Труды Москов. Матем. Общ.
Том 68 (2007)

Trans. Moscow Math. Soc.
2007, Pages 39–66
S 0077-1554(07)00160-4
Article electronically published on October 29, 2007

CLASSIFICATION OF 2-REFLECTIVE HYPERBOLIC LATTICES OF RANK 4

E. B. VINBERG

ABSTRACT. A hyperbolic lattice is said to be 2-reflective if its automorphism group contains a subgroup of finite index generated by 2-reflections. We determine all 2-reflective hyperbolic lattices of rank 4. (For all other values of the rank, this was done by V. V. Nikulin.)

In this paper we classify 2-reflective hyperbolic lattices of rank 4 (for definitions, see below). The author obtained this classification in 1981. While it was mentioned in [7], the proofs were only published as a preprint [14] in 1998. The classification of 2-reflective hyperbolic lattices is a pivotal part of the classification of algebraic $K3$-surfaces with finite automorphism groups [5, 6].

A *quadratic lattice* is a free abelian group endowed with a non-degenerate integral symmetric bilinear form, called the *scalar product*. It is said to be *Euclidean* if the scalar product is positive definite and *hyperbolic* if it has signature $(n, 1)$.

A quadratic lattice L can be viewed as a lattice in a (pseudo-)Euclidean vector space $V = L \otimes \mathbb{R}$. The automorphism group $\mathrm{O}(L)$ of L then becomes a lattice (i.e., a discrete subgroup of finite co-volume) in the (pseudo-)orthogonal group $\mathrm{O}(V)$.

In the hyperbolic case, one of the two connected components of the hyperboloid

$$(x, x) = -1 \tag{0.1}$$

can be viewed as a model of n-dimensional Lobachevsky space L^n such that the group of motions of L^n is the subgroup $\mathrm{O}'(V)$ of index 2 in $\mathrm{O}(V)$ consisting of transformations preserving each connected component of the hyperboloid (0.1). In this model, the planes of L^n are non-empty intersections of the hyperboloid L^n with subspaces of V. The points at infinity in L^n correspond to the isotropic one-dimensional subspaces of V. The group

$$\mathrm{O}'(L) = \mathrm{O}(L) \cap \mathrm{O}'(V)$$

is a discrete group of motions of L^n with finite co-volume.

A primitive vector e of the quadratic lattice L is called a *root* or, more precisely, a *k-root* if $(e, e) = k > 0$ and

$$2(e, x) \in k\mathbb{Z} \quad \forall x \in L. \tag{0.2}$$

Any root e gives rise to an orthogonal reflection

$$R_e : x \mapsto x - \frac{2(e, x)}{(e, e)} e,$$

which preserves L, i.e., belongs to $\mathrm{O}(L)$. In the hyperbolic case, R_e gives rise to a reflection in the hyperplane

$$H_e = \{x \in L^n : (e, x) = 0\}$$

2000 *Mathematics Subject Classification.* Primary 11H06.

©2007 American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

of the space L^n. The reflection defined by a k-root is called a *k-reflection*. Clearly, a root is determined by the corresponding reflection up to sign.

We are especially interested in 2-reflections. Notice that any vector $e \in L$ with $(e,e) = 2$ automatically satisfies (0.2), i.e., it is a root.

Let $\mathrm{O}_r(L)$ (respectively, $\mathrm{O}_r^{(2)}(L)$) denote the (normal) subgroup of $\mathrm{O}(L)$ generated by all reflections (respectively, all 2-reflections) contained in $\mathrm{O}(L)$. A hyperbolic lattice L is said to be *reflective* (respectively, 2-*reflective*) if $\mathrm{O}_r(L)$ (respectively, $\mathrm{O}_r^{(2)}(L)$) is of finite index in $\mathrm{O}(L)$.

If L' is a sublattice of L, then $\mathrm{O}_r^{(2)}(L') \subset \mathrm{O}_r^{(2)}(L)$, and $\mathrm{O}(L')$ and $\mathrm{O}(L)$ are commensurable. Thus if L' is 2-reflective, then L is also 2-reflective.

A quadratic lattice L is said to be *even* if $(x,x) \in 2\mathbb{Z}$ for any $x \in L$; otherwise it is said to be *odd*. Any odd lattice L contains a unique even sublattice L^{even} of index 2, namely,

$$L^{\mathrm{even}} = \{x \in L : (x,x) \in 2\mathbb{Z}\}.$$

Since $\mathrm{O}(L) \subset \mathrm{O}(L^{\mathrm{even}})$ and any 2-root of L^{even} is a 2-root of L, we have

$$\mathrm{O}_r^{(2)}(L) = \mathrm{O}_r^{(2)}(L^{\mathrm{even}}).$$

It follows that an odd hyperbolic lattice L is 2-reflective if and only if L^{even} is 2-reflective. This allows us to consider only even lattices.

Let L be a hyperbolic lattice. Then $\mathrm{O}_r(L)$ and $\mathrm{O}_r^{(2)}(L)$ are discrete subgroups generated by reflections of L^n. Let P and $P^{(2)}$ denote their fundamental polyhedra (in general, infinite). The lattice L is reflective (respectively, 2-reflective) if and only if $\operatorname{vol} P < \infty$ (respectively, $\operatorname{vol} P^{(2)} < \infty$) or, equivalently, if P (respectively, $P^{(2)}$) is finite [10]. There is an algorithm allowing for any given lattice L to recursively find all the faces of P or $P^{(2)}$ and to determine if there are only finitely many of them [10, 12, 2].

V. V. Nikulin [5, 6] classified 2-reflective hyperbolic lattices of rank $\neq 4$. On the other hand, R. Scharlau and C. Walhorn [8] gave a list of all maximal groups of the form $\mathrm{O}_r(L)$, where L is a reflective isotropic hyperbolic lattice of rank 4.

To state our result, we need some notation.

$[C]$ – a quadratic lattice whose scalar product is given, in some basis, by a (symmetric) matrix C,

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ – the standard 2-dimensional hyperbolic lattice,

A_n – the (Euclidean) root lattice of type A_n,

$L \oplus M$ – the orthogonal direct sum of lattices L and M,

$[k]L$ – the quadratic lattice obtained from L by multiplying all scalar products by $k \in \mathbb{Z}$.

Theorem 1. *1) Any isotropic* 2*-reflective even hyperbolic lattice of rank* 4 *is isomorphic to one of the following* 12 *lattices:*

$$\begin{aligned}
&[-8] \oplus [2] \oplus [2] \oplus [2],\\
&[k]U \oplus [2] \oplus [2], \quad k = 1,2,3,4,\\
&[k]U \oplus A_2, \quad k = 1,2,3,6,\\
&\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \oplus A_2,\\
&[-4] \oplus [4] \oplus A_2,\\
&[-4] \oplus A_3.
\end{aligned}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

2) Any anisotropic 2*-reflective even hyperbolic lattice of rank* 4 *is isomorphic to one of the following two lattices:*

$$\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} -12 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

All of the above lattices are indeed 2-reflective and not isomorphic to each other. For more detail see below. We only remark here that the last two lattices are the even sublattices of index 2 in the 2-reflective odd lattices

$$[-7] \oplus [1] \oplus [1] \oplus [1], \quad [-15] \oplus [1] \oplus [1] \oplus [1],$$

respectively.

I thank A. V. Alekseevsky for writing a computer program to calculate determinants for this work.

This work was partially supported by the Alexander von Humboldt Foundation.

1. The key ideas

A vertex of a 3-dimensional convex polyhedron is said to be *simple* if it belongs to exactly three faces.

A convex polyhedron of finite volume in L^3 is just a convex hull of finitely many ordinary points and points at infinity.

Let P be a Coxeter polyhedron of finite volume in L^3. Then any ordinary vertex of P is simple whereas any vertex at infinity is either simple or belongs to four faces, of which any adjacent ones are perpendicular and any non-adjacent ones are parallel.

Let v, e and f denote the number of vertices, edges and faces of P, respectively. By the Euler equation,

$$v - e + f = 2. \tag{1.1}$$

Denote by v_3 and v_4 the numbers of simple and non-simple vertices of P and, for any face F, by $v_3(F)$ and $v_4(F)$ the numbers of simple and non-simple vertices of P belonging to F. Obviously,

$$v = v_3 + v_4, \tag{1.2}$$

$$2e = 3v_3 + 4v_4. \tag{1.3}$$

It follows from (1.1)–(1.3) that

$$f - \frac{1}{2}v_3 - v_4 = 2,$$

whence

$$f > \frac{1}{2}v_3 + v_4 = \sum_F (\frac{1}{6}v_3(F) + \frac{1}{4}v_4(F)).$$

Therefore, there is a face F such that

$$\frac{1}{6}v_3(F) + \frac{1}{4}v_4(F) < 1.$$

Clearly, F cannot have more than five sides. In fact, taking account of all the possibilities, we have

Proposition 1. *Any Coxeter polyhedron of finite volume in L^3 has either a* 3*-face, or a* 4*-face with at most three non-simple vertices, or else a* 5*-face with at most one non-simple vertex.*

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

We apply this to the fundamental polyhedron $P^{(2)}$ of $\mathrm{O}_r^{(2)}(L)$, where L is a 2-reflective hyperbolic lattice of rank 4.

Suppose a face F of $P^{(2)}$ is one of the types listed in the proposition and let $F_1, \dots, F_k$ ($k = 3, 4, 5$) be the adjacent faces. Let $u_1, \dots, u_k, u$ be the 2-roots corresponding to $F_1, \dots, F_k, F$, respectively. We shall see that there are only a few possibilities for the Gram matrix

$$G = G(u_1, \dots, u_k, u)$$

of the vectors $u_1, \dots, u_k, u$ and, therefore, for the lattice L' generated by them.

Clearly, L' is of finite index in L, so L is between L' and the dual lattice $(L')^*$. We also have

$$\big[(L')^* : L'\big] = |d(L')|, \quad \big[L : L'\big]^2 \,|\, d(L'),$$

where $d(L')$ is the discriminant of L. Thus, for a given lattice L' there are only finitely many (in fact, few) possibilities for L.

In such a way we shall obtain a finite (in fact, relatively short) list of lattices that may be 2-reflective.

2. Invariants of a quadratic lattice

In this section we recall some results related to the classification of indefinite quadratic lattices. For further details and proofs, see [3] and [4].

