ERROR ESTIMATES FOR THE AEDG METHOD TO ONE-DIMENSIONAL LINEAR CONVECTION-DIFFUSION EQUATIONS

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ABSTRACT. We study the error estimates for the alternating evolution discontinuous Galerkin (AEDG) method to one-dimensional linear convection-diffusion equations. The AEDG method for general convection-diffusion equations was introduced by H. Liu and M. Pollack [J. Comp. Phys. 307 (2016), 574–592], where stability of the semi-discrete scheme was rigorously proved for linear problems under a CFL-like stability condition $\epsilon < Qh^2$. Here ϵ is the method parameter, and h is the maximum spatial grid size. In this work, we establish optimal L^2 error estimates of order $O(h^{k+1})$ for k-th degree polynomials, under the same stability condition with $\epsilon \sim h^2$. For a fully discrete scheme with the forward Euler temporal discretization, we further obtain the L^2 error estimate of order $O(\tau + h^{k+1})$, under the stability condition $c_0\tau \leq \epsilon < Qh^2$ for time step τ; and an error of order $O(\tau^2 + h^{k+1})$ for the Crank-Nicolson time discretization with any time step τ. Key tools include two approximation spaces to distinguish overlapping polynomials, two bi-linear operators, coupled global projections, and a duality argument adapted to the situation with overlapping polynomials.

1. INTRODUCTION

In this paper, we present a priori error estimates for the alternating evolution discontinuous Galerkin (AEDG) method to the linear convection-diffusion equations

(1.1a) $$\partial_t \phi + \alpha \partial_x \phi = \beta \partial_x^2 \phi, \quad (x,t) \in [a,b] \times (0,T),$$

(1.1b) $$\phi(x,0) = \phi_0(x), \quad x \in [a,b];$$

here $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$ are given constants. We do not pay attention to boundary conditions in this paper, hence the solution is considered to be periodic.

The idea using the alternating evolution (AE) system as a numerical device began in [16] and has been elaborated further in [18,29] using high resolution finite volume and finite difference approximations, respectively. The AEDG method is a grid-based discontinuous Galerkin (DG) method, which was introduced by Liu and Pollack first in [19] for Hamilton-Jacobi equations, and further developed in [20] for nonlinear convection-diffusion equations,

(1.2) $$\partial_t \phi + \nabla_x \cdot f(\phi) = \Delta_x a(\phi),$$

in one- and multi-dimensional settings, where $f(\phi)$ is a given flux function and $a(\phi)$ is a nondecreasing function.

A distinct advantage of AE schemes is that no numerical fluxes are needed in the scheme formulation, instead, the communication with neighboring solution representatives ϕ^{SN} is achieved through an AE formulation

$$\partial_t \phi + \nabla_x \cdot f(\phi^{SN}) = \Delta a(\phi^{SN}) + \frac{1}{\epsilon}(\phi^{SN} - \phi)$$

for the convection-diffusion equation (1.2). Here the abbreviation "SN" stands for "sampling from neighbors" in the sense that $\psi = \phi^{SN}$ will be sampled from neighboring polynomials during the spatial discretization (see [20]). The scheme construction is based on sampling this AE formulation by a polynomial representative near each grid point, and it is carried out by allowing the neighboring polynomials to overlap. It is similar to the central DG methods [23, 24] in the sense that whenever a spatial derivative is evaluated, the neighboring polynomials (or the other representatives in the central DG schemes) are used as in [22]. However, the AEDG method involves only one approximating polynomial near each grid point, independent of the spatial dimension, hence providing high order approximations near each grid point.

The semi-discrete scheme ([20]) in the one-dimensional case thus has the following form:

$$\int_{I_j} (\partial_t \Phi_j + \partial_x f(\Phi_j^{SN}) - \partial_x^2 a(\Phi_j^{SN}))\eta dx$$
$$= \left(-[f(\Phi_j^{SN})]\eta + [\partial_x a(\Phi_j^{SN})]\eta - [a(\Phi_j^{SN})]\partial_x \eta\right)\Big|_{x=x_j}$$
$$+ \frac{1}{\epsilon}\left(\int_{I_j} \Phi_j^{SN}\eta dx - \int_{I_j} \Phi_j \eta dx\right),$$

where x_j is the grid point in cell I_j, in which numerical solution is denoted by Φ_j; Φ_j^{SN} are sampled from neighboring polynomials $\Phi_{j\pm 1}$, with $[g(\Phi_j^{SN})]|_{x_j}$ standing for the difference of two neighboring functions at x_j in the sense that $[g(\Phi_j^{SN})]|_{x_j} = g(\Phi_{j+1}(x_j^+)) - g(\Phi_{j-1}(x_j^-))$. The initial condition is taken as the L^2 projection of the initial condition into the relevant finite element space.

The AEDG scheme is shown to be consistent and conservative. Yet the stability analysis is subtle since the stability property is not obvious from the scheme formulation. For linear convection-diffusion equations (1.1), the L^2 stability of the semi-discrete AEDG method has been proven if $\epsilon \leq Qh^2$, for some Q and mesh size h, while the technical difficulty was resolved in [20] by a special regrouping of mixed terms combined with the use of some inverse inequalities.

The main objective of this work is to obtain the optimal error estimates in L^2 norm based on the stability results established in [20] for the semi-discrete AEDG scheme (2.2). The main result states as follows: for piecewise k-th degree polynomials, if $\epsilon = cQh^2$ with $c \in (0, 1)$, then a priori estimate for the error between the exact smooth solution ϕ and the numerical solution $\Phi_j(x)$ is obtained as

$$\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{|\Phi_{j+1}(x,t) - \phi(x,t)|^2 + |\Phi_j(x,t) - \phi(x,t)|^2}{2} dx \leq Ch^{2k+2},$$

where C is linear in the final time T. This differs from the usual L^2 error since the AEDG method uses overlapping polynomials. These features require new techniques in the error estimate.

In order to distinguish the overlapping polynomials we introduce two approximation spaces $V_h \times U_h$ associated with odd and even grids, respectively, with which the AE scheme is reformulated using two bi-linear operators. The essential novel tool is the two global projections on V_h and U_h, coupled through the ϵ-dependent term dictated by the AEDG formulation. The two coupled projections are shown to be well defined for $\epsilon \leq Qh^2$, which is the sufficient condition for the L^2 stability of the semi-discrete AEDG scheme (see [20]). The optimal L^2 error estimate follows from both the stability estimate and the projection error.

The main task goes to the estimate of the projection error, which is carried out in two steps: First we introduce a novel energy norm of $(v, u) \in V_h \times U_h$, involving a special term of the form $h^{-1}\|u-v\|$, with which we are able to obtain the projection error of order $O(h^k)$ in this energy norm. This estimate already implies the optimal L^2 error of order $O(h^{k+1})$ for the difference of two projection errors. For the sum of two projection errors, we obtain the optimal L^2 error estimate using a duality argument carefully adapted to the case with overlapping polynomials, these together lead to the desired optimal projections errors.

We further investigate the fully discrete scheme with the forward Euler discretization. The stability condition relating ϵ to the time step τ is of the form $c_0 \tau \leq \epsilon < Qh^2$ for some $c_0 > 1$, under which and with an additional constraint on τ the optimal error estimate is established as

$$\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{|\Phi_{j+1}^n(x) - \phi(x,t^n)|^2 + |\Phi_j^n(x) - \phi(x,t^n)|^2}{2} dx \leq C(\tau + h^{k+1})^2,$$

where C is linear in the final time T; again the two global projections and the upper bounds of the corresponding bilinear operators are essentially used. These are the first error estimate results obtained for the AEDG schemes. The main techniques introduced herein may be applied to AEDG schemes in other applications.

We now mention some related results on the *a priori* error estimates for several DG methods when applied to convection-diffusion equations. For smooth solutions of scalar conservation laws, the L^2 error estimate of $O(h^{k+1/2})$ can be obtained for the most general situation [14, 26]. However in many cases the optimal $O(h^{k+1})$ error bound can be proved [11, 15, 27]. For linear convection-diffusion problems, the LDG method introduced in [12], motivated by the work of Bassi and Rebay [4] for the compressible Navier-Stokes equations, when taking the alternating numerical fluxes is shown to be stable and convergent, with the optimal $O(h^{k+1})$ error order in the L^2 norm as proved in [7,8], further proved in [34] for equations with nonlinear convection; a recent advance in the error analysis of the LDG method for linear convection-diffusion equations is found in [9]. For the symmetric interior penalty (SIPG) method, it can be proved that for large enough penalty parameter, the method is stable and has optimal $O(h^{k+1})$ order convergence in L^2 [2, 31]. For the nonsymmetric interior penalty (NIPG) method of Baumann and Oden [5, 25], it is stable and convergent, with a suboptimal $O(h^k)$ order of L^2 errors for even k; however, the optimal error estimate for quadratic polynomials was obtained by Riviere and Wheeler [28] when applied to nonlinear convection-diffusion equations. Suboptimal L^2 error estimates are given in [10] for the so-called ultra weak DG method introduced therein. For the direct DG method introduced in [21], the optimal $O(h^{k+1})$ error bound in L^2 was recently obtained in [17] using a special global projection, dictated by the form of the DDG numerical fluxes. For a unified

error analysis of a class of DG methods when applied to the elliptic problem, we refer
to [3]. For error estimates of fully discrete DG schemes to hyperbolic conservation
laws with third order Runge-Kutta time discretization we refer to [35, 36]. The error
estimate for the fully discrete DG algorithm to solve convection-diffusion equations
is more recent; see, e.g., [32, 33] for the LDG method coupled with a third order
Runge-Kutta time discretization. We would like to mention that the AEDG method
is more complicated to analyze than other DG methods in this context, because
the coupling between overlapping polynomials must be carefully handled.

The article is organized as follows: In section 2 we present both the semi-
discrete and fully discrete AEDG schemes for the one-dimensional linear convection-
diffusion equations, and the main results of error estimates. In section 3 we refor-
mulate the semi-discrete AEDG scheme as a coupled system and define the two
global projections. We further derive the error equations of the coupled system
and prove the main convergence result. In section 4, we derive the projection error
in energy norm, and lift it to L^2 norm by a duality argument. Stability and error
analysis of the fully discrete AEDG method will be presented in section 5.

Throughout this paper, we adopt standard notation for Sobolev spaces such as
$W^{m,p}(D)$ on subdomain $D \subset [a, b]$ equipped with the norm $\| \cdot \|_{m,p,D}$ and semi-
norm $| \cdot |_{m,p,D}$. When $D = [a, b]$, we omit the index D; and if $p = 2$, we set
$W^{m,p}(D) = H^m(D)$, $\| \cdot \|_{m,p,D} = \| \cdot \|_{m,D}$, and $| \cdot |_{m,p,D} = | \cdot |_{m,D}$. We use either
$\|\cdot\|_{0,D}$ or $\|\cdot\|$ when $D = [a, b]$ denote the usual L^2 norm. We use the notation $A \lesssim B$
to indicate that A can be bounded by B multiplied by a constant independent of
the mesh size h. $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$. We will also use C to denote
a positive constant independent of h, which may depend on solutions of (1.1).

2. Alternating evolution DG methods and main results

We recall the AEDG method for the one-dimensional convection diffusion equa-
tion

$$(2.1) \qquad \partial_t \phi + \alpha \partial_x \phi = \beta \partial_x^2 \phi,$$

subject to initial data $\phi_0(x)$ and periodic boundary conditions.

