MATHEMATICS OF COMPUTATION
Volume 86, Number 308, November 2017, Pages 2555–2577
http://dx.doi.org/10.1090/mcom/3208
Article electronically published on March 29, 2017

STABLE SPLITTING OF POLYHARMONIC OPERATORS BY GENERALIZED STOKES SYSTEMS

DIETMAR GALLISTL

ABSTRACT. A stable splitting of $2m$-th order elliptic partial differential equations into $2(m-1)$ problems of Poisson type and one generalized Stokes problem is established for any space dimension $d \geq 2$ and any integer $m \geq 1$. This allows a numerical approximation with standard finite elements that are suited for the Poisson equation and the Stokes system, respectively. For some fourth- and sixth-order problems in two and three space dimensions, precise finite element formulations along with a priori error estimates and numerical experiments are presented.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^d$ be an open, simply-connected and bounded polytope with Lipschitz boundary. This article deals with the numerical approximation of the solutions to partial differential equations (PDEs) of polyharmonic type involving the m-th power Δ^m of the Laplace operator Δ as the highest derivative for an integer $m \geq 1$. The exact solution is customarily sought in an m-th order Sobolev space, for instance $H_0^m(\Omega)$. A model problem is to seek, for a given $f \in L^2(\Omega)$, a function $u \in H_0^m(\Omega)$ such that

$$(-1)^m \Delta^m u = f \quad \text{in } H^{-m}(\Omega). \tag{1.1}$$

The numerical approximation of (1.1) with conforming finite elements, requires the trial functions to belong to the Sobolev space $H_0^m(\Omega)$. This means that the finite element functions must satisfy C^{m-1} continuity across the inter-element boundaries. One prominent instance of (1.1) is the biharmonic equation $\Delta^2 u = f$ for $m = 2$. Even in this case and for two space dimensions, the design and implementation of conforming finite elements [15] is a difficult task and there have been many different attempts to circumvent the use of conforming methods, such as nonconforming finite elements [15, 24] or discontinuous Galerkin methods [4, 10, 18]. For the case $m \geq 3$, besides the conforming finite elements of [32], the following approaches have been suggested: the C^0 interior penalty method of [22] for $m = 3$, the nonconforming finite element method of [31] for arbitrary m with the restriction that $m \geq d$, and the mixed formulation based on Helmholtz-type decompositions [27–29] for the m-th Laplace operator with arbitrary m.

Received by the editor January 4, 2016 and, in revised form, July 7, 2016.
2010 *Mathematics Subject Classification.* Primary 31B30, 35J30, 65N12, 65N15, 65N30.
Key words and phrases. Finite element methods, Stokes system, mixed finite elements, polyharmonic equation.

The method suggested in this work is based on a mixed system. The idea of mixed methods is to reduce a higher-order differential equation to a system of lower-order equations. It is, however, difficult to find stable mixed methods for $m \geq 2$: care has to be taken that the stability of the mixed formulation is independent of the regularity of the solution; see also the discussion in [8, §10.1.1]. In particular, some mixed methods for the biharmonic equation may be divergent in the presence of singularities, e.g., for simply-supported plates [11]. This is related to the fact that the stability of those splittings requires additional regularity on the solution of the PDE. It is therefore desirable that the mixed formulation is stable, independent of regularity assumptions on the solution. This is the case for the splitting proposed in this paper. For the simplest case, the biharmonic equation, this formulation resembles the splitting of [12] for the Reissner-Mindlin model of moderately thick elastic plates, where a singularly perturbed version of a fourth-order problem is reduced to two Poisson equations and a Stokes system. The formulation shows in particular that the biharmonic equation in planar domains can be approximated with any stable finite element for the Stokes equations. It turns out that such an idea can be generalized to any order of differentiability $m \geq 1$ and any space dimension $d \geq 2$ leading to a generalized polyharmonic Stokes equation. The resulting splitting is natural in that it is derived from the principles of exterior calculus and symmetric tensors. In particular, high-order partial differential equations like (1.1) can be numerically approximated with standard, even low-order (independent of m), finite elements for the Poisson and the classical Stokes problem. The method is presented for data $f \in L^2(\Omega)$, but the theory is also valid for the weaker assumption $f \in H^{-1}(\Omega)$. In order to keep the formalism to a minimum, the analysis is carried out for the splitting of the model problem (1.1). The application to more general elliptic systems is, however, possible; an outline is given in Subsection 4.5.

The remaining parts of this article are organized as follows. Section 2 introduces the notation on tensors and their differential operators and proves a result on the existence of generalized scalar potentials. In Section 3, the new split of high-order PDEs is presented. It relies on the study of the polyharmonic Stokes problem with symmetry constraint. Sections 4–5 present applications for the case for $m = 2$ and $m = 3$ in two and three space dimensions along with finite element formulations and a priori error estimates. Numerical computations are presented in Section 6. The remarks from Section 7 conclude the paper.

Standard notation on Lebesgue and Sobolev spaces applies throughout this article. A detailed description of differential operators in tensor-valued Sobolev spaces is given in Section 2. An inequality $A \leq CB$ for some generic mesh-size independent constant C is denoted by $A \lesssim B$ while $A \approx B$ abbreviates $A \lesssim B \lesssim A$. Throughout this paper, m refers to the order of differentiability, n denotes the degree of a tensor, and d is the space dimension. The integration with respect to the d-dimensional Lebesgue measure is indicated by the symbol dx whereas ds denotes integration with respect to the $(d-1)$-dimensional Hausdorff measure; the integral mean is denoted by $⨍$. If not specified otherwise, $\Omega \subseteq \mathbb{R}^d$ is a bounded, open, simply-connected Lipschitz polytope with outer unit normal ν. For a matrix A with transpose A^*, the symmetric and the skew-symmetric part are denoted by

$$\operatorname{sym} A := \frac{1}{2}(A + A^*) \quad \text{and} \quad \operatorname{skw} A := \frac{1}{2}(A - A^*).$$

The trace of A is denoted by $\operatorname{tr} A$.

2. Differential operators for tensor fields

This section introduces the notation for tensor fields and states a result on generalized scalar potentials.

2.1. **Notation.** Let $d \geq 2$. For any nonnegative integer $n \in \mathbb{N}_0$, the set of n-tensors over $\mathbb{R}^d$ is defined as

$$\mathbb{M}_d(n) := (\mathbb{R}^d)^n = \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{n \text{ times}} \quad \text{with the convention } (\mathbb{R}^d)^0 := \mathbb{R}.$$

In particular, $\mathbb{M}_d(0) = \mathbb{R}$, $\mathbb{M}_d(1) = \mathbb{R}^d$, $\mathbb{M}_d(2) = \mathbb{R}^{d\times d}$. Let $\mathfrak{S}(n)$ denote the symmetric group of degree n. An element $\tau \in \mathbb{M}_d(n)$ is referred to as symmetric, if any $(j_1, \dots, j_n) \in \{1, \dots, d\}^n$ and any $\sigma \in \mathfrak{S}(n)$ satisfy

$$\tau_{j_1,\dots,j_n} = \tau_{\sigma(j_1,\dots,j_n)}.$$

This defines the space $\mathbb{S}_d(n)$ of symmetric n-tensors over $\mathbb{R}^d$, namely

$$\mathbb{S}_d(n) := \{\tau \in \mathbb{M}_d(n) : \tau \text{ is symmetric}\}.$$

In particular, $\mathbb{S}_d(0) = \mathbb{R}$, $\mathbb{S}_d(1) = \mathbb{R}^d$, while $\mathbb{S}_d(2)$ is the space of symmetric $d \times d$ matrices.

The inner product of $\sigma, \tau \in \mathbb{M}_d(n)$ is defined as

$$\sigma : \tau = \sum_{(j_1,\dots,j_n)\in\{1,\dots,d\}^n} \sigma_{j_1,\dots,j_n}\tau_{j_1,\dots,j_n}.$$

The space of $\mathbb{M}_d(n)$-valued functions over some domain Ω whose components are square integrable is denoted by $L^2(\Omega; \mathbb{M}_d(n))$. The subspace of symmetric L^2 tensor fields reads $L^2(\Omega; \mathbb{S}_d(n))$. In the scalar case, abbreviate $L^2(\Omega) := L^2(\Omega; \mathbb{R})$. The L^2 inner product and the L^2 norm for fields $\sigma, \tau \in L^2(\Omega; \mathbb{M}_d(n))$ read

$$(\sigma, \tau)_{L^2(\Omega)} := \int_\Omega \sigma : \tau \, dx \quad \text{and} \quad \|\sigma\|_{L^2(\Omega)} := \sqrt{(\sigma, \sigma)_{L^2(\Omega)}}\,.$$

Let $\Omega \subseteq \mathbb{R}^d$ be a simply-connected and bounded Lipschitz domain. For $m \in \mathbb{N}_0$, the m-th order Sobolev space of $L^2(\Omega)$ functions whose generalized derivatives up to order m belong to L^2 and whose boundary traces up to order $(m-1)$ vanish, is denoted by $H^m_0(\Omega)$. Tensor fields with values in $\mathbb{M}_d(n)$ (resp. $\mathbb{S}_d(n)$) whose components belong to $H^m_0(\Omega)$ are denoted by $H^m_0(\Omega; \mathbb{M}_d(n))$ (resp. $H^m_0(\Omega; \mathbb{S}_d(n))$). The following abbreviations will be used:

$$V^{\mathbb{M}}_{m,n} := H^m_0(\Omega; \mathbb{M}_d(n)) \quad \text{and} \quad V^{\mathbb{S}}_{m,n} := H^m_0(\Omega; \mathbb{S}_d(n)).$$

The (total) derivative $Dv \in V^{\mathbb{M}}_{m-1,n+1}$ of an element $v \in V^{\mathbb{M}}_{m,n}$ has the representation

$$(Dv)_{j_1,\dots,j_n,j_{n+1}} = \partial_{j_{n+1}} v_{j_1,\dots,j_n}$$

for any $(j_1, \dots, j_{n+1}) \in \{1, \dots, d\}^{n+1}$. In other words, $(Dv)_{j_1,\dots,j_n,\bullet}$ corresponds to the gradient of the function $v_{j_1,\dots,j_n}$ and $(Dv)_{j_1,\dots,j_{n-1},\bullet,\bullet}$ corresponds to the Jacobian of the vector field $v_{j_1,\dots,j_{n-1},\bullet}$.

Let $n \geq 1$. For any $v \in V^{\mathbb{M}}_{m,n}$, its componentwise exterior derivative $\mathbf{rot}\, v \in V^{\mathbb{M}}_{m-1,n+1}$ is defined by

$$(\mathbf{rot}\, v)_{j_1,\dots,j_{n-1},j_n,j_{n+1}} = \partial_{j_{n+1}} v_{j_1,\dots,j_{n-1},j_n} - \partial_{j_n} v_{j_1,\dots,j_{n-1},j_{n+1}}.$$

In other words, the $d \times d$ matrix field (or 2-form) $(\mathbf{rot}\, v)_{j_1,\dots,j_{n-1},\bullet,\bullet}$ is the exterior derivative of the vector field (or 1-form) $v_{j_1,\dots,j_{n-1},\bullet}$. If $n = 0$, that is, if v is scalar-valued, then, by definition, $\mathbf{rot}\, v = 0$. This is consistent with the fact that the space of alternating 2-forms over $\mathbb{R}$ is the trivial vector space $\{0\}$.

