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ON d-DIMENSIONAL COMPACT HYPERBOLIC COXETER POLYTOPES WITH $d+4$ FACETS

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Abstract. We prove that there are no compact Coxeter polytopes with $d+4$ facets in a hyperbolic space of dimension $d > 7$. This estimate is sharp: examples of such polytopes in dimensions $d \leq 7$ were found by V. O. Bugaenko in 1984. We also show that in dimension 7 there is a unique polytope with 11 facets.

Introduction

A polytope in a hyperbolic space is called a Coxeter polytope if all its dihedral angles are of the form $\frac{\pi}{k_{ij}}$ for some integers $k_{ij} \geq 2$. Any Coxeter polytope is a fundamental domain of the discrete group generated by the reflections in its facets. A complete classifications of compact hyperbolic Coxeter polytopes is not yet known. Vinberg [18] showed that there are no such polytopes in dimensions $d \geq 30$. The known examples are only in dimensions $d \leq 8$ (see [5] and [6]). In dimensions 2 and 3, compact hyperbolic Coxeter polytopes were completely classified by H. Poincaré [15] and Andreev [3]. Compact hyperbolic Coxeter polytopes of the simplest combinatorial type, i.e., simplices were classified by Lannér [14]. Kaplinskaya [13] (see also [19]) classified all simplicial prisms, Esselmann [9] classified the remaining d-dimensional compact hyperbolic Coxeter polytopes with $d+2$ facets. In [8], Esselmann proved that d-dimensional compact Coxeter polytopes with $d+3$ facets exist only in dimensions 8 and lower, and also showed that in dimension 8 there is only one such polytope. Compact d-dimensional Coxeter polytopes with $d+3$ facets in dimensions 4 through 7 were classified by P. Tumarkin [17].

In this paper we investigate the next-in-complexity class of polytopes, namely, d-dimensional compact hyperbolic Coxeter polytopes with $d+4$ facets. We prove that there are no such polytopes in a hyperbolic space of dimension $d \geq 8$. In dimensions $2 \leq d \leq 7$ such polytopes do exist [5]. We also prove that in dimension $d = 7$ there is only one such polytope.

The paper is organized as follows. Section 1 is of auxiliary nature: we recall basic results about Coxeter diagrams and the combinatorics of simple polytopes. We also mention some facts connecting combinatorial (metric) properties of the faces of a polytope and combinatorial (metric) properties of the polytope itself. Section 2 is devoted to Coxeter diagrams that do not contain Lannér subdiagrams of order less than 5. In particular, we show that the Coxeter diagram of any compact hyperbolic Coxeter polytope contains a Lannér subdiagram of order less than 5. In Section 3 we develop a theory of liftings, which connects the combinatorics of a face of a Coxeter polytope and a certain

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subdiagram of the Coxeter diagram of this polytope. In Sections 4 and 5 we apply the obtained results to prove the non-existence of the above polytopes in dimensions $d \geq 8$. Finally, in Section 6 we prove that in a hyperbolic space of dimension 7 there is only one compact Coxeter polytope with 11 facets.

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1. Preliminaries

In this section we collect basic facts about Coxeter diagrams, Gale diagrams, and diagrams of missing faces. In our discussion of Coxeter diagrams we mainly follow [19] and [20]. For details about Gale diagrams, see [12] and [7] (see also [8], where connections between Coxeter polytopes, Gale diagrams, and diagrams of missing faces are explained). At the end of the section, we mention a recent result of D. Allcock [1] showing that most Coxeter polytopes have a Coxeter face, and explaining how to construct the Coxeter diagram of a Coxeter face from the Coxeter diagram of the polytope.

1.1. **Coxeter diagrams. 1.** An *abstract Coxeter diagram* Σ is a finite one-dimensional simplicial complex, whose edges are assigned positive weights w_{ij}, where $w_{ij} = \cos\frac{\pi}{m_{ij}}$ for some integer $m_{ij} \geq 3$ whenever $w_{ij} < 1$. A *subdiagram* of Σ is a subcomplex whose edges are labeled by the same weights as in Σ. The *order* $|\Sigma|$ of the diagram Σ is the number of its nodes.

The union of subdiagrams Σ_1 and Σ_2 of Σ is the subdiagram $\langle \Sigma_1, \Sigma_2 \rangle$ spanned by the nodes of Σ_1 and Σ_2. By definition, the subdiagrams $\Sigma_1 \setminus v$ and $\Sigma_1 \setminus \Sigma_2$ of Σ_1 are spanned by the nodes of Σ_1 without v, respectively, without the nodes of Σ_2.

Given an abstract Coxeter diagram Σ with nodes $v_1, \dots, v_n$ and weights w_{ij} we construct a symmetric $(n \times n)$-matrix $G(\Sigma) = (g_{ij})$ such that $g_{ii} = 1$, and for $i \neq j$, $g_{ij} = -w_{ij}$ if v_i and v_j are joined, and $g_{ij} = 0$ otherwise. We define the determinant $\det(\Sigma)$ and the signature of Σ as the determinant and, respectively, the signature of $G(\Sigma)$.

We draw the edges of a Coxeter diagram in the following way: if the weight equals $\cos\frac{\pi}{m_{ij}}$, then the corresponding nodes are joined by an $(m_{ij}-2)$-fold edge or a simple edge labeled by m_{ij}; if the weight equals 1, the nodes are joined by a bold edge; if the weight is greater than 1, then the nodes are joined by a dashed edge labeled by the weight (or without a label).

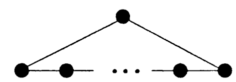
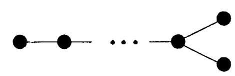
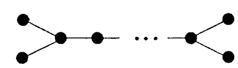
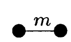
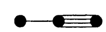
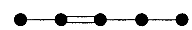
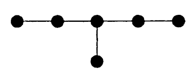
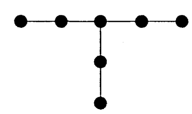
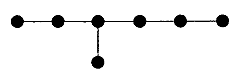
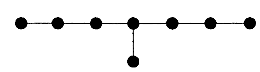
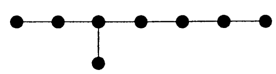
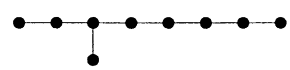
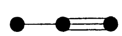
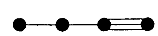
We write $[v_i, v_j] = m_{ij}$ if $w_{ij} = \cos\frac{\pi}{m_{ij}}$, and $[v_i, v_j] = \infty$ if $v_i v_j$ is a dashed edge. The notation $[v_i, v_j] = 2$ indicates that v_i and v_j are not joined.

An abstract Coxeter diagram Σ is said to be *elliptic* if $G(\Sigma)$ is positive definite; Σ is *parabolic* if each indecomposable component of $G(\Sigma)$ is degenerate and positive semi-definite; a connected diagram Σ is said to be *Lannér* if Σ is neither elliptic nor parabolic but any proper subdiagram of Σ is elliptic; Σ is said to be *hyperbolic* if $G(\Sigma)$ is indefinite with negative inertia index equal to 1; Σ is said to be *superhyperbolic* if its negative inertia index is greater than 1; Σ is said to be *admissible* if Σ contains no parabolic subdiagrams and is not superhyperbolic.

Table 1 contains the list of elliptic and connected parabolic Coxeter diagrams in their standard notation. In [20, Table 3] one finds a list of Lannér diagrams. Notice that the order of a Lannér diagram cannot be greater than 5; moreover, there are only finitely many Lannér diagrams of order greater than 3. In Table 2 we reproduce a list of Lannér diagrams of orders 4 and 5 and introduce notation for them.

2. It is convenient to describe Coxeter polytopes via their Coxeter diagrams. Let P be a Coxeter polytope with facets $f_1, \dots, f_r$. The Coxeter diagram $\Sigma(P)$ of P is a diagram

TABLE 1. Coxeter diagrams. The left column lists connected elliptic Coxeter diagrams. The right column lists connected parabolic Coxeter diagrams

A_n $(n \geq 1)$		$\widetilde{A}_1$	
		$\widetilde{A}_n$ $(n \geq 2)$	
$B_n = C_n$ $(n \geq 2)$		$\widetilde{B}_n$ $(n \geq 3)$	
		$\widetilde{C}_n$ $(n \geq 2)$	
D_n $(n \geq 4)$		$\widetilde{D}_n$ $(n \geq 4)$	
$G_2^{(m)}$		$\widetilde{G}_2$	
F_4		$\widetilde{F}_4$	
E_6		$\widetilde{E}_6$	
E_7		$\widetilde{E}_7$	
E_8		$\widetilde{E}_8$	
H_3			
H_4			

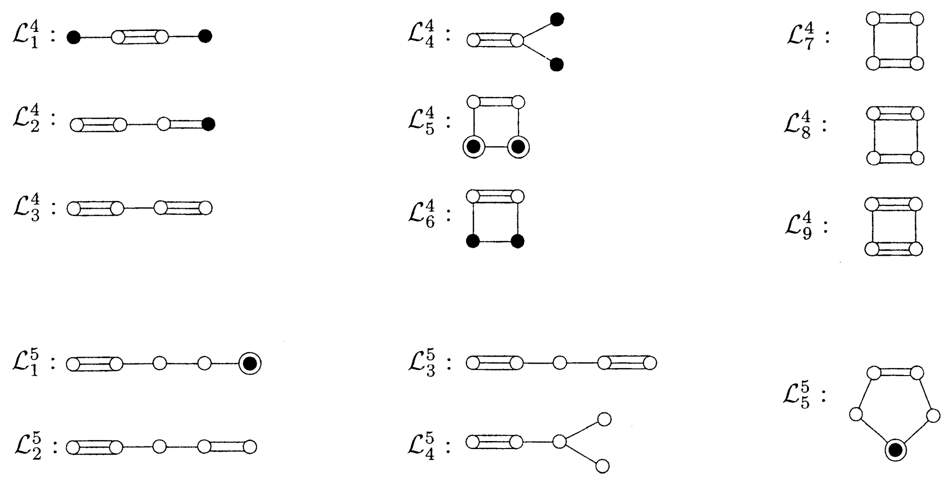
with nodes $v_1, \dots, v_r$; nodes v_i and v_j are not joined if f_i is orthogonal to f_j; v_i and v_j are joined by an edge with weight

$$w_{ij} = \begin{cases} \cos\dfrac{\pi}{k} & \text{if } f_i \text{ and } f_j \text{ form a dihedral angle } \dfrac{\pi}{k}, \\ 1 & \text{if } f_i \text{ is parallel to } f_j, \\ \cosh\rho & \text{if } f_i \text{ and } f_j \text{ diverge and } \rho \text{ is the distance between } f_i \text{ and } f_j. \end{cases}$$

If $\Sigma = \Sigma(P)$, then $G(\Sigma)$ coincides with the Gram matrix of the outward unit normals to the facets of P.

It was shown in [19] that if $\Sigma = \Sigma(P)$ is the Coxeter diagram of a d-dimensional compact hyperbolic polytope P, then Σ is an admissible connected hyperbolic diagram with positive inertia index equal to d. In particular, Σ has no bold edges and parabolic subdiagrams. The elliptic subdiagrams of Σ are in one-to-one correspondence with the

TABLE 2. Lannér diagrams of orders 4 and 5. Open nodes are black, doubly open nodes are encircled. The superscript equals the order of the diagram



faces of P: a k-face F corresponds to the elliptic subdiagram Σ_F of order $d-k$ whose nodes correspond to the facets containing F.

3. Given a Coxeter diagram Σ it is easy to check whether or not it is superhyperbolic. However, when the order of the diagram is big, the computation of the signature could be difficult. When the diagram is the union of two subdiagrams either joined by a single edge or having only one node in common, there is a more effective way to determine if such a diagram is superhyperbolic [18].