Let the spatial domain $[a, b]$ be partitioned into a grid with grid points $\{x_j\}$ such
that $x_1 = a, x_N = b$. We set $I_j = (x_{j-1}, x_{j+1})$ for $j = 1, 2, \ldots, N - 1$, while $I_1 =
(x_0, x_2)$ in which (x_0, x_1) is the periodic shift of (x_{N-1}, x_N) and $h_j = \frac{x_{j+1} - x_{j-1}}{2}$,
and we define the quantities

$$h = \max_{1 \le j \le N-1} h_j \text{ and } \rho = \min_{1 \le j \le N-1} h_j.$$

For simplicity of presentation we would like to assume that the ratio of h and ρ
is upper bounded by a fixed positive constant ν^{-1} when h goes to zero so that
$\nu h \le \rho \le h$. We shall analyze the uniform grid case $\nu = 1$, knowing that the
techniques can be easily carried over to the case $\nu \ne 1$.

Centered at each grid $\{x_j\}$, the numerical approximation is a polynomial $\Phi|_{I_j} =
\Phi_j(x) \in P^k$, where P^k denotes a linear space of all polynomials of degree at most k:

$$P^k := \{p \mid p(x)|_{I_j} = \sum_{0 \le i \le k} a_i (x - x_j)^i, \quad a_i \in \mathbb{R}\}.$$

We denote $v(x^\pm) = \lim_{\epsilon \to 0\pm} v(x + \epsilon)$, and $v_j^\pm = v(x_j^\pm)$. The jump at x_j is $[v]|_{x_j} =
v(x_j^+) - v(x_j^-)$. Note that the solution space here differs from the usual finite

element space since it allows the overlapping of two neighboring polynomials of Φ_j and Φ_{j+1} over $I_j \cap I_{j+1} = [x_j, x_{j+1}] \neq \emptyset$.

The semi-discrete AEDG scheme introduced in [20] is to find $\Phi|_{I_j} \in P^k$ such that for all $\eta \in P^k(I_j)$,

(2.2)
$$\int_{I_j} (\partial_t \Phi_j + \partial_x(\alpha \Phi_j^{SN} - \beta \partial_x \Phi_j^{SN})\eta \, dx$$
$$= \left(-[\alpha \Phi_j^{SN} - \beta \partial_x \Phi_j^{SN}]\eta - \beta[\Phi_j^{SN}]\partial_x \eta \right) \Big|_{x=x_j}$$
$$+ \frac{1}{\epsilon} \int_{I_j} \left(\Phi_j^{SN} - \Phi_j \right) \eta \, dx,$$

where Φ_j^{SN} is defined as

$$\Phi_j^{SN} = \begin{cases} \Phi_{j-1}(x), & x_{j-1} < x < x_j, \\ \Phi_{j+1}(x), & x_j < x < x_{j+1}. \end{cases}$$

With periodic boundary conditions, $\Phi_N(x)$ is regarded to be identical to $\Phi_1(x)$, which is computed over $I_1 = [x_0, x_2] = [a - h, a + h]$. The numerical solution on $[x_{N-1}, x_N]$ is simply taken from Φ_1 over $[x_0, x_1]$. Note that $\Phi_1(x, 0) = \Phi_N(x, 0)$ for initial data.

The initial data for $\Phi_j(x, 0)$ is taken as the L^2 projection of ϕ_0 on I_j for $j = 1, \cdots, N - 1$:

(2.3) $$\int_{I_j} \Phi_j(x, 0)\eta \, dx = \int_{I_j} \phi_0(x)\eta \, dx, \quad \forall \eta \in P^k(I_j), \quad j = 1, \cdots, N - 1.$$

The semi-discrete AEDG scheme is also shown to be conservative and stable in [20].

Theorem 2.1 ([20, Theorem 3.1, Theorem 3.2]). *Let Φ be computed from the AEDG scheme (2.2) for the linear convection-diffusion equation*

$$\partial_t \phi + \alpha \partial_x \phi = \beta \partial_x^2 \phi,$$

with periodic boundary conditions. We have
  (i) *The scheme is conservative in the sense that*

$$\frac{d}{dt} \left(\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\Phi_{j+1} + \Phi_j}{2} dx \right) = 0.$$

  (ii) *The scheme using polynomials of degree $k \geq 1$ is L^2 stable if $\epsilon \leq Qh^2$. Moreover,*

(2.4)

$$\frac{d}{dt} \left(\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\Phi_{j+1}^2 + \Phi_j^2}{2} dx \right) \leq -\beta \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{(\partial_x \Phi_{j+1})^2 + (\partial_x \Phi_j)^2}{2} dx$$
$$+ \left(\frac{1}{Qh^2} - \frac{1}{\epsilon} \right) \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (\Phi_{j+1} - \Phi_j)^2 dx$$

with

(2.5)
$$Q = \frac{1}{\beta(k+1)^2(17(k+1)^2-1)}.$$

Remark 2.2. The dependence of the quantity Q on k stems from the use of inverse inequalities in the stability analysis in [20]. Q is only a bound on the ratio ϵ/h^2 sufficient for justifying (2.4), not necessarily sharp. In fact, in the case $k = 0$, as shown in (2.13) below, $\epsilon = Qh^2$ with $Q = \frac{1}{2\beta}$ is also necessary for stability of the numerical scheme.

Based on these results, we are able to obtain the optimal L^2 error estimates for (2.2), as summarized in the following.

Theorem 2.3. *Let ϕ be a smooth solution to (2.1) subject to initial data $\phi_0(x)$ and periodic boundary conditions, and let $\Phi_j(\cdot, t) \in P^k(I_j)$ ($k \geq 1$) be the numerical solution to (2.2) with $\epsilon = cQh^2$ ($0 < c < 1$), then the following error estimate holds:*

(2.6)
$$\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{|\Phi_{j+1}(x,t) - \phi(x,t)|^2 + |\Phi_j(x,t) - \phi(x,t)|^2}{2} dx \leq Ch^{2k+2}, \quad t \leq T,$$

where C is linear in $T \sup_{t \in [0,T]} \|\phi(\cdot, t)\|_{k+1}$, but is independent of h.

One advantage of the AE framework is to choose time step relating to ϵ properly so that the fully discrete scheme may be made stable. We illustrate this point by considering a class of simple methods in time discretization. Assume that the time interval $[0, T]$ has a uniform partition with time step $\tau = \Delta t$ (which could actually change from step to step but for simplicity is taken as a constant with respect to the time level n), $N_\tau = \lfloor T/\tau \rfloor$, the fully discrete scheme is to find $\Phi^n|_{I_j} \in P^k(I_j)$ such that for all $\eta \in P^k(I_j)$ and $j = 1, \cdots, N$,

(2.7)
$$\int_{I_j} \left(\frac{\Phi_j^{n+1} - \Phi_j^n}{\Delta t} + \partial_x(\alpha\Phi_j^{n+\theta,SN} - \beta\partial_x\Phi_j^{n+\theta,SN}) \right) \eta dx$$
$$= \left(-[\alpha\Phi_j^{n+\theta,SN} - \beta\partial_x\Phi_j^{n+\theta,SN}]\eta - \beta[\Phi_j^{n+\theta,SN}]\partial_x\eta \right) \Big|_{x=x_j}$$
$$+ \frac{1}{\epsilon} \int_{I_j} \left(\Phi_j^{n+\theta,SN} - \Phi_j^{n+\theta} \right) \eta dx,$$

where

(2.8)
$$\Phi^{n+\theta}(x) = \theta\Phi^{n+1}(x) + (1-\theta)\Phi^n(x),$$

with Φ^n denoting the numerical solution at $t^n = n\tau$. Note that Φ^0 is obtained from the projection of $\phi_0(x)$ as defined in (2.3).

For $\theta = 0$, this is the Euler forward discretization; for $\theta = 1$, it is Euler backward, and for $\theta = 1/2$, Crank-Nicolson. The convergence rate result for the fully discrete scheme (2.7) is presented below.

Theorem 2.4. *Let $\phi(x, t)$ be the smooth solution to (2.1) subject to initial data $\phi_0(x)$ and periodic boundary conditions, and let $\Phi_j^n(x) \in P^k(I_j)$ ($k \geq 1$) be the*

numerical solution to (2.7) with $\epsilon = \frac{1}{2}Qh^2$. If either $\theta \geq 1/2$ or $\theta < 1/2$ with τ satisfying

$$(2.9) \qquad (2 + Q\Gamma)^2\tau \leq \epsilon, \quad \tau \leq \frac{\beta h^2}{2\Gamma^2},$$

where Γ is a constant defined in (5.4) of section 5, then scheme (2.7) is stable in the sense that

$$\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (|\Phi_{j+1}^{n+1}(x)|^2 + |\Phi_j^{n+1}(x)|^2)dx \leq \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (|\Phi_{j+1}^n(x)|^2 + |\Phi_j^n(x)|^2)dx.$$

Moreover,

$$(2.10) \qquad \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{|\Phi_{j+1}^n(x) - \phi(x,t^n)|^2 + |\Phi_j^n(x) - \phi(x,t^n)|^2}{2}dx$$

$$\leq C(|1 - 2\theta|\tau + \tau^2 + h^{k+1})^2, \quad n \leq N_\tau,$$

where C is linear in $T \sup_{t \in [0,T]} \|\phi(\cdot, t)\|_{k+7}$, but independent of τ, h.

The proof of Theorem 2.3 is given in section 3, and section 5 is devoted to the proof of Theorem 2.4.

Remark 2.5. In this class of time stepping methods, the scheme is unconditionally stable for $\theta \geq 1/2$, and second order in time accuracy is obtained only when $\theta = 1/2$, i.e., by the Crank-Nicolson time discretization.

Finally, we comment on the case for $k = 0$, for which the AEDG scheme is not consistent in the presence of diffusion. For linear diffusion, the scheme, as pointed out in [20], can be made consistent by choosing $\epsilon = \frac{h^2}{2\beta}$. In such case, the AEDG scheme reduces to

$$(2.11) \qquad \frac{d}{dt}\Phi_j = -\frac{\alpha}{2h}\left(\Phi_{j+1}^n - \Phi_{j-1}^n\right) + \frac{1}{\epsilon}\left(\frac{\Phi_{j-1}^n + \Phi_{j+1}^n}{2} - \Phi_j^n\right).$$

Furthermore, for the forward Euler time discretization of (2.11), we have

$$(2.12) \qquad \frac{\Phi_j^{n+1} - \Phi_j^n}{\tau} = -\frac{\alpha}{2h}\left(\Phi_{j+1}^n - \Phi_{j-1}^n\right) + \frac{1}{\epsilon}\left(\frac{\Phi_{j-1}^n + \Phi_{j+1}^n}{2} - \Phi_j^n\right).$$

Under the condition $|\alpha|\tau/h \leq \tau/\epsilon \leq 1$, this scheme is monotone, hence satisfying the discrete maximum principle as noted in [20]. On the other hand, a direct calculation using (2.12) with summary by parts leads to the identity $\sum_j |\Phi_j^{n+1}|^2 = \sum_j |\Phi_j^n|^2 - F^n$, where

$$F^n = \frac{\tau}{\epsilon}\left(1 - \frac{\tau}{\epsilon}\right)\sum_j |\Phi_{j+1}^n - \Phi_j^n|^2 + \frac{1}{4}\left(\left(\frac{\tau}{\epsilon}\right)^2 - \left(\frac{|\alpha|\tau}{\Delta x}\right)^2\right)\sum_j |\Phi_{j+1}^n - \Phi_{j-1}^n|^2.$$

From this one can verify that the l^2 stability, i.e. $F^n \geq 0$, holds true if and only if

$$\left(\frac{|\alpha|\tau}{h}\right)^2 \leq \frac{\tau}{\epsilon} \leq 1.$$

This condition with $\epsilon = \frac{h^2}{2\beta}$ is equivalent to

$$(2.13) \qquad \tau \leq \epsilon = \frac{h^2}{2\beta}, \quad \tau \leq \frac{2\beta}{|\alpha|^2},$$

which is well known from the von Neumann analysis (see e.g. [13, p. 71]). Therefore, the stability condition (2.9) may be seen as a natural extension of (2.13) when noting that Γ is linear in $|\alpha|h$.