In particular, if v is a vector field ($n = 1$) for $d = 2$ or $d = 3$, the upper triangular part of the skew-symmetric matrix $\mathbf{rot}\, v$ reads as

$$\mathbf{rot}\, v = \begin{pmatrix} 0 & \partial_2 v_1 - \partial_1 v_2 \\ & 0 \end{pmatrix} \text{ or } \mathbf{rot}\, v = \begin{pmatrix} 0 & \partial_2 v_1 - \partial_1 v_2 & \partial_3 v_1 - \partial_1 v_3 \\ & 0 & \partial_3 v_2 - \partial_2 v_3 \\ & & 0 \end{pmatrix} \tag{2.1}$$

and can be identified with the usual "vector proxies"

$$\operatorname{rot} v = \partial_2 v_1 - \partial_1 v_2 \text{ for } d = 2 \quad \text{or} \quad \operatorname{rot} v = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix} \text{ for } d = 3. \tag{2.2}$$

In order to distinguish between alternating quadratic forms and the vector representations, the vector representation of **rot** is denoted by rot without boldface letters.

The divergence of a sufficiently smooth $\mathbb{M}_d(n)$-valued tensor field τ has values in $\mathbb{M}_d(n-1)$ and is defined via

$$(\operatorname{div} \tau)_{j_1,\dots,j_{n-1}} = \operatorname{tr}\left((D\tau)_{j_1,\dots,j_{n-1},\bullet,\bullet}\right) = \sum_{k=1}^{d} \partial_k \tau_{j_1,\dots,j_{n-1},k}.$$

The space of $L^2(\Omega)$ vector fields whose divergence belongs to $L^2(\Omega)$ is denoted by $H(\operatorname{div}, \Omega)$.

2.2. **Exterior calculus of symmetric tensor fields.** The following result serves as the main tool for the stable splitting of higher-order differential operators.

Lemma 1. *Let Ω be simply-connected and let $m \geq 1$ and $n \geq 1$ be integers.*

(a) *Let $v \in V^{\mathbb{M}}_{m,n}$. If $\mathbf{rot}\, v = 0$, then there exists $\eta \in V^{\mathbb{M}}_{m+1,n-1}$ such that $D\eta = v$.*
(b) *Let $v \in V^{\mathbb{M}}_{m,n}$. The matrix $(Dv)_{j_1,\dots,j_{n-1},\bullet,\bullet}$ is symmetric for any index $(j_1,\dots,j_{n-1}) \in \{1,\dots,d\}^{n-1}$ if and only if $\mathbf{rot}\, v = 0$.*
(c) *Let $v \in V^{\mathbb{S}}_{m,n}$. If $\mathbf{rot}\, v = 0$, then there exists $\eta \in V^{\mathbb{S}}_{m+1,n-1}$ such that $D\eta = v$.*
(d) *Let $v \in V^{\mathbb{S}}_{m,n}$. If $\mathbf{rot}\, v = 0$, then for any $k \in \{1,\dots,n\}$ there exists $\eta \in V^{\mathbb{S}}_{m+k,n-k}$ such that $D^k\eta = v$.*

Proof. Assertion (a) is a well-known fact from the exterior calculus on simply-connected domains. Assertion (b) is the observation that $\mathbf{rot}\, v$ contains exactly the skew-symmetric parts of the matrices $(Dv)_{j_1,\dots,j_{n-1},\bullet,\bullet}$. For the proof of (c), let $v \in V^{\mathbb{S}}_{m,n}$. From (a) it is known that there exists $\eta \in V^{\mathbb{M}}_{m+1,n-1}$ such that $D\eta = v$. Now let $\sigma \in \mathfrak{S}(n-1)$ be a permutation. The relation $D\eta = v$ and the symmetry of v imply that for any $(j_1,\dots,j_n) \in \{1,\dots,d\}^n$ the following identity holds:

$$\partial_{j_n} \eta_{j_1,\dots,j_{n-1}} = v_{j_1,\dots,j_n} = v_{\sigma(j_1,\dots,j_{n-1}),j_n} = \partial_{j_n} \eta_{\sigma(j_1,\dots,j_{n-1})}.$$

This means that the difference $\eta_{j_1,\dots,j_{n-1}} - \eta_{\sigma(j_1,\dots,j_{n-1})}$ is constant with respect to the variable x_{j_n} for any $j_n \in \{1,\dots,d\}$. From the boundary condition on η it turns out that $\eta_{j_1,\dots,j_{n-1}} = \eta_{\sigma(j_1,\dots,j_{n-1})}$. Hence, η is symmetric. Assertion (d) follows from the recursive application of (b) and (c). $\square$

Remark 2. Symmetry principles as in Lemma 1 were also studied in [27–29] for the proof of generalized Helmholtz decompositions.

3. Stable splitting of higher-order differential operators

In this section, the splitting of the $2m$-th order PDE in a system of second-order problems is presented. To keep the presentation simple, the polyharmonic operator $(-1)^m\Delta^m$ without lower-order terms is analyzed. Subsection 4.5 will give an outline of the case of more general differential operators. The core part of the split consists of a polyharmonic generalized Stokes equation.

3.1. **The polyharmonic generalized Stokes equation.** Define the space of symmetric and rotation-free H_0^m tensor fields as

$$Z_{m,n} := \{v \in V_{m,n}^{\mathbb{S}} : \mathbf{rot}\, v = 0\}.$$

Let $f \in L^2(\Omega; \mathbb{M}_d(n))$. The polyharmonic generalized Stokes problem seeks $u \in Z_{m,n}$ such that

$$(D^m u, D^m v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in Z_{m,n}. \tag{3.1}$$

This problem is referred to as generalized Stokes equation because it corresponds to the minimization of a quadratic energy functional subject to a linear constraint on the partial derivatives. Sections 4–5 will show how the problem (3.1) can be re-written as a saddle-point problem with Lagrange multipliers. Lemma 1(b) shows that Du is symmetric. With the substitution $w := Du \in Z_{m-1,n+1}$ one arrives at the following alternative formulation: Seek $u \in V_{1,n}^{\mathbb{S}}$, $w \in Z_{m-1,n+1}$, $r \in V_{1,n}^{\mathbb{S}}$ such that

$$(Du, Dv)_{L^2(\Omega)} = (w, Dv)_{L^2(\Omega)} \qquad \text{for all } v \in V_{1,n}^{\mathbb{S}}, \tag{3.2a}$$

$$(D^{(m-1)} w, D^{(m-1)} z)_{L^2(\Omega)} = (Dr, z)_{L^2(\Omega)} \qquad \text{for all } z \in Z_{m-1,n+1}, \tag{3.2b}$$

$$(Dr, Ds)_{L^2(\Omega)} = (f, s)_{L^2(\Omega)} \qquad \text{for all } s \in V_{1,n}^{\mathbb{S}}. \tag{3.2c}$$

The unique solvability of (3.1) and (3.2) is a direct consequence of the obvious coercivity of the involved operators. The next result states the equivalence of (3.1) and (3.2).

Proposition 3. *If $u \in Z_{m,n}$ solves* (3.1)*, then there exists $r \in V_{1,n}^{\mathbb{S}}$ such that $(u, Du, r) \in V_{1,n}^{\mathbb{S}} \times Z_{m-1,n+1} \times V_{1,n}^{\mathbb{S}}$ solves* (3.2)*. If, conversely, $(u, w, r) \in V_{1,n}^{\mathbb{S}} \times Z_{m-1,n+1} \times V_{1,n}^{\mathbb{S}}$ solves* (3.2)*, then u belongs to $Z_{m,n}$ and solves* (3.1) *and $w = Du$.*

Proof. The sufficient condition is verified with the solution r of the Poisson-type problem (3.2c). Indeed, $w := Du$ belongs to $Z_{m-1,n+1}$ and, by Lemma 1(c), any test function $z \in Z_{m-1,n+1}$ can be represented as $z = Dv$ for some $v \in V_{m,n}^{\mathbb{S}}$. Hence,

$$(D^{(m-1)} w, D^{(m-1)} z)_{L^2(\Omega)} = (D^m u, D^m v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} = (Dr, z)_{L^2(\Omega)}.$$

For the verification of the necessary condition, let (u, w, r) solve (3.2). By Lemma 1(c) there exists $\eta \in V_{m,n}^{\mathbb{S}}$ such that $D\eta = w$. Since w is symmetric, η is rotation-free $\mathbf{rot}\, \eta = 0$ (cf. Lemma 1(b)). Hence, $\eta \in Z_{m,n}$. By testing equation (3.2a) with the test function $v := u - \eta$, one obtains

$$\|D(u - \eta)\|_{L^2(\Omega)}^2 = (D(u - \eta), Dv)_{L^2(\Omega)} = (Du - w, Dv)_{L^2(\Omega)} = 0.$$

This implies $\eta = u$ and, thus, $Du = w$. According to Lemma 1(b), for any $v \in Z_{m,n}$ the derivative Dv is symmetric. Hence, (3.2b) and (3.2c) lead to

$$(D^m u, D^m v)_{L^2(\Omega)} = (D^{(m-1)} w, D^{(m-1)} Dv)_{L^2(\Omega)} = (Dr, Dv)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}.$$

Therefore, u solves (3.1). $\square$

System (3.2) consists of two Poisson equations and one generalized symmetric Stokes system. In the present case where no lower-order terms appear in the differential operator, the problem (3.2b) decouples from the equations (3.2a) and (3.2c). However, this decoupling does not hold in the presence of lower-order terms. Examples will be given below.

3.2. **Splitting of polyharmonic equations.** Let $f \in L^2(\Omega)$ and consider the $2m$-th order partial differential equation that seeks $u \in H_0^m(\Omega)$ with

$$(-1)^m \Delta^m u = f.$$

Its weak form is given by

$$(D^m u, D^m v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^m(\Omega). \tag{3.3}$$

The observation that $H_0^m(\Omega) = V_{m,0}^{\mathbb{S}} = Z_{m,0}$ reveals that (3.3) is identical to problem (3.1) for $n = 0$. Hence, by applying recursively the split (3.2), one obtains a splitting of (3.3) into $(2m-1)$ second-order equations, namely $(2m-2)$ Poisson-like problems and one generalized Stokes equation over the symmetric tensors. The resulting system is the following. Seek $(u_0, \dots, u_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$, $w \in Z_{1,m-1}$, and $(r_0, \dots, r_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$ such that, for all $j \in \{0, \dots, m-3\}$,

$$(Du_j, Dv_j)_{L^2(\Omega)} = (u_{j+1}, Dv_j)_{L^2(\Omega)}, \tag{3.4a}$$

$$(Du_{m-2}, Dv_{m-2})_{L^2(\Omega)} = (w, Dv_{m-2})_{L^2(\Omega)}, \tag{3.4b}$$

$$(Dw, Dz)_{L^2(\Omega)} = (Dr_{m-2}, z)_{L^2(\Omega)}, \tag{3.4c}$$

$$(Dr_{m-2-j}, Ds_{m-2-j})_{L^2(\Omega)} = (Dr_{m-3-j}, s_{m-2-j})_{L^2(\Omega)}, \tag{3.4d}$$

$$(Dr_0, Ds_0)_{L^2(\Omega)} = (f, s_0)_{L^2(\Omega)} \tag{3.4e}$$

for all $(v_0, \dots, v_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$, $z \in Z_{1,m-1}$, and $(s_0, \dots, s_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$.

Hence, the following equivalence is valid.

