MATHEMATICS OF COMPUTATION
Volume 86, Number 308, November 2017, Pages 2911–2926
http://dx.doi.org/10.1090/mcom/3189
Article electronically published on May 17, 2017

# COMPUTABLE ABSOLUTELY NORMAL NUMBERS AND DISCREPANCIES

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Abstract. We analyze algorithms that output absolutely normal numbers digit-by-digit with respect to quality of convergence to normality of the output, measured by the discrepancy. We consider explicit variants of algorithms by Sierpinski, by Turing and an adaption of constructive work on normal numbers by Schmidt. There seems to be a trade-off between the complexity of the algorithm and the speed of convergence to normality of the output.

## 1. Introduction

A real number is *normal* to an integer base $b \geqslant 2$ if in its expansion to that base all possible finite blocks of digits appear with the same asymptotic frequency. A real number is *absolutely normal* if it is normal to every integer base $b \geqslant 2$. While the construction of numbers normal to one base has been very successful, no construction of an absolutely normal number by concatenation of blocks of digits is known. However, there are a number of algorithms that output an absolutely normal number digit-by-digit. In this work, we analyze some of these algorithms with respect to the speed of convergence to normality.

The *discrepancy* of a sequence $(x_n)_{n\geqslant 1}$ of real numbers is the quantity

$$D_N(x_n) = \sup_{I\subset[0,1)} \left| \frac{\sharp\{1 \leqslant n \leqslant N \mid x_n \bmod 1 \in I\}}{N} - |I| \right|,$$

where the supremum is over all subintervals of the unit interval. A sequence is *uniformly distributed modulo one*, or equidistributed, if its discrepancy tends to zero as $N$ tends to infinity.

The *speed of convergence to normality* of a real number $x$ (to some integer base $b \geqslant 2$) is the discrepancy of the sequence $(b^n x)_{n\geqslant 0}$. A real $x$ is normal to base $b$ if and only if $(b^n x)_{n\geqslant 0}$ is uniformly distributed modulo one [21]. Consequently, $x$ is absolutely normal if and only if the orbits of $x$ under the multiplication by $b$ map are uniformly distributed modulo one for every integer $b \geqslant 2$. It is thus natural to study the discrepancy of these sequences quantitatively as a measure for the speed of convergence to normality.

A result by Schmidt [18] shows that the discrepancy $D_N(x_n)$ of any sequence $(x_n)_{n\geqslant 1}$ of real numbers satisfies $D_N(x_n) \geqslant c\frac{\log N}{N}$ for infinitely many $N$, where $c$ is some positive absolute constant. The study of sequences whose discrepancy satisfies an upper bound of order $O(\frac{\log N}{N})$, so-called low-discrepancy sequences, is a field in its own right. It is an open problem to give a construction of a normal number to some base that attains discrepancy this low. The best result in this

Received by the editor January 19, 2016 and, in revised form, May 15, 2016.
2010 *Mathematics Subject Classification.* Primary 11K16; Secondary 11Y16.

direction is due to Levin [13] who constructed a number normal to one base with discrepancy $O(\frac{\log^2 N}{N})$. It is known [10], that for almost every real number (with respect to Lebesgue measure), for every integer base $b$, the sequence $(b^n x)_{n\geqslant 0}$ has discrepancy $D_N(b^n x) = O(\frac{\sqrt{\log\log N}}{N^{1/2}})$. For more on normal numbers, discrepancies and uniform distribution modulo one see the books [8], [9] and [11].

A construction for absolutely normal numbers was given by Levin [12] where he constructs a real number $\alpha$ normal to countably many specified real bases $\lambda_i > 1$, $i \geqslant 1$, such that the discrepancy of $(\lambda_i^n \alpha)_{n\geqslant 0}$ satisfies $D_N(\lambda_i^n \alpha) = O(\frac{(\log N)^2 \omega(N)}{N^{1/2}})$. The implied constant depends on $\lambda_i$ and $\omega$ is a function that can grow very slowly (it determines the bases to be considered at each step of the construction). Recently, Alvarez and Becher [1] analyzed Levin's work with respect to computability and discrepancy. They show that Levin's construction can yield a computable absolutely normal number $\alpha$ with discrepancy $O(\frac{(\log N)^3}{N^{1/2}})$. To output the first $N$ digits of $\alpha$, Levin's algorithm takes exponentially many (expensive) mathematical operations. Alvarez and Becher also experimented with small modifications of the algorithm.

In this work the following algorithms are investigated.

**Sierpinski.** Borel's original proof [7] that almost all real numbers with respect to Lebesgue measure are absolutely normal is not constructive. Sierpinski [19] gave a constructive proof of this fact. Becher and Figueira [3] gave a recursive reformulation of Sierpinski's construction. The resulting algorithm outputs the digits to some specified base $b$ of an absolutely normal number $\nu$, depending on $b$, in double exponential time. The sequence $(b^n \nu)_{n\geqslant 0}$ has discrepancy $O(\frac{1}{N^{1/6}})$. The calculation does not appear in [3]. We give it in Section A.1.

**Turing.** Alan Turing gave a computable construction to show that almost all real numbers with respect to Lebesgue measure are absolutely normal. His construction remained unpublished and appeared first in his collected works [20]. Becher, Figueira and Picchi [4] completed his manuscript and showed that Turing's algorithm computes the digits of an absolutely normal number $\alpha$ in double exponential time. The discrepancy of the sequence $(b^n \alpha)_{n\geqslant 0}$, for integer bases $b$, is $O(\frac{1}{N^{1/16}})$. This calculation does not appear in [4], we give it in Section A.4.

**Schmidt.** In [17], Schmidt gave an algorithmic proof that there exist uncountably many real numbers normal to all bases in a given set $R$ and not normal to all bases in a set $S$ where $R$ and $S$ are such that elements of $R$ are multiplicatively independent of elements of $S$ and such that $R \cup S = \mathbb{N}_{\geqslant 2}$. In his construction he requires $S$ to be non-empty. However, in a final remark he points out that it should be possible to modify his construction for $S$ empty.

The main purpose of this paper is to carry out the details of Schmidt's remark explicitly to give an algorithmic construction of an absolutely normal number $\xi$. We show that to output the first $N$ digits of $\xi$ to an integer base $b$ it takes exponentially in $N$ many (expensive) mathematical operations. The discrepancy of $(b^n \xi)_{n\geqslant 0}$ is $O(\frac{\log\log N}{\log N})$. A small modification of the algorithm allows for discrepancy $O(\frac{1}{(\log N)^B})$ for any fixed real number $B > 0$, but the output (i.e. $\xi$) depends on $B$. For $B > 1$ this convergence is simultaneously faster than the speed of convergence to normality of most constructions of normal numbers (to a single base) by concatenations of blocks (see for example [9] and [15]).

Schmidt's main tool is cancellation in a certain trigonometric sum related to multiplicatively independent bases (Hilfssatz 5 in [17] and Lemma 2.1 here). Schmidt's lemma does not make explicit the magnitude of the involved constants. In Lemma 3.1 we present the detailed calculation and make these constants explicit. The elucidation of the constants in Schmidt's lemma can be of interest independent to the present work.

