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HYPERBOLIC COXETER n-POLYTOPES WITH $n + 3$ FACETS

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Abstract. Noncompact hyperbolic Coxeter n-polytopes of finite volume and having $n + 3$ facets are studied in this paper.

Unlike the spherical and parabolic cases, no complete classification exists as yet for hyperbolic Coxeter polytopes of finite volume. It has been shown that the dimension of a bounded Coxeter polytope is at most 29 (Vinberg, 1984), while an upper estimate in the unbounded case is 995 (Prokhorov, 1986). There is a complete classification of simplexes and of Coxeter n-polytopes of finite volume with $n + 2$ facets via the complexity of the combinatorial type.

In 1994, Esselman proved that compact hyperbolic Coxeter n-polytopes with $n+3$ facets can only exist when $n \leq 8$. In dimension 8 there is just one such polytope; it was found by Bugaenko in 1992.

Here we obtain an analogous result for noncompact polytopes of finite volume. There are none when $n > 16$. We prove that there is just one when $n = 16$, and obtain its Coxeter diagram.

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Introduction

A convex polytope in n-dimensional hyperbolic space $\mathbb{H}^n$, Euclidean space $\mathbb{E}^n$ or spherical space $\mathbb{S}^n$ is said to be a *Coxeter polytope* if all of its dihedral angles are integral submultiples of π. The group generated by reflections in the facets of any Coxeter polytope is discrete, with the polytope itself as a fundamental domain.

Hyperbolic Coxeter polytopes of finite volume are of most interest. Unlike in the spherical and parabolic cases, there is no complete classification. It is known [7] that the dimension of a bounded Coxeter polytope is at most 29, with 995 as an upper limit in the unbounded case [13]; at the same time, examples of bounded Coxeter polytopes are known only for $n \leq 8$ [4, 5], while examples of unbounded polytopes of finite volume are known for $n = 21$ [14] and $n \leq 19$ [8, 9]. There is a complete description in [2, 3] of the Coxeter polytopes in $\mathbb{H}^3$; all hyperbolic Coxeter simplexes have been listed [6, 18], and also all Coxeter polytopes with $n + 2$ facets [12, 15, 19].

It is proved in [16] that the dimension of a bounded hyperbolic Coxeter n-polytope with $n + 3$ facets is at most 8. There is an example of an unbounded polytope in $\mathbb{H}^{15}$ with 18 facets (see [8]). The following is the main result of the paper.

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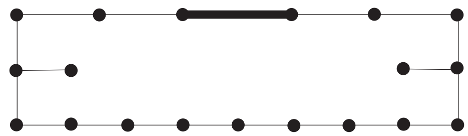
Theorem 1. *There are no Coxeter polytopes of finite volume with $n+3$ facets in hyperbolic space of dimension $n \geq 17$. There is just one such polytope in $\mathbb{H}^{16}$; it has the following Coxeter diagram:*



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1. Gale diagrams and Coxeter diagrams

To every combinatorial type of n-polytope with $n+3$ facets there corresponds a standard two-dimensional *Gale diagram* (see [11] or [17] for example). The standard Gale diagram of an n-polytope with $n+3$ facets which is not a pyramid is a regular $2k$-gon with weighted vertices and centre at the origin of coordinates. The weights satisfy the following conditions:

1) Weights are nonnegative integers whose sum is $n+3$; no two neighbouring vertices (and also no two opposite vertices) have weight zero simultaneously.
2) For every line passing through the origin, the sum of the weights of the vertices lying in each of the open half-hyperplanes bounded by the line is not less than two.

The combinatorial type of a polytope P can be recovered from its Gale diagram like this. To a vertex a_i of the polytope, with weight $\mu(a_i)$, there correspond $\mu(a_i)$ facets $f_{i,1}, \dots, f_{i,\mu(a_i)}$ of P. For an arbitrary subset I of the set of facets of P, the intersection of the set $\{f_{j,\gamma} \mid (j,\gamma) \in I\}$ of facets is a face of P if and only if the origin lies in the convex hull of the set $\widehat{I} = \{a_j \mid (j,\gamma) \notin I\}$. Further, I is a maximal set of facets whose intersection is the given face if and only if the origin lies in the relative interior of the convex hull of $\widehat{I}$.

A convenient way of describing Coxeter polytopes is via *Coxeter diagrams*. Let P be a Coxeter polytope. The vertices of the Coxeter diagram $S(P)$ correspond to the facets of P. Two vertices are joined by an $(m-2)$-fold edge, or a simple edge with label m, if the dihedral angle formed by the corresponding facets is $\frac{\pi}{m}$. If the corresponding facets are parallel, the vertices are joined by a heavy edge. If the facets diverge, the vertices are joined by a dotted edge with label $\cosh d$, where d is the distance between the facets.

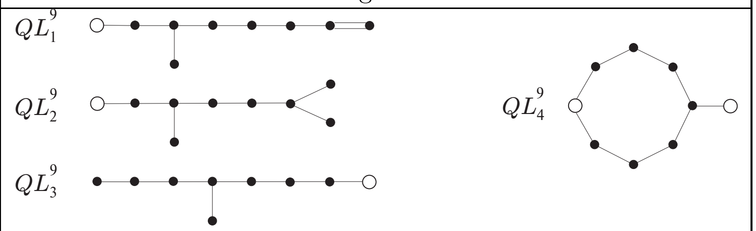
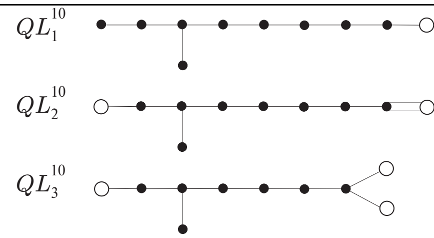
Let S be a diagram with l vertices $v_1, \dots, v_l$. Let $Gr(S)$ be the symmetric $l \times l$ matrix in which $g_{ii} = 1$, for $i = 1, \dots, l$, and $g_{ij} = -\cos\frac{\pi}{m}$ (respectively, $g_{ij} = -1$ or $g_{ij} = -\cosh d$) if v_i and v_j are connected by an edge of label m (respectively a heavy edge or a dotted edge with label $\cosh d$).

If S is the Coxeter diagram for a polytope P, then $Gr(S)$ is the Gram matrix of P (see [8] for details). By the signature, rank and determinant of the diagram S we will understand the signature, rank and determinant of the matrix $Gr(S)$. The order of S (the number of vertices) is denoted by $|S|$.

A Coxeter diagram S is said to be *elliptic* if $Gr(S)$ is positive definite. A connected Coxeter diagram S is said to be *parabolic* if every proper subdiagram is elliptic, but $Gr(S)$ is singular. A nonconnected Coxeter diagram is said to be parabolic if each of its connected components is parabolic. A complete list of the connected elliptic and parabolic diagrams is contained in [8] for example, together with the conventional notation.

Let P be a Coxeter polytope in $\mathbb{H}^n$, $S(P)$ its Coxeter diagram and J the set of facets. If $I \subset J$, we denote by S_I the subdiagram of $S(P)$ corresponding to the facets contained

TABLE 1. The quasi-Lannér Coxeter diagrams of order 9 and 10. We distinguish vertices such that the removal of any one of them yields a parabolic diagram.

order	diagrams
9	
10	

in I. We say that a subset $I \subset J$ determines a face of P if the intersection of all the facets involved in I is a face of P.

The following proposition was proved in [8] as Theorems 3.1 and 3.2.

