MATHEMATICS OF COMPUTATION
Volume 88, Number 317, May 2019, Pages 1061–1090
https://doi.org/10.1090/mcom/3379
Article electronically published on September 10, 2018

FINITE ELEMENT APPROXIMATION OF STEADY FLOWS OF GENERALIZED NEWTONIAN FLUIDS WITH CONCENTRATION-DEPENDENT POWER-LAW INDEX

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Abstract. We consider a system of nonlinear partial differential equations, modeling the motion of a viscous incompressible chemically reacting generalized Newtonian fluid in three space dimensions. The governing system consists of a steady convection-diffusion equation, for the concentration, and a generalized steady power-law-type fluid flow model, for the velocity and the pressure of the fluid, where the viscosity depends on both the shear-rate and the concentration through a concentration-dependent power-law index. The aim of the paper is to perform the mathematical analysis of a finite element approximation of this model. We consider a regularization of the model by introducing an additional term in the momentum equation and construct a finite element approximation of the regularized system. First, the convergence of the finite element method to a weak solution of the regularized model is shown, and we then prove that weak solutions of the regularized problem converge to a weak solution of the original problem.

1. Introduction

The aim of this paper is to establish the convergence of a finite element approximation to a system of nonlinear partial differential equations (PDEs) modeling the rheological response of the synovial fluid. The synovial fluid is a biological fluid found in the cavities of movable joints and is composed of ultrafiltrated blood, called *hyaluronan*. Laboratory experiments have shown that the viscosity of the fluid depends on the concentration of hyaluronan, as well as on the shear-rate. In particular, it was observed in steady shear experiments that the concentration of the hyaluronan is not just a scaling factor of the viscosity (understood as $\nu(c, |Du|) = f(c)\tilde{\nu}(|Du|)$), but it has an influence on the degree of shear-thinning. Therefore, a new mathematical model of the rheological response of the synovial fluid was proposed in [13]. There, the authors considered a power-law-type model for the velocity and the pressure, where the power-law index depends on the concentration, corresponding to the fact that the concentration affects the level of shear-thinning. To close the system, a generalized convection-diffusion equation was assumed to be satisfied by the concentration. For an overview of the rheological background of the model we refer to [13, 16].

Received by the editor September 1, 2017, and, in revised form, April 3, 2018.
2010 *Mathematics Subject Classification.* Primary 65N30; Secondary 74S05, 76A05.
Key words and phrases. Non-Newtonian fluid, variable-exponent spaces, synovial fluid, finite element method.
The first author's work was supported by the UK Engineering and Physical Sciences Research Council [EP/L015811/1].

Based on the description above, we consider the following system of PDEs:

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) = -\nabla p + \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.2}$$

$$\operatorname{div}(c\boldsymbol{u}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) = 0 \qquad \text{in } \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded open Lipschitz domain. In the above system of PDEs, $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$, $c : \overline{\Omega} \to \mathbb{R}_{\geq 0}$ denote the velocity, pressure, and concentration fields, respectively; $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ is a given external force; and $\boldsymbol{D}\boldsymbol{u}$ denotes the symmetric velocity gradient, i.e., $\boldsymbol{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$. To complete the problem, we impose the following Dirichlet boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0}, \qquad c = c_d \qquad \text{on } \partial\Omega, \tag{1.4}$$

where $c_d \in W^{1,s}(\Omega)$ for some $s > d$. By Sobolev embedding, c_d is continuous up to the boundary, and we can therefore define

$$c^- := \min_{x \in \overline{\Omega}} c_d \quad \text{and} \quad c^+ := \max_{x \in \overline{\Omega}} c_d.$$

We further assume that the stress tensor $\boldsymbol{S} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{d \times d}_{\mathrm{sym}} \to \mathbb{R}^{d \times d}_{\mathrm{sym}}$ is a continuous mapping satisfying the following growth, strict monotonicity, and coercivity conditions, respectively: there exist positive constants C_1, C_2, and C_3 such that

$$|\boldsymbol{S}(\xi, \boldsymbol{B})| \leq C_1(|\boldsymbol{B}|^{r(\xi)-1} + 1), \tag{1.5}$$

$$(\boldsymbol{S}(\xi, \boldsymbol{B_1}) - \boldsymbol{S}(\xi, \boldsymbol{B_2})) \cdot (\boldsymbol{B_1} - \boldsymbol{B_2}) > 0 \ \text{ for } \boldsymbol{B_1} \neq \boldsymbol{B_2}, \tag{1.6}$$

$$\boldsymbol{S}(\xi, \boldsymbol{B}) \cdot \boldsymbol{B} \geq C_2(|\boldsymbol{B}|^{r(\xi)} + |\boldsymbol{S}|^{r'(\xi)}) - C_3, \tag{1.7}$$

where $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a Hölder-continuous function satisfying $1 < r^- \leq r(\xi) \leq r^+ < \infty$ and $r'(\xi)$ is defined as its Hölder conjugate, $\frac{r(\xi)}{r(\xi)-1}$. We further assume that the concentration flux vector $\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}_{\mathrm{sym}} \to \mathbb{R}^d$ is a continuous mapping, which is linear with respect to $\boldsymbol{g}$, and it additionally satisfies the following growth and coercivity conditions: there exist positive constants C_4 and C_5 such that

$$|\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B})| \leq C_4 |\boldsymbol{g}|, \tag{1.8}$$

$$\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B}) \cdot \boldsymbol{g} \geq C_5 |\boldsymbol{g}|^2. \tag{1.9}$$

The prototypical examples we have in mind are the following:

$$\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) = \nu(c, |\boldsymbol{D}\boldsymbol{u}|)\boldsymbol{D}\boldsymbol{u}, \qquad \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) = \boldsymbol{K}(c, |\boldsymbol{D}\boldsymbol{u}|)\nabla c,$$

where the viscosity $\nu(c, |\boldsymbol{D}\boldsymbol{u}|)$ is of the form

$$\nu(c, |\boldsymbol{D}\boldsymbol{u}|) \sim \nu_0(\kappa_1 + \kappa_2 |\boldsymbol{D}\boldsymbol{u}|^2)^{\frac{r(c)-2}{2}},$$

where $\nu_0, \kappa_1, \kappa_2$ are positive constants, and $\boldsymbol{K}(\cdot, \cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{d \times d}$ is any continuous function, which is uniformly positive definite and bounded on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

A rigorous mathematical analysis of the existence of global weak solutions to a PDE system, consisting of the generalized Navier–Stokes equations, with a concentration-dependent viscosity coefficient, coupled to a convection-diffusion equation, was initiated in [6]. There, however, the power-law index was fixed, the concentration was a scaling factor of the viscosity, and the authors considered the evolutionary model and established the long-time existence of large-data global weak solutions. Concerning the model (1.1)–(1.9) where the power-law index is concentration-dependent, the mathematical theory was first developed in [7]. The authors established there the existence of weak solutions, provided that $r^- > \frac{3d}{d+2}$, by using generalized monotone operator theory. In [8], with the help of a Lipschitz-truncation technique, the existence of weak solutions with $r^- > \frac{d}{2}$ was proved, and the Hölder continuity of the concentration was shown by using De Giorgi's method.

In [15], the convergence of a finite element approximation to the system (1.1)–(1.9) was shown in two space dimensions, using a discrete De Giorgi regularity result. Because of the absence of an analogous discrete De Giorgi regularity result in three space dimensions, the analysis in [15] was restricted to the case of two space dimensions. The aim of this paper is to extend the analysis developed in [15] to the physically relevant case of three space dimensions. To this end, we exploit a more complicated limiting process here than in [15], so as to avoid reliance on a discrete De Giorgi regularity result. In particular, we consider different meshes for the generalized Navier–Stokes system and the concentration equation, resulting in a two-level Galerkin approximation. This enables us to decouple the passage to the limit in the finite element approximation of the concentration equation from the passage to the limit in the finite element approximation of the generalized Navier–Stokes system, and we thus completely circumvent the need for a discrete De Giorgi estimate.

As a first step, in Section 2 we introduce the necessary notational conventions and auxiliary results, which will be used throughout. In Section 3, we define a suitable regularized problem, which enables us to enlarge the range of the power-law index so as to cover the practically relevant range of values of this index. In Sections 4 and 5, we construct a finite element approximation to the regularized problem and perform a convergence analysis of the numerical method. Finally, in Section 6, we prove that weak solutions of the regularized problem converge to a weak solution of the original problem. The focus of this paper is on theoretical questions; for extensive numerical simulations, based on a $(Q_2, P_1^{\mathrm{disc}})$ mixed finite element approximation of the velocity and the pressure and a Q_2 finite element approximation of the concentration, the reader is referred to Chapters 8–10 in [18].

2. Notation and auxiliary results

In this section, we shall introduce certain function spaces and auxiliary results that will be used throughout the paper. Let $\mathcal{P}$ be the set of all measurable functions $r : \Omega \to [1, \infty]$; we shall call the function $r \in \mathcal{P}(\Omega)$ a variable exponent. We define $r^- := \operatorname{ess\,inf}_{x\in\Omega} r(x)$, $r^+ := \operatorname{ess\,sup}_{x\in\Omega} r(x)$ and consider only the case

$$1 < r^- \leq r^+ < \infty. \tag{2.1}$$

Since we are considering a power-law index depending on the concentration, we need to work with Lebesgue and Sobolev spaces with variable exponents. To be specific, we introduce the following variable-exponent Lebesgue spaces, equipped with the corresponding Luxembourg norms:

$$L^{r(\cdot)}(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_\Omega |u(x)|^{r(x)}\,\mathrm{d}x < \infty \right\},$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} = \|u\|_{r(\cdot)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{r(x)} \mathrm{d}x \leq 1 \right\}.$$

Similarly, we introduce the following generalized Sobolev spaces:

$$W^{1,r(\cdot)}(\Omega) := \left\{ u \in W^{1,1}(\Omega) \cap L^{r(\cdot)}(\Omega) : |\nabla u| \in L^{r(\cdot)} \right\},$$

$$\|u\|_{W^{1,r(\cdot)}(\Omega)} = \|u\|_{1,r(\cdot)} := \inf \left\{ \lambda > 0 : \int_\Omega \left[\left| \frac{u(x)}{\lambda} \right|^{r(x)} + \left| \frac{\nabla u(x)}{\lambda} \right|^{r(x)} \right] \mathrm{d}x \le 1 \right\}.$$

It is easy to show that all of the above spaces are Banach spaces, and because of (2.1), they are all separable and reflexive; see [10].

Furthermore, we recall some function spaces that frequently appear in mathematical models of viscous incompressible fluids. Henceforth, $X(\Omega)^d$ will denote the space of d-component vector-valued functions with components from $X(\Omega)$, and $X(\Omega)^{d\times d}$ will signify the space of $d \times d$ matrix-valued functions with components from $X(\Omega)$. Finally, we define the following spaces:

$$W_0^{1,r(\cdot)}(\Omega)^d := \left\{ \boldsymbol{u} \in W^{1,r(\cdot)}(\Omega)^d : \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega \right\},$$

$$W_{0,\mathrm{div}}^{1,r(\cdot)}(\Omega)^d := \left\{ \boldsymbol{u} \in W_0^{1,r(\cdot)}(\Omega)^d : \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \right\},$$

$$L_0^{r(\cdot)}(\Omega) := \left\{ f \in L^{r(\cdot)}(\Omega) : \int_\Omega f(x)\,\mathrm{d}x = 0 \right\}.$$

Throughout the paper, we shall denote the duality pairing between $f \in X$ and $g \in X^*$ by $\langle g, f\rangle$, and for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}\cdot\boldsymbol{b}$ denotes their scalar product; similarly, for two tensors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}\cdot\mathbf{B}$ signifies their scalar product. Also, for any Lebesgue-measurable set $Q \subset \mathbb{R}^d$, $|Q|$ denotes the Lebesgue measure of the set Q.

Next we introduce the necessary technical tools. First we define the subset $\mathcal{P}^{\log}(\Omega) \subset \mathcal{P}(\Omega)$: it will denote the set of all log-Hölder-continuous functions defined on Ω, that is, the set of all functions $r \in \mathcal{P}(\Omega)$ satisfying

$$(2.2) \qquad |r(x) - r(y)| \le \frac{C_{\log}(r)}{-\log|x-y|} \qquad \forall\, x, y \in \Omega : 0 < |x-y| \le \frac{1}{2}.$$

It is obvious that classical Hölder-continuous functions on Ω automatically belong to this class.

Next we state the following lemma, which summarizes some inequalities involving variable-exponent norms. For proofs, see [10], which is an extensive source of information concerning variable-exponent spaces.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain and let $r \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < r^- \le r^+ < \infty$. Then, the following inequalities hold:*

- *Hölder's inequality, i.e.,*

$$\|fg\|_{s(\cdot)} \le 2\,\|f\|_{r(\cdot)}\,\|g\|_{q(\cdot)}, \ \ \text{with } r, q, s \in \mathcal{P}(\Omega), \ \ \frac{1}{s(x)} = \frac{1}{r(x)} + \frac{1}{q(x)}, \quad x \in \Omega.$$

- *Poincaré's inequality, i.e.,*

$$\|u\|_{r(\cdot)} \le C(d, C_{\log}(r))\,\mathrm{diam}(\Omega)\|\nabla u\|_{r(\cdot)} \qquad \forall\, u \in W_0^{1,r(\cdot)}(\Omega).$$

- *Korn's inequality, i.e.,*

$$\|\nabla \boldsymbol{u}\|_{r(\cdot)} \le C(\Omega, C_{\log}(r))\,\|\boldsymbol{D}\boldsymbol{u}\|_{r(\cdot)} \qquad \forall\, \boldsymbol{u} \in W_0^{1,r(\cdot)}(\Omega)^d,$$

where $C_{\log}(r)$ is the constant in the definition of the class of log-Hölder-continuous functions.

Another important auxiliary result is the existence of the Bogovskiĭ operator in the variable-exponent setting, which is quoted from Section 14.3 in [10].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain and suppose that $r \in \mathcal{P}^{\log}(\Omega)$ with $1 < r^- \le r^+ < \infty$. Then, there exists a bounded linear operator $\mathcal{B} : L_0^{r(\cdot)}(\Omega) \to W_0^{1,r(\cdot)}(\Omega)^d$ such that for all $f \in L_0^{r(\cdot)}(\Omega)$ we have*

$$\operatorname{div}(\mathcal{B}f) = f,$$
$$\|\mathcal{B}f\|_{1,r(\cdot)} \le C\|f\|_{r(\cdot)},$$

where C depends on Ω, r^-, r^+, and $C_{\log}(r)$.

Let us now state the inf-sup condition, which has a crucial role in the mathematical analysis of incompressible fluid flow problems.

Proposition 2.3. *For any s, $s' \in (1,\infty)$, with $\frac{1}{s} + \frac{1}{s'} = 1$, there exists a positive constant $\alpha_s > 0$ such that*

$$\alpha_s \|q\|_{s'} \le \sup_{0 \ne \boldsymbol{v} \in W_0^{1,s}(\Omega)^d} \frac{\langle \operatorname{div} \boldsymbol{v}, q\rangle}{\|\boldsymbol{v}\|_{1,s}} \qquad \forall\, q \in L_0^{s'}(\Omega). \tag{2.3}$$

This is a direct consequence of the existence of the Bogovskiĭ operator in spaces with fixed exponent, which is a special case of Theorem 2.2; see [3,12] for additional details.

