $P_W f = P_W \nabla \cdot \boldsymbol{q}_h^{\text{mix},*}$, property (i) follows from the fact that $\nabla \cdot \boldsymbol{V}^{\text{mix}} = W$. Property (ii) can be proven in a similar way. This completes the proof. $\square$

Finally, we establish estimates for these new approximate fluxes.

Proposition 6.5. *We have that*

$$\begin{aligned}
\|\boldsymbol{q}_{\text{mix},h}^* - \boldsymbol{q}_h\|_K &\leq C_{\widetilde{\boldsymbol{V}}_{\text{mix}}^{\perp}} h_K^{1/2} \|\alpha(u_h - \widehat{u}_h)\|_{\partial K}, \\
\|\boldsymbol{q}_h^{\text{mix},*} - \boldsymbol{q}_h\|_K &\leq C_{(\widetilde{\boldsymbol{V}}^{\text{mix}})^{\perp}} h_K^{1/2} \|\alpha(u_h - \widehat{u}_h)\|_{\partial K}.
\end{aligned}$$

Proof. Let us prove the first inequality. Setting $\boldsymbol{\delta} := \boldsymbol{q}^*_{\text{mix},h} - \boldsymbol{q}_h$, we see by (6.3) that $\boldsymbol{\delta} \in \widetilde{\boldsymbol{V}}(K)^\perp$ is determined by

$$\langle \boldsymbol{\delta} \cdot \boldsymbol{n}, \mu \rangle_{\partial K} = \langle \alpha(u_h - \widehat{u}_h), \mu \rangle_{\partial K} \qquad \forall \mu \in M(\partial K),$$

since $(\widehat{\boldsymbol{q}}_h - \boldsymbol{q}_h) \cdot \boldsymbol{n} = \alpha(u_h - \widehat{u}_h)$. The first inequality follows from taking $\mu = \boldsymbol{\delta} \cdot \boldsymbol{n}$ above and using the definition of $C_{\widetilde{\boldsymbol{V}}^\perp_{\text{mix}}}$. The second inequality can be proven in the same manner. This completes the proof. □

7. Concluding remarks

In [13], superconvergence of DG methods that are *not* hybridizable were obtained. Therein, the HDG$_k$ method on simplicial elements was shown to be superconvergent even if we change the numerical fluxes at the interior faces to

$$\begin{aligned}\widehat{u}_h &= \{u_h\} - \boldsymbol{C}_{12} \cdot [\![u_h]\!] + C_{22}\, [\![\boldsymbol{q}_h]\!],\\ \widehat{\boldsymbol{q}}_h &= \{\boldsymbol{q}_h\} + \boldsymbol{C}_{12}\ [\![\boldsymbol{q}_h]\!] + C_{11}\, [\![u_h]\!],\end{aligned}$$

the resulting method keeps the same superconvergence properties provided the constants $C_{11}, 1/C_{11}, C_{22}, 1/C_{22}, |\boldsymbol{C}_{12}|$ are positive and uniformly bounded. To prove this result, the two sandwiching methods RT$_k$ and BDM$_k$ were used. By using our construction of an HDG method and its two sandwiching mixed methods, such result can be readily extended almost word by word.

Finally, we note that our general theory of M-decompositions for diffusion problems can be *easily* extended to curved elements by following the work done in [15], to HDG and mixed methods for the heat equation by following [5], to the wave equation by following [16], to the velocity gradient-velocity-pressure formulation of the Stokes problem by following [18], and for methods for the equations of linear elasticity with weakly symmetric stress approximations by following [19]. The extension of the theory of M-decompositions to methods for the equations of linear elasticity with strongly symmetric stresses, and to methods for the Maxwell equations constitute subjects of ongoing research.

Appendix A. The properties of the HDG projection and its adjoint

Here, we first show a detailed proof of the approximation properties of the HDG projection, Proposition 3.4, and then sketch the proof of the stability property of its adjoint projection, Proposition 4.2.

A.1. **Proof of Proposition 3.4.** To prove the estimates of Proposition 3.4, the idea is to estimate the quantities $\boldsymbol{\delta}_{\boldsymbol{q}} := \boldsymbol{\Pi}_V \boldsymbol{q} - \boldsymbol{P}_V \boldsymbol{q}$ and $\delta_u := \Pi_W u - P_W u$, and then use the triangle inequality to obtain the desired estimates. We proceed in three steps.

