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LOCAL INVERSE ESTIMATES FOR NON-LOCAL BOUNDARY INTEGRAL OPERATORS

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ABSTRACT. We prove local inverse-type estimates for the four non-local boundary integral operators associated with the Laplace operator on a bounded Lipschitz domain Ω in $\mathbb{R}^d$ for $d \geq 2$ with piecewise smooth boundary. For piecewise polynomial ansatz spaces and $d \in \{2, 3\}$, the inverse estimates are explicit in both the local mesh width and the approximation order. An application to efficiency-type estimates in *a posteriori* error estimation in boundary element methods is given.

1. INTRODUCTION

Inverse estimates are a frequently used tool in the numerical analysis of discretizations of partial differential equations (PDEs). They allow one to bound a stronger (semi)norm of a discrete function by a weaker norm at the expense of negative powers of the mesh width. For example, in the context of finite element methods, it is textbook knowledge that

(1.1)
$$\|h\nabla V_h\|_{L^2(\Omega)} \leq C\,\|V_h\|_{L^2(\Omega)} \quad \text{for all continuous } \mathcal{T}_h\text{-piecewise polynomials } V_h.$$

The constant $C > 0$ depends only on the shape regularity of the underlying triangulation $\mathcal{T}_h$ of $\Omega \subset \mathbb{R}^d$ and the polynomial degree of V_h. Here, $h \in L^\infty(\Omega)$ is the local mesh width function defined by $h|_T := \operatorname{diam}(T)$ for $T \in \mathcal{T}_h$. Inverse estimates have also been derived for fractional order Sobolev spaces [Geo08, GHS05, DFG04]. The usual proof of inverse estimates like (1.1) relies on scaling arguments, i.e., the powers of h arise by elementwise, i.e., *local* considerations and transformations to reference configurations.

In the present work we consider the four classical boundary integral operators (BIOs) associated with the Laplacian. To be specific in this introduction, we focus on the 3D simple-layer integral operator

$$\mathfrak{V}\phi(x) = \frac{1}{4\pi}\int_{\partial\Omega} \frac{1}{|x-y|}\,\phi(y)\,dy \quad \text{for } x \in \partial\Omega, \tag{1.2}$$

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where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain with piecewise C^1-boundary $\partial\Omega$. Let $\Gamma \subseteq \partial\Omega$ be a relatively open subset of the boundary $\partial\Omega$. For $d \in \{2,3\}$ and with the surface gradient $\nabla_\Gamma(\cdot)$, we show the estimate

$$\|h^{1/2}(q+1)^{-1}\nabla_\Gamma \mathfrak{V}\Phi_h\|_{L^2(\Gamma)} \leq C\,\|\Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \qquad \text{for all } \Phi_h \in \mathcal{P}^q(\mathcal{T}_h), \tag{1.3}$$

where $\mathcal{P}^q(\mathcal{T}_h)$ is the space of $\mathcal{T}_h$-piecewise polynomials on Γ of degree $q \in \mathbb{N}_0$; see Section 2.4 for a precise definition. The bound (1.3) can be understood as an inverse estimate: Under appropriate assumptions on Γ, the operator $\mathfrak{V}$ is an isomorphism between $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, so that (1.3) is indeed an inverse estimate for the finite dimensional space $\big\{\mathfrak{V}\Phi_h : \Phi_h \in \mathcal{P}^q(\mathcal{T}_h)\big\}$ when considered with the weighted H^1-seminorm and the natural $H^{1/2}(\Gamma)$-norm. Inverse estimates of the form (1.3) will be shown for all four BIOs associated with the Laplacian and discrete spaces with spatially varying polynomial degree; cf. Corollary 3.2. In fact, in Theorem 3.1 we will show more general results of the form

$$\begin{aligned}&\|w_h\nabla_\Gamma\mathfrak{V}\phi\|_{L^2(\Gamma)}\\ &\quad\leq C\left(\Big\|\frac{w_h}{h^{1/2}}\Big\|_{L^\infty(\Gamma)}\|\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|w_h\phi\|_{L^2(\Gamma)}\right) \qquad \text{for all } \phi \in L^2(\Gamma),\end{aligned} \tag{1.4}$$

where $w_h \in L^\infty(\Omega)$ is a fairly general weight function.

Applications. Inverse estimates of the form (1.3) arise naturally in the analysis of the adaptive BEM (boundary element method) when one generalizes the convergence and quasi-optimality analysis of the adaptive FEM [CKNS08, Ste07, CFPP14] to the adaptive BEM [FKMP13, Gan13]. Indeed, the present results allow us to prove quasi-optimality of adaptive BEM for piecewise smooth geometries and higher (fixed) order discretizations; we refer to [FFK14] and [FFK15], where this application is worked out in detail for weakly singular and hypersingular integral equations on polyhedral surfaces, respectively. While the inverse estimate (1.3) features prominently in the analysis of quasi-optimality of adaptive BEM for symmetric problems, it is also a key ingredient for plain convergence in non-symmetric problems such as FEM-BEM couplings. We refer to [AFF13a] and the precursor preprint [AFF12] of the present work for a convergence proof of the adaptive coupling of FEM and BEM.

In addition, bounds of the form (1.4) allow us to prove novel weak efficiency estimates for the weighted residual error estimators for BEM that are discussed in [Car97, CMS01, CMPS04]. Before detailing this, we emphasize that the optimal convergence behavior of adaptive BEM does not require efficiency of the error estimator, [FFK14, FFK15]; the concept of efficiency is only required to characterize the approximation classes; cf. [CFPP14]. To fix ideas concerning residual error estimators, consider again the weakly singular case and suppose that $\phi \in L^2(\Gamma)$ solves $\mathfrak{V}\phi = f$ for some given $f \in H^1(\Gamma)$. Let Φ_h be the Galerkin approximation of ϕ, where the ansatz space consists of $\mathcal{T}_h$-piecewise polynomials of fixed degree $p \in \mathbb{N}_0$. While the reliability estimate

$$C_{\mathrm{rel}}^{-1}\,\|\phi - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq \eta_{h,\mathfrak{V}} := \|h^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_h)\|_{L^2(\Gamma)} \tag{1.5}$$

is well known for the weighted residual error estimator $\eta_{h,\mathfrak{V}}$ (at least for polyhedral domains Ω), the converse efficiency estimate is not available in the literature (with the exception of some 2D situations, [AFF13b]). However, as a consequence of (1.4), we will see in Corollary 3.4 that

$$C_{\mathrm{eff}}^{-1}\eta_{h,\mathfrak{V}} \le \|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)}, \tag{1.6}$$

which expresses efficiency of $\eta_{h,\mathfrak{V}}$ with respect to the slightly stronger norm $\|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)} \gtrsim \|\phi - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)}$. We refer to Corollary 3.7 for the case of the hypersingular operator. These efficiency bounds are specific instances of new stability estimates for the BIOs in locally weighted L^2-norms detailed in Corollaries 3.3 and 3.6.

Novelty. The results of the present work are required for the convergence analysis of adaptive BEM for both weakly singular and hypersingular integral equations in [FFK14, FFK15]. The discrete inequality (1.3) was first shown independently in [FKMP13] and [Gan13], however, under some restrictions: The work [FKMP13] considers only lowest-order polynomials, i.e., $\mathcal{T}_h$-piecewise constants, but works for polyhedral boundaries Γ. The work [Gan13] proves (1.3) for arbitrary $\mathcal{T}_h$-piecewise polynomials, but its wavelet-based analysis is restricted to $C^{1,1}$-boundaries Γ and the constant $C > 0$ depends on the polynomial degree. Our proof of (1.4) generalizes the works [FKMP13, Gan13] in the following ways: 1) we generalize the analysis of [FKMP13] for the simple-layer operator $\mathfrak{V}$ to all four BIOs associated with the Laplacian (i.e., the double-layer operator $\mathfrak{K}$, its adjoint $\mathfrak{K}'$, and the hypersingular operator $\mathfrak{W}$); 2) we extend our previous analysis from polyhedral domains to piecewise smooth geometries; 3) we lift the restriction to fixed-order polynomial ansatz spaces and permit very general ansatz spaces; 4) for ansatz spaces of piecewise polynomials of arbitrary order, we make the dependence on the polynomial degree in the inverse estimates explicit.

The technical difficulty in the proof of (1.4) and (1.3) lies in the non-locality of the boundary integral operator $\mathfrak{V}$, which precludes simple elementwise considerations. We cope with the non-locality of the BIOs by splitting them into near-field and far-field contributions, each requiring different tools. The analysis of the near-field part relies on local arguments and stability properties of the BIOs. For the far-field part, the key observation is that the BIOs are derived from two volume potentials, namely, the simple-layer potential $\widetilde{\mathfrak{V}}$ and the double-layer potential $\widetilde{\mathfrak{K}}$ by taking appropriate traces. Since these potentials solve elliptic equations, "interior regularity" estimates are available for them and trace inequalities imply corresponding estimates for the BIOs. Section 4 proves the relevant estimates for the simple-layer potential $\widetilde{\mathfrak{V}}$, whereas Section 5 is concerned with the double-layer potential $\widetilde{\mathfrak{K}}$. The final Section 6 then combines these results to give the proof of Theorem 3.1.

Although the present paper considers only the four BIOs associated with the Laplacian, the scope is wider. As just mentioned, the key tool are interior estimates for potentials; such estimates are available for many elliptic equations, for example, the Lamé system, so that we expect that corresponding results can be proved as well for BIOs associated with these problems.

General notation. We close the introduction by stating that $|\cdot|$ denotes, depending on the context, the absolute value of a real number, the Euclidean norm of a vector in $\mathbb{R}^d$, the Lebesgue measure of a subset of $\mathbb{R}^{d-1}$ or $\mathbb{R}^d$ or the $(d-1)$-dimensional surface measure of a subset of $\partial\Omega$. The notation $a \lesssim b$ abbreviates $a \le Cb$ for some constant $C > 0$, and we write $a \simeq b$ to abbreviate $a \lesssim b \lesssim a$.

We write $B_r(x) = \{z \in \mathbb{R}^d : |x - z| < r\}$ or $B_r(x) = \{z \in \mathbb{R}^{d-1} : |x - z| < r\}$ for the open balls with radius r and center x in $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$.

2. Spaces, operators, and meshes

2.1. **Sobolev spaces.** Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with piecewise C^1-boundary $\partial\Omega$, [SS11, Def. 2.2.10], and corresponding exterior domain $\Omega^{\rm ext} := \mathbb{R}^d \setminus \overline{\Omega}$. The exterior unit normal vector field on $\partial\Omega$ is denoted by $\boldsymbol{\nu}$. Throughout, we will assume that either $\Gamma = \partial\Omega$, or that $\Gamma \subseteq \partial\Omega$ is a non-empty, relatively open set that stems from a Lipschitz dissection $\partial\Omega = \Gamma \cup \partial\Gamma \cup (\partial\Omega \setminus \Gamma)$ as described in [McL00, pp. 99].

The non-negative order Sobolev spaces $H^{1/2+s}(\partial\Omega)$ for $s \in \{-1/2, 0, 1/2\}$ are defined as in [McL00, pp. 99] by use of Bessel potentials on $\mathbb{R}^{d-1}$ and liftings via the bi-Lipschitz maps that describe $\partial\Omega$. We also need the spaces $H^{1/2+s}(\Gamma)$ and $\widetilde{H}^{1/2+s}(\Gamma)$. In accordance with [McL00], these are defined as follows:

$$H^{1/2+s}(\Gamma) := \{v|_\Gamma \colon v \in H^{1/2+s}(\partial\Omega)\}, \tag{2.1}$$

$$\widetilde{H}^{1/2+s}(\Gamma) := \{v \colon E_{0,\Gamma} v \in H^{1/2+s}(\partial\Omega)\}, \tag{2.2}$$

where $E_{0,\Gamma}$ denotes the operator that extends a function defined on Γ to a function on $\partial\Omega$ by zero. These spaces are endowed with their natural norms, i.e., the quotient norm $\|v\|_{H^{1/2+s}(\Gamma)} := \inf\{\|V\|_{H^{1/2+s}(\partial\Omega)} \colon V|_\Gamma = v\}$ and $\|v\|_{\widetilde{H}^{1/2+s}(\Gamma)} := \|E_{0,\Gamma} v\|_{H^{1/2+s}(\partial\Omega)}$. Owing to the assumption that $\partial\Omega = \Gamma \cup \partial\Gamma \cup (\partial\Omega \setminus \Gamma)$ is a Lipschitz dissection, we have the following facts (see Appendix B for a sketch of the proof):

Facts 2.1. (i) Since $\partial\Omega$ is piecewise C^1, the surface gradient $\nabla_\Gamma u$ of $u \in H^1(\partial\Omega)$ is defined pointwise a.e. on each C^1 surface piece Γ_i as follows: For a C^1-parametrization $\xi_i : \widehat{\Gamma}_i \to \Gamma_i$ of a surface piece Γ_i the surface gradient $(\nabla_\Gamma u)|_{\Gamma_i}$ is given by the requirement

$$(\nabla_\Gamma u)|_{\Gamma_i} \circ \xi_i = \sum_{k,\ell} g^{k\ell} \partial_k (u \circ \xi_i) \partial_\ell \xi_i, \tag{2.3}$$

where the matrix $(g^{k\ell})_{k,\ell=1}^{d-1}$ is the inverse of the Gramian matrix $G(x) = D\xi_i(x)^\top D\xi_i(x)$.

For $s = 1/2$, we have the norm equivalences $\|u\|^2_{H^1(\partial\Omega)} \simeq \|u\|^2_{L^2(\partial\Omega)} + \|\nabla_\Gamma u\|^2_{L^2(\partial\Omega)}$ and $\|u\|^2_{\widetilde{H}^1(\Gamma)} \simeq \|u\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma u\|^2_{L^2(\Gamma)}$.

(ii) For $s = 0$, the norms $\|u\|_{H^{1/2}(\partial\Omega)}$ and $\|u\|_{\widetilde{H}^{1/2}(\Gamma)}$ can equivalently be described by the Aronstein-Slobodeckii norms of u and $E_{0,\Gamma} u$ (cf. [McL00, (3.18)] for the definition of the Aronstein-Slobodeckii norm).

(iii) For $s = 0$, the spaces $H^{1/2}(\partial\Omega)$ (respectively $\widetilde{H}^{1/2}(\Gamma)$) are equivalently obtained by interpolation with the K-method between the cases $s = -1/2$ (i.e., $L^2(\partial\Omega)$ respectively $L^2(\Gamma)$) and $s = 1/2$ (i.e., $H^1(\partial\Omega)$ respectively $\widetilde{H}^1(\Gamma)$).

Negative order Sobolev spaces are defined by duality, namely, for $s \in \{-1/2, 0, 1/2\}$,

$$\begin{aligned} H^{-1/2}(\partial\Omega) &:= H^{1/2}(\partial\Omega)', \\ \widetilde{H}^{-(1/2+s)}(\Gamma) &:= H^{1/2+s}(\Gamma)', \\ H^{-(1/2+s)}(\Gamma) &:= \widetilde{H}^{1/2+s}(\Gamma)', \end{aligned} \tag{2.4}$$

where duality pairings $\langle\cdot,\cdot\rangle$ are understood to extend the standard L^2-scalar product on $\partial\Omega$ or Γ (indicated by the corresponding index, if necessary). We observe the continuous inclusions

$$\widetilde{H}^{\pm(1/2+s)}(\Gamma) \subseteq H^{\pm(1/2+s)}(\Gamma) \quad \text{as well as} \quad \widetilde{H}^{\pm(1/2+s)}(\partial\Omega) = H^{\pm(1/2+s)}(\partial\Omega).$$

We also note that for $\psi \in L^2(\Gamma)$ the zero extension $E_{0,\Gamma}\psi$ satisfies $E_{0,\Gamma}\psi \in H^{-1/2}(\partial\Omega)$ with

$$\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} = \|E_{0,\Gamma}\psi\|_{H^{-1/2}(\partial\Omega)}. \tag{2.5}$$

We denote by $\gamma_0^{\rm int}(\cdot) : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ the interior trace operator, i.e., $\gamma_0^{\rm int}u$ is the restriction of a function $u \in H^1(\Omega)$ to the boundary $\partial\Omega$. With $H^1_\Delta(\Omega) := \{u \in H^1(\Omega) : -\Delta u \in L^2(\Omega)\}$, the interior conormal derivative operator $\gamma_1^{\rm int} : H^1_\Delta(\Omega) \to H^{-1/2}(\partial\Omega)$ is defined by the first Green's formula, viz.,

$$\langle \gamma_1^{\rm int}u\,,\, v\rangle_{\partial\Omega} = \langle \nabla u\,,\, \nabla v\rangle_\Omega - \langle -\Delta u\,,\, v\rangle_\Omega \quad \text{for all } v \in H^1(\Omega). \tag{2.6}$$

The exterior trace $\gamma_0^{\rm ext}$ and the exterior conormal derivative operator $\gamma_1^{\rm ext}$ are defined analogously to their interior counterparts. To that end, we fix a bounded Lipschitz domain $U \subset \mathbb{R}^d$ with $\overline{\Omega} \subset U$. The exterior trace operator $\gamma_0^{\rm ext} : H^1(U\setminus\overline{\Omega}) \to H^{1/2}(\partial\Omega)$ is defined by restricting to $\partial\Omega$, and the exterior conormal derivative $\gamma_1^{\rm ext}$ is defined as usual by $\langle \gamma_1^{\rm ext}u\,,\, v\rangle_{\partial\Omega} = -\langle \nabla u\,,\, \nabla v\rangle_{U\setminus\overline{\Omega}} - \langle \Delta u\,,\, v\rangle_{U\setminus\overline{\Omega}}$ for all $v \in H^1(U\setminus\overline{\Omega})$ with $\gamma_0^{\rm ext}v = 0$ on ∂U.

For a function u that admits both conormal derivatives or both traces, we define the jumps $[\gamma_1 u] := \gamma_1^{\rm ext}u - \gamma_1^{\rm int}u$ and $[u] = \gamma_0^{\rm ext}u - \gamma_0^{\rm int}u$.

Remark 2.2. The operator $\gamma_1^{\rm int}(\cdot)$ generalizes the classical normal derivative operator: If $u \in H^1_\Delta(\Omega)$ is sufficiently smooth near a boundary point x_0, then $\gamma_1^{\rm int}u$ can be represented near x_0 by a function given by the pointwise defined normal derivative $\partial_{\boldsymbol{\nu}}u$.