Proposition 1. (1) *A subset $I \subset J$ determines a face of the polytope P (apart from an infinitely distant vertex) if and only if the subdiagram S_I is elliptic. In this case the codimension of the corresponding face is the number of elements of I.*

(2) *A subset $I \subset J$ determines an infinitely distant vertex if and only if the subdiagram S_I is not elliptic and there is a subset I' such that $I \subset I' \subset J$ and $S_{I'}$ is parabolic of rank $n-1$.*

Remark. Let I be a subset of the set of facets of a polytope P. It follows immediately from the properties of Gale diagrams and Proposition 1 that the origin lies in the interior of the convex hull of $\widehat{I}$ if and only if the subdiagram $S_I \subset S(P)$ is either elliptic or connected and parabolic of rank $n-1$. In this case I is a minimal set of facets defining the corresponding face.

2. THE ABSENCE OF THE REQUIRED POLYTOPES IN HIGH DIMENSIONS

In §§ 2and 3 we will assume that the polytopes under discussion are not pyramids. The case of pyramids is analysed in the final section.

A Lannér (respectively *quasi-Lannér*) diagram is a connected Coxeter diagram that is neither elliptic nor parabolic but all of whose proper subdiagrams are elliptic (respectively, elliptic or parabolic). A complete list of the Lannér and quasi-Lannér diagrams is to be found in [10]. The quasi-Lannér diagrams of order 9 and 10 are listed in Table 1.

If G is a Gale diagram of a polytope P, we denote by $S_{m,l}$ the subdiagram of the Coxeter diagram $S(P)$ corresponding to the $l-m+1$ (mod $2k$) consecutive vertices $a_m, \dots, a_l$ of G. For $l=m$, we denote this subdiagram by S_m; the weight of the vertex a_i is denoted by $\mu(a_i)$.

Lemma 1. *Let G be the Gale diagram of a polytope P. Suppose that the weights of a_i, a_{k+i} are nonzero. Then*

(1) *the vertices a_i and a_{k+i} have weight 1 and the Coxeter diagrams $S_{i+1,k+i-1}$ and $S_{k+i+1,i-1}$ are connected and parabolic;*
(2) *if a_{i+1} and a_{k+i+1} have nonzero weights, the diagram $S_{i+1,k+i}$ is quasi-Lannér;*
(3) *if a_{i+1} has weight zero, the Coxeter diagram $S_{i+2,k+i}$ is quasi-Lannér.*

Proof. Let A be an infinitely distant vertex of P. By the second item of Proposition 1, the subdiagram of $S(P)$ corresponding to A is parabolic of rank $n-1$. Since P is not a pyramid and has $n+3$ facets, it follows that A lies in at most $n+1$ facets. Thus the corresponding subdiagram has at most two connected components.

1) Suppose now that $S_I = S_{i+1,k+i-1}$. It follows from the remark to Proposition 1 and the preceding proof that every proper subdiagram of S_I is elliptic or parabolic of rank $n-1$. Since the weights of a_i, a_{k+i} and a_{k+i+1} are greater than zero, S_I has order at most $(n+3)-3=n$; that is, none of its proper subdiagrams can be parabolic of rank $n-1$. Therefore every proper subdiagram of S_I is elliptic. Furthermore, the origin lies on the boundary of the convex hull of $\widehat{I}$; this means that S_I is positive semidefinite but not elliptic, so it is parabolic. Similarly $S_{k+i+1,i-1}$ is parabolic. We consider any two facets $f_{i,\gamma}$ and $f_{k+i,\delta}$. By Proposition 1, the subdiagram T of $S(P)$ corresponding to the complement of these two facets is a disconnected union of two connected parabolic diagrams. At the same time, it contains two nonintersecting parabolic subdiagrams $S_{k+i+1,i-1}$ and $S_{i+1,k+i-1}$. Thus, T is the disconnected union of $S_{k+i+1,i-1}$ and $S_{i+1,k+i-1}$, and the vertices a_i and a_{k+i} have weight 1.

2) It follows from the preceding proof that $S_{i+1,k+i}$ has exactly two parabolic subdiagrams, while all the other subdiagrams are elliptic. But $S_{i+1,k+i}$ is connected, and therefore it is quasi-Lannér.

3) Consider the subdiagram $S_{i+2,k+i}$. Its subdiagram $S_{i+2,k+i-1}$ is parabolic, and all its other subdiagrams are elliptic. Thus, if $S_{i+2,k+i}$ is not quasi-Lannér, it is a disconnected union of a parabolic subdiagram and the vertex corresponding to a_{k+i}.

In this last case the diagram $S_{i+2,k+i}$ is positive semidefinite. Thus the facets occurring in $S_{i+2,k+i}$ have as intersection some infinitely distant vertex of P (see [1, Chapter 6, Theorems 1.3 and 1.4]). However, the convex hull of the vertices $a_{k+i+1},\dots,a_{i+1}$ of the Gale diagram does not contain the origin. This contradiction shows that $S_{i+2,k+i}$ is connected, and hence quasi-Lannér. □

Lemma 2. *Let G be the Gale diagram of a polytope P, and suppose that the weights of the vertices a_i and a_{k+i-1} are zero. Then the diagram $S_{i+1,k+i-2}$ is Lannér.*

Proof. Every proper subdiagram of $S_{i+1,k+i-2}$ is elliptic or parabolic of rank $n-1$. It follows from the properties of Gale diagrams that a_{k+i} has weight greater than zero, and the order of the diagram $S_{k+i+1,i-1}$ is not less than 2. Thus $S_{i+1,k+i-2}$ has order at most $(n+3)-1-2=n$, and each of its proper subdiagrams is elliptic. Thus $S_{i+1,k+i-2}$ is itself neither elliptic nor parabolic, and so it is Lannér. □

Lemma 3. *Let G be the Gale diagram of a polytope P. Then $k>2$, and every vertex a_j of G with nonzero weight is contained in some Lannér or quasi-Lannér subdiagram $S_{i,l}$.*

Proof. If $k=1$, then condition 2) governing the weights of vertices in Gale diagrams (§ 1) is clearly not satisfied. If $k=2$, Lemma 1 gives that all vertices have weight 1. Again the conditions on vertex weights are falsified.

We consider three cases.

- Suppose that no vertex in G has zero weight. Then the diagram $S_{j,k+j-1}$ is quasi-Lannér.
- Suppose that some vertex a_i, where $i \neq k+j$, has weight zero. Then the diagrams $S_{i+1,k+i-1}$ and $S_{k+i+1,i-1}$ are Lannér or quasi-Lannér, and all the facets of the form $f_{j,\gamma}$ appear in one of them.
- Suppose that a_{k+j} has zero weight, and all the other vertices have nonzero weight. Then the diagram $S_{j-1,k+j-2}$ is quasi-Lannér. □

Since the number of vertices in a Lannér diagram is at most 5 (see [18]), while quasi-Lannér diagrams have at most 10 vertices [6], the following is a consequence of Lemma 2 and items 2) and 3) of Lemma 1.

Lemma 4. *Let G be the Gale diagram of a polytope P, and suppose that the vertices a_{i-1} and a_{i+1} have weight zero. Then the sum of all the vertex weights is at most* 20*.*

Proof. Suppose that all vertices a_{k+i} have weight zero. By Lemma 2, the Coxeter diagrams $S_{i+2,k+i-1}$ and $S_{k+i+1,i}$ are Lannér of order at most 5. By Lemma 3, it follows that every vertex a_j of G is contained in a Lannér or quasi-Lannér diagram, so that each vertex has weight at most 10. In particular, a_{i+1} has weight at most 10, so that $S(P)$ has no more than $5+5+10 = 20$ vertices.