Let A be a principal ideal ring. A *quadratic A-module* is a free A-module of finite rank endowed with a non-degenerate symmetric bilinear A-valued form, called the *scalar product*. In particular, a quadratic $\mathbb{Z}$-module is called a quadratic lattice.

The *discriminant* $d(L)$ of a quadratic A-module L is the determinant of the Gram matrix of a basis of L. It is defined up to multiplication by an element of the group $(A^*)^2$ (where A^* stands for the group of units of A) and can be viewed as an element of the semigroup $A/(A^*)^2$. A quadratic A-module L is said to be *unimodular*, if $d(A) \in A^*$.

In the case when $2 \notin A^*$, a quadratic A-module L is said to be *even* if $(x, x) \in 2A$ for any $x \in L$, and *odd* otherwise.

We extend to quadratic modules the notation $[C]$, U, A_n, $L \oplus M$, $[k]L$ introduced above for quadratic lattices.

Since $(\mathbb{Z}^*)^2 = \{1\}$, the discriminant $d(L)$ of a quadratic lattice L is just an integer. The invariant factors of the Gram matrix of a basis of L are called the *invariant factors* of L. Their product equals $|d(L)|$. They can be viewed as the invariant factors of L as a subgroup (of finite index) in the group

$$L^* = \{x \in L \otimes \mathbb{Q} : (x, y) \in \mathbb{Z} \quad \forall y \in L\},$$

which is a free abelian group of the same rank.

Any quadratic lattice L gives rise to a quadratic real vector space $L_\infty = L \otimes \mathbb{R}$ and, for any prime p, a quadratic $\mathbb{Z}_p$-module $L_p = L \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers. The signature of L_∞ is called the *signature* of L. Clearly, if quadratic lattices L and M are isomorphic, they have the same signature and $L_p \simeq M_p$ for any prime p. The converse is also true under the following assumptions:

(I) L is indefinite;
(II) for any prime p there are two invariant factors of L divisible by the same power of p.

The structure of quadratic $\mathbb{Z}_p$-modules can be described as follows. Any such module L admits a *Jordan decomposition*

$$L = L^{(0)} \oplus [p]L^{(1)} \oplus [p^2]L^{(2)} \oplus \cdots,$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

where $L^{(0)}, L^{(1)}, L^{(2)}, \dots$ are unimodular quadratic $\mathbb{Z}_p$-modules. Those unimodular modules are determined by L uniquely up to isomorphism, unless $p \neq 2$. When $p = 2$ the rank and the parity of each such module is uniquely determined by L.

Following [4], define the *sign* $s(L)$ of a unimodular quadratic $\mathbb{Z}_p$-module L as follows. For $p \neq 2$, set $s(L) = 1$ if $d(L) \in (\mathbb{Z}_p^*)^2$, and $s(L) = -1$ otherwise. For $p = 2$, set $s(L) = 1$ if $d(L) \equiv \pm 1 \pmod 8$, and $s(L) = -1$ otherwise.

For $p \neq 2$ any unimodular quadratic $\mathbb{Z}_p$-module is isomorphic to

$$[1] \oplus \dots \oplus [1] \oplus [u] \quad \left(u \in \mathbb{Z}_p^*\right)$$

and is, therefore, uniquely determined by its rank and parity.

To any quadratic $\mathbb{Z}_p$-module L ($p \neq 2$) we assign the *symbol*, which is a formal product of the numbers p^k for which $L^{(k)} \neq 0$, with exponents $\operatorname{rk} L^{(k)}$ and with overbars in the cases when $s(L^{(k)}) = -1$. For example, the symbol of

$$L = [1] \oplus [1] \oplus [3]([1] \oplus [1] \oplus [-1]) \oplus [27]$$

is $1^2 \bar{3}^3 27^1$.

It follows from the above that for $p \neq 2$ two quadratic $\mathbb{Z}_p$-modules are isomorphic if and only if their symbols are the same.

When $p = 2$ the situation is more complicated. Following the approach in [4], one can define uniquely and consistently the *oddity* $o(L)$ of a unimodular quadratic $\mathbb{Z}_2$-module L as a residue modulo 8, subject to the following conditions:

- $o([u]) \equiv u \pmod 8$ when $u \in \mathbb{Z}_2^*$;
- $o(L) = 0$ if L is even;
- $o(L \oplus M) = o(L) + o(M)$.

Any even unimodular quadratic $\mathbb{Z}_2$-module is isomorphic to

$$U \oplus \dots \oplus U \oplus K,$$

where $K = U$ or A_2, and is, therefore, uniquely determined by its rank (which is always even) and sign.

Any odd unimodular quadratic $\mathbb{Z}_2$-module L decomposes into a direct sum of an even quadratic $\mathbb{Z}_2$-module and one or two (depending on the parity of $\operatorname{rk} L$) quadratic $\mathbb{Z}_2$-modules of rank 1. It is uniquely determined by its rank, sign and oddity. Clearly

$$o(L) \equiv \operatorname{rk} L \pmod 2.$$

To any quadratic $\mathbb{Z}_2$-module L and a Jordan decomposition

$$L = L^{(0)} \oplus [2]L^{(1)} \oplus [4]L^{(2)} \oplus \cdots \tag{2.1}$$

we assign the *pre-symbol*, which is a formal product of the numbers 2^k for which $L^{(k)} \neq 0$, with exponents $\operatorname{rk} L^{(k)}$, with overbars when $s(L^{(k)}) = -1$, and with subscripts $o(L^{(k)})$ when $L^{(k)}$ is odd. For example, the pre-symbol of

$$L = A_2 \oplus [4](U \oplus [3]) \oplus [-8]$$

is $\bar{1}^2 \bar{4}^3_3 8^1_{-1}$.

In general, the pre-symbol depends on the choice of a Jordan decomposition of L. To explain the situation, we want to use notation introduced in [4].

A *compartment* is a maximal sequence of consecutive (non-trivial) odd modules $L^{(k)}$, $L^{(k+1)}, \dots, L^{(l)}$ in the decomposition (2.1). The *oddity* of a compartment is the sum of the oddities of all of its elements. The *symbol* of L with respect to the decomposition (2.1) is obtained by putting in parentheses the factors of each compartment and replacing their subscripts by a subscript of the whole compartment, which equals its oddity. (We do not parenthesize the compartments consisting of a single element.) For instance, in the above example the symbol is $\bar{1}^2(\bar{4}^3 8^1)_2$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

A pair of consecutive factors of a symbol is said to be *flexible* if:

- they correspond to modules $L^{(k)}, L^{(k+1)}$, at least one of which is odd; or
- they correspond to odd modules $L^{(k-1)}, L^{(k+1)}$ (with $L^{(k)} = 0$).

For example, in the previous symbol $\bar{1}^2(\bar{4}^3 8^1)_2$ the second pair is flexible, whereas the first one is not.

An *elementary transformation* of a symbol is the change of signs of the factors of a flexible pair, combined with adding 4 to the oddities of all (i.e., one or two) compartments containing them. Two symbols are *equivalent* if they can be obtained from each other by applying several elementary operations. For example,

$$\bar{1}^2\big(\bar{4}^3 8^1\big)_2 \sim \bar{1}^2\big(4^3\bar{8}^1\big)_{-2} \nsim 1^2(4^3 8^1)_{-2},$$
$$\bar{1}^2_2 4^1_{-1}\bar{8}^2\overline{16}^2 \sim 1^2_{-2}\bar{4}^1_3\bar{8}^2\overline{16}^2 \sim 1^2_{-2}4^1_{-1}8^2\overline{16}^2 \nsim 1^2_{-2}4^1_{-1}\bar{8}^2 16^2.$$

Two quadratic $\mathbb{Z}_2$-modules are isomorphic if and only if their symbols are equivalent.

Now let L be a quadratic lattice and $L_p = L \otimes \mathbb{Z}_p$ for any prime p. Then $\operatorname{rk} L_p^{(k)}$ equals the number of invariant factors of L exactly divisible by p^k. Now, if $d(L) = p^m u, (p,u) = 1$, then for $p \neq 2$

$$\prod_k s(L_p^{(k)}) = \begin{cases} 1 & \text{if } u \text{ is a quadratic residue modulo } p, \\ -1 & \text{otherwise.} \end{cases}$$

Similarly,

$$\prod_k s(L_2^{(k)}) = \begin{cases} 1 & \text{if } u \equiv \pm 1 \pmod 8, \\ -1 & \text{otherwise.} \end{cases}$$

In particular, if $p \neq 2$ does not divide $d(L)$, then L_p is uniquely determined by $\operatorname{rk} L$ and $d(L)$. Similarly, if $d(L)$ is odd and L is even, then L_2 is uniquely determined by $\operatorname{rk} L$ and $d(L)$.

The symbol of L_p is called the *p-symbol* of L.

In summary, we can say that under conditions (I) and (II) two even quadratic lattices L and M of the same signature are isomorphic if and only if:

- $d(L) = d(M)$;
- for any odd prime divisor p of $d(L)$ the p-symbols of L and M are equal;
- if $d(L)$ is even, then the 2-symbols of L and M are equivalent.

In conclusion we mention the following useful isomorphism:

$$U \oplus [1] \simeq [-1] \oplus [1] \oplus [1]. \tag{2.2}$$

3. Triangular faces with three non-simple vertices

Let F be a triangular face of $P^{(2)}$, all of whose vertices are non-simple, and let u_1, u_2, u_3, u be defined as in Section 1. Then u_1, u_2, u_3 are orthogonal to u and the Gram matrix $G = G(u_1, u_2, u_3, u)$ is of the form

$$G = \begin{pmatrix} 2 & -2 & -2 & 0 \\ -2 & 2 & -2 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

After some simplifications we have:

$$L' \simeq [4]U \oplus [2] \oplus [2],$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

so that the invariant factors of L' are $2, 2, 4, 4$. Therefore, $[L : L'] \le 8$. Finding all even extensions of index 2 of L', then all their even extensions of index 2 and so on, and taking account of the isomorphism (2.2), we have the following possibilities for L:

No.	L	Invariant factors	Symbols
1	$[4]U \oplus [2] \oplus [2]$	$2, 2, 4, 4$	$2_2^2 4^2$
2	$[2]U \oplus [2] \oplus [2]$	$2, 2, 2, 2$	2_2^4
3	$U \oplus [2] \oplus [2]$	$1, 1, 2, 2$	$1^2 2_2^2$

We shall continue this list in the subsequent sections. Let us denote the lattice No. k from the list by $L(k)$.