For sufficiently smooth functions u, the surface gradient of $\gamma_0^{\rm int}u = \gamma_0^{\rm ext}u$ on each C^1 surface piece Γ_i is the tangential component of ∇u; that is, with the outer normal vector $\boldsymbol{\nu}$ one has $(\nabla_\Gamma \gamma_0^{\rm int}u)|_{\Gamma_i} = (\gamma_0^{\rm int}(\nabla u - \boldsymbol{\nu}(\boldsymbol{\nu}\cdot\nabla u))|_{\Gamma_i}$. By smoothness of u, the operator $\gamma_0^{\rm int}$ may be replaced with $\gamma_0^{\rm ext}$ in this identity. ∎

2.2. **Boundary integral operators.** We briefly introduce the pertinent boundary integral operators and refer to the monographs [McL00, HW08, SS11] for further details and proofs. Green's function for the Laplace operator is given by

$$G(x,y) = \begin{cases} -\frac{1}{|\mathbb{S}^1|}\log|x-y|, & \text{for } d = 2, \\ +\frac{1}{|\mathbb{S}^{d-1}|}\,|x-y|^{-(d-2)}, & \text{for } d \geq 3, \end{cases} \tag{2.7}$$

where $|\mathbb{S}^{d-1}|$ denotes the surface measure of the Euclidean sphere in $\mathbb{R}^d$, e.g., $|\mathbb{S}^1| = 2\pi$ and $|\mathbb{S}^2| = 4\pi$. The classical simple-layer potential $\widetilde{\mathfrak{V}}$ and the double-layer potential $\widetilde{\mathfrak{K}}$ are formally defined by

$$(\widetilde{\mathfrak{V}}\psi)(x) := \int_{\partial\Omega} G(x,y)\psi(y)\,dy,$$
$$(\widetilde{\mathfrak{K}}v)(x) := \int_{\partial\Omega} \partial_{\boldsymbol{\nu}(y)} G(x,y)v(y)\,dy, \qquad x \in \mathbb{R}^d \setminus \partial\Omega;$$

here, $\partial_{\boldsymbol{\nu}(y)}$ denotes the (outer) normal derivative with respect to the variable y. These operators are defined for sufficiently smooth functions ψ, v and can be extended to bounded linear operators

$$\widetilde{\mathfrak{V}} \in L\big(H^{-1/2}(\partial\Omega); H^1(U)\big) \quad \text{and} \quad \widetilde{\mathfrak{K}} \in L\big(H^{1/2}(\partial\Omega); H^1(U \setminus \partial\Omega)\big). \tag{2.8}$$

It is well known that $\Delta\widetilde{\mathfrak{V}}\psi = 0 = \Delta\widetilde{\mathfrak{K}}v$ in $U \setminus \partial\Omega$ for all $\psi \in H^{-1/2}(\partial\Omega)$ and $v \in H^{1/2}(\partial\Omega)$. The simple-layer, double-layer, adjoint double-layer, and the hypersingular integral operator are defined as follows:

$$\mathfrak{V} = \gamma_0^{\text{int}}\widetilde{\mathfrak{V}}, \quad \mathfrak{K} = \frac{1}{2} + \gamma_0^{\text{int}}\widetilde{\mathfrak{K}}, \quad \mathfrak{K}' = -\frac{1}{2} + \gamma_1^{\text{int}}\widetilde{\mathfrak{V}}, \text{ and } \quad \mathfrak{W} = -\gamma_1^{\text{int}}\widetilde{\mathfrak{K}}. \tag{2.9}$$

These are bounded linear operators for $s \in \{-1/2, 0, 1/2\}$ as follows:

$$\mathfrak{V} \in L(H^{-1/2+s}(\partial\Omega); H^{1/2+s}(\partial\Omega)), \tag{2.10}$$
$$\mathfrak{K} \in L(H^{1/2+s}(\partial\Omega); H^{1/2+s}(\partial\Omega)), \tag{2.11}$$
$$\mathfrak{K}' \in L(H^{-1/2+s}(\partial\Omega); H^{-1/2+s}(\partial\Omega)), \tag{2.12}$$
$$\mathfrak{W} \in L(H^{1/2+s}(\partial\Omega); H^{-1/2+s}(\partial\Omega)), \tag{2.13}$$

The operators $\widetilde{\mathfrak{V}}$, $\mathfrak{V}$, $\mathfrak{K}'$ will often be applied to functions in $L^2(\Gamma)$. Throughout this paper, we employ the convention that for $\psi \in L^2(\Gamma)$ we implicitly extend by zero, e.g.,

$$\widetilde{\mathfrak{V}}\psi \text{ means } \widetilde{\mathfrak{V}}(E_{0,\Gamma}\psi), \quad \mathfrak{V}\psi \text{ means } \mathfrak{V}(E_{0,\Gamma}\psi), \quad \text{and } \mathfrak{K}'\psi \text{ means } \mathfrak{K}'(E_{0,\Gamma}\psi). \tag{2.14}$$

An analogous extension is obviously used when $\widetilde{\mathfrak{K}}$, $\mathfrak{K}$, $\mathfrak{W}$ are applied to a $v \in \widetilde{H}^{1/2}(\Gamma)$.

Remark 2.3. Ellipticity of $\mathfrak{V}$ and $\mathfrak{W}$ is not used in our analysis of Theorem 3.1 and Corollary 3.2. In particular, there is no need to scale Ω to ensure $\text{diam}(\Omega) < 1$ in 2D or to assume that Γ is connected. ∎

2.3. **Surface simplices and admissible triangulations.** Fix the reference simplex $T_{\text{ref}} := \{x \in \mathbb{R}^{d-1}, 0 < x_1, \ldots, x_{d-1}, \sum_{j=1}^{d-1} x_j < 1\}$, which is the open convex hull of the d vertices $\{0, e_1, \ldots, e_{d-1}\}$ ("0-faces"). The convex hull of any $j+1$ of these vertices is called a "j-face" of T_{ref}.

We require the concept of regular, κ-shape regular triangulations $\mathcal{T}_h$ of Γ.

Definition 2.4 (regular and shape-regular triangulations)**.** A set $\mathcal{T}_h$ of subsets of Γ is called a *regular* triangulation of Γ if the following is true:

(i) The *elements* $T \in \mathcal{T}_h$ are relatively open subsets of Γ and each T is the image of T_{ref} under an *element map* $\gamma_T : \overline{T_{\text{ref}}} \to \overline{T}$. The element map γ_T is assumed to be bijective and C^1 on $\overline{T_{\text{ref}}}$.

(ii) The elements cover Γ: $\bigcup_{T \in \mathcal{T}_h} \overline{T} = \overline{\Gamma}$.

(iii) "no hanging nodes": For each pair $(T, T') \in \mathcal{T}_h \times \mathcal{T}_h$, the intersection $\overline{T} \cap \overline{T'}$ is either empty or there are two j-faces f, $f' \subseteq \partial T_{\rm ref}$ of $T_{\rm ref}$ with $j \in \{0, \ldots, d-2\}$ such that $\overline{T} \cap \overline{T'} = \gamma_T(f) = \gamma_{T'}(f')$.

(iv) Parametrizations of common boundary parts of neighboring elements are compatible: If $\emptyset \neq \overline{T} \cap \overline{T'} = \gamma_T(f) = \gamma_{T'}(f')$, then $\gamma_T^{-1} \circ \gamma_{T'} : f' \to f$ is an affine isomorphism.

We call the images of vertices of $T_{\rm ref}$ under the element maps *nodes* of $\mathcal{T}_h$ and collect them in the set $\mathcal{N}_h$. The images of the $(d-2)$-faces of $T_{\rm ref}$ are called *facets* of $\mathcal{T}_h$ and collected in the set $\mathcal{F}_h$. For each $T \in \mathcal{T}_h$, we set $h(T) := \operatorname{diam}(T) := \sup_{x,y \in T} |x-y|$.

A regular triangulation is called *κ-shape regular*, if the element maps γ_T satisfy the following:

(v) Let $G_T(x) := D\gamma_T(x)^\top D\gamma_T(x) \in \mathbb{R}^{(d-1)\times(d-1)}$ be the symmetric Gramian matrix of γ_T. The triangulation is called κ-shape regular if for all $T \in \mathcal{T}_h$ the extremal eigenvalues $\lambda_{\min}(G_T(x))$ and $\lambda_{\max}(G_T(x))$ of $G_T(x)$ satisfy

$$\sup_{x \in T_{\rm ref}} \left(\frac{h(T)^2}{\lambda_{\min}(G_T(x))} + \frac{\lambda_{\max}(G_T(x))}{h(T)^2} \right) \leq \kappa.$$

(vi) If $d = 2$, we require explicitly that the element sizes of neighboring elements are comparable:

$$h(T) \leq \kappa h(T') \qquad \text{for all } T, T' \text{ with } \overline{T} \cap \overline{T'} \neq \emptyset.$$

With each triangulation $\mathcal{T}_h$, we associate the local mesh size function $h \in L^\infty(\Gamma)$ which is defined elementwise by $h|_T := h(T)$ for all $T \in \mathcal{T}_h$. We note that for a κ-shape regular triangulation we have

$$\max_{T \in \mathcal{T}_h} \frac{h(T)^{d-1}}{|T|} \lesssim 1, \tag{2.15}$$

where the implied constant depends solely on κ.

If Γ is the union of pieces of $(d-1)$-dimensional hyperplanes and the element maps are *affine*, then the Gramians are elementwise constants and Definition 2.4 generalizes the classical concept of a κ-shape regular triangulation of Γ. In the non-affine case, the following example illustrates how triangulations as stipulated in Definition 2.4 can be created; cf. [SS11, Section 4.1.2]:

Example 2.5. Let $\Gamma \subseteq \partial\Omega$ be an open surface piece and assume $\Gamma = \gamma(\widehat{\Gamma})$ for some reference configuration $\widehat{\Gamma} \subseteq \mathbb{R}^{d-1}$ and some sufficiently smooth map γ. Let $\widehat{\mathcal{T}}_h = \{\widehat{T}_1, \ldots, \widehat{T}_N\}$ be a standard, regular, shape-regular triangulation of $\widehat{\Gamma}$ with affine element maps $\widehat{\gamma}_{\widehat{T}_i}$, $i = 1, \ldots, N$. Then, the triangulation with elements $T = \gamma \circ \gamma_{\widehat{T}_i}(T_{\rm ref})$ and element maps $\gamma \circ \gamma_{\widehat{T}_i}$ satisfies the hypotheses of Definition 2.4. This concept generalizes to surfaces consisting of several patches; it is worth emphasizing that in that case the patch parametrizations need to match at patch boundaries. ∎

For an element $T \in \mathcal{T}_h$, we define the element patch $\omega_h(T)$ by

$$\omega_h(T) := \left(\bigcup \left\{ \overline{T'} \,:\, T' \in \mathcal{T}_h \text{ with } \overline{T} \cap \overline{T'} \neq \emptyset \right\} \right)^\circ. \tag{2.16}$$

The assumptions on the element maps of a κ-shape regular triangulation imply that elements of a patch are comparable in size. Furthermore, the fact that Γ results from a Lipschitz dissection of $\partial\Omega$ imposes certain topological restrictions on the patches.

Lemma 2.6. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation. Then, there is a constant $C > 0$ that depends solely on κ and the Lipschitz character of $\partial\Omega$ such that the following holds:*

(i) *$h(T) \leq Ch(T')$ for any two elements T, T' with $\overline{T} \cap \overline{T'} \neq \emptyset$.*
(ii) *The number of elements in an element patch is bounded by C.*
(iii) *For any two elements T, T' in the element patch $\omega_h(T'')$ there is a sequence $T = T_0, \dots, T_n = T'$ of elements $T_i \in \mathcal{T}$, $i = 0, \dots, n$, in $\omega_h(T'')$ such that two successive elements T_i, T_{i+1} share a common facet: $\overline{T_i} \cap \overline{T_{i+1}} \in \mathcal{F}_h$ for $i = 0, \dots, n-1$.*

Sketch of proof. Statement (iii): We first show (iii) for the node patch

$$\omega_h(z) := \Big(\bigcup\{\overline{T} \,:\, T \in \mathcal{T}_h \text{ with } z \in \overline{T}\}\Big)^\circ$$

and any node z of T''. This follows from the fact that Γ results from a Lipschitz dissection and considerations in $\mathbb{R}^{d-1}$ using local charts. Indeed, after a Euclidean change of coordinates, we may assume that $\partial\Omega$ is (locally) a hypograph, i.e., there is a Lipschitz continuous function $\Lambda : B_r(0) \to \mathbb{R}$ with $r > 0$ such that the set $\{(x, \Lambda(x)) \colon x \in B_r(0)\} \subset \partial\Omega$. Without loss of generality, we assume the Euclidean coordinate change is such that $z = (0, \Lambda(0))$. One may also assume (cf. [Ste70, Thm. 3, Sect. VI]) that Λ is defined on $\mathbb{R}^{d-1}$ and Lipschitz continuous so that the map $\widetilde{\Lambda} : \mathbb{R}^d \to \mathbb{R}^d$ given by $(x, t) \mapsto (x, \Lambda(x) + t)$ is bi-Lipschitz.

We distinguish the cases $z \in \Gamma$ and $z \in \partial\Gamma$. Let z be an interior point of Γ. Then, the pull-backs $\widehat{T} := \widetilde{\Lambda}^{-1}(T)$, $T \subseteq \omega_h(z)$, are contained in the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ and (identifying this hyperplane with $\mathbb{R}^{d-1}$) completely cover a neighborhood of $0 \in \mathbb{R}^{d-1}$. This together with (iii) of Definition 2.4 shows the claim. If $z \in \partial\Gamma$, then the fact that the elements are contained in Γ and that Γ results from a Lipschitz dissection implies that near $0 \in \mathbb{R}^{d-1}$, the pull-backs $\widehat{T}$ are all on one side of a Lipschitz graph in $\mathbb{R}^{d-1}$. This together with (iii) of Definition 2.4 again implies the claim. Since $\omega_h(T'')$ is the union of the d node patches $\omega_h(z)$ associated with the d nodes of T'', this concludes the proof of (iii).

Statement (ii): Consider the case of an interior point $z \in \Gamma$. The assumption (iii) of Definition 2.4 and the fact that the map $\widetilde{\Lambda}$ is bi-Lipschitz implies that the solid angles of the elements $\widehat{T}$ at 0 are bounded away from zero by a constant that depends solely on κ and $\widetilde{\Lambda}$. This implies the claim for a node patch $\omega_h(z)$ and thus for $\omega_h(T)$ with $T \in \mathcal{T}_h$.

Statement (i): For $d = 2$, this follows by definition. For $d \geq 3$ we first note that two elements sharing a facet $f \in \mathcal{F}_h$ have comparable size by (iii)–(v) of Definition 2.4. We conclude the proof with the aid of statements (iii) and (ii). ☐

2.4. Admissible weight functions and discrete spaces.

Definition 2.7 (σ-admissible weight functions and polynomial degree distributions)**.** A function $w_h \in L^\infty(\Gamma)$ is σ-*admissible* with respect to $\mathcal{T}_h$ if

$$\|w_h\|_{L^\infty(T)} \leq \sigma\, w_h(x) \quad \text{for almost all } x \in \omega_h(T).$$

A function $q_h \in L^\infty(\Gamma)$ is called a σ-*admissible polynomial degree distribution* with respect to $\mathcal{T}_h$, if q_h is σ-admissible with respect to $\mathcal{T}_h$ and $q_h(T) := q_h|_T \in \mathbb{N}_0$ for all $T \in \mathcal{T}_h$.

We write

$$\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h) := \big\{\Psi_h \in L^2(\Gamma) \,:\, \forall T \in \mathcal{T}_h \quad \Psi_h \circ \gamma_T \text{ is a polynomial of degree } \leq q_h(T)\big\}, \tag{2.17}$$

for the space of (discontinuous) piecewise polynomials of local degree $q_h(T)$. Moreover, we introduce spaces of continuous piecewise polynomials of local degree $q_h(T)+1$ by

$$\mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h) := \mathcal{P}^{\mathbf{q}+1}(\mathcal{T}_h) \cap H^1(\Gamma), \tag{2.18}$$

$$\widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h) := \mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h) \cap \widetilde{H}^1(\Gamma). \tag{2.19}$$

We note the inclusions $\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h) \subset L^2(\Gamma) \subset \widetilde{H}^{-1/2}(\Gamma)$, $\widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h) \subset \widetilde{H}^1(\Gamma) \subset \widetilde{H}^{1/2}(\Gamma)$, and $\mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h) \subset H^1(\Gamma)$, as well as $\widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h) = \mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h)$ in case of $\Gamma = \partial\Omega$.

For $q \in \mathbb{N}_0$, the use of non-boldface superscripts in $\mathcal{P}^q(\mathcal{T}_h)$, $\mathcal{S}^{q+1}(\mathcal{T}_h)$, and $\widetilde{\mathcal{S}}^{q+1}(\mathcal{T}_h)$ indicates that a constant polynomial degree is employed.

Remark 2.8. In Definition 2.4 the conditions on the triangulation are formulated so as to ensure that the spaces $\mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h)$ of continuous functions have good approximation properties. For the spaces $\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$ of functions that may be discontinuous across element boundaries, the conditions (iii) and (iv) in Definition 2.4 could be relaxed. ∎

3. Main result and applications

3.1. **Inverse estimates.** The following Theorem 3.1 is the main result of this work.

Theorem 3.1. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ and let $w_h \in L^\infty(\Gamma)$ be a σ-admissible weight function with respect to $\mathcal{T}_h$. Then,*

$$\begin{aligned}&\|w_h \nabla_\Gamma \mathfrak{V}\psi\|_{L^2(\Gamma)} + \|w_h \mathfrak{K}'\psi\|_{L^2(\Gamma)}\\ &\qquad \leq C_{\rm inv}\big(\|w_h/h^{1/2}\|_{L^\infty(\Gamma)}\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|w_h\psi\|_{L^2(\Gamma)}\big),\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\|w_h \nabla_\Gamma \mathfrak{K}v\|_{L^2(\Gamma)} + \|w_h \mathfrak{W}v\|_{L^2(\Gamma)}\\ &\qquad \leq C_{\rm inv}\big(\|w_h/h^{1/2}\|_{L^\infty(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma)} + \|w_h\nabla_\Gamma v\|_{L^2(\Gamma)}\big),\end{aligned} \tag{3.2}$$

for all functions $\psi \in L^2(\Gamma)$ and all $v \in \widetilde{H}^1(\Gamma)$. The constant $C_{\rm inv} > 0$ depends only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and σ.

In Corollary 3.2 below, we apply the estimates (3.1)–(3.2) of Theorem 3.1 to discrete functions $\Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$ and $V_h \in \widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h)$. We mention that the restriction to $d \in \{2,3\}$ in Corollary 3.2 is due to the fact that the underlying reference [KMR16] restricts to this setting.

Corollary 3.2. *Let $d \in \{2,3\}$ and let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Suppose that q_h is a σ-admissible polynomial degree distribution with respect to $\mathcal{T}_h$. Then, there exists a constant $\widetilde{C}_{\rm inv} > 0$ such that the following estimates hold* *for all discrete functions $\Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$ and $V_h \in \widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h)$:*

(3.3)
$$\|h^{1/2}(q_h+1)^{-1}\,\nabla_\Gamma \mathfrak{V}\Psi_h\|_{L^2(\Gamma)} + \|h^{1/2}(q_h+1)^{-1}\,\mathfrak{K}'\Psi_h\|_{L^2(\Gamma)} \le \widetilde{C}_{\rm inv}\|\Psi_h\|_{\widetilde{H}^{-1/2}(\Gamma)},$$

(3.4)
$$\|h^{1/2}(q_h+1)^{-1}\,\nabla_\Gamma \mathfrak{K}V_h\|_{L^2(\Gamma)} + \|h^{1/2}(q_h+1)^{-1}\,\mathfrak{W}V_h\|_{L^2(\Gamma)} \le \widetilde{C}_{\rm inv}\|V_h\|_{\widetilde{H}^{1/2}(\Gamma)}.$$

The constant $\widetilde{C}_{\rm inv} > 0$ depends only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and the σ-admissibility of q_h, but is otherwise independent of the polynomial degrees and the mesh $\mathcal{T}_h$.

Proof. We start with the following two inverse estimates:

$$\|h^{1/2}(q_h+1)^{-1}\,\Psi_h\|_{L^2(\Gamma)} \lesssim \|\Psi_h\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h), \tag{3.5}$$

$$\|h^{1/2}(q_h+1)^{-1}\nabla_\Gamma V_h\|_{L^2(\Gamma)} \lesssim \|V_h\|_{\widetilde{H}^{1/2}(\Gamma)} \quad \text{for all } V_h \in \widetilde{\mathcal{S}}^{\mathbf{q}+1}(\mathcal{T}_h), \tag{3.6}$$

where the implied constants depend solely on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and the σ-admissibility of q_h. The bound (3.5) is essentially taken from [Geo08, Thm. 3.9]. However, since the non-trivial interpolation argument is not worked out in [Geo08, Thm. 3.9] and since [Geo08, Thm. 3.9] is not concerned with open surfaces Γ, we present the details in Lemma A.1. We remark that its proof employs the characterization of fractional Sobolev norms in terms of the Aronstein-Slobodeckii norm. The bound (3.6) follows also from polynomial inverse estimates and an interpolation argument for spaces of piecewise polynomials, which is non-trivial; see [KMR16] for details. We also refer to [AFF15, Prop. 5] for the h-version of (3.6), in which the dependence on the polynomial degree q_h is left unspecified.

We define a weight function by $w_h := h^{1/2}(q_h+1)^{-1}$. Note that $\|w_h/h^{1/2}\|_{L^\infty(\Gamma)} \le 1$ and that w_h is τ-admissible, where τ depends only on κ and σ. The combination of (3.5) with (3.1) leads to (3.3). The bound (3.6) in conjunction with (3.2) yields (3.4). ☐

3.2. **Application to efficiency of residual error estimation.**

3.2.1. *Weakly singular integral equations.* The next corollary proves that the estimate (3.1) provides stability of $\mathfrak{V}$ and $\mathfrak{K}'$ in weighted norms for subspaces $(1-P_h)L^2(\Gamma) \subseteq L^2(\Gamma)$, where P_h is some projection operator. Note that the following corollary is in particular applicable to the Galerkin projection onto $\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$.