Suppose that a_{k+i+1} has nonzero weight. Then by item (1) of Lemma 1, the vertices a_{i+1} and a_{k+i+1} have weight 1. By item (3) of Lemma 1 the diagrams $S_{i+3,k+i+1}$ and $S_{k+i+1,i-1}$ are quasi-Lannér of order at most 10. Thus, the number of vertices in $S(P)$ is not greater than $1+(10+10-1) = 20$. □

Lemma 5. *Let P be a polytope of dimension more than* 17*. Then its Gale diagram must satisfy the following conditions:*

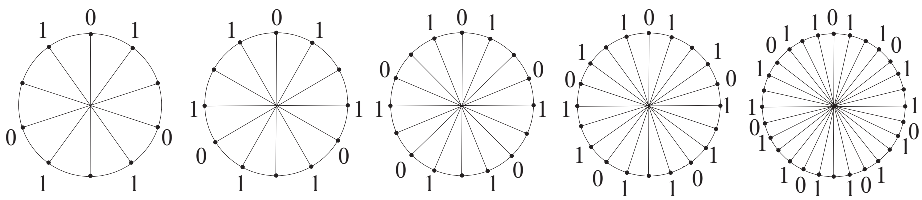
(1) *If the vertex a_i has weight zero, then the weights of $a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}$ are nonzero.*
(2) *If the weights of a_i and a_{i+1} are nonzero, then a_{k+i} or a_{k+i+1} has zero weight.*
(3) *$k \leq 16$.*
(4) *If $k \geq 9$, then between any two vertices of weight zero (on each of the arcs of the circle), there are at least three vertices with positive weights.*

Proof. Condition (1) follows immediately from Lemma 4, and (2) from item (2) of Lemma 1.

Without loss of generality we may assume that a_1 has weight zero. By condition (1) there can be no more than $\frac{k+2}{3}$ vertices with weight zero among the vertices $a_1, \dots, a_k$. At the same time, by item (3) of Lemma 1 and Lemma 2, there are no more than 10 vertices with nonzero weights among $a_2, \dots, a_k$. We thus get the estimate $(k-1)-\left(\frac{k+2}{3}-1\right) \leq 10$, which gives condition (3).

Assume now that a_i and a_{i+3} have weight zero. Then the weight of one of a_{k+i+1}, a_{k+i+2} is zero. Thus by Lemma 2, either $S_{k+i+3,i-1}$ or $S_{i+4,k+i}$ is a Lannér subdiagram. As in the previous discussion, we get $(k-2)-\left(\frac{k+2}{3}-2\right) \leq 5$, since the number of vertices of a Lannér diagram is at most 5. Thus $k < 9$. □

Table 2. The upper index in the notation of a diagram is k. Unit weights are arranged in accordance with item (1) of Lemma 1. All weights not indicated are greater than zero.



$\mathfrak{D}^5$ $\mathfrak{D}^6$ $\mathfrak{D}^8$ $\mathfrak{D}^{10}$ $\mathfrak{D}^{14}$

The Gale diagrams satisfying conditions (1)–(4) are shown in Table 2.

Lemma 6. *Suppose that a Gale diagram in Table 2 corresponds to a hyperbolic Coxeter polytope of finite volume. Then the dimension of the polytope is at most* 15.

Proof. We prove the lemma for each diagram individually. We enumerate the vertices in Gale diagrams clockwise, starting with the one at the top. Then $\mu(a_1) = 0$ for all diagrams.

$\mathfrak{D}^8$**.** Note that $\mu(a_7) = \mu(a_{14}) = 0$. Thus, $S_{8,13}$ is Lannér and its order is at most 5. However, among the vertices $a_8, \dots, a_{13}$ there are exactly 5 with nonzero weights. It follows that these weights are unity. Similarly, $S_{5,10}$ is Lannér and a_6 has unit weight. Further, $S_{15,5}$ is quasi-Lannér and its order is at most 10. Thus the dimension of the Coxeter polytope with this Gale diagram is at most $10 + 1 + 5 - 3 = 13$.

$\mathfrak{D}^{14}$**.** The diagram $S_{2,14}$ is quasi-Lannér and its order is at most 10. At the same time, exactly 10 of the vertices $a_2, \dots, a_{14}$ have nonzero weights. It follows that all these weights are unity. Similarly, all nonzero weights in the diagram are unity. Furthermore, the parabolic diagrams $S_{2,12}$ and $S_{15,27}$ are not joined to each other in $S(P)$. Neither are the parabolic diagrams $S_{6,16}$ and $S_{19,3}$. Thus, the vertex S_2 can be joined only to S_{28}, S_3 and S_4, while S_3 can be joined only to S_2 and S_4. Since $S_{2,14}$ is connected, S_3 is joined to S_4. Similarly, S_3 is joined to S_2. Since the diagram $S_{2,4}$ is elliptic, S_2 is not joined to S_4. The diagram $S_{28,12}$ is quasi-Lannér, which means that it is connected, and so S_2 is joined to S_{28}. We have thus shown that $S_{28,4}$ is linear.

By looking at the remaining vertices in the way described above, we shall find that $S(P)$ is a cycle with 20 vertices. If there is at least one multiple edge, we consider the subdiagram of $S(P)$ that consists of 5 consecutive vertices, in which no end of this edge is a terminal vertex. This diagram should be elliptic, but it is not. Thus, $S(P) = \tilde{A}_{19}$ and is not the diagram of a hyperbolic polytope. Therefore the Gale diagram under discussion does not correspond to any hyperbolic Coxeter polytope.

$\mathfrak{D}^{10}$**.** As in the previous case, we prove that a vertex of the form S_{4l+2} is joined in $S(P)$ only to the vertex S_{4l} and the subdiagram S_{4l+3}; a vertex of the form S_{4l} is joined to the vertex S_{4l+2} and the subdiagram S_{4l-1}; and the subdiagram S_{4l+3} is joined to the vertices S_{4l} and S_{4l+2}. Further, since the diagram $S_{4l-1,4l+7}$ is connected, $S_{4l-1,4l}$ is also connected. Similarly, so is $S_{4l-2,4l-1}$. If the diagram S_{4l-1} were not connected, then the diagram $S_{4l-2,4l}$ would contain a cycle. However, $S_{4l-2,4l}$ is elliptic and thus a diagram of the form S_{4l-1} is connected and is joined to each of the vertices S_{4l-2} and S_{4l} by a single edge.

Every edge of $S(P)$ is contained in some quasi-Lannér subdiagram of order 7 or more. Thus, $S(P)$ has no edges of multiplicity greater than 2. If the subdiagrams S_{4l} and S_{4l+2}

(respectively, S_{4l-1} and S_{4l}, or S_{4l+2} and S_{4l+3}) are joined by an edge of multiplicity 2, then we obtain a contradiction by considering the elliptic diagram $S_{4l-2,4l+3}$ (respectively $S_{4l-2,4l+3}$ or $S_{4l-1,4l+4}$). Thus, $S(P)$ has no multiple edges.

Assume now that the dimension of the corresponding polytope is not less than 16. Then there exists l such that $|S_{4l-1}|+|S_{4l+3}|+|S_{4l+7}|\geq 6$. For definiteness we take $l=0$. Then the order of the connected acyclic parabolic subdiagram $S_{19,7}$ is not less than 10. Thus $S_{19,7}=\tilde{D}_m$ for some $m\geq 9$. Since the diagrams S_{19} and S_7 are connected and $|S_8|=|S_{18}|=1$, we get that both of the diagrams S_{19} and S_7 have order at least three.