Proposition 4. *If $u \in H_0^m(\Omega)$ solves (3.3), then there exist $(u_0, \dots, u_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$, $w \in Z_{1,m-1}$, and $(r_0, \dots, r_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$ such that, with $u_0 := u$, (3.4) is satisfied.*

If, conversely, $(u_0, \dots, u_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$, $w \in Z_{1,m-1}$, and $(r_0, \dots, r_{m-2}) \in V_{1,0}^{\mathbb{S}} \times V_{1,1}^{\mathbb{S}} \times \dots \times V_{1,m-2}^{\mathbb{S}}$ solve (3.4), then $u := u_0$ belongs to $H_0^m(\Omega)$ and solves (3.3) with $D^j u = u_j$ for all $j \in \{1, \dots, m-2\}$ and $D^{(m-1)} u = w$. $\square$

3.3. **Discretization with standard finite elements.** The main advantage of the reformulation of (3.3) as the system (3.4) is that one can use standard finite elements for its discretization. In particular, for the Poisson-like problems in (3.4), standard H_0^1-conforming finite elements can be used without any restriction on the polynomial order. In the absence of lower-order terms, the equations in (3.4) even decouple into a finite sequence of second-order problems. The only equation in (3.4) that needs further investigation is (3.4c). This equation is similar to the classical

Stokes system, but the side constraint is on the rotation (instead on the divergence for the classical Stokes equations). In three space dimensions, this results in a vector valued "pressure" variable. Moreover, the symmetry constraint implies further constraints for the space of multipliers in a saddle-point formulation. Hence, (3.4c) requires modifications of stable finite elements for the second-order generalized Stokes problem. As described in Sections 4–5 below, in the case of space dimensions 2 and 3 some of the known stable finite elements for the classical Stokes problem can be utilized.

4. Application to fourth-order problems

Throughout the remaining sections, the space dimension is $d = 2$ or $d = 3$. If $d = 3$, it is assumed that Ω is contractible.

This section is devoted to the application of the splitting to the biharmonic equation. It establishes a saddle-point formulation of the Stokes-type equation and studies discretization schemes.

4.1. **The mixed system.** The biharmonic equation

$$\Delta^2 u = f \quad \text{in } \Omega \qquad \text{and} \qquad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{4.1}$$

is a classical model for the vertical deflection of a thin elastic plate subject to clamped boundary conditions. Its weak form is given by (3.3) for $m = 2$. In this case, the system from (3.4) seeks $(u, w, r) \in H_0^1(\Omega) \times Z_{1,1} \times H_0^1(\Omega)$ such that

$$(Du, Dv)_{L^2(\Omega)} = (w, Dv)_{L^2(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega), \tag{4.2a}$$

$$(Dw, Dz)_{L^2(\Omega)} = (Dr, z)_{L^2(\Omega)} \qquad \text{for all } z \in Z_{1,1}, \tag{4.2b}$$

$$(Dr, Ds)_{L^2(\Omega)} = (f, s)_{L^2(\Omega)} \qquad \text{for all } s \in H_0^1(\Omega). \tag{4.2c}$$

Similar systems were used and analyzed in [12, 13, 17] for the analysis of the equations of the Reissner-Mindlin plate model in two space dimensions, which can be interpreted as a perturbed biharmonic equation.

For the numerical approximation of (4.2a) and (4.2c), any finite element method for Poisson's equation can be used. The next subsections focus on the approximation of (4.2b).

4.2. **The generalized Stokes equation.** This subsection is devoted to the saddle-point formulation of the problem

$$(Dw, Dz)_{L^2(\Omega)} = (g, z)_{L^2(\Omega)} \quad \text{for all } z \in Z_{1,1} \tag{4.3}$$

with given $g \in L^2(\Omega; \mathbb{R}^d)$. Recall that the space $Z_{1,1}$ consists of the rotation-free vector fields in the first-order Sobolev space $\mathbf{V} := V_{1,1}^{\mathbb{S}} = H_0^1(\Omega; \mathbb{R}^d)$. For $d = 3$, the following space will be employed:

$$H_0(\operatorname{div}^0, \Omega) := \left\{ q \in L^2(\Omega; \mathbb{R}^3) : \operatorname{div} q = 0 \text{ in } \Omega \text{ and } q \cdot \nu|_{\partial\Omega} = 0 \right\}. \tag{4.4}$$

Define the space $\mathbf{Q}$ for the Lagrange multipliers as

$$\mathbf{Q} := \begin{cases} \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\} & \text{if } d = 2, \\ H_0(\operatorname{div}^0, \Omega) & \text{if } d = 3, \end{cases}$$

equipped with the L^2 norm. The saddle-point problem reads as follows: Seek $(w, p) \in \mathbf{V} \times \mathbf{Q}$ such that

$$\begin{aligned} (Dw, Dv)_{L^2(\Omega)} + (\operatorname{rot} v, p)_{L^2(\Omega)} &= (g, v)_{L^2(\Omega)} \qquad && \text{for all } v \in \mathbf{V}, \\ (\operatorname{rot} w, q)_{L^2(\Omega)} &= 0 && \text{for all } q \in \mathbf{Q}. \end{aligned} \tag{4.5}$$

System (4.5) can formally be written as

$$\begin{aligned} -\Delta w + \operatorname{rot}^* p &= g, \\ \operatorname{rot} w &= 0. \end{aligned}$$

Since, $\operatorname{rot} \mathbf{V} \subseteq \mathbf{Q}$, with the standard theory [8] it is readily verified that problems (4.3) and (4.5) are equivalent once the following inf-sup condition is proved.

Proposition 5. *Let $d \in \{2, 3\}$ and assume that, if $d = 3$, the domain Ω is contractible. There is a constant $\beta > 0$ such that*

$$\beta \leq \inf_{q \in \mathbf{Q} \setminus \{0\}} \sup_{v \in \mathbf{V} \setminus \{0\}} \frac{(\operatorname{rot} v, q)_{L^2(\Omega)}}{\|Dv\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}.$$

Proof. Provided the domain Ω has vanishing first and second Betti numbers (which in 3D means that Ω is contractible), the result follows from [25, Prop. A.1] or [1, Theorem 1.2]. For $d = 2$, the result is known as the Ladyzhenskaya lemma [2], and there are no restrictions on the topology of Ω. $\square$

4.3. **Finite elements for the generalized Stokes equation.** System (4.5) resembles the classical Stokes system. Indeed, in two space dimensions, (4.5) can be reformulated as a Stokes system by means of a change of variables. Therefore, any stable finite element method can be used for the approximation of (4.5). In three dimensions, (4.5) differs from the Stokes equations, but the methods that are well-studied for the Stokes equations [8] can nevertheless be used for the discretization of the system. This is exemplified with two choices of discrete spaces: a continuous finite element method, namely a stabilized P_1 element with discontinuous pressure, and the nonconforming P_1 element.

4.3.1. *Notation.* In what follows, $\mathcal{T}$ denotes a shape-regular simplicial triangulation of Ω. The set of $(d-1)$-dimensional hyperfaces is denoted by $\mathcal{F}$. The space of piecewise polynomial functions of maximal degree k is denoted by $P_k(\mathcal{T})$. The space of functions with values in some finite dimensional space X whose components belong to $P_k(\mathcal{T})$ is denoted by $P_k(\mathcal{T}; X)$. Furthermore, denote

$$S^k(\mathcal{T}) := P_k(\mathcal{T}) \cap H^1(\Omega) \quad \text{and} \quad S^k_0(\mathcal{T}) := P_k(\mathcal{T}) \cap H^1_0(\Omega).$$

Similarly, X-valued functions with components in $S^k(\mathcal{T})$ (resp. $S^k_0(\mathcal{T})$) are denoted by $S^k(\mathcal{T}; X)$ (resp. $S^k_0(\mathcal{T}; X)$).

4.3.2. *Continuous finite elements.* The finite element discretization of (4.5) is based on finite-dimensional subspaces $\mathbf{V}_h \subseteq \mathbf{V}$ and $\mathbf{Q}_h \subseteq \mathbf{Q}$ and seeks $(w_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned} (Dw_h, Dv_h)_{L^2(\Omega)} + (\operatorname{rot} v_h, p_h)_{L^2(\Omega)} &= (g, v_h)_{L^2(\Omega)} \qquad && \text{for all } v_h \in \mathbf{V}_h, \\ (\operatorname{rot} w_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in \mathbf{Q}_h. \end{aligned} \tag{4.6}$$

In two space dimensions, this system is equivalent to the Stokes equations after the change of coordinates $(x_1, x_2) \mapsto (-x_2, x_1)$. Therefore, any stable finite element pairing $(\mathbf{V}_h, \mathbf{Q}_h)$ for the Stokes equations (for examples see [8]) can be used for the numerical solution.

Proposition 6. *For $d = 2$, any pairing $\mathbf{V}_h \subseteq \mathbf{V}$, $\mathbf{Q}_h \subseteq \mathbf{Q}$ that is stable for the Stokes equations leads to a unique solution $(w_h, p_h) \in (\mathbf{V}_h, \mathbf{Q}_h)$ to* (4.6)*. It satisfies the error estimate*

$$\|D(w-w_h)\|_{L^2(\Omega)} + \|p-p_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathbf{V}_h} \|D(w-w_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p-p_h\|_{L^2(\Omega)}.$$

□

In the case of three space dimensions, a conforming stable pairing requires the space $\mathbf{Q}_h \subseteq H_0(\mathrm{div}^0, \Omega)$ to be a subset of (pointwise) divergence-free vector fields. The (strong) incorporation of this constraint in a space of continuous discrete vector fields is difficult. The modification of existing Stokes elements with discontinuous pressure can lead to a stable pairing. This is exemplified with a stabilized P_1–P_0 element. The lowest-order Raviart-Thomas space of $H(\mathrm{div})$ conforming vector fields [8] in three dimensions is defined as

$$RT_0(\mathcal{T}) := \left\{ q \in H(\mathrm{div}, \Omega) : \begin{array}{l} \text{for any } T \in \mathcal{T} \text{ there exist } a_T \in \mathbb{R}^3, b_T \in \mathbb{R} \\ \text{with } q|_T(x) = a_T(x) + b_T(x)x \quad \text{for all } x \in T \end{array} \right\}.$$

Let, for any face $F \in \mathcal{F}$, b_F denote the cubic bubble function that vanishes on ∂F and satisfies $\fint_F b_F \, ds = 1$. It satisfies, for any adjacent simplex T, the scaling

$$\|b_F\|_{L^2(T)} \approx \mathrm{diam}(T)^{3/2}. \tag{4.7}$$

The space of three-dimensional vector fields whose components are spanned by the functions $(b_F : F \in \mathcal{F})$ is denoted by $\mathcal{B}_3(\mathcal{F}; \mathbb{R}^3)$. Define the following spaces of discrete functions

$$\mathbf{V}_h := S_0^1(\mathcal{T}; \mathbb{R}^3) \oplus \mathcal{B}_3(\mathcal{F}; \mathbb{R}^3) \qquad \text{and} \qquad \mathbf{Q}_h := RT_0(\mathcal{T}) \cap H_0(\mathrm{div}^0, \Omega). \tag{4.8}$$

This low-order pairing generalizes stabilized finite elements [8] for the Stokes equations.

Remark 7. It is useful to note that $\mathbf{Q}_h = P_0(\mathcal{T}; \mathbb{R}^3) \cap \mathbf{Q}$. More precisely, $\mathbf{Q}_h$ consists exactly of all piecewise constant vector fields that are continuous in the inter-element normal directions and whose normal component vanishes on the boundary $\partial\Omega$. This description allows a straightforward implementation with Lagrange multipliers in the spirit of [3].