Corollary 3.3. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ, and let X_h be such that $\mathcal{P}^0(\mathcal{T}_h) \subseteq X_h \subset L^2(\Gamma)$. Suppose that X_h is a closed subspace of $\widetilde{H}^{-1/2}(\Gamma)$ (and hence also of $L^2(\Gamma)$). Let $\Pi_{X_h} : L^2(\Gamma) \to X_h$ be the L^2-orthogonal projection onto X_h and $\mathbb{P}_h : \widetilde{H}^{-1/2}(\Gamma) \to X_h \subseteq \widetilde{H}^{-1/2}(\Gamma)$ denote an arbitrary $\widetilde{H}^{-1/2}(\Gamma)$-stable projection onto X_h. Then, there is a constant $\widetilde{C}_{\rm inv} > 0$ depending only on the κ-shape regularity of $\mathcal{T}_h$, the stability constant of $\mathbb{P}_h$, $\partial\Omega$, and Γ, such that for $P_h \in \{\Pi_{X_h}, \mathbb{P}_h\}$ and all $\phi \in L^2(\Gamma)$ we have*

(3.7)
$$\|h^{1/2}\nabla_\Gamma \mathfrak{V}(1-P_h)\phi\|_{L^2(\Gamma)} + \|h^{1/2}\mathfrak{K}'(1-P_h)\phi\|_{L^2(\Gamma)} \le \widetilde{C}_{\rm inv}\|h^{1/2}(1-P_h)\phi\|_{L^2(\Gamma)}.$$

Proof. Let $\Pi_h : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_h)$ be the $L^2(\Gamma)$-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$. For arbitrary $w \in H^{1/2}(\partial\Omega)$ we get by transformation to the reference element and standard approximation results that $\|(1-\Pi_h)w\|^2_{L^2(T)} \lesssim h(T)\|w\|^2_{H^{1/2}(T)}$, where we employ the Aronstein-Slobodeckii norm in the definition of $\|\cdot\|_{H^{1/2}(T)}$. Hence, by summation over all $T \in \mathcal{T}_h$ and using the Aronstein-Slobodeckii characterization of $\|\cdot\|_{H^{1/2}(\partial\Omega)}$, we obtain $\|h^{-1/2}(1-\Pi_h)w\|_{L^2(\Gamma)} \lesssim \|w\|_{H^{1/2}(\partial\Omega)}$. Next, using the characterization of the norm $\|\cdot\|_{H^{1/2}(\Gamma)}$, we arrive at

$$\|h^{-1/2}(1-\Pi_h)w\|_{L^2(\Gamma)} \lesssim \|w\|_{H^{1/2}(\Gamma)} \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.8}$$

Orthogonality properties of Π_{X_h}, the inclusion $\mathcal{P}^0(\mathcal{T}_h) \subseteq X_h$, and a duality argument then shows (see [CP06, Theorem 4.1] for the analogous proof on polygonal boundaries)

$$\|(1-\Pi_{X_h})\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|h^{1/2}(1-\Pi_{X_h})\phi\|_{L^2(\Gamma)} \quad \text{for all } \phi \in L^2(\Gamma). \tag{3.9}$$

Combining (3.9) with the inverse estimate (3.1) for $\psi = (1-\Pi_{X_h})\phi$ and $w_h = h^{1/2}$, we get

$$\begin{aligned}\|h^{1/2}\nabla_\Gamma \mathfrak{V}(1-\Pi_{X_h})\phi\|_{L^2(\Gamma)} &+ \|h^{1/2}\mathfrak{K}'(1-\Pi_{X_h})\phi\|_{L^2(\Gamma)} \\ &\lesssim \|h^{1/2}(1-\Pi_{X_h})\phi\|_{L^2(\Gamma)} \text{ for all } \phi \in L^2(\Gamma).\end{aligned}$$

For an $\widetilde{H}^{-1/2}(\Gamma)$-stable projection $\mathbb{P}_h$, we start by noting that the choice $X_h = \mathcal{P}^0(\mathcal{T}_h)$ is admissible and then (3.9) takes the form

$$\|(1-\Pi_h)\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|h^{1/2}(1-\Pi_h)\phi\|_{L^2(\Gamma)} \quad \text{for all } \phi \in L^2(\Gamma). \tag{3.10}$$

We observe that the projection property of $\mathbb{P}_h$ implies $(1-\mathbb{P}_h)(1-\Pi_h) = (1-\mathbb{P}_h)$. This and elementwise stability of Π_h imply, for all $\phi \in L^2(\Gamma)$,

$$\begin{aligned}\|(1-\mathbb{P}_h)\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} &\lesssim \|(1-\Pi_h)\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \\ &\overset{(3.10)}{\lesssim} \|h^{1/2}(1-\Pi_h)\phi\|_{L^2(\Gamma)} \lesssim \|h^{1/2}\phi\|_{L^2(\Gamma)}.\end{aligned}$$

Finally, we use the projection property $(1-\mathbb{P}_h)^2 = (1-\mathbb{P}_h)$ and argue as for Π_{X_h} to obtain

$$\begin{aligned}\|h^{1/2}\nabla_\Gamma \mathfrak{V}(1-\mathbb{P}_h)\phi\|_{L^2(\Gamma)} &+ \|h^{1/2}\mathfrak{K}'(1-\mathbb{P}_h)\phi\|_{L^2(\Gamma)} \\ &\lesssim \|h^{1/2}(1-\mathbb{P}_h)\phi\|_{L^2(\Gamma)} \text{ for all } \phi \in L^2(\Gamma).\end{aligned}$$

This concludes the proof. ∎

One immediate consequence of Corollary 3.3 is a weak efficiency of the weighted residual error estimator $\eta_{h,\mathfrak{V}}$ from [Car97, CMS01]: Suppose that $\mathfrak{V}$ is $\widetilde{H}^{-1/2}(\Gamma)$-elliptic (in the case $d = 2$, this can be enforced, for example, by the scaling requirement $\operatorname{diam}(\Omega) < 1$). For $f \in H^1(\Gamma)$, let $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ be the unique solution of the weakly singular integral equation $\mathfrak{V}\phi = f$. Let $X_h \subset \widetilde{H}^{-1/2}(\Gamma)$ be a closed subspace with $\mathcal{P}^0(\mathcal{T}_h) \subseteq X_h \subset L^2(\Gamma)$, and let $\Phi_h \in X_h$ be the unique Galerkin approximation of ϕ, i.e.,

$$\langle \mathfrak{V}(\phi - \Phi_h)\,,\, \Psi_h\rangle_\Gamma = 0 \quad \text{for all } \Psi_h \in X_h. \tag{3.11}$$

Then [CMS01] proves (strictly speaking, only for polyhedral Γ) the reliability estimate

$$C_{\mathrm{rel}}^{-1}\,\|\phi - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \le \eta_{h,\mathfrak{V}} := \|h^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_h)\|_{L^2(\Gamma)}. \tag{3.12}$$

The constant $C_{\mathrm{rel}} > 0$ depends only on Γ, $\partial\Omega$, and the κ-shape regularity of $\mathcal{T}_h$. The following corollary provides a kind of converse estimate, where the norm is the slightly stronger weighted L^2-norm. We note that the additional assumption $\phi = \mathfrak{V}^{-1} f \in L^2(\Gamma)$ is in particular satisfied for $\Gamma = \partial\Omega$.

Corollary 3.4 (Weak efficiency of $\eta_{h,\mathfrak{V}}$ for weakly singular integral equations)**.** *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Assume $\phi = \mathfrak{V}^{-1} f \in L^2(\Gamma)$ and let $X_h \subset \widetilde{H}^{-1/2}(\Gamma)$ be a closed subspace with $\mathcal{P}^0(\mathcal{T}_h) \subseteq X_h \subset L^2(\Gamma)$. Let $\Phi_h \in X_h$ be given by* (3.11)*. Then the weighted residual error estimator from* (3.12) *satisfies*

$$\eta_{h,\mathfrak{V}} \le C_{\mathrm{eff}} \, \|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)}, \tag{3.13}$$

where $C_{\mathrm{eff}} = \widetilde{C}_{\mathrm{inv}} > 0$ is the constant from Corollary 3.3*.*

Proof. With the Galerkin projection $\mathbb{P}_h : \widetilde{H}^{-1/2}(\Gamma) \to X_h$ and $\Phi_h = \mathbb{P}_h\phi$, Corollary 3.3 yields $\eta_{h,\mathfrak{V}} = \|h^{1/2}\nabla_\Gamma \mathfrak{V}(\phi - \Phi_h)\|_{L^2(\Gamma)} \lesssim \|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)}$. ☐

Remark 3.5 (Stronger efficiency of 2D BEM)*.* While the efficiency estimate (3.13) involves a slightly stronger norm on the right-hand side, particular situations with known singularity expansions (as, e.g., the 2D direct BEM formulation of the Dirichlet problem [AFF13b]) allow us to bound $\|h^{1/2}(\phi - \Phi_h)\|_{L^2(\Gamma)}$ by $\|\phi - \Phi_h\|_{\widetilde{H}^{-1/2}(\Gamma)}$ up to higher-order terms. In [AFF13b], this is achieved by decomposing ϕ in a singular part associated with the vertices of Ω and a regular part; the higher-order terms depend only on the regular part of ϕ. ■

3.2.2. *Hypersingular integral equations.* Results similar to Corollary 3.3 also hold for the double-layer integral operator $\mathfrak{K}$ and the hypersingular integral operator $\mathfrak{W}$. Here, particularly interesting choices for the projection $\mathbb{P}_h$ are Scott-Zhang type projections onto $\widetilde{\mathcal{S}}^{\mathfrak{q}+1}(\mathcal{T}_h)$; see [SZ90] as well as the adaptation to BEM in [AFF15, Section 3.2].

Corollary 3.6. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Let $X_h \subseteq \widetilde{H}^{1/2}(\Gamma)$ be a closed subspace with $\widetilde{\mathcal{S}}^1(\mathcal{T}_h) \subseteq X_h \subseteq \widetilde{H}^1(\Gamma)$. Let $\mathbb{P}_h : \widetilde{H}^{1/2}(\Gamma) \to X_h$ be an $\widetilde{H}^{1/2}(\Gamma)$-stable projection onto X_h. Then, for all $v \in \widetilde{H}^1(\Gamma)$,*

$$\begin{aligned} &\|h^{1/2}\nabla_\Gamma \mathfrak{K}(1 - \mathbb{P}_h)v\|_{L^2(\Gamma)} + \|h^{1/2}\mathfrak{W}(1 - \mathbb{P}_h)v\|_{L^2(\Gamma)} \\ &\qquad \le \widetilde{C}_{\mathrm{inv}} \|h^{1/2}\nabla_\Gamma(1 - \mathbb{P}_h)v\|_{L^2(\Gamma)}. \end{aligned} \tag{3.14}$$

The constant $\widetilde{C}_{\mathrm{inv}} > 0$ depends only on the κ-shape regularity of $\mathcal{T}_h$, the stability constant of $\mathbb{P}_h$, and Γ.

Proof. We argue along the lines of the proof of Corollary 3.3.

Step 1: We construct a modified Scott-Zhang projection $J_h : L^2(\Gamma) \to \widetilde{\mathcal{S}}^1(\mathcal{T}_h)$ with the properties

$$\|(1 - J_h)w\|_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \|h^{1/2}\nabla_\Gamma(1 - J_h)w\|_{L^2(\Gamma)} \qquad \text{for all } w \in \widetilde{H}^1(\Gamma), \tag{3.15}$$

$$\|h^{1/2}\nabla_\Gamma J_h w\|_{L^2(\Gamma)} \lesssim \|h^{1/2}\nabla_\Gamma w\|_{L^2(\Gamma)} \qquad \text{for all } w \in \widetilde{H}^1(\Gamma). \tag{3.16}$$

In order to make the proof self-contained, we sketch the main arguments and refer to [AFF15, Lem. 7] for more details (strictly speaking, [AFF15, Lem. 7] is formulated for polygonal boundaries only, but the proof transfers with minor changes to the present case). To understand the construction of J_h, let us briefly review the classical construction from [SZ90]: There, for any vertex a_i of the triangulation with $a_i \in \partial\Gamma$ a facet f_i with $a_i \in \overline{f_i} \subset \partial\Gamma$ is selected. For the remaining, interior vertices a_i of the triangulation, an element T_i with $a_i \in \overline{T_i}$ is chosen. The classical Scott-Zhang projection $J_h^{\rm SZ} v \in \mathcal{S}^1(\mathcal{T}_h)$ is then determined by the conditions $J_h^{\rm SZ} v(a_i) = v_i$, where v_i is a weighted average of v on f_i (if a_i is a boundary vertex) or T_i (if a_i is an interior vertex); the weight is given by the so-called dual basis. This ensures the projection property $J_h^{\rm SZ} v = v$ for all $v \in \mathcal{S}^1(\mathcal{T}_h)$. Moreover, $J_h^{\rm SZ}$ is then stable in $H^1(\Gamma)$. Note that $J_h^{\rm SZ} v \in \widetilde{H}^1(\Gamma)$ if $v \in \widetilde{H}^1(\Gamma)$ as the contributions of shape functions associated with vertices $a_i \in \partial\Gamma$ vanish. We define J_h in the same way as $J_h^{\rm SZ}$, but simply omit the degrees of freedom on the boundary $a_i \in \partial\Gamma$. It can be checked that J_h is (locally) stable in $L^2(\Gamma)$. Furthermore, $J_h = J_h^{\rm SZ}$ on $\widetilde{H}^1(\Gamma)$. Hence, J_h is (locally) stable in $\widetilde{H}^1(\Gamma)$, and is also a projection onto $\widetilde{\mathcal{S}}^1(\mathcal{T}_h)$. This proves (3.16). By interpolation J_h is stable in $\widetilde{H}^{1/2}(\Gamma)$. The approximation and projection properties of J_h follow from those of $J_h^{\rm SZ}$. An interpolation argument and $(1 - J_h)^2 = (1 - J_h)$ reveals (3.15). The suppressed constant in (3.15) depends only on Γ, $\partial\Omega$, and the κ-shape regularity of $\mathcal{T}_h$.

Step 2: Combining (3.15) with the inverse estimate (3.2) for $v = (1 - J_h)w$ and the weight function $w_h = h^{1/2}$ we arrive at

$$\begin{aligned}\|h^{1/2}\nabla_\Gamma \mathfrak{K}(1 - J_h)w\|_{L^2(\Gamma)} &+ \|h^{1/2}\mathfrak{W}(1 - J_h)w\|_{L^2(\Gamma)} \\ &\lesssim \|h^{1/2}\nabla_\Gamma(1 - J_h)w\|_{L^2(\Gamma)} \quad \text{for all } w \in \widetilde{H}^1(\Gamma).\end{aligned}$$

Similar arguments apply for any $\widetilde{H}^{1/2}$-stable projection $\mathbb{P}_h : \widetilde{H}^{1/2}(\Gamma) \to X_h$. There, additionally $(1 - \mathbb{P}_h)v = (1 - \mathbb{P}_h)(1 - J_h)(1 - \mathbb{P}_h)v$, the stability of $\mathbb{P}_h$, and (3.16) have to be used to bound:

$$\begin{aligned}\|(1 - \mathbb{P}_h)w\|_{\widetilde{H}^{1/2}(\Gamma)} &\lesssim \|(1 - J_h)(1 - \mathbb{P}_h)w\|_{\widetilde{H}^{1/2}(\Gamma)} \\ &\overset{(3.15)}{\lesssim} \|h^{1/2}\nabla_\Gamma(1 - J_h)(1 - \mathbb{P}_h)w\|_{L^2(\Gamma)} \\ &\overset{(3.16)}{\lesssim} \|h^{1/2}\nabla_\Gamma(1 - \mathbb{P}_h)w\|_{L^2(\Gamma)}. \qquad \square\end{aligned}$$

As in Section 3.2.1, an immediate consequence of Corollary 3.6 is a form of efficiency of the weighted residual error estimator $\eta_{h,\mathfrak{W}}$ from [Car97, CMPS04] for the hypersingular integral equation: Suppose that $\widetilde{H}^{1/2}(\Gamma)$ does not contain any non-trivial characteristic function χ_ω with $\omega \subsetneqq \Gamma$ (this is in particular satisfied if $\partial\Omega$ is connected and $\Gamma \subsetneqq \partial\Omega$). Then, $\mathfrak{W} : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is an elliptic isomorphism. For $f \in L^2(\Gamma)$, let $u \in \widetilde{H}^{1/2}(\Gamma)$ be the unique solution of the hypersingular integral equation $\mathfrak{W}u = f$. Let $X_h \subset \widetilde{H}^{1/2}(\Gamma)$ be a closed subspace with $\widetilde{\mathcal{S}}^1(\mathcal{T}_h) \subseteq X_h \subset \widetilde{H}^1(\Gamma)$. In addition, let $U_h \in X_h$ be the unique Galerkin approximation of u, i.e.,

$$\langle \mathfrak{W}(u - U_h)\,,\, V_h\rangle_\Gamma = 0 \quad \text{for all } V_h \in X_h. \tag{3.17}$$

Under these assumptions (and, strictly speaking, for polyhedral Γ), [CMPS04] proves the reliability estimate

$$C_{\rm rel}^{-1}\,\|u - U_h\|_{\widetilde{H}^{1/2}(\Gamma)} \le \eta_{h,\mathfrak{W}} := \|h^{1/2}(f - \mathfrak{W}U_h)\|_{L^2(\Gamma)}. \tag{3.18}$$

The constant $C_{\rm rel} > 0$ depends only on Γ, $\partial\Omega$, and the κ-shape regularity of $\mathcal{T}_h$. The following corollary provides the converse efficiency estimate with respect to some slightly stronger weighted H^1-seminorm.

Corollary 3.7 (Weak efficiency of $\eta_{h,\mathfrak{W}}$ for hypersingular integral equations)**.** *Assume that $\widetilde{H}^{1/2}(\Gamma)$ does not contain any non-trivial characteristic function χ_ω with $\omega \subsetneqq \Gamma$. Assume $u = \mathfrak{W}^{-1}f \in \widetilde{H}^1(\Gamma)$. Let $X_h \subset \widetilde{H}^{1/2}(\Gamma)$ be a closed subspace with $\widetilde{\mathcal{S}}^1(\mathcal{T}_h) \subseteq X_h \subset \widetilde{H}^1(\Gamma)$. Let $U_h \in X_h$ be given by* (3.17)*. Then the weighted residual error estimator from* (3.18) *satisfies*

$$\eta_{h,\mathfrak{W}} \leq C_{\mathrm{eff}} \, \|h^{1/2}\nabla_\Gamma(u - U_h)\|_{L^2(\Gamma)}, \tag{3.19}$$

where $C_{\mathrm{eff}} = \widetilde{C}_{\mathrm{inv}} > 0$ is the constant from Corollary 3.6*.*

Proof. With the Galerkin projection $\mathbb{P}_h : \widetilde{H}^{1/2}(\Gamma) \to X_h$ and $U_h = \mathbb{P}_h u$, Corollary 3.6 yields $\eta_{h,\mathfrak{W}} = \|h^{1/2}\mathfrak{W}(u - U_h)\|_{L^2(\Gamma)} \lesssim \|h^{1/2}\nabla_\Gamma(u - U_h)\|_{L^2(\Gamma)}$. ☐

Remark 3.8. If $\Gamma = \partial\Omega$ is connected, then the kernel of $\mathfrak{W}$ is the space of constant functions on Γ. Therefore, $\mathfrak{W} : H^{1/2}_\star(\partial\Omega) \to H^{-1/2}_\star(\partial\Omega)$ is an elliptic isomorphism, where $H^s_\star(\partial\Omega) := \big\{v \in H^s(\partial\Omega) \,:\, \langle v\,,1\rangle_{\partial\Omega} = 0\big\}$ for $|s| \leq 1$. Note that $\mathfrak{W} : H^s_\star(\partial\Omega) \to H^{s-1}_\star(\partial\Omega)$ is an isomorphism for all $0 \leq s \leq 1$. For $f \in H^0_\star(\partial\Omega)$, the solution $u := \mathfrak{W}^{-1}f$ thus has additional regularity $u \in H^1_\star(\partial\Omega)$, and Corollary 3.7 holds verbatim. ∎

3.2.3. *Remarks on the extension to hp-BEM.* The above efficiency statements are formulated for the h-version BEM. They do generalize to the hp-version.