Furthermore, one can prove the following inf-sup condition in spaces with variable-exponent norms, which will play an important role in the subsequent analysis.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain and let $r \in \mathcal{P}^{\log}(\Omega)$ with $1 < r^- \le r^+ < \infty$. Then, there exists a constant $\alpha_r > 0$ such that*

$$\alpha_r \|q\|_{r'(\cdot)} \le \sup_{0 \ne \boldsymbol{v} \in W_0^{1,r(\cdot)}(\Omega)^d} \frac{\langle \operatorname{div} \boldsymbol{v}, q\rangle}{\|\boldsymbol{v}\|_{1,r(\cdot)}} \qquad \forall\, q \in L_0^{r'(\cdot)}(\Omega).$$

Proposition 2.4 is a direct consequence of Theorem 2.2 and the norm-conjugate formula stated in the following lemma. See Corollary 3.2.14 in [10] for the proof.

Lemma 2.5. *Let $r \in \mathcal{P}(\Omega)$ be a variable exponent with $1 < r^- \le r^+ < \infty$; then we have*

$$\frac{1}{2}\|f\|_{r(\cdot)} \le \sup_{g \in L^{r'(\cdot)}(\Omega),\ \|g\|_{r'(\cdot)} \le 1} \int_\Omega |f||g| \,\mathrm{d}x$$

for all measurable functions $f \in L^{r(\cdot)}(\Omega)$.

Finally, we recall the following well-known result due to De Giorgi and Nash [9,17]; see also [2] for its application to the system of partial differential equations considered in the present paper.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $s > d$ be fixed. Suppose that $\boldsymbol{K} \in L^\infty(\Omega)^{d\times d}$ is uniformly elliptic with ellipticity constant $\lambda > 0$. Then, there exists an $\alpha \in (0,1)$ such that, for any $\boldsymbol{F} \in L^s(\Omega)^d$, $g \in L^{\frac{ds}{d+s}}(\Omega)$, and any $c_d \in W^{1,s}(\Omega)$, there exists a unique $c \in W^{1,2}(\Omega)$ such that $c - c_d \in W_0^{1,2}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ and*

$$\int_\Omega \boldsymbol{K}\nabla c \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \boldsymbol{F}\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega g\varphi\,\mathrm{d}x \qquad \forall\,\varphi \in W_0^{1,2}(\Omega).$$

Furthermore, the following uniform bound holds:

$$\|c\|_{W^{1,2}\cap C^{0,\alpha}} \le C\left(\Omega, \lambda, s, \|\boldsymbol{K}\|_\infty, \|\boldsymbol{F}\|_s, \|g\|_{\frac{ds}{d+s}}, \|c_d\|_{1,s}\right).$$

Using this notation, the weak formulation of the problem (1.1)–(1.9) is as follows.

Problem (Q). For $\boldsymbol{f} \in (W_0^{1,r^-}(\Omega)^d)^*$, $c_d \in W^{1,s}(\Omega)$, $s > d$, and a Hölder-continuous function r, with $1 < r^- \leq r(c) \leq r^+ < \infty$ for all $c \in [c^-, c^+]$, find $(c - c_d) \in W_0^{1,2}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, $\boldsymbol{u} \in W_0^{1,r(c)}(\Omega)^d$, $p \in L_0^{r'(c)}(\Omega)$ such that

$$\int_\Omega \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla \boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \boldsymbol{\psi} \,\mathrm{d}x - \langle \operatorname{div} \boldsymbol{\psi}, p \rangle = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \qquad \forall\, \boldsymbol{\psi} \in W_0^{1,\infty}(\Omega)^d$$

$$\int_\Omega q \operatorname{div} \boldsymbol{u} \,\mathrm{d}x = 0 \qquad \forall\, q \in L_0^{r'(c)}(\Omega),$$

$$\int_\Omega \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla \varphi - c\boldsymbol{u} \cdot \nabla \varphi \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in W_0^{1,2}(\Omega).$$

Thanks to Proposition 2.4, we can restate Problem (Q) in the following (equivalent) divergence-free setting.

Problem (P). For $\boldsymbol{f} \in (W_0^{1,r^-}(\Omega)^d)^*$, $c_d \in W^{1,s}(\Omega)$, $s > d$, and a Hölder-continuous function r, with $1 < r^- \leq r(c) \leq r^+ < \infty$ for all $c \in [c^-, c^+]$, find $(c - c_d) \in C^{0,\alpha}(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$, $\boldsymbol{u} \in W_{0,\mathrm{div}}^{1,r(c)}(\Omega)^d$, such that

$$\int_\Omega \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla \boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \boldsymbol{\psi} \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \qquad \forall\, \boldsymbol{\psi} \in W_{0,\mathrm{div}}^{1,\infty}(\Omega)^d$$

$$\int_\Omega \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla \varphi - c\boldsymbol{u} \cdot \nabla \varphi \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in W_0^{1,2}(\Omega).$$

From now on, for simplicity, we shall restrict ourselves to the case of $d = 3$. Our results can be however easily extended to the case of any $d \geq 2$. We note in passing that since no uniqueness result is currently known or is expected to hold, for large-data weak solutions of Problem (P) under the general assumptions on the data adopted here, we can only prove that a subsequence of the sequence of discrete solutions converges to a weak solution of the problem.

3. Regularization of the problem

Before constructing the approximation of Problem (Q) we shall formulate a regularized problem; it will then be the regularized problem that will be approximated by a finite element method. We shall show that the sequence of finite element approximations converges to a weak solution of the regularized problem and that solutions of the regularized problem, in turn, converge to a weak solution of Problem (Q). The reason for proceeding in this way is that direct approximation of Problem (Q), which bypasses the use of the regularized problem, necessitates the imposition of an unnaturally strong condition on the variable exponent r in the convergence analysis of the finite element method. More precisely, if no regularization were used, then the condition $r^- > 2$ would need to be assumed in order to be able to apply Theorem 2.6, which would be an overly restrictive hypothesis as it would exclude even the case of a Newtonian fluid, corresponding to r being identically equal to 2. The procedure that we describe below does not suffer from this shortcoming.

Motivated by [5], we shall utilize the following regularized problem, involving the regularization parameter $k \in \mathbb{N}$. We choose $t > 2$, so that $r^- > \frac{3}{2} > \frac{t}{t-2}$. We then seek a weak solution $(\boldsymbol{u}, p, c) := (\boldsymbol{u}^k, p^k, c^k)$ to

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{3.1}$$

$$\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) + \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u} = -\nabla p + \boldsymbol{f} \qquad \text{in } \Omega, \tag{3.2}$$

$$\operatorname{div}(c\boldsymbol{u}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) = 0 \qquad \text{in } \Omega. \tag{3.3}$$

Therefore, we consider the following regularized weak formulation.

Problem (Q*). For $\boldsymbol{f} \in (W_0^{1,r^-}(\Omega)^3)^*$, $c_d \in W^{1,s}(\Omega)$, $s > 3$, and a Hölder-continuous function r, with $1 < r^- \leq r(c) \leq r^+ < \infty$ for all $c \in [c^-, c^+]$, and $r^- > \frac{3}{2} > \frac{t}{t-2}$, $t > 2$, find $(c - c_d) := (c^k - c_d) \in W_0^{1,2}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, $\boldsymbol{u} := \boldsymbol{u}^k \in W_0^{1,r(c^k)}(\Omega)^3$, $p := p^k \in L_0^{r'(c^k)}(\Omega)$ such that

$$\int_\Omega \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla\boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla\boldsymbol{\psi} + \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u} \cdot \boldsymbol{\psi} \,\mathrm{d}x - \langle \operatorname{div} \boldsymbol{\psi}, p\rangle = \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle \qquad \forall\, \boldsymbol{\psi} \in W_0^{1,\infty}(\Omega)^3, \tag{3.4}$$

$$\int_\Omega q \operatorname{div} \boldsymbol{u} \,\mathrm{d}x = 0 \qquad \forall\, q \in L_0^{r'(c)}(\Omega), \tag{3.5}$$

$$\int_\Omega \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla\varphi - c\boldsymbol{u} \cdot \nabla\varphi \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in W_0^{1,2}(\Omega). \tag{3.6}$$

Again, by using Proposition 2.4, we can restate Problem (Q*) in the following (equivalent) divergence-free setting.

Problem (P*). For $\boldsymbol{f} \in (W_0^{1,r^-}(\Omega)^3)^*$, $c_d \in W^{1,s}(\Omega)$, $s > 3$, and a Hölder-continuous function r, with $1 < r^- \leq r(c) \leq r^+ < \infty$ for all $c \in [c^-, c^+]$, and $r^- > \frac{3}{2} > \frac{t}{t-2}$, $t > 2$, find $(c - c_d) := (c^k - c_d) \in C^{0,\alpha}(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$, $\boldsymbol{u} := \boldsymbol{u}^k \in W_{0,\mathrm{div}}^{1,r(c^k)}(\Omega)^3$, such that

$$\int_\Omega \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla\boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla\boldsymbol{\psi} + \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u} \cdot \boldsymbol{\psi} \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle \qquad \forall\, \boldsymbol{\psi} \in W_{0,\mathrm{div}}^{1,\infty}(\Omega)^3, \tag{3.7}$$

$$\int_\Omega \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla\varphi - c\boldsymbol{u} \cdot \nabla\varphi \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in W_0^{1,2}(\Omega). \tag{3.8}$$

We shall formulate the finite element approximation of the regularized problem Problem (Q*) in a three-dimensional domain; the convergence analysis of the method is presented in Sections 4 and 5. In Section 6, we will prove that a sequence of weak solution triples $\{(\boldsymbol{u}^k, p^k, c^k)\}_{k\geq 1}$ of the regularized problem converges to a weak solution triple $(\boldsymbol{u}, p, c)$ of Problem (Q). The latter result is recorded in our next theorem. As will become clear from the subsequent analysis, the condition $r^- > \frac{3}{2} > \frac{t}{t-2}$ is necessary in order for us to be able to obtain a uniform bound on the sequence of approximate pressures (i.e., to prove (5.24)) and to apply Theorem 2.6 in the final step in our passage to the limit (i.e., to prove (6.7)).

Theorem 3.1. *Suppose that $\Omega \subset \mathbb{R}^3$ is a convex polyhedral domain and that $c_d \in W^{1,s}(\Omega)$ for some $s > 3$. Let us further assume that $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a Hölder-continuous function with $r^- > \frac{3}{2} > \frac{t}{t-2}$, $t > 2$, and suppose that $\boldsymbol{f} \in (W_0^{1,r^-}(\Omega)^3)^*$.*

Let $(\boldsymbol{u}^k, p^k, c^k)$ be a weak solution of the regularized problem (3.1)–(3.3). Then, as $k \to \infty$, (a subsequence, not indicated, of) the sequence $\{(\boldsymbol{u}^k, p^k, c^k)\}_{k\geq 1}$ converges to $(\boldsymbol{u}, p, c)$ in the following sense:

$$\begin{aligned}
\boldsymbol{u}^k &\rightharpoonup \boldsymbol{u} && \text{weakly in } W^{1,r^-}_{0,\mathrm{div}}(\Omega)^3,\\
c^k &\rightharpoonup c && \text{weakly in } W^{1,2}(\Omega),\\
c^k &\to c && \text{strongly in } C^{0,\alpha}(\overline{\Omega}) \qquad \text{for some } \alpha \in (0,1),\\
p^k &\rightharpoonup p && \text{weakly in } L^{j'}(\Omega) \qquad \forall\, j > \max\{r^+, 2\}.
\end{aligned}$$

Furthermore, $(\boldsymbol{u}, p, c)$ is a weak solution of Problem (Q) stated in (1.1)–(1.3).

4. Finite element approximation

4.1. **Finite element spaces.** Let $\{\mathcal{G}_n\}_{n=1}^{\infty}$, $\{\mathcal{H}_m\}_{m=1}^{\infty}$ be families of shape-regular partitions of $\overline{\Omega}$ such that the following properties hold:

- **Affine equivalence**: For each element $E \in \mathcal{G}_n$ (or $E \in \mathcal{H}_m$), there exists an invertible affine mapping
$$\boldsymbol{F}_E : E \to \hat{E},$$
where $\hat{E}$ is the standard reference 3-simplex in $\mathbb{R}^3$.
- **Shape-regularity**: For any element $E \in \mathcal{G}_n$ (or $E \in \mathcal{H}_m$), the ratio of $\mathrm{diam}(E)$ to the radius of the inscribed ball is bounded below uniformly by a positive constant, with respect to all $\mathcal{G}_n$ (or $\mathcal{H}_m$) and $n \in \mathbb{N}$ (or $m \in \mathbb{N}$).

For given partitions $\mathcal{G}_n$ and $\mathcal{H}_m$, the finite element spaces are defined by

$$\begin{aligned}
\mathbb{V}^n = \mathbb{V}(\mathcal{G}_n) &:= \{\boldsymbol{V} \in C(\overline{\Omega})^3 : \boldsymbol{V}_{|E} \circ \boldsymbol{F}_E^{-1} \in \hat{\mathbb{P}}_{\mathbb{V}}, E \in \mathcal{G}_n, \text{ and } \boldsymbol{V}_{|\partial\Omega} = \boldsymbol{0}\},\\
\mathbb{Q}^n = \mathbb{Q}(\mathcal{G}_n) &:= \{Q \in L^\infty(\Omega) : Q_{|E} \circ \boldsymbol{F}_E^{-1} \in \hat{\mathbb{P}}_{\mathbb{Q}}, E \in \mathcal{G}_n\},\\
\mathbb{Z}^m = \mathbb{Z}(\mathcal{H}_m) &:= \{Z \in C(\overline{\Omega}) : Z_{|E} \circ \boldsymbol{F}_E^{-1} \in \hat{\mathbb{P}}_{\mathbb{Z}}, E \in \mathcal{H}_m, \text{ and } Z_{|\partial\Omega} = 0\},
\end{aligned}$$

where $\hat{\mathbb{P}}_{\mathbb{V}} \subset W^{1,\infty}(\hat{E})^3$, $\hat{\mathbb{P}}_{\mathbb{Q}} \subset L^\infty(\hat{E})$, and $\hat{\mathbb{P}}_{\mathbb{Z}} \subset W^{1,\infty}(\hat{E})$ are finite-dimensional linear subspaces.

We assume that $\mathbb{V}^n$ and $\mathbb{Z}^m$ have finite and locally supported bases; for example, for each $n \in \mathbb{N}$ and $m \in \mathbb{N}$, there exists an $N_n \in \mathbb{N}$ and an $N_m \in \mathbb{N}$ such that

$$\begin{aligned}
\mathbb{V}^n &= \mathrm{span}\{\boldsymbol{V}_1^n, \dots, \boldsymbol{V}_{N_n}^n\},\\
\mathbb{Z}^m &= \mathrm{span}\{Z_1^m, \dots, Z_{N_m}^m\},
\end{aligned}$$

and for each basis function $\boldsymbol{V}_i^n$, Z_j^m, we have that if there exists an $E \in \mathcal{G}_n$ (respectively, $\mathcal{H}_m$), with $\boldsymbol{V}_i^n \neq 0$ (respectively, $Z_j^m \neq 0$) on E, then

$$\mathrm{supp}\, \boldsymbol{V}_i^n \subset \bigcup \{E' \in \mathcal{G}_n : E' \cap E \neq \emptyset\} =: S_E,$$

$$\mathrm{supp}\, Z_j^m \subset \bigcup \{E' \in \mathcal{H}_m : E' \cap E \neq \emptyset\} =: T_E.$$

For the pressure space $\mathbb{Q}^n$, we assume that $\mathbb{Q}^n$ has a basis consisting of discontinuous piecewise polynomials; i.e., for each $n \in \mathbb{N}$, there exists an $\tilde{N}_n \in \mathbb{N}$ such that

$$\mathbb{Q}^n = \mathrm{span}\{Q_1^n, \dots, Q_{\tilde{N}_n}^n\},$$

and for each basis function Q_i^n, we have that

$$\mathrm{supp}\, Q_i^n = E \qquad \text{for some } E \in \mathcal{G}_n.$$

We assume further that $\mathbb{V}^n$ contains continuous piecewise linear functions and $\mathbb{Q}^n$ contains piecewise constant functions.