Suppose T is a subdiagram of Σ such that $\det(\Sigma\setminus T)\neq 0$. A *local determinant* of Σ on T is $\det(\Sigma,T)=\frac{\det(\Sigma)}{\det(\Sigma\setminus T)}$.

Proposition 1.1 ([18, Prop. 12]). *If a Coxeter diagram Σ consists of subdiagrams Σ_1 and Σ_2 that intersect in a single node v, then*

$$\det(\Sigma,v)=\det(\Sigma_1,v)+\det(\Sigma_2,v)-1.$$

Proposition 1.2 ([18, Prop. 13]). *If a Coxeter diagram Σ consists of non-intersecting subdiagrams Σ_1 and Σ_2 joined by a single edge v_1v_2, then*

$$\det(\Sigma,\langle v_1,v_2\rangle)=\det(\Sigma_1,v_1)\det(\Sigma_2,v_2)-w_{12}^2,$$

where w_{12} is the weight of the edge v_1v_2.

Proposition 1.3 ([18, Prop. 15]). *Suppose that a Coxeter diagram Σ consists of two non-intersecting hyperbolic subdiagrams Σ_1 and Σ_2 joined by a single edge v_1v_2 such that $\Sigma_1\setminus v_1$ and $\Sigma_2\setminus v_2$ are elliptic. Assume also that one of the following conditions holds:*

1) *v_1v_2 is a simple edge and* $\det(\Sigma_1,v_1)\det(\Sigma_2,v_2)>\frac{1}{4}$;
2) *v_1v_2 is a double edge, and* $\det(\Sigma_1,v_1)\det(\Sigma_2,v_2)>\frac{1}{2}$.

Then Σ is superhyperbolic.

In [18, Table 2] one can find some useful determinants. When we need to check if a certain diagram is superhyperbolic, we use Propositions 1.1–1.3 and Table 2 of [18] without further stipulation. We also use the fact that local determinants of Lannér diagrams of orders 4 and 5 on their open nodes (see Table 2 and the definition of an open node below) do not exceed 0.95 — this can be checked directly. In particular, we use the following consequence of Proposition 1.2:

Proposition 1.4. *Suppose Σ consists of two disjoint Lannér diagrams L_1 and L_2, each of order 5, joined by a single non-dashed edge v_1v_2 such that $L_i \setminus v_i$ (where $v_i \in L_i$, $i = 1, 2$) are of type H_4 or F_4. Then Σ is superhyperbolic.*

4. Suppose Σ is an abstract Coxeter diagram and v is a node of Σ. Suppose we can add a node x to Σ so that $x \notin \Sigma$, x is joined with v in $\langle \Sigma, x\rangle$, and x belongs to neither a Lannér nor a parabolic subdiagram of $\langle \Sigma, x\rangle$. Then v is called an *open* node of Σ.

A node v is said to be *doubly open* if a node x can be added to Σ so that x be joined with v in $\langle \Sigma, x\rangle$ and x be open in $\langle \Sigma, x\rangle$.

Notice that for any non-open node v of a Lannér diagram Σ and any node $x \notin \Sigma$ joined with v there exists a non-elliptic subdiagram $M \subset \langle \Sigma, x\rangle$ containing both x and v. Indeed, if x is not joined with $\Sigma \setminus v$, the assertion is obvious. When we add edges joining x with $\Sigma \setminus v$, non-elliptic subdiagrams remain non-elliptic.

In [8, Table 2] Esselmann listed all Lannér diagrams of orders 4 and 5 containing open nodes. In Table 2, the open nodes are black, and the doubly open nodes are encircled.

As a direct consequence of the classification of Lannér diagrams, we have

Proposition 1.5. *Any Lannér diagram of order 4 contains at least two subdiagrams of type H_3 or B_3.*

Proposition 1.6 ([18, Prop. 2])**.** *Any two Lannér subdiagrams of the diagram Σ of a hyperbolic Coxeter polytope are joined by at least one edge.*

This is obvious: if it is not true, then Σ is a superhyperbolic diagram.

1.2. **Gale diagrams and diagrams of missing faces. 1.** As was shown in [19, Corollary to Th. 3.1], a compact hyperbolic Coxeter polytope is simple (i.e., a k-face of a d-dimensional polytope belongs to exactly $d - k$ facets). Now we want to list some combinatorial properties of simple polytopes. For brevity, a d-dimensional polytope will be called a "d-polytope".

Each combinatorial type of a simple d-polytope can be represented by its *Gale diagram.* It consists of $d + k$ points $a_1, \dots, a_{d+k}$ on the $(k-2)$-dimensional sphere $\mathbb{S}^{k-2} \subset \mathbb{R}^{k-1}$ centered at the origin. Each a_i corresponds to a facet f_i of P. The combinatorial type of a simple convex polytope can be read off its Gale diagram as follows: for any subset $J \subset \{1, \dots, d+k\}$ the intersection of the faces $\{f_j \mid j \in J\}$ is a face of P if and only if the origin is an interior point of $\operatorname{conv}\{a_j \mid j \notin J\}$ (where $\operatorname{conv} X$ is the convex hull of the set X).

A set of points $a_1, \dots, a_{d+k} \in \mathbb{S}^{k-2}$ is the Gale diagram of a d-polytope P with $d+k$ facets if and only if each open half-space H^+ of $\mathbb{R}^{k-1}$ bounded by a hyperplane passing through the origin contains at least two of the points $a_1, \dots, a_{d+k}$.

Two Gale diagrams are said to be *isomorphic* if the corresponding polytopes are combinatorially equivalent.

Let P be a simple polytope. The facets $f_1, \dots, f_m$ of P form a *missing face* if $\bigcap_{i=1}^m f_i = \emptyset$, but any proper subset of the facets $\{f_1, \dots, f_m\}$ has a non-empty intersection. Clearly, any set of facets with an empty intersection contains at least one missing face. Whence:

Lemma 1.1. *Let P be a simple d-polytope with $d + k$ facets, let $a_1, \dots, a_{d+k} \in \mathbb{S}^{k-2}$ be the Gale diagram of P, and let H^+ be an open half-space bounded by a hyperplane passing through the origin. Then H^+ contains a set $I \subset \{a_1, \dots, a_{d+k}\}$ corresponding to a missing face of P.*

2. When $k = 2$, the Gale diagram of P is 1-dimensional, i.e., the points a_i lie on a zero-dimensional unit sphere. In other words, each point a_i lies on the number line and coincides with either -1 or 1. Whence:

Proposition 1.7 ([12, 7]). *A simple d-polytope with $d+2$ facets is the direct product of two simplices $\Delta^{n-k} \times \Delta^k$ (where $0 \le k \le [n/2]$ and Δ^m stands for an m-simplex).*

As was shown in [9], a compact Coxeter d-polytope with $d+2$ facets is either a simplicial prism or the product of two triangles. Coxeter prisms are listed in [13], the remaining Coxeter polytopes of this type (there are seven of them) can be found in [9]. We call those seven polytopes the *Esselmann polytopes*.

When $k = 3$ the Gale diagram of P is two-dimensional, i.e., the points a_i lie on a unit circle.

A *standard Gale diagram* of a simple d-polytope with $d+3$ facets consists of labeled vertices $v_1, \dots, v_m$ of a regular m-gon (m is odd) in $\mathbb{R}^2$ centered at the origin such that:

1) each label is a natural number and the sum of the labels equals $d+3$;
2) the sum of the labels of the vertices lying in a half-plane bounded by a line passing through the origin is at least 2.

It is not difficult to check that each two-dimensional Gale diagram is isomorphic to some standard diagram (see, for example, [12]). Two d-polytopes with $d+3$ facets are combinatorially equivalent if and only if their standard Gale diagrams are congruent (i.e., coincide up to a motion of the plane).

When $k > 3$, there is no definition of a standard Gale diagram. To describe the combinatorics of simple polytopes we shall use another kind of diagrams.

A *diagram of missing faces* is a finite set D with a specified collection $\mathcal{M}_D$ of subsets of D such that $M' \not\subset M$ for any $M, M' \in \mathcal{M}_D$. The order $|M|$ of M is defined as the cardinality of M. The elements of $\mathcal{M}_D$ are called the *missing faces* of D.

A diagram $D_1 \subset D$ of missing faces is a *subdiagram* of D if for any $M \subset D_1$ we have that $M \in \mathcal{M}_{D_1}$ if and only if $M \in \mathcal{M}_D$.

It is convenient to visualize diagrams of missing faces as follows: for each element of D mark a point (vertex) on the plane, and then encircle the set of points corresponding to a subset M (i.e., draw a closed curve about the points) if and only if $M \in \mathcal{M}_D$.

3. With a simple polytope P we associate a diagram of missing faces $D(P)$ as follows: the elements of $D(P)$ correspond to the facets of P; a set of elements is a missing face of $D(P)$ if and only if the corresponding facets form a missing face of P.

The combinatorial structure of P can be recovered from $D(P)$: a collection of facets has a non-empty intersection if and only if the corresponding subset of $D(P)$ contains no missing faces.

Lemma 1.2. *Let P be a simple polytope and $D(P)$ its diagram of missing faces. For any missing face $M \in \mathcal{M}_{D(P)}$ there is a missing face $M' \in \mathcal{M}_{D(P)}$ such that $M \cap M' = \emptyset$.*

Proof. The Gale diagram $G(P)$ of P consists of several points on the d-sphere $\mathbb{S}^d$. Let $\overline{M}$ be the points of $G(P)$ corresponding to the elements of M. Since M is a missing face, $\mathrm{conv}(G(P) \setminus \overline{M})$ does not contain the origin. In other words, there is a hyperplane H passing through the origin with a half-space H^+ containing $G(P) \setminus \overline{M}$. Let H^- be the other half-space relative to H. By Lemma 1.1, H^- contains the points corresponding to some missing face. Since those points belong to $\overline{M}$ and a missing face cannot contain another missing face, we have that H separates the points of $\overline{M}$ from the other points of $G(P)$, i.e., $H^- \cap G(P) = \overline{M}$. By Lemma 1.1, H^+ contains a subset of points corresponding to a missing face of M'. Clearly, M does not intersect M', which proves the lemma. ☐

We also need the following two results.

Proposition 1.8 ([8, Lemma 1.6])**.** *Let P be a simple polytope and f a facet of P. Let $\{f_1, \dots, f_k\}$ be the set of the facets of P different from f and such that $f_i \cap f \neq 0$. Set $f_i' = f_i \cap f$ for each $i = 1, \dots, k$, and for any subset $G \subseteq \{f_1, \dots, f_k\}$, set $G' = \{f_i' \mid f_i \in G\}$.*

Then G' is a missing face of the facet f if and only if

1) *either $\{f\} \cup G$ is a missing face of P, or*
2) *G is a missing face of P and G contains no proper subsets G_0 such that $\{f\} \cup G_0$ is a missing face of P.*

Proposition 1.9 ([8, Lemma 1.9])**.** *For any facet f of a simple polytope P there is a missing face of P containing f.*

1.3. **Lannér diagrams and missing faces.** Let P be a compact Coxeter polytope in $\mathbb{H}^n$, $\Sigma(P)$ its Coxeter diagram, and L a Lannér subdiagram of $\Sigma(P)$. By the definition of Lannér subdiagrams, the facets corresponding to L form a missing face of P (and any missing face of P corresponds to some Lannér subdiagram of $\Sigma(P)$). Thus, the diagram of missing faces $D(P)$ can easily be recovered from $\Sigma(P)$: in $\Sigma(P)$, encircle all Lannér subdiagrams and remove all edges.

In the same way we construct a *diagram of missing faces* $D(\Sigma)$ for any admissible Coxeter diagram Σ.

The correspondence "Lannér diagram $\longleftrightarrow$ missing face" shows in particular that a compact hyperbolic Coxeter polytope has no missing faces of order greater than 5.