4. Triangular faces with two non-simple vertices

Let F be a triangular face of $P^{(2)}$ with two non-simple vertices. Then u_1, u_2, u_3 are also orthogonal to u and the matrix G has one of the following forms:

$$G_1 = \begin{pmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -2 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

In the former case

$$L' \simeq [3]U \oplus [2] \oplus [2],$$

with invariant factors $1, 1, 6, 6$. The lattice L' has no even extensions of index 2, and all of its extensions of index 3 are isomorphic to $L(3)$. Thus, we get nothing new, except for L' itself.

In the latter case $L' \simeq L(2)$, so we get nothing new.

Thus, we add only one lattice to our list:

No.	L	Invariant factors	Symbols
4	$[3]U \oplus [2] \oplus [2]$	$1, 1, 6, 6$	$1^2 2_2^2, 1^2 \overline{3}^2$

5. Triangular faces with one non-simple vertex

Let F be a triangular face of $P^{(2)}$ with one non-simple vertex. Then at least two of the vectors u_1, u_2, u_3 are orthogonal to u and the matrix G has one of the following forms:

$$G_1 = \begin{pmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

In the first case, replacing u_1 by $u_1 + 2u_2 + 2u_3 + u_4$, we obtain

$$L' \simeq [-4] \oplus A_3,$$

with invariant factors $1, 1, 4, 4$.

In the second case L' has invariant factors $1, 1, 2, 8$. It also contains a primitive sublattice of type A_2. Since A_2 is unimodular over $\mathbb{Z}_2$, we have the following isomorphism over $\mathbb{Z}_2$:

$$L'_2 \simeq A_2 \oplus [2] \oplus [-24].$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Therefore, the symbol of L_2' is $\bar{1}^2 2^1_1 \bar{8}^1_{-3} \sim 1^2 2^1_1 8^1_1$ and therefore

$$L' \simeq U \oplus [2] \oplus [8].$$

In both of the preceding cases L' admits only one even extension of index 2. It is generated by $\frac{1}{2}(u_1+u_2)$ and is isomorphic to $L(3)$.

In the two remaining cases L' itself is isomorphic to $L(3)$.

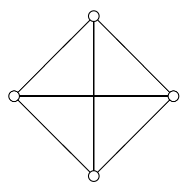
Thus, we add two lattices to the list:

No.	L	Invariant factors	Symbols
5	$[-4] \oplus A_3$	$1,1,4,4$	$\bar{1}^1\bar{4}^2_2$
6	$U \oplus [2] \oplus [8]$	$1,1,2,8$	$1^2 2^1_1 8^1_1$

6. Triangular faces without non-simple vertices

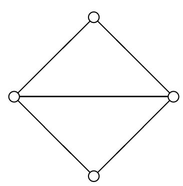
Let F be a triangular face of $P^{(2)}$ all of whose vertices are simple. Then all scalar products of vectors u_1, u_2, u_3, u are either -1 or 0. The Coxeter diagram Σ for the vectors u_1, u_2, u_3, u and their Gram matrix can only have one of the following forms:

Σ_1



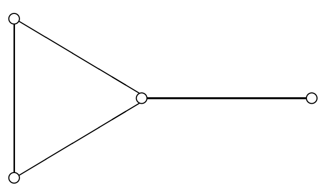
$$G_1 = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix},$$

Σ_2



$$G_2 = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix},$$

Σ_3



$$G_3 = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

In each of those cases Σ defines a Coxeter polyhedron of finite volume, which must coincide with $P^{(2)}$ [12]. Therefore, L' and all its extensions are 2-reflective.

It is easy to see that the lattices determined by the matrices G_1, G_2, G_3 are isomorphic to $[k]U \oplus A_2$, where $k = 3,2,1$ respectively. All the even extensions of the first two of them are isomorphic to the third one, whereas the last one does not admit any extensions.

Thus, we add three new lattices to our list (and we already know that they are 2-reflective):

No.	L	Invariant factors	Symbols
7	$[3]U \oplus A_2$	$1,3,3,3$	$\bar{1}^1 3^3$
8	$[2]U \oplus A_2$	$1,1,2,6$	$\bar{1}^2 2^2, 1^3 \bar{3}^1$
9	$U \oplus A_2$	$1,1,1,3$	$1^3 \bar{3}^1$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

7. Quadrilateral faces with three non-simple vertices

Let F be a 4-face of $P^{(2)}$ with three non-simple vertices and suppose u_1, u_2, u_3, u_4, u are defined as in Section 1. Then u_1, u_2, u_3, u_4 are orthogonal to u and the Gram matrix $G = G(u_1, u_2, u_3, u_4, u)$ is of the form

$$G = \begin{pmatrix} 2 & -2 & -x & -\varepsilon & 0 \\ -2 & 2 & -2 & -y & 0 \\ -x & -2 & 2 & -2 & 0 \\ -\varepsilon & -y & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

where $\varepsilon = 0$ or 1, and x and y are some integers ≥ 3.

Since $\operatorname{rk} G = 4$, we have $\det G = 2(x+2)(y+2)[(x-2)(y-2) - 4(\varepsilon+2)] = 0$, whence the following possibilities for ε, x, y (up to permutation of x and y):

ε	1	1	1	0	0
x	3	4	5	3	4
y	14	8	6	10	6

It is easy to see that in all the cases, except the last one, $L' \simeq L(3)$; in the last case $L' \simeq L(2)$. Thus, we obtain nothing new.

8. Quadrilateral faces with two opposite non-simple vertices

Let F be a 4-face of $P^{(2)}$ with two opposite non-simple vertices. Then u_1, u_2, u_3, u_4 are orthogonal to u and the matrix G is of the form

$$G = \begin{pmatrix} 2 & -2 & -x & -\varepsilon & 0 \\ -2 & 2 & -\eta & -y & 0 \\ -x & -\eta & 2 & -2 & 0 \\ -\varepsilon & -y & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

where $\varepsilon, \eta = 0$ or 1, and x and y are some integers ≥ 3.

We have

$$\det G = 2[(x+2)(y+2) - (2-\varepsilon)(2-\eta)][(x-2)(y-2) - (\varepsilon+2)(\eta+2)] = 0,$$

whence the following possibilities for ε, η, x, y (up to permutation of ε and η, and of x and y):

ε	1	1	0	0	0	0
η	1	1	1	1	0	0
x	3	5	3	4	3	4
y	11	5	8	5	6	4

In the third and fourth cases $L' \simeq L(3)$.

In all other cases $u_2 + u_4 = k(u_1 + u_3) \quad (k = 1, 2, 3)$, so that u_1, u_2, u_3, u is a basis of L'.

In the first case the invariant factors of L' are $1, 1, 2, 32$. Replacing u_1 by $u_1 + u_2$ and u_3 by $-u_3$, we have

$$L' \simeq \begin{bmatrix} 0 & 0 & 4 \\ 0 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix} \oplus [2].$$

The lattice L' has a unique extension of index 2. It is generated by $\frac{1}{2}(u_1 + u_2)$ and is isomorphic to $L(6)$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

In the second case

$$L' \simeq [-24] \oplus A_2 \oplus [2],$$

with invariant factors $1, 1, 6, 24$. The lattice L' has two extensions of index 3, each isomorphic to $L(6)$, and one extension of index 2, which is isomorphic to

$$[-6] \oplus A_2 \oplus [2] \simeq L(4).$$

In the fifth case

$$L' \simeq \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \oplus [2] \oplus [2],$$

with invariant factors $1, 1, 2, 18$. The lattice L' has no even extensions of index 2 and has only one extension of index 3, which is isomorphic to $L(3)$.

In the sixth case

$$L' \simeq [-8] \oplus [2] \oplus [2] \oplus [2].$$

The lattice L' has one even extension of index 2, which is isomorphic to $L(2)$.

Thus, we add four new lattices to our list:

No.	L	Invariant factors	Symbols
10	$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix} \oplus [2]$	$1, 1, 2, 32$	$\bar{1}^2 2_1^1 \overline{32}_{-3}^1$
11	$[-24] \oplus A_2 \oplus [2]$	$1, 1, 6, 24$	$\bar{1}^2 2_1^1 \bar{8}_{-3}^1, 1^2 \bar{3}^2$
12	$\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \oplus [2] \oplus [2]$	$1, 1, 2, 18$	$1^2 2_2^2, \bar{1}^3 9^1$
13	$[-8] \oplus [2] \oplus [2] \oplus [2]$	$2, 2, 2, 8$	$2_3^3 8_{-1}^1$

9. Quadrilateral faces with two adjacent non-simple vertices

Let F be a 4-face of $P^{(2)}$ with two adjacent non-simple vertices. Then at least three of the vectors u_1, u_2, u_3, u_4 are orthogonal to u and the matrix G is of the form

$$G = \begin{pmatrix} 2 & -2 & -x & -\varepsilon & 0 \\ -2 & 2 & -2 & -y & 0 \\ -x & -2 & 2 & -\eta & 0 \\ -\varepsilon & -y & -\eta & 2 & -\theta \\ 0 & 0 & 0 & -\theta & 2 \end{pmatrix},$$

where $\varepsilon, \eta, \theta = 0$ or 1, $x \geq 3, y \geq 2$ and if $\theta = 0$, then $y \geq 3$.

We have

$$\det G = 2(x+2)[(x-2)y^2 - (4-\theta)(x+2) - 4(\varepsilon+\eta)y - 4\varepsilon\eta] = 0,$$

whence

$$(x-2)y^2 - (4-\theta)(x+2) - 4(\varepsilon+\eta)y - 4\varepsilon\eta = 0. \tag{9.1}$$

Let $\theta = 0$. Set $(x-2)(y-2) = k$ and rewrite equation (9.1) as

$$[k - 4(\varepsilon+\eta)]y = 2[2(4+\varepsilon\eta) - k].$$

Since $y > 0$, we have that k is strictly between $4(\varepsilon+\eta)$ and $2(4+\varepsilon\eta)$. For each pair of values of ε, η only a few values of k satisfy this condition. Analyzing those possibilities, one can easily show that in this case equation (9.1) has no solutions.