Using the assumed shape-regularity we can easily verify that

$$\exists X \in \mathbb{N} : |S_E| \leq X|E| \quad \text{for all } E \in \mathcal{G}_n,$$
$$\exists Y \in \mathbb{N} : |T_E| \leq Y|E| \quad \text{for all } E \in \mathcal{H}_m,$$

where X is independent of n and Y is independent of m. We denote by g_E the diameter of $E \in \mathcal{G}_n$ and by h_E the diameter of $E \in \mathcal{H}_m$.

We also introduce the subspace $\mathbb{V}^n_{\mathrm{div}}$ of discretely divergence-free functions. More precisely, we define

$$\mathbb{V}^n_{\mathrm{div}} := \{\boldsymbol{V} \in \mathbb{V}^n : \langle \operatorname{div} \boldsymbol{V}, Q\rangle = 0 \;\; \forall\, Q \in \mathbb{Q}^n\},$$

and the subspace of $\mathbb{Q}^n$ consisting of vanishing integral mean-value approximations:

$$\mathbb{Q}^n_0 := \{Q \in \mathbb{Q}^n : \int_\Omega Q \,\mathrm{d}x = 0\}.$$

Throughout this paper, we assume that the finite element spaces introduced above have the following minimal approximation properties.

Assumption 1 (Approximability)**.** For all $s \in [1, \infty)$,

$$\begin{aligned}
&\inf_{\boldsymbol{V}\in\mathbb{V}^n} \|\boldsymbol{v} - \boldsymbol{V}\|_{1,s} \to 0 && \forall\, \boldsymbol{v} \in W^{1,s}_0(\Omega)^3 \quad \text{as } n \to \infty,\\
&\inf_{Q\in\mathbb{Q}^n} \|q - Q\|_{s} \to 0 && \forall\, q \in L^s(\Omega) \quad \text{as } n \to \infty,\\
&\inf_{Z\in\mathbb{Z}^m} \|z - Z\|_{1,s} \to 0 && \forall\, z \in W^{1,s}_0(\Omega) \quad \text{as } m \to \infty.
\end{aligned}$$

For these, a necessary condition is that the maximal mesh size vanishes, i.e., that $\max_{E\in\mathcal{G}_n} g_E \to 0$ as $n \to \infty$ and $\max_{E\in\mathcal{H}_m} h_E \to 0$ as $m \to \infty$.

Assumption 2 (Existence of a projection operator Π^n_{div})**.** For each $n \in \mathbb{N}$, there exists a linear projection operator $\Pi^n_{\mathrm{div}} : W^{1,1}_0(\Omega)^3 \to \mathbb{V}^n$ such that:

- Π^n_{div} preserves the divergence structure in the dual of the discrete pressure space; in other words, for any $\boldsymbol{v} \in W^{1,1}_0(\Omega)^3$, we have

$$\langle \operatorname{div} \boldsymbol{v}, Q\rangle = \langle \operatorname{div} \Pi^n_{\mathrm{div}}\boldsymbol{v}, Q\rangle \qquad \forall\, Q \in \mathbb{Q}^n.$$

- Π^n_{div} is locally $W^{1,1}$-stable, i.e., there exists a constant $c_1 > 0$, independent of n, such that

$$\fint_E |\Pi^n_{\mathrm{div}}\boldsymbol{v}| + g_E|\nabla \Pi^n_{\mathrm{div}}\boldsymbol{v}| \,\mathrm{d}x \leq c_1 \fint_{S_E} |\boldsymbol{v}| + g_E|\nabla \boldsymbol{v}| \,\mathrm{d}x \qquad \forall\, \boldsymbol{v} \in W^{1,1}_0(\Omega)^3 \text{ and } \forall\, E \in \mathcal{G}_n. \tag{4.1}$$

Note that the local $W^{1,1}(\Omega)^3$-stability of Π^n_{div} implies its local and global $W^{1,s}(\Omega)^3$-stability for $s \in [1, \infty]$. In other words, for any $s \in [1, \infty]$ we have

$$\|\Pi^n_{\mathrm{div}}\boldsymbol{v}\|_{1,s} \leq c_s \|\boldsymbol{v}\|_{1,s} \qquad \forall\, \boldsymbol{v} \in W^{1,s}_0(\Omega)^3, \tag{4.2}$$

with a constant $c_s > 0$ independent of $n > 0$.

Note further that the approximability (Assumption 1) and inequality (4.2) imply the convergence of $\Pi^n_{\mathrm{div}}\boldsymbol{v}$ to $\boldsymbol{v}$. In fact,

$$\|\boldsymbol{v} - \Pi^n_{\mathrm{div}}\boldsymbol{v}\|_{1,s} \to 0 \qquad \forall\, \boldsymbol{v} \in W^{1,s}_0(\Omega)^3 \quad \text{as } n \to \infty \quad \forall\, s \in [1, \infty); \tag{4.3}$$

see [15] for the details of the proof.

Assumption 3 (Existence of a projection operator $\Pi_{\mathbb{Q}}^n$)**.** For each $n \in \mathbb{N}$, there exists a linear projection operator $\Pi_{\mathbb{Q}}^n : L^1(\Omega) \to \mathbb{Q}^n$ such that $\Pi_{\mathbb{Q}}^n$ is locally L^1-stable; i.e., there exists a constant $c_2 > 0$, independent of n, such that

$$\fint_E |\Pi_{\mathbb{Q}}^n q| \, \mathrm{d}x \le c_2 \fint_{S_E} |q| \, \mathrm{d}x \tag{4.4}$$

for all $q \in L^1(\Omega)$ and all $E \in \mathcal{G}_n$.

Again, we have the following global stability and convergence property:

$$\|\Pi_{\mathbb{Q}}^n q\|_{s'} \le c_{s'} \|q\|_{s'} \qquad \forall\, q \in L^{s'}(\Omega) \quad \forall\, s' \in (1, \infty), \tag{4.5}$$

and

$$\|q - \Pi_{\mathbb{Q}}^n q\|_{s'} \to 0, \qquad \text{as } n \to \infty \quad \text{for all } q \in L^{s'}(\Omega) \text{ and all } s' \in (1, \infty). \tag{4.6}$$

Remark 4.1. According to [1], the following pairs of velocity-pressure finite element spaces satisfy Assumptions 1, 2, and 3, for example:

- The conforming Crouzeix–Raviart Stokes element, i.e., continuous piecewise quadratic plus cubic bubble velocity and discontinuous piecewise linear pressure approximation (compare, e.g., with [4]).
- The space of continuous piecewise quadratic polynomials for the velocity and piecewise constant pressure approximation; see [4].

Our final assumption is the existence of a projection operator for the concentration space.

Assumption 4 (Existence of a projection operator $\Pi_{\mathbb{Z}}^m$)**.** For each $m \in \mathbb{N}$, there exists a linear projection operator $\Pi_{\mathbb{Z}}^m : W_0^{1,1}(\Omega) \to \mathbb{Z}^m$ such that

$$\fint_E |\Pi_{\mathbb{Z}}^m z| + h_E |\nabla \Pi_{\mathbb{Z}}^m z| \, \mathrm{d}x \le c_3 \fint_{T_E} |z| + h_E |\nabla z| \, \mathrm{d}x \qquad \forall\, z \in W_0^{1,1}(\Omega) \text{ and } \forall\, E \in \mathcal{H}_m,$$

where c_3 does not depend on m.

Similarly as above, the projection operator $\Pi_{\mathbb{Z}}^m$ is globally $W^{1,s}$-stable for $s \in [1, \infty]$, and thus, by approximability,

$$\|\Pi_{\mathbb{Z}}^m z - z\|_{1,s} \to 0 \qquad \text{as } m \to \infty \quad \text{for all } z \in W_0^{1,s}(\Omega) \text{ and all } s \in [1, \infty). \tag{4.7}$$

Finally, we state a discrete inf-sup condition, which holds in our finite element setting. It is a direct consequence of (2.3) and the existence of Π_{div}^n; see [1] for further details.

Proposition 4.2. *For s, $s' \in (1, \infty)$ satisfying $\frac{1}{s} + \frac{1}{s'} = 1$, there exists a positive constant $\beta_s > 0$, which is independent of n, such that*

$$\beta_s \|Q\|_{s'} \le \sup_{0 \ne \boldsymbol{V} \in \mathbb{V}^n} \frac{\langle \operatorname{div} \boldsymbol{V}, Q \rangle}{\|\boldsymbol{v}\|_{1,s}} \qquad \forall\, Q \in \mathbb{Q}_0^n \quad \text{and} \quad \forall\, n \in \mathbb{N}.$$

4.2. **The finite element approximation.** In this section, we shall construct the finite element approximation of the problem (3.1)–(3.3). An important property of the incompressible Navier–Stokes equations is that the convective term in the momentum equation is skew-symmetric; this is a consequence of the velocity field $\boldsymbol{u}$ being divergence-free. However, in the discretized problem, we might lose the skew-symmetry because we are considering only discretely divergence-free finite element functions from the finite element space for the velocity. Thus we need to modify the finite element approximation of the convective term in order to ensure that the skew-symmetry is preserved under discretization. We therefore define the following modified convective terms:

$$B_u[\boldsymbol{v},\boldsymbol{w},\boldsymbol{h}] := \frac{1}{2}\int_\Omega ((\boldsymbol{v}\otimes\boldsymbol{h})\cdot\nabla\boldsymbol{w} - (\boldsymbol{v}\otimes\boldsymbol{w})\cdot\nabla\boldsymbol{h})\,\mathrm{d}x,$$
$$B_c[b,\boldsymbol{v},z] := \frac{1}{2}\int_\Omega (z\boldsymbol{v}\cdot\nabla b - b\boldsymbol{v}\cdot\nabla z)\,\mathrm{d}x$$

for all $\boldsymbol{v},\boldsymbol{w},\boldsymbol{h}\in W^{1,\infty}_0(\Omega)^3$, $b,z\in W^{1,\infty}(\Omega)$. These trilinear forms then coincide with the corresponding trilinear forms appearing in the weak formulations of the momentum equation and the concentration equation, provided that we are considering pointwise divergence-free velocity fields. Furthermore, thanks to their skew symmetry, these two trilinear forms now also vanish for discretely divergence-free functions when $\boldsymbol{w}=\boldsymbol{h}$ and $b=z$, respectively. Explicitly, we have

(4.8)
$$\begin{aligned} B_u[\boldsymbol{v},\boldsymbol{v},\boldsymbol{v}] = 0 \quad\text{and}\quad B_c[z,\boldsymbol{v},z]=0 \qquad & \forall\,\boldsymbol{v}\in W^{1,\infty}_0(\Omega)^3,\ \ z\in W^{1,\infty}(\Omega),\\ B_u[\boldsymbol{v},\boldsymbol{w},\boldsymbol{h}] = -\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{w})\cdot\nabla\boldsymbol{h}\,\mathrm{d}x \qquad & \forall\,\boldsymbol{v},\boldsymbol{w},\boldsymbol{h}\in W^{1,\infty}_{0,\mathrm{div}}(\Omega)^3,\\ B_c[b,\boldsymbol{v},z] = -\int_\Omega b\boldsymbol{v}\cdot\nabla z\,\mathrm{d}x \qquad & \forall\,\boldsymbol{v}\in W^{1,\infty}_{0,\mathrm{div}}(\Omega)^3,\ \ b,z\in W^{1,\infty}(\Omega). \end{aligned}$$

Moreover, the trilinear form $B_u[\cdot,\cdot,\cdot]$ is bounded. Indeed, if $\boldsymbol{v},\boldsymbol{w},\boldsymbol{h}\in W^{1,\infty}_0(\Omega)^3$, then, by Hölder's inequality,

$$\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{w})\cdot\nabla\boldsymbol{h}\,\mathrm{d}x \le \|\boldsymbol{v}\|_{2(r^-)'}\|\boldsymbol{w}\|_{2(r^-)'}\|\boldsymbol{h}\|_{1,r^-}$$

and

$$\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{h})\cdot\nabla\boldsymbol{w}\,\mathrm{d}x \le \|\boldsymbol{v}\|_{2(r^-)'}\|\boldsymbol{h}\|_{2(r^-)'}\|\boldsymbol{w}\|_{1,r^-}.$$

Therefore, we obtain the bound

$$|B_u[\boldsymbol{v},\boldsymbol{w},\boldsymbol{h}]| \le \|\boldsymbol{v}\|_{2(r^-)'}\|\boldsymbol{w}\|_{2(r^-)'}\|\boldsymbol{h}\|_{1,r^-} + \|\boldsymbol{v}\|_{2(r^-)'}\|\boldsymbol{w}\|_{1,r^-}\|\boldsymbol{h}\|_{2(r^-)'}. \tag{4.9}$$

Now, we first fix the regularization parameter $k\in\mathbb{N}$ and perform the finite element convergence analysis, for a fixed $k\in\mathbb{N}$, of the regularized problem. We shall denote by n the discretization parameter associated with the velocity and the pressure, and by m the discretization parameter for the concentration (cf. the discussion at the beginning of Section 4), and will pass to the limit, first as $m\to\infty$ and then as $n\to\infty$. Since $k\in\mathbb{N}$ is fixed throughout this section and the next section, it will be omitted from our notation: for example, we shall write $\boldsymbol{U}^{n,m}$ instead of $\boldsymbol{U}^{n,m,k}$, $P^{n,m}$ instead of $P^{n,m,k}$, $C^{n,m}$ instead of $C^{n,m,k}$, and $(\boldsymbol{u},p,c)$ instead of $(\boldsymbol{u}^k,p^k,c^k)$ for a solution triple of the regularized problem. Having passed to the limits $m,n\to\infty$ with $k\in\mathbb{N}$ fixed, we shall reinstate the index k in our notation at the start of Section 6 in preparation for the final passage to the limit $k\to\infty$ with the regularization parameter.

For each $n, m \in \mathbb{N}$, we call a triple $(\boldsymbol{U}^{n,m}, P^{n,m}, C^{n,m}) \in \mathbb{V}^n \times \mathbb{Q}_0^n \times (\mathbb{Z}^m + c_d)$ a discrete solution to the Galerkin approximation if it satisfies

$$\int_\Omega \boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \cdot \boldsymbol{D}\boldsymbol{V} + \frac{1}{k}|\boldsymbol{U}^{n,m}|^{t-2}\, \boldsymbol{U}^{n,m} \cdot \boldsymbol{V} \,\mathrm{d}x + B_u[\boldsymbol{U}^{n,m}, \boldsymbol{U}^{n,m}, \boldsymbol{V}] \\ -\langle \operatorname{div} \boldsymbol{V}, P^{n,m}\rangle = \langle \boldsymbol{f}, \boldsymbol{V}\rangle \quad \forall\, \boldsymbol{V} \in \mathbb{V}^n, \tag{4.10}$$

$$\int_\Omega Q \operatorname{div} \boldsymbol{U}^{n,m} \,\mathrm{d}x = 0 \qquad \forall\, Q \in \mathbb{Q}^n, \tag{4.11}$$

$$\int_\Omega \boldsymbol{q}_c(C^{n,m}, \nabla C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \cdot \nabla Z \,\mathrm{d}x + B_c[C^{n,m}, \boldsymbol{U}^{n,m}, Z] = 0 \qquad \forall\, Z \in \mathbb{Z}^m, \tag{4.12}$$

where $c_d \in W^{1,s}(\Omega)$ with $s > 3$ and $\boldsymbol{f} \in (W_0^{1,r^-}(\Omega)^3)^*$.