The diagram $S_{12,20}$ is quasi-Lannér. Thus, $|S_{19}|+|S_{15}|\leq 10-5=5$ and $|S_{15}|\leq 2$. Similarly, $|S_{11}|\leq 2$. Since $S_{10,16}$ is parabolic, we get that $|S_{11}|=|S_{15}|=2$, $|S_{19}|=|S_7|=3$, and $S_{10,16}=\tilde{D}_7$. It follows that $S(P)$ is a cycle with four "twigs". The corresponding terminal vertices lie in S_{19}, S_7, S_{11} and S_{15}. At the same time, the origin is contained in the interior of the convex hull of the set $\{a_{19},a_7,a_{11}\}$. Consequently, $S(P)$ minus the three terminal vertices must be either elliptic or parabolic, which is obviously not the case. This contradiction shows that the dimension of the hyperbolic Coxeter polytope with this Gale diagram is not more than 15.

$\mathfrak{D}^6$**.** In a similar way to the preceding case, it can be shown that a subdiagram of the form S_{4l+3} is joined in $S(P)$ only to the vertices S_{4l} and S_{4l+2}. There is a difference, namely that $S_{4l+2,4l+4}$ is parabolic, not elliptic, so it could be a cycle. A vertex of the form S_{4l+2} as well as one of the form S_{4l} and a subdiagram S_{4l+3} can be joined to the vertex S_{4l+4}. Further, since the diagram $S_{4l-1,4l+3}$ is connected, $S_{4l-1,4l}$ is connected. Similarly, $S_{4l-2,4l-1}$ is connected. If S_{4l-1} is not connected, then $S_{4l-2,4l}$ is a cycle without multiple edges. Analogously, if a vertex S_{4l-2} is joined to the vertex S_{4l}, then the diagram $S_{4l-2,4l}$ is a cycle without multiple edges.

The diagram $S_{4l+2,4l+6}$ is quasi-Lannér, so that $|S_{4l+3}|\leq 7$. Suppose that $|S_3|=7$. Then $|S_{2,6}|=10$, and $S_{2,6}$ has exactly one parabolic subdiagram, which means that $S_{2,6}=QL_1^{10}$. Similarly, $S_{12,4}=E_{10}$, where the vertices S_{12} and S_6 must be joined to one and the same vertex of the parabolic diagram $S_{2,4}=\tilde{E}_8$. But S_{12} is joined to S_2, and S_6 to S_4. Thus the order of the diagrams S_3, S_7 and S_{11} is not more than 6.

We now assume that the dimension of the polytope is not less than 16. Then $|S_3|+|S_7|+|S_{11}|\geq 13$. For definiteness we assume that $|S_3|\geq|S_7|\geq|S_{11}|$. Then $|S_3|+|S_7|\geq 9$, and $|S_7|+|S_{11}|\geq 7$. The diagram $S_{3,7}$ is parabolic but not a cycle, and its order is not less than 11. Thus, it is either $\tilde{B}_m$, $\tilde{C}_m$ or $\tilde{D}_m$. Since $|S_3|\geq|S_7|\geq 4$, we get that $S_{2,4}$ and $S_{6,8}$ are not cycles. For the same reason $S_{3,7}=\tilde{D}_m$. Otherwise there is an edge of multiplicity 2, neither of whose ends is a terminal vertex, in the connected parabolic diagram $S_{2,4}$ (or $S_{6,8}$) of order 6 or more.

Consider now the diagram $S_{2,4}$. Since $S_{3,7}=\tilde{D}_m$, it follows that $S_{3,4}=D_l$. The complement in D_l of a single vertex (S_2) must be a parabolic diagram. Thus either $S_{2,4}=\tilde{E}_8$, or $S_{2,4}=\tilde{D}_l$. In the first case $|S_{2,4}|=9$ and $|S_3|=7$, which is impossible, as proved above. So $S_{2,4}=\tilde{D}_l$, and the vertices S_2 and S_4 are joined to one and the same vertex u in the diagram $S_3=D_{l-1}$. Similarly, $S_{6,8}=\tilde{D}_r$, and the vertices S_6 and S_8 are joined to one and the same vertex v in the diagram $S_7=D_{r-1}$.

The origin lies in the interior of the convex hull of the set $\{a_3,a_7,a_{11}\}$. Thus the diagram obtained from $S(P)$ by removing the subdiagram $S_{10,12}$, together with all terminal vertices of the diagrams S_3 and S_7 that are terminal vertices in $S_{2,4}$ and $S_{6,8}$ respectively, must be elliptic. But it has at least two vertices of valency 3, namely u and v. This contradiction shows that the dimension of a hyperbolic Coxeter polytope with the Gale diagram under discussion is at most 15.

$\mathfrak{D}^5$**.** The diagram $S_{3,6}$ is connected and parabolic. If the dimension of the polytope is sufficiently large (at least 12), then the order of $S_{3,6}$ is at least 4. Thus, its subdiagram $S_{5,6}$ does not contain edges of multiplicity more than 2. Similarly, $S_{6,7}$ does not contain edges of multiplicity more than 2. Further, no vertex of the diagram $S_{3,6}$ is joined in $S(P)$ to a vertex of $S_{9,10}$ (Lemma 1), and in addition $S_{6,9}$ is connected. It follows that $S_{6,7}$ and $S_{7,9}$ are connected; likewise, so are $S_{5,6}$ and $S_{3,5}$.

The diagram $S_{5,7}$ is Lannér. If there are no cycles in it, then the vertices S_5 and S_7 of the diagram $S(P)$ are not joined to each other. Thus we get that the Lannér diagram $S_{5,7}$ contains no cycles nor edges of multiplicity more than 2; but this is impossible. So $S_{5,7}$ is a cycle. To complete the proof, we need the following

Assertion 1. $|S_{5,7}| = 3$.

Proof. Suppose that $S_{5,7}$ has order greater than 3. Two cases are possible:

1) The vertices S_5 and S_7 of the diagram $S(P)$ are joined by an edge. Then the diagram S_6 is connected and linear. Consider the following subdiagram of $S(P)$: $S^{2,6,10} = S(P) \setminus \{v_2, v_6, v_{10}\}$, where $v_i \in S_i$ are vertices in $S(P)$. Since the origin lies in the interior of the convex hull of the set $\{a_2, a_6, a_{10}\}$, the diagram $S^{2,6,10}$ is elliptic or connected and parabolic. This diagram $S^{2,6,10}$ is connected, since $S_{7,9}$ and $S_{3,5}$ are connected. At the same time, it has a vertex of valency 3 (namely S_5 or S_7). We may assume that the valency of S_7 is 3. If the dimension of the polytope is sufficiently large (more than 10), then $S^{2,6,10} = D_l, \tilde{B}_l$ or $\tilde{D}_l$ for some l. The vertex S_7 is joined to S_5, $S_6 \setminus v_6$ and S_9. The vertex S_5 is not a terminal vertex in $S^{2,6,10}$, since $S_{3,5}$ is connected. Thus, there are two terminal vertices adjacent to S_7, namely $S_6 \setminus v_6$ and S_9. In particular, $|S_9| = 1$, and the order of $S(P)$ is at most 16.