1.4. **Faces of Coxeter polytopes.** Let P be a compact hyperbolic Coxeter d-polytope, Σ its Coxeter diagram, and S_0 an elliptic subdiagram of Σ. As was shown in [19, Th. 3.1], S_0 corresponds to a face of P of dimension $d - |S_0|$. Denote that face $P(S_0)$. It is an acute-angled polytope [2], but it need not be a Coxeter polytope. R. Borcherds obtained the following sufficient condition for $P(S_0)$ to be a Coxeter polytope.

Proposition 1.10 ([4, Example 5.6])**.** *Let P be a compact hyperbolic Coxeter polytope with Coxeter diagram Σ, and S_0 an elliptic subdiagram of Σ having no connected components of types A_n and D_5. Then $P(S_0)$ is a Coxeter polytope.*

The facets of $P(S_0)$ correspond to those nodes of Σ that form elliptic subdiagrams with S_0. The dihedral angles of $P(S_0)$ can be determined using the following result of D. Allcock.

Let a and b be the facets of $P(S_0)$ determined by facets A and B of P, i.e., $a = A \cap P(S_0)$ and $b = B \cap P(S_0)$. Let v_A and v_B be the nodes of Σ corresponding to A and B. The angles of $P(S_0)$ can now be found as follows.

Proposition 1.11 ([1, Th. 2.2])**.** *Under the assumptions of Proposition* 1.10*,*

(1) *If neither v_A nor v_B is joined with S_0, then $\angle ab = \angle AB$.*
(2) *If exactly one of the nodes v_A and v_B is joined with S_0, say, via the connected component S_0^i, then*
    (a) *if $A \perp B$, then $a \perp b$;*
    (b) *if v_A and v_B are joined by a simple edge and adjoining v_A and v_B to S_0^i yields a diagram of type B_k (resp., D_k, E_8, or H_4), then $\angle ab = \pi/4$ (resp., $\pi/4$, $\pi/6$ or $\pi/10$);*
    (c) *otherwise, a and b do not intersect.*
(3) *If v_A and v_B are joined with different components of S_0, then*
    (a) *if $A \perp B$, then $a \perp b$;*
    (b) *otherwise, a and b do not intersect.*

(4) *If v_A and v_B are joined with the same connected component of S_0, say S_0^i, then*
   (a) *if $A \perp B$ and $S_0^i \cup \{A, B\}$ is of type E_6 (resp., E_8 or F_4), then $\angle ab = \pi/3$ (resp., $\pi/4$ or $\pi/4$);*
   (b) *otherwise, a and b do not intersect.*

We shall say that $w \in \Sigma$ is a *neighbor* of S_0 if w is joined with S_0 by an edge. A neighbor w is said to be *good* if $\langle S_0, w\rangle$ is an elliptic diagram, and *bad* otherwise. Let $\bar{S}_0$ be the subdiagram of Σ consisting of the nodes corresponding to the facets of $P(S_0)$. Then $\bar{S}_0$ is spanned by the good neighbors of S_0 and by the nodes which are not neighbors of S_0. If $P(S_0)$ is a Coxeter polytope, denote its diagram Σ_{S_0}. By a simple edge we understand a 1-fold edge. By an empty edge we understand two nodes which are not joined. By an ordinary edge we understand non-dashed edges, including empty ones.

For each node $v \in \Sigma$ which is not a bad neighbor of S_0 (i.e., it belongs to $\bar{S}_0$) the corresponding node of Σ_{S_0} will be denoted $\widetilde{v}$.

In the case when Σ_{S_0} does not differ from $\bar{S}_0$ we shall view Σ_{S_0} as a subdiagram of Σ, and, accordingly, identify v and $\widetilde{v}$.

Corollary 1.1. *Under the assumptions of Proposition* 1.10*,*

(a) *If S_0 is of type H_4, F_4, $G_2^{(m)}$ for $m \geq 6$, or any other diagram with no good neighbors, then $\bar{S}_0 = \Sigma_{S_0}$.*
(b) *If S_0 is of type H_3, then $\bar{S}_0$ can be obtained from Σ_{S_0} by replacing some dashed edges by ordinary ones.*
(c) *If S_0 is of type $G_2^{(5)}$, then $\bar{S}_0$ can be obtained from Σ_{S_0} by replacing some edges labeled by* 10 *with simple edges, and some dashed edges with ordinary ones.*
(d) *If S_0 is of type B_n, $n \geq 3$, then $\bar{S}_0$ can be obtained from Σ_{S_0} by replacing some double edges by simple ones, and some dashed edges by ordinary ones.*
(e) *If S_0 is of type $B_2 = G_2^{(4)}$, then $\bar{S}_0$ can be obtained from Σ_{S_0} by replacing some double edges by simple ones, and some dashed edges by ordinary or empty edges.*

The corollary follows directly from Proposition 1.11. We just remark that all neighbors of the diagrams mentioned in (a) are bad.

Here is another direct consequence of Proposition 1.11.

Corollary 1.2. *Under the assumptions of Proposition* 1.10*, let $S_1 \subset \Sigma_{S_0}$ be a subdiagram of type $G_2^{(m)}$, where $m \neq 4, 10$, and let S_1' be the corresponding subdiagram $\bar{S}_0$. Then no node of $S_1' \subset \Sigma$ is a good neighbor of S_0.*

In particular, any subdiagram $S_2 \subset \Sigma_{S_0}$ of type F_4, H_4, H_3 or $G_2^{(m)}$, where $m \neq 4, 10$, corresponds to a subdiagram of $\bar{S}_0$ of the same type.

Any Lannér subdiagram $L \subset \Sigma_{S_0}$ of order 5 *corresponds to a Lannér subdiagram of $\bar{S}_0$.*

Lemma 1.3. *Suppose that S_0 is an elliptic subdiagram and $|S_0| < d$. Then S_0 has at most $|\Sigma| - d - 1$ bad neighbors. In particular, if P has $d + 4$ facets, then any elliptic subdiagram of Σ of order less than d has at most three bad neighbors.*

Proof. The lemma follows from the fact that a k-polytope has at least $k + 1$ facets. □

Lemma 1.4. *Let $S \subset \Sigma$ be an elliptic subdiagram containing no components of types A_n and D_5, and let a be a bad neighbor of S. Then a is joined with each Lannér subdiagram of $\bar{S}$.*

This is a direct consequence of Proposition 1.6.

2. Admissible Coxeter diagrams without small Lannér subdiagrams

A Lannér diagram (or a missing face) L is said to be *small* if $|L| < 5$. A missing face M is said to be *large* if $|M| > 5$.

Lemma 2.1. *Let Σ be a connected admissible Coxeter diagram without small Lannér subdiagrams. Suppose that each node of Σ belongs to some Lannér subdiagram of Σ. Then $|\Sigma| \leq 10$. If $|\Sigma| = 10$, then Σ is one of the three diagrams shown in Figure 1.*

If, in addition, $\det(\Sigma) = 0$, *then $\Sigma = \Theta_1$ (see Figure 1).*

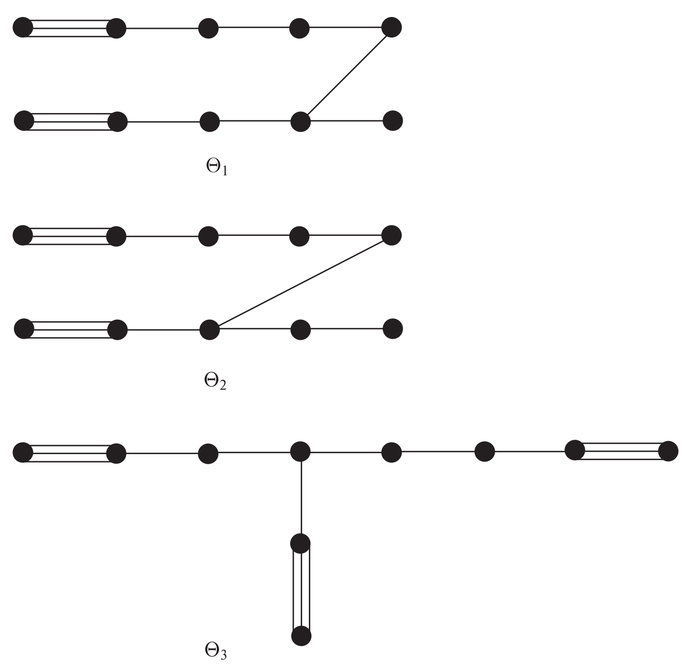


Figure 1. Diagrams without small Lannér subdiagrams

Proof. Since Σ is connected and does not contain small Lannér subdiagrams, any edge of Σ is either simple, or double, or triple. Consider two cases: either Σ contains $\mathcal{L}_5^5$ or it does not (for notation, see Table 2).

Case 1. Suppose that Σ contains $\mathcal{L}_5^5$. It is clear that only the open node of $\mathcal{L}_5^5$ can be joined with some other nodes of Σ; otherwise Σ would contain either a parabolic subdiagram of type $\widetilde{B}_4$ or $\widetilde{B}_5$, or a small Lannér subdiagram. For the same reason, $\Sigma \setminus \mathcal{L}_5^5$ is linear and contains no double edges. Since each node of Σ belongs to some Lannér subdiagram, $\Sigma \setminus \mathcal{L}_5^5$ contains a triple edge, and for $|\Sigma| > 9$ such a diagram either contains a small Lannér subdiagram or is superhyperbolic.

Case 2. Suppose that Σ does not contain $\mathcal{L}_5^5$.

- *For any triple edge of Σ, one of its nodes is a leaf of Σ and the valency of the other node is two.*

Proof. This follows immediately from the absence of small Lannér subdiagrams. □

- *Σ has no double edges.*

Proof. Let v_1v_2 be a double edge of Σ. Suppose that v_1 is a leaf of Σ. By assumption, v_1 belongs to some Lannér diagram L. It is clear that L contains v_2 and is a linear Lannér diagram. Let v be a node of $\Sigma \setminus L$ joined with L. Then $\langle L, v\rangle$ contains either a parabolic or a small Lannér subdiagram.

Thus, neither v_1 nor v_2 is a leaf of Σ. Hence Σ contains edges of the form v_0v_1 and v_2v_3 (where $v_0 \neq v_3$, since otherwise $\langle v_0, v_1, v_2\rangle$ would contain a Lannér or a parabolic diagram). These edges are simple; otherwise Σ would contain a parabolic or a small Lannér subdiagram. Therefore, $\langle v_0, v_1, v_2, v_3\rangle$ is of type F_4. If v_4 is an arbitrary node of Σ joined with $\langle v_0, v_1, v_2, v_3\rangle$, then either $\langle v_0, v_1, v_2, v_3, v_4\rangle$ is of type $\mathcal{L}_5^5$ or it contains a parabolic or a small Lannér subdiagram, contrary to the assumption. □

- Σ *is a tree.*

Proof. The two preceding results imply that any minimal cycle of Σ is a parabolic subdiagram of type $\widetilde{A}_n$. □

Now we remove from Σ all leaves that belong to triple edges and denote the obtained diagram Σ'. As was shown above, Σ' is a tree with simple edges. All such trees without parabolic subdiagrams are elliptic diagrams of type A_n, D_n, E_6, E_7, or E_8.

Next we attach triple edges to some leaves of Σ'. If $\Sigma' = A_n$ or D_{n+1} for $n \geq 9$, then Σ has a node that does not belong to any Lannér subdiagram. In the remaining cases either $|\Sigma| < 10$, or Σ is superhyperbolic, or Σ is one of the diagrams in Figure 1.