Proposition 8. *Let $d = 3$ and let Ω be contractible. The system* (4.6) *with the choice* (4.8) *of discrete spaces has a unique solution $(w_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$. It satisfies the error estimate*

$$\|D(w-w_h)\|_{L^2(\Omega)} + \|p-p_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathbf{V}_h} \|D(w-w_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p-p_h\|_{L^2(\Omega)}.$$

Proof. The proof follows the usual reasoning for stabilized Stokes elements. Let $q_h \in \mathbf{Q}_h$. According to Proposition 5, there exists $v \in \mathbf{V}$ such that

$$\|q_h\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} \lesssim (\mathrm{rot}\, v, q_h)_{L^2(\Omega)}.$$

Let $I_h : V \to V \cap P_1(\mathcal{T}; \mathbb{R}^3)$ denote a quasi-interpolation operator [9] satisfying, for any $v \in V$ and any $T \in \mathcal{T}$, the stability estimate

$$\mathrm{diam}(T)^{-1} \|v - I_h v\|_{L^2(T)} + \|D I_h v\|_{L^2(T)} \lesssim \|Dv\|_{L^2(\bigcup\{K \in \mathcal{T} : K \cap T \neq \emptyset\})}. \tag{4.9}$$

Define $v_h \in \mathbf{V}_h$ by

$$v_h = I_h v + \sum_{F\in\mathcal{F}} b_F \fint_F (v - I_h v)\, ds.$$

The combination of the scaling and stability properties (4.7) and (4.9) readily proves

$$\|Dv_h\|_{L^2(\Omega)} \lesssim \|Dv\|_{L^2(\Omega)}.$$

On the other hand, since the elements in $\mathbf{Q}_h$ are piecewise constant and by the definition of v_h, one has for any $T \in \mathcal{T}$ (with outward-pointing unit normal ν_T) that

$$(\operatorname{rot}(v - v_h), q_h)_{L^2(T)} = \int_{\partial T} ((v - v_h) \wedge \nu_T) \cdot q_h \, ds = 0$$

(the wedge $\wedge$ denotes the exterior product in $\mathbb{R}^3$). Hence $(\operatorname{rot} v_h, q_h)_{L^2(\Omega)} = (\operatorname{rot} v, q_h)_{L^2(\Omega)}$. This establishes the discrete stability condition

$$\beta \lesssim \inf_{q_h\in\mathbf{Q}_h\setminus\{0\}} \sup_{v_h\in\mathbf{V}_h\setminus\{0\}} \frac{(\operatorname{rot} v_h, q_h)_{L^2(\Omega)}}{\|Dv_h\|_{L^2(\Omega)}\|q_h\|_{L^2(\Omega)}}.$$

The stability and the error estimate follow from the usual theory for saddle-point problems [8]. □

Remark 9. From the proof of Proposition 8 it can be seen that a smaller choice of $\mathbf{V}_h$ with stabilization in the tangential directions only would also suffice for a stable pairing. This would correspond to an analogue of the SMALL element [8] for the usual Stokes system.

4.3.3. *Nonconforming P_1 element.* Nonconforming methods where $\mathbf{V}_h^{NC} \not\subseteq \mathbf{V}$ may have favorable stability properties with trial functions based on low-order polynomials. This subsection describes an application of the nonconforming P_1 finite element [16] for the generalized Stokes system (4.5). Define the spaces

$$\mathbf{V}_h^{NC} := \left\{ v \in \mathcal{P}_1(\mathcal{T};\mathbb{R}^d) \;\middle|\; \begin{array}{l} v \text{ is continuous in the interior hyper-faces'} \\ \text{midpoints and vanishes in the midpoints} \\ \text{of hyper-faces on the boundary} \end{array} \right\}$$

and

$$\mathbf{Q}_h := \begin{cases} P_0(\mathcal{T}) \cap \mathbf{Q} & \text{if } d = 2, \\ RT_0(\mathcal{T}) \cap \mathbf{Q} & \text{if } d = 3. \end{cases}$$

Again, it will be made use of the fact that $RT_0(\mathcal{T}) \cap \mathbf{Q} = P_0(\mathcal{T};\mathbb{R}^3) \cap \mathbf{Q}$. The $\mathcal{T}$-piecewise action of the differential operators D and rot is denoted by D_h and rot_h. The nonconforming finite element method seeks $(w_h, p_h) \in \mathbf{V}_h^{NC} \times \mathbf{Q}_h$ such that

$$\begin{aligned} (D_h w_h, D_h v_h)_{L^2(\Omega)} + (\operatorname{rot}_h v_h, p_h)_{L^2(\Omega)} &= (g, v_h)_{L^2(\Omega)} && \text{for all } v_h \in \mathbf{V}_h^{NC}, \\ (\operatorname{rot}_h w_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in \mathbf{Q}_h. \end{aligned} \tag{4.10}$$

Proposition 10. *The system* (4.10) *has a unique solution* $(w_h, p_h) \in (\mathbf{V}_h^{NC}, \mathbf{Q}_h)$. *It satisfies the error estimate*

$$\begin{aligned} &\|D_h(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\ &\qquad \lesssim \inf_{v_h\in\mathbf{V}_h^{NC}} \|D_h(w - w_h)\|_{L^2(\Omega)} + \inf_{q_h\in\mathbf{Q}_h} \|p - p_h\|_{L^2(\Omega)} \\ &\qquad\quad + \sqrt{\sum_{T\in\mathcal{T}} \operatorname{diam}(T)^2 \inf_{c\in\mathbb{R}} \|g - c\|^2_{L^2(T)}}. \end{aligned}$$

Proof. The nonconforming interpolation operator $I_h^{NC} : \mathbf{V} \to \mathbf{V}_h^{NC}$ is defined for any $v \in \mathbf{V}$ via the condition

$$\int_F (v - I_h^{NC} v)\, ds = 0 \quad \text{for all hyper-faces } F \in \mathcal{F}.$$

It is well known [16] that, for any $v \in \mathbf{V}$, the function $D_h I_h^{NC} v$ equals the L^2 projection of Dv onto the piecewise constants. In particular, for any $q_h \in \mathbf{Q}_h$, any $v \in \mathbf{V}$ satisfies $(\operatorname{rot} v, q_h)_{L^2(\Omega)} = (\operatorname{rot}_h I_h^{NC} v, q_h)_{L^2(\Omega)}$. This and $\|D_h I_h^{NC} v\|_{L^2(\Omega)} \leq \|Dv\|_{L^2(\Omega)}$ establish the discrete inf-sup condition

$$\beta \lesssim \inf_{v_h \in \mathbf{V}_h^{NC} \setminus \{0\}} \sup_{q_h \in \mathbf{Q}_h \setminus \{0\}} \frac{(\operatorname{rot}_h v_h, q_h)_{L^2(\Omega)}}{\|D_h v_h\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}}.$$

The a priori error estimate can be deduced with the techniques of [14] for $d = 2$ and [19] for $d = 3$. □

4.4. **Numerical methods for the biharmonic equation.** This subsection describes the numerical approximation of the biharmonic system (4.2). Let $\mathbf{U}_h \subseteq H_0^1(\Omega)$ be a finite-dimensional subspace and let $\mathbf{V}_h$, $\mathbf{Q}_h$ be one of the admissible pairings from §4.3.2. Then the discrete problem seeks $(u_h, w_h, p_h, r_h) \in \mathbf{U}_h \times \mathbf{V}_h \times \mathbf{Q}_h \times \mathbf{U}_h$

$$\begin{aligned} (Du_h, Dv_h)_{L^2(\Omega)} &= (w_h, Dv_h)_{L^2(\Omega)} && \text{for all } v_h \in \mathbf{U}_h,\\ (Dw_h, D\xi_h)_{L^2(\Omega)} + (\operatorname{rot} \xi_h, p_h)_{L^2(\Omega)} &= (Dr_h, \xi_h)_{L^2(\Omega)} && \text{for all } \xi_h \in \mathbf{V}_h,\\ (\operatorname{rot} w_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in \mathbf{Q}_h,\\ (Dr_h, Ds_h)_{L^2(\Omega)} &= (f, s_h)_{L^2(\Omega)} && \text{for all } s_h \in \mathbf{U}_h. \end{aligned} \tag{4.11}$$

Standard a priori error estimates for the Poisson problem together with the triangle inequality and the a priori error estimates from Propositions 6 and 8 give the following quasi-optimal error estimate

$$\begin{aligned} &\|D(u - u_h)\|_{L^2(\Omega)} + \|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} + \|D(r - r_h)\|_{L^2(\Omega)}\\ &\lesssim \inf_{v_h \in \mathbf{U}_h, \xi_h \in \mathbf{V}_h, q_h \in \mathbf{Q}_h, s_h \in \mathbf{U}_h} \big[\|D(u - v_h)\|_{L^2(\Omega)} + \|D(w - \xi_h)\|_{L^2(\Omega)}\\ &\qquad\qquad + \|p - q_h\|_{L^2(\Omega)} + \|D(r - s_h)\|_{L^2(\Omega)}\big]. \end{aligned}$$

It is emphasized that this error estimate can dispense with any regularity assumptions on the solution. In particular, if the solution is smooth enough such that Du, Dw, Dr, p have H^s regularity for some $0 < s \leq 1$ (this is satisfied $u \in H^{2+s}(\Omega)$ and the Poisson equation in Ω has $H^{1+s}(\Omega)$ regularity), this results in the following convergence rate

$$\begin{aligned} &\|D(u - u_h)\|_{L^2(\Omega)} + \|D(w - w_h)\|_{L^2(\Omega)}\\ &\qquad + \|p - p_h\|_{L^2(\Omega)} + \|D(r - r_h)\|_{L^2(\Omega)} \lesssim h^s \|f\|_{L^2(\Omega)} \end{aligned}$$

for the maximal mesh-size h. With standard duality techniques, the following convergence rates in weaker norms can be proven

$$\|u - u_h\|_{L^2(\Omega)} + \|w - w_h\|_{L^2(\Omega)} + \|r - r_h\|_{L^2(\Omega)} \lesssim h^{2s} \|f\|_{L^2(\Omega)}.$$

Analogous estimates can be obtained for the nonconforming discretization.

Remark 11. With the nonconforming method from §4.3.3, the Hessian is approximated with piecewise derivatives of nonconforming P_1 finite element functions.

Hence, the method may be interpreted as a variant of the Morley finite element [15].

4.5. **More general fourth-order operators.** This section describes the application of the new methodology to more general fourth-order problems. For simplicity, constant material coefficients are assumed in the model problem: Let $\delta \geq 0$ be a nonnegative number, let $\boldsymbol{\gamma} \in \mathbb{S}_d(2)$ be a symmetric and positive definite $d \times d$ matrix and let $\mathbf{B} \in \mathbb{S}_d(4)$ be a symmetric and positive definite fourth-order tensor. Given $f \in L^2(\Omega)$, the fourth-order model problem under consideration reads

$$\operatorname{div}\operatorname{div}\mathbf{B}D^2 u - \operatorname{div}\boldsymbol{\gamma}Du + \delta u = f \text{ in } \Omega \qquad \text{and} \qquad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

That is, $u \in H^2_0(\Omega)$ solves, for all $v \in H^2_0(\Omega)$,

$$(\mathbf{B}D^2 u, D^2 v)_{L^2(\Omega)} + (\boldsymbol{\gamma}Du, Dv)_{L^2(\Omega)} + (\delta u, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}.$$

With the split from Section 3, this leads to the following equivalent system

$$\begin{array}{llllll} -\operatorname{div}\mathbf{B}Dw + \boldsymbol{\gamma}w & +\operatorname{rot}^* p & & -Dr & = & 0,\\ \operatorname{rot} w & & & & = & 0,\\ & & \delta u & -\Delta r & = & f,\\ \operatorname{div} w & & -\Delta u & & = & 0. \end{array} \tag{4.12}$$

The rows and columns have been rearranged for better readibility. This is a symmetric problem in saddle-point form. The last row in (4.12) asks for $w = Du$ while the third row implies $-\operatorname{div}(Dr) = f - \delta u$. Hence, taking the divergence of the first row leads to $\operatorname{div}\operatorname{div}\mathbf{B}D^2 u - \operatorname{div}\boldsymbol{\gamma}Du + \delta u = f$. Let

$$a(w, v) := (\mathbf{B}Dw, Dv)_{L^2(\Omega)} + (\boldsymbol{\gamma}w, v)_{L^2(\Omega)} \quad \text{for any } w, v \in H^1_0(\Omega; \mathbb{R}^d).$$

The weak form of the the mixed system seeks $(w, u, r) \in Z_{1,1} \times H^1_0(\Omega) \times H^1_0(\Omega)$ such that, for all $(z, s, v) \in Z_{1,1} \times H^1_0(\Omega) \times H^1_0(\Omega)$,

$$\begin{array}{llll} a(w,z) & & -(Dr,z)_{L^2(\Omega)} & = 0,\\ & (\delta u, s)_{L^2(\Omega)} & + (Dr, Ds)_{L^2(\Omega)} & = (f, s)_{L^2(\Omega)},\\ -(w, Dv)_{L^2(\Omega)} & +(Du, Dv)_{L^2(\Omega)} & & = 0. \end{array}$$

The well-posedness of this system and the stability of the numerical methods suggested in 5.3 follow immediately, provided u and r are discretized with the same finite element spaces. The discretization from Subsection 4.4 is stable and leads to the a priori error estimates from that subsection.