If we restrict the test functions $\boldsymbol{V}$ to $\mathbb{V}^n_{\mathrm{div}}$, then the above problem is transformed to the following: find $(\boldsymbol{U}^{n,m}, C^{n,m}) \in \mathbb{V}^n_{\mathrm{div}} \times (\mathbb{Z}^m + c_d)$ satisfying

$$\int_\Omega \boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \cdot \boldsymbol{D}\boldsymbol{V} + \frac{1}{k}|\boldsymbol{U}^{n,m}|^{t-2}\, \boldsymbol{U}^{n,m} \cdot \boldsymbol{V} \,\mathrm{d}x + B_u[\boldsymbol{U}^{n,m}, \boldsymbol{U}^{n,m}, \boldsymbol{V}] \\ = \langle \boldsymbol{f}, \boldsymbol{V}\rangle \quad \forall\, \boldsymbol{V} \in \mathbb{V}^n_{\mathrm{div}}, \tag{4.13}$$

$$\int_\Omega \boldsymbol{q}_c(C^{n,m}, \nabla C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \cdot \nabla Z \,\mathrm{d}x + B_c[C^{n,m}, \boldsymbol{U}^{n,m}, Z] = 0 \qquad \forall\, Z \in \mathbb{Z}^m. \tag{4.14}$$

If $\frac{3}{2} < r^-$, the existence of the discrete solution pair $(\boldsymbol{U}^{n,m}, C^{n,m}) \in \mathbb{V}^n_{\mathrm{div}} \times (\mathbb{Z}^m + c_d)$ follows from a fixed point argument combined with an iteration scheme. Let us briefly summarize the proof of the existence of the pair $(\boldsymbol{U}^{n,m}, C^{n,m}) \in \mathbb{V}^n_{\mathrm{div}} \times (\mathbb{Z}^m + c_d)$. Let $\{\boldsymbol{w}_i\}_{i=1}^{N_n}$ be a basis of $\mathbb{V}^n_{\mathrm{div}} \subset W_0^{1,\infty}(\Omega)^3$ such that $\int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j \,\mathrm{d}x = \delta_{ij}$ and let $\{z_j\}_{j=1}^{N_m}$ be a basis of $\mathbb{Z}^m \subset W_0^{1,2}(\Omega)$ such that $\int_\Omega z_i z_j \,\mathrm{d}x = \delta_{ij}$. Then, for fixed $n, m \in \mathbb{N}$, we define the Galerkin approximations

$$\boldsymbol{U}^{n,m} := \sum_{i=1}^{N_n} \alpha_i^{n,m} \boldsymbol{w}_i, \qquad C^{n,m} := \sum_{i=1}^{N_m} \beta_i^{n,m} z_i + c_d, \tag{4.15}$$

which satisfy (4.13) and (4.14).

First we define $C_1^{n,m} := c_d \in \mathbb{Z}^m + c_d$. Then, for any $\ell \in \mathbb{N}$, we define $\boldsymbol{U}_\ell^{n,m} \in \mathbb{V}^n_{\mathrm{div}}$ as a solution of the finite-dimensional problem

$$\int_\Omega \boldsymbol{S}(C_\ell^{n,m}, \boldsymbol{D}\boldsymbol{U}_\ell^{n,m}) \cdot \boldsymbol{D}\boldsymbol{V} + \frac{1}{k}|\boldsymbol{U}_\ell^{n,m}|^{t-2}\, \boldsymbol{U}^{n,m} \cdot \boldsymbol{V} \,\mathrm{d}x + B_u[\boldsymbol{U}_\ell^{n,m}, \boldsymbol{U}_\ell^{n,m}, \boldsymbol{V}] \\ = \langle \boldsymbol{f}, \boldsymbol{V}\rangle \qquad \forall\, \boldsymbol{V} \in \mathbb{V}^n_{\mathrm{div}},$$

and $C_\ell^{n,m} \in \mathbb{Z}^m + c_d$ as a solution of the finite-dimensional problem

$$\int_\Omega \boldsymbol{q}_c(C_\ell^{n,m}, \nabla C_\ell^{n,m}, \boldsymbol{D}\boldsymbol{U}_{\ell-1}^{n,m}) \cdot \nabla Z \,\mathrm{d}x + B_c[C_\ell^{n,m}, \boldsymbol{U}_{\ell-1}^{n,m}, Z] = 0 \qquad \forall\, Z \in \mathbb{Z}^m.$$

The existence of the functions $\boldsymbol{U}_\ell^{n,m} \in \mathbb{V}^n_{\mathrm{div}}$ and $C_\ell^{n,m} \in \mathbb{Z}^m + c_d$ is easily shown by means of Brouwer's fixed point theorem. Furthermore, for each $n, m \in \mathbb{N}$, the sequences of functions $\{\boldsymbol{U}_\ell^{n,m}\}_{\ell=1}^\infty$ and $\{C_\ell^{n,m}\}_{\ell=1}^\infty$ satisfy the following uniform bounds:

$$\|\boldsymbol{U}_\ell^{n,m}\|_{1,r^-} + \|\boldsymbol{U}_\ell^{n,m}\|_t \le C_1, \qquad \|\nabla C_\ell^{n,m}\|_2 \le C_2,$$

where C_1 and C_2 are positive constants, independent of ℓ. Thus, by the Bolzano–Weierstrass theorem we deduce the existence of limits $\boldsymbol{U}^{n,m} \in \mathbb{V}^n_{\mathrm{div}}$ and $C^{n,m} \in \mathbb{Z}^m + c_d$ for $\boldsymbol{U}^{n,m}_\ell$ and $C^{n,m}_\ell$, respectively, as $\ell \to \infty$, and these limits form a solution pair for the Galerkin approximation (4.13) and (4.14). For further details, see [15]. This establishes the existence of a solution to the Galerkin approximations (4.13) and (4.14) for any fixed pair of integers $n, m \in \mathbb{N}$. The existence of a discrete solution triple for (4.10)–(4.12) then follows by the discrete inf-sup condition stated in Proposition 4.2, and we write $P^{n,m} = \sum_{i=1}^{\tilde{N}_n} \gamma_i^{n,m} y_i$ where $\{y_i\}_{i=1}^{\tilde{N}_n}$ is a basis of $\mathbb{Q}^n_0$.

We are now ready to state and prove our main theorem in this section. It asserts that, as $n, m \to \infty$, the sequence of discrete solution triples converges to a weak solution triple of the regularized problem.

Theorem 4.3. *Suppose that $\Omega \subset \mathbb{R}^3$ is a convex polyhedral domain and that $c_d \in W^{1,s}(\Omega)$ for some $s > 3$. Let us assume that $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a Hölder-continuous function with $r^- > \frac{3}{2} > \frac{t}{t-2}$, $t > 2$, and let $\boldsymbol{f} \in (W^{1,r^-}_0(\Omega)^3)^*$. Let $(\boldsymbol{U}^{n,m}, P^{n,m}, C^{n,m}) \in \mathbb{V}^n_{\mathrm{div}} \times \mathbb{Q}^n_0 \times (\mathbb{Z}^m + c_d)$ be a discrete solution triple defined by the finite element approximation (4.10)–(4.12). Then, the following convergence results hold.*

- *At the first level of Galerkin approximation, there exist subsequences (not relabeled) with respect to m such that (as $m \to \infty$)*
$$\begin{aligned} \boldsymbol{U}^{n,m} &\to \boldsymbol{U}^n && \textit{uniformly on } \overline{\Omega}, \\ \boldsymbol{D}\boldsymbol{U}^{n,m} &\to \boldsymbol{D}\boldsymbol{U}^n && \textit{uniformly on } \overline{\Omega}, \\ P^{n,m} &\to P^n && \textit{uniformly on } \overline{\Omega}, \\ C^{n,m} &\rightharpoonup C^n && \textit{weakly in } W^{1,2}(\Omega), \end{aligned}$$
where $\boldsymbol{U}^n \in \mathbb{V}^n$, $P^n \in \mathbb{Q}^n_0$.
- *At the second level of Galerkin approximation, there exist subsequences (not relabeled) with respect to n such that (as $n \to \infty$)*
$$\begin{aligned} \boldsymbol{U}^n &\rightharpoonup \boldsymbol{u} && \textit{weakly in } W^{1,r^-}_0(\Omega)^3, \\ P^n &\rightharpoonup p && \textit{weakly in } L^{j'}(\Omega) \qquad \forall\, j > \max\{r^+, 2\}, \\ C^n &\rightharpoonup c && \textit{weakly in } W^{1,2}(\Omega), \\ C^n &\to c && \textit{strongly in } C^{0,\alpha}(\overline{\Omega}), \end{aligned}$$
where $(\boldsymbol{u}, p, c) = (\boldsymbol{u}^k, p^k, c^k)$ is a weak solution triple of the regularized problem (3.4)–(3.6).

5. Proof of Theorem 4.3

5.1. **The limit $m \to \infty$.** First, we shall derive some uniform bounds, independent of $m \in \mathbb{N}$, and let m tend to infinity by using the weak compactness properties in the corresponding reflexive spaces. For simplicity, we shall denote $\boldsymbol{S}^{n,m} := \boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m})$, $\boldsymbol{q}^{n,m}_c := \boldsymbol{q}_c(C^{n,m}, \nabla C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m})$.

We test with $\boldsymbol{U}^{n,m} \in \mathbb{V}^n_{\mathrm{div}}$ in (4.10); then, thanks to the skew symmetry of $B_u[\cdot,\cdot,\cdot]$, we have
$$\int_\Omega \boldsymbol{S}^{n,m} \cdot \nabla \boldsymbol{U}^{n,m} + \frac{1}{k}|\boldsymbol{U}^{n,m}|^t \,\mathrm{d}x = \int_\Omega \boldsymbol{S}^{n,m} \cdot \boldsymbol{D}\boldsymbol{U}^{n,m} + \frac{1}{k}|\boldsymbol{U}^{n,m}|^t \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{U}^{n,m} \rangle.$$

By (1.7) and Young's inequality, we have

$$\int_\Omega |\nabla \boldsymbol{U}^{n,m}|^{r(C^{n,m})} + |\boldsymbol{S}^{n,m}|^{r'(C^{n,m})} + |\boldsymbol{U}^{n,m}|^t \,\mathrm{d}x \le C_1, \tag{5.1}$$

where C_1 is independent of m, n, and k.

Next, we test with $C^{n,m} - c_d \in \mathbb{Z}^m$ in (4.12) and deduce that

$$\int_\Omega \boldsymbol{q}_c(C^{n,m}, \nabla C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \cdot \nabla (C^{n,m} - c_d)\,\mathrm{d}x = B_c[C^{n,m}, \boldsymbol{U}^{n,m}, c_d].$$

By (1.8), (1.9), Hölder's inequality, and Young's inequality,

$$\begin{aligned} \|\nabla C^{n,m}\|_2^2 &\le \int_\Omega |\nabla C^{n,m}||\nabla c_d|\,\mathrm{d}x + B_c[C^{n,m}, \boldsymbol{U}^{n,m}, c_d] \\ &\le \varepsilon \|\nabla C^{n,m}\|_2^2 + C(\varepsilon)\|\nabla c_d\|_2^2 + B_c[C^{n,m}, \boldsymbol{U}^{n,m}, c_d]. \end{aligned}$$

Then, by Sobolev embedding,

$$\begin{aligned} B_c[C^{n,m}, \boldsymbol{U}^{n,m}, c_d] &= \frac{1}{2}\int_\Omega c_d \boldsymbol{U}^{n,m} \cdot \nabla C^{n,m}\,\mathrm{d}x - \frac{1}{2}\int_\Omega C^{n,m}\boldsymbol{U}^{n,m}\cdot \nabla c_d\,\mathrm{d}x \\ &= \int_\Omega c_d \boldsymbol{U}^{n,m}\cdot \nabla C^{n,m}\,\mathrm{d}x + \frac{1}{2}\int_\Omega C^{n,m}(\operatorname{div} \boldsymbol{U}^{n,m}) c_d\,\mathrm{d}x \\ &\le \|c_d\|_\infty \|\boldsymbol{U}^{n,m}\|_2 \|\nabla C^{n,m}\|_2 + \frac{1}{2}\|c_d\|_\infty \|C^{n,m}\|_{(r^-)'} \|\operatorname{div} \boldsymbol{U}^{n,m}\|_{r^-} \\ &\le C\|\boldsymbol{U}^{n,m}\|_{1,r^-}\|\nabla C^{n,m}\|_2 + C\|\boldsymbol{U}^{n,m}\|_{1,r^-}\|\nabla C^{n,m}\|_{\frac{3r^-}{4r^- -3}} \\ &\le C(\varepsilon)\|\boldsymbol{U}^{n,m}\|_{1,r^-}^2 + \varepsilon\|\nabla C^{n,m}\|_2^2. \end{aligned}$$

Hence, by (1.8) and (5.1), we have

$$\int_\Omega |\nabla C^{n,m}|^2 + |\boldsymbol{q}_c^{n,m}|^2\,\mathrm{d}x \le C_2, \tag{5.2}$$

where C_2 is independent of m, n, and k.

Next, we shall derive a uniform bound on the pressure. By Proposition 4.2 together with (4.10), (4.9), and the equivalence of norms in the finite-dimensional spaces, we have

$$\begin{aligned} \beta_r \|P^{n,m}\|_{(r^+)'} &\le \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{\langle \operatorname{div} \boldsymbol{V}, P^{n,m}\rangle}{\|\boldsymbol{V}\|_{1,r^+}} \\ &\le \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{|\int_\Omega \boldsymbol{S}^{n,m}\cdot \boldsymbol{D}\boldsymbol{V}\,\mathrm{d}x|}{\|\boldsymbol{V}\|_{1,r^+}} + C \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{|B_u[\boldsymbol{U}^{n,m}, \boldsymbol{U}^{n,m}, \boldsymbol{V}] - \langle \boldsymbol{f}, \boldsymbol{V}\rangle|}{\|\boldsymbol{V}\|_{1,r^-}} \\ &\le C \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{\|\boldsymbol{S}^{n,m}\|_{(r^+)'}\|\boldsymbol{D}\boldsymbol{V}\|_{r^+}}{\|\boldsymbol{V}\|_{1,r^+}} \\ &\quad + C(n) \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{\|\boldsymbol{U}^{n,m}\|_{2(r^-)'}^2 \|\boldsymbol{V}\|_{1,r^-} + \|\boldsymbol{f}\|_{-1}\|\boldsymbol{V}\|_{1,r^-}}{\|\boldsymbol{V}\|_{1,r^-}}, \end{aligned}$$

where C and $C(n)$ are independent of m. Therefore, by (5.1), we deduce that

$$\|P^{n,m}\|_{(r^+)'} \le C(n). \tag{5.3}$$

Now we are ready to let m tend to infinity. By (5.1) and (5.3) with the equivalence of norms in finite-dimensional spaces, we have $|\boldsymbol{\alpha}^{n,m}| \le C_1(n)$ and $|\boldsymbol{\gamma}^{n,m}| \le C_2(n)$, where $C_1(n)$ and $C_2(n)$ are independent of m. Then, together with the uniform estimates (5.2), we can extract (not relabeled) subsequences such that

$$\boldsymbol{\alpha}^{n,m} \to \boldsymbol{\alpha}^n \qquad \text{strongly in } \mathbb{R}^{N_n}, \tag{5.4}$$

$$\boldsymbol{\gamma}^{n,m} \to \boldsymbol{\gamma}^n \qquad \text{strongly in } \mathbb{R}^{\tilde{N}_n}, \tag{5.5}$$

$$C^{n,m} \rightharpoonup C^n \qquad \text{weakly in } W^{1,2}(\Omega). \tag{5.6}$$

From (5.4), (5.5), (5.6), and compact embedding, we have

$$\boldsymbol{U}^{n,m} \to \boldsymbol{U}^n \qquad \text{uniformly on } \overline{\Omega}, \tag{5.7}$$

$$\boldsymbol{D}\boldsymbol{U}^{n,m} \to \boldsymbol{D}\boldsymbol{U}^n \qquad \text{uniformly on } \overline{\Omega}, \tag{5.8}$$

$$P^{n,m} \to P^n \qquad \text{uniformly on } \overline{\Omega}, \tag{5.9}$$

$$C^{n,m} \to C^n \qquad \text{strongly in } L^2(\Omega). \tag{5.10}$$

By (5.4) and (5.5), note that

$$\boldsymbol{U}^n \in \mathbb{V}^n \qquad \text{and} \qquad P^n \in \mathbb{Q}_0^n.$$