Since the corresponding reliability estimates have only been formulated for closed surfaces $\Gamma = \partial\Omega$ and affine element maps in [KM15], we restrict the following result to that setting:

Corollary 3.9. *Let $d \in \{2, 3\}$, $\Gamma = \partial\Omega$, and let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Assume that the element maps are affine. Let q_h be a σ-admissible polynomial degree distribution. Then there exists $C > 0$ depending only on $\partial\Omega$, the κ-shape regularity of $\mathcal{T}_h$, and σ such that the following hold:*

(i) *Let $\mathfrak{V} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ be an isomorphism. Let $\phi = \mathfrak{V}^{-1}f$ for some $f \in H^1(\partial\Omega)$. Set $X_h := \mathcal{P}^{\mathfrak{q}}(\mathcal{T}_h)$ and let $\Phi_{hp} \in X_h$ be the Galerkin solution given by* (3.11)*. Then:*

$$C^{-1}\|\phi - \Phi_{hp}\|_{H^{-1/2}(\partial\Omega)} \leq \eta_{hp,\mathfrak{V}} := \|(h/(1+q_h))^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_{hp})\|_{L^2(\partial\Omega)}, \tag{3.20}$$

$$\eta_{hp,\mathfrak{V}} \leq C\|(h/(1+q_h))^{1/2}(\phi - \Phi_{hp})\|_{L^2(\partial\Omega)}. \tag{3.21}$$

(ii) *Let $\Gamma = \partial\Omega$ be connected and $u = \mathfrak{W}^{-1}f$ for some $f \in H^0_\star(\partial\Omega)$. Set $X_h := \mathcal{S}^{\mathfrak{q}+1}(\mathcal{T}_h) \cap H^1_\star(\partial\Omega)$ and let $U_{hp} \in X_h$ be the Galerkin solution given by* (3.17)*. Then:*

$$C^{-1}\|u - U_{hp}\|_{H^{1/2}(\partial\Omega)} \leq \eta_{hp,\mathfrak{W}} := \|(h/(1+q_h))^{1/2}(f - \mathfrak{W}U_{hp})\|_{L^2(\partial\Omega)}, \tag{3.22}$$

$$\eta_{hp,\mathfrak{W}} \leq C\Big[\|(h/(1+q_h))^{1/2}\nabla_\Gamma(u - U_{hp})\|_{L^2(\partial\Omega)} + \|(h/(1+q_h))^{1/2}(u - U_{hp})\|_{L^2(\partial\Omega)}\Big]. \tag{3.23}$$

Proof. The reliability bounds (3.20), (3.22) are taken from [KM15, Cor. 3.11, Cor. 3.14].

For the proof of (3.21), we let Π_{hp} be the $L^2(\partial\Omega)$-projection and $\mathbb{P}_{hp}$ be the Galerkin projection. We note that the analogue of (3.10) is

(3.24)
$$\begin{aligned}\|(1-\mathbb{P}_{hp})\psi\|_{H^{-1/2}(\partial\Omega)} &\lesssim \|(1-\Pi_{hp})\psi\|_{H^{-1/2}(\partial\Omega)}\\ &\lesssim \|(h/(1+q_h))^{1/2}\psi\|_{L^2(\partial\Omega)} \qquad \text{for all } \psi\in L^2(\partial\Omega);\end{aligned}$$

this is shown in the same way as (3.10), but exploits that in a high order context the estimate (3.8) can be replaced with the improved estimate

$$\|((q_h+1)/h)^{1/2}(1-\Pi_{hp})v\|_{L^2(\Gamma)} \lesssim \|v\|_{H^{1/2}(\Gamma)}.$$

This latter estimate is obtained from elementwise considerations, uses standard estimate (see, e.g., [Mel05, Prop. A.2]) for integer order Sobolev spaces, and an interpolation argument on the reference element. Hence, proceeding as in the proof of Corollary 3.3 with $w_h = (h/(1+q_h))^{1/2}$ we get

$$\begin{aligned}&\|w_h\nabla_\Gamma\mathfrak{V}(1-\mathbb{P}_{hp})\phi\|_{L^2(\partial\Omega)}\\ &\quad\overset{(3.1)}{\lesssim} \|(1+q_h)^{-1/2}\|_{L^\infty(\partial\Omega)}\|(1-\mathbb{P}_{hp})^2\phi\|_{H^{-1/2}(\partial\Omega)} + \|w_h(1-\mathbb{P}_{hp})\phi\|_{L^2(\partial\Omega)}\\ &\quad\overset{(3.24)}{\lesssim} \|(1+q_h)^{-1/2}\|_{L^\infty(\partial\Omega)}\|w_h(1-\mathbb{P}_{hp})\phi\|_{L^2(\partial\Omega)} + \|w_h(1-\mathbb{P}_{hp})\phi\|_{L^2(\partial\Omega)}.\end{aligned}$$

The proof of (3.23) proceeds along similar lines. The key is the analog of (3.15). By [KM15, Lem. 3.12] there exists an operator $J'_{hp}: H^1(\partial\Omega)\to\mathcal{S}^{\mathbf{q}+1}(\mathcal{T}_h)$ with

$$\|(1-J'_{hp})v\|_{H^{1/2}(\partial\Omega)} \lesssim \|(h/(1+q_h))^{1/2}\nabla_\Gamma v\|_{L^2(\partial\Omega)} + \|(h/(1+q_h))^{1/2}v\|_{L^2(\partial\Omega)}.$$

Finally, an operator $J_{hp}: H^1_\star(\partial\Omega)\to X_h$ is then obtained by setting $J_{hp}v := J'_{hp}v - \overline{J'_{hp}v}$, where the overbar denotes the average over $\partial\Omega$. It is clear that $\|\overline{v-J'_{hp}v}\|_{L^2(\partial\Omega)} \le \|v-J'_{hp}v\|_{L^2(\partial\Omega)}$, and norm equivalence on finite dimensional spaces then yields $\|\overline{v-J'_{hp}v}\|_{H^{1/2}(\partial\Omega)} \lesssim \|(1-J'_{hp})v\|_{L^2(\partial\Omega)}$. Hence, for $v\in H^1_\star(\partial\Omega)$,

$$\begin{aligned}\|(1-J_{hp})v\|_{H^{1/2}(\partial\Omega)} &\le \|(1-J'_{hp})v\|_{H^{1/2}(\partial\Omega)} + \|\overline{(1-J'_{hp})v}\|_{H^{1/2}(\partial\Omega)}\\ &\lesssim \|(1-J'_{hp})v\|_{H^{1/2}(\partial\Omega)},\end{aligned}$$

so that J_{hp} has the same approximation properties as J'_{hp} on the space $H^1_\star(\partial\Omega)$. With $\mathbb{P}_{hp}$ again denoting the Galerkin projection, we have

$$(1-\mathbb{P}_{hp}) = (1-\mathbb{P}_{hp})(1-J_{hp})(1-\mathbb{P}_{hp}),$$

and hence

$$\begin{aligned}&\|(1-\mathbb{P}_{hp})u\|_{H^{1/2}(\partial\Omega)} \lesssim \|(1-J_{hp})(1-\mathbb{P}_{hp})u\|_{H^{1/2}(\partial\Omega)}\\ &\quad\lesssim \|(1-J'_{hp})(1-\mathbb{P}_{hp})u\|_{H^{1/2}(\partial\Omega)}\\ &\quad\lesssim \|(h/(1+q_h))^{1/2}\nabla_\Gamma(1-\mathbb{P}_{hp})u\|_{L^2(\partial\Omega)} + \|(h/(1+q_h))^{1/2}(1-\mathbb{P}_{hp})u\|_{L^2(\partial\Omega)}.\end{aligned}$$

Combining this with (3.22) and the inverse estimate (3.2) shows (3.23). $\square$

4. Far-field and near-field estimates for the simple-layer potential

The proof of Theorem 3.1 is based on decomposing the pertinent potentials into "far-field" and "near-field" contributions. In the present section, we analyze the decomposition for the simple-layer potential and provide inverse estimates for both components. Section 4.2 is concerned with inverse estimates for the near-field parts, which essentially follow from scaling arguments, whereas Section 4.3 deals with the far-field part. Throughout the section, we let

$$\psi \in L^2(\Gamma) \text{ and assume that } \psi \text{ is extended by zero to } \partial\Omega \setminus \overline{\Gamma}, \tag{4.1}$$

i.e., we identify ψ with $E_{0,\Gamma}\psi$.

4.1. **Decomposition into near-field and far-field.** For a parameter $\delta > 0$, we define for each element $T \in \mathcal{T}_h$ the neighborhood U_T of T by

$$T \subset U_T := \bigcup_{x \in T} B_{2\delta h(T)}(x). \tag{4.2}$$

Since $\partial\Omega$ is Lipschitz and Γ stems from a Lipschitz dissection and by κ-shape regularity of $\mathcal{T}_h$, we can fix the parameter $\delta > 0$ and find $M \in \mathbb{N}$ (both δ and M are independent of $\mathcal{T}_h$) such that the following two conditions are satisfied:

(a) $\Gamma \cap U_T$ is contained in the patch $\omega_h(T)$ of T (see (2.16) for the definition of $\omega_h(T)$), i.e.,

$$\Gamma \cap U_T \subseteq \omega_h(T). \tag{4.3}$$

(b) The covering $\Gamma \subseteq \bigcup_{T \in \mathcal{T}_h} U_T$ is locally finite with a uniform bound, i.e.,

$$\sup_{x \in \mathbb{R}^d} \#\big\{U_T \,:\, T \in \mathcal{T}_h \text{ and } x \in U_T\big\} \leq M. \tag{4.4}$$

Finally, we fix a bounded domain $U \subset \mathbb{R}^d$ such that

$$U_T \subset U \quad \text{for all } T \in \mathcal{T}_h. \tag{4.5}$$

It will be important that U is chosen independently of $\mathcal{T}_h$. To deal with the non-locality of the integral operators, we define for $T \in \mathcal{T}_h$ the near-field $u_{\mathfrak{V},T}^{\text{near}}$ and the far-field $u_{\mathfrak{V},T}^{\text{far}}$ of the simple-layer potential $u_{\mathfrak{V}} := \widetilde{\mathfrak{V}}\psi$ by

$$u_{\mathfrak{V},T}^{\text{near}} := \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\cap U_T}) \quad \text{and} \quad u_{\mathfrak{V},T}^{\text{far}} := \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\setminus U_T}), \tag{4.6}$$

where χ_ω denotes the characteristic function of the set $\omega \subseteq \mathbb{R}^d$. We have the obvious identity

$$u_{\mathfrak{V}} = \widetilde{\mathfrak{V}}\psi = u_{\mathfrak{V},T}^{\text{near}} + u_{\mathfrak{V},T}^{\text{far}} \quad \text{for all } T \in \mathcal{T}_h. \tag{4.7}$$

In our analysis, we will treat $u_{\mathfrak{V},T}^{\text{near}}$ and $u_{\mathfrak{V},T}^{\text{far}}$ separately, starting with the simpler case of $u_{\mathfrak{V},T}^{\text{near}}$.

4.2. **Inverse estimates for the near-field part $u_{\mathfrak{V},T}^{\text{near}}$.** The near-field parts of a potential can be treated with local arguments and the stability properties of the associated boundary integral operators.

Lemma 4.1. *There exists a constant $\widetilde{C}_{\text{near}} > 0$ depending only on $\partial\Omega$, Γ, and the κ-shape regularity of $\mathcal{T}_h$ such that for arbitrary $T \in \mathcal{T}_h$ and $\Psi_h^T \in \mathcal{P}^0(\mathcal{T}_h)$ with $\operatorname{supp}\big(\Psi_h^T\big) \subseteq \overline{\omega_h(T)}$ it holds that*

$$\|\nabla\widetilde{\mathfrak{V}}\Psi_h^T\|_{L^2(U_T)} \leq \widetilde{C}_{\text{near}}\|h^{1/2}\Psi_h^T\|_{L^2(\omega_h(T))}.$$

Proof. We fix an element $T \in \mathcal{T}_h$. We recall that Ψ_h^T is piecewise constant and compute

$$(\nabla\widetilde{\mathfrak{V}}\Psi_h^T)(x) = \sum_{T' \in \omega_h(T)} \Psi_h^T|_{T'} \int_{T'} \nabla_x G(x,y)\,dy \quad \text{for all } x \in \mathbb{R}^d \setminus \Gamma.$$

The number of elements T' in the patch $\omega_h(T)$ is bounded in terms of the shape regularity constant κ (cf. Lemma 2.6). With some constant that depends only on κ and $\partial\Omega$, we estimate

$$|(\nabla\widetilde{\mathfrak{V}}\Psi_h^T)(x)|^2 \lesssim \sum_{T' \in \omega_h(T)} |\Psi_h^T|_{T'}|^2 \Big(\int_{T'} \big|\nabla_x G(x,y)\big|\,dy \Big)^2. \tag{4.8}$$

Next, we show for elements $T' \subseteq \omega_h(T)$

$$\int_{U_T} \Big(\int_{T'} \big|\nabla_x G(x,y)\big|\,dy \Big)^2 dx \lesssim h(T)^d. \tag{4.9}$$

This follows from a local Lipschitz parametrization of $\partial\Omega$. We assume (after a Euclidean change of coordinates if necessary) that $\{(x', \Lambda(x'))\colon x' \in B_{2r}(0)\}$ is a part of $\partial\Omega$ that contains $\omega_h(T)$. The function Λ is Lipschitz continuous, and we remark in passing that by [Ste70, Thm. 3, Sect. VI] we may assume that $\Lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz continuous. (If such a local consideration is not possible, then, since the number of local charts is finite by definition of bounded Lipschitz domains, we must have $\operatorname{diam}(\omega_h(T)) = O(1)$ so that (4.9) is trivially true.) We may also assume that $U_T \subset \{(x', \Lambda(x') + t) \,|\, x' \in B_{2r}(0), t \in \mathbb{R}\}$. The key observation is that the mapping $\widetilde{\Lambda} : \mathbb{R}^d \to \mathbb{R}^d$ given by $(x', t) \mapsto (x', \Lambda(x') + t)$ is bi-Lipschitz. We conclude for elements $T' \subset \omega_h(T)$ that $\widetilde{\Lambda}^{-1} U_T =: \widetilde{U}_T$ and $\widetilde{\Lambda}^{-1} T' =: \widetilde{T}' \subseteq B_{2r}(0) \times \{0\}$ satisfy, for some $x_0 \in B_{2r}(0)$ and some $c > 0$, which depends solely on the bi-Lipschitz mapping $\widetilde{\Lambda}$,

$$\widetilde{U}_T \subseteq B_{ch(T)}(x_0) \times [-ch(T), ch(T)], \qquad \widetilde{T}' \subseteq B_{ch(T)}(x_0) \times \{0\}.$$

Finally, using $\nabla_x G(x,y) \simeq |x-y|^{-(d-1)}$, the definition of the surface integral, and the change of variables formula for bi-Lipschitz mappings from [EG92, Sec. 3.3.3], we get

$$\begin{aligned}
&\int_{x \in U_T} \Big(\int_{y \in T'} |\nabla_x G(x,y)|\,dy \Big)^2 dx \\
&\quad \simeq \int_{\widetilde{x} \in \widetilde{U}_T} \Big(\int_{\widetilde{y} \in \widetilde{T}'} |\widetilde{x} - \widetilde{y}|^{-(d-1)}\,d\widetilde{y} \Big)^2 d\widetilde{x} \\
&\quad \lesssim \int_{\xi \in B_{ch(T)}(x_0)} \int_{t=-ch(T)}^{ch(T)} \Big(\int_{\eta \in B_{ch(T)}(x_0)} \big(|\xi - \eta|^2 + t^2\big)^{-(d-1)/2}\,d\eta \Big)^2 dt\,d\xi \\
&\quad \simeq h(T)^d \int_{\xi \in B_c(x_0)} \int_{t=-c}^{c} \Big(\int_{\eta \in B_c(x_0)} \big(|\xi - \eta|^2 + t^2\big)^{-(d-1)/2}\,d\eta \Big)^2 dt\,d\xi \\
&\quad \simeq h(T)^d,
\end{aligned}$$

where the last estimate follows by a direct estimation of the integrals, which is independent of $h(T)$. We have thus shown (4.9). Inserting (4.9) in (4.8) gives

$$\int_{U_T} |(\nabla\widetilde{\mathfrak{V}}\Psi_h^T)(x)|^2\,dx \lesssim \sum_{T'\in\omega_h(T)} |\Psi_h^T|_{T'}|^2 h(T)^d \simeq \|h^{1/2}\Psi_h^T\|^2_{L^2(\omega_h(T))}. \qquad \square$$

Proposition 4.2 (Near-field bound for $\widetilde{\mathfrak{V}}$)**.** *Let w_h be a σ-admissible weight function. There exists a constant $C_{\rm near} > 0$ depending only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and σ, such that the near-field part $u^{\rm near}_{\mathfrak{V},T}$ satisfies $u^{\rm near}_{\mathfrak{V},T} \in H^1(U)$ and $\gamma_0^{\rm int} u^{\rm near}_{\mathfrak{V},T} \in H^1(\Gamma)$ together with*

$$\sum_{T\in\mathcal{T}_h} \|w_h\nabla_\Gamma\gamma_0^{\rm int}u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(U_T)} \le C_{\rm near}\,\|w_h\psi\|^2_{L^2(\Gamma)}. \tag{4.10}$$

Proof. The stability (2.10) of $\mathfrak{V} : L^2(\partial\Omega) \to H^1(\partial\Omega)$ proved in [Ver84] together with Fact 2.1(i) gives, for each $T \in \mathcal{T}_h$,

$$\|\nabla_\Gamma\gamma_0^{\rm int}u^{\rm near}_{\mathfrak{V},T}\|_{L^2(T)} \le \|\mathfrak{V}(\psi\chi_{U_T\cap\Gamma})\|_{H^1(\partial\Omega)} \lesssim \|\psi\chi_{U_T\cap\Gamma}\|_{L^2(\partial\Omega)} = \|\psi\|_{L^2(U_T\cap\Gamma)}.$$

Summing the last estimate over all $T \in \mathcal{T}_h$ and using (4.3)–(4.4), together with the σ-admissibility of w_h, we arrive at

$$\sum_{T\in\mathcal{T}_h}\|w_h\nabla_\Gamma\gamma_0^{\rm int}u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(T)} \lesssim \sum_{T\in\mathcal{T}_h}\|w_h\|^2_{L^\infty(T)}\|\psi\|^2_{L^2(U_T\cap\Gamma)} \simeq \|w_h\psi\|^2_{L^2(\Gamma)}, \tag{4.11}$$

where all estimates depend only on $\partial\Omega$, the κ-shape regularity of $\mathcal{T}_h$, and the admissibility constant σ. This bounds the first term on the left-hand side of (4.10). To bound the second term, let Π_h denote the $L^2(\Gamma)$-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$. We decompose the near-field as $u^{\rm near}_{\mathfrak{V},T} = \widetilde{\mathfrak{V}}(\Pi_h(\psi\chi_{\Gamma\cap U_T})) + \widetilde{\mathfrak{V}}\big((1-\Pi_h)\psi\chi_{\Gamma\cap U_T}\big)$. The condition $\operatorname{supp}(\psi\chi_{\Gamma\cap U_T}) \subseteq \overline{\omega_h(T)}$ implies $\operatorname{supp}(\Pi_h(\psi\chi_{\Gamma\cap U_T})) \subseteq \overline{\omega_h(T)}$ and therefore, taking $\Psi_h^T = \Pi_h(\psi\chi_{\Gamma\cap U_T})$ in Lemma 4.1 we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}&\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla\widetilde{\mathfrak{V}}(\Pi_h(\psi\chi_{\Gamma\cap U_T}))\|^2_{L^2(U_T)}\\ &\lesssim \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|h^{1/2}\Pi_h(\psi\chi_{\Gamma\cap U_T})\|^2_{L^2(\omega_h(T))}\\ &\lesssim \|w_h\psi\|^2_{L^2(\Gamma)},\end{aligned} \tag{4.12}$$

where we used the local L^2-stability of Π_h in the last estimate. Recalling the stability $\widetilde{\mathfrak{V}} : H^{-1/2}(\partial\Omega) \to H^1(U)$ of (2.8), the equality (2.5), and the approximation property (3.9) of Π_h, we get

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}&\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla\widetilde{\mathfrak{V}}\big((1-\Pi_h)\psi\chi_{\Gamma\cap U_T}\big)\|^2_{L^2(U_T)}\\ &\overset{(2.8),(2.5)}{\lesssim} \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|(1-\Pi_h)\psi\chi_{\Gamma\cap U_T}\|^2_{\widetilde{H}^{-1/2}(\Gamma)}\\ &\overset{(3.9)}{\lesssim} \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|h^{1/2}(\psi\chi_{\Gamma\cap U_T})\|^2_{L^2(\Gamma)} \simeq \|w_h\psi\|^2_{L^2(\Gamma)}.\end{aligned} \tag{4.13}$$

Combining (4.11)–(4.13) gives (4.10). $\square$

4.3. **Estimates for the far-field part $u^{\rm far}_{\mathfrak{V},T}$.** The following lemma is taken from [FKMP13]. For the convenience of the reader and since the same argument underlies the proof of the analogous lemma for the double-layer potential (Lemma 5.3 below), we recall its proof here.