Remark 12. For simplicity, the analysis of this paper focuses on the boundary condition $u \in H^m_0(\Omega)$, which in the context of plate theory corresponds to clamped boundary conditions. However, the arguments from the stability proofs apply equally well for more general boundary conditions. In particular, the discrete stability does not rely on additional smoothness assumptions. Subsection 6.3 shows an application for simply-supported plates.

5. Application to sixth-order problems

This section deals with the case $m = 3$. The section is structured similarly as Section 4. The main difference lies in the identification of the multiplier space $\mathbf{Q}$, which requires proper side restrictions in order to allow for the inf-sup condition. For better readability, the following notation is used:

$$\mathbb{M}_{d\times d} = \mathbb{R}^{d\times d} = \mathbb{M}_d(2) \qquad \text{and} \qquad \mathbb{S}_{d\times d} = \mathbb{S}_d(2).$$

Again, the convention (2.2) for the rotation is used. If v is a matrix-valued function, the rot operator is applied row-wise.

5.1. **The mixed system.** The triharmonic equation

$$-\Delta^3 u = f \quad \text{in } \Omega \qquad \text{and} \qquad u = \frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} = 0 \quad \text{on } \partial\Omega \tag{5.1}$$

is given in weak form by (3.3) for $m = 3$. Differential operators of this type arise, for example, in mathematical models of silicon oxidation [5, 23]. The system from (3.4) seeks

$$(u_0, u_1, w, r_1, r_0) \in H^1_0(\Omega) \times H^1_0(\Omega; \mathbb{R}^d) \times Z_{1,2} \times H^1_0(\Omega; \mathbb{R}^d) \times H^1_0(\Omega)$$

such that

$$(Du_0, Dv_0)_{L^2(\Omega)} = (u_1, Dv_0)_{L^2(\Omega)} \qquad \text{for all } v_1 \in H^1_0(\Omega), \tag{5.2a}$$

$$(Du_1, Dv_1)_{L^2(\Omega)} = (w, Dv_1)_{L^2(\Omega)} \qquad \text{for all } v_1 \in H^1_0(\Omega; \mathbb{R}^d), \tag{5.2b}$$

$$(Dw, Dz)_{L^2(\Omega)} = (Dr_1, z)_{L^2(\Omega)} \qquad \text{for all } z \in Z_{1,2}, \tag{5.2c}$$

$$(Dr_1, Ds_1)_{L^2(\Omega)} = (Dr_0, s_1)_{L^2(\Omega)} \qquad \text{for all } s_1 \in H^1_0(\Omega; \mathbb{R}^d), \tag{5.2d}$$

$$(Dr_0, Ds_0)_{L^2(\Omega)} = (f, s_0)_{L^2(\Omega)} \qquad \text{for all } s_0 \in H^1_0(\Omega). \tag{5.2e}$$

The subsequent subsections will study the saddle-point formulation and numerical approximation of (5.2c).

5.2. **The generalized Stokes equation.** Let $g \in L^2(\Omega; \mathbb{R}^d)$. This subsection is devoted to the saddle-point formulation of the problem

$$(Dw, Dz)_{L^2(\Omega)} = (g, z)_{L^2(\Omega)} \quad \text{for all } z \in Z_{1,2}. \tag{5.3}$$

Recall that the space $Z_{1,2}$ consists of the rotation-free symmetric tensor fields in the first-order Sobolev space $\mathbf{V} := V^{\mathbb{S}}_{1,2} = H^1_0(\Omega; \mathbb{S}_{d\times d})$. Thus, (5.3) is a generalized tensor-valued Stokes equation with a symmetry constraint. Recall the definition of the space $H_0(\operatorname{div}^0, \Omega)$ from (4.4) let $H_0(\operatorname{div}^0, \Omega; \mathbb{M}_{3\times 3})$ denote the space of 3×3 tensor fields whose rows belong to $H_0(\operatorname{div}^0, \Omega)$. Let $\xi : \Omega \to \mathbb{R}$; $x \mapsto x$ denote the identity mapping. Define the space of multipliers as

$$\mathbf{Q} := \begin{cases} \big\{q \in L^2(\Omega; \mathbb{R}^2) : \int_\Omega q \, dx = 0 \text{ and } \int_\Omega q \cdot \xi \, dx = 0\big\} & \text{if } d = 2, \\ \big\{q \in H_0(\operatorname{div}^0, \Omega; \mathbb{M}_{3\times 3}) : \operatorname{tr} q = 0 \text{ a.e. in } \Omega\big\} & \text{if } d = 3. \end{cases}$$

For a stable saddle-point formulation, the following result is required.

Proposition 13. *Let $d \in \{2, 3\}$ and assume that, if $d = 3$, the domain Ω is contractible. For any $q \in \mathbf{Q}$ there exists $v \in \mathbf{V}$ with* $\operatorname{rot} v = q$ *and* $\|Dv\|_{L^2(\Omega)} \lesssim \|q\|_{L^2(\Omega)}$.

Proof. Let $d = 2$. Let $q \in \mathbf{Q}$. By the classical Ladyzhenskaya lemma (applied row-wise) there exists a tensor field $\tau \in H_0^1(\Omega; \mathbb{M}_{2\times 2})$ with $\operatorname{rot}\tau = q$ and $\|D\tau\|_{L^2(\Omega)} \lesssim \|q\|_{L^2(\Omega)}$. Its skew-symmetric part is determined by the entry $\varphi := \tau_{12} - \tau_{21}$. Recall the identity mapping $\xi(x) = x$. By the definition of $\mathbf{Q}$ and integration by parts,

$$0 = (\operatorname{rot}\tau, \xi)_{L^2(\Omega)} = (\tau, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))_{L^2(\Omega)} = \int_\Omega \varphi\, dx.$$

Hence, by the Ladyzhenskaya lemma [2], there exists $\alpha \in H_0^2(\Omega; \mathbb{R}^2)$ such that $\varphi = \operatorname{rot}\alpha$ and $\|\alpha\|_{H^2(\Omega)} \lesssim \|D\varphi\|_{L^2(\Omega)}$. This implies

$$\operatorname{skw} D\alpha = \frac{1}{2}\begin{pmatrix} 0 & \varphi \\ -\varphi & 0 \end{pmatrix} = \operatorname{skw}\tau.$$

With $\operatorname{rot} D\alpha = 0$, this leads to

$$\operatorname{rot}\operatorname{sym} D\alpha = -\operatorname{rot}\operatorname{skw} D\alpha = -\operatorname{rot}\operatorname{skw}\tau.$$

Thus, from the split in symmetric and skew-symmetric parts one obtains

$$q = \operatorname{rot}(\operatorname{sym}\tau + \operatorname{skw}\tau) = \operatorname{rot}(\operatorname{sym}\tau - \operatorname{sym} D\alpha).$$

Hence, the assertion follows with $v := \operatorname{sym}\tau - \operatorname{sym} D\alpha$.

For the case $d = 3$, let $q \in \mathbf{Q}$. Then, by [25, Prop. A.1] or [1, Theorem 1.2] there exists $\tau \in H_0^1(\Omega; \mathbb{M}_{3\times 3})$ with $\operatorname{rot}\tau = q$ and $\|D\tau\|_{L^2(\Omega)} \lesssim \|q\|_{L^2(\Omega)}$. Abbreviate

$$\rho := \begin{pmatrix} \tau_{23} - \tau_{32} \\ \tau_{31} - \tau_{13} \\ \tau_{12} - \tau_{21} \end{pmatrix}.$$

Recall that $q \in \mathbf{Q}$ and, thus, $\operatorname{tr} q = 0$ almost everywhere in Ω. Thus, for any $\phi \in H_0^1(\Omega)$, the integration by parts plus elementary manipulations reveal, with the 3×3 identity matrix $I_{3\times 3}$, that

$$0 = (q, \phi I_{3\times 3})_{L^2(\Omega)} = (\operatorname{rot}\tau, \phi I_{3\times 3})_{L^2(\Omega)} = (\rho, D\phi)_{L^2(\Omega)}.$$

This means that $\operatorname{div}\rho = 0$ in Ω and, hence $\rho \in H_0(\operatorname{div}^0, \Omega) \cap H_0^1(\Omega; \mathbb{R}^3)$. Thus, by [25, Prop. A.1] or [1, Theorem 1.2] there exists $\alpha \in H_0^2(\Omega; \mathbb{R}^3)$ with $\operatorname{rot}\alpha = \rho$ and $\|\alpha\|_{H^2(\Omega)} \lesssim \|D\rho\|_{L^2(\Omega)}$. The representation (2.1) of $\mathbf{rot}$ therefore shows

$$2 \operatorname{skw} D\alpha = \mathbf{rot}\,\alpha = \begin{pmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{pmatrix} = -2 \operatorname{skw}\tau.$$

Since $\operatorname{rot}\operatorname{skw} D\alpha = -\operatorname{rot}\operatorname{sym} D\alpha$, we infer

$$\operatorname{rot}\tau = \operatorname{rot}\operatorname{sym}\tau + \operatorname{rot}\operatorname{skw}\tau = \operatorname{rot}\operatorname{sym}\tau + \operatorname{rot}\operatorname{sym} D\alpha.$$

The claim thus follows with the choice $v := \operatorname{sym}\tau + \operatorname{sym} D\alpha$. ☐

The saddle-point formulation seeks $(u, p) \in \mathbf{V} \times \mathbf{Q}$ such that

$$\begin{aligned} (Du, Dv)_{L^2(\Omega)} + (\operatorname{rot} v, p)_{L^2(\Omega)} &= (g, v)_{L^2(\Omega)} && \text{for all } v \in \mathbf{V}, \\ (\operatorname{rot} u, q)_{L^2(\Omega)} &= 0 && \text{for all } q \in \mathbf{Q}. \end{aligned} \tag{5.4}$$

Proposition 13 and the inclusion $\operatorname{rot}\mathbf{V} \subseteq \mathbf{Q}$ imply that (5.4) is equivalent to (5.3), provided the domain Ω is contractible for $d = 3$.

5.3. Finite elements for the generalized Stokes equation with symmetry.

This subsection presents finite elements for problem (5.4). Recall the notation for finite element spaces from §4.3.1.