Now let $\theta = 1$. Set $(x-2)(y-\sqrt{3}) = k$ and rewrite equation (9.1) as

$$[k - 4(\varepsilon+\eta)]y = 12 + 4\varepsilon\eta - k\sqrt{3}.$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

We see that k is strictly between $4(\varepsilon+\eta)$ and $\frac{12+4\varepsilon\eta}{\sqrt{3}}$. Analyzing all the possibilities one can show that in this case equation (9.1) has the following solutions:

ε	1	0	0	0	0
η	1	1	0	1	0
x	34	22	14	6	4
y	2	2	2	3	3

In the first case

$$L' \simeq U \oplus [2] \oplus [18].$$

The lattice L' has no even extensions of index 2 and has one extension of index 3, which is isomorphic to $L(3)$.

In the second case $L' \simeq L(3)$.

In the third case $L' \simeq L(6)$.

In the fourth case $L' \simeq L(3)$.

In the fifth case $L' \simeq L(4)$.

Thus we add only one new lattice to our list:

No.	L	Invariant factors	Symbols
14	$U \oplus [2] \oplus [18]$	$1, 1, 2, 18$	$1^2 2_2^2, 1^3 \bar{9}^1$

10. Quadrilateral and pentagonal faces with at most one non-simple vertex

Let F be a 4- or 5-face of $P^{(2)}$ with at most one non-simple vertex, and let F_1, F_2, F_3, F_4 be adjacent faces. (If F is a 5-face, we omit one of the adjacent faces.)

The Gram matrix $G = G(u_1, u_2, u_3, u_4, u)$ is of the form

$$G = \begin{pmatrix} 2 & -\varepsilon_{12} & -x & -z & -\varepsilon_1 \\ -\varepsilon_{12} & 2 & -\varepsilon_{23} & -y & -\varepsilon_2 \\ -x & -\varepsilon_{23} & 2 & -\varepsilon_{34} & -\varepsilon_3 \\ -z & -y & -\varepsilon_{34} & 2 & -\varepsilon_4 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & -\varepsilon_4 & 2 \end{pmatrix},$$

where

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 0 \quad \text{or} \quad 1, \tag{10.1}$$

$$\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{34} = 0, 1 \quad \text{or} \quad 2, \tag{10.2}$$

$$x, y \geq 2, \quad z \geq 0.$$

Because of our assumption about non-simple vertices,

$$\text{at most one of the } \varepsilon_{ij} \text{ equals } 2, \tag{10.3}$$

and

$$\text{if} \quad \varepsilon_{ij} = 2, \quad \text{then} \quad \varepsilon_i = \varepsilon_j = 0. \tag{10.4}$$

We may assume that x and y do not equal 2 simultaneously, since otherwise F_1, F_2, F_3, F_4 would have a common vertex at infinity and $P^{(2)}$ would have triangular faces. We may also assume that $x \geq y$.

If F is a 4-face and $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{34} = z = 0$, then again F_1, F_2, F_3, F_4 have a common vertex at infinity [1] and $P^{(2)}$ is a quadrilateral pyramid. As we do not need to consider this possibility, we may assume that $z \geq 1$. (This automatically holds if F is a 5-face.)

If F is a 5-face, then we can choose the omitted face such that z will have the smallest possible value and, in particular, $z \leq x, y$. (This automatically holds if F is a 4-face.)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

As a result, we have the following restrictions on x, y, z:

$$x \geq y \geq z, \quad x \geq 3, \quad y \geq 2, \quad z \geq 1. \tag{10.5}$$

Moreover, since F_2, F_4 and F have no common vertex at infinity,

$$y = 2 \quad \text{implies} \quad \varepsilon_2 + \varepsilon_4 > 0. \tag{10.6}$$

Furthermore, if $y = 2$ and $\varepsilon_{23} = \varepsilon_{34} = 0$, then F_2, F_4 and F_3 have a common vertex at infinity and hence F_3 is triangular. Therefore, we may assume that

$$y = 2 \quad \text{implies} \quad \varepsilon_{23} + \varepsilon_{34} > 0. \tag{10.7}$$

We now consider the condition $\det G = 0$. We have

$$\begin{aligned}
\frac{1}{2}\det G = {} & x^2y^2 - xyz(2\varepsilon_{23} + \varepsilon_2\varepsilon_3) \\
& + x^2\big(\varepsilon_2\varepsilon_4 y + \varepsilon_2^2 + \varepsilon_4^2 - 4\big) + y^2\big(\varepsilon_1\varepsilon_3 x + \varepsilon_1^2 + \varepsilon_3^2 - 4\big) + z^2\big(\varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_{23}^2 + \varepsilon_2\varepsilon_3\varepsilon_{23} - 4\big) \\
& - xy(\varepsilon_1\varepsilon_4\varepsilon_{23} + \varepsilon_1\varepsilon_2\varepsilon_{34} + \varepsilon_3\varepsilon_4\varepsilon_{12} + 2\varepsilon_{12}\varepsilon_{34}) \\
& - xz\big(\varepsilon_2\varepsilon_4\varepsilon_{23} + 2\varepsilon_3\varepsilon_4 + \big(4 - \varepsilon_2^2\big)\varepsilon_{34}\big) - yz\big(\varepsilon_1\varepsilon_3\varepsilon_{23} + 2\varepsilon_1\varepsilon_2 + \big(4 - \varepsilon_3^2\big)\varepsilon_{12}\big) \\
& - x\big(\big(4 - \varepsilon_4^2\big)\varepsilon_{12}\varepsilon_{23} + 2\varepsilon_1\varepsilon_2\varepsilon_{23} + 2\varepsilon_2\varepsilon_3\varepsilon_{12} + 2\varepsilon_1\varepsilon_4\varepsilon_{34} + \varepsilon_2\varepsilon_4\varepsilon_{12}\varepsilon_{34} + 4\varepsilon_1\varepsilon_3\big) \\
& - y((4 - \varepsilon_1^2)\varepsilon_{23}\varepsilon_{34} + 2\varepsilon_3\varepsilon_4\varepsilon_{23} + 2\varepsilon_2\varepsilon_3\varepsilon_{34} + 2\varepsilon_1\varepsilon_4\varepsilon_{12} + \varepsilon_1\varepsilon_3\varepsilon_{12}\varepsilon_{34} + 4\varepsilon_2\varepsilon_4) \\
& - z\big(\big(4 - \varepsilon_{23}^2\big)\varepsilon_1\varepsilon_4 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{34} + 2\varepsilon_1\varepsilon_3\varepsilon_{34} + 2\varepsilon_2\varepsilon_4\varepsilon_{12} \\
& \qquad + \varepsilon_3\varepsilon_4\varepsilon_{12}\varepsilon_{23} + \varepsilon_2\varepsilon_3\varepsilon_{12}\varepsilon_{34} + \varepsilon_1\varepsilon_2\varepsilon_{23}\varepsilon_{34}\big) \\
& + 16 - 4\big(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{34}^2\big) - 4(\varepsilon_1\varepsilon_2\varepsilon_{12} + \varepsilon_3\varepsilon_4\varepsilon_{34} + \varepsilon_2\varepsilon_3\varepsilon_{23}) \\
& - \varepsilon_1\varepsilon_4\varepsilon_{12}\varepsilon_{23}\varepsilon_{34} + (\varepsilon_1\varepsilon_{23} - \varepsilon_3\varepsilon_{12})^2 + (\varepsilon_2\varepsilon_{34} - \varepsilon_4\varepsilon_{23})^2 \\
& + \varepsilon_1^2\varepsilon_{34}^2 + \varepsilon_4^2\varepsilon_{12}^2 + \varepsilon_{12}^2\varepsilon_{34}^2 + \varepsilon_1\varepsilon_2\varepsilon_{12}\varepsilon_{34}^2 + \varepsilon_3\varepsilon_4\varepsilon_{34}\varepsilon_{12}^2.
\end{aligned}$$

Let $y = 2$ (and $z = 1$ or 2). Then $\varepsilon_2 + \varepsilon_4 > 0$. If $\varepsilon_2 = \varepsilon_4 = 1$, then one can show that

$$\frac{1}{2}\det G \geq 4x^2 - 45x - 101,$$

whence $x \leq 13$. If $\varepsilon_2 + \varepsilon_4 = 1$, then one can show that

$$\frac{1}{2}\det G \geq x^2 - 33x - 69,$$

whence $x \leq 34$.

Suppose now that $y \geq 3$. Then

$$\begin{aligned}
\frac{1}{2}\det G \geq {} & x^2y^2 - 4xyz - 4x^2 - 4y^2 - 4z^2 - 6xy - 8xz - 8yz \\
& + \varepsilon_2\varepsilon_4(x^2y - x\varepsilon_{12}\varepsilon_{34} - 4y - 2z\varepsilon_{12}) + \varepsilon_1\varepsilon_3(xy^2 - 4x - y\varepsilon_{12}\varepsilon_{34} - 2z\varepsilon_{34}) \\
& + \varepsilon_2\varepsilon_{34}z(x - \varepsilon_1\varepsilon_{23}) + \varepsilon_3\varepsilon_{12}z(y - \varepsilon_4\varepsilon_{23}) \\
& - x(4\varepsilon_{12}\varepsilon_{23} + 2\varepsilon_1\varepsilon_2\varepsilon_{23} + 2\varepsilon_2\varepsilon_3\varepsilon_{12} + 2\varepsilon_1\varepsilon_4\varepsilon_{34}) \\
& - y(4\varepsilon_{23}\varepsilon_{34} + 2\varepsilon_3\varepsilon_4\varepsilon_{23} + 2\varepsilon_2\varepsilon_3\varepsilon_{34} + 2\varepsilon_1\varepsilon_4\varepsilon_{12}) \\
& - z(4\varepsilon_1\varepsilon_4 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{34} + \varepsilon_2\varepsilon_3\varepsilon_{12}\varepsilon_{34}) \\
& + 16 + (y^2 - 4)\varepsilon_1 + (x^2 + z^2 - 4)\varepsilon_2 + (y^2 + z^2 - 4)\varepsilon_3 + (x^2 - 4)\varepsilon_4 \\
& - 4\varepsilon_{12}^2 - (4 - z^2)\varepsilon_{23}^2 - 4\varepsilon_{34}^2 - 4\varepsilon_1\varepsilon_2\varepsilon_{12} - 4\varepsilon_3\varepsilon_4\varepsilon_{34} - 4\varepsilon_2\varepsilon_3\varepsilon_{23} - \varepsilon_1\varepsilon_4\varepsilon_{12}\varepsilon_{23}\varepsilon_{34}.
\end{aligned}$$