Finally, from (5.10), we can extract a further subsequence (not relabeled) such that

$$C^{n,m} \to C^n \qquad \text{a.e. in } \Omega. \tag{5.11}$$

Note that since $\boldsymbol{S}$ is continuous, by (5.11) and (5.8), we have that

$$\boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \to \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) \qquad \text{a.e. in } \Omega.$$

Now, by (5.8), we have that, for sufficiently large $m \in \mathbb{N}$,

$$|\boldsymbol{D}\boldsymbol{U}^{n,m}| < 1 + |\boldsymbol{D}\boldsymbol{U}^n| \qquad \text{for a.e. } x \in \Omega.$$

Thus, by (1.5), we have, for sufficiently large $m \in \mathbb{N}$, that

$$\begin{aligned} |\boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m})| &\le C|\boldsymbol{D}\boldsymbol{U}^{n,m}|^{r(C^{n,m})-1} + C \\ &\le C(1 + |\boldsymbol{D}\boldsymbol{U}^n|)^{r(c^{n,m})-1} + C \\ &\le C(1 + |\boldsymbol{D}\boldsymbol{U}^n|)^{r^+-1} + C, \end{aligned}$$

and $C(1 + |\boldsymbol{D}\boldsymbol{U}^n|)^{r^+-1} + C \in L^{(r^+)'}(\Omega)$, where C is independent of m. Therefore, by the Dominated Convergence Theorem, we have

$$\boldsymbol{S}^{n,m} \to \boldsymbol{S}^n := \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) \qquad \text{strongly in } L^{(r^+)'}(\Omega)^{3\times 3}. \tag{5.12}$$

Furthermore, by (5.11) and (5.8), together with the Dominated Convergence Theorem,

$$\boldsymbol{K}(C^{n,m}, |\boldsymbol{D}\boldsymbol{U}^{n,m}|) \to \boldsymbol{K}(C^n, |\boldsymbol{D}\boldsymbol{U}^n|) \qquad \text{strongly in } L^q(\Omega) \qquad \forall\, q \in (1, \infty).$$

Therefore, together with (5.6), we have

$$\boldsymbol{q}_c^{n,m} \rightharpoonup \boldsymbol{q}_c^n := \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \qquad \text{weakly in } L^2(\Omega)^3. \tag{5.13}$$

Now we are ready to pass m to infinity in the Galerkin approximation (4.10)–(4.12). First, by (5.7) and (5.8),

$$B_u[\boldsymbol{U}^{n,m}, \boldsymbol{U}^{n,m}, \boldsymbol{V}] \to B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}] \qquad \forall\, \boldsymbol{V} \in \mathbb{V}^n,$$

$$\frac{1}{k}|\boldsymbol{U}^{n,m}|^{t-2}\, \boldsymbol{U}^{n,m} \cdot \boldsymbol{V} \to \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\, \boldsymbol{U}^n \cdot \boldsymbol{V} \qquad \forall\, \boldsymbol{V} \in \mathbb{V}^n.$$

Furthermore, from (5.12) and (5.9),

$$\int_\Omega \boldsymbol{S}^{n,m} \cdot \boldsymbol{D}\boldsymbol{V}\,\mathrm{d}x \to \int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}\boldsymbol{V}\,\mathrm{d}x \qquad \forall\, \boldsymbol{V} \in \mathbb{V}^n,$$
$$\langle \operatorname{div} \boldsymbol{V}, P^{n,m}\rangle \to \langle \operatorname{div} \boldsymbol{V}, P^n\rangle \qquad \forall\, \boldsymbol{V} \in \mathbb{V}^n.$$

Therefore, we have

(5.14)
$$\int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}\boldsymbol{V} + \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\boldsymbol{U}^n \cdot \boldsymbol{V}\,\mathrm{d}x + B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}] - \langle \operatorname{div} \boldsymbol{V}, P^n\rangle = \langle \boldsymbol{f}, \boldsymbol{V}\rangle \qquad \forall\, \boldsymbol{V} \in \mathbb{V}^n.$$

Moreover, from (4.11) and (5.8),

$$\int_\Omega Q \operatorname{div} \boldsymbol{U}^n\,\mathrm{d}x = 0 \qquad \forall\, Q \in \mathbb{Q}^n. \tag{5.15}$$

Next, let us investigate the limit of the concentration equation (4.12). We fix an arbitrary $Z \in W_0^{1,2}(\Omega)$ and define $Z^m := \Pi_{\mathbb{Z}}^m Z \in \mathbb{Z}^m$. Thanks to (5.7) and (5.10),

$$\begin{aligned}\|C^{n,m}\boldsymbol{U}^{n,m} - C^n\boldsymbol{U}^n\|_2 &\le \|(\boldsymbol{U}^{n,m} - \boldsymbol{U}^n)C^{n,m}\|_2 + \|\boldsymbol{U}^n(C^{n,m} - C^n)\|_2 \\ &\le \|\boldsymbol{U}^{n,m} - \boldsymbol{U}^n\|_\infty \|C^{n,m}\|_2 + \|\boldsymbol{U}^n\|_\infty \|C^{n,m} - C^n\|_2 \to 0.\end{aligned}$$

Also, thanks to (5.7) and (4.7),

$$\begin{aligned}\|Z^m\boldsymbol{U}^{n,m} - Z\boldsymbol{U}^n\|_2 &\le \|(\boldsymbol{U}^{n,m} - \boldsymbol{U}^n)Z^m\|_2 + \|\boldsymbol{U}^n(Z^m - Z)\|_2 \\ &\le \|\boldsymbol{U}^{n,m} - \boldsymbol{U}^n\|_\infty \|Z^m\|_2 + \|\boldsymbol{U}^n\|_\infty \|Z^m - Z\|_2 \to 0.\end{aligned}$$

In other words, we have

$$C^{n,m}\boldsymbol{U}^{n,m} \to C^n\boldsymbol{U}^n \quad \text{strongly in } L^2(\Omega)^3, \tag{5.16}$$
$$Z^m\boldsymbol{U}^{n,m} \to Z\boldsymbol{U}^n \quad \text{strongly in } L^2(\Omega)^3. \tag{5.17}$$

By (5.17) and (5.6),

$$\begin{aligned}&\left|\int_\Omega Z^m\boldsymbol{U}^{n,m} \cdot \nabla C^{n,m}\,\mathrm{d}x - \int_\Omega Z\boldsymbol{U}^n \cdot \nabla C^n\,\mathrm{d}x\right| \\ &\quad \le \int_\Omega |Z^m\boldsymbol{U}^{n,m} - Z\boldsymbol{U}^n||\nabla C^{n,m}|\,\mathrm{d}x + \left|\int_\Omega Z\boldsymbol{U}^n(\nabla C^{n,m} - \nabla C^n)\,\mathrm{d}x\right| \to 0.\end{aligned}$$

Moreover, from (5.16) and (4.7),

$$\begin{aligned}&\left|\int_\Omega C^{n,m}\boldsymbol{U}^{n,m} \cdot \nabla Z^m\,\mathrm{d}x - \int_\Omega C^n\boldsymbol{U}^n \cdot \nabla Z\,\mathrm{d}x\right| \\ &\quad \le \|C^{n,m}\boldsymbol{U}^{n,m}\|_2 \|Z^m - Z\|_{1,2} + \|Z\|_{1,2}\|C^{n,m}\boldsymbol{U}^{n,m} - C^n\boldsymbol{U}^n\|_2 \to 0.\end{aligned}$$

Therefore, we have

$$\lim_{m\to\infty} B_c[C^{n,m}, \boldsymbol{U}^{n,m}, Z^m] = B_c[C^n, \boldsymbol{U}^n, Z].$$

Finally, from (5.13),

$$\int_\Omega \boldsymbol{q}_c^{n,m} \cdot \nabla Z^m\,\mathrm{d}x \to \int_\Omega \boldsymbol{q}_c^n \cdot \nabla Z\,\mathrm{d}x \qquad \text{as } m \to \infty.$$

Altogether, we have

$$\int_\Omega \boldsymbol{q}_c^n \cdot \nabla Z\,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, Z] = 0 \qquad \forall\, Z \in W_0^{1,2}(\Omega). \tag{5.18}$$

5.2. **The limit $n \to \infty$.** Now we shall derive further uniform estimates, independent of $n \in \mathbb{N}$, and let n pass to infinity. First, we test with $\boldsymbol{U}^n$ in (5.14). Then, by (4.8) and (5.15), we have

$$\int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}\boldsymbol{U}^n + \frac{1}{k}|\boldsymbol{U}^n|^t \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{U}^n \rangle.$$

By using (1.7) and Young's inequality, we have

$$\int_\Omega |\boldsymbol{D}\boldsymbol{U}^n|^{r(C^n)} + |\boldsymbol{S}^n|^{r'(C^n)} + \frac{1}{k}|\boldsymbol{U}^n|^t \,\mathrm{d}x \le C_1, \tag{5.19}$$

where C_1 is independent of n and k, which leads us to

$$\|\boldsymbol{U}^n\|_{1,r^-}^{r^-} + \|\boldsymbol{S}^n\|_{(r^+)'}^{(r^+)'} + \frac{1}{k}\|\boldsymbol{U}^n\|_t^t \le C_1, \tag{5.20}$$

where C_1 is independent of n and k.

Next, we test with $C^n - c_d$ in (5.18), and by (4.8) we obtain

$$\int_\Omega \boldsymbol{q}_c^n \cdot \nabla C^n \,\mathrm{d}x = \int_\Omega \boldsymbol{q}_c^n \cdot \nabla c_d \,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, c_d].$$

From (1.8), (1.9), Hölder's inequality, and Young's inequality we have

$$\begin{aligned}\|\nabla C^n\|_2^2 &\le C \int_\Omega |\nabla C^n||\nabla c_d| \,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, c_d] \\ &\le \varepsilon \|\nabla C^n\|_2^2 + C(\varepsilon)\|\nabla c_d\|_2^2 + B_c[C^n, \boldsymbol{U}^n, c_d].\end{aligned}$$

Furthermore, by Sobolev embedding,

$$\begin{aligned}B_c[C^n, \boldsymbol{U}^n, c_d] &= \frac{1}{2}\int_\Omega c_d \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x - \frac{1}{2}\int_\Omega C^n \boldsymbol{U}^n \cdot \nabla c_d \,\mathrm{d}x \\ &= \int_\Omega c_d \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x + \frac{1}{2}\int_\Omega C^n (\operatorname{div} \boldsymbol{U}^n) c_d \,\mathrm{d}x \\ &\le \|c_d\|_\infty \|\boldsymbol{U}^n\|_2 \|\nabla C^n\|_2 + \frac{\|c_d\|_\infty}{2}\|C^n\|_{(r^-)'}\|\operatorname{div} \boldsymbol{U}^n\|_{r^-} \\ &\le C\|\boldsymbol{U}^n\|_{1,r^-}\|\nabla C^n\|_2 + C\|\boldsymbol{U}^n\|_{1,r^-}\|\nabla C^n\|_{\frac{3r^-}{4r^- - 3}} \\ &\le C(\varepsilon)\|\boldsymbol{U}^n\|_{1,r^-}^2 + \varepsilon\|\nabla C^n\|_2^2.\end{aligned}$$

Hence, from (1.8) and (5.19),

$$\int_\Omega |\nabla C^n|^2 + |\boldsymbol{q}_c^n|^2 \,\mathrm{d}x \le C_2, \tag{5.21}$$

where C_2 is independent of n and k. Thus we have

$$\|C^n\|_{1,2}^2 + \|\boldsymbol{q}_c^n\|_2^2 \le C_2, \tag{5.22}$$

where C_2 is independent of n and k.

Now, since $\frac{3}{2} > \frac{t}{t-2}$ for $t > 2$, by Sobolev embedding and the uniform estimates (5.19) and (5.21), for $s > 3$ sufficiently close to 3,

$$\|C^n \boldsymbol{U}^n\|_s \leq \|C^n\|_6 \|\boldsymbol{U}^n\|_{\frac{6s}{6-s}} \leq C\|C^n\|_{1,2}\|\boldsymbol{U}^n\|_t \leq C,$$

where C is independent of n. Also, for $s > 3$ sufficiently close to 3, we have

$$\|\nabla C^n \cdot \boldsymbol{U}^n\|_{\frac{3s}{s+3}} \leq \|\nabla C^n\|_2 \|\boldsymbol{U}^n\|_{\frac{6s}{6-s}} \leq C\|C^n\|_{1,2}\|\boldsymbol{U}^n\|_t \leq C,$$

where C is independent of n. Note that if we considered the original equation (1.2) instead of the regularized problem (3.2), we would only be able to deduce that $\|\boldsymbol{U}^n\|_{3r^-/(3-r^-)} \leq C$, which would force us to assume the strong condition $r^- > 2$ so as to be able to prove the above estimates.

Therefore, we can apply Theorem 2.6 with $\boldsymbol{F} = C^n\boldsymbol{U}^n$ and $g = \nabla C^n \cdot \boldsymbol{U}^n$. Hence, there exists an $\alpha_1 \in (0,1)$ such that

$$\|C^n\|_{C^{0,\alpha_1}(\overline{\Omega})} \leq C_3, \tag{5.23}$$

where C_3 is independent of n. Since $C^{0,\alpha_1}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{0,\tilde{\alpha}_1}(\overline{\Omega})$ for all $\tilde{\alpha}_1 \in (0,\alpha_1)$, we have by (5.23) that

$$C^n \to c \qquad \text{strongly in } C^{0,\tilde{\alpha}_1}(\overline{\Omega}),$$

which implies that

$$r \circ C^n \to r \circ c \qquad \text{strongly in } C^{0,\beta_1}(\overline{\Omega}),$$

for some $\beta_1 \in (0,1)$.

We now apply Proposition 4.2. For a given $r^+ > 0$, choose $j > \max\{r^+, 2\}$. Then, since $r^- > \frac{3}{2}$, we have that $W_0^{1,j}(\Omega)^3 \hookrightarrow L^{2(r^-)'}(\Omega)^3$ by Sobolev embedding. Furthermore, since $\frac{t}{t-2} < r^-$, we have that $2(r^-)' < t$. Now, from (4.10) and (4.9), we have

$$\begin{aligned}
\beta_r\|P^n\|_{j'} &\leq \sup_{0\neq \boldsymbol{V}\in\mathbb{V}^n} \frac{\langle \operatorname{div} \boldsymbol{V}, P^n\rangle}{\|\boldsymbol{V}\|_{1,j}} \\
&\leq \sup_{0\neq \boldsymbol{V}\in\mathbb{V}^n} \frac{|\int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}\boldsymbol{V}\,\mathrm{d}x + B_u[\boldsymbol{U}^n,\boldsymbol{U}^n,\boldsymbol{V}] - \langle \boldsymbol{f},\boldsymbol{V}\rangle|}{\|\boldsymbol{V}\|_{1,j}} \\
&\leq C \sup_{0\neq \boldsymbol{V}\in\mathbb{V}^n} \frac{\|\boldsymbol{S}^n\|_{(r^+)'}\|\boldsymbol{V}\|_{1,r^+}}{\|\boldsymbol{V}\|_{1,r^+}} \\
&\quad + C \sup_{0\neq \boldsymbol{V}\in\mathbb{V}^n} \frac{\|\boldsymbol{U}^n\|_t^2\|\boldsymbol{V}\|_{1,r^-} + \|\boldsymbol{f}\|_{-1}\|\boldsymbol{V}\|_{1,r^-}}{\|\boldsymbol{V}\|_{1,r^-}} \\
&\quad + C \sup_{0\neq \boldsymbol{V}\in\mathbb{V}^n} \frac{\|\boldsymbol{U}^n\|_{2(r-)'}\|\boldsymbol{V}\|_{2(r-)'}\|\boldsymbol{U}^n\|_{1,r-}}{\|\boldsymbol{V}\|_{2(r-)'}},
\end{aligned}$$

where C is independent of n and k. Hence, by noting (5.19), we have

$$\|P^n\|_{j'} \leq C_4, \tag{5.24}$$

where C_4 is independent of n.