A direct calculation shows that $\det \Sigma = 0$ only when $\Sigma = \Theta_1$. □

Corollary 2.1. *The Coxeter diagram $\Sigma(P)$ of any compact Coxeter polytope P contains a small Lannér subdiagram.*

Proof. Suppose that $\Sigma(P)$ does not contain small Lannér diagrams. By Lemma 1.2, $\Sigma(P)$ contains two disjoint Lannér subdiagrams L_1 and L_2 of order 5, and therefore, $|\Sigma(P)| \geq 10$. The subdiagram $\langle L_1, L_2\rangle$ is connected because otherwise it would be superhyperbolic. It follows from Lemma 2.1 that $\Sigma(P)$ is one of the diagrams from Figure 1. None of those diagrams is the Coxeter diagram of a compact polytope: in each of the three cases, $\Sigma(P)$ contains a Lannér subdiagram that intersects any other Lannér subdiagram of $\Sigma(P)$, which is impossible by Lemma 1.2. □

3. Liftings

Let D be a diagram of missing faces, and Σ an admissible Coxeter diagram. The diagram Σ is called a 0-*lifting* of D if there is a bijection $\phi : D \to \Sigma$ such that $M \in \mathcal{M}_D$ if and only if $\phi(M)$ is a Lannér subdiagram; ϕ is called a *lifting bijection*.

Σ is called a k-*lifting* of D ($k \in \mathbb{N}$) if Σ contains a subset A of "additional nodes" such that:

1) $|A| = k$;
2) there is an injection $\phi : D \to \Sigma$ sending D bijectively to $\Sigma \setminus A$; ϕ is called a *lifting injection*;
3) for any Lannér diagram $L \subset \Sigma$ the set $\phi^{-1}(L \setminus A)$ contains a missing face of D;
4) for any missing face $M \in \mathcal{M}_D$ there is a Lannér subdiagram $L \subset \langle \phi(M), A\rangle$ containing $\phi(M)$;
5) for any set $\{a_1, \dots, a_r\} \subset A$ the subdiagram $\Sigma \setminus \{a_1, \dots, a_r\}$ is not a $(k-r)$-lifting of D.

We remark that in [8] a 0-lifting of D was called a "hyperbolic realization" of D. Also, a 0-lifting satisfies the general definition of a k-lifting. When the value of k is not important, we write "lifting" instead of "k-lifting".

Let Σ be a lifting of D, and $L \subset \Sigma$ a Lannér subdiagram such that for any missing face $M \in \mathcal{M}_D$ we have $L \not\subseteq \langle \phi(M), A \rangle$. We then say that L is an *additional* Lannér subdiagram of the lifting Σ.

Let D be an abstract diagram of missing faces, Σ an abstract Coxeter diagram, and $\phi : D \to \Sigma$ an injection. Let $\Sigma' \subset \Sigma$ be a non-elliptic subdiagram not containing $\phi(M)$ for any $M \in \mathcal{M}_D$. We then say that Σ' is a *conflicting* subdiagram; in that case Σ (with the given injection ϕ) is not a lifting of D.

Notice that $\phi(M)$ is a subset of the set of nodes of Σ. When we mean the Coxeter diagram spanned by the nodes of $\phi(M)$, we write $\langle \phi(M) \rangle$, in accordance with the notation introduced in 1.1.

Lemma 3.1. *Let D be a diagram of missing faces, and $D_1 \subset D$ a subdiagram of it. If Σ is a k-lifting of D, then Σ contains a subdiagram Σ_1 which is a k_1-lifting of D_1 for some $k_1 \leq k$.*

Proof. Let A be the set of additional nodes of Σ. Consider $\phi\big|_{D_1}$ as a lifting injection of D_1. By the definition of lifting, $\langle \phi(D_1), A \rangle$ contains some k_1-lifting of D_1 with $k_1 \leq k$. □

The proof of the following lemma is based on a repeated application of Proposition 1.8.

Lemma 3.2. *Let $P \subset \mathbb{H}^d$ be a simple hyperbolic Coxeter polytope, and f an m-face of it. Let $D(f)$ be the diagram of missing faces of f. Then $\Sigma(P)$ contains a subdiagram Σ_0 which is a k-lifting of $D(f)$ for some $k \leq d - m$.*

Proof. Let $f_1, \dots, f_{d-m}$ be the facets of P containing f. Set $F_1 = f_1$ and define inductively $F_i = F_{i-1} \cap f_i$. It is clear that F_i is a facet of F_{i-1} and $F_{d-m} = f$.

When $m = d$, the lemma is trivial. Assuming that the assertion holds for any face of dimension greater than m we shall prove it for the m-face f. By the induction assumption, $\Sigma(P)$ contains a subdiagram Σ_{r-1} which is a k-lifting of $D(F_{d-m-1})$, where $k \leq d - (m+1)$. The following claim then completes the proof.

Claim. Either Σ_{r-1} contains a k-lifting of f or $\langle \Sigma_{r-1}, v_{d-m} \rangle$ contains a $(k+1)$-lifting of f, where v_{d-m} is the node of $\Sigma(P)$ corresponding to the facet f_{d-m}.

Proof of the claim. Let $\Pi_1, \dots, \Pi_s$ be the facets of F_{d-m-1}. Denote by J the set of indices i such that $\Pi_i \cap f_{d-m} \neq \emptyset$ and set $\pi_i = \Pi_i \cap f_{d-m}$, $i \in J$. Then $\{\pi_i\}$ is the set of facets of $f = F_{d-m}$.

Let ϕ_1 be a lifting injection for F_{d-m-1}; in particular, ϕ_1 sends $\{\Pi_i\}$ to $\Sigma_{r-1} \setminus A$, where A is the set of additional nodes. Denote by ψ the map $\pi_i \to \Pi_i$ and consider $\phi = \phi_1 \circ \psi$. Then ϕ is an injection from $\{\pi_i\}$ to $\Sigma \setminus A$. Let $\{\Pi_i \,|\, i \in I\}$ be a missing face of $D(F_{d-m})$ (where $I \subset J$ is some indexing set). By Proposition 1.8, either $\{\pi_i \,|\, i \in I\}$ or $\{f_{d-m}\} \cup \{\pi_i \,|\, i \in I\}$ is a missing face of $D(F_{d-m-1})$ for some i. This establishes condition 4) in the definition of lifting for either Σ_{r-1} or $\langle \Sigma_{r-1}, v_{d-m} \rangle$. By the same proposition, if $\{\Pi_i \,|\, i \in K\}$ is not a missing face of $D(F_{d-m})$, then neither $\{\pi_i \,|\, i \in K\}$ nor $\{f_{d-m}\} \cup \{\pi_i \,|\, i \in K\}$ is a missing face of $D(F_{d-m-1})$, whence condition 3).

Thus, either A or $A \cup \{f_{d-m}\}$ contains a set of additional nodes for some k-lifting of $D(F_{d-m-1})$. Hence $k \leq d - (m+1) + 1 = d - m$, and the lemma is proved. □

Corollary 3.1. *Let $P \subset \mathbb{H}^d$ be a compact Coxeter polytope, and let f be a face of P. Then $D(f)$ does not contain large missing faces. In particular, if* $\dim f > 4$, *then f is not a simplex.*

Proof. By the definition of lifting, for any missing face M of D there is a Lannér subdiagram L of Σ such that $|L| \geq |M|$. Now the assertion follows from the fact that a Lannér diagram contains at most five nodes. □

Lemma 3.3. *Let D be a diagram of missing faces such that $|M| = 5$ for any $M \in \mathcal{M}_D$. Then any lifting of D is a 0-lifting.*

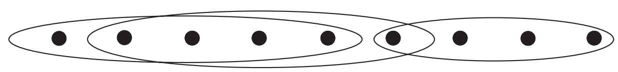
Proof. Suppose that D is a k-lifting, where $k > 0$, and ϕ is a lifting injection. Remove from D all additional nodes and denote the obtained diagram D_1. We shall show that D_1 is a 0-lifting and ϕ is the lifting bijection. Indeed, conditions 2), 3), and 5) in the definition of k-lifting clearly hold.

For condition 4), consider an arbitrary missing face M of D. By definition, $\phi(M)$ belongs to some Lannér subdiagram of Σ. Notice that the order of $\langle\phi(M)\rangle$ equals 5. Since the order of a Lannér diagram is at most 5, $\langle\phi(M)\rangle$ is a Lannér diagram. □

Next we want to establish several properties of liftings which will be needed later.

Notation.

- Let D be a diagram of missing faces and $N_1, \dots, N_r \subset D$. We write $D = \bigcup_{i=1}^r N_i$ if for any node of D there is a set $N_i, i \in \{1, \dots, r\}$, containing that node.
- Let $D = \bigcup_{i=1}^r N_i$ be a diagram of missing faces such that $N_i \cap N_j = \emptyset$ for every $i \neq j$. Suppose that there is a $k \in \{1, \dots, r\}$ such that $M \in \mathcal{M}_D$ if and only if $M = \bigcup_{t=i}^{i+k-1} N_t$ for some $i \in \{1, \dots, r-k+1\}$. In this case we write $D = \lfloor N_1, N_2, \dots, N_r \rfloor_k$.
- When we are interested only in the combinatorial type of D rather than the actual subdiagrams N_i, we write $\lfloor |N_1|, \dots, |N_r| \rfloor_k$. For example, $\lfloor 1, 4, 1, 3 \rfloor_2$ indicates the diagram



- If there is a $k \in \{1, \dots, r\}$ such that $M \in \mathcal{M}_D$ if and only if $M = \bigcup_{t=i}^{i+k-1} N_t$ for some $i \in 1, \dots, r$, where t is taken modulo r, we write $D = (N_1, N_2, \dots, N_r)_k$ or $(|N_1|, \dots, |N_r|)_k$.
- We write $\Sigma \approx \lfloor N_1, N_2, \dots, N_r \rfloor_k$ if Σ contains no parabolic subdiagrams and the structure of Lannér subdiagrams of Σ corresponds to the diagram of missing faces $\lfloor N_1, N_2, \dots, N_r \rfloor_k$; i.e., Σ consists of disjoint subdiagrams $L_1, \dots, L_r$, $|L_i| = |N_i|$, and $L \subset \Sigma$ is a Lannér subdiagram if and only if $L = \langle L_i, L_{i+1}, \dots, L_{i+k-1} \rangle$ for some $i \in \{1, \dots, r-k+1\}$.

  Similarly, we use the notation

$$\Sigma \approx \lfloor |N_1|, \dots, |N_r| \rfloor_k, \quad \Sigma \approx (N_1, N_2, \dots, N_r)_k, \quad \textit{and} \quad \Sigma \approx (|N_1|, \dots, |N_r|)_k.$$

Proposition 3.1 ([8, Lemma 4.7])**.** *Let $D = \lfloor M, N \rfloor_1$ be a diagram of missing faces with $|M| \geq 3$ and $|N| \geq 4$. Then D has no 0-liftings.*

Lemma 3.4. *The diagram Θ_1 of Figure 1 is the only lifting of $D = \lfloor 4, 5 \rfloor_1$. The additional node of this lifting is the leaf of Θ_1 which does not belong to a triple edge.*

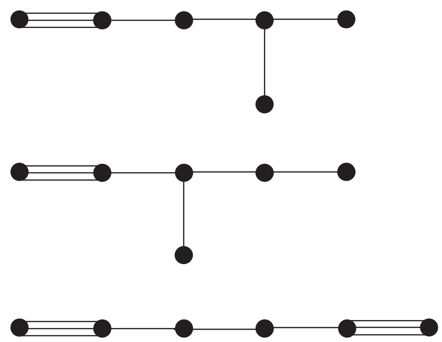
Proof. Let Σ be a lifting of $D = \lfloor M_4, M_5 \rfloor_1$ with $|M_5| = 5$ and $|M_4| = 4$, and let ϕ be a lifting injection. Suppose that $\langle\phi(M_4)\rangle$ is a Lannér subdiagram of Σ. By Proposition 3.1, $\Sigma \approx \lfloor \phi(M_4), \phi(M_5) \rfloor_1$ is superhyperbolic. Hence $\langle\phi(M_4)\rangle$ is elliptic and, by condition 4), Σ contains an additional node a such that $\langle\phi(M_4), a\rangle$ is Lannér. Then $\langle a, \phi(M_4), \phi(M_5)\rangle$ satisfies conditions 1)–4) in the definition of lifting, and, by condition 5), a is the only additional node and Σ is a 1-lifting. By condition 3), the order of any Lannér subdiagram of Σ is 5. Since $|\Sigma| = \left|\langle a, \phi(M_4), \phi(M_5)\rangle\right| = 10$, it follows from Lemma 2.1 that Σ is one of the diagrams Θ_1, Θ_2, and Θ_3 of Figure 1. By Lemma 1.2, Σ contains two disjoint Lannér diagrams $\langle\phi(M_4), a\rangle$ and $\langle\phi(M_5)\rangle$, and therefore Σ is either Θ_1 or Θ_2.