All the expressions in parentheses are non-negative, so

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$\begin{aligned}\frac{1}{2}\det G \geq{}& x^2y^2 - 4xyz - 4x^2 - 4y^2 - 4z^2 - 6xy - 8xz - 8yz - 10x - 10y - 8z\\ &+ 16 + 5\varepsilon_1 + 6\varepsilon_2 + 6\varepsilon_3 + 5\varepsilon_4 - 4\varepsilon_{12}^2 - 3\varepsilon_{23}^2 - 4\varepsilon_{34}^2\\ &- 4\varepsilon_1\varepsilon_2\varepsilon_{12} - 4\varepsilon_3\varepsilon_4\varepsilon_{34} - 4\varepsilon_2\varepsilon_3\varepsilon_{23} - \varepsilon_1\varepsilon_4\varepsilon_{12}\varepsilon_{23}\varepsilon_{34}\\ \geq{}& x^2y^2 - 4xyz - 4x^2 - 4y^2 - 4z^2 - 6xy - 8xz - 8yz - 10x - 10y - 8z - 7\\ \geq{}& x^2y^2 - 4xy^2 - 4x^2 - 16y^2 - 14xy - 10x - 18y - 7.\end{aligned}$$

Therefore,

$$x^2(y^2-4) - 2x(2y^2+7y+5) - (16y^2+18y+7) \leq 0.$$

Assigning different values to y, one can get from this upper bounds for x. Taking account of the case $y = 2$ considered above, we have a table:

y	2	3	4	5	6	7	8
$x \leq$	34	19	12	10	9	9	8

There are no admissible values for x when $y \geq 9$.

This leaves less than 40,000 possibilities for G. The determinants of all those matrices were calculated using a computer program written by A. V. Alekseevsky. As a result, all matrices G satisfying conditions (10.1)–(10.7) with $\det G = 0$ were found. Some of them differ only in the numbering of the faces F_1, F_2, F_3, F_4. In view of this, there are only 75 essentially different matrices. They are listed below. In each case when the corresponding lattice L' is not isomorphic to any of the lattices already on the list, we assign a number to it and indicate an explicit form of it in a suitable basis, along with its invariant factors and symbols.

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & 0\\ 0 & 2 & -1 & -2 & -1\\ -3 & -1 & 2 & 0 & 0\\ -1 & -2 & 0 & 2 & -1\\ 0 & -1 & 0 & -1 & 2\end{pmatrix}$$

$L' = L(15) \simeq U \oplus \begin{bmatrix} 2 & 1\\ 1 & 10\end{bmatrix}$
Invariant factors: $1, 1, 1, 19$
Symbols : $1^3\overline{19}^1$

$$G = \begin{pmatrix} 2 & 0 & -5 & -1 & 0\\ 0 & 2 & 0 & -2 & -1\\ -5 & 0 & 2 & -1 & 0\\ -1 & -2 & -1 & 2 & 0\\ 0 & -1 & 0 & 0 & 2\end{pmatrix}$$

$L' = L(16) \simeq U \oplus \begin{bmatrix} 2 & 1\\ 1 & 4\end{bmatrix}$
Invariant factors: $1, 1, 1, 7$
Symbols : $\bar{1}^3 7$

$$G = \begin{pmatrix} 2 & 0 & -5 & -1 & -1\\ 0 & 2 & -1 & -2 & -1\\ -5 & -1 & 2 & 0 & -1\\ -1 & -2 & 0 & 2 & -1\\ -1 & -1 & -1 & -1 & 2\end{pmatrix}$$

$L' = L(17) \simeq U \oplus \begin{bmatrix} 2 & 1\\ 1 & 14\end{bmatrix}$
Invariant factors: $1, 1, 1, 27$
Symbols : $1^3\overline{27}^1$

$$G = \begin{pmatrix} 2 & 0 & -6 & -1 & 0\\ 0 & 2 & 0 & -2 & 0\\ -6 & 0 & 2 & -1 & 0\\ -1 & -2 & -1 & 2 & -1\\ 0 & 0 & 0 & -1 & 2\end{pmatrix}$$

$L' = L(18) \simeq U \oplus [2] \oplus [4]$
Invariant factors: $1, 1, 2, 4$
Symbols : $1^2(2^1 4^1)_2$

$$G = \begin{pmatrix} 2 & -1 & -6 & -1 & 0\\ -1 & 2 & -1 & -2 & -1\\ -6 & -1 & 2 & -1 & 0\\ -1 & -2 & -1 & 2 & -1\\ 0 & -1 & 0 & -1 & 2\end{pmatrix}$$

$L' = L(19) \simeq [-8] \oplus A_3$
Invariant factors: $1, 1, 4, 8$
Symbols : $\bar{1}^2(\bar{4}^1 8^1)_2$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$G = \begin{pmatrix} 2 & 0 & -7 & -1 & -1 \\ 0 & 2 & -1 & -2 & -1 \\ -7 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & 0 & -9 & -1 & -1 \\ 0 & 2 & 0 & -2 & -1 \\ -9 & 0 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix} \qquad L' = L(20) \simeq U \oplus \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

Invariant factors: $1, 1, 1, 11$
Symbols : $1^3\overline{11}^1$

$$G = \begin{pmatrix} 2 & 0 & -10 & -1 & -1 \\ 0 & 2 & 0 & -2 & 0 \\ -10 & 0 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' = L(21) \simeq U \oplus [2] \oplus [6]$$

Invariant factors: $1, 1, 2, 6$
Symbols : $1^2\bar{2}^2_4, 1^3\bar{3}^1$

$$G = \begin{pmatrix} 2 & -1 & -10 & -1 & -1 \\ -1 & 2 & -1 & -2 & -1 \\ -10 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' = L(22) \simeq [4]U \oplus A_2$$

Invariant factors: $1, 1, 4, 12$
Symbols : $\bar{1}^2 4^2, 1^3\bar{3}^1$

$$G = \begin{pmatrix} 2 & 0 & -16 & -1 & -1 \\ 0 & 2 & -1 & -2 & 0 \\ -16 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -17 & -1 & 0 \\ -1 & 2 & -1 & -2 & 0 \\ -17 & -1 & 2 & -1 & 0 \\ -1 & -2 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(15)$$

$$G = \begin{pmatrix} 2 & -1 & -25 & -1 & -1 \\ -1 & 2 & -1 & -2 & 0 \\ -25 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(17)$$

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & 0 \\ 0 & 2 & -1 & -3 & 0 \\ -3 & -1 & 2 & 0 & 0 \\ -1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' = L(23) \simeq U \oplus [4] \oplus [6]$$

Invariant factors: $1, 1, 2, 12$
Symbols : $1^2(\bar{2}^1 4^1)_4$

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & 0 \\ 0 & 2 & -1 & -3 & 0 \\ -3 & -1 & 2 & 0 & -1 \\ -1 & -3 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' = L(24) \simeq \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Invariant factors: $1, 1, 2, 14$
Symbols : $1^2\bar{2}^2, 1^3\bar{7}^1$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & 0 \\ 0 & 2 & -1 & -3 & -1 \\ -3 & -1 & 2 & 0 & -1 \\ -1 & -3 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}$$

$L' = L(25) \simeq \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \oplus A_2$
Invariant factors: $1, 1, 3, 9$
Symbols : $1^2\bar{3}^1 9^1$

$$G = \begin{pmatrix} 2 & 0 & -3 & -1 & -1 \\ 0 & 2 & -1 & -3 & -1 \\ -3 & -1 & 2 & 0 & -1 \\ -1 & -3 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' = L(26) \simeq [-4] \oplus [4] \oplus A_2$
Invariant factors: $1, 1, 4, 12$
Symbols : $\bar{1}^2 4_0^2, 1^3\bar{3}^1$

$$G = \begin{pmatrix} 2 & -1 & -4 & -1 & 0 \\ -1 & 2 & -1 & -3 & -1 \\ -4 & -1 & 2 & -1 & 0 \\ -1 & -3 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

$L' = L(27) \simeq \begin{bmatrix} -12 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$
Invariant factors: $1, 1, 2, 30$
Symbols : $\bar{1}^2\bar{2}^2, 1^3 3^1, 1^3\bar{5}^1$

$$G = \begin{pmatrix} 2 & 0 & -4 & -1 & 0 \\ 0 & 2 & -1 & -3 & -1 \\ -4 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & 0 & -4 & -1 & -1 \\ 0 & 2 & -1 & -3 & 0 \\ -4 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & 0 & -4 & -1 & -1 \\ 0 & 2 & -1 & -3 & -1 \\ -4 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & -1 & -5 & -1 & -1 \\ -1 & 2 & -1 & -3 & 0 \\ -5 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & 0 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$L' = L(28) \simeq U \oplus \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$
Invariant factors: $1, 1, 1, 35$
Symbols : $1^3\bar{5}^1, 1^3 7^1$

$$G = \begin{pmatrix} 2 & -1 & -6 & -1 & 0 \\ -1 & 2 & -1 & -3 & 0 \\ -6 & -1 & 2 & -1 & 0 \\ -1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$L' = L(29) \simeq U \oplus [4] \oplus [10]$
Invariant factors: $1, 1, 2, 20$
Symbols : $1^2(\bar{2}^1 4^1)_{-2}, 1^3\bar{5}^1$

$$G = \begin{pmatrix} 2 & -1 & -6 & -1 & -1 \\ -1 & 2 & -1 & -3 & 0 \\ -6 & -1 & 2 & -1 & -1 \\ -1 & -3 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(26)$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$G=\begin{pmatrix} 2 & -1 & -4 & -1 & 0\\ -1 & 2 & -1 & -4 & 0\\ -4 & -1 & 2 & -1 & 0\\ -1 & -4 & -1 & 2 & 0\\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$L'=L(30)\simeq[-12]\oplus[2]\oplus A_2$
Invariant factors: $1,1,6,12$
Symbols : $\bar{1}^2(2^1\bar{4}^1)_{-2},1^23^2$