3. ERROR ESTIMATES

3.1. **Scheme reformulation.** In order to distinguish the overlapping polynomials, we introduce two solution spaces of piecewise polynomials as
(3.1)
$$V_h = \{\eta \in L^2, \eta \in P^k(I_j), \quad j = \text{odd}\}, \quad U_h = \{\eta \in L^2, \eta \in P^k(I_j), \quad j = \text{even}\}.$$

Note that for N odd, the set $\{j = \text{even}\} = \{2, 4, \cdots, N - 1\}$, and $\{j = \text{odd}\} = \{1, 3, \cdots, N - 2\}$; for N even, the set $\{j = \text{even}\} = \{2, 4, \cdots, N - 2\}$ and $\{j = \text{odd}\} = \{1, 3, \cdots, N - 1\}$. This way the periodic boundary condition is always satisfied through $\Phi_1 = \Phi_N$, with $\Phi_1 \in V_h$, no matter if N is odd or even.

Hence the AEDG scheme (2.2), when added over $j = \text{odd}$ and $j = \text{even}$, respectively, leads to a coupled system

(3.2)
$$\langle \partial_t v, \xi \rangle + A_{21}(u, \xi) = \frac{1}{\epsilon}\langle u - v, \xi \rangle, \quad \xi \in V_h,$$

(3.3)
$$\langle \partial_t u, \eta \rangle + A_{12}(v, \eta) = \frac{1}{\epsilon}\langle v - u, \eta \rangle, \quad \eta \in U_h,$$

where the two bilinear operators are defined by

$$A_{21}(u, \xi) = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \partial_x J(u)\xi dx + \sum_{j=\text{odd}} ([J(u)]\xi + \beta[u]\partial_x\xi)|_{x_j}, (u, \xi) \in U_h \times V_h,$$

$$A_{12}(v, \eta) = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \partial_x J(v)\eta dx + \sum_{j=\text{even}} ([J(v)]\eta + \beta[v]\partial_x\eta)|_{x_j}, (v, \eta) \in V_h \times U_h$$

with $J(w) = \alpha w - \beta \partial_x w$, and inner product is defined as $\langle w, \xi \rangle = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} w\xi dx$. Note that for N odd, $\int_{x_{N-1}}^{x_N} \partial_x J(v)\eta dx$ is defined by $\int_{x_0}^{x_1} \partial_x J(v)\eta dx$ and for N even, $\int_{x_{N-1}}^{x_N} \partial_x J(u)\xi dx$ is defined by $\int_{x_0}^{x_1} \partial_x J(v)\xi dx$, using the periodicity of the numerical solution. We remark that the subscripts in the operator A_{12} or A_{21} indicate the odd and even (or even and odd) spaces to which the corresponding arguments belong. In what follows this notation will be used; also, the two operators are reformulated in section 4 for the convenience of analysis therein.

The stability analysis in [20] ensures that the following inequality holds.

Lemma 3.1. *For any $(v, u) \in V_h \times U_h$, we have*

$$A_{21}(u, v) + A_{12}(v, u) \geq \frac{\beta}{2}\int_a^b ((\partial_x u)^2 + (\partial_x v)^2)dx - \frac{1}{Qh^2}\int_a^b (u - v)^2 dx,$$

where Q is defined in (2.5).

Here and in what follows we use the notation $\|\partial_x v\|^2 := \int_a^b (\partial_x v)^2 dx$ and $\|\partial_x u\|^2 := \int_a^b (\partial_x u)^2 dx$ to denote

$$\sum_{j=\text{odd}} \int_{x_{j-1}}^{x_{j+1}} |\partial_x v|^2 dx, \quad \sum_{j=\text{even}} \int_{x_{j-1}}^{x_{j+1}} |\partial_x u|^2 dx,$$

respectively, if $(v, u) \in V_h \times U_h$, unless otherwise stated.

3.2. **Projection and projection errors.** Let w be a smooth periodic function. We define two projections $(\Pi_v w, \Pi_u w) \in V_h \times U_h$ as follows:

$$(3.4) \qquad \langle \Pi_v w - w, \xi \rangle + A_{21}(\Pi_u w - w, \xi) = \frac{1}{\epsilon}(\Pi_u w - \Pi_v w, \xi), \quad \xi \in V_h,$$

$$(3.5) \qquad \langle \Pi_u w - w, \eta \rangle + A_{12}(\Pi_v w - w, \eta) = \frac{1}{\epsilon}(\Pi_v w - \Pi_u w, \eta), \quad \eta \in U_h.$$

Here, we again construct $\Pi_v w$ over the extended cell $I_1 = [x_0, x_2]$ and set

$$\Pi_v w|_{[x_{N-1}, x_N]} = \Pi_v w|_{[x_0, x_1]}, \quad N = \text{odd},$$
$$\Pi_u w|_{[x_{N-1}, x_N]} = \Pi_v w|_{[x_0, x_1]}, \quad N = \text{even}.$$

Upon periodic extension for both $\Pi_u w$ and $\Pi_u w$, so that they become periodic.

Lemma 3.2. *For $\epsilon \leq Qh^2$, the above two projections (3.4) and (3.5) are uniquely defined.*

Proof. For a finite dimensional problem, existence is implied by uniqueness. Projections listed in (3.4) and (3.5) are well defined if $w = 0$ implies $(\Pi_v w, \Pi_u w) \equiv (0, 0)$. To this aim, let $\xi = \Pi_v w, \eta = \Pi_u w$, and so we have

$$\langle \xi, \xi \rangle + A_{21}(\eta, \xi) = \frac{1}{\epsilon}(\eta - \xi, \xi), \quad \xi \in V_h,$$

$$\langle \eta, \eta \rangle + A_{12}(\xi, \eta) = \frac{1}{\epsilon}(\xi - \eta, \eta), \quad \eta \in U_h.$$

Summing these two relations together we have

$$\|\xi\|^2 + \|\eta\|^2 + A_{21}(\eta, \xi) + A_{12}(\xi, \eta) = -\frac{1}{\epsilon}\int_a^b (\xi - \eta)^2 dx.$$

This with the inequality in Lemma 3.1 yields

$$\|\xi\|^2 + \|\eta\|^2 + \frac{\beta}{2}\int_a^b ((\partial_x \xi)^2 + (\partial_x \eta)^2) dx \leq \left(\frac{1}{Qh^2} - \frac{1}{\epsilon}\right)\int_a^b (\xi - \eta)^2 dx.$$

If $\epsilon \leq Qh^2$, then $\|\xi\|^2 + \|\eta\|^2 \leq 0$, and so we must have $(\xi, \eta) \equiv 0$. $\qquad\square$

Theorem 3.3. *Let w be a smooth periodic function that belongs to H^m, and Π_v, Π_u are two projection operators defined in (3.4), (3.5). If $\epsilon = cQh^2$ for $0 < c < 1$, then we have*

$$(3.6) \qquad \|\Pi_v w - w\| + \|\Pi_u w - w\| \leq Ch^{\min\{k+1, m\}}|w|_m,$$

where C is a constant independent of h. Here $\|\cdot\|$ denotes the L^2 norm in $[a, b]$.

We defer the proof of Theorem 3.3 to section 4.

3.3. **Optimal error estimates.** With the above projections and the projection error estimates, we proceed to carry out the main error estimate, between the exact solution and the numerical solution. The main result is stated as follows.

Theorem 3.4. *Let ϕ be a smooth solution to (2.1) subject to initial data $\phi_0(x)$ and periodic boundary conditions, and let $(v, u) \in V_h \times U_h$ be the numerical solution to (3.2), (3.3) with $\epsilon = cQh^2$ $(0 < c < 1)$, then the following error estimate holds:*

$$(3.7) \qquad \|\phi(\cdot, t) - v(\cdot, t)\| + \|\phi(\cdot, t) - u(\cdot, t)\| \leq Ch^{k+1}, \quad t \leq T,$$

where C depends on T, ϕ and its derivatives, but is independent of h.

Proof. It is shown in [20] that the AEDG scheme is consistent in the sense that the exact solution ϕ also satisfies (3.2), (3.3), i.e.,

(3.8)
$$\begin{cases} \langle \partial_t \phi, \xi \rangle + A_{21}(\phi, \xi) = \dfrac{1}{\epsilon} \langle \phi - \phi, \xi \rangle, \\[2mm] \langle \partial_t \phi, \eta \rangle + A_{12}(\phi, \eta) = \dfrac{1}{\epsilon} \langle \phi - \phi, \eta \rangle. \end{cases}$$

Upon subtraction from the global formulation (3.2), (3.3), we obtain

(3.9)
$$\begin{cases} \langle \partial_t(\phi - v), \xi \rangle + A_{21}(\phi - u, \xi) = \dfrac{1}{\epsilon} \langle v - u, \xi \rangle, \\[2mm] \langle \partial_t(\phi - u), \eta \rangle + A_{12}(\phi - v, \eta) = \dfrac{1}{\epsilon} \langle u - v, \eta \rangle. \end{cases}$$

Set

$$e_1 = \Pi_v \phi - v, \ \epsilon_1 = \Pi_v \phi - \phi,$$
$$e_2 = \Pi_u \phi - u, \ \epsilon_2 = \Pi_u \phi - \phi,$$

so that

(3.10)
$$\phi - v = e_1 - \epsilon_1, \quad \phi - u = e_2 - \epsilon_2.$$

Taking $\xi = e_1, \eta = e_2$ in (3.9) and substituting (3.10) into (3.9) gives

(3.11) $\langle \partial_t e_1, e_1 \rangle + A_{21}(e_2, e_1) = \langle \partial_t \epsilon_1, e_1 \rangle + A_{21}(\epsilon_2, e_1) + \dfrac{1}{\epsilon} \langle v - u, e_1 \rangle,$

(3.12) $\langle \partial_t e_2, e_2 \rangle + A_{12}(e_1, e_2) = \langle \partial_t \epsilon_2, e_2 \rangle + A_{12}(\epsilon_1, e_2) + \dfrac{1}{\epsilon} \langle u - v, e_2 \rangle.$

Also taking $\xi = e_1, \eta = e_2$ in (3.4), (3.5), we have

$$\langle \epsilon_1, e_1 \rangle + A_{21}(\epsilon_2, e_1) = \dfrac{1}{\epsilon} \langle \epsilon_2 - \epsilon_1, e_1 \rangle,$$
$$\langle \epsilon_2, e_2 \rangle + A_{12}(\epsilon_1, e_2) = \dfrac{1}{\epsilon} \langle \epsilon_1 - \epsilon_2, e_2 \rangle.$$