Becher, Heiber, Slaman [5] gave an algorithm that computes the digits of an absolutely normal number $X$ to some designated base $b$ in polynomial time. The algorithm depends on a parameter function $f$ that controls the speed of convergence to normality. Becher, Heiber and Slaman optimize in $f$ to achieve a polynomial time algorithm. The resulting discrepancy of $(b^n X)_{n\geqslant 0}$ was not analyzed but has been recently presented in [14].

*Notation.* For a real number $x$, we denote by $\lfloor x \rfloor$ the largest integer not exceeding $x$. The fractional part of $x$ is denoted as $\{x\}$, hence $x = \lfloor x \rfloor + \{x\}$. Two functions $f$ and $g$ are $f = O(g)$ or equivalently $f \ll g$ if there is a $x_0$ and a positive constant $C$ such that $f(x) \leqslant Cg(x)$ for all $x \geqslant x_0$. We mean $\lim_{x\to\infty} f(x)/g(x) = 1$ when we say $f \sim g$ and $g \neq 0$. We abbreviate $e(x) = \exp(2\pi i x)$. Two integers $r$, $s$ are multiplicatively dependent, $r \sim s$, when they are rational powers of each other.

In our terminology, *mathematical operations* include addition, subtraction, multiplication, division, comparison, exponentiation and logarithm. *Elementary operations* take a fixed amount of time to be computed. When we include the evaluation of a complex number of the form $\exp(2\pi i x)$ as a mathematical operation we refer to it as being *expensive*.

## 2. Schmidt's algorithm

In this section we present an algorithm to compute an absolutely normal number. We derived this algorithm from Schmidt's work [17]. Schmidt's construction employs Weyl's criterion for uniform distribution and as such uses exponential sums. The following estimate for trigonometric series is his main tool.

**Lemma 2.1** (Hilfssatz 5 in [17])**.** *Let $r$ and $s$ be integers greater than 1 such that $r \not\sim s$. Let $K, l$ be positive integers such that $l \geqslant s^K$. Then*

$$\sum_{n=0}^{N-1} \prod_{k=K+1}^{\infty} |\cos(\pi r^n l/s^k)| \leqslant 2N^{1-a_{20}} \tag{2.1}$$

*for some positive constant $a_{20}$ only dependent on $r$ and $s$.*

In Section 3 we give an explicit version of Lemma 2.1.

2.1. **The Algorithm.** We begin by stating Schmidt's algorithm. In Schmidt's notation we are specializing to the case $R = \mathbb{N}_{\geqslant 2}$ and $S = \varnothing$. We considered Schmidt's indications on how to modify the construction to produce absolutely normal numbers.

*Setup.* Let $\mathcal{R} = (r_i)_{i\geqslant 1} = \mathbb{N}_{\geqslant 2}$ (in non-decreasing order) and let $\mathcal{S} = (s_j)_{j\geqslant 1}$ be a sequence of integers $s$ greater than 2 such that $s_m \leqslant m s_1$ and such that for each $r \in \mathcal{R}$ there is an index $m_0(r)$ such that $r \not\sim s_m$ for all $m \geqslant m_0(r)$. Let $\beta_{i,j} = a_{20}(r_i, s_j)$ from Lemma 2.1 and denote by $\beta_k = \min_{1\leqslant i,j\leqslant k} \beta_{i,j}$. We can assume that $\beta_k < \frac{1}{2}$. Let $\gamma_k = \max(r_1, \ldots, r_k, s_1, \ldots, s_k)$.

Schmidt assumes that the sequences $\mathcal{R}$ and $\mathcal{S}$ are such that $\beta_k \geqslant \beta_1/k^{1/4}$ and that $\gamma_k \leqslant \gamma_1 k$ holds. This can be achieved by repeating the values of the sequences $\mathcal{R}$ and $\mathcal{S}$ sufficiently many times. Set $\varphi(1) = 1$ and let $\varphi(k)$ be the largest integer $\varphi$ such that the conditions

$$\varphi \leqslant \varphi(k-1) + 1, \quad \beta_\varphi \geqslant \frac{\beta_1}{k^{1/4}} \quad \text{and} \quad \gamma_\varphi \leqslant \gamma_1 k$$

hold. Then modify the sequences $\mathcal{R}$ and $\mathcal{S}$ according to $r_i' = r_{\varphi(i)}$, $s_i' = s_{\varphi(i)}$. Note that (up to suitable repetition) $\mathcal{S}$ can be chosen to be the set of positive integers bigger than 2 that are not perfect powers. In principle, using the explicit version of Hilfssatz 5, Lemma 3.1, one could write down $\mathcal{R}$ and $\mathcal{S}$ explicitly.

Following Schmidt, we introduce the following symbols where $m$ is a positive integer. Let $\langle m \rangle = \lfloor e^{\sqrt{m}} + 2s_1 m^3 \rfloor$, denote $\langle m; x \rangle = \lfloor \langle m \rangle / \log x \rfloor$ for $x > 1$ and let $a_m = \langle m; s_m \rangle$, $b_m = \langle m+1; s_m \rangle$.

**Algorithm.** Step 0: Put $\xi_0 = 0$.

Step $m$: Compute $a_m, b_m, s_m$. We have from the previous step $\xi_{m-1}$. Let $\sigma_m(\xi_{m-1})$ be the set of all numbers

$$\eta_m(\xi_{m-1}) + c_{a_m+1}^{s_m} s_m^{-a_m-1} + \ldots + c_{b_m-2}^{s_m} s_m^{-b_m+2}$$

where the digits $c$ are 0 or 1, and where $\eta_m(\xi_{m-1})$ is the smallest of the numbers $\eta = g s_m^{-a_m}$, $g$ an integer, that satisfy $\xi_{m-1} \leqslant \eta$.

Let $\xi_m$ be the smallest of the numbers in $\sigma_m(\xi_{m-1})$ that minimize

$$A_m'(x) = \sum_{\substack{t=-m \\ t \neq 0}}^{m} \sum_{\substack{i \leqslant m \\ m_0(r_i) \leqslant m}} \left| \sum_{j=\langle m; r_i \rangle + 1}^{\langle m+1; r_i \rangle} e(r_i^j t x) \right|^2. \tag{2.2}$$

The following lemma establishes cancellation in the sums $A_m'$ in order for Weyl's criterion to apply.

**Lemma 2.2.** *There exists a positive absolute constant $\delta_1'$ such that*

$$A_m'(\xi) \leqslant \delta_1' m^2 (\langle m+1 \rangle - \langle m \rangle)^{2-\beta_m}. \tag{2.3}$$

*Proof.* Schmidt's proof of Hilfssatz 7 in [17] can directly be adopted. The inner sum in $A_m'$ over $j$ is essentially the same as in Schmidt's function $A_m$. The outer sums over $r_i$ and $t$ are evaluated trivially and contribute a constant factor times $m^2$. □

*Remark* 2.3. Following the constants in Schmidt's argument shows that $\delta_1' = 36$ is admissible.