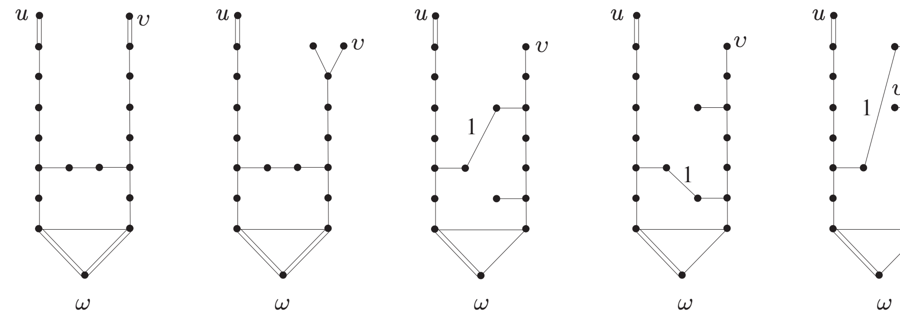
2) The vertices S_5 and S_7 in the diagram $S(P)$ are not joined to each other. Then the subdiagram S_6 is not connected. There is a vertex $u_6 \in S_6$, incident to an edge of multiplicity 2. We consider now the diagram $S^{2,6,10} = S(P) \setminus \{v_2, v_6, v_{10}\}$, where $v_6 \in S_6$ and $v_6 \neq u_6$. This diagram is elliptic, its order is not less than 5 (if the dimension of the polytope is at least 5), and it is connected (since $S_{7,9}$ and $S_{3,5}$ are connected) and contains an edge of multiplicity 2. Thus, $S^{2,6,10} = B_l$ for some l. On the other hand, no vertex incident to an edge of multiplicity 2 is a terminal vertex in $S^{2,6,10}$. □

The contradiction thus obtained shows that $|S_6| = 1$, and the vertices S_5 and S_7 are joined by a simple edge (see the preceding discussion). We may assume that the multiplicity of an edge joining S_6 and S_7 is 2. Then, the multiplicity of an edge joining S_5 and S_6 is 1 or 2. Further, it is not hard to show that the vertex S_{10} is joined in $S(P)$ only to the vertex S_2 and the subdiagram S_9, and the subdiagram S_9 is joined to the vertices S_{10} and S_7.

We assume that the dimension of the polytope is at least 16, so that $|S_3| + |S_9| \geq 14$. The diagram $S_{2,5}$ is quasi-Lannér, so that $|S_3| \leq 8$. Similarly, $|S_9| \leq 8$.

Let us assume that $|S_9| = 8$. Then $S_{7,10} = QL_1^{10}$ and $S_{9,10} = \tilde{E}_8$. Since the diagram $S_{6,9}$ is parabolic and has order 10, while $S_{7,10} = QL_1^{10}$ does not contain multiple edges, we find that $S_{6,9} = \tilde{B}_9$ and $S_9 = D_8$, that is, $S_{9,10} \setminus S_{10} = D_8$. But then $S_{9,2} \neq QL_1^{10}$, which is impossible since $S_{9,2}$ is quasi-Lannér with a unique parabolic subdiagram and has order 10. Thus $|S_9| \leq 7$. If the multiplicity of an edge joining S_5 and S_6 is 2, then $|S_3| \leq 7$ also. Otherwise, that is if the edge is simple, the parabolic diagram $S_{3,6}$ does not contain multiple edges and nor is it $\tilde{D}_l$. Thus, its order is at most 9, and the order of S_3 is at most 7. Since $|S_3| + |S_9| \geq 14$, we get $|S_3| = |S_9| = 7$.

Consider the diagram $S_{6,9}$. It is parabolic, has order 9 and contains an edge of multiplicity 2, that is, $S_{6,9} = \tilde{B}_8$ or $\tilde{C}_8$. Suppose that $S_{6,9} = \tilde{C}_8$. Then $S(P)$ is one of the following diagrams:



Since $u \in S_9$, $v \in S_3$ and $w \in S_6$, the diagram $S(P) \setminus \{u, v, w\}$ must be elliptic or parabolic. Obviously, this is false for all the diagrams presented. The case where $S_{6,9} = \tilde{B}_8$ is dealt with in the same sort of way. $\square$

We have shown that for $n > 17$ there are no Coxeter polytopes of finite volume in $\mathbb{H}^n$ with $n+3$ facets, not combinatorially equivalent to pyramids.

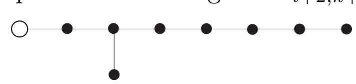
3. Polytopes of dimensions 16 and 17

We now consider polytopes in $\mathbb{H}^{16}$ and $\mathbb{H}^{17}$.

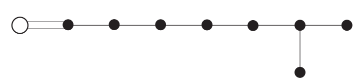
Lemma 7. *Suppose that the dimension of a hyperbolic Coxeter polytope of finite volume is* 16 *or* 17*. Then at least one vertex in every quartet a_i, a_{i+1}, a_{k+i} and a_{k+i+1} of vertices of the Gale diagram has weight zero.*

Proof. Suppose that P is a polytope whose Gale diagram G has points a_i, a_{i+1}, a_{k+i}, and a_{k+i+1} with nonzero weights. Then the diagrams $S_{i+1,k+i}$ and $S_{k+i+1,i}$ are quasi-Lannér (Lemma 1), and each of them has exactly two parabolic subdiagrams. Since $S(P)$ has at least 19 vertices, the order of each of the diagrams $S_{i+1,k+i}$ and $S_{k+i+1,i}$ is 9 or 10. Thus, one of them is QL_2^{10} and the other is QL_2^{10} or QL_4^9 (see Table 1). Assume for definiteness that $S_{i+1,k+i} = QL_2^{10}$.

Assume that the weight of a_{k+i+2} is nonzero. Then the diagram $S_{i+2,k+i+1}$ is quasi-Lannér and has order 10. Its parabolic subdiagram $S_{i+2,k+i}$ has the form



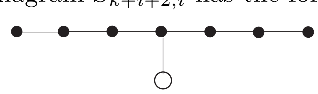
or



where we have used the white circles to distinguish the vertices that must be joined to the vertex S_{k+i+1}. But the addition of the vertex S_{k+i+1} as shown does not yield a quasi-Lannér diagram. The contradiction thus obtained shows that a_{k+i+2} has weight zero. Similarly a_{i-1} has weight zero.

Further, $S_{k+i+1,i} = QL_4^9$. For, if we assume that $S_{k+i+1,i} = QL_2^{10}$, applying the arguments described above, and using the fact that the weights of a_{i+2} and a_{k+i-1} are nonzero, we again arrive at a contradiction.

One of the parabolic subdiagrams $S_{k+i+2,i}$ and $S_{k+i+1,i-1}$ is a cycle of order 8. We may assume that $S_{k+i+1,i-1}$ is such a cycle. The diagram $S_{k+i+2,i+1}$ is quasi-Lannér of order 9 and its parabolic subdiagram $S_{k+i+2,i}$ has the form



where the vertex to which S_{i+1} must be joined is distinguished by the white circle. But we do not get a quasi-Lannér diagram by adding the vertex S_{i+1} in the way indicated, and the lemma follows from this contradiction. □

In particular it follows from Lemma 7 that $N \geq \frac{k}{2}$, where N is the number of points with zero weight in the Gale diagram.

Lemma 8. *Suppose that $k \geq 6$. Then $S(P)$ can contain at most one Lannér subdiagram.*

Proof. Suppose that $\mu(a_i) = \mu(a_{k+i-1}) = 0$. Then the diagram $S_{i+1,k+i-2}$ is Lannér and its order is at most 5. We consider two cases.

1) Suppose that $\mu(a_{k+i+1}) = 0$; then $S_{k+i+1,i-1}$ is Lannér. If the dimension of the polytope is at least 16, then $|S_{k+i}| \geq (16+3) - 5 - 5 = 9$. Since $k \geq 6$, at least 3 of the points $a_{i+2}, \dots, a_{k+i}$ have nonzero weights. Thus the quasi-Lannér (or Lannér) diagram $S_{i+2,k+i}$ has order 11 or more, and this is impossible.