This together with $v - u = e_2 - e_1 - (\epsilon_2 - \epsilon_1)$ gives

$$A_{21}(\epsilon_2, e_1) + A_{12}(\epsilon_1, e_2) = \dfrac{1}{\epsilon} \langle \epsilon_2 - \epsilon_1, e_1 - e_2 \rangle - \langle \epsilon_1, e_1 \rangle - \langle \epsilon_2, e_2 \rangle$$
$$= -\dfrac{1}{\epsilon} \int_a^b (e_1 - e_2)^2 dx - \dfrac{1}{\epsilon} \langle v - u, e_1 - e_2 \rangle - \langle \epsilon_1, e_1 \rangle - \langle \epsilon_2, e_2 \rangle,$$

which is equivalent to

(3.13)
$$A_{21}(\epsilon_2, e_1) + A_{12}(\epsilon_1, e_2) + \dfrac{1}{\epsilon} \langle v - u, e_1 - e_2 \rangle$$
$$= -\dfrac{1}{\epsilon} \int_a^b (e_1 - e_2)^2 dx - \langle \epsilon_1, e_1 \rangle - \langle \epsilon_2, e_2 \rangle.$$

By Lemma 3.1,

(3.14) $A_{21}(e_2, e_1) + A_{12}(e_1, e_2) \geq \dfrac{\beta}{2} \int_a^b (|\partial_x e_1|^2 + |\partial_x e_2|^2) dx - \dfrac{1}{Qh^2} \int_a^b (e_1 - e_2)^2 dx.$

Summing (3.11) and (3.12), and using (3.13), (3.14), we obtain

$$\frac{d}{dt} \int_a^b \frac{e_1^2 + e_2^2}{2} dx \leq \langle \partial_t \epsilon_1, e_1 \rangle + \langle \partial_t \epsilon_2, e_2 \rangle - \langle \epsilon_1, e_1 \rangle - \langle \epsilon_2, e_2 \rangle$$

$$- \left(\frac{1}{\epsilon} - \frac{1}{Qh^2} \right) \int_a^b (e_1 - e_2)^2 dx$$

$$\leq (\|\partial_t \epsilon_1\| + \|\epsilon_1\|)\|e_1\| + (\|\partial_t \epsilon_2\| + \|\epsilon_2\|)\|e_2\|,$$

$$\leq (\|\partial_t \epsilon_1\| + \|\epsilon_1\| + \|\partial_t \epsilon_2\| + \|\epsilon_2\|) \left(\int_a^b (e_1^2 + e_2^2) dx \right)^{\frac{1}{2}},$$

where we have used $\epsilon \leq Qh^2$.

Using the approximation result in Theorem 3.3, we have

$$(3.15) \qquad \frac{d}{dt} \left(\int_a^b (e_1^2 + e_2^2) dx \right)^{\frac{1}{2}} \leq 2Ch^{k+1}.$$

Integration gives

$$\left(\int_a^b (e_1^2 + e_2^2) dx \right)^{\frac{1}{2}} \leq \left(\int_a^b (e_1^2(x,0) + e_2^2(x,0)) dx \right)^{\frac{1}{2}} + 2CTh^{k+1}.$$

From the choice of the initial data in (2.3), we have

$$(3.16) \quad \|e_1(\cdot, 0)\| = \|\Pi_v \phi_0 - v(\cdot, 0)\| \leq \|\Pi_v \phi_0 - \phi_0\| + \|\phi_0 - v(\cdot, 0)\| \leq C_0 h^{k+1};$$

similarly also $\|e_2(\cdot, 0)\| \leq C_0 h^{k+1}$. Hence

$$\|e_i(\cdot, t)\| \leq \left(\int_a^b (e_1^2 + e_2^2) dx \right)^{\frac{1}{2}} \leq 2(C_0 + CT)h^{k+1}, \quad i = 1, 2,$$

for all $t \leq T$. Thus estimate (3.7) follows from using the triangle inequality as

$$\|\phi(\cdot, t) - v(\cdot, t)\| + \|\phi(\cdot, t) - u(\cdot, t)\| \leq \sum_{i=1}^2 (\|e_i(\cdot, t)\| + \|\epsilon_i(\cdot, t)\|). \qquad \square$$

4. Projection error analysis

In this section, we estimate the projection error in two steps: We first obtain the error of order h^k in some energy norm, and then we lift to achieve the optimal L^2 error, as claimed in Theorem 3.3.

We first present some local approximation results (see, e.g.,[6, Lemma 4.3.8]), which will be used for the energy estimates.

Lemma 4.1. *If $w \in H^m(\Omega)$ is a periodic function, then there exists polynomials $(v_I, u_I) \in V_h \times U_h$ that satisfy optimal approximation properties, i.e., $v_I \in V_h$, $u_I \in U_h$ are polynomials in I_j, for $j = $ odd and $j = $ even, respectively,*

$$(4.1a) \qquad |w - v_I|_{s, I_j} \leq Ch^{\min\{m, k+1\} - s} |w|_{m, I_j}, j = \text{odd},$$

$$(4.1b) \qquad |w - u_I|_{s, I_j} \leq C'h^{\min\{m, k+1\} - s} |w|_{m, I_j}, j = \text{even},$$

for $0 \leq s \leq \min\{m, k+1\}$, where C, C' are two constants independent of mesh size h.

Lemma 4.2. *Let $I = [c, d] \subset [a, b]$ be an interval of length $|I|$, and $v \in P^m(I)$, then*

(4.2a) $\max\{|v(c)|, |v(d)|\} \leq (m+1)|I|^{-1/2}\|v\|_{0,I},$

(4.2b) $\|\partial_x v\|_{0,I} \leq (m+1)\sqrt{m(m+2)}|I|^{-1}\|v\|_{0,I},$

(4.2c) $\|v(\cdot)\|_{\infty,I}^2 \leq \dfrac{\sqrt{5}+1}{2}(|I|^{-1}\|v\|_{0,I}^2 + |I|\|\partial_x v\|_{0,I}^2), \quad if \quad v \in H^1(I).$

Proof. The first bound is well known; see e.g. [30]. The second inequality may be found in [20, Lemma 3.1]. The third one follows from the inequality of the form[*]

$$\|v(\cdot)\|_{\infty,I}^2 \leq (1+\delta)|I|^{-1}\|v\|_{0,I}^2 + \delta^{-1}|I|\|\partial_x v\|_{0,I}^2, \quad \forall \delta > 0,$$

by taking $\delta = \frac{\sqrt{5}-1}{2}$ such that $1 + \delta = \delta^{-1} = \frac{\sqrt{5}+1}{2}$.   □

In the sequel, we shall use the following mesh-dependent norm

(4.3) $\|w\|_{E,h}^2 = h^{-2}\|w\|^2 + \|\partial_x w\|^2 + h^2\|\partial_x^2 w\|^2$

and

$$\|(v, u)\|_{E,h}^2 = \|v\|_{E,h}^2 + \|u\|_{E,h}^2.$$

For piecewise smooth functions in

$$(V(h), U(h)) = (V_h, U_h) + H_p^2(\Omega),$$

where H_p^2 is the space of periodic H^2-functions, the bilinear operators A_{12} and A_{21} may be reformulated, as summarized below.

Lemma 4.3. *For $(\tilde{v}, \tilde{u}) \in V(h) \times U(h)$, the bilinear operators $A_{12}(\tilde{v}, \tilde{u})$ and $A_{21}(\tilde{u}, \tilde{v})$ can be rewritten as*

(4.4) $A_{12}(\tilde{v}, \tilde{u}) = -\sum\limits_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} J(\tilde{v})\partial_x \tilde{u}\,dx - \sum\limits_{j=\text{odd}} [\tilde{u}]J(\tilde{v})|_{x_j} + \beta \sum\limits_{j=\text{even}} \partial_x \tilde{u}[\tilde{v}]|_{x_j},$

(4.5) $A_{21}(\tilde{u}, \tilde{v}) = -\sum\limits_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} J(\tilde{u})\partial_x \tilde{v}\,dx - \sum\limits_{j=\text{even}} [\tilde{v}]J(\tilde{u})|_{x_j} + \beta \sum\limits_{j=\text{odd}} \partial_x \tilde{v}[\tilde{u}]|_{x_j}.$

Proof. The reformulation follows from a straightforward calculation using integration by parts once. Recall $J(\tilde{v}) = \alpha\tilde{v} - \beta\partial_x\tilde{v}$, we illustrate the reformulation of

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[*]This one-dimensional inequality may be derived by simply integrating over I twice: For $x, y \in I$ one has

$$v^2(x) = v^2(y) + 2\int_y^x v(z)\partial_x v(z)\,dz\,dz \leq v^2(y) + 2\|v\|_{0,I}\|\partial_x v\|_{0,I} \quad \text{(by Cauchy-Schwarz)},$$

which upon integration over $y \in I$ gives

$$|I|v^2(x) \leq \|v\|_{0,I}^2 + 2|I|\|v\|_{0,I}\|\partial_x v\|_{0,I} \leq (1+\delta)\|v\|_{0,I}^2 + \frac{|I|^2}{\delta}\|\partial_x v\|_{0,I}^2 \quad \text{(by Young's inequality)}.$$

In the multi-dimensional case, it is valid only in the form of trace inequalities; see e.g. [1, Theorem 3.10].

$A_{12}(\tilde{v}, \tilde{u})$ as follows:

$$A_{12}(\tilde{v}, \tilde{u}) = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \partial_x J(\tilde{v})\tilde{u}dx + \sum_{j=\text{even}} ([J(\tilde{v})]\tilde{u} + \beta[\tilde{v}]\partial_x\tilde{u})|_{x_j}$$

$$= -\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} J(\tilde{v})\partial_x\tilde{u}dx + \beta \sum_{j=\text{even}} \partial_x\tilde{u}[\tilde{v}]|_{x_j} - \sum_{j=\text{odd}} [\tilde{u}]J(\tilde{v})|_{x_j}.$$

The reformulation of $A_{21}(\tilde{u}, \tilde{v})$ is entirely similar. $\qquad\square$

4.1. **Projection error in energy norm.** For convenience, we define the energy norm of $(v, u) \in V_h \times U_h$ as

$$(4.6) \qquad \|(v, u)\|_E^2 \triangleq \|v\|^2 + \|u\|^2 + \int_a^b ((\partial_x u)^2 + (\partial_x v)^2)dx + \frac{1}{h^2}\int_a^b (v - u)^2 dx.$$

Using this energy norm we have the following estimate.

Theorem 4.4. *For projections Π_v, Π_u defined in (3.4), (3.5) with $\epsilon = cQh^2$ for $c \in (0, 1)$, the following estimate holds:*

$$(4.7) \qquad \|(\Pi_v w - w, \Pi_u w - w)\|_E \leq Ch^{\min\{k,m\}}|w|_m,$$

where C is a constant independent of h.