Schmidt shows that the sequence $(\xi_m)_{m \geqslant 1}$ has a limit $\xi$ that is normal to all bases in the set $\mathcal{R}$, i.e., absolutely normal. We have the approximations

$$\xi_m \leqslant \xi < \xi_m + s_m^{-b_m+2}. \tag{2.4}$$

2.2. **Complexity.** We given an estimate for the number of (expensive) mathematical operations Schmidt's algorithm takes to compute the first $N$ digits of the absolutely normal number $\xi$ to some given base $r \geqslant 2$.

Note that from inequality (2.4), the representation of $\xi_M$ in base $s_M$ agrees on the first $b_M - 2$ digits with the base $s_M$ representation of $\xi$. These $b_M - 2$ digits of $\xi$ to base $s_M$ determine the first $(b_M - 2)\frac{\log s_M}{\log r}$ digits of $\xi$ to base $r$. Thus we want to find $M$ such that

$$(b_M - 2)\frac{\log s_M}{\log r} \geqslant N. \tag{2.5}$$

We find that $b_M - 2 > \frac{e^{\sqrt{M}}}{\log s_M}$ for $M$ large enough. Thus, for $N$ large enough, any $M$ that satisfies

$$e^{\sqrt{M}} > N^2$$

also satisfies inequality (2.5). Hence, to compute first $N$ digits of $\xi$ to base $r$, $N$ large enough, it is enough to carry out $4(\log N)^2$ many steps of the algorithm.

Naively finding the minimum of $A'_m$ in each step $m \leqslant M$ by calculating all values $A'_m(x)$ for $x$ in the set $\sigma_m(\xi_{m-1})$ costs $O(e^{m^{1/2}})$ computations of a complex number of the form $e(r_i^j tx)$ for each of the $2^{b_m - a_m - 2} = O(2^{e^{m^{1/2}}})$ elements $x$ in $\sigma_m(\xi_{m-1})$. Hence in each step $m$ we need to perform $e^{m^{1/2}} \cdot 2^{e^{m^{1/2}}} = O(N2^N)$ mathematical operations. Carrying out $M = 4(\log N)^2$ many steps, these are in total

$$O(N2^N 4(\log N)^2) = O(e^N)$$

many (expensive) mathematical operations.

2.3. **Discrepancy.** We fix a base $r \geqslant 2$ and $t$ shall denote a non-zero integer. For a large natural number $N$, using Schmidt's Hilfssatz 7, the Erdős-Turán inequality, and via approximating $N$ by a suitable value $\langle M; r\rangle$ we can find an upper bound for the discrepancy $D_N(\{r^n\xi\})$.

**Theorem 2.4.** *The discrepancy of Schmidt's absolutely normal number $\xi$ is*

$$D_N(\{r^n\xi\}) \ll \frac{\log\log N}{\log N} \tag{2.6}$$

*where the implied constant and 'N large enough' depend on the base $r$.*

*Proof.* For a given $N$ large enough, let $M$ be such that $\langle M; r\rangle \leqslant N < \langle M+1; r\rangle$. Such an $M$ satisfies a lower bound of the form $M \gg (\log N)^2$ if $N$ is large enough.

We split the Weyl sum $\sum_{n=1}^{N} e(r^n t\xi)$ according to

$$\sum_{n=1}^{N} e(r^n\xi t) = \sum_{n=1}^{\langle M;r\rangle} e(r^n\xi t) + \sum_{n=\langle M;r\rangle+1}^{N} e(r^n\xi t). \tag{2.7}$$

An estimate for the first sum $\sum_{n=1}^{\langle M;r\rangle} e(r^n\xi t)$ in equation (2.7) can be obtained from equation (2.2) and yields

$$\begin{aligned}\sum_{n=1}^{\langle M;r\rangle} e(r^n t\xi) &= \sum_{m=m_0(r)}^{M-1} \sum_{n=\langle m;r\rangle+1}^{\langle m+1;r\rangle} e(r^n t\xi) + O(1)\\ &\ll \sum_{m=m_0(r)}^{M-1} m(\langle m+1\rangle - \langle m\rangle)^{1-\frac{\beta_m}{2}}\\ &\leqslant M \sum_{m=1}^{M-1} (\langle m+1\rangle - \langle m\rangle)^{1-\frac{\beta_M}{2}}\\ &< M^2 \left(\sum_{m=1}^{M-1} \langle m+1\rangle - \langle m\rangle\right)^{1-\frac{\beta_M}{2}},\end{aligned}$$

which is equal to $M^2\langle M\rangle^{1-\frac{b_M}{2}}$. Using the decay property $\beta_M \geqslant \beta_1 M^{-1/2}$, the first sum in equation (2.7) is thus

$$\ll M^2 e^{M^{1/2}-\frac{\beta_1}{2}M^{1/4}}. \tag{2.8}$$

For the error of approximation of $N$ via $\langle M;r\rangle$ we calculate for fixed $r, s_1$, and $M$ large enough,

$$\langle M+1;r\rangle - \langle M;r\rangle \ll e^{\sqrt{M}}\left(e^{\frac{1}{2\sqrt{M}}} - 1 + \frac{M^2}{e^{\sqrt{M}}}\right) \tag{2.9}$$

where the implied constant depends on $s_1$ and $r$. We used $\sqrt{M+1}-\sqrt{M} = 1/(\sqrt{M+1}+\sqrt{M}) \leqslant 1/2\sqrt{M}$. For $M$ large enough we have $e^{1/2\sqrt{M}} \leqslant 1 + \frac{1}{\sqrt{M}}$, hence the right-hand side of estimate (2.9) is

$$\leqslant e^{\sqrt{M}}\left(\frac{1}{\sqrt{M}} + \frac{M^2}{e^{\sqrt{M}}}\right).$$

Thus,

$$\langle M+1;r\rangle - \langle M;r\rangle = e^{\sqrt{M}} \cdot O\left(\frac{1}{\sqrt{M}}\right) = \langle M;r\rangle \cdot O\left(\frac{1}{\sqrt{M}}\right). \tag{2.10}$$

By the choice of $M$, $\langle M;r\rangle \leqslant N$ and $M \gg (\log N)^2$. Thus equation (2.10) is

$$\ll \frac{N}{\log N},$$

hence the second term in equation (2.7) dominates the first.

The Erdős-Turán inequality applied to the sequence $\{r^n\xi\}_{n\geqslant 0}$ is

$$D_N(\{r^n\xi\}) \ll \frac{1}{H} + \sum_{t=1}^{H} \frac{1}{t}\left|\frac{1}{N}\sum_{n=1}^{N} e(r^n\xi t)\right| \tag{2.11}$$

where $H$ is a natural number. Splitting the exponential sum as before and upon putting $H = \log N$, we thus obtain

$$D_N(\{r^n\xi\}) \ll \frac{\log\log N}{\log N}$$

where the implied constant depends on the base $r$. □

2.4. **Modifying Schmidt's algorithm.** We show that it is possible to modify Schmidt's algorithm for a given real number $B > 0$ to output an absolutely normal number $\xi$, depending on $B$, with discrepancy $D_N(\{r^n\xi\}) = O_r(\frac{\log\log N}{(\log N)^B})$ to base $r$, where the implied constant depends on $r$, thus exponentially lowering the discrepancy associated to Schmidt's algorithm by an exponent of $B$.