Lemma 4.3 (Caccioppoli inequality for $u^{\rm far}_{\mathfrak{V},T}$)**.** *Let $\delta > 0$ be fixed as in the beginning of Section 4.1. With $\Omega^{\rm ext} = \mathbb{R}^d \setminus \overline{\Omega}$, the function $u^{\rm far}_{\mathfrak{V},T}$ from (4.6) satisfies $u^{\rm far}_{\mathfrak{V},T}|_\Omega \in C^\infty(\Omega)$, $u^{\rm far}_{\mathfrak{V},T}|_{\Omega^{\rm ext}} \in C^\infty(\Omega^{\rm ext})$, and $u^{\rm far}_{\mathfrak{V},T}|_{U_T} \in C^\infty(U_T)$. Moreover, there exists a constant $C_{\rm cacc} > 0$ depending only on $\partial\Omega$, Γ, and the κ-shape regularity of $\mathcal{T}_h$ such that the Hessian matrix $D^2 u^{\rm far}_{\mathfrak{V},T}$ satisfies*

$$\|D^2 u^{\rm far}_{\mathfrak{V},T}\|_{L^2(B_{3\delta h(T)/4}(x))} \le C_{\rm cacc}\,\frac{1}{h(T)}\,\|\nabla u^{\rm far}_{\mathfrak{V},T}\|_{L^2(B_{\delta h(T)}(x))} \quad \forall x \in T \in \mathcal{T}_h. \tag{4.14}$$

Proof. The statements $u^{\rm far}_{\mathfrak{V},T}|_\Omega \in C^\infty(\Omega)$ and $u^{\rm far}_{\mathfrak{V},T}|_{\Omega^{\rm ext}} \in C^\infty(\Omega^{\rm ext})$ are taken from [SS11, Theorem 3.1.1], and we therefore focus on $u^{\rm far}_{\mathfrak{V},T}|_{U_T} \in C^\infty(U_T)$ and the estimate (4.14). According to [SS11, Proposition 3.1.7], [SS11, Theorem 3.1.16], and [SS11, Theorem 3.3.1], the function $u^{\rm far}_{\mathfrak{V},T} \in H^1_{\ell oc}(\mathbb{R}^d) := \big\{v \in \mathbb{R}^d \to \mathbb{R} : \varphi\cdot v \in H^1(\mathbb{R}^d) \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^d)\big\}$ solves the transmission problem

$$\begin{aligned} -\Delta u^{\rm far}_{\mathfrak{V},T} &= 0 && \text{in } \Omega \cup \Omega^{\rm ext}, \\ [u^{\rm far}_{\mathfrak{V},T}] &= 0 && \text{in } H^{1/2}(\partial\Omega), \\ [\gamma_1 u^{\rm far}_{\mathfrak{V},T}] &= -\psi\chi_{\Gamma\setminus U_T} && \text{in } H^{-1/2}(\partial\Omega). \end{aligned} \tag{4.15}$$

In particular, (4.15) states that the jump of $u^{\rm far}_{\mathfrak{V},T}$ as well as the jump of the normal derivative vanish on $\partial\Omega \cap U_T$. This implies that $u^{\rm far}_{\mathfrak{V},T}$ is harmonic in U_T by the following classical argument: First, we observe that $u^{\rm far}_{\mathfrak{V},T}$ is distributionally harmonic in U_T, since a two-fold integration by parts that uses these jump conditions shows for $v \in C_0^\infty(U_T)$ that $\langle u^{\rm far}_{\mathfrak{V},T}, -\Delta v\rangle_\Omega = 0$. Weyl's lemma (see, e.g., [Mor08, Theorem 2.3.1]) then implies that $u^{\rm far}_{\mathfrak{V},T}$ is strongly harmonic and $u^{\rm far}_{\mathfrak{V},T} \in C^\infty(U_T)$.

The Caccioppoli inequality (4.14) expresses interior regularity for elliptic problems. Indeed, for each $u \in H^1(B_{r+\varepsilon})$ such that $u \in H^2(B_r)$ and $\Delta u = f$ on $B_{r+\varepsilon}$ with balls $B_r \subseteq B_{r+\varepsilon}$ with radii $0 < r < r+\varepsilon$ and some $f \in L^2(B_{r+\varepsilon})$, [Mor08, Lemma 5.7.1] shows

$$\|D^2 u\|_{L^2(B_r)} \lesssim \Big(\|f\|_{L^2(B_{r+\varepsilon})} + \frac{1}{\varepsilon}\,\|\nabla u\|_{L^2(B_{r+\varepsilon})} + \frac{1}{\varepsilon^2}\|u\|_{L^2(B_{r+\varepsilon})}\Big). \tag{4.16}$$

The suppressed constant depends solely on the spatial dimension and is independent of $r, \varepsilon > 0$, and u, f. We apply (4.16) with $r = 3\delta h(T)/4$, $\varepsilon = \delta h(T)/4$, $f = 0$, and $u = u^{\rm far}_{\mathfrak{V},T} - c_T$, where $c_T = \frac{1}{|B_{\delta h(T)}(x)|}\int_{B_{\delta h(T)}(x)} u^{\rm far}_{\mathfrak{V},T}(y)\,dy$. An additional Poincaré inequality finally leads to (4.14). Note that δ and hence $C_{\rm cacc}$ depend only on $\partial\Omega$, Γ, and the κ-shape regularity of $\mathcal{T}_h$. ◻

The non-local character of the operator $\widetilde{\mathfrak{V}}$ is represented by the far-field part. Lemma 4.3 allows us to show a local inverse estimate for the far-field part of the simple-layer operator.

Lemma 4.4 (Local far-field bound for $\widetilde{\mathfrak{V}}$)**.** *For all $T \in \mathcal{T}_h$, it holds that*

$$\|h^{1/2}\nabla_\Gamma \gamma_0^{\rm int} u^{\rm far}_{\mathfrak{V},T}\|_{L^2(T)} \le \|h^{1/2}\nabla u^{\rm far}_{\mathfrak{V},T}\|_{L^2(T)} \le C_{\rm far}\,\|\nabla u^{\rm far}_{\mathfrak{V},T}\|_{L^2(U_T)}. \tag{4.17}$$

The constant $C_{\rm far} > 0$ depends only on Γ, $\partial\Omega$, and the κ-shape regularity constant of $\mathcal{T}_h$.

Proof. By Lemma 4.3 we have $u^{\mathrm{far}}_{\mathfrak{V},T} \in C^\infty(U_T)$. The first estimate in (4.17) follows from the fact that, for smooth functions, the surface gradient $\nabla_\Gamma(\cdot)$ is the orthogonal projection of the gradient $\nabla(\cdot)$ onto the tangent plane, i.e., $\nabla_\Gamma \gamma_0^{\mathrm{int}} u(x) = \nabla u(x) - \big(\nabla u(x) \cdot \boldsymbol{\nu}(x)\big)\, \boldsymbol{\nu}(x)$; see [Ver84]. The second estimate in (4.17) is proved with a trace inequality and the Caccioppoli inequality (4.14) in the following way. We fix an element $T \in \mathcal{T}_h$.

Step 1: We provide a trace inequality. Let $B = B_r(x)$ be a ball with center $x \in T \subseteq \partial\Omega$ and radius $r > 0$. Let $B' = B_{3r/2}(x)$ and $B'' = B_{5r/4}(x)$. We define a smooth cut-off function $\widetilde{\chi}_B \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \widetilde{\chi}_B \subseteq B'$ and $\widetilde{\chi}_B \equiv 1$ on B by

$$\widetilde{\chi}_B := \chi_{B''} \star \rho_{r/4},$$

where ρ_ε is a standard mollifier of the form $\rho_\varepsilon(x) = \varepsilon^{-d}\rho_1(x/\varepsilon)$ for a fixed $\rho_1 \in C_0^\infty(\mathbb{R}^d)$ with $\rho_1 \geq 0$, $\operatorname{supp}\rho_1 \subseteq B_1(0)$ and $\int_{\mathbb{R}^d} \rho_1(x)\,dx = 1$. We note that for a $C > 0$ depending solely on the choice of ρ_1, we have

$$\|\nabla \widetilde{\chi}_B\|_{L^\infty(\mathbb{R}^d)} \leq C r^{-1}.$$

With this cut-off function and the standard multiplicative trace inequality for $\partial\Omega$, we estimate for sufficiently regular functions v:

$$\begin{aligned}\|v\|^2_{L^2(B\cap\partial\Omega)} &\leq \|\widetilde{\chi}_B v\|^2_{L^2(\partial\Omega)} \lesssim \|\widetilde{\chi}_B v\|^2_{L^2(\Omega)} + \|\widetilde{\chi}_B v\|_{L^2(\Omega)}\|\nabla(\widetilde{\chi}_B v)\|_{L^2(\Omega)} \\ &\lesssim r^{-1}\|v\|^2_{L^2(B')} + \|v\|_{L^2(B')}\|\nabla v\|_{L^2(B')}.\end{aligned} \tag{4.18}$$

Step 2: The set $\mathcal{F} := \Big\{\overline{B_{\delta h(T)/2}(x)} \mid x \in T\Big\}$ is a closed cover of T. By Besicovitch's covering theorem (cf. [EG92, Sect. 1.5.2]) there is a constant N_d, which depends only on the spatial dimension d, as well as countable subsets $\mathcal{G}_j \subseteq \mathcal{F}$, $j = 1, \dots, N_d$, the elements of every $\mathcal{G}_j$ being pairwise disjoint, such that $T \subseteq \bigcup_{j=1}^{N_d} \bigcup_{B\in\mathcal{G}_j} B$. Let $\widehat{\mathcal{G}}_j$ be the set of balls obtained by doubling the radius of the balls of $\mathcal{G}_j$, i.e., $\widehat{\mathcal{G}}_j := \big\{B_{\delta h(T)}(x) \mid B_{\delta h(T)/2}(x) \in \mathcal{G}_j\big\}$. As the elements of $\mathcal{G}_j$ are pairwise disjoint and all balls have the same radius $\delta h(T)/2$, there is a constant $\widehat{N}_d$, also depending only on the spatial dimension d, such that each element of $\widehat{\mathcal{G}}_j$ intersects at most $\widehat{N}_d$ other elements of $\widehat{\mathcal{G}}_j$. If we abbreviate $B = B_{\delta h(T)/2}(x)$, $B' = B_{3/4\delta h(T)}(x)$, and $\widehat{B} = B_{\delta h(T)}(x)$, the multiplicative trace inequality (4.18) and the Caccioppoli inequality (4.14) show that

$$\begin{aligned}\|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(B\cap T)} &\overset{(4.18)}{\lesssim} \frac{1}{h(T)}\|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(B')} + \|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|_{L^2(B')}\|D^2 u^{\mathrm{far}}_{\mathfrak{V},T}\|_{L^2(B')} \\ &\overset{(4.14)}{\lesssim} \frac{1}{h(T)}\|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(\widehat{B})}.\end{aligned}$$

Step 3: We use the last estimate to get

$$\begin{aligned}\|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(T)} &\leq \sum_{j=1}^{N_d}\sum_{B\in\mathcal{G}_j} \|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(B\cap T)} \lesssim \frac{1}{h(T)}\sum_{j=1}^{N_d}\sum_{\widehat{B}\in\widehat{\mathcal{G}}_j} \|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(\widehat{B})} \\ &\lesssim \frac{N_d \widehat{N}_d}{h(T)}\, \|\nabla u^{\mathrm{far}}_{\mathfrak{V},T}\|^2_{L^2(U_T)}.\end{aligned}$$

This concludes the proof of (4.17). ∎

Summation of the elementwise estimates of Lemma 4.4 yields the following result.

Proposition 4.5 (Far-field bound for $\widetilde{\mathfrak{V}}$). *There is a constant $C_{\rm far} > 0$ depending only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and the σ-admissibility of the weight function w_h such that*

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \gamma_0^{\rm int} u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(T)} &\le \sum_{T\in\mathcal{T}_h} \|w_h \nabla u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(T)} \\ &\le C_{\rm far}\left(\|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\|\psi\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + \|w_h\psi\|^2_{L^2(\Gamma)}\right).\end{aligned}$$

Proof. We use the local far-field bound (4.17) of Lemma 4.4 and $u^{\rm far}_{\mathfrak{V},T} = \widetilde{\mathfrak{V}}\psi - u^{\rm near}_{\mathfrak{V},T}$ to see

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \gamma_0^{\rm int} u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(T)} \\ &\le \sum_{T\in\mathcal{T}_h} \|w_h \nabla u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(T)} \overset{(4.17)}{\lesssim} \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|\nabla u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(U_T)} \\ &\overset{(4.7)}{\lesssim} \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|\nabla \widetilde{\mathfrak{V}}\psi\|^2_{L^2(U_T)} + \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|\nabla u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(U_T)}.\end{aligned} \tag{4.19}$$

The first term on the right-hand side in (4.19) is estimated by stability of $\widetilde{\mathfrak{V}}$, the finite overlap property (4.4), and (2.5):

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|\nabla \widetilde{\mathfrak{V}}\psi\|^2_{L^2(U_T)} &\overset{(4.4)}{\lesssim} \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)} \|\nabla \widetilde{\mathfrak{V}}\psi\|^2_{L^2(U)} \\ &\lesssim \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)} \|\psi\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

The second term in (4.19) is bounded with the near-field bound (4.10). ☐

5. Far-field and near-field estimates for the double-layer potential

Section 4 studied far-field and near-field estimates for the simple-layer potential. Corresponding results for the double-layer potential are derived in the present section. Throughout this section, let

$$v \in \widetilde{H}^1(\Gamma) \subset H^1(\partial\Omega).$$

In particular, $v \in \widetilde{H}^{1/2}(\Gamma)$ with $\|v\|_{\widetilde{H}^{1/2}(\Gamma)} = \|v\|_{H^{1/2}(\partial\Omega)}$, where we identify $v = E_{0,\Gamma}v$. Since functions from $H^{1/2}(\Gamma)$ may not be discontinuous across element boundaries, the splitting into near-field and far-field contribution of the double-layer potential $u_{\mathfrak{K}} := \widetilde{\mathfrak{K}}v$ cannot be achieved by characteristic functions, but requires smoother cut-off functions and greater technical care.

5.1. **Decomposition into near-field and far-field.** We use the notation introduced in Section 4.1 concerning the neighborhoods U_T. In order to define the near-field and far-field parts for the double-layer potential, we need appropriate

cut-off functions: For each $T \in \mathcal{T}_h$, we define $\eta_T \in C_0^\infty(\mathbb{R}^d)$ with the aid of the standard mollifier ρ_ε that was already used in the proof of Lemma 4.4:
(5.1)

$$\eta_T := \chi_{\widetilde{U}_T} \star \rho_{\delta h(T)/4}, \qquad \widetilde{U}_T := \bigcup_{x \in T} B_{\delta h(T)/2}(x), \qquad U'_T := \bigcup_{x \in T} B_{\delta h(T)/4}(x).$$

This function satisfies:

$$\text{supp}\,\eta_T \subseteq U_T, \quad \eta_T|_{U'_T} \equiv 1, \quad \|\eta_T\|_{L^\infty(\mathbb{R}^d)} \le 1, \quad \|\nabla \eta_T\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{h(T)}, \tag{5.2}$$

where the implied constant depends on the κ-shape regularity of the triangulation through the parameter δ. We note that the assumptions on U_T imply $(\text{supp}\,\eta_T) \cap \Gamma \subseteq \overline{\omega_h(T)}$.

The following lemma may be viewed as an extension of [DS80, Thm. 7.1] to the case of curved elements.

Lemma 5.1 (Poincaré-Friedrichs inequality on patches)**.** *Let $v \in \widetilde{H}^1(\Gamma)$. For each $T \in \mathcal{T}_h$, there is a constant $v_T \in \mathbb{R}$ such that $(v - v_T)\eta_T \in \widetilde{H}^1(\Gamma)$, $(v-v_T)(1-\eta_T) \in H^1(\partial\Omega)$, and*

$$\|v - v_T\|_{L^2(\omega_h(T))} \le C_1 \|h \nabla_\Gamma v\|_{L^2(\omega_h(T))}, \tag{5.3}$$

$$\|(v - v_T)\eta_T\|_{H^{1/2}(\partial\Omega)} \le C_1 \|h^{1/2} \nabla_\Gamma v\|_{L^2(\omega_h(T))}, \tag{5.4}$$

$$\|(v - v_T)\eta_T\|_{H^1(\partial\Omega)} \le C_1 \|\nabla_\Gamma v\|_{L^2(\omega_h(T))}. \tag{5.5}$$

Furthermore, $v_T = 0$ if $\partial\omega_h(T) \cap \partial\Gamma$ contains a facet of the triangulation. The constant $C_1 > 0$ depends only on $\partial\Omega$, Γ, and the κ-shape regularity constant of $\mathcal{T}_h$.

Proof. It is clear that $(v - v_T)(1 - \eta_T) \in H^1(\partial\Omega)$, since $v \in \widetilde{H}^1(\Gamma)$ and η_T is smooth. The remaining statements require more care.

Step 1: For $v \in \widetilde{H}^1(\Gamma)$ and a facet $f \in \mathcal{F}_h$ of the triangulation $\mathcal{T}_h$ (recall that facets are images of $(d-2)$-faces of T_{ref} under the element map) denote by $\ell_f(v)$ the average of v on f. As $\ell_f(1) = 1$, we can use the Deny-Lions lemma on the reference element, and the assumptions on the element maps then imply

$$\|v - \ell_f(v)\|_{L^2(T)} \lesssim h(T) \|\nabla_\Gamma v\|_{L^2(T)} \quad \text{if } f \text{ is a facet of } T, \tag{5.6}$$

$$|\ell_{f_1}(v) - \ell_{f_2}(v)| \lesssim h(T)^{1-(d-1)/2} \|\nabla_\Gamma v\|_{L^2(T)} \quad \text{if } f_1, f_2 \text{ are two facets of } T. \tag{5.7}$$

Step 2: Fix an element $T \in \mathcal{T}_h$.

- If $\eta_T|_{\partial\Gamma} \equiv 0$, then select an arbitrary facet f_T of the element patch $\omega_h(T)$.
- If $\eta_T|_{\partial\Gamma} \not\equiv 0$, then we claim that there exists a facet f of $\omega_h(T)$ with $f \subseteq \partial\Gamma$. To see this, let $x_0 \in \partial\Gamma$ with $\eta_T(x_0) \ne 0$. By continuity of η_T and since $\partial\Gamma$ is covered by the closure of facets of the triangulation, we may assume that x_0 is in the interior of a boundary facet f_T. This facet belongs to a unique element T_f of the triangulation; by continuity of η_T, we may assume $\text{supp}\,\eta_T \cap T_f \ne \emptyset$. Since $(\text{supp}\,\eta_T) \cap \overline{\Gamma} \subseteq \overline{\omega_h(T)}$, we conclude $T_f \subseteq \omega_h(T)$ and thus the boundary facet f_T is a facet of $\omega_h(T)$.