It is not difficult to find the only lifting injection for Θ_1: a is the only leaf of Σ which does not belong to a triple edge, and $\phi(M_4)$ consists of the remaining nodes in the bottom row (see Figure 1).

For Θ_2 there are no lifting injections, since $\langle\phi(M_4)\rangle$ belongs to each Lannér subdiagram different from $\langle\phi(M_5)\rangle$, whereas Θ_2 has no such quadruples of nodes. □

Lemma 3.5. *Let $D = N \cup \{x_1, x_2, x_3\}$ be a diagram of missing faces with $|N| = 4$. If $\mathcal{M}_D = \{N \cup x_1, N \cup x_2, N \cup x_3\}$, then D has no liftings.*

Proof. Suppose that Σ is a lifting of D and ϕ is a lifting injection. By Lemma 3.3, Σ is a 0-lifting. Consider $\Sigma_{ij} = \big\langle\phi(N \cup \{x_i, x_j\})\big\rangle$, $i \neq j$. It is clear that $\Sigma_{ij} \approx \lfloor 1,4,1 \rfloor_2$. By [8, Lemma 5.3], $\lfloor 1,4,1 \rfloor_2$ is one of the following diagrams:



Viewing each of these subdiagrams as Σ_{12} and trying to add x_3 to form Σ_{13}, we obtain in each case a conflicting subdiagram, which proves the lemma. □

Lemma 3.6. *The diagram $D = \lfloor 4,4 \rfloor_1$ has no 0-liftings. Any 1-lifting of D is one of the diagrams of Figure 2.*

Proof. By Proposition 3.1, D has no 0-liftings. Let Σ be a 1-lifting of D, ϕ a lifting injection, and a the additional node.

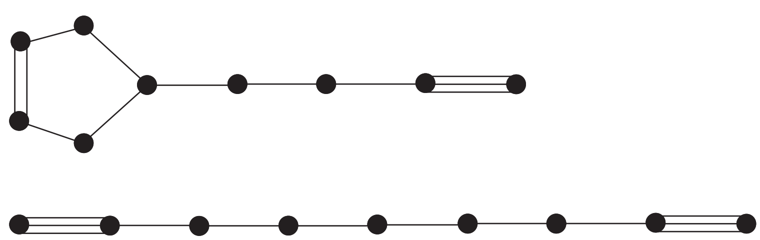


FIGURE 2. 1-liftings of $\lfloor 4,4 \rfloor_1$

Let M_1 and M_2 be the missing faces of D. We may assume that the diagram $\langle\phi(M_1)\rangle$ is elliptic and the diagram $\langle\phi(M_1), a\rangle$ is Lannér. We consider two cases.

Case 1. Suppose that $\langle\phi(M_2)\rangle$ is a Lannér diagram. By Proposition 3.1, Σ contains an additional Lannér subdiagram $L = \langle\phi(M_1), x\rangle$, where $x \in \phi(M_2)$. By Lemma 3.5, that additional subdiagram is unique. Thus,

$$\Sigma \approx \lfloor a, \phi(M_1), x, \phi(M_2) \setminus x \rfloor_2 = \lfloor 1,4,1,3 \rfloor_2 \,,$$

contrary to [8, Cor. 5.10].

Case 2. Suppose that both $\langle\phi(M_1)\rangle$ and $\langle\phi(M_2)\rangle$ are elliptic. Then $\langle\phi(M_1), a\rangle$ and $\langle\phi(M_2), a\rangle$ are Lannér. Notice that a is an open node of $\langle\phi(M_1), a\rangle$, since otherwise Σ would contain a conflicting Lannér subdiagram. Similarly, a is an open node of $\langle\phi(M_2), a\rangle$. Hence each of the diagrams $\langle\phi(M_1), a\rangle$ and $\langle\phi(M_2), a\rangle$ coincides with $\mathcal{L}_1^5$ or $\mathcal{L}_5^5$.

If Σ has an edge joining $\langle\phi(M_1)\rangle$ with $\langle\phi(M_2)\rangle$, then Σ has a conflicting subdiagram. Thus, the subdiagrams $\langle\phi(M_1)\rangle$ and $\langle\phi(M_2)\rangle$ are not joined in Σ, and Σ is one of the two diagrams of Figure 2 (if $\langle\phi(M_1), a\rangle = \langle\phi(M_2), a\rangle = \mathcal{L}_5^5$, then Σ contains a parabolic subdiagram). □

Let P_8 denote the only compact hyperbolic Coxeter 8-polytope with eleven facets (see [5, 8]), and let $\Sigma(P_8)$ be its Coxeter diagram (see Figure 3).

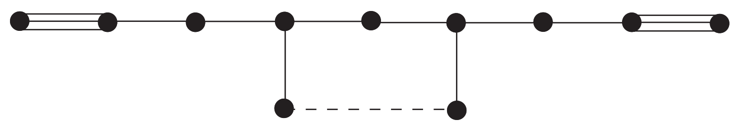


FIGURE 3. The only compact hyperbolic Coxeter 8-polytope with 11 facets

Lemma 3.7. *Let $D = (4, 4, 2)_1$, and let Σ be a* 0- *or* 1-*lifting of D that does not contain Lannér subdiagrams of order* 3. *If the positive inertia index of Σ is at most* 8, *then $\Sigma = \Sigma(P_8)$.*

Proof. By Lemmas 3.1 and 3.6, D has no 0-liftings. Let Σ be a 1-lifting of D, ϕ a lifting injection, and a the additional node. Let M_1, M_2, and M_3 be the missing faces of D, where $|M_1| = |M_2| = 4$ and $|M_3| = 2$. Let $M_3 = \langle u_1, u_2\rangle$. By Lemma 3.6, $\langle\phi(M_1), \phi(M_2), a\rangle$ is one of the two diagrams of Figure 2. Since Σ contains only one additional node, the non-existence of Lannér subdiagrams of order 3 implies that $\langle\phi(M_3)\rangle$ is Lannér. Hence the order of any Lannér subdiagram of Σ different from $\langle\phi(M_3)\rangle$ is 5.

Suppose that u_1 does not belong to any additional Lannér subdiagram of Σ. Then $\langle\phi(M_1), \phi(M_2), u_1\rangle$ is elliptic of order 9, contrary to the assumption.

Thus u_1 and u_2 belong to some additional Lannér subdiagrams. The subdiagram $X_1 = \big\langle u_1, \phi(M_1), \phi(M_2), a\big\rangle$ is of order 10 and does not contain small Lannér subdiagrams. Moreover, each node of X_1 belongs to some Lannér diagram. By Lemma 2.1, X_1 is one of the diagrams Θ_1, Θ_2, and Θ_3 of Figure 1. Since the positive inertia index of Σ is at most 8, we have $\det(X_1) = 0$ and $X_1 = \Theta_1$ (Lemma 2.1). By Lemma 3.6, $\big\langle\phi(M_1), \phi(M_2), a\big\rangle$ is a linear subdiagram of Θ_1. Hence u_1 is the only leaf of Θ_1 not incident to a triple edge. Similarly, $X_2 = \big\langle u_2, \phi(M_1), \phi(M_2), a\big\rangle$ also coincides with Θ_1. Moreover, u_1 and u_2 are joined with different nodes of $\big\langle\phi(M_1), \phi(M_2), a\big\rangle$, since otherwise the diagram $\big\langle\phi(M_i), a, u_1, u_2\big\rangle$ would be superhyperbolic. Thus $\Sigma = \Sigma(P_8)$ and the lemma is proved. □

Lemma 3.8. *Let $D = (1, 4, 1, 3, 1)_2$, and let Σ be a* 0- *or* 1-*lifting of D that does not contain Lannér subdiagrams of order* 3. *If the positive inertia index of Σ is at most* 8, *then $\Sigma = \Sigma(P_8)$.*

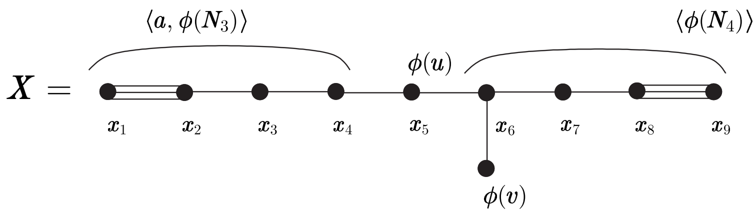
Proof. Let ϕ be a lifting injection. We use the following notation for the subdiagrams of D: $(1, 4, 1, 3, 1)_2 = (v, N_4, u, N_3, w)_2$.

Suppose that $\langle\phi(N_3), \phi(u)\rangle$ is a Lannér diagram. Then

$$\big\langle\phi(v), \phi(N_4), \phi(u), \phi(N_3)\big\rangle \approx \lfloor 1, 4, 1, 3\rfloor_2\,,$$

which is impossible by [8, Cor. 5.10]. Hence $\langle\phi(N_3), \phi(u)\rangle$ is elliptic and Σ cannot be a 0-lifting of D.

Now assume that Σ is a 1-lifting and a is the only additional node of Σ. Then $X = \langle a, \phi(D \setminus \{w\})\rangle$ satisfies the assumptions of Lemma 2.1. Since $|X| = 10$, X is one of the diagrams Θ_1, Θ_2, and Θ_3 of Figure 1. By assumption, the positive inertia index of Σ is at most 8 and therefore $\det(X) = 0$ and $X = \Theta_1$:

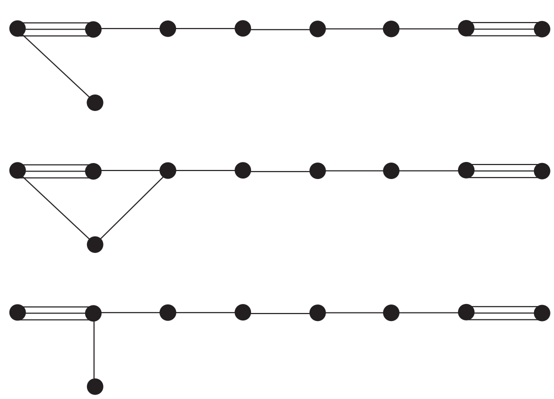


Consider the diagram $\langle X, \phi(w)\rangle$. Since $\langle \phi(N_4), \phi(w)\rangle$ is elliptic, $\phi(w)$ is not joined with $\langle x_6, x_7, x_8, x_9\rangle = \langle \phi(N_4)\rangle$. Furthermore, $\phi(w)$ is not joined with $\langle x_1, x_2\rangle$, otherwise $\langle x_1, x_2, x_3, \phi(w)\rangle$ would be conflicting. Hence, besides $\phi(v)$, $\phi(w)$ may be joined only with x_3, x_4, or x_5. Any edge joining w with x_3, x_4, x_5 is simple, since otherwise one of the diagrams $\langle x_2, x_3, x_4, \phi(w)\rangle$, $\langle x_3, x_4, x_5, \phi(w)\rangle$, and $\langle x_4, x_5, x_6, \phi(w)\rangle$ would be conflicting. To avoid parabolic subdiagrams, $\phi(w)$ must be joined with at most one of the nodes x_3, x_4, x_5 (except $\phi(v)$).