Proof. Without loss of generality, we prove only the case $m = k$. Fix $\epsilon = \frac{1}{2}Qh^2$ in the proof to follow. Recall the definition of projections Π_v, Π_u. We have

$$(4.8) \qquad \langle\Pi_v w - w, \xi\rangle + A_{21}(\Pi_u w - w, \xi) = \frac{1}{\epsilon}(\Pi_u w - \Pi_v w, \xi), \quad \xi \in V_h,$$

$$(4.9) \qquad \langle\Pi_u w - w, \eta\rangle + A_{12}(\Pi_v w - w, \eta) = \frac{1}{\epsilon}(\Pi_v w - \Pi_u w, \eta), \quad \eta \in U_h.$$

Set

$$(\xi, \eta) = (\Pi_v w - v_I, \Pi_u w - u_I) \in V_h \times U_h,$$

where (v_I, u_I) is the polynomial pair of (w, w) in $V_h \times U_h$, satisfying (4.1). Hence,

$$\Pi_v w = \xi + v_I, \quad \Pi_u w = \eta + u_I.$$

This when inserted in (4.8), (4.9) gives

(4.10a)

$$\langle\xi, \xi\rangle + A_{21}(\eta, \xi) - \frac{1}{\epsilon}\langle\eta - \xi, \xi\rangle = \langle w - v_I, \xi\rangle + A_{21}(w - u_I, \xi) - \frac{1}{\epsilon}\langle v_I - u_I, \xi\rangle,$$

(4.10b)

$$\langle\eta, \eta\rangle + A_{12}(\xi, \eta) - \frac{1}{\epsilon}\langle\xi - \eta, \eta\rangle = \langle w - u_I, \eta\rangle + A_{12}(w - v_I, \eta) - \frac{1}{\epsilon}\langle u_I - v_I, \eta\rangle.$$

The sum of the left-hand sides of (4.10) is

$$\|\xi\|^2 + \|\eta\|^2 + A_{21}(\eta, \xi) + A_{12}(\xi, \eta) + \frac{1}{\epsilon}\int_a^b (\xi - \eta)^2 dx \geq \min(1, \frac{\beta}{2}, \frac{1}{Q})\|(\xi, \eta)\|_E^2,$$

provided $\epsilon = \frac{1}{2}Qh^2$, due to Lemma 3.1.

Denote $w - v_I = \tilde{v}$, $w - u_I = \tilde{u}$, from (4.1) it follows that

$$(4.11) \qquad \|(\tilde{v}, \tilde{u})\|_{E,h} \leq Ch^k|w|_{k+1}.$$

Note that the upper bound will become $Ch^{\min\{k+1,m\}}|w|_m$ for $w \in H_p^m$. The constant C may vary from line to line.

It is left to estimate each term on the right side of (4.10). First, we have

$$(4.12) \qquad \langle \tilde{v}, \xi \rangle + \langle \tilde{u}, \eta \rangle \le \|\tilde{v}\| \|\xi\| + \|\tilde{u}\| \|\eta\| \le \|(\xi, \eta)\|_E (\|\tilde{v}\| + \|\tilde{u}\|).$$

For terms involving ϵ, since $-(v_I - u_I) = \tilde{v} - \tilde{u}$, we have

$$(4.13) \quad \frac{1}{\epsilon} \langle \tilde{v} - \tilde{u}, \xi - \eta \rangle \le \frac{2}{Qh^2} \|\xi - \eta\| (\|\tilde{v}\| + \|\tilde{u}\|) \le \frac{2}{Q} h^{-1} \|(\xi, \eta)\|_E (\|\tilde{v}\| + \|\tilde{u}\|).$$

From (4.5), we have

$$A_{21}(\tilde{u}, \xi) = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (-\alpha \tilde{u} + \beta \partial_x \tilde{u}) \, \partial_x \xi dx$$

$$+ \sum_{j=\text{even}} (-\alpha \tilde{u} + \beta \partial_x \tilde{u})[\xi]|_{x_j} + \beta \sum_{j=\text{odd}} \partial_x \xi [\tilde{u}]|_{x_j}.$$

The integral terms are bounded as

$$(4.14) \qquad \begin{aligned} \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (-\alpha \tilde{u} + \beta \partial_x \tilde{u}) \, \partial_x \xi dx &\le (|\alpha| \|\tilde{u}\| + \beta \|\partial_x \tilde{u}\|) \|\partial_x \xi\| \\ &\le (|\alpha| h + \beta) \|\partial_x \xi\| \|\tilde{u}\|_{E,h}. \end{aligned}$$

For terms evaluated at even grid points x_j, we need to use (4.2a) so that

$$|[\eta - \xi]^2|_{x_j} \le 2h^{-1}(k+1)^2 \left(\|\eta - \xi\|_{0,\tau_{j-1}}^2 + \|\eta - \xi\|_{0,\tau_j}^2 \right)$$

$$= 2h^{-1}(k+1)^2 \|\eta - \xi\|_{0,I_j}^2, \quad \tau_j = (x_j, x_{j+1}),$$

and (4.2c), so that

$$\left| \sum_{j=\text{even}} J(\tilde{u})[\xi]|_{x_j} \right| = \left| \sum_{j=\text{even}} J(\tilde{u})[\xi - \eta]|_{x_j} \right|$$

$$\le \left(\sum_{j=\text{even}} J^2(\tilde{u})|_{x_j} \right)^{1/2} \left(\sum_{j=\text{even}} [\xi - \eta]^2|_{x_j} \right)^{1/2}$$

$$\le \left(\sum_{j=\text{even}} \frac{\sqrt{5}+1}{2} (h^{-1} \|J(\tilde{u})\|_{0,I_j}^2 + h \|\partial_x J(\tilde{u})\|_{0,I_j}^2) \right)^{1/2}$$

$$\times \left(\sum_{j=\text{even}} 2(k+1)^2 h^{-1} \|\xi - \eta\|_{0,I_j}^2 \right)^{1/2}$$

$$(4.15) \qquad \le 2(k+1)h^{-1} \|\xi - \eta\| (\|J(\tilde{u})\|^2 + h^2 \|\partial_x J(\tilde{u})\|^2)^{1/2}.$$

Note that $J(\tilde{u}) = \alpha \tilde{u} - \beta \tilde{u}_x$, hence

$$\left| \sum_{j=\text{even}} J(\tilde{u})[\xi]|_{x_j} \right| \le 2(k+1)\sqrt{2(\alpha^2 h^2 + \beta^2)} h^{-1} \|\xi - \eta\| \cdot \|\tilde{u}\|_{E,h}.$$

For terms evaluated at odd grid points x_j, from (4.2a) we have

$$|\xi_x|_{x_j} \le (k+1)h^{-1/2} \min\{\|\partial_x \xi\|_{0,\tau_{j-1}}, \|\partial_x \xi\|_{0,\tau_j}\}.$$

This together with the use of (4.2c) yields

$$\sum_{j=\text{odd}} \partial_x \xi [\tilde{u}]|_{x_j} \le \left(\sum_{j=\text{odd}} (\partial_x \xi)^2|_{x_j} \right)^{1/2} \left(\sum_{j=\text{odd}} [\tilde{u}]^2|_{x_j} \right)^{1/2}$$

$$\le \left(\sum_{j=\text{odd}} h^{-1}(k+1)^2 \|\partial_x \xi\|_{0,\tau_j}^2 \right)^{1/2}$$

$$\times \left(2 \sum_{j=\text{odd}} \frac{\sqrt{5}+1}{2} (h^{-1}\|\tilde{u}\|_{0,\tau_{j-1}\cup\tau_j}^2 + h\|\partial_x\tilde{u}\|_{0,\tau_{j-1}\cup\tau_j}^2) \right)^{1/2}$$

$$(4.16) \qquad \le 2(k+1)\|\partial_x\xi\| \cdot \|\tilde{u}\|_{E,h}.$$

Collecting the above estimates for three terms in $A_{21}(\tilde{u},\xi)$, we obtain

$$(4.17) \qquad A_{21}(\tilde{u},\xi) \le C(\|\partial_x\xi\| + h^{-1}\|\xi - \eta\|)\|\tilde{u}\|_{E,h} \le \sqrt{2}C\|(\xi,\eta)\|_E\|\tilde{u}\|_{E,h},$$

where $C = |\alpha|h + \beta + 2(k+1)\sqrt{2(\alpha^2 h^2 + \beta^2)} + 2(k+1)$.

In an entirely similar manner, $A_{12}(\tilde{v},\eta)$ can be estimated so that

$$(4.18) \qquad A_{12}(\tilde{v},\eta) \le C(\|\partial_x\eta\| + h^{-1}\|\xi - \eta\|)\|\tilde{u}\|_{E,h} \le \sqrt{2}C\|(\xi,\eta)\|_E\|\tilde{v}\|_{E,h}.$$

Adding (4.12), (4.13), (4.17) and (4.18) gives an upper bound of the sum of the right-hand side of (4.10) as

$$(2h + \frac{2}{Q} + 2\sqrt{2}C)\|(\xi,\eta)\|_E\|(\tilde{v},\tilde{u})\|_{E,h}.$$

Therefore,

$$\|(\xi,\eta)\|_E^2 \lesssim \|(\xi,\eta)\|_E\|(\tilde{v},\tilde{u})\|_{E,h}.$$

This when combined with (4.11) gives

$$\|(\xi,\eta)\|_E \le \|(\tilde{v},\tilde{u})\|_{E,h} \le Ch^k|w|_{k+1}.$$

That is,

$$\|(\Pi_v w - v_I, \Pi_u w - u_I)\|_E \le Ch^k|w|_{k+1}.$$

This and (4.11) combined with the triangle inequality give (4.7). The proof is complete. $\qquad \square$

Remark 4.5. One main difference in the analysis of the AEDG method and other DG methods lies in the use of an additional term $\|\xi - \eta\|$ in the energy norm.

4.2. **Projection error in L^2 norm.** Now we turn to recover the L^2 error from the error in energy norm, by using a duality argument.

Proof of Theorem 3.3. Set $\theta_v = \Pi_v w - w$, $\quad \theta_u = \Pi_u w - w$. From (4.7) we see that

$$\|\theta_v - \theta_u\| \le h\|(\theta_v, \theta_u)\|_E.$$

We only need to prove that

$$(4.19) \qquad \|\theta_v + \theta_u\| \le Ch\|(\theta_v, \theta_u)\|_E.$$

These together with

$$\|\Pi_v w - w\| + \|\Pi_u w - w\| \leq \|\theta_v - \theta_u\| + \|\theta_v + \theta_u\|,$$

and (4.7) conclude the claimed error in (3.6).

The rest is devoted to the proof of (4.19).

Step 1 (Auxiliary problem). We define the auxiliary function ψ as the solution of the following problem:

$$(4.20) \qquad \begin{cases} \psi - \alpha \partial_x \psi - \beta \partial_x^2 \psi = \theta_v + \theta_u, & \text{in } (a, b), \\ \psi(a) = \psi(b), \partial_x \psi(a) = \partial_x \psi(b). \end{cases}$$

For piecewise continuous function $\theta_v + \theta_u$ in $a \leq x \leq b$, the unique solution to problem (4.20) is ensured since the corresponding homogeneous problem admits only the trivial solution.