2) Suppose that $\mu(a_{k+i+1}) > 0$; then $S_{k+i+1,i-1}$ is quasi-Lannér. Since we have assumed that the dimension of the polytope is at least 16, we have $|S_{k+i}| \geq (16+3) - 5 - 10 = 4$. Similarly, $|S_{i-1}| \geq 4$. Since $\mu(a_{i+1}) > 0$, we get that $\mu(a_{k+i+1}) = \mu(a_{i+1}) = 1$. Further, we may assume that $\mu(a_{i-2}) > 0$ (otherwise we again get the preceding case) and that $\mu(a_{i-2}) = \mu(a_{k+i-2}) = 1$. If $S(P)$ has another Lannér subdiagram, then it contains $S_{i-2,i+1}$ or $S_{k+i-2,k+i+1}$. But the order of both of these is at least 6. □

It follows from Lemma 8 that $N \leq \frac{k+1}{2}$ for $k \geq 6$. Earlier we showed that $N \geq \frac{k}{2}$; thus if $k \geq 6$, then $N = \left[\frac{k+1}{2}\right]$.

It was shown in § 2 that the dimension of a hyperbolic Coxeter polytope of finite volume and having $n+3$ facets is at most 17. It follows that at most 20 vertices in its Gale diagram have nonzero weights. Applying the bound obtained above for the number N of vertices in the Gale diagrams with zero weights, we find that $2k - \left[\frac{k+1}{2}\right] \leq 20$, so that $k \leq 13$.

Thus, suppose that the dimension of the polytope P is 16 or 17. Then its Gale diagram must satisfy the following conditions:

(1) $k \leq 13$;
(2) at least one vertex in every quartet $a_i,\ a_{i+1},\ a_{k+i},\ a_{k+i+1}$ has nonzero weight;
(3) $N \geq \frac{k}{2}$, and if $k \geq 6,\ N = \left[\frac{k+1}{2}\right]$;
(4) if $\mu(a_i) = \mu(a_{k+i-1}) = 0$ (that is, $S_{i+1,k+i-2}$ is Lannér), then at most 5 of the vertices $a_{i+1}, \dots, a_{k+i-2}$ have nonzero weights;
(5) if $\mu(a_i) = 0$, and $\mu(a_{k+i-1}) > 0$ (that is, $S_{i+1,k+i-1}$ is quasi-Lannér), then at most 10 of the vertices $a_{i+1}, \dots, a_{k+i-2}$ have nonzero weights.

The Gale diagrams satisfying conditions (1)–(5) are listed in Table 3.

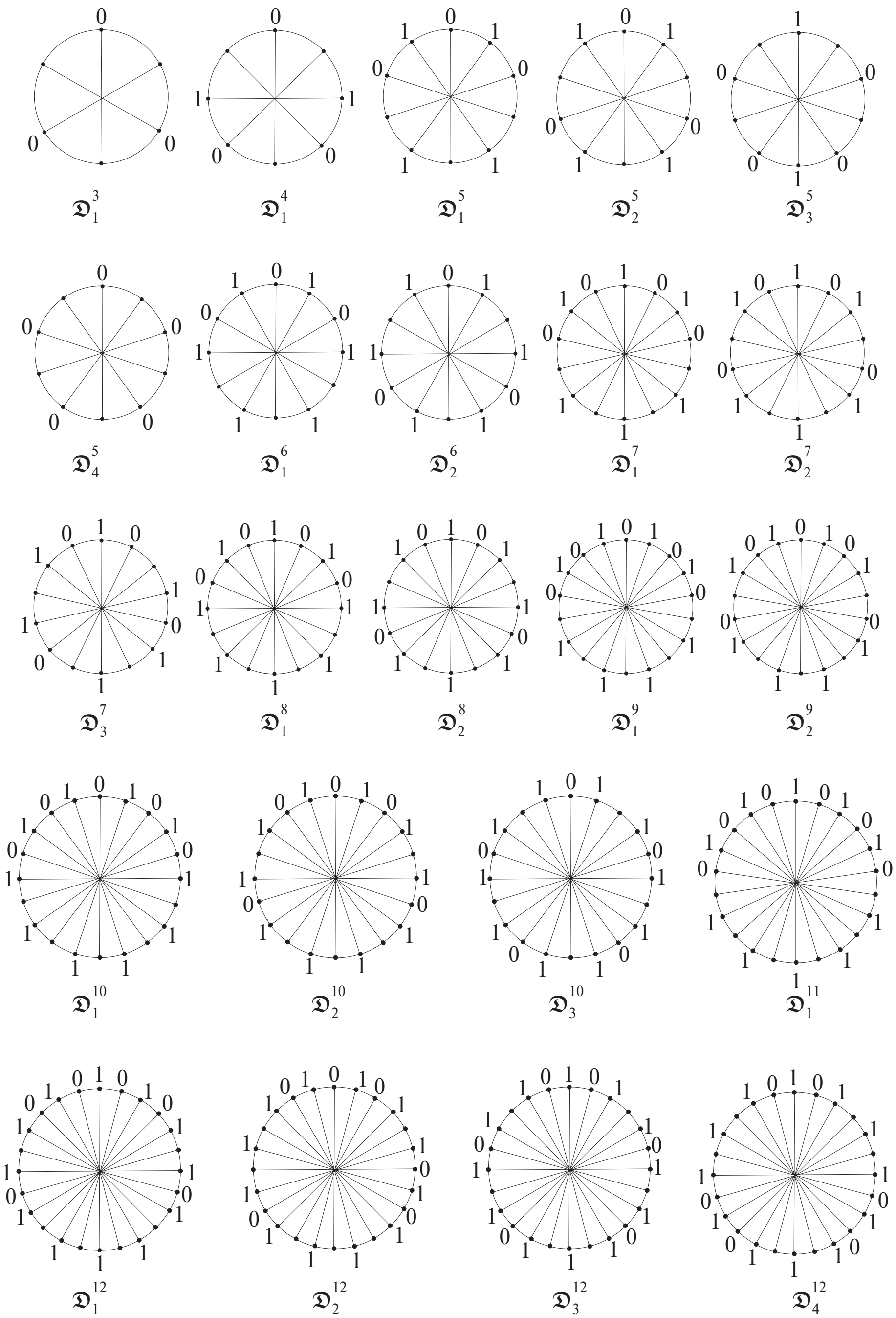
Lemma 9. *Let P be a Coxeter polytope of finite volume in $\mathbb{H}^n$, other than a pyramid. If* $n = 16$ *or* 17*, then $S(P)$ is the diagram shown in Theorem* 1*.*

Proof. We number the vertices in the Gale diagram clockwise, starting at the top. We consider each of the diagrams listed in Table 3 separately.

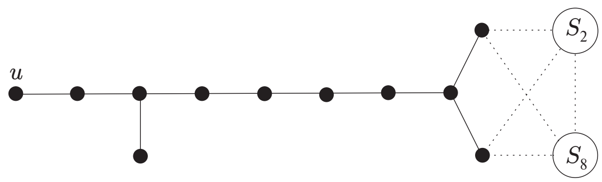
$\mathfrak{D}_1^3$**.** The diagrams S_1, S_3 and S_5 are Lannér, that is, the order of $S(P)$ is at most 5+5+5=15.

$\mathfrak{D}_1^4$**.** The diagrams $S_{2,3}$ and $S_{7,8}$ are Lannér, that is, the orders of S_2 and S_8 are at most 4. The diagram $S_{3,5}$ is quasi-Lannér, so that $|S_5| \leq 9$. If $|S(P)| \geq 19$, then $|S_2| = |S_8| = 4$, and $|S_5| = 9$. Since the quasi-Lannér diagrams $S_{3,5}$ and $S_{5,7}$ of order 10 contain exactly

TABLE 3. The upper index in the notation of a diagram is k. Unit weights are arranged in accordance with item (1) of Lemma 1. All the weights that are not shown are nonzero.