If $\phi(w)$ is joined with x_3, then the subdiagram $\langle \Sigma \setminus \phi(v)\rangle$ is superhyperbolic. If $\phi(w)$ is joined with x_5, then $\phi(N_3 \cup \{w\})$ does not belong to any Lannér subdiagram, contrary to the definition of lifting. Hence $\phi(w)$ is joined with x_4. By assumption, Σ does not contain Lannér subdiagrams of order 3, therefore the diagram $\langle \phi(v), \phi(w)\rangle$ is Lannér, and $\Sigma = \Sigma(P_8)$. ☐

Lemma 3.9. *The diagram $D = \lfloor 1, 3, 2, 3 \rfloor_2$ has no* 0*-liftings.*

Proof. Suppose that Σ is a 0-lifting of $\lfloor 1, 3, 2, 3 \rfloor_2$. Comparing the 0-liftings of $\lfloor 3, 2, 3 \rfloor_2$ (see [8, Lemma 5.12]) with the 0-liftings of $\lfloor 2, 3, 1 \rfloor_2$ (see [17, Table 4.8]), we conclude that Σ is one of the following diagrams:

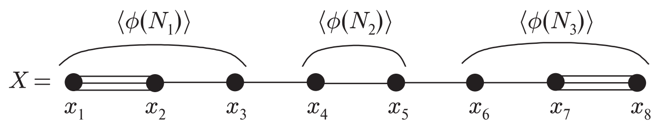


These diagrams are superhyperbolic, which proves the lemma. ☐

Lemma 3.10. *Let $D = (3, 2, 3, 1, 1)_2$. Then there are no* 0*- or* 1*-liftings of D of positive inertia index smaller than or equal to* 8 *and not containing Lannér subdiagrams of order* 3*.*

Proof. Suppose there is such a lifting Σ and let ϕ be a lifting injection. Denote the subsets of D as follows: $(3, 2, 3, 1, 1)_2 = (N_1, N_2, N_3, v, u)$. Set $X = \langle \phi(N_1), \phi(N_2), \phi(N_3)\rangle$.

The subdiagram X of $\langle\phi(D)\rangle$ does not contain small Lannér subdiagrams. Hence $X \approx \lfloor 3,2,3 \rfloor_2$, and by [8, Lemma 5.12], we have



By the definition of lifting, the diagram $\langle\phi(N_1),\phi(u)\rangle$ belongs to some Lannér subdiagram of Σ. If $\langle\phi(N_1),\phi(u)\rangle$ is Lannér, then

$$\langle X,\phi(u)\rangle \approx \lfloor \phi(u),\phi(N_1),\phi(N_2),\phi(N_3) \rfloor_2 = \lfloor 1,3,2,3 \rfloor_2 \,,$$

which is impossible by Lemma 3.9.

Hence $\langle\phi(N_1),\phi(u)\rangle$ is elliptic and Σ contains exactly one additional node a. Similarly, $\langle\phi(N_3),\phi(v)\rangle$ is also elliptic. Since Σ does not contain Lannér subdiagrams of order 3, $\langle\phi(u),\phi(v)\rangle$ is a Lannér diagram. The remaining small missing faces are $\langle\phi(N_1),\phi(u)\rangle$ and $\langle\phi(N_3),\phi(v)\rangle$. Hence each additional subdiagram of Σ contains either $\langle\phi(N_1),\phi(u)\rangle$ or $\langle\phi(N_3),\phi(v)\rangle$, and the diagram $Y = \langle X,a,\phi(u)\rangle$ satisfies the assumptions of Lemma 2.1. Since $|Y| = 10$, $|Y|$ is one of the diagrams Θ_1,Θ_2, and Θ_3 of Figure 1. The diagram Θ_1 does not contain $\Sigma_0 \approx \lfloor 3,2,3 \rfloor_2$. The diagrams Θ_2 and Θ_3 have positive inertia index 9, contrary to the assumption. □

Lemma 3.11. *The diagrams* $\lfloor 5,3 \rfloor_1$ *and* $\lfloor 5,3,2 \rfloor_1$ *have neither* 0- *nor* 1-*liftings.*

Proof. We shall prove the lemma for $D = \lfloor 5,3 \rfloor_1$; the assertion for $\lfloor 5,3,2 \rfloor_1$ will then immediately follow.

By Proposition 3.1, D has no 0-liftings. Suppose that Σ is a 1-lifting of D, ϕ is a lifting injection, and a is the additional node. Let M_5 and M_3 be the missing faces of D of order 5 and 3, respectively. Then $\langle\phi(M_3),a\rangle$ is a Lannér diagram. By Proposition 3.1, Σ has at least one additional Lannér subdiagram. By the definition of lifting, any additional Lannér subdiagram of Σ contains $\phi(M_3)$.

Suppose that $L = \langle\phi(M_3),x\rangle$ is an additional Lannér subdiagram of Σ of order 4, $x \in \phi(M_5)$. Then x is an open node of L, since otherwise Σ would contain a conflicting subdiagram. It is not difficult to see that x is a doubly open node of L, whence $L = \mathcal{L}_5^4$. On the other hand, $\langle L,a\rangle \approx \lfloor x,M_3,a \rfloor_2 = \lfloor 1,3,1 \rfloor_2$. By [17, Table 4.8], no diagram $\Sigma_0 \approx \lfloor 1,3,1 \rfloor_2$ contains subdiagrams $\mathcal{L}_5^4$. It follows that Σ has no additional Lannér subdiagrams of order 4.

Thus Σ has at least one additional Lannér subdiagram L of order 5, $L = \langle\phi(M_3), x_1,x_2\rangle$, $x_1,x_2 \in \phi(M_5)$. Since L is connected, we may assume that x_1 is joined with some node $y \in \phi(M_3)$ (up to a transposition of x_1 and x_2). Then x_1 is an open node of $\langle\phi(M_5)\rangle$ and y is an open node of $\langle\phi(M_3),a\rangle$. Since no Lannér diagram of order 5 has more than one open node, x_2 cannot be joined with $\langle\phi(M_3)\rangle$. Hence x_2 is joined with x_1. For the same reason, any additional Lannér subdiagram L' contains x_1; moreover, if $x_k \in L'$ and $x_k \neq x_1$, then x_k is joined with x_1 and is not joined with $\langle\phi(M_3)\rangle$.

If L is the only additional Lannér subdiagram of Σ, then

$$\Sigma \approx \big\lfloor \langle\phi(M_5)\setminus\{x_1,x_2\}\rangle, \langle x_1,x_2\rangle, \langle\phi(M_3)\rangle, a \big\rfloor_2 = \lfloor 3,2,3,1 \rfloor_2 \,,$$

which is impossible by Lemma 3.9. Therefore, Σ contains at least one more additional Lannér subdiagram L'. Suppose that $L' = \langle\phi(M_3),x_1,x_3\rangle$, $x_3 \in \phi(M_5)$, $x_3 \neq x_2$. As was shown above, the open node x_1 of $\langle\phi(M_5)\rangle$ is joined with two other nodes, x_2 and x_3, which implies that $\langle\phi(M_5)\rangle = \mathcal{L}_5^5$. In particular, the edges x_1x_2 and x_1x_3 are simple.

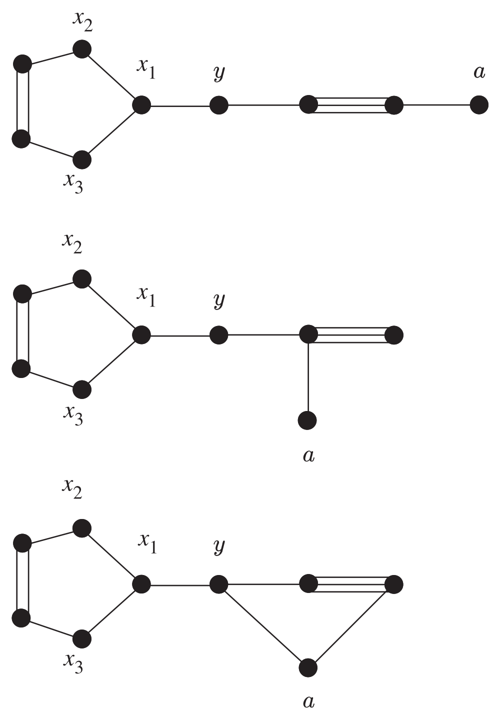
Keeping in mind that

$$\langle L, a\rangle \approx \lfloor a, \langle \phi(M_3)\rangle, \langle x_1, x_2\rangle \rfloor_2 = \lfloor 1, 3, 2\rfloor_2$$

and

$$\langle L', a\rangle \approx \lfloor a, \langle \phi(M_3)\rangle, \langle x_1, x_3\rangle \rfloor_2 = \lfloor 1, 3, 2\rfloor_2\,,$$

we check all possibilities for $\lfloor 1, 3, 2\rfloor_2$ such that x_1x_2 and x_1x_3 are simple edges (see [17, Table 4.8]). We then have that either Σ contains a conflicting subdiagram or Σ is one of the following diagrams:



All these diagrams are superhyperbolic, which proves the lemma. □

4. Non-existence of polytopes in dimensions ≥ 9

Suppose that there is a compact hyperbolic Coxeter d-polytope P with $d+4$ facets. Let Σ be the Coxeter diagram of P.

For the study of polytopes with $d+4$ facets we shall refer to the classification of compact Coxeter d-polytopes with $d+1$, $d+2$, and $d+3$ facets (see [14, 9, 13, 17]). In particular, recall that P_8 is the only 8-polytope with 11 facets, and $\Sigma(P_8)$ stands for its Coxeter diagram (see Figure 3).

Henceforth, by a polytope we understand a compact hyperbolic Coxeter polytope, and by the diagram of a polytope, the Coxeter diagram of it.

Lemma 4.1. *Suppose that $d \geq 9$. Then:*

(1) *any node v of Σ is incident to at most one non-simple edge;*
(2) *Σ has no Lannér subdiagrams of order 3;*
(3) *Σ has no edges of multiplicity ≥ 4.*

Proof. If the lemma is not true, then Σ has either a node incident to two dashed edges or a subdiagram $S_0 \subset \Sigma$ of type $G_2^{(m)}$, $m \geq 4$, with a bad neighbor.

Suppose that Σ has a node v incident to two dashed edges. Then the facet $f = P(v)$ is a simple (possibly non-Coxeter) $(d-1)$-polytope with at most $d+1 = (d-1)+2$ facets, i.e., either a simplex or a product of two simplices. If f is a simplex, then P has a large missing face, contrary to Corollary 3.1. If f is a product of two simplices and has no large missing faces, then it is a product of two 4-simplices (since $d \geq 9$), and therefore $D(f) = \lfloor 5,5 \rfloor_1$. By Lemma 3.3 and Proposition 3.1, the diagram $\lfloor 5,5 \rfloor_1$ has no liftings, contrary to Lemma 3.2.