Multiplying (4.20) by $\partial_x^2 \psi$ and integrating over $[a, b]$ we obtain

$$\beta \|\partial_x^2 \psi\|^2 = -\|\partial_x \psi\|^2 - \int_a^b \partial_x^2 \psi (\theta_v + \theta_u) dx \leq \|\partial_x^2 \psi\| \|\theta_v + \theta_u\|,$$

which gives

$$(4.21) \qquad \|\partial_x^2 \psi\| \leq \frac{1}{\beta} \|\theta_v + \theta_u\|.$$

Step 2 (L^2 norm reformulation). We now evaluate the L^2 norm of $\theta_v + \theta_u$ by

$$(4.22) \qquad \|\theta_v + \theta_u\|^2 = \int_a^b (\psi - \alpha \partial_x \psi - \beta \partial_x^2 \psi)(\theta_v + \theta_u) dx.$$

Rewriting those terms including derivatives of ψ on the right-hand side, we have

$$(4.23)$$

$$\int_a^b (-\alpha \partial_x \psi - \beta \partial_x^2 \psi) \theta_v dx$$

$$= \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (-\alpha \theta_v + \beta \partial_x \theta_v) \partial_x \psi dx - \beta \sum_{j=1}^{N-1} (\partial_x \psi \theta_v) \big|_{x_j^+}^{x_{j+1}^-}$$

$$= \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (-\alpha \theta_v + \beta \partial_x \theta_v) \partial_x \psi dx + \beta \sum_{j=\text{even}} (\partial_x \psi [\theta_v]) \big|_{x_j}$$

$$= - \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} J(\theta_v) \partial_x \psi dx - \sum_{j=\text{odd}} ([\psi] J(\theta_v)) \big|_{x_j} + \beta \sum_{j=\text{even}} (\partial_x \psi [\theta_v]) \big|_{x_j}$$

$$= A_{12}(\theta_v, \psi),$$

where we have used the fact that $[\psi] = 0$ at $x = x_j$, $\partial_x \psi(a) = \partial_x \psi(b)$ and periodicity of θ_v as so constructed in (3.4), (3.5).

In an entirely similar manner, we have

$$\int_a^b (-\alpha \partial_x \psi - \beta \partial_x^2 \psi) \theta_u dx = A_{21}(\theta_u, \psi).$$

Thus (4.22) can be rewritten as

$$(4.24) \qquad \|\theta_v + \theta_u\|^2 = \langle \psi, \theta_v + \theta_u \rangle + A_{21}(\theta_u, \psi) + A_{12}(\theta_v, \psi).$$

Recall the projection defined in (3.4), (3.5),

(4.25a) $$0 = \langle \theta_v, \xi \rangle + A_{21}(\theta_u, \xi) - \frac{1}{\epsilon} \langle \theta_u - \theta_v, \xi \rangle,$$

(4.25b) $$0 = \langle \theta_u, \eta \rangle + A_{12}(\theta_v, \eta) - \frac{1}{\epsilon} \langle \theta_v - \theta_u, \eta \rangle.$$

Therefore, (4.24) subtracting (4.25a) and (4.25b) gives

(4.26) $$\|\theta_v + \theta_u\|^2 = \langle \theta_v, \psi - \xi \rangle + \langle \theta_u, \psi - \eta \rangle$$
$$+ A_{21}(\theta_u, \psi - \xi) + A_{12}(\theta_v, \psi - \eta) + \frac{1}{\epsilon} \langle \theta_v - \theta_u, \eta - \xi \rangle.$$

To simplify notation, we denote $\psi - \xi = w_1 \in V(h)$ and $\psi - \eta = w_2 \in U(h)$, so that

(4.27) $$\|\theta_v + \theta_u\|^2 = \langle \theta_v, w_1 \rangle + \langle \theta_u, w_2 \rangle + A_{21}(\theta_u, w_1) + A_{12}(\theta_v, w_2)$$
$$+ \frac{1}{\epsilon} \langle \theta_v - \theta_u, w_1 - w_2 \rangle.$$

Step 3 (Approximation and estimates). In order to bound w_1, w_2, we choose ξ (or η) a piecewise linear polynomial interpolating the smooth function ψ at odd (or even) grid points, then the standard approximation results imply

(4.28) $$\|\partial_x^q w_i\| \leq C_1 h^{2-q} \|\partial_x^2 \psi\|, \quad i = 1, 2, \ q = 0, 1, 2,$$

where C_1 is a positive constant independent of h. Also, both ξ and η are continuous functions so that for $i = 1, 2$, we have

(4.29) $$[w_i]|_{x_j} = 0, \quad j = 1, \cdots, N - 1.$$

Using the energy norm definition in (4.6), we proceed to estimate terms on the right-hand side of (4.27). The first two terms are bounded by

(4.30) $$\langle \theta_v, w_1 \rangle + \langle \theta_u, w_2 \rangle \leq \|\theta_v\|\|w_1\| + \|\theta_u\|\|w_2\| \leq \|(\theta_v, \theta_u)\|_E (\|w_1\| + \|w_2\|).$$

For the last term we have

(4.31) $$\frac{1}{\epsilon} \langle \theta_v - \theta_u, w_1 - w_2 \rangle \leq \frac{2}{Q h^2} \|\theta_u - \theta_v\| (\|w_1\| + \|w_2\|)$$
$$\leq \frac{2}{Q} h^{-1} \|(\theta_v, \theta_u)\|_E (\|w_1\| + \|w_2\|).$$

In virtue of (4.29), terms involving $[w_1]$ at x_j in $A_{21}(\theta_u, w_1)$ vanish. For the integral term in $A_{21}(\theta_u, w_1)$, we have

$$\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} J(\theta_u) \partial_x w_1 dx \leq (\alpha \|\theta_u\| + \beta \|\partial_x \theta_u\|) \|\partial_x w_1\|.$$

For the jump terms at odd grid points in $A_{21}(\theta_u, w_1)$ of (4.27), using (4.2c) we have

$$\sum_{j=\text{odd}} \partial_x w_1 [\theta_u]|_{x_j} = \sum_{j=\text{odd}} \partial_x w_1 [\theta_u - \theta_v]|_{x_j}$$

$$\leq \sum_{j=\text{odd}} h^{-\frac{1}{2}} (k+1) \|\theta_u - \theta_v\|_{L^2(\tau_{j-1} \cup \tau_j)} \left(\frac{\sqrt{5}+1}{2} \right)^{\frac{1}{2}}$$

$$\cdot \left(h^{-1} \|\partial_x w_1\|_{L^2(I_j)}^2 + h \|\partial_x^2 w_1\|_{L^2(I_j)}^2 \right)^{\frac{1}{2}}$$

$$\leq 2(k+1) h^{-1} \|\theta_u - \theta_v\| \cdot \|w_1\|_{E,h}.$$

Collecting these two terms, we obtain

$$(4.32) \qquad A_{21}(\theta_u, w_1) \le C_2 \|(\theta_v, \theta_u)\|_E \|w_1\|_{E,h},$$

where $C_2 = (\max\{|\alpha|, \beta\} + 2(k+1)Q)$. Similarly,

$$(4.33) \qquad A_{12}(\theta_v, w_2) \le C_2 \|(\theta_v, \theta_u)\|_E \|w_2\|_{E,h}.$$

Step 4 (Final substitution). Insertion of (4.30), (4.33), (4.32) and (4.31) into (4.27), using (4.28) and (4.21), yields

$$
\begin{aligned}
\|\theta_v + \theta_u\|^2 &\le \|(\theta_v, \theta_u)\|_E \left(\left(1 + \frac{2}{Q}h^{-1}\right) \|w_1\| + \|w_2\| + C_2(\|w_1\|_{E,h} + \|w_2\|_{E,h}) \right) \\
&\le Ch \|(\theta_v, \theta_u)\|_E \|\partial_x^2 \psi\| \\
&\le \frac{Ch}{\beta} \|(\theta_v, \theta_u)\|_E \|\theta_v + \theta_u\|.
\end{aligned}
$$

Hence

$$\|\theta_v + \theta_u\| \lesssim h \|(\theta_v, \theta_u)\|_E.$$

This ends the proof of (4.19). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Box$

Remark 4.6. From the above analysis we see that the regularity of problem (1.1) requires $\beta > 0$ as indicated in (4.21). When $\beta = 0$, the convection diffusion equation degenerates to a hyperbolic equation, we then have

$$A_{21}(u, v) + A_{12}(v, u) = 0 \quad \forall (u, v) \in U_h \times V_h,$$

with which the projection error of order $O(h^k)$ in energy norm can still be obtained. Unfortunately the above analysis for recovery of the optimal L^2 error does not seem to work in this case. Yet, the AEDG method still works for the $\beta = 0$ case as well numerically; see [20] for related numerical tests.

5. TIME DISCRETIZATION

For time dependent problems, it is generally of interest to know how the solution errors grow with time. To this end, stability estimates are used. Rewriting the fully discrete scheme (2.7) we have

$$(5.1)$$
$$
\begin{cases}
\langle \dfrac{v^{n+1} - v^n}{\tau}, \xi \rangle + A_{21}(u^{n+\theta}, \xi) = \dfrac{1}{\epsilon}\langle u^{n+\theta} - v^{n+\theta}, \xi \rangle, & (\xi, \eta) \in V_h \times U_h, \\
\langle \dfrac{u^{n+1} - u^n}{\tau}, \eta \rangle + A_{12}(v^{n+\theta}, \eta) = \dfrac{1}{\epsilon}\langle v^{n+\theta} - u^{n+\theta}, \eta \rangle, & (v^n, u^n) \in V_h \times U_h,
\end{cases}
$$

where $w^{n+\theta} = \theta w^{n+1} + (1-\theta)w^n$ for $w = u$ or $w = v$, and $0 \le \theta \le 1$. Here v^n, u^n denote the numerical solution at $t^n = n\tau$ for $n \le N_\tau$, and the initial data (v^0, u^0) is taken as (2.3).

5.1. Stability analysis.
We seek a sufficient condition in relating time step τ to ϵ so that the fully discrete scheme is stable.

We begin to prepare the following bound of two bilinear operators A_{21} and A_{12}.

Lemma 5.1. *For any* $(\xi, \eta), (v, u) \in V_h \times U_h$, *it holds that*

$$(5.2) \qquad A_{21}(u, \xi) \le h^{-1}\Gamma(\|u_x\| + h^{-1}\|v - u\|)\|\xi\|,$$

$$(5.3) \qquad A_{12}(v, \eta) \le h^{-1}\Gamma(\|v_x\| + h^{-1}\|v - u\|)\|\eta\|,$$

where for $\gamma_k = (k + 1)\sqrt{k(k + 2)}$, *we have*

$$(5.4) \ \Gamma := \max\{|\alpha|h + \beta\gamma_k + 2(k+1)(1+\gamma_k^2)^{1/2}, 2(k+1)(1+\gamma_k^2)^{1/2}(|\alpha|h + \beta\gamma_k)\}.$$