Now, by (5.19)–(5.24), thanks to the reflexivity of the relevant spaces and by compact embedding, we can extract (not relabeled) subsequences such that

$$\boldsymbol{U}^n \rightharpoonup \boldsymbol{u} \quad \text{weakly in } W_0^{1,r^-}(\Omega)^3 \cap L^t(\Omega)^3, \tag{5.25}$$

$$\boldsymbol{U}^n \to \boldsymbol{u} \quad \text{strongly in } L^\sigma(\Omega)^3 \quad \forall\, \sigma \in [1,t), \tag{5.26}$$

$$|\boldsymbol{U}^n|^{t-2}\boldsymbol{U}^n \rightharpoonup |\boldsymbol{u}|^{t-2}\boldsymbol{u} \quad \text{weakly in } L^{\frac{t}{t-1}}(\Omega)^3, \tag{5.27}$$

$$C^n \rightharpoonup c \quad \text{weakly in } W^{1,2}(\Omega), \tag{5.28}$$

$$C^n \to c \quad \text{strongly in } C^{0,\tilde{\alpha}_1}(\overline{\Omega}), \tag{5.29}$$

$$P^n \rightharpoonup p \quad \text{weakly in } L^{j'}(\Omega) \quad \forall\, j > \max\{r^+, 2\}, \tag{5.30}$$

$$\boldsymbol{S}^n \rightharpoonup \bar{\boldsymbol{S}} \quad \text{weakly in } L^{(r^+)'}(\Omega)^{3\times 3}, \tag{5.31}$$

$$\boldsymbol{q}_c^n \rightharpoonup \bar{\boldsymbol{q}}_c \quad \text{weakly in } L^2(\Omega)^3. \tag{5.32}$$

Before proceeding further, we note that these limits, together with weak lower semicontinuity and (5.19), in conjunction with Korn's inequality, imply that

$$\int_\Omega |\nabla \boldsymbol{u}|^{r(c)} + |\bar{\boldsymbol{S}}|^{r'(c)}\,\mathrm{d}x \le C, \tag{5.33}$$

where C is independent of k; hence the limit function $\boldsymbol{u}$ is, in fact, contained in the space $W_0^{1,r(c)}(\Omega)^3$; see [15] for the details of the proof of this.

Next, we shall prove that the limit function $\boldsymbol{u}$ is pointwise divergence-free. For an arbitrary $q \in C_0^\infty(\Omega)$, by (5.15),

$$\begin{aligned} 0 &= \int_\Omega (\Pi_{\mathbb{Q}}^n q)\,\mathrm{div}\,\boldsymbol{U}^n\,\mathrm{d}x \\ &= \int_\Omega (\Pi_{\mathbb{Q}}^n q - q)\,\mathrm{div}\,\boldsymbol{U}^n\,\mathrm{d}x + \int_\Omega q\,(\mathrm{div}\,\boldsymbol{U}^n - \mathrm{div}\,\boldsymbol{u})\,\mathrm{d}x + \int_\Omega q\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}x. \end{aligned}$$

The first term tends to zero by (5.19) and (4.6), and the second term converges to zero by (5.25). Therefore,

$$\int_\Omega q\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}x = 0 \qquad \text{for any } q \in C_0^\infty(\Omega),$$

which implies that $\mathrm{div}\,\boldsymbol{u} = 0$ a.e. on Ω.

Now, we shall identify the limit of the convective term $B_u[\cdot,\cdot,\cdot]$ as follows. For an arbitrary $\boldsymbol{v} \in W_0^{1,\infty}(\Omega)^3$, we define $\boldsymbol{V}^n := \Pi_{\mathrm{div}}^n \boldsymbol{v} \in \mathbb{V}^n$. Then, by (4.3), we have

$$\boldsymbol{V}^n \to \boldsymbol{v} \qquad \text{strongly in } W_0^{1,\sigma}(\Omega)^2 \text{ for } \sigma \in (1,\infty). \tag{5.34}$$

By (5.26),

$$\boldsymbol{U}^n \otimes \boldsymbol{U}^n \to \boldsymbol{u} \otimes \boldsymbol{u} \qquad \text{strongly in } L^{1+\varepsilon}(\Omega)^{3\times 3}.$$

Hence, we can identify the second part of the convective term

$$-\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{U}^n) \cdot \nabla \boldsymbol{V}^n\,\mathrm{d}x \to -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \boldsymbol{v}\,\mathrm{d}x \qquad \text{as } n \to \infty.$$

Also, we assert that $\boldsymbol{U}^n \cdot \boldsymbol{V}^n \to \boldsymbol{u} \cdot \boldsymbol{v}$ strongly in $L^{(r^-)'}(\Omega)$. Indeed,

$$\begin{aligned} \|\boldsymbol{U}^n \cdot \boldsymbol{V}^n - \boldsymbol{u} \cdot \boldsymbol{v}\|_{(r^-)'} &\le \|(\boldsymbol{V}^n - \boldsymbol{v})\boldsymbol{U}^n + (\boldsymbol{U}^n - \boldsymbol{u})\boldsymbol{v}\|_{(r^-)'} \\ &\le \|\boldsymbol{V}^n - \boldsymbol{v}\|_\sigma \|\boldsymbol{U}^n\|_{t-\varepsilon} + \|\boldsymbol{U}^n - \boldsymbol{u}\|_{t-\varepsilon} \|\boldsymbol{v}\|_\sigma \end{aligned}$$

for some $\sigma \in (1,\infty)$. The first term tends to zero thanks to (5.34) and (5.26), and the second term tends to zero by (5.26). Therefore, since $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{V}^n) \cdot \nabla \boldsymbol{U}^n \,\mathrm{d}x = -\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{U}^n) \cdot \nabla \boldsymbol{V}^n \,\mathrm{d}x + \int_\Omega (\operatorname{div} \boldsymbol{U}^n)\, \boldsymbol{U}^n \cdot \boldsymbol{V}^n \,\mathrm{d}x$$
$$\to -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \boldsymbol{v} \,\mathrm{d}x \qquad \text{as } n \to \infty.$$

Altogether, we then deduce that

$$\lim_{n\to\infty} B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}^n] = -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \boldsymbol{v} \,\mathrm{d}x. \tag{5.35}$$

Now, we are ready to pass n to infinity in the Navier–Stokes equations. Since Π^n_{div} is linear, by noting (5.14), we have

$$\begin{aligned}\langle \operatorname{div} \boldsymbol{v}, P^n\rangle &= \langle \operatorname{div} \boldsymbol{V}^n, P^n\rangle + \langle \operatorname{div}(\boldsymbol{v} - \boldsymbol{V}^n), P^n\rangle \\ &= \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) \cdot \boldsymbol{D}\boldsymbol{V}^n + \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\, \boldsymbol{U}^n \cdot \boldsymbol{V}^n \,\mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{V}^n\rangle \\ &\quad + B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}^n] + \langle \operatorname{div}(\boldsymbol{v} - \boldsymbol{V}^n), P^n\rangle \\ &\to \int_\Omega \bar{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{v} + \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u} \cdot \boldsymbol{v} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\end{aligned}$$

where we have used (5.30), (5.31), (5.27), (5.34), and (5.35). Also, by (5.30) again,

$$\langle \operatorname{div} \boldsymbol{v}, P^n\rangle \to \langle \operatorname{div} \boldsymbol{v}, p\rangle.$$

Collecting all the limits gives us

$$\int_\Omega \bar{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{v} + \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u} \cdot \boldsymbol{v} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}x - \langle \operatorname{div} \boldsymbol{v}, p\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \qquad \forall\, \boldsymbol{v} \in W^{1,\infty}_0(\Omega)^3. \tag{5.36}$$

With the same argument as above, we also have that

$$\int_\Omega \bar{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{v} + \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u} \cdot \boldsymbol{v} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \boldsymbol{v} \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \qquad \forall\, \boldsymbol{v} \in W^{1,\infty}_{0,\mathrm{div}}(\Omega)^3. \tag{5.37}$$

Note that by Proposition 2.4 and (5.33), we have

$$p \in L^{r'(c)}_0(\Omega).$$

Now, let us investigate the limit of the convection-diffusion equation (5.18). For an arbitrary but fixed $z \in W^{1,2}_0(\Omega)$, we define $Z^n := \Pi^n_{\mathbb{Z}} z \in \mathbb{Z}^n$. Thanks to (5.26) and (5.29),

$$\begin{aligned}\|C^n \boldsymbol{U}^n - c\boldsymbol{u}\|_2 &\le \|(C^n - c)\boldsymbol{U}^n\|_2 + \|c(\boldsymbol{U}^n - \boldsymbol{u})\|_2 \\ &\le \|C^n - c\|_\infty \|\boldsymbol{U}^n\|_2 + \|c\|_\infty \|\boldsymbol{U}^n - \boldsymbol{u}\|_2 \to 0.\end{aligned}$$

Moreover, by (5.26), (4.7), and Sobolev embedding,

$$\begin{aligned}\|Z^n \boldsymbol{U}^n - z\boldsymbol{u}\|_2 &\le \|(Z^n - z)\boldsymbol{U}^n\|_2 + \|z(\boldsymbol{U}^n - \boldsymbol{u})\|_2 \\ &\le \|Z^n - z\|_6 \|\boldsymbol{U}^n\|_3 + \|z\|_6 \|\boldsymbol{U}^n - \boldsymbol{u}\|_3 \\ &\le C\|Z^n - z\|_{1,2} \|\boldsymbol{U}^n\|_3 + C\|z\|_{1,2} \|\boldsymbol{U}^n - \boldsymbol{u}\|_3 \to 0.\end{aligned}$$

In other words,

$$C^n \boldsymbol{U}^n \to c\boldsymbol{u} \qquad \text{strongly in } L^2(\Omega)^3, \tag{5.38}$$

$$Z^n \boldsymbol{U}^n \to z\boldsymbol{u} \qquad \text{strongly in } L^2(\Omega)^3. \tag{5.39}$$

From (5.28) and (5.39),

$$\left|\int_\Omega Z^n \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x - \int_\Omega z\boldsymbol{u} \cdot \nabla c \,\mathrm{d}x\right| \\ \leq \int_\Omega |Z^n\boldsymbol{U}^n - z\boldsymbol{u}||\nabla C^n| \,\mathrm{d}x + \left|\int_\Omega z\boldsymbol{u} \cdot (\nabla C^n - \nabla c) \,\mathrm{d}x\right| \to 0.$$

Therefore, as $\operatorname{div}\boldsymbol{u} = 0$ a.e. on Ω, we obtain

$$\int_\Omega Z^n \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x \to \int_\Omega z\boldsymbol{u} \cdot \nabla c \,\mathrm{d}x = -\int_\Omega c\boldsymbol{u} \cdot \nabla z \,\mathrm{d}x \qquad \text{as } n \to \infty.$$

Moreover, by (5.38) and (4.7),

$$\left|\int_\Omega C^n \boldsymbol{U}^n \cdot \nabla Z^n \,\mathrm{d}x - \int_\Omega c\boldsymbol{u} \cdot \nabla z \,\mathrm{d}x\right| \\ \leq \|C^n\boldsymbol{U}^n\|_2 \|Z^n - z\|_{1,2} + \|C^n\boldsymbol{U}^n - c\boldsymbol{u}\|_2 \|z\|_{1,2} \to 0.$$

Altogether, we have

$$\lim_{n\to\infty} B_c[C^n, \boldsymbol{U}^n, Z^n] = -\int_\Omega c\boldsymbol{u} \cdot \nabla z \,\mathrm{d}x.$$

Finally, by (5.32) and (4.7), we have

$$\int_\Omega \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \cdot \nabla Z^n \,\mathrm{d}x \to \int_\Omega \bar{\boldsymbol{q}}_c \cdot \nabla z \,\mathrm{d}x \qquad \text{as } n \to \infty.$$

By collecting all the limits, we obtain that

$$\int_\Omega \bar{\boldsymbol{q}}_c \cdot \nabla z - c\boldsymbol{u} \cdot \nabla z \,\mathrm{d}x = 0 \qquad \forall\, z \in W_0^{1,2}(\Omega). \tag{5.40}$$

As we can see from (5.36) and (5.40), what we now need to prove is the identification of the limits:

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \quad \text{and} \quad \bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}).$$

To this end, we require the following lemma.

Lemma 5.1. *The sequences $\{\boldsymbol{D}\boldsymbol{U}^n\}_{n\in\mathbb{N}}$ and $\{C^n\}_{n\in\mathbb{N}}$ satisfy the following equality:*

$$\lim_{n\to\infty} \int_\Omega ((\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{u})) \cdot (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u}))^{\frac{1}{4}} \,\mathrm{d}x = 0. \tag{5.41}$$

The detailed proof of Lemma 5.1 is presented in Section 4.2 in [15]. Here, we shall briefly summarize the key steps of the proof, as we shall require a similar, but more involved, argument in the next section. The strategy is to decompose the integral into two terms and to estimate them separately. More precisely, if we define $\Omega_\chi := \{x \in \Omega : |\boldsymbol{D}\boldsymbol{u}| > \chi\}$ for an arbitrarily, fixed, $\chi > 0$, it is easy to see by Hölder's inequality that the integral of the integrand in (5.41) over Ω_χ tends to 0 as $\chi \to \infty$. To control the integral of the integrand over $\Omega \setminus \Omega_\chi$, we introduce the matrix-truncation function $T_\chi : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ by

$$T_\chi(\boldsymbol{M}) = \begin{cases} \boldsymbol{M} & \text{for } |\boldsymbol{M}| \leq \chi, \\ \chi \frac{\boldsymbol{M}}{|\boldsymbol{M}|} & \text{for } |\boldsymbol{M}| > \chi. \end{cases}$$

The essential step relies on using a discrete Lipschitz truncation technique to control integrals of functions over $\Omega\setminus\Omega_\chi$. Lipschitz truncation, which is widely used in the calculus of variations and in the mathematical analysis of partial differential equations, is based on the idea of approximating a Sobolev function by a sequence of Lipschitz functions, its key property being that one can control the measure of the 'bad set', on which the Sobolev function and its Lipschitz approximation are not equal to each other, in terms of the Hardy–Littlewood maximal function of the gradient of the Sobolev function. In [11], the authors constructed a discrete counterpart of the Lipschitz truncation method, as the composition of a 'continuous' Lipschitz truncation and the projection operator onto the finite element space for the velocity field, so that the discretely Lipschitz-truncated functions are again contained in the finite element space. The key observation here is that one can still control the measure of the 'bad set' for the discretely Lipschitz-truncated functions. In [15] a version of the discrete Lipschitz truncation method in variable-exponent norms was presented; see Theorem 3.15 in [15], with $\boldsymbol{U}_j^n$ denoting the discrete Lipschitz truncation of the function of $\boldsymbol{U}^n$.

Then, the most important and most difficult part of the proof of (5.41) is to show that

$$\lim_{\chi\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega (\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)-\boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u})))\cdot(\boldsymbol{D}\boldsymbol{U}_j^n - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x \le 0 \tag{5.42}$$

(see eq. (4.23) in [15]). It is in the proof of this inequality that the choice of the exponent 1/4 appearing in the statement of Lemma 5.1 becomes essential. The reader is referred to Section 4.2 in [15] for the details.