Set $v_T := \ell_{f_T}(v)$. An immediate consequence of $v \in \widetilde{H}^1(\Gamma)$ is that $v_T = 0$ if η_T does not vanish on $\partial\Gamma$. Since η_T is smooth, we conclude $(v - v_T)\eta_T \in \widetilde{H}^1(\Gamma)$. In fact, viewed as a function on $\partial\Omega$, we have

$$\text{supp}((v - v_T)\eta_T) \subseteq \overline{\omega_h(T)}. \tag{5.8}$$

Step 3: The bounds (5.6), (5.7) in conjunction with Lemma 2.6 and a finite number of applications of the triangle inequality implies

$$\|v - v_T\|_{L^2(\omega_h(T))} \lesssim \|h\nabla_\Gamma v\|_{L^2(\omega_h(T))}, \tag{5.9}$$

$$\|\nabla_\Gamma(v - v_T)\|_{L^2(\omega_h(T))} = \|\nabla_\Gamma v\|_{L^2(\omega_h(T))}, \tag{5.10}$$

where (5.9) is already the claim (5.3). The product rule, (5.2), (5.8), the estimate (5.9), the trivial bound $h(T) \lesssim |T|^{1/(d-1)} \le |\Gamma|^{1/(d-1)} \lesssim 1$ yield

$$\begin{aligned}\|\nabla_\Gamma\big((v - v_T)\eta_T\big)\|_{L^2(\partial\Omega)} &\le \|(v - v_T)\nabla_\Gamma\eta_T\|_{L^2(\omega_h(T))} + \|\eta_T\nabla_\Gamma(v - v_T)\|_{L^2(\omega_h(T))} \\ &\lesssim \|\nabla_\Gamma v\|_{L^2(\omega_h(T))},\end{aligned}$$

which proves (5.5). It remains to verify (5.4). To that end, we recall the interpolation inequality $\|u\|^2_{H^{1/2}(\partial\Omega)} \lesssim \|u\|_{L^2(\partial\Omega)}\|u\|_{H^1(\partial\Omega)}$ for all $u \in H^1(\partial\Omega)$. Since $\|(v - v_T)\eta_T\|_{L^2(\partial\Omega)} \le \|v - v_T\|_{L^2(\omega_h(T))}$, we get

$$\begin{aligned}\|(v - v_T)\eta_T\|_{H^{1/2}(\partial\Omega)} &\lesssim \|(v - v_T)\eta_T\|^{1/2}_{L^2(\partial\Omega)}\|(v - v_T)\eta_T\|^{1/2}_{H^1(\partial\Omega)} \\ &\lesssim \|h\nabla_\Gamma v\|^{1/2}_{L^2(\omega_h(T))}\|\nabla_\Gamma v\|^{1/2}_{L^2(\omega_h(T))} \\ &\simeq \|h^{1/2}\nabla_\Gamma v\|_{L^2(\omega_h(T))},\end{aligned}$$

where the last estimate hinges on κ-shape regularity of $\mathcal{T}_h$ (cf. Lemma 2.6(i)). ☐

Let $v \in \widetilde{H}^1(\Gamma)$. For each $T \in \mathcal{T}_h$, let v_T be the constant from Lemma 5.1. For each $T \in \mathcal{T}_h$ we define the near-field and the far-field part of the double-layer potential $u_{\mathfrak{K}} := \widetilde{\mathfrak{K}}v$ by

$$u^{\text{near}}_{\mathfrak{K},T} := \widetilde{\mathfrak{K}}\big((v - v_T)\eta_T\big) \quad \text{and} \quad u^{\text{far}}_{\mathfrak{K},T} := \widetilde{\mathfrak{K}}\big((v - v_T)(1 - \eta_T)\big). \tag{5.11}$$

Note that $(v - v_T)\eta_T \in \widetilde{H}^1(\Gamma) \subseteq H^1(\partial\Omega)$ and $(v - v_T)(1 - \eta_T) \in H^1(\partial\Omega)$ so that $u^{\text{near}}_{\mathfrak{K},T}, u^{\text{far}}_{\mathfrak{K},T} \in H^1(U\backslash\partial\Omega)$ are well defined. Since $\widetilde{\mathfrak{K}}1 \equiv -1$ in Ω and $\widetilde{\mathfrak{K}}1 \equiv 0$ in Ω^{ext}, we have, for every $T \in \mathcal{T}_h$, the identities

$$u_{\mathfrak{K}} + v_T = u^{\text{near}}_{\mathfrak{K},T} + u^{\text{far}}_{\mathfrak{K},T} \quad \text{in } \Omega \quad \text{and} \quad u_{\mathfrak{K}} = u^{\text{near}}_{\mathfrak{K},T} + u^{\text{far}}_{\mathfrak{K},T} \quad \text{in } \Omega^{\text{ext}}. \tag{5.12}$$

5.2. Inverse estimates for the near-field part $u^{\text{near}}_{\mathfrak{K},T}$.

The following proposition provides an estimate for the near-field part of the double-layer potential.

Proposition 5.2 (Near-field bound for $\widetilde{\mathfrak{K}}$)**.** *Let w_h be a σ-admissible weight function. There exists a constant $C_{\text{near}} > 0$ depending only on $\partial\Omega$, Γ, the κ-shape regularity of $\mathcal{T}_h$, and σ such that the near-field part $u^{\text{near}}_{\mathfrak{K},T}$ satisfies $\gamma_0^{\text{int}}u^{\text{near}}_{\mathfrak{K},T} \in H^1(\Gamma)$, $u^{\text{near}}_{\mathfrak{K},T}|_\Omega \in H^1(\Omega)$, and $u^{\text{near}}_{\mathfrak{K},T}|_{U\setminus\overline{\Omega}} \in H^1(U \setminus \overline{\Omega})$ with*

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)}\Big(&\|h^{1/2}\nabla_\Gamma\gamma_0^{\text{int}}u^{\text{near}}_{\mathfrak{K},T}\|^2_{L^2(T)} \\ &+ \|\nabla u^{\text{near}}_{\mathfrak{K},T}\|^2_{L^2(U_T\cap\Omega)} + \|\nabla u^{\text{near}}_{\mathfrak{K},T}\|^2_{L^2(U_T\cap\Omega^{\text{ext}})}\Big) \\ \le C_{\text{near}}\,\|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)}.&\end{aligned} \tag{5.13}$$

Proof. Recall the stability (2.11) of $\gamma_0^{\rm int}\widetilde{\mathfrak{K}} = \mathfrak{K} - \frac{1}{2} : H^1(\partial\Omega) \to H^1(\partial\Omega)$. Taking into account (5.2) and the Poincaré-type estimate (5.5), we observe

$$\begin{aligned}\|\nabla_\Gamma \gamma_0^{\rm int} u^{\rm near}_{\mathfrak{K},T}\|_{L^2(T)} \le \|\nabla_\Gamma \gamma_0^{\rm int} u^{\rm near}_{\mathfrak{K},T}\|_{L^2(\Gamma)} &\overset{(2.11)}{\lesssim} \|(v - v_T)\eta_T\|_{H^1(\partial\Omega)} \\ &\overset{(5.5)}{\lesssim} \|\nabla_\Gamma v\|_{L^2(\omega_h(T))}.\end{aligned}$$

Summation over all $T \in \mathcal{T}_h$ and σ-admissibility of the weight w_h shows that

$$\sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \|h^{1/2}\nabla_\Gamma \gamma_0^{\rm int} u^{\rm near}_{\mathfrak{K},T}\|^2_{L^2(T)} \lesssim \|w_h \nabla_\Gamma v\|^2_{L^2(\Gamma)}. \tag{5.14}$$

Next, we use the continuity of $\widetilde{\mathfrak{K}} : H^{1/2}(\partial\Omega) \to H^1(U \setminus \partial\Omega)$ from (2.8) and get

$$\begin{aligned}\|\nabla u^{\rm near}_{\mathfrak{K},T}\|^2_{L^2(U_T\cap\Omega)} + \|\nabla u^{\rm near}_{\mathfrak{K},T}\|^2_{L^2(U_T\cap\Omega^{\rm ext})} &\overset{(2.8)}{\lesssim} \|(v - v_T)\eta_T\|^2_{H^{1/2}(\partial\Omega)} \\ &\overset{(5.4)}{\lesssim} \|h^{1/2}\nabla_\Gamma v\|^2_{L^2(\omega_h(T))}.\end{aligned}$$

Summation over all $T \in \mathcal{T}_h$ and σ-admissibility of w_h gives

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|^2_{L^\infty(T)} \Big(\|\nabla u^{\rm near}_{\mathfrak{K},T}\|^2_{L^2(U_T\cap\Omega)} + \|\nabla u^{\rm near}_{\mathfrak{K},T}\|^2_{L^2(U_T\cap\Omega^{\rm ext})} \Big) \\ \lesssim \|w_h \nabla_\Gamma v\|^2_{L^2(\Gamma)}.\end{aligned} \tag{5.15}$$

Combining (5.14)–(5.15), we conclude the proof. □

5.3. **Estimates for the far-field part $u^{\rm far}_{\mathfrak{K},T}$.** As for the simple-layer potential, we have a Caccioppoli inequality for the double-layer potential, which underlies the analysis of the far-field contribution. For the next result, recall U'_T from (5.1).

Lemma 5.3 (Caccioppoli inequality for $u^{\rm far}_{\mathfrak{K},T}$). *The functions $u^{\rm far}_{\mathfrak{K},T}$ of* (5.11) *satisfy $u^{\rm far}_{\mathfrak{K},T}|_\Omega \in C^\infty(\Omega)$, $u^{\rm far}_{\mathfrak{K},T}|_{\Omega^{\rm ext}} \in C^\infty(\Omega^{\rm ext})$, and $u^{\rm far}_{\mathfrak{K},T}|_{U'_T} \in C^\infty(U'_T)$. Furthermore, there exists a constant $C'_{\rm cacc}$ depending only on $\partial\Omega$, Γ, and the κ-shape regularity of $\mathcal{T}_h$ such that the Hessian matrix $D^2 u^{\rm far}_{\mathfrak{K},T}$ satisfies*

$$\|D^2 u^{\rm far}_{\mathfrak{K},T}\|_{L^2(B_{\delta h(T)/8}(x))} \le C'_{\rm cacc} \frac{1}{h(T)} \|\nabla u^{\rm far}_{\mathfrak{K},T}\|_{L^2(B_{\delta h(T)/4}(x))} \quad \forall x \in T \in \mathcal{T}_h. \tag{5.16}$$

Proof. The proof is very similar to that of Lemma 4.3. One observes that the far-field $u^{\rm far}_{\mathfrak{K},T}$ solves the transmission problem

$$\begin{aligned} -\Delta u^{\rm far}_{\mathfrak{K},T} &= 0 && \text{in } \Omega \cup \Omega^{\rm ext}, \\ [u^{\rm far}_{\mathfrak{K},T}] &= (v - v_T)(1 - \eta_T) && \text{in } H^{1/2}(\partial\Omega), \\ [\gamma_1 u^{\rm far}_{\mathfrak{K},T}] &= 0 && \text{in } H^{-1/2}(\partial\Omega). \end{aligned}$$

We note that $(1 - \eta_T)|_{\Gamma\cap U'_T} = 0$ by construction of η_T in (5.2). Hence, the same reasoning as in the proof of Lemma 4.3 can be applied to reach the conclusion (5.16). □

Lemma 5.4 (Local far-field bound for $\widetilde{\mathfrak{K}}$). *For all $T \in \mathcal{T}_h$,*

$$\|h^{1/2}\nabla_\Gamma \gamma_0^{\rm int} u^{\rm far}_{\mathfrak{K},T}\|_{L^2(T)} \le \|h^{1/2}\nabla u^{\rm far}_{\mathfrak{K},T}\|_{L^2(T)} \le C_{\rm far} \|\nabla u^{\rm far}_{\mathfrak{K},T}\|_{L^2(U'_T)}. \tag{5.17}$$

The constant $C_{\rm far} > 0$ depends only on $\partial\Omega$, Γ, and the κ-shape regularity constant of $\mathcal{T}_h$.

Proof. The lemma is shown in exactly the same way as the corresponding bound for the simple-layer potential $\mathfrak{V}$ in Lemma 4.4, appealing to the Caccioppoli inequality (5.16) instead of (4.14). $\square$

Proposition 5.5 (Far-field bound for $\widetilde{\mathfrak{K}}$)**.** *Let w_h be a σ-admissible weight function. There is a constant $C_{\text{far}} > 0$ depending only on $\partial\Omega$, Γ, the κ-shape regularity constant of $\mathcal{T}_h$, and σ, such that*

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} &\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|h^{1/2}\nabla_\Gamma\gamma_0^{\text{int}}u^{\text{far}}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\le \sum_{T\in\mathcal{T}_h}\|w_h\nabla u^{\text{far}}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\le C_{\text{far}}\left(\|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)} + \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)}\right).\end{aligned}$$

Proof. Lemma 5.4 implies

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\|w_h\nabla_\Gamma\gamma_0^{\text{int}}u^{\text{far}}_{\mathfrak{K},T}\|^2_{L^2(T)} &\le \sum_{T\in\mathcal{T}_h}\|w_h\nabla u^{\text{far}}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\lesssim \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla u^{\text{far}}_{\mathfrak{K},T}\|^2_{L^2(U'_T)}\\ &= \sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla u^{\text{far}}_{\mathfrak{K},T}\|^2_{L^2(U'_T\cap\Omega)}\\ &\quad+\sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\|\nabla u^{\text{far}}_{\mathfrak{K},T}\|^2_{L^2(U'_T\cap\Omega^{\text{ext}})}.\end{aligned} \tag{5.18}$$

With the identities (5.12), our definition $u_{\mathfrak{K}} = \widetilde{\mathfrak{K}}v$, and a triangle inequality, we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}&\|w_h\nabla u^{\text{far}}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\overset{(5.18)}{\lesssim}\sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\left(\|\nabla(\widetilde{\mathfrak{K}}v+v_T)\|^2_{L^2(U'_T\cap\Omega)}+\|\nabla\widetilde{\mathfrak{K}}v\|^2_{L^2(U'_T\cap\Omega^{\text{ext}})}\right)\\ &\quad+\sum_{T\in\mathcal{T}_h}\|w_h/h^{1/2}\|^2_{L^\infty(T)}\left(\|\nabla u^{\text{near}}_{\mathfrak{K},T}\|^2_{L^2(U'_T\cap\Omega)}+\|\nabla u^{\text{near}}_{\mathfrak{K},T}\|^2_{L^2(U'_T\cap\Omega^{\text{ext}})}\right).\\ &\overset{(5.13)}{\lesssim}\|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\sum_{T\in\mathcal{T}_h}\left(\|\nabla\widetilde{\mathfrak{K}}v\|^2_{L^2(U'_T\cap\Omega)}+\|\nabla\widetilde{\mathfrak{K}}v\|^2_{L^2(U'_T\cap\Omega^{\text{ext}})}\right)\\ &\quad+\|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)}\\ &\overset{(2.8),\,(4.4)}{\lesssim}\|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)}+\|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)},\end{aligned}$$

where we have used additionally $\|v\|_{\widetilde{H}^{1/2}(\Gamma)} = \|v\|_{H^{1/2}(\partial\Omega)}$. $\square$

6. Proof of Theorem 3.1

We are in position to prove the inverse estimates (3.1), (3.2) of Theorem 3.1.

Proof of the inverse estimate (3.1). Let $\psi \in L^2(\Gamma)$, extend ψ by zero to the entire boundary $\partial\Omega$, and recall the notation from Section 4.1. First, we treat the simple-layer integral operator $\mathfrak{V}$. With the bounds of Propositions 4.2 and 4.5 we get

(6.1)
$$\begin{aligned}\|w_h \nabla_\Gamma \mathfrak{V}\psi\|^2_{L^2(\Gamma)} &= \sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \mathfrak{V}\psi\|^2_{L^2(T)} \\ &\lesssim \sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \gamma_0^{\rm int} u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h} \|w_h \nabla_\Gamma \gamma_0^{\rm int} u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(T)} \\ &\lesssim \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)} \|\psi\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + \|w_h\psi\|^2_{L^2(\Gamma)}.\end{aligned}$$

The estimate for the adjoint double-layer integral operator $\mathfrak{K}'$ follows by similar arguments. We split the left-hand side into near-field and far-field contributions to obtain

$$\begin{aligned}\|w_h \mathfrak{K}'\psi\|^2_{L^2(\Gamma)} \lesssim &\sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{U_T\cap\Gamma})\|^2_{L^2(T)} \\ &+ \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{\Gamma\setminus U_T})\|^2_{L^2(T)}.\end{aligned} \tag{6.2}$$

The continuity $\mathfrak{K}' : L^2(\partial\Omega) \to L^2(\partial\Omega)$ stated in (2.12) yields for the near-field contribution

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{U_T\cap\Gamma})\|^2_{L^2(T)} &\le \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{U_T\cap\Gamma})\|^2_{L^2(\partial\Omega)} \\ &\overset{(2.12)}{\lesssim} \sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\psi\|^2_{L^2(U_T\cap\Gamma)} \\ &\lesssim \|w_h\psi\|^2_{L^2(\Gamma)}.\end{aligned}$$

For the far-field contribution, we write $u^{\rm far}_{\mathfrak{V},T} = \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\setminus U_T})$ and note that $\mathfrak{K}' = -1/2 + \gamma_1^{\rm int}\widetilde{\mathfrak{V}}$ and clearly $(\psi\chi_{\Gamma\setminus U_T})|_T = 0$. Therefore, on T we have $\mathfrak{K}'(\psi\chi_{\Gamma\setminus U_T}) = \gamma_1^{\rm int} u^{\rm far}_{\mathfrak{V},T}$. Furthermore, by the smoothness of $u^{\rm far}_{\mathfrak{V},T}$ near T (see Lemma 4.3), we have $\gamma_1^{\rm int} u^{\rm far}_{\mathfrak{V},T} = \partial_{\boldsymbol\nu} u^{\rm far}_{\mathfrak{V},T}$ on T (cf. Remark 2.2) and get

$$\|\mathfrak{K}'(\psi\chi_{\Gamma\setminus U_T})\|_{L^2(T)} = \|\gamma_1^{\rm int} u^{\rm far}_{\mathfrak{V},T}\|_{L^2(T)} = \|\partial_{\boldsymbol\nu} u^{\rm far}_{\mathfrak{V},T}\|_{L^2(T)} \lesssim \|\nabla u^{\rm far}_{\mathfrak{V},T}\|_{L^2(T)}.$$

The far-field contribution in (6.2) can therefore be bounded by Proposition 4.5 via

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|w_h\|^2_{L^\infty(T)} \|\mathfrak{K}'(\psi\chi_{\Gamma\setminus U_T})\|^2_{L^2(T)} &\lesssim \sum_{T\in\mathcal{T}_h} \|w_h \nabla u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(T)} \\ &\lesssim \|w_h\psi\|^2_{L^2(\Gamma)} + \|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)} \|\psi\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

Altogether, this gives

$$\|w_h \mathfrak{K}'\psi\|_{L^2(\Gamma)} \lesssim \|w_h\psi\|_{L^2(\Gamma)} + \|w_h/h^{1/2}\|_{L^\infty(\Gamma)} \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}. \qquad \square$$

Proof of inverse estimate (3.2). First, we treat the double-layer integral operator $\mathfrak{K}$. Let $v \in \widetilde{H}^1(\Gamma)$, extend v by zero to $v \in H^1(\partial\Omega)$, and recall the notation from

Section 5.1. We recall the stability of $\mathfrak{K} = \frac{1}{2} + \gamma_0^{\rm int}\widetilde{\mathfrak{K}} : H^1(\partial\Omega) \to H^1(\partial\Omega)$, from which we conclude $\gamma_0^{\rm int}\widetilde{\mathfrak{K}}v \in H^1(\Gamma)$. Therefore,

$$
\begin{aligned}
\|w_h \nabla_\Gamma \mathfrak{K} v\|_{L^2(\Gamma)} &= \|w_h \nabla_\Gamma \big(\tfrac{1}{2} + \gamma_0^{\rm int}\widetilde{\mathfrak{K}}\big) v\|_{L^2(\Gamma)} \\
&\le \frac{1}{2}\|w_h \nabla_\Gamma v\|_{L^2(\Gamma)} + \|w_h \nabla_\Gamma \gamma_0^{\rm int} u_{\mathfrak{K}}\|_{L^2(\Gamma)}
\end{aligned} \tag{6.3}
$$

with $u_{\mathfrak{K}} = \widetilde{\mathfrak{K}}v$. It holds that $u_{\mathfrak{K}} + v_T = u_{\mathfrak{K},T}^{\rm near} + u_{\mathfrak{K},T}^{\rm far}$ in Ω; cf. (5.12). For the second term on the right-hand side in (6.3), we obtain

$$
\begin{aligned}
\|w_h \nabla_\Gamma \gamma_0^{\rm int} u_{\mathfrak{K}}\|_{L^2(\Gamma)}^2 &\le \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|_{L^\infty(T)}^2 \|h^{1/2}\nabla_\Gamma \gamma_0^{\rm int}(u_{\mathfrak{K}} + v_T)\|_{L^2(T)}^2 \\
&\overset{(5.12)}{\lesssim} \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|_{L^\infty(T)}^2 \|h^{1/2}\nabla_\Gamma \gamma_0^{\rm int} u_{\mathfrak{K},T}^{\rm near}\|_{L^2(T)}^2 \\
&\quad + \sum_{T\in\mathcal{T}_h} \|w_h/h^{1/2}\|_{L^\infty(T)}^2 \|h^{1/2}\nabla_\Gamma \gamma_0^{\rm int} u_{\mathfrak{K},T}^{\rm far}\|_{L^2(T)}^2.
\end{aligned} \tag{6.4}
$$

The first sum can be bounded by Proposition 5.2, whereas the second sum can be bounded by Proposition 5.5. Altogether, this yields

$$
\|w_h \nabla_\Gamma \mathfrak{K} v\|_{L^2(\Gamma)} \lesssim \|w_h/h^{1/2}\|_{L^\infty(\Gamma)} \|v\|_{\widetilde{H}^{1/2}(\Gamma)} + \|w_h \nabla_\Gamma v\|_{L^2(\Gamma)}
$$

and concludes the first part of the proof.