Step 1: The equations for $\boldsymbol{\delta}_{\boldsymbol{q}}$ and δ_u. By the equations defining the projection in Definition 3.1, we have that

$$(\boldsymbol{\delta}_{\boldsymbol{q}}, \boldsymbol{v})_K = 0 \qquad \forall\, \boldsymbol{v} \in \widetilde{\boldsymbol{V}}, \tag{A.1a}$$

$$(\delta_u, w)_K = 0 \qquad \forall\, w \in \widetilde{W}, \tag{A.1b}$$

$$\langle \boldsymbol{\delta}_{\boldsymbol{q}} \cdot \boldsymbol{n} + \alpha(\delta_u), \mu \rangle_{\partial K} = \langle \boldsymbol{I}_{\boldsymbol{q}} \cdot \boldsymbol{n} + \alpha(I_u), \mu \rangle_{\partial K} \quad \forall\, \mu \in M. \tag{A.1c}$$

Here $\boldsymbol{I_q} := \boldsymbol{q} - \boldsymbol{P}_V\boldsymbol{q}$ and $I_u := P_M u - P_W u$. The first equation implies $\boldsymbol{\delta_q} \in \widetilde{\boldsymbol{V}}^{\perp}$, and the second equation implies $\delta_u \in \widetilde{W}^{\perp}$. Therefore

$$(\nabla \cdot \boldsymbol{P}_V\boldsymbol{q}, \delta_u)_K = 0, \qquad \langle \boldsymbol{\delta_q} \cdot \boldsymbol{n}, \delta_u\rangle_{\partial K} = 0, \tag{A.2}$$

where the last equality follows by Proposition 2.2.

Step 2: The estimate of δ_u. Next, we obtain an estimate of δ_u. Taking $\mu = \delta_u$ in (A.1c), and using (A.2), integration by parts and the fact that $\delta_u \in \widetilde{W}^{\perp}$, it follows that

$$\begin{aligned}
\langle \alpha(\delta_u), \delta_u\rangle_{\partial K} &= \langle \boldsymbol{I_q} \cdot \boldsymbol{n} + \alpha(I_u), \delta_u\rangle_{\partial K} \\
&= (\boldsymbol{I_q}, \nabla\delta_u)_K + (\nabla \cdot \boldsymbol{I_q}, \delta_u)_K + \langle \alpha(I_u), \delta_u\rangle_{\partial K} \\
&= \big((Id - P_{\widetilde{W}})\nabla \cdot \boldsymbol{q}, \delta_u\big)_K + \langle \alpha(I_u), \delta_u\rangle_{\partial K} \\
&\le \|(Id - P_{\widetilde{W}})\nabla \cdot \boldsymbol{q}\|_K \, \|\delta_u\|_K + \|\alpha\| \, \|I_u\|_{\partial K} \, \|\delta_u\|_{\partial K}.
\end{aligned}$$

By the definition of the constants $C_{\widetilde{W}^{\perp}}$, $a_{\widetilde{W}^{\perp}}$, and $\|\alpha\|$, we get

$$\|\delta_u\|_{\partial K} \le \frac{C_{\widetilde{W}^{\perp}}}{a_{\widetilde{W}^{\perp}}} \, h_K^{1/2} \, \|(Id - P_{\widetilde{W}})\nabla \cdot \boldsymbol{q}\|_K + \frac{\|\alpha\|}{a_{\widetilde{W}^{\perp}}} \, \|I_u\|_{\partial K}.$$

Since $\delta_u \in \widetilde{W}^{\perp}$, we have that $\|\delta_u\|_K \le C_{\widetilde{W}^{\perp}} \, h_K^{1/2} \, \|\delta_u\|_{\partial K}$, by the definition of the constant $C_{\widetilde{W}^{\perp}}$. As a consequence, we get that

$$\|\delta_u\|_K \le \frac{C^2_{\widetilde{W}^{\perp}}}{a_{\widetilde{W}^{\perp}}} \, h_K \, \|(Id - P_{\widetilde{W}})\nabla \cdot \boldsymbol{q}\|_K + \frac{C_{\widetilde{W}^{\perp}}}{a_{\widetilde{W}^{\perp}}} \|\alpha\| \, h_K^{1/2} \|I_u\|_{\partial K}.$$

Step 3: The estimate of $\boldsymbol{\delta_q}$. Finally, let us estimate $\boldsymbol{\delta_q}$. Taking $\mu = \boldsymbol{\delta_q} \cdot \boldsymbol{n}$ in the boundary equation (A.1c) and applying the Cauchy-Schwarz inequality, we obtain

$$\|\boldsymbol{\delta_q} \cdot \boldsymbol{n}\|_{\partial K} \le \|\boldsymbol{I_q} \cdot \boldsymbol{n}\|_{\partial K} + \|\alpha\| \, \|I_u\|_{\partial K} + \|\alpha\| \, \|\delta_u\|_{\partial K}.$$