**Proposition 2.5.** *Fix* $0 < c < 1$. *Schmidt's algorithm still holds when the function* $\langle m\rangle$ *is replaced by the function*

$$\langle m\rangle = \lfloor e^{m^c}\rfloor.$$

Note that the functions $\langle m; r\rangle$, $a_m$ and $b_m$ and also the construction of the sets $\sigma_m$ have to be modified accordingly. The algorithm works in exactly the same way, but the output depends on $c$.

*Proof.* We need to show that the estimate (2.3) for $A'_m$ is still valid with this choice of $\langle m\rangle$. In the course of the proof of this estimate, Schmidt evaluates the inner sum over $j$ in $A'_m$ trivially on a range of size $O(m)$. This range constitutes only a minor part of the full sum over $j$ since $m \leqslant \delta(\langle m+1\rangle - \langle m\rangle)^{1-\varepsilon}$ for some $\delta > 0$ and some $0 < \varepsilon < 1$. This can be seen from

$$\begin{aligned} e^{(m+1)^c} - e^{m^c} &= e^{m^c}\left(e^{(m+1)^c - m^c} - 1\right) \\ &\geqslant e^{m^c}\left((m+1)^c - m^c\right) \\ &\gg cm^{c\alpha}m^{c-1}, \end{aligned}$$

since $e^{m^c} \gg m^{c\alpha}$ for any $\alpha > 0$. Choosing $\alpha = \frac{2}{c} - 1 + \eta$ for some $\eta > 0$ gives

$$\langle m+1\rangle - \langle m\rangle \gg cm^{c\alpha}m^{c-1} = cm^{1+\eta},$$

which establishes our claim. □

The discrepancy of $\xi = \xi_c$ can be estimated the same way as before. Note that any $N$ large enough can now be approximated by the function $\langle M\rangle$ with error

$$O(\langle M+1\rangle - \langle M\rangle) \ll e^{M^c}\frac{1}{M^{1-c}},$$

which with $M$ of order $(\log N)^{1/c}$ is

$$\frac{N}{(\log N)^{\frac{1-c}{c}}}.$$

Hence the discrepancy of the sequence $\{r^n\xi\}_{n\geqslant 0}$ satisfies

$$D_N(\{r^n\xi\}) \ll \frac{\log\log N}{(\log N)^B} \tag{2.12}$$

with $0 < B = \frac{1-c}{c} < \infty$.

## 3. The constants $a_{20}$ in Schmidt's Hilfssatz 5

In this section we prove the following explicit variant of Schmidt's Hilfssatz 5 in [17].

**Lemma 3.1** (Explicit variant of Lemma 2.1)**.** *Let $r$ and $s$ be integers greater than 1 such that $r \not\sim s$. Let $K, l$ be positive integers such that $l \geqslant s^K$ and denote $m = \max(r, s)$. Then for*

$$N \geqslant N_0(r,s) = \exp(288 \cdot (12m(\log m)^4 + 8(\log m)^3 + (\log m)^2)) \tag{3.1}$$

*we have*

$$\sum_{n=0}^{N-1} \prod_{k=K+1}^{\infty} |\cos(\pi r^n l/s^k)| \leqslant 2N^{1-a_{20}} \tag{3.2}$$

*for some positive constant $a_{20}$ as specified in equation* (3.25) *that satisfies*

$$a_{20} = \frac{1}{6}\frac{\pi^2}{2} \cdot 0.007 \cdot \frac{1}{s^4}\frac{1}{\log s}\left(\frac{1}{\log s} - \frac{1}{s}\right). \tag{3.3}$$

*Remark* 3.2. The statement of Lemma 3.1 holds true *for all* $N$ with

$$a_{20} = \min\left(\frac{1}{N_0 \log N_0}, \frac{-\log\cos(\frac{\pi}{s^2})}{2\log N_0}\right) \tag{3.4}$$

as specified in equation (3.27) where $N_0 = N_0(r,s)$ as in equation (3.1).

This enables us in principle to give an explicit description of the sequences $(r_i)_{i\geqslant 1}$ and $(s_j)_{j\geqslant 1}$ *after* the repetition of the entries via the function $\varphi$ as suggested by Schmidt. Lemma 3.1 might also be of independent interest as its non-explicit variant has been used by several authors; see e.g. [2] and [6]. We do not claim optimality of the bounds in Lemma 3.1.

*Proof.* The proof is basically a careful line-by-line checking of Schmidt's proof of Lemma 2.1. The reader might find it helpful having a copy of both [16] and [17] at hand.

We follow Schmidt's notation and his argument in [17]. Let $h$ be the number of distinct prime divisors of $rs$ and let

$$r = p_1^{d_1} \cdot \ldots \cdot p_h^{d_h},$$
$$s = p_1^{e_1} \cdot \ldots \cdot p_h^{e_h}$$

be the prime factorizations of $r$ and $s$ with $d_i$ and $e_i$ not both equal to zero. We assume the $p_i$ to be ordered such that

$$\frac{d_1}{e_1} \geqslant \ldots \geqslant \frac{d_h}{e_h},$$

with the convention that $\frac{d}{0} = +\infty$. This implies that $d_k e_l - d_l e_k \geqslant 0$ for all $k \geqslant l$.

Let $b = \max_i(d_i) \cdot \max_i(e_i)$. Schmidt denotes by $l_i$ numbers not divisible by $p_i^{2b}$.

For $1 \leqslant i \leqslant h$, let

$$u_i = (p_1^{d_1} \ldots p_i^{d_i})^{e_i}(p_1^{e_i} \ldots p_i^{e_i})^{-d_i},$$
$$v_i = (p_{i+1}^{e_{i+1}} \ldots p_h^{e_h})^{d_i}(p_{i+1}^{d_{i+1}} \ldots p_h^{d_h})^{-e_i},$$

where empty products (for $i = h$) are 1. These numbers are integers, and $t_i = \frac{u_i}{v_i}$ is not equal to 1 since $r \not\sim s$. We have $t_i = \frac{r^{e_i}}{s^{d_i}}$, hence, when writing $t_i$ in lowest terms, the prime $p_i$ has been cancelled.