To show (5.42), we define the following discretely divergence-free approximations with zero trace on $\partial\Omega$:

$$\begin{aligned}\boldsymbol{\Psi}_j^n &:= \mathcal{B}^n(\operatorname{div}\boldsymbol{U}_j^n),\\ \boldsymbol{\Phi}_j^n &:= \boldsymbol{U}_j^n - \boldsymbol{\Psi}_j^n.\end{aligned}$$

Here $\mathcal{B}^n$ is a discrete Bogovskiĭ operator defined in Section 3.4 of [15]. It is then clear that $\boldsymbol{\Phi}_j^n$ has zero trace on $\partial\Omega$ and, by construction, $\boldsymbol{\Phi}_j^n \in \mathbb{V}^n_{\mathrm{div}}$. Moreover, it can be easily verified, by using basic properties of the discrete Lipschitz truncation and the discrete Bogovskiĭ operator, that

$$\boldsymbol{\Phi}_j^n \rightharpoonup \boldsymbol{U}_j - \mathcal{B}(\operatorname{div}\boldsymbol{U}_j) =: \boldsymbol{\Phi}_j \qquad \text{weakly in } W_0^{1,\sigma}(\Omega)^3, \tag{5.43}$$

$$\boldsymbol{\Phi}_j^n \to \boldsymbol{\Phi}_j \qquad \text{strongly in } L^\sigma(\Omega)^3, \tag{5.44}$$

as $n\to\infty$, where $\sigma\in(1,\infty)$ is arbitrary. We can then rewrite (5.42) above in terms of this approximation to obtain

$$\begin{aligned}&\int_\Omega (\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)-\boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u})))\cdot(\boldsymbol{D}\boldsymbol{U}_j^n - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x\\ &= \int_\Omega \boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)\cdot(\boldsymbol{D}\boldsymbol{\Phi}_j^n + \boldsymbol{D}\boldsymbol{\Psi}_j^n)\,\mathrm{d}x\\ &\quad - \int_\Omega \boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)\cdot T_\chi(\boldsymbol{D}\boldsymbol{u})\,\mathrm{d}x - \int_\Omega \boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u}))\cdot(\boldsymbol{D}\boldsymbol{U}_j^n - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x\\ &=: B_{\chi,j}^{n,1} - B_{\chi,j}^{n,2} - B_{\chi,j}^{n,3}.\end{aligned}$$

Now we use (5.14) with $\boldsymbol{V} = \boldsymbol{\Phi}_j^n \in \mathbb{V}_{\mathrm{div}}^n$ and pass to the limit; thus we have, by (5.37), that

(5.45)
$$\lim_{n\to\infty} \int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}\boldsymbol{\Phi}_j^n \,\mathrm{d}x = -\lim_{n\to\infty} B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{\Phi}_j^n] - \int_\Omega \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\boldsymbol{U}^n \cdot \boldsymbol{\Phi}_j^n \,\mathrm{d}x + \lim_{n\to\infty} \langle \boldsymbol{f}, \boldsymbol{\Phi}_j^n \rangle$$

$$= \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) \cdot \nabla\boldsymbol{\Phi_j} - \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u}\cdot\boldsymbol{\Phi_j}\,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\Phi_j}\rangle \tag{5.46}$$

$$= \int_\Omega \bar{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{\Phi_j}\,\mathrm{d}x. \tag{5.47}$$

Furthermore, with the help of Lipschitz truncation, we can show that

$$\lim_{n\to\infty} \int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}\boldsymbol{\Psi}_j^n \,\mathrm{d}x \le \left(\frac{C}{2^{j/r^+}}\right)^{\gamma(r^-,r^+)}, \tag{5.48}$$

$$\int_\Omega \bar{\boldsymbol{S}} \cdot \boldsymbol{D}\mathcal{B}(\operatorname{div}\boldsymbol{U}_j)\,\mathrm{d}x \le \left(\frac{C}{2^{j/r^+}}\right)^{\gamma(r^-,r^+)}. \tag{5.49}$$

Altogether, we have

$$\lim_{\chi\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\left(B_{\chi,j}^{n,1} - B_{\chi,j}^{n,2} - B_{\chi,j}^{n,3}\right) \le \lim_{\chi\to\infty}\int_\Omega (\bar{\boldsymbol{S}} - \boldsymbol{S}(c, T_\chi(\boldsymbol{D}\boldsymbol{u}))) \cdot (\boldsymbol{D}\boldsymbol{U}_j - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x.$$

The last limit is equal to zero by using the Dominated Convergence Theorem. That completes the proof of (5.42) and thereby also of the most technical step in the proof of the lemma.

Now we are ready to identify the limits. In the above lemma, since the integrand is nonnegative, (5.41) also holds with Ω replaced by the set $Q_\gamma \subset \Omega$ defined by

$$Q_\gamma := \{x \in \Omega : |\boldsymbol{D}\boldsymbol{u}| \le \gamma\},$$

with a given $\gamma > 0$; thus, from the sequence of integrands featured in (5.41), we can extract a subsequence (again not relabeled), which converges to zero almost everywhere in Q_γ. Then, by Egoroff's theorem, for an arbitrary $\varepsilon > 0$, there exists a subset $Q_\gamma^\varepsilon \subset Q_\gamma \subset \Omega$ satisfying $|Q_\gamma \setminus Q_\gamma^\varepsilon| < \varepsilon$, where the convergence of integrands is uniform. Note that, thanks to the choice of Q_γ^ε, we have

$$\lim_{\gamma\to\infty}\lim_{\varepsilon\to 0}|\Omega\setminus Q_\gamma^\varepsilon| = \lim_{\gamma\to\infty}\lim_{\varepsilon\to 0}\left[|\Omega\setminus Q_\gamma| + |Q_\gamma\setminus Q_\gamma^\varepsilon|\right] = 0.$$

Moreover, we have from the uniform convergence of the integrands that

$$\lim_{n\to\infty}\int_{Q_\gamma^\varepsilon} (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{u})) \cdot (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u})\,\mathrm{d}x = 0. \tag{5.50}$$

Since $\boldsymbol{D}\boldsymbol{u}$ is bounded on Q_γ^ε, by the Dominated Convergence Theorem we have $\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{u}) \to \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$ strongly in $L^q(\Omega)^{3\times 3}$ for any $q \in [1,\infty)$. Hence, from the above L^q-convergence, (5.4), and (5.50), we obtain

$$\lim_{n\to\infty}\int_{Q_\gamma^\varepsilon} \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) \cdot (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u})\,\mathrm{d}x = 0.$$

Thus, by the boundedness of $\boldsymbol{Du}$ on Q_γ^ε and (5.31), we have

$$\lim_{n\to\infty}\int_{Q_\gamma^\varepsilon} \boldsymbol{S}(C^n,\boldsymbol{DU}^n)\cdot \boldsymbol{DU}^n \,\mathrm{d}x = \int_{Q_\gamma^\varepsilon} \bar{\boldsymbol{S}}\cdot \boldsymbol{Du}\,\mathrm{d}x. \tag{5.51}$$

Now, let $\boldsymbol{B}\in L^\infty(Q_\gamma^\varepsilon)^{3\times 3}$ be arbitrary but fixed. From the monotonicity (1.6), (5.51), the L^q-convergence of $\boldsymbol{S}(C^n,\boldsymbol{B})\to \boldsymbol{S}(c,\boldsymbol{B})$, and the weak convergence (5.4), we have

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\int_{Q_\gamma^\varepsilon} (\boldsymbol{S}(C^n,\boldsymbol{DU}^n)-\boldsymbol{S}(C^n,\boldsymbol{B}))\cdot(\boldsymbol{DU}^n-\boldsymbol{B})\,\mathrm{d}x \\ &= \int_{Q_\gamma^\varepsilon} \bar{\boldsymbol{S}}\cdot(\boldsymbol{Du}-\boldsymbol{B})\,\mathrm{d}x - \int_{Q_\gamma^\varepsilon} \boldsymbol{S}(c,\boldsymbol{B})\cdot(\boldsymbol{Du}-\boldsymbol{B})\,\mathrm{d}x \\ &= \int_{Q_\gamma^\varepsilon} (\bar{\boldsymbol{S}}-\boldsymbol{S}(c,\boldsymbol{B}))\cdot(\boldsymbol{Du}-\boldsymbol{B})\,\mathrm{d}x. \end{aligned}$$

Now we are ready to use Minty's trick. First, we choose $\boldsymbol{B}=\boldsymbol{Du}\pm\lambda\boldsymbol{A}$ with $\lambda>0$ and $\boldsymbol{A}\in L^\infty(Q_\gamma^\varepsilon)^{3\times 3}$. Then, passing to the limit $\lambda\to 0$, the continuity of $\boldsymbol{S}$ gives us

$$\int_{Q_\gamma^\varepsilon} (\bar{\boldsymbol{S}}-\boldsymbol{S}(c,\boldsymbol{Du}))\cdot\boldsymbol{A}\,\mathrm{d}x = 0.$$

Hence, we have that

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c,\boldsymbol{Du}) \quad \text{a.e. on } Q_\gamma^\varepsilon.$$

Now we pass $\varepsilon\to 0$ and then $\gamma\to\infty$ to conclude that

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c,\boldsymbol{Du}) \quad \text{a.e. on } \Omega. \tag{5.52}$$

Finally, since $\boldsymbol{S}$ is strictly monotonic and $C^n\to c$ in $C^{0,\tilde{\alpha}_1}(\overline{\Omega})$, by (5.41) we deduce that

$$\boldsymbol{DU}^n\to\boldsymbol{Du} \quad \text{a.e. on } \Omega. \tag{5.53}$$

By the Dominated Convergence Theorem, with (5.28), (5.29), and (5.53), we obtain that

$$\boldsymbol{q}_c(C^n,\nabla C^n,\boldsymbol{DU}^n)\rightharpoonup \boldsymbol{q}_c(c,\nabla c,\boldsymbol{Du}) \qquad \text{weakly in } L^2(\Omega)^3.$$

Therefore, by the uniqueness of the weak limit, we can identify

$$\bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c,\nabla c,\boldsymbol{Du}). \tag{5.54}$$

6. Proof of Theorem 3.1

6.1. **Minimum and maximum principles.** Having passed to the limits $m,n\to\infty$, we shall reinstate the index k for the regularization parameter in our notation in preparation for passage to the limit $k\to\infty$.

Before we proceed, let us prove minimum and maximum principles for the concentration. Let $\varphi_1^k=(c^k-\min_{x\in\partial\Omega}c_d)_-$ and $\varphi_2^k=(c^k-\max_{x\in\partial\Omega}c_d)_+$. Since $c^k=c_d$ on $\partial\Omega$, it is clear that $\varphi_1^k,\varphi_2^k\in W_0^{1,2}(\Omega)$, so we can test with φ_1^k and φ_2^k in (5.40). Therefore, we have

$$-\int_\Omega \boldsymbol{u}^k c^k\cdot\nabla\varphi_1^k\,\mathrm{d}x + \int_\Omega \bar{\boldsymbol{q}}_c\cdot\nabla\varphi_1^k\,\mathrm{d}x = 0, \tag{6.1}$$

$$-\int_\Omega \boldsymbol{u}^k c^k \cdot \nabla\varphi_2^k \,\mathrm{d}x + \int_\Omega \bar{\boldsymbol{q}}_c \cdot \nabla\varphi_2^k \,\mathrm{d}x = 0. \tag{6.2}$$

We first consider (6.1). From (1.9) with integration by parts we obtain

$$\int_{\Omega^-} \boldsymbol{u}^k \cdot \nabla c^k \varphi_1^k \,\mathrm{d}x + \int_{\Omega^-} C|\nabla c^k|^2 \,\mathrm{d}x \le 0,$$

where $\Omega^- = \{x \in \Omega : \varphi_1^k(x) < 0\}$, since $\operatorname{div} \boldsymbol{u}^k = 0$ and $\boldsymbol{u}^k = 0$ on $\partial\Omega$. By using the fact that $\nabla c^k = \nabla\varphi_1^k$ on Ω^- and the extension of ∇c^k from Ω^- to the whole domain Ω by using the negative part, we have

$$\int_\Omega \boldsymbol{u}^k \cdot \nabla\varphi_1^k\varphi_1^k \,\mathrm{d}x + \int_\Omega C|\nabla\varphi_1^k|^2 \,\mathrm{d}x \le 0.$$

Note that

$$\int_\Omega \boldsymbol{u}^k \cdot \nabla\varphi_1^k\varphi_1^k \,\mathrm{d}x = \frac{1}{2}\int_\Omega \boldsymbol{u}^k \cdot \nabla|\varphi_1^k|^2 \,\mathrm{d}x = -\frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{u}^k)|\varphi_1^k|^2 \,\mathrm{d}x = 0,$$

and thus,

$$\varphi_1^k = (c^k - \min_{x\in\partial\Omega} c_d)_- = \text{constant a.e. in } \Omega.$$

In the same way, we can also show that

$$\varphi_2^k = (c^k - \max_{x\in\partial\Omega} c_d)_+ = \text{constant a.e. in } \Omega.$$

By combining the above results we finally obtain that

$$\min_{x\in\partial\Omega} c_d \le c^k \le \max_{x\in\partial\Omega} c_d \qquad \text{a.e. in } \Omega. \tag{6.3}$$

6.2. **The limit $k \to \infty$.** First, note that by weak lower semicontinuity of the norm-function, and by (5.20), (5.22), and (5.24), we obtain the following uniform bounds, independent of $k \in \mathbb{N}$:

$$\|\boldsymbol{u}^k\|_{1,r^-}^{r^-} + \|\boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k)\|_{(r^+)'}^{(r^+)'} + \frac{1}{k}\|\boldsymbol{u}^k\|_t^t \le C_1, \tag{6.4}$$

$$\|c^k\|_{1,2}^2 + \|\boldsymbol{q}_c(c^k, \nabla c^k, \boldsymbol{D}\boldsymbol{u}^k)\|_2^2 \le C_2, \tag{6.5}$$

$$\|p^k\|_{j'}^{j'} \le C_3, \tag{6.6}$$

for some positive constants C_1, C_2, and C_3, which are independent of $k \in \mathbb{N}$.

Now, since $r^- > \frac{3}{2}$, by the min/max principle (6.3), Sobolev embedding, and the uniform estimate (6.4), for $s > 3$ sufficiently close to 3,

$$\|c^k\boldsymbol{u}^k\|_s \le \|c^k\|_\infty\|\boldsymbol{u}^k\|_s \le C\|\boldsymbol{u}^k\|_{1,r^-} \le C,$$

where C is independent of $k \in \mathbb{N}$.

Therefore, we can again apply Theorem 2.6 with $\boldsymbol{F} = c^k\boldsymbol{u}^k$ and $g = 0$. Hence, there exists an $\alpha_2 \in (0, 1)$ such that

$$\|c^k\|_{C^{0,\alpha_2}(\overline{\Omega})} \le C_4, \tag{6.7}$$

for some positive constant C_4 independent of $k \in \mathbb{N}$. Since $C^{0,\alpha_2}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{0,\tilde{\alpha}_2}(\overline{\Omega})$ for all $\tilde{\alpha}_2 \in (0, \alpha_2)$, we have

$$c^k \to c \qquad \text{strongly in } C^{0,\tilde{\alpha}_2}(\overline{\Omega}),$$

which implies that

$$r \circ c^k \to r \circ c \qquad \text{strongly in } C^{0,\beta_2}(\overline{\Omega}),$$

for some $\beta_2 \in (0,1)$.