$$G=\begin{pmatrix} 2 & -1 & -4 & -1 & 0\\ -1 & 2 & -1 & -4 & 0\\ -4 & -1 & 2 & -1 & -1\\ -1 & -4 & -1 & 2 & -1\\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L'=L(31)\simeq U\oplus\begin{bmatrix}6 & 3\\ 3 & 14\end{bmatrix}$
Invariant factors: $1,1,1,75$
Symbols : $1^3\bar{3}^1,1^3\overline{25}^1$

$$G=\begin{pmatrix} 2 & -1 & -4 & -1 & -1\\ -1 & 2 & -1 & -4 & -1\\ -4 & -1 & 2 & -1 & -1\\ -1 & -4 & -1 & 2 & -1\\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L'=L(32)\simeq[6]U\oplus A_2$
Invariant factors: $1,3,6,6$
Symbols : $\bar{1}^22^2,\bar{1}^13^3$

$$G=\begin{pmatrix} 2 & -1 & -9 & -2 & -1\\ -1 & 2 & 0 & -2 & -1\\ -9 & 0 & 2 & -1 & -1\\ -2 & -2 & -1 & 2 & -1\\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}\qquad L'\simeq L(9)$$

$$G=\begin{pmatrix} 2 & 0 & -12 & -2 & -1\\ 0 & 2 & -1 & -2 & 0\\ -12 & -1 & 2 & 0 & 0\\ -2 & -2 & 0 & 2 & -1\\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}\qquad L'\simeq L(16)$$

$$G=\begin{pmatrix} 2 & -1 & -13 & -2 & -1\\ -1 & 2 & -1 & -2 & -1\\ -13 & -1 & 2 & -1 & -1\\ -2 & -2 & -1 & 2 & -1\\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}\qquad L'\simeq L(9)$$

$$G=\begin{pmatrix} 2 & 0 & -14 & -2 & 0\\ 0 & 2 & 0 & -2 & -1\\ -14 & 0 & 2 & -2 & 0\\ -2 & -2 & -2 & 2 & 0\\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}\qquad L'\simeq L(6)$$

$$G=\begin{pmatrix} 2 & 0 & -22 & -2 & 0\\ 0 & 2 & -1 & -2 & -1\\ -22 & -1 & 2 & -2 & 0\\ -2 & -2 & -2 & 2 & 0\\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}\qquad L'\simeq L(3)$$

$$G=\begin{pmatrix} 2 & -1 & -22 & -2 & -1\\ -1 & 2 & -1 & -2 & -1\\ -22 & -1 & 2 & 0 & -1\\ -2 & -2 & 0 & 2 & 0\\ -1 & -1 & -1 & 0 & 2 \end{pmatrix}\qquad L'\simeq L(21)$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$G = \begin{pmatrix} 2 & -1 & -34 & -2 & 0 \\ -1 & 2 & -1 & -2 & -1 \\ -34 & -1 & 2 & -2 & 0 \\ -2 & -2 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(14)$$

$$G = \begin{pmatrix} 2 & -1 & -34 & -2 & -1 \\ -1 & 2 & -1 & -2 & -1 \\ -34 & -1 & 2 & -1 & -1 \\ -2 & -2 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & 0 & -4 & -2 & 0 \\ 0 & 2 & 0 & -3 & -1 \\ -4 & 0 & 2 & -2 & 0 \\ -2 & -3 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(4)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -5 & 0 & 2 & -1 & 0 \\ -2 & -3 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -5 & 0 & 2 & -1 & -1 \\ -2 & -3 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & 0 & -6 & -2 & 0 \\ 0 & 2 & -2 & -3 & 0 \\ -6 & -2 & 2 & 0 & 0 \\ -2 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(12)$$

$$G = \begin{pmatrix} 2 & 0 & -6 & -2 & 0 \\ 0 & 2 & -1 & -3 & -1 \\ -6 & -1 & 2 & -2 & 0 \\ -2 & -3 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -2 & -1 \\ -1 & 2 & -1 & -3 & -1 \\ -7 & -1 & 2 & -1 & -1 \\ -2 & -3 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & 0 & -8 & -2 & 0 \\ 0 & 2 & -2 & -3 & 0 \\ -8 & -2 & 2 & -1 & 0 \\ -2 & -3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$G = \begin{pmatrix} 2 & -1 & -8 & -2 & -1 \\ -1 & 2 & -1 & -3 & -1 \\ -8 & -1 & 2 & -1 & -1 \\ -2 & -3 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -11 & -2 & 0 \\ -1 & 2 & -2 & -3 & 0 \\ -11 & -2 & 2 & -1 & 0 \\ -2 & -3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(10)$$

$$G = \begin{pmatrix} 2 & 0 & -4 & -2 & 0 \\ 0 & 2 & -2 & -4 & 0 \\ -4 & -2 & 2 & 0 & 0 \\ -2 & -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(13)$$

$$G = \begin{pmatrix} 2 & 0 & -4 & -2 & -1 \\ 0 & 2 & -2 & -4 & 0 \\ -4 & -2 & 2 & 0 & 0 \\ -2 & -4 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(10)$$

$$G = \begin{pmatrix} 2 & 0 & -5 & -2 & 0 \\ 0 & 2 & -2 & -4 & 0 \\ -5 & -2 & 2 & -1 & 0 \\ -2 & -4 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & 0 & -5 & -2 & -1 \\ 0 & 2 & -2 & -4 & 0 \\ -5 & -2 & 2 & -1 & 0 \\ -2 & -4 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -5 & -1 & 2 & -1 & 0 \\ -2 & -4 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -5 & -1 & 2 & -1 & -1 \\ -2 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & 0 \\ -1 & 2 & -2 & -5 & 0 \\ -5 & -2 & 2 & -1 & 0 \\ -2 & -5 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(11)$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$G = \begin{pmatrix} 2 & -1 & -5 & -2 & -1 \\ -1 & 2 & -2 & -5 & 0 \\ -5 & -2 & 2 & -1 & 0 \\ -2 & -5 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$L' = L(33) \simeq [-36] \oplus A_3$
Invariant factors: $1, 1, 4, 36$
Symbols : $\bar{1}^2\bar{4}^2_2, 1^3\bar{9}^1$

$$G = \begin{pmatrix} 2 & 0 & -6 & -3 & -1 \\ 0 & 2 & 0 & -3 & -1 \\ -6 & 0 & 2 & -2 & 0 \\ -3 & -3 & -2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$L' \simeq L(3)$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -6 & 0 & 2 & -1 & 0 \\ -3 & -3 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$L' \simeq L(25)$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -6 & 0 & 2 & -1 & -1 \\ -3 & -3 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(9)$

$$G = \begin{pmatrix} 2 & -1 & -7 & -3 & 0 \\ -1 & 2 & -1 & -3 & 0 \\ -7 & -1 & 2 & 0 & 0 \\ -3 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$L' \simeq L(23)$

$$G = \begin{pmatrix} 2 & -1 & -7 & -3 & -1 \\ -1 & 2 & -1 & -3 & 0 \\ -7 & -1 & 2 & 0 & -1 \\ -3 & -3 & 0 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L' \simeq L(24)$

$$G = \begin{pmatrix} 2 & -1 & -7 & -3 & -1 \\ -1 & 2 & -1 & -3 & -1 \\ -7 & -1 & 2 & 0 & -1 \\ -3 & -3 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix}$$

$L' \simeq L(25)$

$$G = \begin{pmatrix} 2 & 0 & -9 & -3 & -1 \\ 0 & 2 & -1 & -3 & -1 \\ -9 & -1 & 2 & -2 & 0 \\ -3 & -3 & -2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$L' = L(34) \simeq [-16] \oplus A_3$
Invariant factors: $1, 1, 4, 16$
Symbols : $\bar{1}^2\bar{4}^1_3 16^1_{-1}$

$$G = \begin{pmatrix} 2 & -1 & -10 & -3 & -1 \\ -1 & 2 & -1 & -3 & -1 \\ -10 & -1 & 2 & -1 & -1 \\ -3 & -3 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 2 \end{pmatrix}$$

$L' \simeq L(8)$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$G = \begin{pmatrix} 2 & -1 & -4 & -3 & -1 \\ -1 & 2 & 0 & -4 & 0 \\ -4 & 0 & 2 & -1 & -1 \\ -3 & -4 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & -1 & -4 & 0 \\ -6 & -1 & 2 & -1 & -1 \\ -3 & -4 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -6 & -1 & 2 & -1 & 0 \\ -3 & -4 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -3 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -6 & -1 & 2 & -1 & -1 \\ -3 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & -2 & -4 & -4 & 0 \\ -2 & 2 & 0 & -4 & 0 \\ -4 & 0 & 2 & 0 & -1 \\ -4 & -4 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -4 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -5 & -1 & 2 & 0 & 0 \\ -4 & -4 & 0 & 2 & -1 \\ -1 & -1 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -4 & -1 \\ -1 & 2 & 0 & -4 & -1 \\ -5 & 0 & 2 & -1 & -1 \\ -4 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & 0 & -6 & -4 & 0 \\ 0 & 2 & -1 & -4 & -1 \\ -6 & -1 & 2 & -2 & 0 \\ -4 & -4 & -2 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$L' = L(35) \simeq U \oplus [2] \oplus [50]$
Invariant factors: $1, 1, 2, 50$
Symbols : $1^2 2_2^2, \bar{1}^3 \overline{25}^1$

$$G = \begin{pmatrix} 2 & 0 & -6 & -4 & -1 \\ 0 & 2 & -2 & -4 & 0 \\ -6 & -2 & 2 & 0 & 0 \\ -4 & -4 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(3)$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$G = \begin{pmatrix} 2 & -1 & -7 & -4 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -7 & -1 & 2 & -1 & 0 \\ -4 & -4 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -4 & -1 \\ -1 & 2 & -1 & -4 & -1 \\ -7 & -1 & 2 & -1 & -1 \\ -4 & -4 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(7)$$

$$G = \begin{pmatrix} 2 & -1 & -5 & -5 & 0 \\ -1 & 2 & -1 & -5 & -1 \\ -5 & -1 & 2 & 0 & -1 \\ -5 & -5 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -5 & 0 \\ -1 & 2 & -1 & -5 & -1 \\ -6 & -1 & 2 & -1 & -1 \\ -5 & -5 & -1 & 2 & -1 \\ 0 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(9)$$

$$G = \begin{pmatrix} 2 & -1 & -6 & -5 & -1 \\ -1 & 2 & -1 & -5 & 0 \\ -6 & -1 & 2 & -1 & -1 \\ -5 & -5 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(8)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -7 & 0 \\ -1 & 2 & -2 & -7 & 0 \\ -7 & -2 & 2 & -1 & 0 \\ -7 & -7 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \qquad L' \simeq L(10)$$

$$G = \begin{pmatrix} 2 & -1 & -7 & -7 & -1 \\ -1 & 2 & -2 & -7 & 0 \\ -7 & -2 & 2 & -1 & 0 \\ -7 & -7 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad L' \simeq L(34)$$

As the result, we have 21 new lattices $L(15)$–$L(35)$. Below we describe all their even extensions of prime indices.