Let $f_i = p_i - 1$ if $p_i$ is odd, and $f_i = 2$ otherwise. There are well-defined integers $g_i$ such that

$$t_i^{f_i} \equiv 1 + q_i p_i^{g_i - 1} \pmod{p_i^{g_i}}$$

with $p_i \nmid q_i$ (especially $q_i \neq 0$). We have $g_i > 1$ by the small Fermat theorem and for $p_i = 2$ we even have $g_i > 2$ since squares are congruent 1 modulo 4. To give an upper bound for $g_i$, note that $p_i^{g_i}$ can be at most equal to $t_i^{f_i}$. Hence $g_i \leqslant \lfloor f_i \frac{\log t_i}{\log p_i} \rfloor + 1$. Since naively $\log p_i \geqslant \log 2$, $\log t_i = e_i \log r - d_i \log s \leqslant e_i \log r \leqslant \frac{\log r \log s}{\log 2}$ and $p_i \leqslant \max(r,s)$, a trivial upper bound on $g_i$, valid for all $i$, is

$$g_i \leqslant 12 \max(r,s) \log r \log s.$$

Let $a_1 = \max(g_1, \ldots, g_h)$. Then

$$2 \leqslant a_1 \leqslant 12 \max(r,s) \log r \log s. \tag{3.5}$$

Assume $k \geqslant a_1$, $e_i > 0$. The constant $a_2$ is such that at most $a_2(s/2)^k$ of the numbers $l_i t_i^n$ fall in the same residue class modulo $s^k$ if $n$ runs through a set of representatives modulo $s^k$ (Hilfssatz 1 in [17]). At most $p_i^{2b} p_i^{g_i}$ of the numbers $t_i^n l_i$ fall in the same residue class modulo $p_i^k$ if $n$ runs through a set of representatives modulo $p_i^k$. If $p_i | s$, then there are at most $(s/2)^k$ elements in a set of representatives modulo $s^k$ that are congruent to each other. Hence $a_2 = \max_{i, e_i > 0} p_i^{2b+g_i}$. Naive upper and lower bounds on $a_2$ are thus

$$8 \leqslant a_2 \leqslant \max(r,s)^{8 \log(\max(r,s)) + 12 \max(r,s) \log r \log s}. \tag{3.6}$$

The constant $a_4$ (named $\alpha_3$ in [16]) is chosen such that

$$(s^2 - 2)^{a_4} < 2^{1/4 + 2a_4} a_4^{a_4} (1 - 2a_4)^{1/2 - a_4}. \tag{3.7}$$

The right-hand side of inequality (3.7) as a function of $a_4$ (denote it by $f(a_4)$) can be numerically analyzed. It is a strictly decreasing continuous function on the interval $(0, 1/16]$ with values $f(0^+) = \sqrt[4]{2} \approx 1.19 > f(1/16) \approx 1.028 > 1$. Hence any $a_4$ in $(0, 1/16)$ that satisfies

$$(s^2 - 2)^{a_4} \leqslant f(1/16) \tag{3.8}$$

also satisfies inequality (3.7). Note that $a_4 < 1/16$ is no proper restriction as $a_4(2) \approx 0.055 < 1/16$ and since $a_4$ is decreasing in $s$. Now, inequality (3.8) is easy to solve and gives

$$a_4(s) \geqslant \frac{c}{\log(s^2 - 2)}$$

for $c = \log(f(1/16)) \approx 0.028$. This constitutes a non-trivial (i.e., positive) lower bound on the values of $a_4$ that are admissible. To simplify matters we continue with this value for $a_4$, i.e., we put

$$a_4 = \frac{0.028}{\log(s^2 - 2)}. \tag{3.9}$$

The constant $a_3$ also comes from the earlier Schmidt paper [16] and was called $\alpha_4$ there. Schmidt counts the number of blocks of digits in base $s$ with few 'nice' digit pairs. These are successive digits not both equal to zero or $s - 1$. He derives the proof of Lemma 3 in [16] that the number of combinations of $k$ base $s$ digits with less than $\alpha_3 k (= a_4 k)$ nice digit pairs, counting only non-overlapping pairs of digits, does not exceed

$$k \binom{\frac{k}{2}}{\lfloor a_4 k \rfloor} (s^2 - 2)^{\lfloor a_4 k \rfloor} 2^{k/2 - \lfloor a_4 k \rfloor}. \tag{3.10}$$

With the approximation

$$\sqrt{2\pi} n^{n+1/2} e^{-n} \leqslant n! \leqslant e n^{n+1/2} e^{-n}$$

we find that the quantity (3.10) is

$$\leqslant k\frac{e}{2\pi}\frac{\left(\frac{k}{2}\right)^{k/2+1/2}}{(a_4k)^{a_4k+1/2}\left(\frac{k}{2}-a_4k\right)^{k/2-a_4k+1/2}}(s^2-2)^{a_4k}2^{k/2-a_4k}2^{-1}$$

$$= \frac{e}{2\pi}\frac{1}{\sqrt{a_4}\sqrt{1-2a_4}}\frac{1}{2}\cdot\frac{1}{(2a_4)^{a_4k}(1-2a_4)^{(1/2-a_4)k}}. \tag{3.11}$$

In [16], Schmidt denotes the constant factor by $\alpha_5$,

$$\alpha_5 = \frac{e}{4\pi\sqrt{a_4(1-2a_4)}}.$$

Using $a_4 \leqslant 0.055$ and $a_4 \geqslant \frac{c}{\log(s^2-2)}$ with $c \approx 0.028$ we obtain the upper bound

$$\alpha_5 \leqslant 1.87\sqrt{\log s} \approx 2\sqrt{\log s}. \tag{3.12}$$

Finally, $a_3$ is such that if $k \geqslant a_3$, and respecting the choice of $a_4$, then

$$\alpha_5 k\frac{(s^2-2)^{a_4k}2^{(1/2-a_4)k}}{(2a_4)^{a_4k}(1-2a_4)^{(1/2-a_4)k}} < 2^{3/4\cdot k}$$

holds. The left-hand side is equal to

$$\alpha_5 k\left(\frac{(s^2-2)^{a_4}}{f(a_4)}2^{3/4}\right)^k = \alpha_5 k\left(\frac{f(1/16)}{f(a_4)}2^{3/4}\right)^k \leqslant \alpha_5 k 2^{0.74k}$$

by the choice of $a_4$ and since $f(a_4) \geqslant f(0.055) > f(1/16)$. Using $\log(x) \leqslant x^{1/2}$ for all $x \geqslant 0$,

$$\alpha_5 k 2^{0.74k} < 2^{3/4\cdot k}$$

is satisfied for all $k$ larger than

$$a_3 = 120\sqrt{\log(s)}. \tag{3.13}$$

Let $N \geqslant \max(s^{a_1}, s^{a_3+1})$ (hence certainly $N \geqslant s^2$ since $a_1 \geqslant 2$) and let $k$ be such that $s^k \leqslant N < s^{k+1}$. The constants $a_7$ and $a_5$ in Hilfssatz 2 in [17] are such that

$$a_2(s/2)^k s 2^{3k/4} < a_7 N^{1-a_5}.$$

With $k > \frac{\log N}{\log s} - 1$ the left-hand side is

$$< a_2 s N^{1-\frac{\log 2}{4}\left(\frac{1}{\log s}-\frac{1}{\log N}\right)}$$

which is

$$\leqslant a_2 s N^{1-\frac{\log 2}{8\log s}} \tag{3.14}$$

due to $N \geqslant s^2$. Hence $a_7 = a_2 s$. We want quantity (3.14) to be $< N^{1-\frac{\log 2}{16\log s}}$, hence

$$a_5 = \frac{\log 2}{16\log s} > 0. \tag{3.15}$$

This happens, if

$$\frac{\log(a_2 s)}{\log N} < \frac{\log 2}{16\log s},$$

which is satisfied when $N \geqslant N_0^{\mathrm{HS2}}$, where

$$\log N_0^{\mathrm{HS2}} = 288m(\log m)^4 + 192(\log m)^3 + 24(\log m)^2 \tag{3.16}$$

where we denoted $m = \max(r, s)$. Note that in particular $N$ is much larger than $e^s$.