5.3.1. *Continuous finite elements.* The finite element discretization of (5.4) is based on finite-dimensional subspaces $\mathbf{V}_h \subseteq \mathbf{V}$ and $\mathbf{Q}_h \subseteq \mathbf{Q}$ and seeks $(w_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ such that

$$(5.5)\qquad \begin{aligned} (Dw_h, Dv_h)_{L^2(\Omega)} + (\operatorname{rot} v_h, p_h)_{L^2(\Omega)} &= (g, v_h)_{L^2(\Omega)} &&\text{for all } v_h \in \mathbf{V}_h,\\ (\operatorname{rot} w_h, q_h)_{L^2(\Omega)} &= 0 &&\text{for all } q_h \in \mathbf{Q}_h. \end{aligned}$$

In two space dimensions, some stable Stokes elements can be used "row-wise", but the additional constraints of $\mathbf{V}$ and $\mathbf{Q}$ must be taken into account. The following choice serves as an example of how the established stabilized schemes can be adapted to this situation. Let, for any $T \in \mathcal{T}$, b_T denote the cubic bubble function that vanishes on ∂T and satisfies $\fint_T b_T\,dx = 1$. The space of symmetric tensors that are componentwise spanned by these functions is denoted by $\mathcal{B}_3(\mathcal{T}; \mathbb{S}_{2\times 2})$. As a generalization of the stabilized P_1 or MINI element [8] define the discrete spaces

$$\mathbf{V}_h := \mathbf{V} \cap (P_1(\mathcal{T}; \mathbb{S}_{2\times 2}) \oplus \mathcal{B}_3(\mathcal{T}; \mathbb{S}_{2\times 2})) \quad \text{and} \quad \mathbf{Q}_h := \mathbf{Q} \cap S^1(\mathcal{T}; \mathbb{R}^2).$$

"Row-wise" this is the MINI element, but with the symmetry constraint of $\mathbf{V}$ and the integral constraint of $\mathbf{Q}$.

Proposition 14. *Let $d = 2$. Then the choice $(\mathbf{V}_h, \mathbf{Q}_h)$ leads to a unique solution $(w_h, p_h) \in (\mathbf{V}_h, \mathbf{Q}_h)$ to* (5.5). *It satisfies the error estimate*

$$\|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathbf{V}_h} \|D(w - w_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p - p_h\|_{L^2(\Omega)}.$$

Proof. The proof follows the usual design of a suitable Fortin interpolation (see, e.g., [8, §4.8.2]). The details are omitted. ☐

For three dimensions, the stabilized finite element suggested in (4.8) can be generalized to the tensor case. Let $RT_0(\mathcal{T}; \mathbb{M}_{3\times 3})$ denote the space of 3×3 tensor fields whose rows belong to $RT_0(\mathcal{T})$. Furthermore, let $\mathcal{B}_3(\mathcal{F}; \mathbb{S}_{3\times 3})$ denote the space of symmetric 3×3 tensor fields whose rows belong to $\mathcal{B}_3(\mathcal{F}; \mathbb{R}^3)$ (for the definition, cf. §4.3.2). Define the following spaces of discrete functions

$$(5.6)\qquad \mathbf{V}_h := S^1_0(\mathcal{T}; \mathbb{S}_{3\times 3}) \oplus \mathcal{B}_3(\mathcal{F}; \mathbb{S}_{3\times 3}) \quad \text{and} \quad \mathbf{Q}_h := RT_0(\mathcal{T}; \mathbb{M}_{3\times 3}) \cap \mathbf{Q}.$$

Proposition 15. *Let $d = 3$ and let Ω be contractible. The system* (5.5) *with the choice* (5.6) *of discrete spaces has a unique solution. It satisfies the error estimate*

$$\|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim \inf_{v_h \in \mathbf{V}_h} \|D(w - w_h)\|_{L^2(\Omega)} + \inf_{q_h \in \mathbf{Q}_h} \|p - p_h\|_{L^2(\Omega)}.$$

Proof. The proof is analogous to that of Proposition 8. ☐

5.3.2. *Nonconforming P_1 element.* Similarly as in §4.3.3, the nonconforming P_1 finite element can be used for the saddle-point problem (5.4), where the trial functions from §4.3.3 are are taken row-wise and are combined with the additional symmetry and integral constraints on $\mathbf{V}$ and $\mathbf{Q}$. The analysis is analogous to the foregoing paragraph and §4.3.3. The details are omitted.

5.4. **Numerical methods for triharmonic problems.** Analogous to Subsection 4.4,the mixed system (5.2) can be approximated with standard finite elements for the Poisson equations (5.2a), (5.2b), (5.2c),(5.2d). The Stokes-like system (5.2c) is approximated with one of the methods described in Section 5.3. Let $\mathbf{U}_h \subseteq H^1_0(\Omega)$ be a finite-dimensional subspace and let $\mathbf{V}_h$, $\mathbf{Q}_h$ be one of the admissible pairings from §5.3.1. Then the discrete problem seeks

$$(u_{h,0}, u_{h,1}, w_h, p_h, r_{h,1}, r_{h,0}) \in \mathbf{U}_h \times [\mathbf{U}_h]^d \times \mathbf{V}_h \times \mathbf{Q}_h \times [\mathbf{U}_h]^d \times \mathbf{U}_h$$

such that

$$
\begin{aligned}
(Du_{h,0}, Dv_{h,0})_{L^2(\Omega)} &= (u_{h,1}, Dv_{h,0})_{L^2(\Omega)} && \text{for all } v_{h,0} \in \mathbf{U}_h,\\
(Du_{h,1}, Dv_{h,1})_{L^2(\Omega)} &= (w_h, Dv_{h,1})_{L^2(\Omega)} && \text{for all } v_{h,1} \in [\mathbf{U}_h]^d,\\
(Dw_h, D\xi_h)_{L^2(\Omega)} + (\operatorname{rot} \xi_h, p_h)_{L^2(\Omega)} &= (Dr_{h,1}, \xi_h)_{L^2(\Omega)} && \text{for all } \xi_h \in \mathbf{V}_h,\\
(\operatorname{rot} w_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in \mathbf{Q}_h,\\
(Dr_{h,1}, Ds_{h,1})_{L^2(\Omega)} &= (Dr_{h,0}, s_{h,1})_{L^2(\Omega)} && \text{for all } s_{h,1} \in [\mathbf{U}_h]^d,\\
(Dr_{h,0}, Ds_{h,0})_{L^2(\Omega)} &= (f, s_{h,0})_{L^2(\Omega)} && \text{for all } s_{h,0} \in \mathbf{U}_h.
\end{aligned}
$$

Standard a priori error estimates for the Poisson problem together with the triangle inequality and the a priori error estimates from Propositions 14 and 15 give quasi-optimal error estimates for the quantity

$$
\sum_{j=0}^{1} \Big(\|D(u_j - u_{h,j})\|_{L^2(\Omega)} + \|D(r_j - r_{h,j})\|_{L^2(\Omega)} \Big) + \|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)}.
$$

In particular, if the solution is smooth enough such that Du_0, Du_1 Dw, Dr_1, Dr_0, p have H^s regularity for some $0 < s \leq 1$ (which is the case if $u \in H^{3+s}(\Omega)$ and the Poisson equation in Ω has $H^{1+s}(\Omega)$ regularity), one obtains the convergence rate

$$
\sum_{j=0}^{1} \Big(\|D(u_j - u_{h,j})\|_{L^2(\Omega)} + \|D(r_j - r_{h,j})\|_{L^2(\Omega)} \Big) + \|D(w - w_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim h^s \|f\|_{L^2(\Omega)}
$$

for the maximal mesh-size h. With standard duality techniques, the following convergence rates in the L^2 norms can be proven

$$
\sum_{j=0}^{1} \Big(\|u_j - u_{h,j}\|_{L^2(\Omega)} + \|r_j - r_{h,j}\|_{L^2(\Omega)} \Big) + \|w - w_h\|_{L^2(\Omega)} \lesssim h^{2s} \|f\|_{L^2(\Omega)}.
$$

Analogous estimates can be obtained for the nonconforming discretization. Also, more general systems can be considered. Let, for example, $\delta \geq 0$ be a nonnegative number, $\boldsymbol{\gamma} \in \mathbb{S}_d(2)$ a symmetric and positive definite $d \times d$ matrix, $\mathbf{B} \in \mathbb{S}_d(4)$ be a symmetric and positive definite fourth-order tensor, and $\mathcal{A} \in \mathbb{S}_d(8)$ a symmetric eighth-order tensor. Consider, for given $f \in L^2(\Omega)$, the equation

$$
-\operatorname{div}^3 \mathcal{A} D^3 u + \operatorname{div}^2 \mathbf{B} D^2 u - \operatorname{div} \boldsymbol{\gamma} Du + \delta u = f \text{ in } \Omega
$$

subject to the boundary condition $u = \partial u / \partial \nu = \partial^2 u / \partial \nu^2 = 0$ on $\partial\Omega$. The weak solution $u \in H^3_0(\Omega)$ is characterized by

$$
\begin{aligned}
&(\mathcal{A} D^3 u, D^3 v)_{L^2(\Omega)} + (\mathbf{B} D^2 u, D^2 v)_{L^2(\Omega)}\\
&\qquad + (\boldsymbol{\gamma} Du, Dv)_{L^2(\Omega)} + (\delta u, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \qquad \text{for all } v \in H^3_0(\Omega).
\end{aligned}
$$

Following the lines of Section 3, this problem can be split into a symmetric mixed system and the finite element discretizations of Sections 5.3–5.4 yield quasi-optimal a priori error estimates.

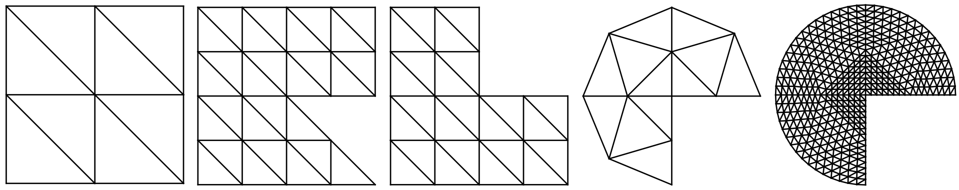

FIGURE 1. Initial meshes in 2D: the square, cusp, L, and sector domains. The last plot displays the triangulation of the sector domain after three refinements.

TABLE 1. Convergence history for the biharmonic equation on the square.

mesh-size h	$\|Dw_h - D^2u\|_{L^2(\Omega)}$	rate	$\|u_h - u\|_{L^2(\Omega)}$	rate
7.0711e-01	4.8604e-02	—	1.4042e-03	—
3.5355e-01	2.8576e-02	0.76	8.8575e-04	0.66
1.7678e-01	1.4373e-02	0.99	3.0992e-04	1.51
8.8388e-02	6.9823e-03	1.04	8.3667e-05	1.88
4.4194e-02	3.4409e-03	1.02	2.1225e-05	1.97
2.2097e-02	1.7098e-03	1.00	5.3134e-06	1.99
1.1049e-02	8.5257e-04	1.00	1.3272e-06	2.00
5.5243e-03	4.2576e-04	1.00	3.3152e-07	2.00

6. Numerical experiments

In this section, numerical computations with the new method are presented for fourth- and sixth-order problems. In all examples, uniform mesh-refinement is used. The mesh-size is quantified by the parameter $h := \max_{T\in\mathcal{T}} \operatorname{diam}(T)$. The rate α of algebraic convergence h^α for some error quantity error(j) (with respect to the mesh-size $h(j)$) in the j-th row of the table ($j \geq 2$) is computed with the formula

$$\alpha = \frac{\log(\operatorname{error}(j-1)/\operatorname{error}(j))}{\log(h(j-1)/h(j))}.$$

The domains and their initial partitions are displayed in Figure 1 for 2D and Figure 2 for 3D.

6.1. Biharmonic equation with smooth solution.

Consider the square domain $\Omega = (0,1)^2$ and the biharmonic equation (4.1) with the exact solution

$$u(x) = (x_1 - x_1^2)^2(x_2 - x_2^2)^2 \qquad \text{for } f = \Delta^2 u.$$

For the $H_0^1(\Omega)$ variables, standard P_1 conforming finite elements are employed. The variables in the Stokes-like system are approximated with the MINI finite element [8]. Table 1 displays the convergence history. For this smooth solution, the optimal convergence order of h and h^2 for the error $\|Dw_h - D^2u\|_{L^2(\Omega)}$ in the energy norm and the error $\|u_h - u\|_{L^2(\Omega)}$ in the L^2 norm, respectively, is observed.