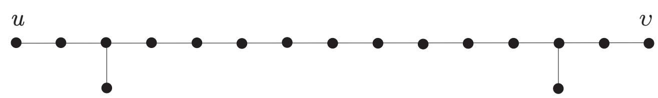


one parabolic subdiagram each, $S(P)$ has the form



where the dots denote possible joinings. On considering the Gale diagram, it is easy to see that $S(P)\setminus\{u, S_{8,2}\}$ should be elliptic (it cannot be parabolic since $|S_2|+|S_8| = 8 > 2$). But it is parabolic.

$\mathfrak{D}_1^5$. The diagram $S_{4,7}$ is quasi-Lannér, that is, $|S_4| \le 7$. If $|S(P)| \ge 19$, then $|S_4| = |S_8| = 7$, and $|S_6| = 1$. Then the diagram $S_{8,2} = S(P)\setminus S_{10,2}$ has the form

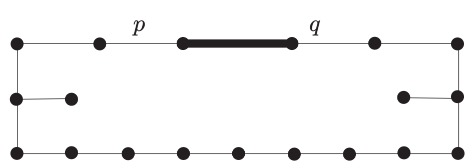


Since $u \in S_8$ and $v \in S_4$, the diagram $S(P)\setminus\{u, v, S_{10,2}\}$ must be elliptic, while it is actually parabolic.

$\mathfrak{D}_3^5$. The diagrams $S_{4,6}$, $S_{6,8}$ and $S_{10,2}$ are Lannér, that is $S(P)$ has order at most $(5+5-1)+5 = 14$.

$\mathfrak{D}_4^5$. All the diagrams of the form $S_{2l,2l+2}$ are Lannér. Thus, $S(P)$ has order at most $\left[\frac{5\times 5}{2}\right] = 12$.

$\mathfrak{D}_1^6$. The diagrams $S_{4,8}$ and $S_{6,10}$ are quasi-Lannér. If $|S(P)| \ge 19$, then $|S_5| = |S_9| = 6$ and $|S_7| = 1$. Then $S(P)$ has the form



Since $S_{2,5}$ is parabolic, $q = 3$. Similarly $p = 3$. It is not hard to check that the signature of $S(P)$ is $(16, 1, 2)$, where the positive index of inertia is the first component, the negative index of inertia is the second and the dimension of the kernel is the third. By Theorem 2.1 of [8], we have obtained the Coxeter diagram of a 16-dimensional hyperbolic polytope. The fact that it has finite volume can be established using Theorem 4.2 from [8].

$\mathfrak{D}_1^7$. The diagram $S_{5,10}$ is quasi-Lannér, that is, $|S_5| \le 5$. On the other hand $S_{6,11}$ is also quasi-Lannér. Thus the order of $S(P)$ is at most $10+5+3 = 18$.

$\mathfrak{D}_2^7$. The diagram $S_{6,10}$ is Lannér of order 5, and is not a cycle. At the same time, each of its edges is included in some quasi-Lannér diagram of order 5 or greater. Thus $S_{6,10}$ has no edges of multiplicity greater than 2, and this is impossible.

$\mathfrak{D}_3^7$. The diagram $S_{11,1}$ is Lannér of order 4 or 5 and is not a cycle. If $|S_3| > 1$, then for every edge of $S_{11,1}$ there is a quasi-Lannér diagram of order 5 or greater containing it. In this case $S_{11,1}$ has no edges of multiplicity greater than 2, and this is impossible. Thus, $|S_3| = 1$. Since $S_{4,9}$ is quasi-Lannér and is of order at most 10, the order of $S(P)$ is at most $10+5+1 = 16$.

$\mathfrak{D}_1^8$**.** The diagrams $S_{5,11}$ and $S_{7,13}$ are quasi-Lannér. Their intersection consists of at least 5 vertices, and so the order of $S(P)$ does not exceed $(10+10-5)+3=18$.

$\mathfrak{D}_2^8$**.** The diagrams $S_{3,9}$ and $S_{9,15}$ are quasi-Lannér, that is, $|S_4|+|S_8|\le 6$ and $|S_{10}|+|S_{14}|\le 6$. If the dimension of the polytope is at least 16, we may assume that $|S_{10}|+|S_{14}|=6$ (and $S_{9,15}=QL_1^{10}$), while $|S_4|+|S_8|=5$ or 6. Using the same sort of arguments as for the diagram $\mathfrak{D}^{14}$ (§ 2), it is easy to show that the subdiagram S_i is joined in $S(P)$ only to the subdiagrams S_{i+1} (or S_{i+2}, if $\mu(a_{i+1})=0$) and S_{i-1} (or S_{i-2}, if $\mu(a_{i-1})=0$).

If $|S_4|+|S_8|=6$, then $S_{3,9}$ and $S_{9,15}$ are two copies of QL_1^{10}, intersecting in the distinguished vertex (see Table 1). Then the diagram $S_{8,14}$ (whose order is not less than 11 and not greater than 15) is elliptic. On the other hand, it follows from the Gale diagram that $S_{8,14}$ must be parabolic.

Suppose that $|S_4|+|S_8|=5$. Since $S_{8,14}$ is parabolic and of order at least 11, and S_{14} has a vertex of valency 3, the diagram $S_{8,9}$ is one of B_l or D_l for some $l\le 5$. It follows from the classification of quasi-Lannér diagrams of order 9 given in Table 1, that $S_{8,9}=D_5$ and $|S_8|=4$. Then $|S_4|=1$ and $S_{3,7}$ is linear, so that $S_{3,9}=QL_1^9$. Further, $S_{1,5}$ is parabolic and its order is 4 since $|S_4|=1$. However, it contains an edge of multiplicity 2 both ends of which are terminal vertices, and this is impossible.

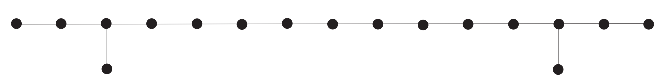
$\mathfrak{D}_1^9$**.** The diagram $S_{6,13}$ is quasi-Lannér, so that $|S_6|\le 3$. On the other hand $S_{7,14}$ is also quasi-Lannér, and thus the order of $S(P)$ is at most $10+3+4=17$.

$\mathfrak{D}_2^9$**.** The diagram $S_{13,2}$ is Lannér of order 5 and is not a cycle. At the same time, each of its edges is included in some quasi-Lannér subdiagram of order 5 or greater. Thus, $S_{13,2}$ has no edges of multiplicity greater than 2, and this is impossible.

$\mathfrak{D}_1^{10}$**.** The diagram $S_{6,14}$ is quasi-Lannér, so that $|S_7|\le 2$. But $S_{8,16}$ is also quasi-Lannér, so the order of $S(P)$ is at most $10+2+5=17$.

$\mathfrak{D}_2^{10}$**.** The diagrams $S_{4,12}$ and $S_{10,18}$ are quasi-Lannér. If the dimension of the polytope is at least 16, then $|S_{11}|=1$, $|S_5|+|S_9|=|S_{13}|+|S_{17}|=4$, and $S_{4,12}$ and $S_{10,18}$ are both copies of QL_1^{10}.