The constant $a_6$ is such that $a_4 k > a_6 \log N$. With $a_4 \geqslant \frac{0.028}{\log(s^2-2)} > \frac{0.014}{\log s}$ and $k > \frac{\log N}{\log s} - 1$ and due to $N \geqslant e^s$, we have

$$a_4 k > \log N \frac{0.014}{\log s}\left(\frac{1}{\log s} - \frac{1}{s}\right).$$

This is a positive value for all $s$. Hence

$$a_6 = \frac{0.014}{\log s}\left(\frac{1}{\log s} - \frac{1}{s}\right) > 0 \tag{3.17}$$

is an admissible choice for $a_6$.

Recall that $h$ was defined as the number of distinct prime divisors in $rs$ so that $r = p_1^{d_1} \cdot \ldots \cdot p_h^{d_h}$, $s = p_1^{e_1} \cdot \ldots \cdot p_h^{e_h}$ are the prime factorizations of $r$ and $s$ with $d_i$ and $e_i$ not both equal to zero. Recall that $b = \max_i(d_i) \cdot \max_i(e_i)$.

In Hilfssatz 3 in [17], Schmidt divides the set numbers $lr^n$ in at most $hb$ subsets each of which contains a certain number of consecutive $lr^n$. If the number of elements in such a subset is $\leqslant N^{1/2}$, he counts trivially. If the number of elements in such a subset is larger than $N^{1/2}$, he uses Hilfssatz 2 with this $N$. Hence the $N$ in Hilfssatz 3 needs to be large enough such that $N^{1/2}$ is large enough for Hilfssatz 2. Thus, for

$$N \geqslant N_0^{\mathrm{HS3}} = (N_0^{\mathrm{HS2}})^2 = \exp(2\cdot(288m(\log m)^4 + 192(\log m)^3 + 24(\log m)^2)) \tag{3.18}$$

there are at most $hbN^{1-a_5}$ numbers of the $lr^n$ having less than $a_6 \log\sqrt{N}$ nice digit pairs. Hence

$$a_9 = \frac{a_6}{2} = \frac{0.007}{\log s}\left(\frac{1}{\log s} - \frac{1}{s}\right). \tag{3.19}$$

We have the trivial bound is $h \leqslant \log_2(rs) \leqslant \frac{2\log m}{\log 2}$ with $m = \max(r,s)$. Another trivial bound is $b = \max_i(d_i) \cdot \max_i(e_i) \leqslant (\log_2(m))^2$. Thus with $N \geqslant e^{288m}$, we have

$$hbN^{1-a_5} \leqslant 7(\log m)^3 N^{1-a_5} \leqslant N^{1-a_8}$$

with

$$a_8 = a_5 - \frac{\log 7 + 3\log\log m}{288m} = \frac{\log 2}{16\log s} - \frac{\log 7 + 3\log\log m}{288m}. \tag{3.20}$$

Recall from Schmidt's paper that $z_K(x)$ denotes the number of nice digit pairs $c_{i+1}c_i$ of $x$ with $i \geqslant K$ where the $c$ are the digits of $x$ in base $s$.

In Hilfssatz 4, Schmidt begins with the restriction $n \geqslant N^{2/3}\log s/\log r$ which reduces $a_{14}$ to a value less than $1/3$. The remaining numbers $lr^n$ are divided in at most $2N^{2/3}$ many intervals of length $\lfloor N^{1/3}\rfloor$ which are analyzed separately. The restriction $n \geqslant N^{2/3}\log s/\log r$ implies $lr^n \geqslant s^{K+\lfloor N^{1/3}\rfloor^2}$.

Denote by $n_0$ a number $N^{2/3}\log s/\log r \leqslant n_0 < N$. Schmidt wants to apply Hilfssatz 3 to intervals $N^{2/3}\log s/\log r \leqslant n_0 \leqslant n < n_0 + \lfloor N^{1/3}\rfloor$ of length $\lfloor N^{1/3}\rfloor$. However, he makes one further preliminary reduction in showing that one can assume that $z_K(l)$ is less than $\frac{a_9}{2}\log N$.

Denote by $n_1$ the least $n \geqslant N^{2/3}\log s/\log r$ such that $z_K(lr^n) < \frac{a_9}{2}\log N$. Replace $lr^n$ for $n \geqslant n_1$ by $l^* r^{n-n_1}$ where $l^* = lr^{n_1}$. All $lr^n$ with $N^{2/3}\log s/\log r \leqslant n < n_1$ are by the choice of $n_1$ such that $z_K(lr^n) \geqslant \frac{a_9}{2}\log N$. As Schmidt's version is not explicit, he can assume $N$ to be large enough, and apply Hilfssatz 3 to the interval $n_1 \leqslant n < N$ (or $0 \leqslant n < N - n_1$ for numbers $l^* r^n$).

To make things explicit, we distinguish three cases for the size of $n_1$. We write $M = \lfloor N^{1/3} \rfloor$ for the number of $lr^n$ under consideration. We want to find explicit lower bounds on $M$ such that we can apply Hilfssatz 3.

Case 0: $n_1$ does not exist at all. Then the number of $lr^n$ with $z_K$ less than $a_9 \log M$ is trivially less than $M^{1-a}$ for any $0 < a < 1$.

Case 1: $n_1$ is large such that the number of $lr^n$ with $z_K(lr^n) < a_9 \log M$ can be trivially estimated by $M - n_1 \leqslant M^{1-a_8}$. This is the case when $n_1 \geqslant M - M^{1-a_8}$.

Case 2: $n_1 < M - M^{1-a_8}$. We need the interval $M - n_1$ to be large enough to be able to apply Hilfssatz 3 to obtain cancellation, i.e., $M - n_1 \geqslant N_0^{\mathrm{HS3}}$ which holds if $M \geqslant M_0 = (N_0^{\mathrm{HS3}})^{\frac{1}{1-a_8}}$. Thus by Hilfssatz 3 the number of $lr^n$, $n_0 \leqslant n < n_0 + M$, with $z_K(lr^n) < a_9 \log N$ is at most $(M - n_1)^{1-a_8} \leqslant M^{1-a_8}$.

Schmidt uses a reduction to count only $z_K$ instead of all nice digit pairs. This reduction looses at one point 2 digit pairs, i.e. after an application of Hilfssatz 3 one finds numbers with at most $a_9 \log M - 2$ nice digit pairs. This is $\leqslant \frac{a_9}{2} \log M$ for

$$\log M > \frac{4}{a_9}. \tag{3.21}$$

Also, Schmidt's reduction works if

$$M \frac{\log r}{\log s} < \left\lfloor \frac{M^2 - 1}{\frac{a_9}{2} \log M} \right\rfloor - 1. \tag{3.22}$$

Note that inequalities (3.21) and (3.22) do not pose further restrictions on $M$.