Suppose that Σ has a subdiagram $S_0 \subset \Sigma$ of type $G_2^{(m)}$ with $m \geq 4$ and a bad neighbor. Then $P(S_0)$ is a Coxeter $(d-2)$-polytope with at most $(d-2)+3$ facets (see Proposition 1.11), therefore either $d-2 = 8$ or $d-2 \leq 6$ (see [17]). If $d-2 \leq 6$, then $d \leq 8$, contrary to the assumption. If $d-2 = 8$, then $d = 10$ and $P(S_0)$ is the only Coxeter 8-polytope with 11 facets (see Figure 3). Notice that Σ_{S_0} has a subdiagram of type H_4 with two neighbors joined with it by simple edges. In view of Corollary 1.2, the corresponding diagram $S_1 \subset \Sigma$ is also of type H_4 with at least two neighbors. Hence $P(S_1)$ is a Coxeter 6-polytope with at most eight facets, which is impossible (see [9, 13]). ☐

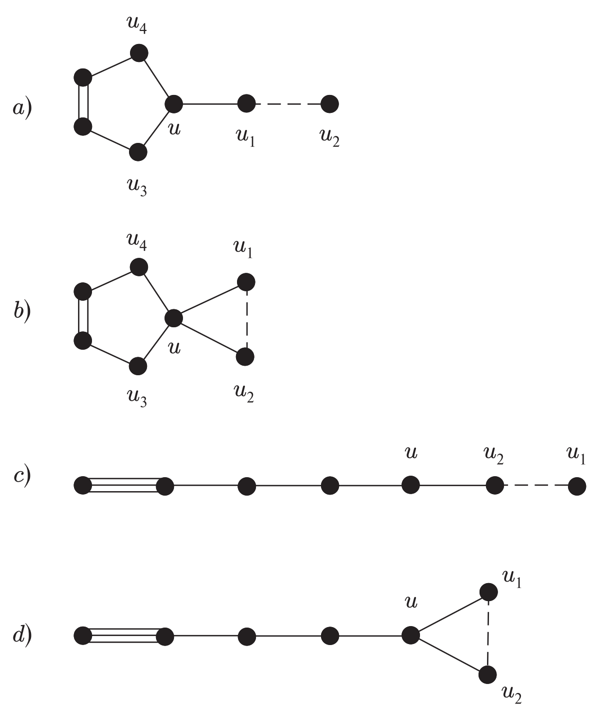


FIGURE 4. The 5-prisms that have property (1) in Lemma 4.1

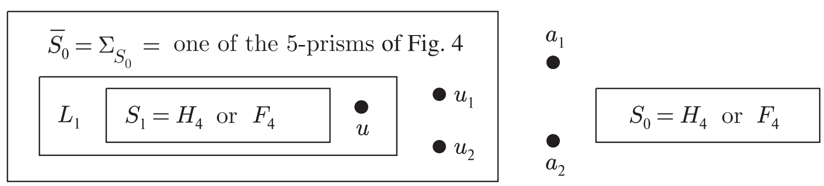
Lemma 4.2. *Suppose $d \geq 9$. Then Σ does not contain Lannér subdiagrams of order* 5.

Proof. Suppose that there is a Lannér subdiagram $L_0 \subset \Sigma$ of order 5. The classification of Lannér diagrams shows that there is a subdiagram $S_0 \subset L_0$ of type H_4 or F_4. Then $P(S_0)$ is a Coxeter $(d-4)$-polytope and $\Sigma_{S_0} = \bar{S}_0$ is a subdiagram of Σ (see Corollary 1.1). Since S_0 has at least one neighbor (the node $L_0 \setminus S_0$) and a neighbor of a diagram of type H_4 or F_4 cannot be good, the order of Σ_{S_0} is either $d-3$, or $d-2$, or $d-1$.

Case 1. $P(S_0)$ has $d-3$ facets. Then $P(S_0)$ is a Coxeter simplex and $d-4 \le 4$, i.e., $d \le 8$, contrary to the assumption.

Case 2. $P(S_0)$ has $d-2$ facets, i.e., $P(S_0)$ is a Coxeter $(d-4)$-polytope with $(d-4)+2$ facets, and therefore (see [8, 13]) $d-4 \le 5$, i.e., $d \le 9$. Hence $d = 9$ and $P(S_0)$ is a 5-prism. Since $\Sigma_{S_0} = \bar{S}_0$, Lemma 4.1(1) together with [13] implies that $\bar{S}_0$ is one of the four diagrams of Figure 4.

Let L_1 be the only Lannér subdiagram of $\bar{S}_0$, $S_1 \subset L_1$ the only subdiagram of type H_4 or F_4, and $u = L_1 \setminus S_1$. The diagram Σ consists of the following parts:

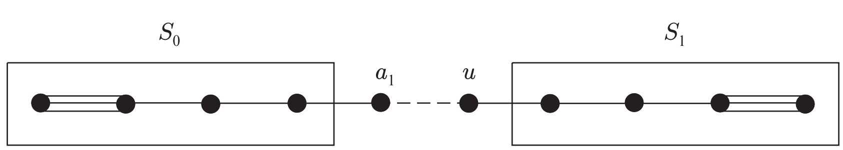


where a_1 and a_2 are neighbors of S_0, and S_0 is not joined with $\bar{S}_0$. Since any two indefinite subdiagrams must be joined in Σ, each of the nodes a_1 and a_2 is joined with L_1. If each of these nodes is joined with S_1, then S_1 has three neighbors, which is impossible (see Case 1). Hence one of these nodes, say a_1, is joined with $u = L_1 \setminus S_1$ (see Figure 4) and is not joined with S_1.

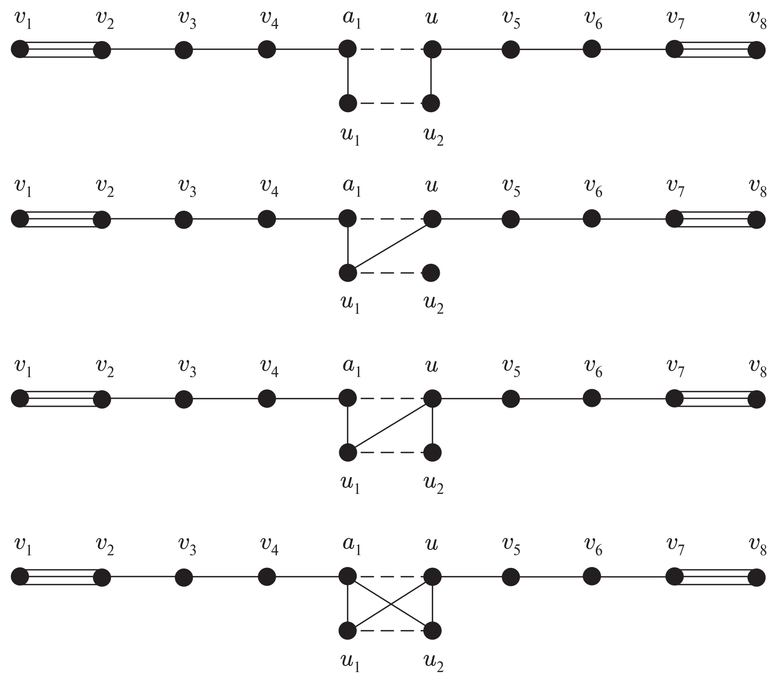
Consider diagrams *a*) and *b*) in Figure 4, with the nodes indexed as is shown there. Let $S_2 = \Sigma_{S_0} \setminus \langle u_2, u_4 \rangle$. Then S_2 is of type B_5 with at least three bad neighbors (u_4, u_2, and a_1). By Lemma 1.3, a_2 cannot be a neighbor of S_2. Similarly, examining $S_3 = \Sigma_{S_0} \setminus \langle u_2, u_3 \rangle$ we have that a_2 is not joined with u_4. Hence a_2 is not joined with L_1, i.e., the indefinite subdiagram $\langle S_0, a_2 \rangle$ is not joined with the Lannér diagram L_1, which is impossible.

Suppose now that Σ_{S_0} is one of the diagrams *c*) and *d*) in Figure 4. There are two possibilities: either a_2 is joined with S_1 or not. Suppose a_2 is not joined with S_1. Then $P(S_1)$ is a Coxeter 5-polytope with eight facets. By Corollary 1.1, $\Sigma_{S_1} = \bar{S}_1 \subset \Sigma$. But property (1) in Lemma 4.1 is not shared by any Coxeter 5-polytope with eight facets (see [17]). Hence a_2 is joined with S_1, and S_1 has exactly two neighbors, u and a_2 (it cannot have more neighbors by Case 1). Thus, $P(S_1)$ is a Coxeter 5-prism. We may assume that the diagram Σ_{S_1} of $P(S_1)$ is of type *c*) or *d*) in Figure 4 (in the case of prisms *a*) and *b*) use the arguments from the previous paragraph). Notice that $\Sigma_{S_1} = \bar{S}_1$ contains S_0; hence S_0 is a diagram of type H_4.

Furthermore, Σ_{S_1} contains u_1, u_2, and a_1. Since $P(S_1)$ is a 5-prism, Σ_{S_1} contains a Lannér subdiagram of order 5. On the other hand, $\Sigma_{S_1} = \langle S_0, u_1, u_2, a_1 \rangle$, where u_1 and u_2 are not joined with S_0, and therefore $\langle S_0, a_1 \rangle$ is a Lannér diagram. Hence $\langle S_0, a_1 \rangle$ is joined with the Lannér diagram L_1, i.e., a_1 is joined with u. By Lemma 1.4, a_1 is joined with u by a dashed edge. Thus, $\langle S_0, a_1, u, S_1 \rangle$ is of the form



Recall that u_1 and u_2 are joined by a dashed edge. Since $u_1, u_2 \in \bar{S}_0$, each of these nodes is joined in $\Sigma \setminus a_2$ with u and a_1 only. For $\Sigma \setminus a_2$ we have four possibilities:



By assumption, a_2 is joined with $S_1 = \langle v_5, v_6, v_7, v_8 \rangle$. Since $a_2 \notin \Sigma_{S_0}$, a_2 is joined with $S_0 = \langle v_1, v_2, v_3, v_4 \rangle$. Furthermore, a_2 is joined with $\langle u_1, u_2 \rangle$, since otherwise the indefinite diagram $\langle S_0, a_2 \rangle$ would not be joined with the Lannér diagram $\langle u_1, u_2 \rangle$. Since each of the diagrams $\langle v_2, v_3, v_4, a_1, u_1 \rangle$, $\langle v_7, v_6, v_5, u, u_1 \rangle$, and $\langle v_7, v_6, v_5, u, u_2 \rangle$ has three bad neighbors, no dashed edge ends in a_2(see Lemma 4.1(1)). Examining the possible multiplicities of the edges (i.e., simple, double, triple, and empty), we see that there is always either a Lannér subdiagram of order 3, or a parabolic subdiagram, or a subdiagram of type H_4 with at least three neighbors (the latter is in fact impossible; see Case 1).

Case 3. $P(S_0)$ has $d-1$ facets. Then $P(S_0)$ is a Coxeter $(d-4)$-polytope with $(d-4)+3$ facets. Since $d \geq 9$, we have $d - 4 \geq 5$. By [17], the Coxeter diagrams of such polytopes either do not have properties (1)–(3) of Lemma 4.1 or have a subdiagram of type H_4 with at least two neighbors, which is impossible by the previous cases. $\square$

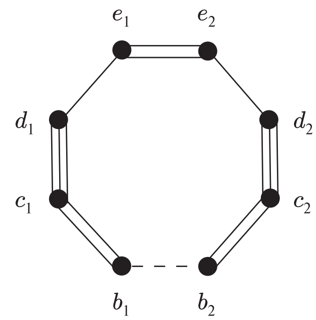
Lemma 4.3. *Suppose that one of the following holds:*

1) $d \geq 9$*;*
2) $d = 8$*,* Σ *has properties* (1)–(3) *of Lemma* 4.1*, and* Σ *does not contain Lannér subdiagrams of order* 5*.*

Then Σ *has no Lannér subdiagrams of order* 4*.*

Proof. Suppose that L_0 is a Lannér subdiagram of Σ of order 4. Let $S_0 \subset L_0$ be a subdiagram of type H_3 (if any) or B_3 (otherwise). Then $P(S_0)$ is a Coxeter $(d-3)$-polytope with at most $(d-3)+3$ facets. By Lemmas 4.2 and 3.2, Σ_{S_0} has no Lannér subdiagrams of order 5. If $P(S_0)$ is a simplex or a product of two simplices, then $d-3 \leq 3$ or $d-3 \leq 4$, respectively; hence $d \leq 7$, contrary to the assumption.