Using arguments like those in the investigation of the diagram $\mathfrak{D}^{14}$ in § 2, it is easy to deduce that the subdiagram S_i is joined in $S(P)$ only to the subdiagrams S_{i+1} (or S_{i+2}, if $\mu(a_{i+1})=0$) and S_{i-1} (or S_{i-2}, if $\mu(a_{i-1})=0$). Thus the diagram $S(P)\setminus S_{20,2}$ has the form



Its subdiagram $S(P)\setminus\{S_{20,2},S_{4,6},S_{16,18}\}$ is elliptic, since $|S_{4,6}|\ge 3$, $|S_{16,18}|\ge 3$, and $S_{4,6}$ and $S_{16,18}$ are connected diagrams. On the other hand $S(P)\setminus\{S_{20,2},S_{4,6},S_{16,18}\}=S_{8,14}$, and it follows from the Gale diagram that $S_{8,14}$ is parabolic.

$\mathfrak{D}_1^{11}$**.** The diagram $S_{7,16}$ is quasi-Lannér, that is, the weights of all the vertices in the diagram are unity. The order of $S(P)$ is $22-6=16$.

$\mathfrak{D}_1^{12}$**.** The diagram $S_{5,15}$ is quasi-Lannér, that is, the weights of all vertices in the diagram are unity. The order of $S(P)$ is $24-6=18$.

$\mathfrak{D}_2^{12}$**.** The diagrams $S_{10,20}$ and $S_{12,22}$ are quasi-Lannér, that is, the weights of all vertices except a_5 are unity. By the same arguments as used for $\mathfrak{D}^{14}$ (see § 2), it is easy to deduce that the subdiagram S_i is joined in $S(P)$ only to the subdiagrams S_{i+1} (or S_{i+2}, if $\mu(a_{i+1})=0$) and S_{i-1} (or S_{i-2}, if $\mu(a_{i-1})=0$). Then the subdiagram $S_{10,20}$ is linear, which means that it is not quasi-Lannér.

$\mathfrak{D}_3^{12}$**.** The diagrams $S_{3,11}$ and $S_{11,23}$ are quasi-Lannér. If the dimension of the polytope is not less than 16, we may assume that $S_{11,23} = QL_1^{10}$. As in the preceding case, the subdiagram S_i is joined in $S(P)$ only to the subdiagrams S_{i+1} (or S_{i+2}, if $\mu(a_{i+1}) = 0$) and S_{i-1} (or S_{i-2}, if $\mu(a_{i-1}) = 0$). In our case, it follows that $S_{11,23} = QL_1^{10}$ appears as 9 subdiagrams joined in sequence; 8 of these have order 1. If we number these subdiagrams (starting with the distinguished vertex), then there should be a subdiagram of order 2 third from the end. But S_{22} is second from the end, S_{18} is fifth from the end and S_{14} eight from the end.

$\mathfrak{D}_4^{12}$**.** The discussion is exactly like that of the preceding case.

The diagrams $\mathfrak{D}_2^5$, $\mathfrak{D}_2^6$ and $\mathfrak{D}_3^{10}$ were analysed in § 2 (Lemma 6). □

4. Pyramids

It remains to consider the case where P is a pyramid.

Lemma 10. *If a hyperbolic Coxeter n-polytope of finite volume with $n + 3$ facets is a pyramid, then it is a pyramid over the product of three simplexes.*

Proof. Suppose that the polytope P is a pyramid over a polytope P'. Let A be the apex of P opposite to the facet P' . Then A lies in $n + 2$ facets, so it is an infinitely distant vertex of P. We consider a sufficiently small horosphere h with centre A. Then the combinatorial type of the intersection $h \cap P$ coincides with that of P'.

On the other hand, $h \cap P$ is a bounded Euclidean Coxeter $(n - 1)$-polytope with $n + 2$ facets. Every bounded Euclidean Coxeter polytope is a product of simplexes. The number of facets in the product of m simplexes of dimension l is $l + m$. It follows that P' is combinatorially equivalent to the product of three simplexes. □

The proof of the following lemma is like that of Lemma 4 in [19].

Lemma 11. *Let P be a hyperbolic Coxeter pyramid over the product of three simplexes, and $S(P)$ its Coxeter diagram. Then $S(P)$ satisfies the following three conditions:*

(1) *$S(P)$ is the union of three quasi-Lannér subdiagrams L_1, L_2 and L_3, intersecting in a single vertex v. The subdiagrams $L_1\backslash v$, $L_2\backslash v$ and $L_3\backslash v$ are not joined by any edge.*
(2) *The subdiagrams $L_1\backslash v$, $L_2\backslash v$ and $L_3\backslash v$ are parabolic. All the other subdiagrams of L_1, L_2 and L_3 are elliptic.*
(3) *For arbitrary $v_1 \in L_1\backslash v$, $v_2 \in L_2\backslash v$ and $v_3 \in L_3\backslash v$ the diagram $S(P)\backslash\{v_1, v_2, v_3\}$ is elliptic or parabolic.*

Every Coxeter diagram satisfying conditions (1)–(3) defines a hyperbolic Coxeter polytope combinatorially equivalent to a pyramid over the product of three simplexes.

The Coxeter diagrams satisfying conditions (1)–(3), are listed in Tables 4 and 5. The corresponding polytopes have dimensions at most 11.

All the Coxeter diagrams presented in Tables 4 and 5 are quasi-Lannér diagrams with a single parabolic subdiagram (obtained by the removal of the distinguished black dot). The pasting together of any three (not necessarily distinct) diagrams in Table 5 at the distinguished vertex gives the Coxeter diagram of a pyramid over the product of three simplexes. Each diagram in the left-hand column of Table 4 can be attached to any two diagrams in the right-hand column.

TABLE 4. Any two diagrams in the right-hand column can be attached to any diagram in the left-hand column.

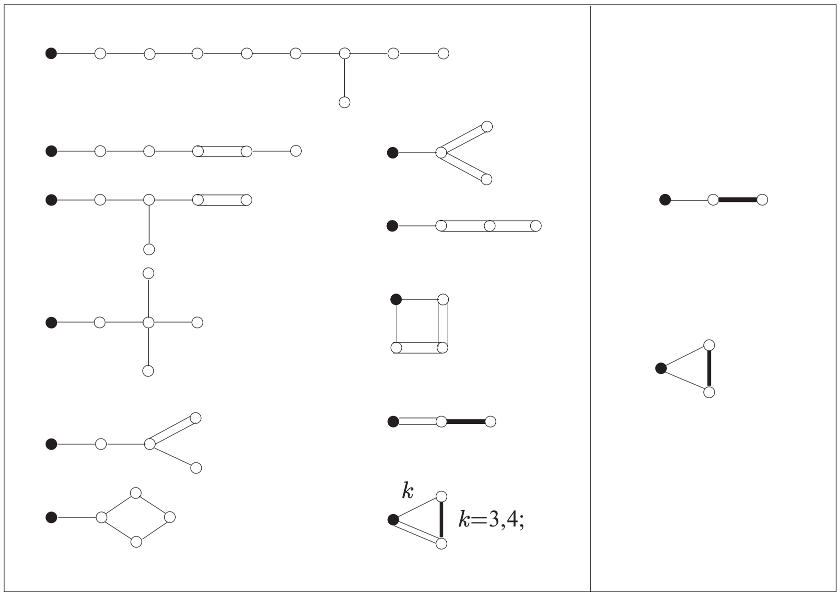
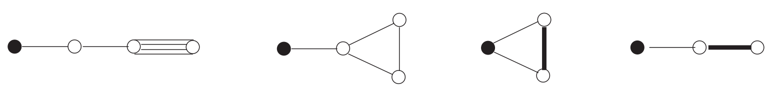


TABLE 5. The pasting together of any three (not necessarily distinct) diagrams at the distinguished vertex gives the Coxeter diagram of a pyramid over the product of three simplexes.



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