Finally, from $M \geqslant (N_0^{\mathrm{HS3}})^{\frac{1}{1-a_8}}$, $M = \lfloor N^{1/3} \rfloor$, and since we may assume that $a_8 < \frac{1}{2}$, the requirement

$$N \geqslant N_0^{\mathrm{HS4}} = (N_0^{\mathrm{HS3}})^6 = \exp(288 \cdot (12m(\log m)^4 + 8(\log m)^3 + (\log m)^2)) \tag{3.23}$$

for the original $N$ follows. We established that in each subsequence of length $\lfloor N^{1/3} \rfloor$ there are at most $\lfloor N^{1/3} \rfloor^{1-a_8}$ elements $lr^n$ with $z_K(lr^n) < \frac{a_9}{2} \log \lfloor N^{1/3} \rfloor$.

In total, since there are at most $2N^{2/3}$ many intervals for $n$ of length $\lfloor N^{1/3} \rfloor$, we obtain (for $\log N \geqslant \frac{6 \log 3}{a_8}$) that there are at most

$$N^{2/3} \frac{\log s}{\log r} + 2N^{2/3} \cdot \lfloor N^{1/3} \rfloor^{1-a_8} \leqslant 3N^{1-a_8/3} \leqslant N^{1-a_8/6}$$

elements $lr^n$, $0 \leqslant n < N$, with $z_K(lr^n) < \frac{a_9}{2} \log \lfloor N \rfloor^{1/3} \leqslant \frac{a_9}{6} \log N$. Thus

$$a_{14} = \frac{a_8}{6}, \quad a_{15} = \frac{a_9}{6}. \tag{3.24}$$

From Hilfssatz 5 it follows that $a_{20} = \min(a_{14}, a_{22})$ where $a_{22} = -a_{15} \log a_{21}$ with $a_{21} = \cos(\pi/s^2)$. We have $-\log a_{21} = -\log \cos(\frac{\pi}{s^2}) \geqslant \frac{\pi^2}{2} \frac{1}{s^4}$. Plugging in the values of $a_{14}$ and $a_{15}$ shows that $\min(a_{14}, a_{22}) = a_{22}$. Hence

$$a_{20} = \frac{1}{6} \frac{\pi^2}{2} \cdot 0.007 \cdot \frac{1}{s^4} \frac{1}{\log s} \left( \frac{1}{\log s} - \frac{1}{s} \right) \tag{3.25}$$

where the constant factor is approximately 0.0057.

To find $a_{20}$ such that Lemma 3.1 holds *for all* $N$, we need to replace $a_{14}$ and $a_{15}$ by sufficiently small constants such that Hilfssatz 4 holds for all $N$. This can be achieved by redefining

$$a_{14} = \min(a_{14}, 1 - \frac{\log(N_0 - 1)}{\log N_0}), \quad \text{and} \quad a_{15} = \min(a_{15}, \frac{1}{2 \log N_0})$$

where we denoted $N_0 = N_0^{\mathrm{HS4}}$. We have $a_{14}^{\mathrm{old}} \approx 0.007\frac{1}{\log s}$ and $1 - \frac{\log(N_0-1)}{\log N_0} \leqslant \frac{2}{N_0 \log N_0}$ which decays worse than exponentially in $m$. Furthermore, $a_{15}^{\mathrm{old}} \approx 0.001\frac{1}{\log s}$ and $\frac{1}{2\log N_0}$ is worse than linear with $m$ a large constant. Note also $1 - \frac{\log(N_0-1)}{\log N_0} \geqslant \frac{1}{N_0 \log N_0}$. Hence Hilfssatz 4 holds true for all $N$ with constants

$$a_{14} = \frac{1}{N_0 \log N_0} \quad \text{and} \quad a_{15} = \frac{1}{2\log N_0} \tag{3.26}$$

with $N_0 = N_0^{\mathrm{HS4}}$ as in equation (3.23).

The constant $a_{20}$ then modifies according to

$$a_{20} = \min(a_{14}, a_{22}) = \min\left(\frac{1}{N_0 \log N_0}, \frac{-\log\cos(\frac{\pi}{s^2})}{2\log N_0}\right). \tag{3.27}$$

For large $m$, $a_{20}$ equals $a_{14}$ but since $a_{22} \geqslant \frac{\pi^2}{4}\frac{1}{s^4 \log N_0}$, for small $m$ we have $a_{20} = a_{22}$. Explicitly, with $a_{14} \leqslant \frac{1}{e^m 1728m}$ we have

$$a_{20} = \frac{1}{N_0 \log N_0} \tag{3.28}$$

for $m \geqslant 7$ where we denoted $m = \max(r, s)$ and $N_0 = N_0^{\mathrm{HS4}}$ as given in equation (3.23). □

## Appendix A. Algorithms by Sierpinski and Turing

A.1. **Sierpinski's algorithm.** In this section we estimate the runtime and discrepancy of the effective version of Sierpinsk's algorithm [19] by Becher and Figueira [3]. This algorithm outputs the digits to some specified base $b$ of an absolutely normal number $\nu$, depending on $b$, in double exponential time such that the sequence $(b^n\nu)_{n\geqslant 0}$ has discrepancy $O(\frac{1}{N^{1/6}})$.

Let $0 < \varepsilon \leqslant \frac{1}{2}$ be a rational (or computable real) number that remains fixed throughout the algorithm. We also choose in advance a base $b \geqslant 2$. The algorithm computes the digits to base $b$ of an absolutely normal number $\nu$. The output (i.e., $\nu$) depends on the choice of $\varepsilon$ and $b$.

**Notation.** Let $m$, $q$, $p$ be integers such that $m \geqslant 1$, $q \geqslant 2$ and $0 \leqslant p \leqslant q-1$ and put $n_{m,q} = \lfloor \frac{24m^6q^2}{\varepsilon} \rfloor + 2$.

Let $\Delta_{q,m,n,p}$ be the interval $(\frac{0.b_1\ldots b_{n-1}(b_n-1)}{q^n}, \frac{0.b_1\ldots b_{n-1}(b_n+2)}{q^n})$ where the string $b_1\ldots b_n$ is such that the digit $p$ appears too often, i.e., $|\frac{N_p(b_1\ldots b_n)}{n} - \frac{1}{q}| \geqslant \frac{1}{m}$ where $N_p(b_1\ldots b_n)$ denotes the number of occurrences of the digit $p$ amongst the $b_i$.

Let

$$\Delta = \bigcup_{q=2}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{n=n_{m,q}}^{\infty}\bigcup_{p=0}^{q-1}\Delta_{q,m,n,p},$$

and denote a truncated version of $\Delta$ by

$$\Delta_k = \bigcup_{q=2}^{k+1}\bigcup_{m=1}^{k}\bigcup_{n=n_{m,q}}^{k\cdot n_{m,q}}\bigcup_{p=0}^{q-1}\Delta_{q,m,n,p}.$$

The complement of $\Delta$ in $[0,1)$ is

$$E = [0,1) \smallsetminus \Delta.$$

Sierpinski's algorithm computes the digits to base $b$ of a number $\nu \in E$. This number is absolutely normal as shown by Sierpinski and in Theorem 7 in [3].

The truncated sets $\Delta_k$ approximate $\Delta$ in the sense that if a number is not in $\Delta_k$ for large enough $k$, then it is also not in $\Delta$. Becher and Figueira's algorithm computes the digits of $\nu$ such that the $n$-th digit ensures that $\nu$ is not in some $\Delta_{p_n}$, where $p_n \to \infty$.