Therefore, by the reflexivity of the relevant spaces and compact embedding, there exist subsequences (not relabeled) such that

$$\boldsymbol{u}^k \rightharpoonup \boldsymbol{u} \qquad \text{weakly in } W^{1,r^-}_{0,\mathrm{div}}(\Omega)^3, \tag{6.8}$$

$$\boldsymbol{u}^k \to \boldsymbol{u} \qquad \text{strongly in } L^{2(1+\varepsilon)}(\Omega)^3, \tag{6.9}$$

$$c^k \rightharpoonup c \qquad \text{weakly in } W^{1,2}(\Omega), \tag{6.10}$$

$$c^k \to c \qquad \text{strongly in } C^{0,\tilde{\alpha}_2}(\overline{\Omega}), \tag{6.11}$$

$$p^k \rightharpoonup p \qquad \text{weakly in } L^{j'}(\Omega) \qquad \forall\, j > \max\{r^+, 2\}, \tag{6.12}$$

$$\boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k) \rightharpoonup \hat{\boldsymbol{S}} \qquad \text{weakly in } L^{(r^+)'}(\Omega)^{3\times 3}, \tag{6.13}$$

$$\boldsymbol{q}_c(c^k, \nabla c^k, \boldsymbol{D}\boldsymbol{u}^k) \rightharpoonup \hat{\boldsymbol{q}}_c \qquad \text{weakly in } L^2(\Omega)^3. \tag{6.14}$$

Again, by the weak lower semicontinuity of the norm function, (5.33), and (6.11), together with Korn's inequality, we have that

$$\int_\Omega |\nabla \boldsymbol{u}|^{r(c)} + |\hat{\boldsymbol{S}}|^{r'(c)} \,\mathrm{d}x \le C, \tag{6.15}$$

and thus the weak solution $\boldsymbol{u}$ is in the desired space $W^{1,r(c)}_0(\Omega)^3$.

Now we shall let $k \to \infty$ in (5.36), with $\boldsymbol{v} \in W^{1,\infty}_0(\Omega)^3$ chosen arbitrarily. By (6.9),

$$\boldsymbol{u}^k \otimes \boldsymbol{u}^k \to \boldsymbol{u} \otimes \boldsymbol{u} \qquad \text{strongly in } L^{1+\varepsilon}(\Omega)^{3\times 3}.$$

Thus, we can identify the limit of the convective term

$$-\int_\Omega (\boldsymbol{u}^k \otimes \boldsymbol{u}^k) \cdot \nabla \boldsymbol{v} \,\mathrm{d}x \to -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \boldsymbol{v} \,\mathrm{d}x \qquad \text{as } k \to \infty \qquad \forall\, \boldsymbol{v} \in W^{1,\infty}_0(\Omega)^3.$$

Next, by (6.4), we have that

$$\frac{1}{k}\|\boldsymbol{u}^k\|_t^{t-1} \to 0 \qquad \text{as } k \to \infty.$$

Therefore, we have

$$\frac{1}{k}\left|\int_\Omega |\boldsymbol{u}^k|^{t-2}\boldsymbol{u}^k \cdot \boldsymbol{v} \,\mathrm{d}x\right| \le \frac{1}{k}\|\boldsymbol{u}^k\|_t^{t-1}\|\boldsymbol{v}\|_t \to 0 \qquad \text{as } k \to \infty \qquad \forall\, \boldsymbol{v} \in W^{1,\infty}_0(\Omega)^3.$$

We recall from the identification asserted in (5.52) that $\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$ a.e. on Ω; more precisely, with the index k reinstated in our notation, $\bar{\boldsymbol{S}}^k = \boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k)$ a.e. on Ω. Hence, from (6.13) and (6.12), we obtain

$$\langle \operatorname{div} \boldsymbol{v}, p^k\rangle \to \langle \operatorname{div} \boldsymbol{v}, p\rangle \quad \text{and} \quad \int_\Omega \bar{\boldsymbol{S}}^k \cdot \boldsymbol{D}\boldsymbol{v} \,\mathrm{d}x \to \int_\Omega \hat{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{v} \,\mathrm{d}x \quad \text{as } k \to \infty \qquad \forall\, \boldsymbol{v} \in W^{1,\infty}_0(\Omega)^3.$$

Altogether, we have

$$\int_\Omega \hat{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{v} + (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \boldsymbol{v} \,\mathrm{d}x - \langle \operatorname{div} \boldsymbol{v}, p\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \qquad \forall\, \boldsymbol{v} \in W^{1,\infty}_0(\Omega)^3. \tag{6.16}$$

Furthermore, it is clear that

$$\int_\Omega \hat{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{v} + (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla \boldsymbol{v} \, \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall \, \boldsymbol{v} \in W^{1,\infty}_{0,\mathrm{div}}(\Omega)^3. \tag{6.17}$$

Note that by Proposition 2.4 and (6.15) we have

$$p \in L_0^{r'(c)}(\Omega).$$

Now, let us investigate the limit of the concentration equation (5.40). Let us choose an arbitrary, but fixed, $z \in W_0^{1,2}(\Omega)$. By (6.9) and (6.11),

$$\|c^k \boldsymbol{u}^k - c\boldsymbol{u}\|_2 \le \|(c^k - c)\boldsymbol{u}^k\|_2 + \|c(\boldsymbol{u}^k - \boldsymbol{u})\|_2 \le \|c^k - c\|_\infty \|\boldsymbol{u}^k\|_2 + \|c\|_\infty \|\boldsymbol{u}^k - \boldsymbol{u}\|_2 \to 0.$$

In other words,

$$c^k \boldsymbol{u}^k \to c\boldsymbol{u} \qquad \text{strongly in } L^2(\Omega)^3.$$

Hence we have

$$\int_\Omega c^k \boldsymbol{u}^k \cdot \nabla z \, \mathrm{d}x \to \int_\Omega c\boldsymbol{u} \cdot \nabla z \, \mathrm{d}x.$$

Recalling the identification (5.54) and reinstating the index k, we have $\bar{\boldsymbol{q}}_c^k := \boldsymbol{q}_c(c^k, \nabla c^k, \boldsymbol{D}\boldsymbol{u}^k)$; hence, by (6.14), we get

$$\int_\Omega \bar{\boldsymbol{q}}_c^k \cdot \nabla z \, \mathrm{d}x \to \int_\Omega \hat{\boldsymbol{q}}_c \cdot \nabla z \, \mathrm{d}x \qquad \text{as } k \to \infty.$$

By collecting the above limits, we deduce that

$$\int_\Omega \hat{\boldsymbol{q}}_c \cdot \nabla z - c\boldsymbol{u} \cdot \nabla z \, \mathrm{d}x = 0 \qquad \forall \, z \in W_0^{1,2}(\Omega). \tag{6.18}$$

As a final step, we need to identify the limits:

$$\hat{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \quad \text{and} \quad \hat{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}).$$

To this end, analogously as before, we need to prove the following equality:

$$\lim_{k \to \infty} \int_\Omega ((\boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k) - \boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u})) \cdot (\boldsymbol{D}\boldsymbol{u}^k - \boldsymbol{D}\boldsymbol{u}))^{\frac{1}{4}} \, \mathrm{d}x = 0. \tag{6.19}$$

The proof is similar to the one presented in the previous section. The only part of the argument that we shall give here in detail is the proof of the analogue of (5.45)–(5.47) since we now have a different weak formulation at this level. The other parts of the proof proceed as in Section 4.2 in [15].

First we define a divergence-free approximation with zero trace as follows:

$$\boldsymbol{\Phi}_j^k := \boldsymbol{u}_j^k - \mathcal{B}(\operatorname{div} \boldsymbol{u}_j^k),$$

where $\mathcal{B}$ is the Bogovskiĭ operator introduced in Theorem 2.2. Then, as before, we have

$$\boldsymbol{\Phi}_j^k \rightharpoonup \boldsymbol{u}_j - \mathcal{B}(\operatorname{div} \boldsymbol{u}_j) =: \boldsymbol{\Phi}_j \qquad \text{weakly in } W_0^{1,\sigma}(\Omega)^3, \tag{6.20}$$

$$\boldsymbol{\Phi}_j^k \to \boldsymbol{\Phi}_j \qquad \text{strongly in } L^\sigma(\Omega)^3, \tag{6.21}$$

as $k \to \infty$, where $\sigma \in (1, \infty)$ is arbitrary.

Let us further define $\boldsymbol{\chi}_{1,j}^{n,k} := \Pi^n_{\mathrm{div}} \boldsymbol{\Phi}_j^k$. Then, by (4.3),

$$\boldsymbol{\chi}_{1,j}^{n,k} \to \boldsymbol{\Phi}_j^k \qquad \text{strongly in } W_0^{1,\sigma}(\Omega)^3 \ \forall \, \sigma \in (1, \infty).$$

Now, by (5.14),

$$\int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}\boldsymbol{\chi}_{1,j}^{n,k}\,\mathrm{d}x = -B_u[\boldsymbol{U}^n,\boldsymbol{U}^n,\boldsymbol{\chi}_{1,j}^{n,k}] - \int_\Omega \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\,\boldsymbol{U}^n\cdot\boldsymbol{\chi}_{1,j}^{n,k}\,\mathrm{d}x + \langle \boldsymbol{f},\boldsymbol{\chi}_{1,j}^{n,k}\rangle.$$

If we take $n\to\infty$ in the above equality, we have

$$\int_\Omega \boldsymbol{S}(c^k,\boldsymbol{D}\boldsymbol{u}^k)\cdot\boldsymbol{D}\boldsymbol{\Phi}_j^k\,\mathrm{d}x = \int_\Omega (\boldsymbol{u}^k\otimes\boldsymbol{u}^k)\cdot\nabla\boldsymbol{\Phi}_j^k - \frac{1}{k}|\boldsymbol{u}^k|^{t-2}\boldsymbol{u}^k\cdot\boldsymbol{\Phi}_j^k\,\mathrm{d}x + \langle\boldsymbol{f},\boldsymbol{\Phi}_j^k\rangle. \tag{6.22}$$

Next, we define $\boldsymbol{\chi}_{2,j}^{n,k} := \Pi_{\mathrm{div}}^n\boldsymbol{\Phi}_j$, and then we have

$$\boldsymbol{\chi}_{2,j}^{n,k}\to\boldsymbol{\Phi}_j \qquad \text{strongly in } W_0^{1,\sigma}(\Omega)^3 \ \ \forall\,\sigma\in(1,\infty).$$

Again, by (5.14),

$$\int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}\boldsymbol{\chi}_{2,j}^{n,k}\,\mathrm{d}x = -B_u[\boldsymbol{U}^n,\boldsymbol{U}^n,\boldsymbol{\chi}_{2,j}^{n,k}] - \int_\Omega \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\,\boldsymbol{U}^n\cdot\boldsymbol{\chi}_{2,j}^{n,k}\,\mathrm{d}x + \langle \boldsymbol{f},\boldsymbol{\chi}_{2,j}^{n,k}\rangle.$$

If we take $n\to\infty$, we have

$$\int_\Omega \boldsymbol{S}(c^k,\boldsymbol{D}\boldsymbol{u}^k)\cdot\boldsymbol{D}\boldsymbol{\Phi}_j\,\mathrm{d}x = \int_\Omega (\boldsymbol{u}^k\otimes\boldsymbol{u}^k)\cdot\nabla\boldsymbol{\Phi}_j - \frac{1}{k}|\boldsymbol{u}^k|^{t-2}\boldsymbol{u}^k\cdot\boldsymbol{\Phi}_j\,\mathrm{d}x + \langle\boldsymbol{f},\boldsymbol{\Phi}_j\rangle.$$

Subsequently, if we pass k to infinity, we obtain

$$\int_\Omega \hat{\boldsymbol{S}}\cdot\boldsymbol{D}\,\mathrm{d}x = \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})\cdot\nabla\boldsymbol{\Phi}_j + \langle\boldsymbol{f},\boldsymbol{\Phi}_j\rangle. \tag{6.23}$$

Therefore, from (6.22) and (6.23), we deduce that

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \boldsymbol{S}(c^k,\boldsymbol{D}\boldsymbol{u}^k)\cdot\boldsymbol{D}\boldsymbol{\Phi}_j^k\,\mathrm{d}x &= \lim_{k\to\infty}\int_\Omega (\boldsymbol{u}^k\otimes\boldsymbol{u}^k)\cdot\nabla\boldsymbol{\Phi}_j^k - \frac{1}{k}|\boldsymbol{u}^k|^{t-2}\boldsymbol{u}^k\cdot\boldsymbol{\Phi}_j^k\,\mathrm{d}x\\ &\quad + \lim_{k\to\infty}\langle\boldsymbol{f},\boldsymbol{\Phi}_j^k\rangle\\ &= \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})\cdot\nabla\boldsymbol{\Phi}_j\,\mathrm{d}x + \langle\boldsymbol{f},\boldsymbol{\Phi}_j\rangle\\ &= \int_\Omega \hat{\boldsymbol{S}}\cdot\boldsymbol{D}\boldsymbol{\Phi}_j\,\mathrm{d}x,\end{aligned}$$

which is the desired analogue of (5.45)–(5.47) corresponding to the limit $k\to\infty$, and thereby the proof of (6.19) has been completed.

We can then use the same argument as the one we employed in the previous section to identify $\bar{\boldsymbol{S}} = \bar{\boldsymbol{S}}^k = \boldsymbol{S}(c^k,\boldsymbol{D}\boldsymbol{u}^k)$ and $\bar{\boldsymbol{q}}_c = \bar{\boldsymbol{q}}_c^k = \boldsymbol{q}_c(c^k,\nabla c^k,\boldsymbol{D}\boldsymbol{u}^k)$ (cf. (5.52) and (5.54), with the index k reinstated), and thus we can again identify $\hat{\boldsymbol{S}} = \boldsymbol{S}(c,\boldsymbol{D}\boldsymbol{u})$, $\hat{\boldsymbol{q}}_c = \boldsymbol{q}_c(c,\nabla c,\boldsymbol{D}\boldsymbol{u})$. That completes the proof of the convergence theorem.

7. Conclusions

We considered a system of nonlinear elliptic partial differential equations that arises in mathematical models of the motion of an incompressible chemically reacting generalized Newtonian fluid in three space dimensions. The governing system involves the coupling of a steady convection-diffusion equation, for the concentration, and a generalized steady power-law-type fluid flow model, for the velocity and the pressure of the fluid, where the viscosity depends on both the shear-rate and the concentration through a concentration-dependent power-law index. We have constructed a numerical approximation of the model by performing a regularization of the momentum equation in the generalized Navier–Stokes system, which was then approximated by a mixed finite element method by using different triangulations for the generalized Navier–Stokes system and the concentration equation. We proved that a sequence of finite element approximations, defined by such a two-grid finite element method, converges to a weak solution of the regularized model, and we then showed that a sequence of weak solutions to the regularized model converges to a weak solution of the original problem. The construction of the numerical method enabled us to completely avoid invoking a discrete version of De Giorgi's regularity result, which we had to rely on in the finite element approximation of the model in two space dimensions [15], but which is not available (at least not in the form that would be required here) in the case of three space dimensions. Hence, whereas in [15] we were forced to confine ourselves to the case of $d = 2$, the constructions and proofs presented herein apply for both $d = 2$ and $d = 3$, although, for the sake of brevity and ease of exposition, we restricted ourselves here to the physically more relevant case of $d = 3$.

Steady flow fields modeled by equation (1.2) can be expected in practice if in the nondimensionalized form of the unsteady counterpart of equation (1.2) the Strouhal number, defined as the ratio of the representative dimensional length-scale and the product of the representative dimensional time with the representative dimensional speed of flow, is small compared to the other nondimensional numbers appearing in the nondimensionalized equation. Similarly, the time derivative term has been ignored in the concentration equation (1.3) by assuming that concentration of the fluid is constant in time on the representative time-scale considered. For further details concerning the derivation of the model we refer to Chapters 2–6 in [18]. The mathematical reason for restricting ourselves to such steady flows here is that the analysis of the unsteady counterpart of the model (cf. [14]) would have introduced further technical complexities, including Lebesgue and Sobolev spaces of functions defined on a space-time domain, with integrability indices that vary in space and in time, for which the counterparts of the analytical tools we have used here have not yet been developed. The numerical analysis of the unsteady model will therefore be explored elsewhere.

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