6.2. Clamped plate.

Consider the equation

$$\Delta^2 u + u = f \quad \text{in } \Omega \quad \text{and} \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

in the the planar cusp domain $\Omega = (-1,1)^2 \setminus (\operatorname{conv}\{(0,0),(1,-1),(1,0)\})$. Define $\omega := 7\pi/4$ and $\alpha := 0.50500969$ as a noncharacteristic root of $\sin^2(\alpha\omega) = \alpha^2 \sin^2(\omega)$.

Table 2. Convergence history for the clamped plate with cusp.

mesh-size h	$\|Dw_h - D^2u\|_{L^2(\Omega)}$	rate	$\|u_h - u\|_{L^2(\Omega)}$	rate
7.0711e-01	3.0092e+00	—	1.6664e-01	—
3.5355e-01	1.8534e+00	0.69	8.1044e-02	1.04
1.7678e-01	1.0189e+00	0.86	2.6797e-02	1.59
8.8388e-02	5.7615e-01	0.82	8.2287e-03	1.70
4.4194e-02	3.4537e-01	0.73	2.7135e-03	1.60
2.2097e-02	2.1886e-01	0.65	1.0120e-03	1.42
1.1049e-02	1.4476e-01	0.59	4.2341e-04	1.25
5.5243e-03	9.8488e-02	0.55	1.9163e-04	1.14

The exact singular solution from [21, p. 107] reads in polar coordinates

$$u(r,\theta) = (r^2\cos^2\theta - 1)^2\,(r^2\sin^2\theta - 1)^2\, r^{1+\alpha}\, g(\theta)$$

for

$$\begin{aligned} g(\theta) = &\left[\frac{\sin((\alpha-1)\omega)}{\alpha-1} - \frac{\sin((\alpha+1)\omega)}{\alpha+1}\right]\big(\cos((\alpha-1)\theta) - \cos((\alpha+1)\theta)\big)\\ &- \left[\frac{\sin((\alpha-1)\theta)}{\alpha-1} - \frac{\sin((\alpha+1)\theta)}{\alpha+1}\right]\big(\cos((\alpha-1)\omega) - \cos((\alpha+1)\omega)\big). \end{aligned} \tag{6.1}$$

The right-hand side f is computed according to the exact solution. The derivation of the mixed system follows Subsection 4.5 with the identity tensor $\mathbf{B}$ and $\boldsymbol{\gamma} = 0$, $\delta = 1$. As in the previous example, standard P_1 conforming finite elements are used for the $H^1_0(\Omega)$ variables and the MINI finite element is used for the Stokes-like system. The convergence history of the energy error $\|Dw_h - D^2u\|_{L^2(\Omega)}$ and the L^2 error $\|u_h - u\|_{L^2(\Omega)}$ is displayed in Table 2. The solution is not smoother than $H^{2+\alpha}(\Omega)$ [6, 26] and the expected convergence order for the energy error is $h^{0.505}$. This rate is approached by the values in Table 2. The error in the L^2 norm is observed to decrease with the double rate.

6.3. **Vibrations of a simply-supported plate.** Consider the planar L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1]^2$. The fundamental frequency of a vibrating thin elastic plate subject to simply-supported boundary conditions is described by the first eigenvalue λ of

$$\Delta^2 u = \lambda u \quad \text{in } \Omega \quad \text{and} \quad u = \Delta u = 0 \quad \text{on } \partial\Omega.$$

The weak formulation seeks $u \in H^1_0(\Omega) \cap H^2(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$(D^2u, D^2v)_{L^2(\Omega)} = \lambda (u,v)_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega) \cap H^2(\Omega). \tag{6.2}$$

We proceed with a brief description of how to adapt the functional setting of Section 4 to this boundary condition. While u and r from (4.2) belong to $H^1_0(\Omega)$ as in prior sections, $Z_{1,1}$ from (4.2b) is replaced by

$$Z^t_{1,1} := \{v \in H^1_t(\Omega;\mathbb{R}^d) : \operatorname{rot} v = 0\},$$

where $H^1_t(\Omega;\mathbb{R}^d)$ denotes the space of $H^1(\Omega)$ vector fields with vanishing tangential trace on $\partial\Omega$. For the verification that, in generalization of Lemma 1, this equals the space of gradients of $H^1_0(\Omega) \cap H^2(\Omega)$, let $v \in Z_{1,1}$. Since $\operatorname{rot} v = 0$, it satisfies $v = D\eta$ for some $\eta \in H^2(\Omega)$. The tangential boundary condition on v assures that the restriction of η to $\partial\Omega$ equals a constant, which can be chosen to be zero; hence $\eta \in H^1_0(\Omega) \cap H^2(\Omega)$. With this modification, the splitting (4.2) and consequently its

TABLE 3. Convergence history for the first eigenvalue of the simply-supported L-shaped plate. The reference value is $\lambda = 163.731$.

mesh-size h	λ_h	error $\frac{\lvert\lambda-\lambda_h\rvert}{\lambda}$	rate	error $\frac{\lvert\lambda_{h/2}/\lambda_h\rvert}{\lambda_{h/2}}$	rate
7.0711e-01	3.7632e+02	1.2984e+00	—	7.3878e-01	—
3.5355e-01	2.1643e+02	3.2185e-01	2.0123	1.9461e-01	1.92
1.7678e-01	1.8117e+02	1.0651e-01	1.5954	5.9931e-02	1.69
8.8388e-02	1.7093e+02	4.3946e-02	1.2772	2.1696e-02	1.46
4.4194e-02	1.6730e+02	2.1777e-02	1.0129	9.4819e-03	1.19
2.2097e-02	1.6573e+02	1.2180e-02	0.8383	4.8714e-03	0.96
1.1049e-02	1.6492e+02	7.2731e-03	0.7438	2.7746e-03	0.81
5.5243e-03	1.6447e+02	4.4861e-03	0.6971	—	—

equivalence with (6.2) in the sense of Proposition 4 can be proven. For a formulation as a saddle-point problem, choose $\mathbf{Q}$ as in Subsection 4.3 and $\mathbf{V} := H^1_t(\Omega;\mathbb{R}^d)$. The inf-sup condition directly follows from Proposition 5 because the space $\mathbf{V}$ therein is a subset of $H^1_t(\Omega;\mathbb{R}^d)$ and the range of $\operatorname{rot}(H^1_t(\Omega;\mathbb{R}^d))$ equals $\mathbf{Q}$ (due to the tangential boundary condition). With these modifications, the eigenvalue problem corresponds to the system from Subsection 4.5 where $\mathbf{B}$ is the identity, $\boldsymbol{\gamma} = 0$ and $\delta = 0$, while $f = \lambda u$. As in the previous example, standard P_1 finite elements are combined with the MINI element. The simply-supported boundary condition is realized by demanding homogeneous Dirichlet boundary conditions for u and for the tangential component of w on $\partial\Omega$.

A reference value $\lambda = 163.731$ for the first eigenvalue was computed with the adaptive Morley finite element method, which is known to converge at optimal order [20]. For comparison we mention the eigenvalue $\hat\lambda = 2\,663.3927$ obtained in [11] with a discontinuous Galerkin method on the domain $\widehat\Omega = (0,1)^2 \setminus ([1/2,1] \times [0,1/2])$, which after scaling results in the reference $\lambda = 166.46$. The relative error between these values is less than 2%.

The convergence history of the first eigenvalue is displayed in Table 3. In this case, the solution is not smoother than $H^{7/3}(\Omega)$ [6, 26] and the expected convergence order for the energy error is $h^{1/3}$. Accordingly, the convergence order of the eigenvalue error approaches $h^{2/3}$. This doubled rate of convergence is typical for symmetric eigenvalue problems; see, e.g., [7, 30]. Table 3 also displays the convergence of an alternative error quantity (used in [11]) where the reference value is $\lambda_{h/2}$, the computed eigenvalue on the uniformly refined grid. It cannot be expected in general that the first eigenfunction is smoother than $H^{7/3}(\Omega)$. Indeed, this is a setting where some of the conventional mixed methods do not converge; see the examples in [11]. However, the method in this paper is stable also in the case of low smoothness.

6.4. **Tri-Laplacian on the square domain.** Let $\Omega = (0,1)^2$ be the unit square and consider the sixth-order problem (5.1) with exact solution [22]

$$u(x) = (x_1^2 + x_2^2)^{\frac{7.1}{4}}(x_1 - x_1^2)^3(x_2 - x_2^2)^3.$$

The Poisson-type problems (5.2a), (5.2b), (5.2d), (5.2e) are approximated with standard conforming P_1 finite elements and the Stokes-type system (5.2c) is discretized with the MINI element from §5.3.1. The convergence history of the energy

TABLE 4. Convergence history for the Tri-Laplacian on the square domain.

mesh-size h	$\|Dw_h - D^3u\|_{L^2(\Omega)}$	rate	$\|u_h - u\|_{L^2(\Omega)}$	rate
7.0711e-01	3.8378e-02	—	8.9477e-05	—
3.5355e-01	3.3265e-02	0.20	6.9306e-05	0.36
1.7678e-01	1.9384e-02	0.77	3.6861e-05	0.91
8.8388e-02	9.3635e-03	1.04	1.2082e-05	1.60
4.4194e-02	4.4908e-03	1.06	3.2260e-06	1.90
2.2097e-02	2.2048e-03	1.02	8.1679e-07	1.98
1.1049e-02	1.0946e-03	1.01	2.0443e-07	1.99
5.5243e-03	5.4572e-04	1.00	5.1071e-08	2.00

TABLE 5. Convergence history for the Tri-Laplacian on the sector domain.

mesh-size h	$\|Dw_h - D^3u\|_{L^2(\Omega)}$	rate	$\|u_h - u\|_{L^2(\Omega)}$	rate
7.6537e-01	2.4008e+00	—	9.4515e-03	—
4.2033e-01	1.6271e+00	0.64	6.5989e-03	0.59
2.2193e-01	9.2028e-01	0.89	2.8347e-03	1.32
1.1373e-01	5.2651e-01	0.83	9.3696e-04	1.65
5.7536e-02	3.1540e-01	0.75	2.9592e-04	1.69
2.8933e-02	1.9484e-01	0.70	9.7135e-05	1.62
1.4507e-02	1.2196e-01	0.67	3.4079e-05	1.51
7.2637e-03	7.6700e-02	0.67	1.2786e-05	1.41
3.6344e-03	4.8305e-02	0.66	5.0664e-06	1.33

error $\|Dw_h - D^3u\|_{L^2(\Omega)}$ and the L^2 error $\|u_h - u\|_{L^2(\Omega)}$ is displayed in Table 4. The exact solution does not belong to $H^5(\Omega)$. However, the H^4 regularity implies the convergence order h for this first-order scheme, as confirmed by the numerical experiment. The error in the L^2 norm is observed to converge with order h^2.

6.5. **Tri-Laplacian on the sector domain.** Let $\Omega = \{(r,\theta) : r \in (0,1), \theta \in (0, 3\pi/2)\}$ be the sector domain (defined in polar coordinates (r,θ)) and consider the sixth-order problem (5.1) with the exact solution given in polar coordinates as

$$u(r,\theta) = r^{8/3}(1-r)^3 \sin(2\theta/3)^3.$$

As in the previous example, standard conforming P_1 finite elements and the MINI element from §5.3.1 are used. The curved boundary is approximated with polygonal domains. The convergence history is displayed in Table 5. The exact solution has regularity $H^{11/3}(\Omega)$ and the expected convergence order is $h^{2/3}$ for the energy error and $h^{4/3}$ for the L^2 error. The observed convergence rates correspond to these predictions.