**The algorithm.** First digit: Split the unit interval in subintervals $c_d^1 = [\frac{d}{b}, \frac{d+1}{b})$ for $0 \leqslant d < b$. Put $p_1 = 5 \cdot (b-1)$. Compute the Lebesgue measure of $\Delta_{p_1} \cap c_d^1$ for all $d$. The first digit $b_1$ of $\nu$ is chosen such that it is (the smallest among) the $d$ such that the Lebesgue measure of $\Delta_{p_1} \cap c_{b_1}^1$ is minimal among the $\Delta_{p_1} \cap c_d^1$.

$n$-th digit: Split the interval $[0.b_1 \dots b_{n-1}, 0.b_1 \dots (b_{n-1}+1))$ in subintervals

$$c_d^n = [0.b_1 \dots b_{n-1} d, 0.b_1 \dots b_{n-1}(d+1))$$

for all $0 \leqslant d < b$. Put $p_n = 5 \cdot (b-1) \cdot 2^{2n-2}$. The $n$-th digit $b_n$ of $\nu$ is the (smallest of the) $d$ that minimize the Lebesgue measure of the $\Delta_{p_n} \cap c_d^n$.

A.2. **Runtime.** For fixed $q$, $m$, $n$ and $p$, writing down all strings $b_1 \dots b_n$ of length $n$ of digits $0 \leqslant b_i < b$ that satisfy the conditions of $\Delta_{q,m,n,p}$ takes exponential time in $n$. Naively estimating gives the complexity of computing $\Delta_k$ as being exponential in $k$. So, since $p_n$ grows exponentially in $n$, the computation of $\Delta_{p_n}$ takes doubly exponentially many elementary operations.

A.3. **Discrepancy.** We give an estimate for the discrepancy of $(q^n \nu)_{n \geqslant 1}$, valid for any $\nu \in E$ and any base $q \geqslant 2$, not taking into account that the algorithm might in fact construct an element with better distributional properties.

The family of intervals $\bigcup_{p=0}^{q-1} \Delta_{b,m,n,p}$ contains all real numbers with expansion to base $b$ not simply normal regarding the first $n$ digits. The union

$$\Delta_{q,m} = \bigcup_{n=n_{m,q}}^{\infty} \bigcup_{p=0}^{q-1} \Delta_{q,m,n,p}$$

contains all real numbers whose base-$q$ expansion is not simply normal regarding any large enough number of digits. Hence any $\nu$ not in $\Delta_{q,m}$ satisfies

$$\left| \frac{\sharp\{n \leqslant N \mid \{q^n \nu\} \in I\}}{N} - |I| \right| < \frac{1}{m}$$

for all $N \geqslant n_{m,q}$ and $I$ of the form $I = [\frac{p}{q}, \frac{p+1}{q})$ , $p = 0, \dots, q-1$.

Inverting the relation between $N$ and $m$ and using Sierpinski's choice for $n_{m,q} = \lfloor \frac{24 m^6 q^2}{\varepsilon} \rfloor + 2 \approx \frac{24 m^6 q^2}{\varepsilon}$, we find that

$$\sup_{p=0,\dots q-1} \left| \frac{\sharp\{n \leqslant N \mid \{q^n \nu\} \in [\frac{p}{q}, \frac{p+1}{q})\}}{N} - |I| \right|$$
$$\leqslant \left(\frac{24}{\varepsilon}\right)^{1/6} \frac{q^{1/3}}{N^{1/6}} + O\left(\frac{1}{N^{1/3}}\right) \ll_\varepsilon q^{1/3} \frac{1}{N^{1/6}}$$

where the implied constant depends on $\varepsilon$ but not on $q$.

Fix $I \subset [0,1)$, $\delta > 0$ and $k$ such that $\frac{2}{q^k} < \delta$. Choose $l, m$ such that $I \subset [\frac{l}{q^k}, \frac{m}{q^k})$ and $|I| < \frac{m-l}{q^k} + \frac{2}{q^k}$. Then

$$\begin{aligned}\frac{\sharp\{n \leqslant N \mid \{q^n\nu\} \in I\}}{N} &\leqslant \frac{\sharp\{n \leqslant N \mid \{q^n\nu\} \in [\frac{l}{q^k}, \frac{m}{q^k})\}}{N} \\ &\ll \frac{m-n}{q^k} + O\left((q^k)^{1/3}\frac{1}{N^{1/6}}\right) \\ &< |I| + \delta + O\left((q^k)^{1/3}\frac{1}{N^{1/6}}\right) \\ &= |I| + \delta + O_\delta\left(\frac{1}{N^{1/6}}\right).\end{aligned}$$

Since $\delta$ and $I$ are arbitrary, this shows that

$$D_N(\{q^n\nu\}) \ll \frac{1}{N^{1/6}}$$

for any $\nu \in E$ and any base $q$.

A.4. **Turing's algorithm.** Since Turing's algorithm has been very well studied in [4], we restrict ourselves to presenting their result in our terminology. Becher, Figueira and Picchi [4] show that Turing's algorithm computes the digits of an absolutely normal number $\alpha$ in double exponential time. With respect to the speed of convergence to normality Becher, Figueira and Picchi note (Remark 23 in [4]) that for each initial segment of $\alpha$ of length $N = k2^{2n+1}$ expressed to each base up to $e^L$ all words of length up to $L = \sqrt{\log N}/4$ occur with the expected frequency plus or minus $e^{-L^2}$. Here, $k$ is a positive integer parameter, and $n$ is the step of the algorithm.

The discrepancy of $\{b^n\alpha\}$ for some base $b \geqslant 2$ can then be calculated as follows. Fix some arbitrary $\varepsilon > 0$ and a subinterval $I \subset [0,1)$. Let $n$ be large enough, such that $\frac{2}{b^L} < \varepsilon$. Approximate $I$ by a $b^L$-adic interval $[\frac{c}{b^L}, \frac{d}{b^L})$ such that $[\frac{c-1}{b^L}, \frac{d+1}{b^L}) \supset I \supset [\frac{c}{b^L}, \frac{d}{b^L})$. Then

$$\begin{aligned}\frac{\sharp\{0 \leqslant m < N \mid \{b^m\alpha\} \in I\}}{N} &< \frac{\sharp\{0 \leqslant m < N \mid \{b^m\alpha\} \in [\frac{c}{b^L}, \frac{d}{b^L})\}}{N} \\ &\leqslant \frac{d-c+2}{b^L} + O(e^{-L^2})) \\ &\leqslant |I| + \varepsilon + O\left(\frac{1}{N^{1/16}}\right).\end{aligned}$$

Since $I$ and $\varepsilon$ are arbitrary this means that $\{b^n\alpha\}$ is uniformly distributed modulo one with discrepancy bounded by $O(\frac{1}{N^{1/16}})$.

## Acknowledgements

This research was supported by the Austrian Science Fund (FWF): I 1751-N26; W1230, Doctoral Program "Discrete Mathematics", and SFB F 5510-N26. The author would like to thank Manfred Madritsch and Robert Tichy for many discussions on the subject and for reading several versions of this manuscript. The author is indebted to the anonymous referee for many valuable comments.

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