The result for the hypersingular integral operator $\mathfrak{W}$ is shown with similar arguments. Again let $v \in \widetilde{H}^1(\Gamma)$ and v_T as in Lemma 5.1. Note that $\mathfrak{W}v_T = 0$. Now splitting into near-field and far-field yields

$$
\begin{aligned}
\|w_h \mathfrak{W} v\|_{L^2(\Gamma)}^2 &= \sum_{T\in\mathcal{T}_h} \|w_h \mathfrak{W}(v - v_T)\|_{L^2(T)}^2 \\
&\lesssim \sum_{T\in\mathcal{T}_h} \|w_h \mathfrak{W}((v - v_T)\eta_T)\|_{L^2(T)}^2 \\
&\quad + \sum_{T\in\mathcal{T}_h} \|w_h \mathfrak{W}((v - v_T)(1 - \eta_T))\|_{L^2(T)}^2.
\end{aligned} \tag{6.5}
$$

The near-field contribution is bounded by the stability of $\mathfrak{W} : H^1(\partial\Omega) \to L^2(\partial\Omega)$ stated in (2.13) and the Poincaré-type estimate (5.5):

$$
\|\mathfrak{W}((v - v_T)\eta_T)\|_{L^2(T)}^2 \overset{(2.13)}{\lesssim} \|(v - v_T)\eta_T\|_{H^1(\omega_h(T))}^2 \overset{(5.5)}{\lesssim} \|\nabla_\Gamma v\|_{L^2(\omega_h(T))}^2.
$$

The sum over all elements gives

$$
\sum_{T\in\mathcal{T}_h} \|w_h \mathfrak{W}((v - v_T)\eta_T)\|_{L^2(T)}^2 \lesssim \sum_{T\in\mathcal{T}_h} \|w_h\|_{L^\infty(T)}^2 \|\nabla_\Gamma v\|_{L^2(\omega_h(T))}^2 \lesssim \|w_h \nabla_\Gamma v\|_{L^2(\Gamma)}^2.
$$

It remains to bound the second term on the right-hand side in (6.5). In view of the support properties of η_T, the potential $u_{\mathfrak{K},T}^{\rm far} = \widetilde{\mathfrak{K}}((v - v_T)(1 - \eta_T))$ is smooth near T (cf. Lemma 5.3) so that $\gamma_1^{\rm int} u_{\mathfrak{K},T}^{\rm far} = \partial_{\boldsymbol{\nu}} u_{\mathfrak{K},T}^{\rm far}$ on T. Furthermore, since $\mathfrak{W} = -\gamma_1^{\rm int}\widetilde{\mathfrak{K}}$ we see

$$
\|\mathfrak{W}((v - v_T)(1 - \eta_T))\|_{L^2(T)}^2 = \|\gamma_1^{\rm int} u_{\mathfrak{K},T}^{\rm far}\|_{L^2(T)}^2 = \|\partial_{\boldsymbol{\nu}} u_{\mathfrak{K},T}^{\rm far}\|_{L^2(T)}^2 \le \|\nabla u_{\mathfrak{K},T}^{\rm far}\|_{L^2(T)}^2.
$$

We use Proposition 5.5 to conclude

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} &\|w_h\mathfrak{W}((v-v_T)(1-\eta_T))\|^2_{L^2(T)}\\ &\leq \sum_{T\in\mathcal{T}_h}\|w_h\nabla u^{\rm far}_{\mathfrak{K},T}\|^2_{L^2(T)}\\ &\lesssim \|w_h\nabla_\Gamma v\|^2_{L^2(\Gamma)}+\|w_h/h^{1/2}\|^2_{L^\infty(\Gamma)}\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)}.\end{aligned}$$

Altogether, we obtain

$$\|w_h\mathfrak{W}v\|_{L^2(\Gamma)}\lesssim \|w_h\nabla_\Gamma v\|_{L^2(\Gamma)}+\|w_h/h^{1/2}\|_{L^\infty(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma)}. \qquad \square$$

Appendix A. A polynomial inverse estimate

Lemma A.1. *Let $\mathcal{T}_h$ be a regular, κ-shape regular triangulation of Γ. Suppose that $d\geq 2$ and that q_h is a σ-admissible polynomial degree distribution with respect to $\mathcal{T}_h$. Then, there exists a constant $\widetilde{C}_{\rm inv}>0$ which depends solely on $\partial\Omega$, the κ-shape regularity of $\mathcal{T}_h$, and σ, such that*

$$\|h^{1/2}(q_h+1)^{-1}\,\Psi_h\|_{L^2(\Gamma)}\leq \widetilde{C}_{\rm inv}\|\Psi_h\|_{H^{-1/2}(\Gamma)}\quad \text{for all } \Psi_h\in\mathcal{P}^{\mathbf{q}}(\mathcal{T}_h). \tag{A.1}$$

Proof. Step 1: For each $T\in\mathcal{T}_h$, we claim the existence of a function $\widehat{\chi}_{T,q_h(T)}\in C^\infty(\mathbb{R}^{d-1})$ with the following properties for some fixed $\delta>0$ (see, e.g., the proofs of [Geo08, Lem. 3.7, Prop. 3.8] or the arguments below):

$$\operatorname{supp}\widehat{\chi}_{T,q_h(T)}\subseteq\{x\in T_{\rm ref}\colon\ \operatorname{dist}(x,\partial T_{\rm ref})>\delta/(q_h(T)+1)^2\}, \tag{A.2}$$

$$0\leq\widehat{\chi}_{T,q_h(T)}\leq 1\quad\text{in } T_{\rm ref},\qquad \|\nabla\widehat{\chi}_{T,q_h(T)}\|_{L^\infty(T_{\rm ref})}\lesssim (q_h(T)+1)^{-2}, \tag{A.3}$$

$$\widehat{\chi}_{T,q_h(T)}\equiv 1\quad\text{in } \{x\in T_{\rm ref}\colon\ \operatorname{dist}(x,\partial T_{\rm ref})>3\delta/(q_h(T)+1)^2\}, \tag{A.4}$$

$$\|\pi\|_{L^2(T_{\rm ref})}\leq C\|\pi\widehat{\chi}_{T,p(T)}\|_{L^2(T_{\rm ref})}\quad \text{for all polynomials } \pi \text{ of degree } q_h(T), \tag{A.5}$$

$$\|\pi\widehat{\chi}_{T,q_h(T)}\|_{H^1(T_{\rm ref})}\leq C(1+q_h(T))^2\|\pi\|_{L^2(T_{\rm ref})}\quad \text{for all polynomials } \pi \text{ of degree } q_h(T). \tag{A.6}$$

The function $\widehat{\chi}_{T,q_h(T)}$ is obtained from a mollification with length-scale $\delta/(q_h(T+1))^2$ of the characteristic function of $T_{\rm ref}\setminus S_{2\delta/(q_h(T)+1)^2}$, where $S_\varepsilon:=\{x\in T_{\rm ref}\colon\ \operatorname{dist}(x,\partial T_{\rm ref})<\varepsilon\}$. The parameter $\delta>0$ is dictated by the requirement (A.5): For this, we introduce the shorthand $\varepsilon(\delta)=3\delta/(q_h(T)+1)^2$. Observe that the radius of mollification is chosen such that $\widehat{\chi}_{T,q_h(T)}\equiv 1$ on $T_{\rm ref}\setminus S_{\varepsilon(\delta)}$, so that we are done once we have established $\|\pi\|_{L^2(S_{\varepsilon(\delta)})}\lesssim\|\pi\|_{L^2(T_{\rm ref}\setminus S_{\varepsilon(\delta)})}$. [LMWZ10, Lemma 2.1] and the polynomial inverse estimate $\|\pi\|_{H^1(T_{\rm ref})}\lesssim (q_h(T)+1)^2\|\pi\|_{L^2(T_{\rm ref})}$, yield

$$\begin{aligned}\|\pi\|^2_{L^2(S_{\varepsilon(\delta)})}&\lesssim\varepsilon(\delta)\|\pi\|_{L^2(T_{\rm ref})}\|\pi\|_{H^1(T_{\rm ref})}\lesssim\varepsilon(\delta)(q_h(T)+1)^2\|\pi\|^2_{L^2(T_{\rm ref})}\\ &=\varepsilon(\delta)(q_h(T)+1)^2\left[\|\pi\|^2_{L^2(T_{\rm ref}\setminus S_{\varepsilon(\delta)})}+\|\pi\|^2_{L^2(S_{\varepsilon(\delta)})}\right]\\ &=3\delta\left[\|\pi\|^2_{L^2(T_{\rm ref}\setminus S_{\varepsilon(\delta)})}+\|\pi\|^2_{L^2(S_{\varepsilon(\delta)})}\right].\end{aligned}$$

Taking δ sufficiently small produces $\|\pi\|_{L^2(S_{\varepsilon(\delta)})}\lesssim\|\pi\|_{L^2(T_{\rm ref}\setminus S_{\varepsilon(\delta)})}$ as desired.

Step 2: Define $\chi_{T,q_h(T)}$ with $\operatorname{supp}\chi_{T,q_h(T)} \subseteq T$ by $\chi_{T,q_h(T)} \circ \gamma_T = \widehat{\chi}_{T,q_h(T)}$. Given $\Psi_h \in \mathcal{P}^{\mathbf{q}}(\mathcal{T}_h)$, define

$$v_T|_T := \frac{h(T)}{(1+q_h(T))^2}(\Psi_h|_T)\chi_{T,q_h(T)}, \tag{A.7}$$

and extend v_T by zero to Γ. Note that $v_T \in \widetilde{H}^1(\Gamma)$ by the support properties of $\chi_{T,q_h(T)}$. An interpolation inequality and the estimates (A.5), (A.6) on the reference element give

$$\begin{aligned}\|v_T\|^2_{\widetilde{H}^{1/2}(\Gamma)} &= \|v_T\|^2_{\widetilde{H}^{1/2}(\partial\Omega)} \lesssim \|v_T\|_{L^2(\partial\Omega)}\|v_T\|_{H^1(\partial\Omega)} \\ &= \|v_T\|_{L^2(T)}\|v_T\|_{H^1(T)} \\ &\overset{(A.6),(A.5)}{\lesssim} \frac{(1+q_h(T))^2}{h(T)}\|v_T\|^2_{L^2(T)}.\end{aligned} \tag{A.8}$$

For $v := \sum_{T\in\mathcal{T}_h} v_T$, there holds $v \in \widetilde{H}^1(\Gamma)$. With $\operatorname{supp}(v_T) \subseteq T$ and [SS11, Lemma 4.1.49], we have

$$\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)} = \Big\|\sum_{T\in\mathcal{T}_h} v_T\Big\|^2_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \sum_{T\in\mathcal{T}_h}\|v_T\|^2_{\widetilde{H}^{1/2}(\Gamma)} \overset{(A.8)}{\lesssim} \Big\|\frac{1+q_h}{h^{1/2}}\, v\Big\|^2_{L^2(\Gamma)}. \tag{A.9}$$

Finally, we estimate

$$\begin{aligned}\Big\|\frac{h^{1/2}}{1+q_h}\Psi_h\Big\|^2_{L^2(\Gamma)} &= \sum_{T\in\mathcal{T}_h}\Big\|\frac{h(T)^{1/2}}{1+q_h(T)}\Psi_h\Big\|^2_{L^2(T)} = \sum_{T\in\mathcal{T}_h}(v_T,\Psi_h)_{L^2(T)} = (v,\Psi_h)_{L^2(\Gamma)} \\ &\le \|\Psi_h\|_{H^{-1/2}(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma)} \overset{(A.9)}{\lesssim} \|\Psi_h\|_{H^{-1/2}(\Gamma)}\Big\|\frac{1+q_h}{h^{1/2}}\, v\Big\|_{L^2(\Gamma)} \\ &\overset{(A.7)}{\lesssim} \|\Psi_h\|_{H^{-1/2}(\Gamma)}\Big\|\frac{h^{1/2}}{1+q_h}\Psi_h\Big\|_{L^2(\Gamma)}.\end{aligned}$$

□

Appendix B. Norm equivalences (Proof of Facts 2.1)

B.1. **Preliminaries.** For an open set $\omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, and $\theta \in (0,1)$, we define the Aronstein-Slobodeckii norm by

$$\begin{aligned}\|u\|^2_{AS,s,\omega} &:= \|u\|^2_{L^2(\omega)} + |u|^2_{AS,s,\omega}, \\ |u|^2_{AS,\theta,\omega} &:= \int_{x\in\omega}\int_{y\in\omega}\frac{|u(x)-u(y)|^2}{|x-y|^{2\theta+n}}\,dy\,dx.\end{aligned} \tag{B.1}$$

Lemma B.1. *Let $n \in \mathbb{N}$ and $\theta \in (0,1)$. Introduce for $h \in \mathbb{R}^n$ the notation $(\Delta_h u)(x) := u(x+h) - u(x)$. Fix a non-negative function $\rho \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\rho \subset B_2(0)\setminus B_1(0)$ and $\int_{\mathbb{R}^n}\rho(x)\,dx = 1$. Set $\rho_t(x) := t^{-n}\rho(x/t)$. For $u \in L^2(\mathbb{R}^n)$ and $t > 0$ define the convolution $u_t := u \star \rho_t \in C^\infty(\mathbb{R}^n)$. Then, for a constant $C > 0$ depending solely on ρ, it holds that*

$$\begin{aligned}k(u,t) &:= \|u-u_t\|_{L^2(\mathbb{R}^n)} + t\|u_t\|_{H^1(\mathbb{R}^n)} \\ &\le C\left(\int_{t\le|h|\le 2t}\|\Delta_h u\|^2_{L^2(\mathbb{R}^n)}|h|^{-n}\,dh\right)^{1/2} + t\|u\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{B.2}$$

If $\theta \in (0,1)$, then there exists $C_\theta > 0$ (depending only on ρ, θ, and n) such that for u with $\|u\|_{AS,\theta,\mathbb{R}^n} < \infty$, one has additionally

$$\int_{t=0}^{1} \left(t^{-\theta} k(u,t)\right)^2 \frac{dt}{t} \le C_\theta \|u\|^2_{AS,\theta,\mathbb{R}^n}. \tag{B.3}$$

Proof. The bound (B.2) follows from inspecting [AF03, Thm. 7.47, proof of (c) $\Rightarrow$ (a)]. The bound (B.3) follows from (B.2). ∎

The special feature of Lemma B.1 is that it provides a decomposition $u = (u - u_t) + u_t$ that is suitable for use in connection with the K-functional. Additional properties of u_t can be enforced by judiciously choosing ρ:

Lemma B.2. *Let $n \in \mathbb{N}$ and $\theta \in (0,1)$. Let $\zeta : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz with Lipschitz constant L. For $\varepsilon \ge 0$, define the sets $\Omega^+_\varepsilon := \{(x', y) \,|\, x' \in \mathbb{R}^{n-1}, y > \zeta(x') - \varepsilon\}$. Then one can select a function ρ such that the following is true for all $u \in \{u\colon \|u\|_{AS,\theta,\mathbb{R}^n} < \infty, \quad u|_{\Omega^+_0} = 0\}$:*

(i) *The estimates of Lemma* B.1 *hold.*
(ii) $u_t|_{\Omega^+_{t/2}} = 0$.

Proof. For $x \in \mathbb{R}^n$ and $c > 0$, introduce the (infinite) cones $C_{x,c} := x + \{(x', y) \,|\, x' \in \mathbb{R}^{n-1}, c|x'| < y\}$. Note that, since ζ is Lipschitz, there exists $c' > 0$ (depending solely on L and n) such that $C_{x,c'} \subset \Omega^+_0$ for all $x \in \Omega^+_0$. We select the non-negative function $\rho \in C^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$ with the support property $\operatorname{supp} \rho \subset (B_2(0) \setminus B_1(0)) \cap (-C_{(\mathbf{0},1/2),c'})$. For any $t > 0$ and any $x \in \Omega^+_{t/2}$, these support properties ensure $\operatorname{supp} \rho_t(x - \cdot) \subset \overline{\Omega^+_0}$. Hence, $u_t|_{\Omega^+_{t/2}} = 0$ if $u|_{\Omega^+_0} = 0$. This shows (ii). The statement (i) follows directly from Lemma B.1. ∎

Lemma B.3. *Let ω, $\omega' \subset \mathbb{R}^n$ be open. Let $\Phi : \overline{\omega} \to \overline{\omega'}$ be bi-Lipschitz and $u \in H^1_{\ell oc}(\omega')$. Then, the composed function $v := u \circ \Phi$ satisfies $v \in H^1_{\ell oc}(\omega)$, and the chain rule $(\nabla v)^\top(x) = ((\nabla u)^\top \circ \Phi(x)) D\Phi(x)$ holds almost everywhere in ω.*

Proof. We follow essentially [Zie89, Thm. 2.2.2]. First, we note that Φ may be extended as a Lipschitz function to a function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, [Ste70, Thm. 3, Chap. VI]. (This extension may not be invertible as a map $\mathbb{R}^n \to \mathbb{R}^n$ but it suffices for our purposes that $\Phi : \overline{\omega} \to \overline{\omega'}$ is bi-Lipschitz.) We conclude from [Fed69, Thm. 3.2.5] or [EG92, Thm. 2, Sec. 3.4.3] that for any function $g \in L^1(\omega')$, it holds that

$$\int_\omega (g \circ \Phi)(x) J(x)\, dx = \int_{\omega'} g(y)\, dy, \qquad J(x) = |\det D\Phi(x)|. \tag{B.4}$$

Inspection of the proof of [Zie89, Thm. 2.2.2] shows that (B.4) can take the role of [Zie89, eqn. (2.2.9)] in the proof of [Zie89, Thm. 2.2.2]. The result then follows. ∎

B.2. **Proof of Facts 2.1.** We fix notation, following [McL00, p. 96ff]. The boundary $\partial\Omega$ is described by $N \in \mathbb{N}$ Lipschitz continuous functions $\zeta_i : \mathbb{R}^{d-1} \to \mathbb{R}$ and Euclidean transformations Q_i (i.e., translations and rotations). In terms of the functions ζ_i, we define functions $Z_i : \mathbb{R}^d \to \mathbb{R}^d$ given by $(x', t) \mapsto Q_i(x', \zeta_i(x') + t)$, which are bi-Lipschitz. It is convenient to introduce the maps $\widehat{\zeta}_i : \mathbb{R}^{d-1} \to \mathbb{R}^d$ by $\widehat{\zeta}_i(x') := Z_i(x', 0)$. The setting of [McL00, p. 96ff] is as follows: There are N domains Ω_i that are Lipschitz hypographs described by the maps Z_i; in particular, $\widehat{\zeta}_i(\mathbb{R}^{d-1}) = Z_i(\mathbb{R}^{d-1}, 0) = \partial\Omega_i$. We note that bi-Lipschitz continuity of

$Z_i : \mathbb{R}^d \to \mathbb{R}^d$ implies that $\widehat{\zeta}_i : \mathbb{R}^{d-1} \to \partial\Omega_i$ is bi-Lipschitz. The setting of Lipschitz domains is such that, locally, Ω "coincides" with one of the hypographs Ω_i; that is, there are N Lipschitz domains $\widehat{\omega}_i \subset \mathbb{R}^{d-1}$ such that the sets $\omega_i := \widehat{\zeta}_i(\widehat{\omega}_i) \subset \partial\Omega$ are an open cover of $\partial\Omega$. For any function $v \in L^2(\partial\Omega)$, we define its pull-back $\widehat{v}_i$ to $\widehat{\omega}_i$ by $\widehat{v}_i := v|_{\omega_i} \circ \widehat{\zeta}_i$. A key property of Lipschitz domains Ω is that $H^1(\partial\Omega)$-functions feature the expected transformation rule under change of charts:

Theorem B.4. *Let $\emptyset \neq \omega = \omega_i \cap \omega_j$ for some i, j. Set $\omega_i' := \widehat{\zeta}_i^{-1}(\omega)$. Then, for $u \in H^1(\partial\Omega)$, it holds that*

$$\widehat{u}_i = \widehat{u}_j \circ (\widehat{\zeta}_j^{-1} \circ \widehat{\zeta}_i) \qquad \text{a.e. on } \omega_i', \tag{B.5}$$

$$(\nabla\widehat{u}_i)^\top = ((\nabla\widehat{u}_j)^\top \circ (\widehat{\zeta}_j^{-1} \circ \widehat{\zeta}_i))D(\widehat{\zeta}_j^{-1} \circ \widehat{\zeta}_i) \qquad \text{a.e. on } \omega_i'. \tag{B.6}$$

Proof. (B.5) is clear. The transformation rule (B.6) follows from Lemma B.3. ☐

We employ the partition of unity $(\varphi_i)_{i=1}^N$ associated with the cover $(\omega_i)_{i=1}^N$ that is described in [McL00, p. 98ff]. With $\widehat{\varphi}_i$ being the pull-back of φ_i, we note the compact inclusion $\operatorname{supp}\widehat{\varphi}_i \subset\subset \widehat{\omega}_i$. The Sobolev norms $\|\cdot\|_{H^\theta(\partial\Omega)}$, $0 \le \theta \le 1$ are defined in [McL00, p. 99] as

$$\|u\|^2_{H^\theta(\partial\Omega)} := \sum_{i=1}^N \|\widehat{\varphi}_i\widehat{u}_i\|^2_{H^\theta(\mathbb{R}^{d-1})}, \tag{B.7}$$

with the norms on the right-hand side defined by Bessel potentials. According to [McL00, Thm. 3.16], we can alternatively use the definition by norms of distributional derivatives. Then, by the support properties of the functions $\widehat{\varphi}_i$, one may replace $\|\cdot\|_{H^\theta(\mathbb{R}^{d-1})}$ by $\|\cdot\|_{H^\theta(\widehat{\omega}_i)}$ for $\theta \in \{0, 1\}$.