Each of the lattices $L(19), L(26), L(33), L(34)$ has one even extension of index 2, which is isomorphic to $L(18), L(8), L(14), L(5)$, respectively. The lattice $L(22)$ has three even extensions of index 2, two of which are isomorphic to $L(8)$ and one to $L(21)$. The lattice $L(32)$ has two even extensions of index 2, which are isomorphic to $L(7)$.

Each of the lattices $L(17), L(25), L(33)$ has one extension of index 3, which is isomorphic to $L(9), L(9), L(5)$, respectively. The lattice $L(32)$ has four extensions of index 3, which are isomorphic to $L(8)$.

Each of the lattices $L(31), L(35)$ has one extension of index 5, which is isomorphic to $L(9), L(3)$, respectively.

Thus, there are no new lattices among these extensions, so all 2-reflective lattices of signature $(3, 1)$ are among the lattices $L(1)$–$L(35)$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

11. Non-2-reflectivity of some isotropic lattices

To show that some of the lattices $L(1)$–$L(35)$ are not 2-reflective, we apply a method going back to [11]. Notice that this method only applies to isotropic lattices.

A Euclidean lattice M is said to be 2-*reflective* if its 2-roots span the space $M \otimes \mathbb{R}$.

Let L be a hyperbolic lattice of rank $n+1$ and $v \in L$ a primitive isotropic vector. Then

$$L_v = (\mathbb{Z}v)^\perp / \mathbb{Z}v.$$

Lemma 1. *If L is 2-reflective, then L_v is also 2-reflective.*

Proof. Suppose L is 2-reflective and let q be the point at infinity in the Lobachevsky space L^n determined by v. Since $v \in L$, we have that q is a vertex of a fundamental polyhedron of $P^{(2)}$ of the group $\mathrm{O}_r^{(2)}(L)$ (see, for example, [11]). The 2-roots of L corresponding to the walls of $P^{(2)}$ passing through q are orthogonal to v. The images of these roots in the Euclidean space $(\mathbb{R}v)^\perp/\mathbb{R}v$ are 2-roots of L_v. Since the intersection of $P^{(2)}$ with the horosphere centered at q is compact, these roots of L_v span the space

$$(\mathbb{R}v)^\perp/\mathbb{R}v = L_v \otimes \mathbb{R}.$$

□

Notice that there are only two 2-reflective Euclidean lattices of rank 2: $[2] \oplus [2]$ and A_2.

This immediately implies that the lattices $L(6)$, $L(14)$, $L(15)$, $L(16)$, $L(17)$, $L(18)$, $L(20)$, $L(21)$, $L(23)$, $L(28)$, $L(29)$, $L(31)$, $L(35)$ are not 2-reflective. Therefore their sublattices $L(10)$, $L(11)$, $L(19)$, $L(22)$, $L(33)$ are not 2-reflective either.

Comparing the symbols one has

$$L(12) \simeq U \oplus \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}.$$

This shows that $L(12)$ is not 2-reflective.

Finally, the lattice $L(34)$ has the Gram matrix

$$\begin{pmatrix} -16 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

in some basis $\{v_0, v_1, v_2, v_3\}$. The vector

$$v = v_0 + 2(v_1 - v_3)$$

is isotropic and

$$(\mathbb{Z}v)^\perp = \langle v, v_2, v_1 + v_3 \rangle.$$

Therefore,

$$(\mathbb{Z}v)^\perp/\mathbb{Z}v \simeq \langle v_2, v_1 + v_3 \rangle = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}.$$

This shows that $L(34)$ is not 2-reflective.

12. The remaining lattices

In order to determine if a hyperbolic lattice L is 2-reflective, it is useful to consider all reflections in $\mathrm{O}(L)$. In the notation of the introduction we have the following.

Lemma 2. $\mathrm{O}_r(L) = \mathrm{O}_r^{(2)}(L) \rtimes \Delta$, *where Δ is the subgroup of $\mathrm{O}_r(L)$ generated by the reflections in the walls of P which are not 2-reflections.*

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Proof. Decompose the set S of all reflections in the walls of P into two parts: the set S_2 of all 2-reflections and the set S' of all other reflections in S. Since any reflection in $\mathrm{O}_r(L)$ is conjugate to some reflection in S, the group $\mathrm{O}_r^{(2)}(L)$ is the smallest normal subgroup of $\mathrm{O}_r(L)$ containing S_2.

Notice that the exponent of the Coxeter relation between any $s \in S_2$ and $s' \in S'$ is even, since otherwise s and s' would be conjugate, which is impossible. Therefore, the defining relations for the factor group $\mathrm{O}_r(L)/\mathrm{O}_r^{(2)}(L)$ are exactly the same as those for Δ. This means that Δ is isomorphic to $\mathrm{O}_r(L)/\mathrm{O}_r^{(2)}(L)$. The assertion of the lemma now follows. (Cf. [9, Lemma 1].) $\square$

Thus, L is 2-reflective if and only if it is reflective and the group Δ is finite.

Usually it is easier to find P than $P^{(2)}$. To determine if Δ is finite is not difficult since the classification of finite Coxeter groups is well known.

Using this method we now investigate the remaining lattices $L(1)$–$L(5)$, $L(13)$, $L(24)$–$L(27)$, $L(30)$, $L(32)$. It turns out that in all the cases L is reflective. To determine P, we shall use the algorithm described in [10, 12].

The following remark is useful for our calculations: L can contain k-roots only if k divides the doubled last invariant factor ([13, Prop. 24]).

Consider the lattice $L(1)$. In some coordinates u, v, x_1, x_2 the scalar square in it is given by the quadratic form

$$2\bigl(-4uv + x_1^2 + x_2^2\bigr).$$

By the above remark, our lattice can contain k-roots only for $k \in \{2, 4, 8\}$. Their coordinates are determined by the conditions

$$x_1^2 + x_2^2 = 4uv + \frac{k}{2}, \quad 2x_1 \equiv 2x_2 \equiv 0 \pmod{\frac{k}{2}}.$$

Taking $(1, 0, 0, 0)$ as the base point of the algorithm, we find the first four simple roots:

No.	u	v	x_1	x_2	k
1	0	0	-1	1	4
2	0	0	0	-1	2
3	1	0	2	0	8
4	-1	1	0	0	8

Their Coxeter diagram

defines a tetrahedron with one vertex at infinity. Therefore, the lattice is reflective and P coincides with this tetrahedron.

The group Δ generated by reflections in the walls of P which are not 2-reflections (the corresponding vertices of the Coxeter diagram are represented by black circles) is a finite Coxeter group of type B_3. Therefore, $L(1)$ is 2-reflective.

Since the lattices $L(2)$ and $L(3)$ contain $L(1)$, they are also 2-reflective.

Similarly, the scalar square in $L(4)$ is given by the quadratic form

$$2\bigl(-3uv + x_1^2 + x_2^2\bigr).$$

The coordinates of the roots are determined by the conditions

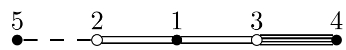
$$x_1^2 + x_2^2 = 3uv + \frac{k}{2}, \quad 3u \equiv 3v \equiv 2x_1 \equiv 2x_2 \equiv 0 \pmod{\frac{k}{2}},$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

where $k \in \{2, 4, 6, 12\}$. The first five simple roots are as follows:

No.	u	v	x_1	x_2	k
1	0	0	−1	1	4
2	0	0	0	−1	2
3	1	0	1	0	2
4	−1	1	0	0	6
5	2	2	3	3	12

Their Coxeter diagram



defines a triangular prism with one vertex at infinity, so the lattice is reflective and P coincides with this prism. The group Δ is a finite Coxeter group of type $3A_1$. Therefore, the lattice $L(4)$ is 2-reflective.

Consider the lattice $L(32)$. It is convenient to realize it as a sublattice defined by the equation

$$x_1 + x_2 + x_3 = 0 \tag{12.1}$$

in the lattice of rank 5 with scalar square

$$-12uv + x_1^2 + x_2^2 + x_3^2.$$

The coordinates of the roots are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 = 12uv + k, \quad x_1 \equiv x_2 \equiv x_3 (\mathrm{mod}\, \frac{k}{2}),$$

where $k \in \{2, 4, 6, 12\}$. The first four simple roots are as follows:

No.	u	v	x_1	x_2	x_3	k
1	0	0	−1	2	−1	6
2	0	0	0	−1	1	2
3	1	0	2	−1	−1	6
4	−1	1	0	0	0	12

Their Coxeter diagram

defines a tetrahedron with one vertex at infinity, so the lattice is reflective, and P coincides with this tetrahedron. (In particular, we see that there are no 4-roots.) The group Δ is a finite Coxeter group of type B_3. Therefore, the lattice $L(32)$ is 2-reflective.

Similarly, the lattice $L(25)$ can be realized as the sublattice defined by equation (12.1) in the lattice of rank 5 with scalar square

$$-6x_0u + 2x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

The coordinates of the roots are defined by the conditions

$$2x_0^2 + x_1^2 + x_2^2 + x_3^2 = 6x_0u + k,$$

$$x_0 \equiv 0 \quad (\mathrm{mod}\, \frac{k}{2}), \quad x_1 \equiv x_2 \equiv x_3 \quad (\mathrm{mod}\, \frac{k}{2}),$$

where $k \in \{2, 6\}$. The first four simple roots are as follows:

No.	u	x_0	x_1	x_2	x_3	k
1	0	0	−1	2	−1	6
2	0	0	0	−1	1	2
3	1	0	2	−1	−1	6
4	0	1	0	0	0	2

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Their Coxeter diagram

defines a tetrahedron with two vertices at infinity, so the lattice is reflective and P coincides with this tetrahedron. The group Δ is a finite Coxeter group of type A_2. Therefore, the lattice $L(25)$ is 2-reflective.