Proof. From (4.4), we do integration by parts in terms containing α so that

$$A_{12}(v, \eta) = -\sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}}(\alpha v - \beta\partial_x v)\partial_x\eta dx$$

$$-\sum_{j=\text{odd}}(\alpha v - \beta\partial_x v)[\eta]|_{x_j} + \beta\sum_{j=\text{even}}\partial_x\eta[v]|_{x_j}$$

$$(5.5) \qquad = \sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}}(\alpha\eta + \beta\partial_x\eta)\partial_x v dx$$

$$+\sum_{j=\text{odd}}\beta\partial_x v[\eta]|_{x_j} + \sum_{j=\text{even}}(\alpha\eta + \beta\partial_x\eta)[v]|_{x_j}$$

$$=: \widetilde{A}_{21}(\eta, v).$$

Note that the estimate of $\widetilde{A}_{21}(\eta, v)$ is the same as that of $A_{21}(\eta, v)$, for which we recall the estimates performed in (4.14), (4.15) and (4.16) to obtain

$$\widetilde{A}_{21}(\eta, v) \le \|\partial_x v\|(|\alpha|\|\eta\| + \beta\|\partial_x\eta\|)$$

$$(5.6) \qquad + 2(k+1)h^{-1}\|v - u\|(\|J(\eta)\|^2 + h^2\|\partial_x J(\eta)\|^2)^{1/2}$$

$$+ 2(k+1)\|\partial_x v\|(h^{-2}\|\eta\|^2 + \|\partial_x\eta\|^2)^{1/2}.$$

Using (4.2b) with $m = k$, we have

$$\|\partial_x\eta\| \le \gamma_k h^{-1}\|\eta\|.$$

This leads to the following estimates:

$$\|J(\eta)\| \le |\alpha|\|\eta\| + \beta\|\partial_x\eta\| \le (|\alpha|h + \beta\gamma_k)h^{-1}\|\eta\|,$$

$$\|\partial_x J(\eta)\| \le \gamma_k h^{-1}\|J(\eta)\| \le \gamma_k(|\alpha|h + \beta\gamma_k)h^{-2}\|\eta\|.$$

Substitution of these into (5.6) yields

$$A_{12}(v, \eta) \le (|\alpha|h + \beta\gamma_k)h^{-1}\|\partial_x v\|\|\eta\|$$

$$+ 2(k+1)h^{-1}\|v - u\|(1 + \gamma_k^2)^{1/2}(|\alpha|h + \beta\gamma_k)h^{-1}\|\eta\|$$

$$+ 2(k+1)(1 + \gamma_k^2)^{1/2}h^{-1}\|\partial_x v\|\|\eta\|$$

$$\le h^{-1}\Gamma(\|\partial_x v\| + h^{-1}\|v - u\|)\|\eta\|,$$

where we have used (5.4). Similarly, we have

$$A_{21}(u, \xi) \le h^{-1}\Gamma(\|\partial_x u\| + h^{-1}\|v - u\|)\|\xi\|.$$

This ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now present the stability result for the fully discrete scheme (5.1) as follows.

Theorem 5.2. *Let v^n and u^n be the numerical solution computed from* (5.1) *with* $\epsilon = cQh^2, c \in (0,1)$.

(i) *For $\theta \geq 1/2$, scheme* (5.1) *is unconditionally stable.*

(ii) *For $\theta < 1/2$, if*

$$\frac{2(1-2\theta)(1+cQ\Gamma)^2}{1-c}\tau \leq \epsilon = cQh^2 \ and \ \tau \leq \frac{\beta h^2}{2(1-2\theta)\Gamma^2},$$

then there exists $C_1 \geq 0$ and $C_2 \geq 0$ such that

$$
\begin{aligned}
(5.7) \quad & \|v^{n+1}\|^2 + \|u^{n+1}\|^2 + C_1\tau\|(\partial_x v^{n+\theta}, \partial_x u^{n+\theta})\|^2 + \frac{C_2\tau}{\epsilon}\|v^{n+\theta} - u^{n+\theta}\|^2 \\
& \leq \|v^n\|^2 + \|u^n\|^2.
\end{aligned}
$$

Proof. We divide the proof into two steps.

Step 1. The summation of two equations in (5.1), with $(\xi, \eta) = (v^{n+\theta}, u^{n+\theta}) \in V_h \times U_h$, when using the fact that

$$\langle v^{n+1} - v^n, v^{n+\theta}\rangle = \frac{1}{2}(\|v^{n+1}\|^2 - \|v^n\|^2) + (\theta - \frac{1}{2})(\|v^{n+1} - v^n\|^2),$$

gives

$$
\begin{aligned}
& \|v^{n+1}\|^2 + \|u^{n+1}\|^2 + 2\tau\left(A_{21}(u^{n+\theta}, v^{n+\theta}) + A_{12}(v^{n+\theta}, u^{n+\theta})\right) \\
& = \|v^n\|^2 + \|u^n\|^2 + (1-2\theta)\|v^{n+1} - v^n\|^2 + (1-2\theta)\|u^{n+1} - u^n\|^2 \\
& \quad - \frac{2\tau}{\epsilon}\|v^{n+\theta} - u^{n+\theta}\|^2.
\end{aligned}
$$

By Lemma 3.1, we have

(5.8)

$$
\|v^{n+1}\|^2 + \|u^{n+1}\|^2 + \beta\tau\|(\partial_x v^{n+\theta}, \partial_x u^{n+\theta})\|^2 + 2\tau\left(\frac{1}{\epsilon} - \frac{1}{Qh^2}\right)\|v^{n+\theta} - u^{n+\theta}\|^2
$$
$$
\leq \|v^n\|^2 + \|u^n\|^2 + (1-2\theta)\|v^{n+1} - v^n\|^2 + (1-2\theta)\|u^{n+1} - u^n\|^2.
$$

Step 2. (i) If $\theta \geq 1/2$, stability estimate (5.7) follows from (5.8) in a straightforward manner by taking $(C_1, C_2) = (\beta, 2(1-c))$.

(ii) If $\theta < 1/2$, we need to bound $\|v^{n+1} - v^n\|^2 + \|u^{n+1} - u^n\|^2$ using (5.1) again. Taking $\xi = (v^{n+1} - v^n)/\tau$ and $\eta = (u^{n+1} - u^n)/\tau$ in (5.1), we have

$$(5.9a) \qquad \|\xi\|^2 + A_{21}(u^{n+\theta}, \xi) = \frac{1}{\epsilon}(u^{n+\theta} - v^{n+\theta}, \xi),$$

$$(5.9b) \qquad \|\eta\|^2 + A_{12}(v^{n+\theta}, \eta) = \frac{1}{\epsilon}(v^{n+\theta} - u^{n+\theta}, \eta).$$

From (5.9a), using Lemma 5.1, it follows that

$$
\begin{aligned}
\|\xi\|^2 & = -A_{21}(u^{n+\theta}, \xi) + \frac{1}{\epsilon}(u^{n+\theta} - v^{n+\theta}, \xi) \\
& \leq h^{-1}\Gamma(\|\partial_x u^{n+\theta}\| + h^{-1}\|v^{n+\theta} - u^{n+\theta}\|)\|\xi\| + \frac{1}{\epsilon}\|v^{n+\theta} - u^{n+\theta}\|\|\xi\|.
\end{aligned}
$$

That is, with $\epsilon = cQh^2$,

$$\|\xi\| \leq h^{-1}\Gamma\|\partial_x u^{n+\theta}\| + \frac{1}{\epsilon}(1 + cQ\Gamma)\|v^{n+\theta} - u^{n+\theta}\|.$$

Similarly we have

$$\|\eta\| \leq h^{-1}\Gamma\|\partial_x v^{n+\theta}\| + \frac{1}{\epsilon}(1 + cQ\Gamma)\|v^{n+\theta} - u^{n+\theta}\|.$$

Thus

$$\|v^{n+1} - v^n\|^2 + \|u^{n+1} - u^n\|^2 = \tau^2(\|\xi\|^2 + \|\eta\|^2)$$
$$\leq \frac{2\Gamma^2\tau^2}{h^2}(\|\partial_x v^{n+\theta}\|^2 + \|\partial_x u^{n+\theta}\|^2) + \frac{4\tau^2}{\epsilon^2}(1 + cQ\Gamma)^2\|v^{n+\theta} - u^{n+\theta}\|^2.$$

This when inserted into (5.8) gives

$$\|v^{n+1}\|^2 + \|u^{n+1}\|^2 + C_1\tau\|(\partial_x v^{n+\theta}, \partial_x u^{n+\theta})\|^2 + \frac{C_2\tau}{\epsilon}\|v^{n+\theta} - u^{n+\theta}\|^2$$
$$\leq \|v^n\|^2 + \|u^n\|^2,$$

where

$$C_1 = \beta - (1 - 2\theta)\frac{2\Gamma^2\tau}{h^2}, \quad C_2 = 2\left(1 - c - (1 - 2\theta)\frac{2\tau}{\epsilon}(1 + cQ\Gamma)^2\right).$$

It is left to specify the mesh size h relating to ϵ so that $C_1 \geq 0$ and $C_2 \geq 0$. Clearly, $C_1 \geq 0$ if

$$\tau \leq \frac{\beta h^2}{2(1 - 2\theta)\Gamma^2},$$

and $C_2 \geq 0$ if

$$\epsilon \geq \frac{2(1 - 2\theta)(1 + cQ\Gamma)^2}{1 - c}\tau.$$

This ends the proof.   □

Remark 5.3. The choice of c will affect the computational cost. From

$$\frac{2(1 - 2\theta)(1 + cQ\Gamma)^2}{1 - c}\tau \leq \epsilon = cQh^2,$$

we see that

$$\tau \leq \frac{c(1 - c)Q}{2(1 - 2\theta)(1 + cQ\Gamma)^2}h^2,$$

for which the optimal choice should be $c = 1/2$. For such choice, the stability condition becomes

$$(1 - 2\theta)(2 + Q\Gamma)^2\tau \leq \epsilon = \frac{1}{2}Qh^2.$$

5.2. **Error estimates.** Based on the stability result, we set out to derive the error estimates of scheme (5.1). The result is summarized as follows.

Theorem 5.4. *Let ϕ be the smooth solution of (2.1) subject to initial data $\phi_0(x)$ and periodic boundary conditions, $(v^n, u^n) \in V_h \times U_h$ be the numerical solution computed through the fully discrete scheme (5.1), then we have the error estimate*

$$(5.10) \qquad \|\phi(\cdot, t^n) - v^n(\cdot)\| + \|\phi(\cdot, t^n) - u^n(\cdot)\| \leq C(h^{k+1} + \tau|1 - 2\theta| + \tau^2),$$

where C in linear in $T\sup_{t \in [0,T]} \|\phi(\cdot, t)\|_{k+7}$ $(k \geq 1)$, but independent of τ, h.

Proof. To simplify notation, let $\phi^n = \phi(x, t^n)$ and $v^n = v^n(x)$. The consistency of the AEDG scheme, as given in (3.8), when evaluated at t^n and t^{n+1}, respectively, upon further linear combination gives

(5.11)
$$\begin{cases} \langle \partial_t \phi^{n+\theta}, \xi \rangle + A_{21}(\phi^{n+\theta}, \xi) = \frac{1}{\epsilon} \langle \phi^{n+\theta} - \phi^{n+\theta}, \xi \rangle, \\ \langle \partial_t \phi^{n+\theta}, \eta \rangle + A_{12}(\phi^{n+\theta}, \eta) = \frac{1}{\epsilon} \langle \phi^{n+\theta} - \phi^{n+\theta}, \eta \rangle. \end{cases}$$

Here we use the notation $g^{n+\theta} := \theta g^{n+1} + (1-\theta) g^n$ with $0 \le \theta \le 1$, for $g = \phi$, and also for $g = e_i$ later in (5.18).