Since $\delta_{\boldsymbol{q}} \in \widetilde{\boldsymbol{V}}^{\perp}$, we have that $\|\delta_{\boldsymbol{q}}\|_K \le C_{\widetilde{\boldsymbol{V}}^{\perp}} \, h_K^{1/2} \, \|\delta_{\boldsymbol{q}} \cdot \boldsymbol{n}\|_{\partial K}$, by the definition of the constant $C_{\widetilde{\boldsymbol{V}}^{\perp}}$. As a consequence, we get that

$$\begin{aligned}
\|\boldsymbol{\delta_q}\|_K \le{}& C_{\widetilde{\boldsymbol{V}}^{\perp}} \, h_K^{1/2} \, \|\boldsymbol{I_q} \cdot \boldsymbol{n}\|_{\partial K} + \left(\frac{C_{\widetilde{W}^{\perp}}}{a_{\widetilde{W}^{\perp}}} \, C_{\widetilde{\boldsymbol{V}}^{\perp}} \, \|\alpha\|\right) h_K \, \|(Id - P_{\widetilde{W}})\nabla \cdot \boldsymbol{q}\|_K \\
&+ \left(1 + \frac{\|\alpha\|}{a_{\widetilde{W}^{\perp}}}\right) C_{\widetilde{\boldsymbol{V}}^{\perp}} \, \|\alpha\| \, h_K^{1/2} \, \|I_u\|_{\partial K}.
\end{aligned}$$

This completes the proof.

A.2. **Proof of Proposition 4.2.** Here we sketch the proof of Proposition 4.2 since it is quite similar to the proof of Proposition 3.4. We first estimate the quantities $\delta^*_w d := \Pi^*_W d - d_w$ and $\boldsymbol{\delta}^*_v d := \boldsymbol{\Pi}^*_V d - \boldsymbol{d}_v$, and then use the triangle inequality to obtain the estimates for $\Pi^*_W d$ and $\boldsymbol{\Pi}^*_V d$.

First, by the equations defining the ajoint projection in Definition 4.1, we have that

$$\begin{aligned}
(\boldsymbol{\delta}^*_v d, \boldsymbol{v})_K &= 0 && \forall\, \boldsymbol{v} \in \widetilde{\boldsymbol{V}}, \\
(\delta^*_w d, w)_K &= 0 && \forall\, w \in \widetilde{W}, \\
\langle \boldsymbol{\delta}^*_v d \cdot \boldsymbol{n} - \alpha(\delta^*_w d), \mu\rangle_{\partial K} &= \langle d_\mu - \boldsymbol{d}_v \cdot \boldsymbol{n} + \alpha(d_w), \mu\rangle_{\partial K} && \forall\, \mu \in M.
\end{aligned}$$

The first equation implies $\boldsymbol{\delta}_v^* d \in \widetilde{\boldsymbol{V}}^\perp$, and the second equation implies $\delta_w^* d \in \widetilde{W}^\perp$ since $\boldsymbol{d}_v \in \nabla W$ and $d_w \in \nabla \cdot \boldsymbol{V}$.

Next, we obtain an estimate of $\delta_w^* d$. Take $\mu = \delta_w^* d$ in the previous boundary equation, use the $L^2(\partial K)$-orthogonality of $\delta_w^* d$ and $\boldsymbol{\delta}_v^* d \cdot \boldsymbol{n}$, and apply the Cauchy-Schwarz inequality, we obtain

$$\langle \alpha(\delta_w^* d), \delta_w^* d\rangle_{\partial K} \le \big(\|d_\mu\|_{\partial K} + \|\boldsymbol{d}_v \cdot \boldsymbol{n}\|_{\partial K} + \|\alpha\|\, \|d_w\|_{\partial K}\big)\|\delta_w^* d\|.$$

Now, the desired estimate for $\Pi_W^* d$ comes from using inverse inequalities and the triangle inequality.

Finally, taking $\mu = \boldsymbol{\delta}_v^* d \cdot \boldsymbol{n}$ in the boundary equation and applying the Cauchy-Schwarz inequality, inverse inequalities, and the triangle inequality, we get the estimate for $\boldsymbol{\Pi}_V^* d$. This completes the proof.

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