The lattice $L(26)$ is realized as the sublattice defined by equation (12.1) in the lattice of rank 5 with scalar square

$$-4x_0^2 + 4y^2 + x_1^2 + x_2^2 + x_3^2.$$

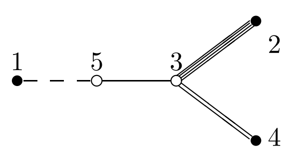
The coordinates of the roots are defined by the conditions

$$\begin{gathered} 4y^2 + x_1^2 + x_2^2 + x_3^2 = 4x_0^2 + k, \\ 4x_0 \equiv 4y \equiv 0 \pmod{\frac{k}{2}}, \quad x_1 \equiv x_2 \equiv x_3 \pmod{\frac{k}{2}}, \end{gathered}$$

where $k \in \{2, 4, 6\}$. Taking $(1, 0, 0, 0, 0)$ as the base point of the algorithm, we find the first five simple roots:

No.	x_0	y	x_1	x_2	x_3	k
1	0	−1	0	0	0	4
2	0	0	−1	2	−1	6
3	0	0	0	−1	1	2
4	1	0	2	0	−2	4
5	1	1	1	0	−1	2

Their Coxeter diagram



defines a triangular prism with one vertex at infinity, so the lattice is reflective and P coincides with this prism. The group Δ is a finite Coxeter group of type $3A_1$. Therefore, the lattice $L(26)$ is 2-reflective.

Similarly, the lattice $L(30)$ is realized as the sublattice defined by equation (12.1) in the lattice of rank 5 with scalar square

$$-12x_0^2 + 2y^2 + x_1^2 + x_2^2 + x_3^2.$$

The coordinates of the roots are defined by the conditions

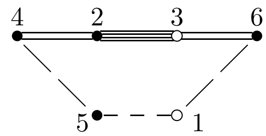
$$\begin{gathered} 2y^2 + x_1^2 + x_2^2 + x_3^2 = 12x_0^2 + k, \\ 2y \equiv 0 \pmod{\frac{k}{2}}, \quad x_1 \equiv x_2 \equiv x_3 \pmod{\frac{k}{2}}, \end{gathered}$$

where $k \in \{2, 4, 6, 12\}$. The first six simple roots are as follows:

No.	x_0	y	x_1	x_2	x_3	k
1	0	−1	0	0	0	2
2	0	0	−1	2	−1	6
3	0	0	0	−1	1	2
4	1	0	4	−2	−2	12
5	1	3	0	0	0	6
6	1	2	2	0	−2	4

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Their Coxeter diagram



defines a bounded polyhedron (a tetrahedron with two truncated vertices), so the lattice is reflective. However, the group Δ is infinite. Therefore, the lattice $L(30)$ is not 2-reflective.

The lattice $L(5)$ is realized as the sublattice defined by the equation

$$x_1 + x_2 + x_3 + x_4 = 0$$

in the lattice of rank 5 with scalar square

$$-4x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

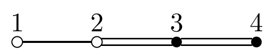
The coordinates of the roots are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4x_0^2 + k, \quad x_1 \equiv x_2 \equiv x_3 \equiv x_4 \quad (\mathrm{mod}\ \frac{)}{k}2,$$

where $k \in \{2, 4, 8\}$. The first four simple roots are as follows:

No.	x_0	x_1	x_2	x_3	x_4	k
1	0	0	-1	1	0	2
2	0	0	0	-1	1	2
3	0	-1	1	1	-1	4
4	1	3	-1	-1	-1	8

Their Coxeter diagram



defines a tetrahedron with one vertex at infinity. The group Δ is a finite Coxeter group of type B_2. Therefore, the lattice $L(5)$ is 2-reflective.

The scalar square in $L(13)$ is of the form

$$2\left(-4x_0^2 + x_1^2 + x_2^2 + x_3^2\right).$$

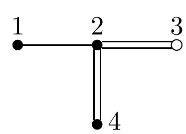
The coordinates of the roots are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 = 4x_0^2 + \frac{k}{2}, \quad 2x_1 \equiv 2x_2 \equiv 2x_3 \equiv 0 \pmod{\frac{k}{2}},$$

where $k \in \{2, 4, 8, 16\}$. The first four simple roots are as follows:

No.	x_0	x_1	x_2	x_3	k
1	0	-1	1	0	4
2	0	0	-1	1	4
3	0	0	0	-1	2
4	1	2	2	0	8

Their Coxeter diagram



defines a tetrahedron with one vertex at infinity. The group Δ is a finite Coxeter group of type B_3. Therefore, the lattice $L(13)$ is 2-reflective.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

To investigate the lattice $L(24)$, we remark that it has an odd extension $\tilde{L}(24)$ of index 2 (with discriminant -7), so

$$L(24) = \tilde{L}(24)^{\text{even}}.$$

Since the 7-symbols of the lattices $\tilde{L}(24)$ and $L(24)$ must be the same, we have

$$\tilde{L}(24) \simeq [-7] \oplus [1] \oplus [1] \oplus [1].$$

The scalar square in $\tilde{L}(24)$ is given by the quadratic form

$$-7x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

and the sublattice $L(24)$ is defined by the congruence

$$x_0 + x_1 + x_2 + x_3 \equiv 0 \pmod 2. \tag{12.2}$$

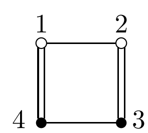
The coordinates of the roots of $L(24)$ are defined by the conditions

$$x_1^2 + x_2^2 + x_3^2 = 7x_0^2 + k, \quad x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod{\frac{k}{2}},$$

where $k \in \{2, 4\}$. The first four simple roots are as follows:

No.	x_0	x_1	x_2	x_3	k
1	0	-1	1	0	2
2	0	0	-1	1	2
3	0	0	0	-2	4
4	1	3	1	1	4

Their Coxeter diagram



defines a tetrahedron, so the lattice is reflective and P coincides with this tetrahedron. The group Δ is a finite Coxeter group of type A_2. Therefore, the lattice $L(24)$ is 2-reflective.

Similarly,

$$L(27) = \tilde{L}(27)^{\text{even}},$$

where

$$\tilde{L}(27) \simeq [-15] \oplus [1] \oplus [1] \oplus [1].$$

The scalar square in $\tilde{L}(27)$ is given by the quadratic form

$$-15x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

and the sublattice $L(27)$ is defined by the congruence (12.2).

The coordinates of the roots of $L(27)$ are defined by the conditions

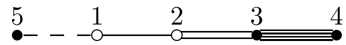
$$x_1^2 + x_2^2 + x_3^2 = 15x_0^2 + k, \quad 15x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod{\frac{k}{2}},$$

where $k \in \{2, 4, 10, 12\}$. The first five simple roots are as follows:

No.	x_0	x_1	x_2	x_3	k
1	0	-1	1	0	2
2	0	0	-1	1	2
3	0	0	0	-2	4
4	1	3	3	3	12
5	1	5	0	0	10

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Their Coxeter diagram



defines a triangular prism, so the lattice is reflective and P coincides with this prism. The group Δ is a finite Coxeter group of type $G_2 + A_1$. Therefore, the lattice $L(27)$ is 2-reflective.

References

[1] Andreev, E. M., *Convex polyhedra of finite volume in Lobachevsky space.* (Russian), Mat. Sb. (N.S.) 83(125) (1970), 256–260. MR0273510 (42:8388)

[2] Bugaenko, V. O., *Arithmetic crystallographic groups generated by reflections and reflective hyperbolic lattices.* Adv. Sov. Math. 1992. Vol. 8, E. B. Vinberg, ed., pp. 33–55. MR1155663 (93g:20094)

[3] Cassels, J.W.S., *Rational quadratic forms.* London Math. Soc. Monographs, 13. Academic Press, London, New York, 1978. MR522835 (80m:10019)

[4] Conway, J. H., Sloane, N. J. A., *Sphere packings, lattices and groups.* Third edition. Springer-Verlag, New York, 1999. MR1662447 (2000b:11077)

[5] Nikulin, V, V., *Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by* 2*-reflections. Algebro-geometric applications.* (Russian) Current problems in mathematics, Vol. 18, pp. 3–114, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981 MR633160 (83c:10030)

[6] ———, *K*3 *surfaces with a finite group of automorphisms and a Picard group of rank three.* (Russian) Algebraic geometry and its applications. Trudy Mat. Inst. Steklov. 165 (1984), 119–142. MR0752938 (86e:14018)

[7] ———, *Discrete reflection groups in Lobachevsky spaces and algebraic surfaces.* Proc. Int. Congr. Math. Berkeley. 1986. Vol. 1., pp. 654–671. MR0934268 (89d:11032)

[8] Scharlau R., Walhorn C., *Integral lattices and hyperbolic reflection groups.* Astérisque. 1992. Vol. 209, pp. 279–291. MR1211022 (94j:11057)

[9] Vinberg, E. B., *Some examples of crystallographic groups in Lobachevsky spaces.* (Russian) Mat. Sb. (N.S.) 78 (120) (1969), 633–639. MR0246193 (39:7497)

[10] ———, *The groups of units of certain quadratic forms.* (Russian) Mat. Sb. (N.S.) 87(129) (1972), 18–36. MR0295193 (45:4261)

[11] ———, *The unimodular integral quadratic forms.* (Russian) Funkcional. Anal. i Priložen. 6 (1972), no. 2, 24–31. MR0299557 (45:8605)

[12] ———, *Some arithmetical discrete groups in Lobachevsky spaces.* Proc. Int. Coll. on Discrete Subgroups of Lie Groups and Appl. to Moduli. Pap. Bombay Coll. 1975. pp. 323–348. MR0422505 (54:10492)

[13] ———, *Absence of crystallographic groups of reflections in Lobachevsky spaces of large dimension.* (Russian) Trudy Moskov. Mat. Obshch. 47 (1984), 68–102, 246. MR774946 (86i:22020)

[14] ———, *Classification of* 2*-reflective hyperbolic lattices of rank* 4. Preprint 98–113. Universität Bielefeld. 1998.

Department of Mechanics and Mathematics, Moscow State University, Moscow 119992, GSP-2, Russia

E-mail address: `vinberg@ebv.pvt.msu.su`

Translated by ALEX MARTSINKOVSKY

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use