To proceed, we first replace $\partial_t \phi^{n+\theta}$ by the following finite difference approximation, so that

(5.12)
$$\partial_t \phi^{n+\theta} = \frac{\phi^{n+1} - \phi^n}{\tau} - F(n; x),$$

where $F(n : x)$ is a remainder from using Taylor's expansion at t^n and t^{n+1}, respectively,

$$\phi^{n+1} = \phi^n + \tau \partial_t \phi^n + \frac{\tau^2}{2} \partial_t^2 \phi^n + \frac{1}{2} \int_{t^n}^{t^{n+1}} (t^{n+1} - s)^2 \partial_t^3 \phi(x, s) ds,$$

$$\phi^n = \phi^{n+1} - \tau \partial_t \phi^{n+1} + \frac{\tau^2}{2} \partial_t^2 \phi^{n+1} + \frac{1}{2} \int_{t^n}^{t^{n+1}} (t^n - s)^2 \partial_t^3 \phi(x, s) ds,$$

and upon an additional linear combination, we have

(5.13)
$$F(n; x) = \tau \left((1-\theta) \partial_t^2 \phi^n - \theta \partial_t^2 \phi^{n+1} \right)$$
$$+ \frac{1}{2\tau} \int_{t_n}^{t^{n+1}} \partial_t^3 \phi(x, s) \left((1-\theta)(t^{n+1} - s)^2 - \theta(s - t_n)^2 \right) ds.$$

Hence

$$|F(n; x)| \le \tau |1 - 2\theta| |\partial_t^2 \phi(x, t^n)| + \left(\frac{\theta}{2} + \frac{1}{6} \right) \tau^2 \sup_{s \in [0,T]} |\partial_t^3 \phi(x, s)|.$$

Using equation (1.1) we see that

$$\partial_t^3 \phi = \partial_t^2 (\beta \partial_x^2 \phi - \alpha \partial_x \phi) = \partial_t (\beta^2 \partial_x^4 \phi - 2\alpha\beta \partial_x^3 \phi + \alpha^2 \partial_x^2 \phi)$$
$$= \beta^3 \partial_x^6 \phi + 3\alpha^2 \beta \partial_x^4 \phi - 3\alpha\beta^2 \partial_x^5 \phi - \alpha^3 \partial_x^3 \phi.$$

Hence we have

(5.14)
$$\|F(n, \cdot)\| \le \tau |1 - 2\theta| \|\partial_t^2 \phi(\cdot, t^n)\| + C\tau^2 \sup_{t \in [0,T]} \|\phi(\cdot, t)\|_6,$$

where C depends on α, β, θ, but independent of τ and h.

Substitution of (5.13) into (5.11) gives

(5.15)
$$\begin{cases} \langle \frac{\phi^{n+1} - \phi^n}{\tau}, \xi \rangle + A_{21}(\phi^{n+\theta}, \xi) = \frac{1}{\epsilon} \langle \phi^{n+\theta} - \phi^{n+\theta}, \xi \rangle + \langle F(n; x), \xi \rangle, \\ \langle \frac{\phi^{n+1} - \phi^n}{\tau}, \eta \rangle + A_{12}(\phi^{n+\theta}, \eta) = \frac{1}{\epsilon} \langle \phi^{n+\theta} - \phi^{n+\theta}, \eta \rangle + \langle F(n; x), \eta \rangle. \end{cases}$$

Each equation in scheme (5.1) subtracted from the corresponding one in (5.15) leads to

(5.16)
$$
\begin{cases}
\langle \dfrac{\phi^{n+1} - v^{n+1} - (\phi^n - v^n)}{\tau}, \xi \rangle + A_{21}(\phi^{n+\theta} - u^{n+\theta}, \xi) \\
\qquad\qquad = \dfrac{1}{\epsilon} \langle v^{n+\theta} - u^{n+\theta}, \xi \rangle + \langle F(n;x), \xi \rangle, \\
\langle \dfrac{\phi^{n+1} - u^{n+1} - (\phi^n - u^n)}{\tau}, \eta \rangle + A_{12}(\phi^{n+\theta} - v^{n+\theta}, \eta) \\
\qquad\qquad = \dfrac{1}{\epsilon} \langle u^{n+\theta} - v^{n+\theta}, \eta \rangle + \langle F(n;x), \eta \rangle.
\end{cases}
$$

Next we represent the error by

(5.17)
$$
\phi^n - v^n = e_1^n - \epsilon_1^n, \qquad \phi^n - u^n = e_2^n - \epsilon_2^n,
$$

where

$$
e_1^n = \Pi_v \phi^n - v^n, \; \epsilon_1^n = \Pi_v \phi^n - \phi^n,
$$
$$
e_2^n = \Pi_u \phi^n - u^n, \; \epsilon_2^n = \Pi_u \phi^n - \phi^n.
$$

Substituting (5.17) into (5.16), and then taking $\xi = e_1^{n+\theta}, \eta = e_2^{n+\theta}$ in (5.16), we have

(5.18a)
$$
\langle \frac{e_1^{n+1} - e_1^n}{\tau}, e_1^{n+\theta} \rangle + A_{21}(e_2^{n+\theta}, e_1^{n+\theta})
$$
$$
= \langle \frac{\epsilon_1^{n+1} - \epsilon_1^n}{\tau}, e_1^{n+\theta} \rangle + A_{21}(\epsilon_2^{n+\theta}, e_1^{n+\theta})
$$
$$
+ \frac{1}{\epsilon} \langle v^{n+\theta} - u^{n+\theta}, e_1^{n+\theta} \rangle + \langle F(n;x), e_1^{n+\theta} \rangle,
$$

(5.18b)
$$
\langle \frac{e_2^{n+1} - e_2^n}{\tau}, e_2^{n+\theta} \rangle + A_{12}(e_1^{n+\theta}, e_2^{n+\theta})
$$
$$
= \langle \frac{\epsilon_2^{n+1} - \epsilon_2^n}{\tau}, e_2^{n+\theta} \rangle + A_{12}(\epsilon_1^{n+\theta}, e_2^{n+\theta})
$$
$$
+ \frac{1}{\epsilon} \langle u^{n+\theta} - v^{n+\theta}, e_2^{n+\theta} \rangle + \langle F(n;x), e_2^{n+\theta} \rangle.
$$

From (3.13) we have

$$
A_{21}(\epsilon_2^{n+\theta}, e_1^{n+\theta}) + A_{12}(\epsilon_1^{n+\theta}, e_2^{n+\theta}) + \frac{1}{\epsilon} \langle v^{n+\theta} - u^{n+\theta}, e_1^{n+\theta} - e_2^{n+\theta} \rangle
$$
$$
= -\frac{1}{\epsilon} \| e_1^{n+\theta} - e_2^{n+\theta} \|^2 - \langle \epsilon_1^{n+\theta}, e_1^{n+\theta} \rangle - \langle \epsilon_2^{n+\theta}, e_2^{n+\theta} \rangle.
$$

From the stability analysis in leading to (5.7) it follows that

(5.19)
$$
\| (e_1^{n+1}, e_2^{n+1}) \|^2 + C_1 \tau \| (\partial_x e_1^{n+\theta}, \partial_x e_2^{n+\theta}) \|^2
$$
$$
+ C_2 \frac{\tau}{\epsilon} \| e_1^{n+\theta} - e_2^{n+\theta} \|^2 \le \| (e_1^n, e_2^n) \|^2 + 2\tau G,
$$

where

$$
G = \langle \frac{\epsilon_1^{n+1} - \epsilon_1^n}{\tau}, e_1^{n+\theta} \rangle + \langle \frac{\epsilon_2^{n+1} - \epsilon_2^n}{\tau}, e_2^{n+\theta} \rangle - \langle \epsilon_1^{n+\theta}, e_1^{n+\theta} \rangle - \langle \epsilon_2^{n+\theta}, e_2^{n+\theta} \rangle
$$
$$
+ \langle F(n;x), e_1^{n+\theta} + e_2^{n+\theta} \rangle.
$$

We now estimate each term in G as follows:

$$\frac{\epsilon_1^{n+1} - \epsilon_1^n}{\tau} = \Pi_v\left(\frac{\phi^{n+1} - \phi^n}{\tau}\right) - \frac{\phi^{n+1} - \phi^n}{\tau} = \Pi_v(\partial_t\phi(\cdot, t^*)) - \partial_t\phi(\cdot, t^*),$$

where t^* is an intermediate value between t^n and t^{n+1}. Using the projection error estimate (3.6) we have

$$(5.20) \qquad \left\|\frac{\epsilon_1^{n+1} - \epsilon_1^n}{\tau}\right\| \leq Ch^{\min\{k+1, m\}}|\partial_t\phi(\cdot, t^*)|_m$$

$$\leq Ch^{\min\{k+1, m\}}\|\phi(\cdot, t)\|_{m+2},$$

where we have used equation (1.1). The same estimate holds true for $\epsilon_2^{n+1} - \epsilon_2^n$. For $w \in L^2$, we have

$$(5.21) \qquad \langle w, e_i^{n+\theta}\rangle = \theta\langle w, e_i^{n+1}\rangle + (1-\theta)\langle w, e_i^n\rangle$$

$$\leq \delta(\|e_i^{n+1}\|^2 + \|e_i^n\|^2) + \frac{1}{4\delta}\|w\|^2.$$

This when applied to each term in G gives

$$(5.22) \qquad |G| \leq 3\delta(\|(e_1^{n+1}, e_2^{n+1})\|^2 + \|(e_1^n, e_2^n)\|^2)$$

$$+ \frac{1}{4\delta}\sum_{i=1}^2\left(\left\|\frac{\epsilon_1^{n+1} - \epsilon_1^n}{\tau}\right\|^2 + 2\|\epsilon_i^{n+1}\|^2 + 2\|\epsilon_i^n\|^2 + \|F(n; \cdot)\|^2\right).$$

Hence together with (5.20), (3.6), and (5.14) when taking $m = k+1$, we obtain

$$(5.23) \quad |G| \leq 3\delta(\|(e_1^{n+1}, e_2^{n+1})\|^2 + \|(e_1^n, e_2^n)\|^2) + \frac{C}{\delta}(h^{2(k+1)} + \tau^4 + \tau^2(1-2\theta)^2),$$

where $\sup_{t\in[0,T]}\|\phi(\cdot, t)\|_{k+3}$, $\|\partial_t^2\phi^n\|$, and $\sup_{t\in[0,T]}\|\phi(\cdot, t)\|_6$ have been absorbed into the constant C for easy presentation below.

We choose δ to minimize the right side of G_2 so that

$$(5.24) \quad |G| \leq 2\sqrt{3C}(\|(e_1^{n+1}, e_2^{n+1})\| + \|(e_1^n, e_2^n)\|)(h^{2(k+1)} + \tau^4 + \tau^2(1-2\theta)^2)^{1/2}.$$

Plugging (5.24) into (5.19) and dividing by the common factor $(\|(e_1^{n+1}, e_2^{n+1})\| + \|(e_1^n, e_2^n)\|)$ yields

$$\|(e_1^{n+1}, e_2^{n+1})\| \leq \|(e_1^n, e_2^n)\| + 4\tau\sqrt{3C}(h^{2(k+1)} + \tau^4 + \tau^2(1-2\theta)^2)^{1/2}.$$

This gives

$$\|(e_1^n, e_2^n)\| \leq \|(e_1^0, e_2^0)\| + 4T\sqrt{3C}(h^{2(k+1)} + \tau^4 + \tau^2(1-2\theta)^2)^{1/2},$$

since $n\tau \leq T$. Further we use the initial error as given in (3.16), so that

$$\|(e_1^n, e_2^n)\| \leq C(h^{k+1} + \tau|1-2\theta| + \tau^2),$$

where C is linear in T. This together with the projection error for ϵ_i^n when inserted into (5.17) yields the desired estimate (5.10). $\qquad\square$

Acknowledgments

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