Lemma B.5. *For $\theta \in \{0, 1\}$ and $u \in H^\theta(\partial\Omega)$, one has the norm equivalence $\|u\|^2_{H^\theta(\partial\Omega)} \simeq \sum_{i=1}^N \|\widehat{u}_i\|^2_{H^\theta(\widehat{\omega}_i)}$.*

Proof. The estimate $\|u\|^2_{H^\theta(\partial\Omega)} \lesssim \sum_{i=1}^N \|\widehat{u}_i\|^2_{H^\theta(\widehat{\omega}_i)}$ follows readily from the definition of $\|\cdot\|_{H^\theta(\partial\Omega)}$. The converse estimate results from the fact that $(\varphi_i)_{i=1}^N$ is a partition of unity on $\partial\Omega$. To see this, fix i and define for $j = 1, \dots, N$, the sets $\widehat{\omega}_{ij} := \{x \in \widehat{\omega}_i : |\varphi_j(\widehat{\zeta}_i(x))| > 1/(2N)\}$. Since $(\varphi_j)_{j=1}^N$ is a partition of unity, we have $\bigcup_{j=1}^N \widehat{\omega}_{ij} = \widehat{\omega}_i$. Since $|\varphi_j(\widehat{\zeta}_j(x))| \ge 1/(2N)$ for $x \in (\widehat{\zeta}_j^{-1} \circ \widehat{\zeta}_i)(\widehat{\omega}_{ij})$, we can infer with (B.5) of Theorem B.4 that $\|\widehat{u}_i\|_{L^2(\widehat{\omega}_{ij})} \lesssim \|\widehat{\varphi}_j\widehat{u}_j\|_{L^2(\widehat{\omega}_j)}$, and hence $\|\widehat{u}_i\|_{L^2(\widehat{\omega}_i)} \lesssim \|u\|_{L^2(\partial\Omega)}$. For $\theta = 1$, we additionally note $\nabla(\widehat{\varphi}_j\widehat{u}_j) = \widehat{\varphi}_j\nabla\widehat{u}_j + \widehat{u}_j\nabla\widehat{\varphi}_j$. With (B.5) and (B.6) of Theorem B.4 we infer that $\|\nabla\widehat{u}_i\|_{L^2(\widehat{\omega}_{ij})} \lesssim \|\nabla(\widehat{\varphi}_j\widehat{u}_j)\|_{L^2(\widehat{\omega}_j)} + \|\widehat{u}_j\|_{L^2(\widehat{\omega}_j)}$. Hence,

$$\|\nabla\widehat{u}_i\|_{L^2(\widehat{\omega}_i)} \lesssim \sum_{j=1}^N \|\widehat{\varphi}_j\widehat{u}_j\|_{H^1(\widehat{\omega}_j)} + \|u\|_{L^2(\partial\Omega)}.$$ ☐

Lemma B.6. *Let N functions $\widetilde{u}_i \in \widetilde{H}^1(\widehat{\omega}_i)$, $i = 1, \dots, N$, be given and let each function u_i be defined on ω_i as the push-forward of $\widetilde{u}_i$, i.e., u_i is characterized by $u_i \circ \widehat{\zeta}_i = \widetilde{u}_i$. Then the function $u = \sum_{i=1}^N \chi_{\omega_i} u_i$, where χ_A denote the characteristic function of the set $A \subset \partial\Omega$, is in $H^1(\partial\Omega)$ and*

$$\|u\|_{H^1(\partial\Omega)} \lesssim \sum_{i=1}^N \|\widetilde{u}_i\|_{H^1(\widehat{\omega}_i)}.$$

Proof. It suffices to consider the case of a fixed i and $\widetilde{u}_j \equiv 0$ for $j \neq i$. Let $\widetilde{u}_i \in C_0^\infty(\widehat{\omega}_i)$. The essential step is to show that the function u (as defined in the statement of the lemma) is in $H^1(\partial\Omega)$. The stated bound then follows from Lemma B.5, using also Theorem B.4 and the fact that there are finitely many domains $\widehat{\omega}_i$. The general case of $\widetilde{u}_i \in \widetilde{H}^1(\widehat{\omega}_i)$ follows from that of $\widetilde{u}_i \in C_0^\infty(\widehat{\omega}_i)$ by a density argument.

The assumption $\widetilde{u}_i \in C_0^\infty(\widehat{\omega}_i)$ implies $u \in C(\partial\Omega)$ and $\operatorname{supp} u \subset \omega_i$. In order to see $u \in H^1(\partial\Omega)$ we write $\widehat{u}_j := u \circ \widehat{\zeta}_j$ and claim $\widehat{u}_j \in H^1(\widehat{\omega}_j)$. To that end, consider j with $\omega_{ji} := \omega_j \cap \omega_i \neq \emptyset$ and introduce the open set $\widehat{\omega}_{ji} := \widehat{\zeta}_j^{-1}(\omega_{ji}) \subset \widehat{\omega}_j$. By Lemma B.3, we have $\widehat{u}_j \in H^1_{\ell oc}(\widehat{\omega}_{ji})$. Our proof will be complete once we have shown $\operatorname{supp}(\widehat{u}_j\widehat{\varphi}_j) \subset\subset \widehat{\omega}_{ji}$. This last assertion follows from the above observation $\operatorname{supp} u \subset \omega_i$, the observation $\operatorname{supp} \varphi_j \subset \omega_j$ and $\operatorname{supp}(\widehat{u}_j\widehat{\varphi}_j) = \widehat{\zeta}_j^{-1}(\operatorname{supp}(u\varphi_j)) \subset \widehat{\zeta}_j^{-1}(\operatorname{supp}\varphi_j \cap \operatorname{supp} u) = \widehat{\zeta}_j^{-1}(\omega_j \cap \omega_i) = \zeta_j^{-1}(\omega_{ji}) = \widehat{\omega}_{ji}$. □

Proposition B.7. *Facts* 2.1(i) *holds with equivalent norms.*

Proof. It suffices to consider the case $\Gamma = \partial\Omega$ since $u|_{\partial\Omega\setminus\overline{\Gamma}} = 0$ implies $(\nabla_\Gamma u)|_{\partial\Omega\setminus\overline{\Gamma}} = 0$. We exploit that $\partial\Omega$ is piecewise C^1 as defined in [SS11, Def. 2.2.10]. Recall from (2.3) the definition of the surface gradient on a surface piece Γ_ℓ with parametrization $\xi_\ell : \widehat{\Gamma}_\ell \to \Gamma_\ell$. Since the Gramian matrix $G = D\xi_\ell^\top D\xi_\ell$ and its inverse G^{-1} are uniformly symmetric positive definite on $\overline{\widehat{\Gamma}_\ell}$, we infer the existence of a constant $C > 0$ (depending only $\partial\Omega$, Γ_ℓ, and ξ_ℓ) such that

$$C^{-1}|(\nabla_\Gamma u)|_{\Gamma_\ell} \circ \xi_\ell| \leq |\nabla(u \circ \xi_\ell)| \leq C|(\nabla_\Gamma u)|_{\Gamma_\ell} \circ \xi_\ell| \qquad \text{a.e. on } \widehat{\Gamma}_\ell. \tag{B.8}$$

For each ω_i, we get from Theorem B.4 for a constant $\widetilde{C}$ (depending only on $\partial\Omega$, Γ_ℓ, and ξ_ℓ) that

$$\widetilde{C}^{-1}|\nabla(u \circ \xi_\ell)| \leq |\nabla\widehat{u}_i \circ \widehat{\zeta}_i^{-1} \circ \xi_\ell| \leq \widetilde{C}|\nabla(u \circ \xi_\ell)| \qquad \text{a.e. on } \xi_\ell^{-1}(\Gamma_\ell \cap \omega_i). \tag{B.9}$$

Since $\partial\Omega = \bigcup_i \omega_i$ and $\partial\Omega = \bigcup_\ell \overline{\Gamma_\ell}$, the estimates (B.8) and (B.9) and Lemma B.5 imply $\|\nabla_\Gamma u\|^2_{L^2(\partial\Omega)} \lesssim \sum_\ell \|\nabla_\Gamma u\|^2_{L^2(\Gamma_\ell)} \lesssim \sum_i \|\nabla\widehat{u}_i\|^2_{L^2(\widehat{\omega}_i)} \lesssim \|u\|^2_{H^1(\partial\Omega)}$. For the reverse estimate, we again note that the surface patches Γ_ℓ form a partition of $\partial\Omega$, that is, $\omega_i = \bigcup_\ell \overline{\Gamma_\ell} \cap \omega_i$. Hence, (B.8) and (B.9) imply

$$|\nabla\widehat{u}_i \circ \widehat{\zeta}_i^{-1}| \leq C'|\nabla_\Gamma u| \quad \text{ a.e. on } \omega_i$$

for a constant $C' > 0$ depending only on $\partial\Omega$ and the (finitely many) surface patches Γ_ℓ with their parametrizations ξ_ℓ. This in turn implies $\sum_i \|\nabla\widehat{u}_i\|^2_{L^2(\widehat{\omega}_i)} \lesssim \|\nabla_\Gamma u\|^2_{L^2(\partial\Omega)}$, and an appeal to Lemma B.5 finishes the proof. □

Proposition B.8. *Facts* 2.1(ii) *holds with equivalent norms.*

Proof. The Aronstein-Slobodeckii norm on $\partial\Omega$ is defined as in (B.1), where the integration over ω is replaced with that over the surface $\partial\Omega$ and $|x - y|$ in the denominator is replaced with the Euclidean distance between x and y. (Since $\partial\Omega$ is Lipschitz, the distance could alternatively be taken as the geodesic distance on $\partial\Omega$.) Recall the partition of unity $(\varphi_i)_{i=1}^N$. Using the fact that the maps $\widehat{\zeta}_i$ are bi-Lipschitz and the equivalence of the Aronstein-Slobodeckii norm to the Sobolev norm (cf., e.g., [McL00, Thm. 3.16]), we calculate for $\theta \in (0,1)$,

$$\|u\|_{AS,\theta,\partial\Omega} = \|\textstyle\sum_{i=1}^N \varphi_i u\|_{AS,\theta,\partial\Omega} \le \displaystyle\sum_{i=1}^N \|\varphi_i u\|_{AS,\theta,\partial\Omega} \lesssim \sum_{i=1}^N \|\widehat{\varphi}_i \widehat{u}_i\|_{AS,\theta,\mathbb{R}^{d-1}}$$
$$\simeq \left(\sum_{i=1}^N \|\widehat{\varphi}_i \widehat{u}_i\|^2_{H^\theta(\mathbb{R}^{d-1})}\right)^{1/2} \overset{(B.7)}{=} \|u\|_{H^\theta(\partial\Omega)}.$$

For the converse estimate, we compute, using again the equivalence of the Aronstein-Slobodeckii norm and the Sobolev norm as well as the facts that the $\widehat{\zeta}_i$ are bi-Lipschitz and that the functions $\widehat{\varphi}_i$ are Lipschitz continuous with $\operatorname{supp}\widehat{\varphi}_i \subset \widehat{\omega}_i$,

$$\|u\|^2_{H^\theta(\partial\Omega)} \overset{(B.7)}{=} \sum_{i=1}^N \|\widehat{\varphi}_i \widehat{u}_i\|^2_{H^\theta(\mathbb{R}^{d-1})} \simeq \sum_{i=1}^N \|\widehat{\varphi}_i \widehat{u}_i\|^2_{AS,\theta,\mathbb{R}^{d-1}}$$
$$\lesssim \sum_{i=1}^N \|\widehat{u}_i\|^2_{AS,\theta,\widehat{\omega}_i} \lesssim \sum_{i=1}^N \|u\|^2_{AS,\theta,\omega_i} \le N\|u\|^2_{AS,\theta,\partial\Omega}. \qquad \square$$

Proposition B.9. *Facts* 2.1(iii) *holds with equivalent norms.*

Proof. First, we note that [McL00, Thm. B.11] shows for $\theta \in (0,1)$ that the interpolation spaces $(L^2(\partial\Omega), H^1(\partial\Omega))_\theta$ and the Sobolev spaces $H^\theta(\partial\Omega)$ are equal (with equivalent norms), i.e., $(L^2(\partial\Omega), H^1(\partial\Omega))_\theta = H^\theta(\partial\Omega)$. We next establish $(L^2(\Gamma), \widetilde{H}^1(\Gamma))_{1/2} = \widetilde{H}^{1/2}(\Gamma)$.

Proof of $(L^2(\Gamma), \widetilde{H}^1(\Gamma))_{1/2} \subseteq \widetilde{H}^{1/2}(\Gamma)$*:* The operator $E_{0,\Gamma}$ is a bounded and linear operator $L^2(\Gamma) \to L^2(\partial\Omega)$ and $\widetilde{H}^1(\Gamma) \to H^1(\partial\Omega)$. Interpolation theory provides that $E_{0,\Gamma} : (L^2(\Gamma), \widetilde{H}^1(\Gamma))_\theta \to (L^2(\partial\Omega), H^1(\partial\Omega))_\theta = H^\theta(\partial\Omega)$ is a bounded linear operator. In particular, $\|u\|_{\widetilde{H}^{1/2}(\Gamma)} = \|E_{0,\Gamma} u\|_{H^{1/2}(\partial\Omega)} \lesssim \|u\|_{(L^2(\Gamma),\widetilde{H}^1(\Gamma))_{1/2}}$.

Proof of $\widetilde{H}^{1/2}(\Gamma) \subseteq (L^2(\Gamma), \widetilde{H}^1(\Gamma))_{1/2}$*:* We show that for some $\delta > 0$ we can construct for each $t \in (0,\delta]$ and each $u \in \widetilde{H}^{1/2}(\Gamma)$ a function $u_t \in \widetilde{H}^1(\Gamma)$ such that the functional $k(u,t) := \|u - u_t\|_{L^2(\Gamma)} + t\|u_t\|_{\widetilde{H}^1(\Gamma)}$ satisfies

$$\int_{t=0}^{\delta} \left(t^{-1/2} k(u,t)\right)^2 \frac{dt}{t} \le C\|u\|^2_{\widetilde{H}^{1/2}(\Gamma)}.$$

By definition of the real K-method of interpolation, the latter estimate yields $\|u\|_{(L^2(\Gamma),\widetilde{H}^1(\Gamma))_{1/2}} \lesssim \|u\|_{\widetilde{H}^{1/2}(\Gamma)}$ and hence concludes this step, since, as it is shown in [DL93, Chap. 6, Sec. 7], we may replace the integral over $(0,\infty)$ by an integral over $(0,\delta)$ for fixed $\delta > 0$ in the definition of the interpolation spaces. We start with the case $\Gamma = \partial\Omega$ in order to illustrate the main ideas. Using the partition of unity $(\varphi_i)_{i=1}^N$ we write the function $u \in H^{1/2}(\partial\Omega)$ as $u = \sum_{i=1}^N \varphi_i u$. The pull-backs $\widehat{\varphi}_i \widehat{u}_i$ satisfy $\widehat{\varphi}_i \widehat{u}_i \in H^{1/2}(\mathbb{R}^{d-1})$ and $\operatorname{supp}(\widehat{\varphi}_i \widehat{u}_i) \subset \widehat{\omega}_i$. Select a function ρ as in Lemma B.1. Set $\widehat{u}_{i,1,t} := (\widehat{\varphi}_i \widehat{u}_i) \star \rho_t$ and $\widehat{u}_{i,0,t} := \widehat{\varphi}_i \widehat{u}_i - \widehat{u}_{i,1,t}$. Noting that the function ρ in Lemma B.1 has compact support, we obtain from Lemma B.1 that for sufficiently small δ and all $0 < t \le \delta$, we have the properties $\operatorname{supp}\widehat{u}_{i,1,t} \subset \widehat{\omega}_i$ and $\operatorname{supp}\widehat{u}_{i,0,t} \subset \widehat{\omega}_i$ as well as

$$\int_{t=0}^{\delta} \left(t^{-1/2} k_i(u,t)\right)^2 \frac{dt}{t} \le C\|\widehat{\varphi}_i \widehat{u}_i\|^2_{H^{1/2}(\mathbb{R}^{d-1})},$$

where $k_i(u,t) := \|\widehat{u}_{i,0,t}\|_{L^2(\mathbb{R}^{d-1})} + t\|\widehat{u}_{i,1,t}\|_{H^1(\mathbb{R}^{d-1})}$. (The parameter $\delta > 0$ depends solely on the sets $\widehat{\omega}_i$ and the support properties of the functions $\widehat{\varphi}_i$.) Define $u_{i,0,t}$ and $u_{i,1,t}$ on ω_i as the push-forwards of $\widehat{u}_{i,0,t}$ and $\widehat{u}_{i,1,t}$, respectively. Decompose u as $u = (u - u_t) + u_t =: \sum_{i=1}^N \chi_{\omega_i} u_{i,0,t} + \sum_{i=1}^N \chi_{\omega_i} u_{i,1,t}$. In view of Lemma B.6 we have $u_t \in H^1(\partial\Omega)$. Furthermore,

$$\int_{t=0}^{\delta} \left\{ t^{-1/2} \left(\|u - u_t\|_{L^2(\partial\Omega)} + t\|u_t\|_{H^1(\partial\Omega)} \right) \right\}^2 \frac{dt}{t}$$
$$\lesssim \sum_{i=1}^{N} \|\widehat{\varphi}_i \widehat{u}_i\|^2_{H^{1/2}(\mathbb{R}^{d-1})} \simeq \|u\|^2_{H^{1/2}(\partial\Omega)}.$$

This concludes the proof for $\Gamma = \partial\Omega$. For the case $\Gamma \neq \partial\Omega$, we proceed along the same lines, but use the more careful choice of the smoothing function ρ given in Lemma B.2 for those indices i with $\omega_i \cap \partial\Gamma \neq \emptyset$ so as to ensure $u_t \in \widetilde{H}^1(\Gamma)$. For these indices i, one has to use the fact that Γ stems from a Lipschitz dissection as discussed in [McL00, p. 99]. Lemma B.2 is formulated so as to be applicable in this situation. In particular, the mollifier ρ can be selected such that the push-forwards $u_{i,0,t}$ and $u_{i,1,t}$ satisfy the additional constraints $\operatorname{supp} u_{i,1,t} \subset \Gamma$ and $\operatorname{supp} u_{i,0,t} \subset \overline{\Gamma}$ if $\operatorname{supp} u \subset \overline{\Gamma}$. These support properties are those required to conclude the proof for the case that $\Gamma \subsetneqq \partial\Omega$ stems from a Lipschitz dissection of $\partial\Omega$. □

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