6.6. **Bi-Laplacian on the cube domain.** Let $\Omega = (0,1)^3$ be the unit cube. Consider the biharmonic equation (4.1) with exact solution

$$u(x) = \prod_{j=1}^{3} (x_j(1-x_j))^2.$$

This problem is approximated with the stabilized P_1 finite element from (4.8) for (4.2b), while (4.2a) and (4.2c) are discretized with standard P_1 finite elements. Table 6 displays the convergence history of the energy error $\|Dw_h - D^2u\|_{L^2(\Omega)}$ and

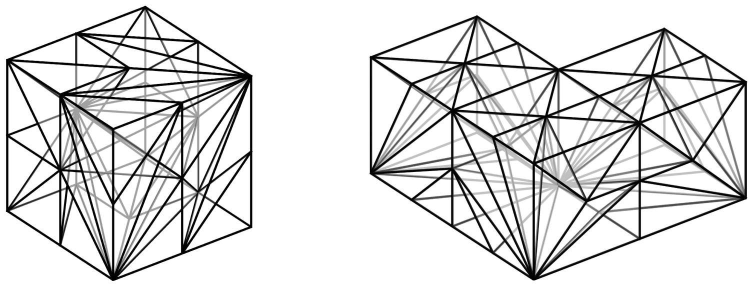

FIGURE 2. Initial meshes in 3D: cube and tensor product L-domain.

TABLE 6. Convergence history for the Bi-Laplacian on the cube domain.

mesh-size h	$\|Dw_h - D^2u\|_{L^2(\Omega)}$	rate	$\|u_h - u\|_{L^2(\Omega)}$	rate
1.2247e+00	3.0507e-03	—	6.3507e-05	—
7.9057e-01	2.7227e-03	0.25	6.0334e-05	0.11
3.9528e-01	1.7855e-03	0.60	3.5170e-05	0.77
1.9764e-01	9.2940e-04	0.94	1.2177e-05	1.53
9.8821e-02	4.6011e-04	1.01	3.2859e-06	1.88

TABLE 7. Convergence history for the Bi-Laplacian on the tensor product L-domain.

mesh-size h	$\|Dw_h - D^2u\|_{L^2(\Omega)}$	rate	$\|u_h - u\|_{L^2(\Omega)}$	rate
1.4142e+00	7.2084e-01	—	2.1545e-02	—
7.9057e-01	6.5507e-01	0.16	1.9770e-02	0.14
3.9528e-01	4.2226e-01	0.63	1.0651e-02	0.89
1.9764e-01	2.2298e-01	0.92	3.4186e-03	1.63
9.8821e-02	1.1287e-01	0.98	9.1808e-04	1.89

the L^2 error $\|u_h - u\|_{L^2(\Omega)}$. The convergence order for the energy error is h. The observed convergence order for the L^2 error is between h and h^2.

6.7. **Bi-Laplacian on the tensor product L-domain.** Let

$$\Omega = \Big((-1,1)^2 \setminus ([0,1] \times [-1,0])\Big) \times (0,1)$$

be the tensor product of a planar L-shaped domain with the unit interval. Consider the biharmonic equation (4.1). Define $\omega := 3\pi/2$ and $\alpha := 0.5444837$ as a noncharacteristic root of $\sin^2(\alpha\omega) = \alpha^2 \sin^2(\omega)$. The exact singular solution is given in cylindrical coordinates by

$$u(r,\theta,z) = (z - z^2)^2\,(r^2\cos^2\theta - 1)^2\,(r^2\sin^2\theta - 1)^2\,r^{1+\alpha}\,g(\theta)$$

for the function g from (6.1). Again, the stabilized P_1 finite element is combined with standard P_1 finite elements. The exact solution is not smoother than $H^{2+\alpha}(\Omega)$ and the expected convergence order is $h^{0.5444}$ for the energy error and $h^{1.0888}$ for the error in the L^2 norm. The empirical rates from Table 7 are slightly higher, which may be a pre-asymptotic phenomenon.

7. Conclusive remarks

(a) The new splitting of polyharmonic equations presented in this work is stable, irrespective of the regularity of the solution. This stands in stark contrast with traditional mixed formulations which require additional regularity for a stable splitting. Therefore, the new method is also suitable for problems where the solution has only low regularity; see, e.g., the simply-supported plate problem from Subsection 6.3.
(b) The new mixed formulation also covers the case of more general polyharmonic operators involving lower-order terms; see the outline in Subsection 4.5 and the numerical experiments.
(c) The new formulation allows a finite element approximation of polyharmonic functions with lowest-order standard finite elements. The computer implementation is straightforward. The three-dimensional formulation requires divergence-free functions for the pressure-like variable. This side constraint can be strongly incorporated with Raviart-Thomas finite elements.
(d) The numerical experiments show that the pre-asymptotic regime is small. This suggests that the constants in the a priori error estimates are of moderate size.

References

[1] C. Amrouche, P. G. Ciarlet, and P. Ciarlet Jr., *Vector and scalar potentials, Poincaré's theorem and Korn's inequality*, C. R. Math. Acad. Sci. Paris **345** (2007), no. 11, 603–608, DOI 10.1016/j.crma.2007.10.020. MR2371475
[2] C. Amrouche and V. Girault, *Problèmes généralisés de Stokes*, Portugal. Math. **49** (1992), no. 4, 463–503. MR1300580
[3] D. N. Arnold and F. Brezzi, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, RAIRO Modél. Math. Anal. Numér. **19** (1985), no. 1, 7–32. MR813687
[4] G. A. Baker, *Finite element methods for elliptic equations using nonconforming elements*, Math. Comp. **31** (1977), no. 137, 45–59. MR0431742
[5] J. W. Barrett, S. Langdon, and R. Nürnberg, *Finite element approximation of a sixth order nonlinear degenerate parabolic equation*, Numer. Math. **96** (2004), no. 3, 401–434, DOI 10.1007/s00211-003-0479-4. MR2028722
[6] H. Blum and R. Rannacher, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Methods Appl. Sci. **2** (1980), no. 4, 556–581, DOI 10.1002/mma.1670020416. MR595625
[7] D. Boffi, *Finite element approximation of eigenvalue problems*, Acta Numer. **19** (2010), 1–120, DOI 10.1017/S0962492910000012. MR2652780
[8] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer Series in Computational Mathematics, vol. 44, Springer, Heidelberg, 2013. MR3097958
[9] D. Braess, *Finite Elements: Theory, Fast Solvers, and Applications in Elasticity Theory*, 3rd ed., Cambridge University Press, Cambridge, 2007. MR2322235
[10] S. C. Brenner, *C^0 interior penalty methods*, Frontiers in numerical analysis—Durham 2010, Lect. Notes Comput. Sci. Eng., vol. 85, Springer, Heidelberg, 2012, pp. 79–147, DOI 10.1007/978-3-642-23914-4_2. MR3051409
[11] S. C. Brenner, P. Monk, and J. Sun, *C^0 interior penalty Galerkin method for biharmonic eigenvalue problems*, Spectral and High Order Methods for Partial Differential Equations ICOSAHOM 2014 (R. M. Kirby, M. Berzins, and J. S. Hesthaven, eds.), Lecture Notes in Computational Science and Engineering, vol. 106, Springer International Publishing, 2015, pp. 3–15.
[12] F. Brezzi and M. Fortin, *Numerical approximation of Mindlin-Reissner plates*, Math. Comp. **47** (1986), no. 175, 151–158, DOI 10.2307/2008086. MR842127

[13] F. Brezzi, M. Fortin, and R. Stenberg, *Error analysis of mixed-interpolated elements for Reissner-Mindlin plates*, Math. Models Methods Appl. Sci. **1** (1991), no. 2, 125–151, DOI 10.1142/S0218202591000083. MR1115287

[14] C. Carstensen, K. Köhler, D. Peterseim, and M. Schedensack, *Comparison results for the Stokes equations*, Appl. Numer. Math. **95** (2015), 118–129, DOI 10.1016/j.apnum.2013.12.005. MR3349689

[15] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4. MR0520174

[16] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **7** (1973), no. R-3, 33–75. MR0343661

[17] P. Destuynder and T. Nevers, *Une modification du modèle de Mindlin pour les plaques minces en flexion présentant un bord libre*, RAIRO Modél. Math. Anal. Numér. **22** (1988), no. 2, 217–242. MR945123

[18] G. Engel, K. Garikipati, T. J. R. Hughes, M. G. Larson, L. Mazzei, and R. L. Taylor, *Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity*, Comput. Methods Appl. Mech. Engrg. **191** (2002), no. 34, 3669–3750, DOI 10.1016/S0045-7825(02)00286-4. MR1915664

[19] D. Gallistl, *Adaptive nonconforming finite element approximation of eigenvalue clusters*, Comput. Methods Appl. Math. **14** (2014), no. 4, 509–535, DOI 10.1515/cmam-2014-0020. MR3259027

[20] D. Gallistl, *Morley finite element method for the eigenvalues of the biharmonic operator*, IMA J. Numer. Anal. **35** (2015), no. 4, 1779–1811, DOI 10.1093/imanum/dru054. MR3407244

[21] P. Grisvard, *Singularities in Boundary Value Problems*, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], vol. 22, Masson, Paris; Springer-Verlag, Berlin, 1992. MR1173209

[22] T. Gudi and M. Neilan, *An interior penalty method for a sixth-order elliptic equation*, IMA J. Numer. Anal. **31** (2011), no. 4, 1734–1753, DOI 10.1093/imanum/drq031. MR2846773

[23] J. R. King, *The isolation oxidation of silicon: the reaction-controlled case*, SIAM J. Appl. Math. **49** (1989), no. 4, 1064–1080, DOI 10.1137/0149064. MR1005497

[24] P. Lascaux and P. Lesaint, *Some nonconforming finite elements for the plate bending problem*, Rev. Française Automat. Informat. Recherche Operationnelle Sér. Rouge Anal. Numér. **9** (1975), no. R-1, 9–53. MR0423968

[25] Z. Lou and A. McIntosh, *Hardy space of exact forms on* $\mathbb{R}^N$, Trans. Amer. Math. Soc. **357** (2005), no. 4, 1469–1496, DOI 10.1090/S0002-9947-04-03535-4. MR2115373

[26] H. Melzer and R. Rannacher, *Spannungskonzentrationen in Eckpunkten der Kirchhoffschen Platte*, Bauingenieur **55** (1980), 181–184.

[27] M. Schedensack, *A class of mixed finite element methods based on the Helmholtz decomposition*, Oberwolfach Reports **12** (2015), no. 3, 2555–2556.

[28] M. Schedensack, *A class of mixed finite element methods based on the Helmholtz decomposition in computational mechanics*, Doctoral dissertation, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2015.

[29] M. Schedensack, *A new discretization for mth-Laplace equations with arbitrary polynomial degrees*, SIAM J. Numer. Anal. **54** (2016), no. 4, 2138–2162, DOI 10.1137/15M1013651. MR3519555

[30] G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1973. Prentice-Hall Series in Automatic Computation. MR0443377

[31] M. Wang and J. Xu, *Minimal finite element spaces for* $2m$*-th-order partial differential equations in* R^n, Math. Comp. **82** (2013), no. 281, 25–43, DOI 10.1090/S0025-5718-2012-02611-1. MR2983014

[32] A. Ženíšek, *Interpolation polynomials on the triangle*, Numer. Math. **15** (1970), 283–296. MR0275014

Institut für Angewandte und Numerische Mathematik, Karlsruher Institut für Technologie, Englerstr. 2, 76131 Karlsruhe, Germany