Suppose that $P(S_0)$ is a polytope with $(d-3)+3$ facets. When $d \geq 8$, the Coxeter diagram of almost all $(d-3)$-polytopes with d facets contains a Lannér subdiagram of order 5 (see [17]). The only exception is a 5-polytope with eight facets and Coxeter diagram Σ_{S_0} of the form



By Corollary 1.2, no node of $\bar{S}_0 \setminus \langle b_1, b_2\rangle$ is a good neighbor of S_0. The node b_1 is a good neighbor; otherwise Σ would contain the Lannér subdiagram $\langle b_1, c_1, d_1\rangle$ of order 3, contrary to the assumption. Thus, $\langle b_1, c_1\rangle$ is a simple edge of Σ and $\langle d_1, c_1, b_1, x_1, x_2\rangle$ is a Lannér diagram of order 5 (here x_1 and x_2 are the ends of the simple edge of S_0). This contradicts the assumption. □

Theorem 1. *There are no compact hyperbolic Coxeter d-polytopes with $d+4$ facets with $d \geq 9$.*

Proof. It follows from Lemmas 4.1, 4.2, and 4.3 that the order of any Lannér subdiagram of Σ is 2. It was shown in [8, Prop. 6.9] that such a polytope with $d+4$ facets may only exist when $d \leq 4$. □

5. Non-existence of polytopes in dimension 8

In this section we show that there are no compact Coxeter 8-polytopes with 12 facets. Assuming that P is such a polytope and Σ is its Coxeter diagram, we show that the properties of Σ are similar to those proved for polytopes in large dimensions, which eventually leads to a contradiction. However, the proofs in the 8-dimensional case are much more complicated than in larger dimensions.

Lemma 5.1 ([11, Lemma 1])**.** *Let $\Sigma(P)$ be the Coxeter diagram of a Coxeter d-polytope P. Then no proper subdiagram of $\Sigma(P)$ is the diagram of a Coxeter d-polytope of finite volume.*

In particular, Σ does not contain $\Sigma(P_8)$ as a proper subdiagram.

Lemma 5.2. *Suppose $\langle v_1, v_2\rangle \subset \Sigma$ is a subdiagram of type $G_2^{(k)}$, $k \geq 4$. Then:*

(1) *$\langle v_1, v_2\rangle$ has at most one bad neighbor;*
(2) *$k \leq 5$;*
(3) *if $\langle v_1, v_2\rangle$ has a bad neighbor, then it also has a good neighbor.*

Proof. Let $S_0 = \langle v_1, v_2\rangle$. If S_0 has two bad neighbors, then $P(S_0)$ is a Coxeter 6-polytope with at most eight facets, which is impossible by [13, 8].

Suppose that either $k \geq 6$ or S_0 has a bad neighbor and no good ones. In particular, all neighbors of S_0 are bad and, by Corollary 1.1, $\bar{S}_0 = \Sigma_{S_0}$. In addition, $P(S_0)$ is a Coxeter 6-polytope with at most nine facets. By [17], the diagram of such a polytope is one of the three diagrams of Figure 5. The Coxeter diagram of each of those polytopes contains a multiple edge with two bad neighbors. This is impossible, as we have shown before. □

Lemma 5.3. (1) *Σ does not contain Lannér subdiagrams of order* 3.

(2) *No node incident to a dashed edge is incident to a triple edge.*

(3) *If Σ has exactly one dashed edge, then neither of its ends is incident to a double edge.*

Proof. Suppose that the lemma is not true. Let L_0 be a Lannér subdiagram of order 3 or a subdiagram with three nodes that contains a dashed edge and a multiple edge. Consider two cases.

Case 1. L_0 contains a triple edge.

Let S_0 be the subdiagram consisting of that edge and its ends. Then $P(S_0)$ is a Coxeter 6-polytope with at most nine facets (since L_0 contains a bad neighbor of S_0). Then Σ_{S_0} is one of the three diagrams of Figure 5. By Corollary 1.1, $\bar{S}_0 \subset \Sigma$ can be obtained from Σ_{S_0} by replacing (if necessary) some dashed edges with ordinary edges and some edges labeled by 10, with simple edges. Applying this procedure to P_6^1 and P_6^3 (Figure 5), we see that the resulting diagrams contain either a parabolic subdiagram or a multiple edge with two bad neighbors.

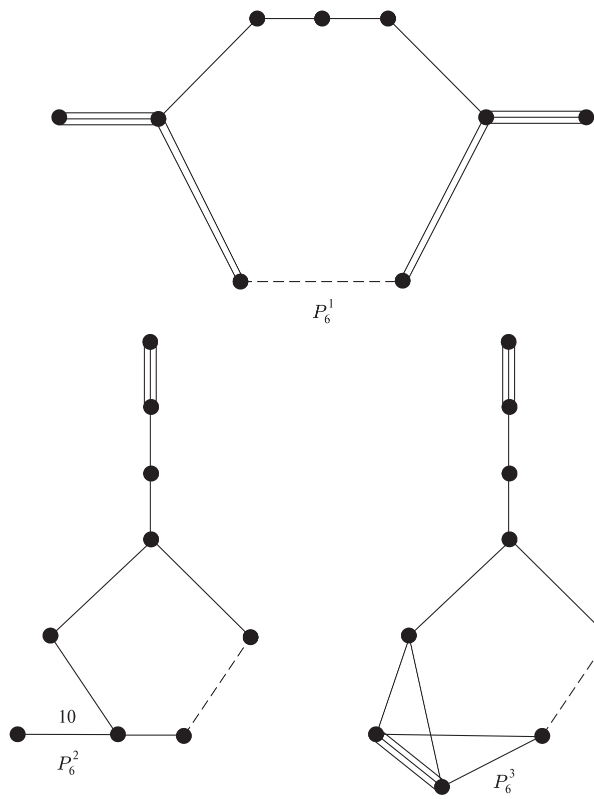


FIGURE 5. Compact hyperbolic Coxeter 6-polytopes with nine facets

Suppose that $P(S_0) = P_6^2$. Let u be the bad neighbor of S_0 (it is unique by Lemma 5.2). In view of Corollary 1.1, $\bar{S}_0$ can be obtained from Σ_{S_0} by replacing some edges labeled by 10 with simple edges and some dashed edges with ordinary ones. By Corollary 1.2, the dashed edge of Σ_{S_0} remains dashed in $\bar{S}_0$. By Lemma 5.2, the edge of Σ_{S_0} labeled by 10 is a simple edge in $\bar{S}_0$. Hence the leaf of Σ_{S_0} incident to the edge labeled by 10 is a good neighbor of S_0 in Σ. The remaining nodes of Σ_{S_0} cannot be good neighbors of

S_0 by Corollary 1.2. Thus, $\Sigma \setminus u$ is the Coxeter diagram $\Sigma(P_8)$, which is impossible by Lemma 5.1.

Case 2. L_0 has no triple edges.

Let S_0 be the subdiagram of L_0 consisting of a double edge and its ends. As before, $P(S_0)$ is a 6-polytope with nine facets. By Corollary 1.2, $\bar{S}_0 \subset \Sigma$ can be obtained from Σ_{S_0} by replacing (if necessary) some dashed edges with ordinary ones and some double edges, with simple or empty ones. Applying this procedure to the diagrams of the polytopes P_6^3 and P_6^2, we conclude that the resulting diagrams contain multiple edges with at least two bad neighbors, which is impossible by Lemma 5.2.

Suppose that $P(S_0) = P_6^1$. Then both double edges of Σ_{S_0} become simple or empty in $\bar{S}_0$; otherwise Σ would have a Lannér subdiagram with a triple edge, which is impossible by Case 1. Moreover, by Corollary 1.2, only one end of each double edge can be a good neighbor of S_0. Therefore, by Proposition 1.11, each double edge of Σ_{S_0} is simple in $\bar{S}_0$. Moreover, the only dashed edge remains dashed, since otherwise $\bar{S}_0$ would be superhyperbolic. Hence in this case we may assume that L_0 has no dashed edges (otherwise all assertions of the lemma are obviously true). By Proposition 1.11, the ends of the dashed edge of $\bar{S}_0$ are good neighbors of S_0, whereas the remaining nodes of $\bar{S}_0$ are not. Consider the subdiagram $S_1 \subset \bar{S}_0$ of type H_4 consisting of a triple edge, a simple edge adjacent to it and obtained from a double edge, and a simple edge joining S_0 with its good neighbor. The diagram S_1 has two neighbors in $\bar{S}_0$ and two neighbors in S_0 (since L_0 is a Lannér diagram with no edges of multiplicity greater than 2). This means that S_1 has at least four bad neighbors, contrary to Lemma 1.3. □

The next lemma is the main result of [11].

Lemma 5.4 ([11, Th. A]). *Let P be a compact Coxeter polytope in the d-dimensional hyperbolic space, and $\Sigma(P)$ its Coxeter diagram. If $d > 4$, then $\Sigma(P)$ contains a dashed edge.*

Lemma 5.5. *Let $S \subset \Sigma$ be an elliptic subdiagram of order* 3 *having no components of type A_n. Then S has at most two bad neighbors.*

Proof. Suppose that S has three or more bad neighbors. Then Σ_S is a 5-polytope with at most six facets, which is impossible. □

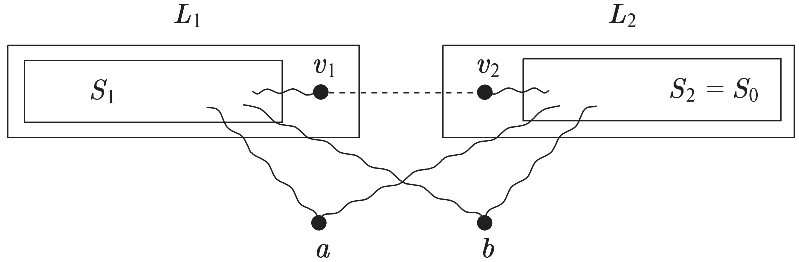
Lemma 5.6. *Suppose that Σ contains a Lannér subdiagram L_0 of order* 5 *and a unique dashed edge. Then Σ contains a subdiagram S of type F_4 or H_4 such that $P(S)$ is not a simplex.*

Proof. Suppose that the lemma is not true, i.e., for any diagram $S' \subset \Sigma$ of type H_4 or F_4 the face $P(S')$ is a simplex.

Let $S_0 \subset L_0$ be a diagram of type H_4 or F_4. Then $P(S_0)$ is a Coxeter 4-simplex. The diagram Σ_{S_0} is a Lannér diagram of order 5, we denote it L_1. Let S_1 be a subdiagram of L_1 of type H_4 or F_4. By assumption, $P(S_1)$ is a Coxeter simplex, and $\bar{S}_1 = \Sigma_{S_1}$ is a Lannér diagram of order 5, we denote it L_2. Notice that $S_0 \subset L_2$ and let $S_2 = S_0$. Let $v_i = L_i \setminus S_i$, $i = 0, 1, 2$.

Since $S_2 = S_0$ is not joined with L_1, v_2 is joined with L_1. On the other hand, L_2 is not joined with S_1. Hence $\langle L_1, L_2 \rangle$ consists of two Lannér subdiagrams L_1 and L_2 joined by a single edge v_1v_2 such that $L_i \setminus v_i$ is of type H_4 or F_4. By Lemma 1.4, v_1v_2 is a dashed edge.

Let a and b be nodes not contained in $\langle L_1, L_2 \rangle$. Notice that if $v_2 \neq v_0$, then either a or b coincides with v_0. Both a and b are joined with S_1 and S_2. The diagram Σ consists of the following parts:



where the wavy line indicates that the node is joined with the subdiagram by a non-dashed edge; the subdiagram $\langle a, b, v_1, v_2\rangle$ may contain some other (non-dashed) edges.

By Lemma 5.3, v_1 and v_2 are joined with S_1 and S_2 only by simple edges. Hence L_1 (as well as L_2) is one of the diagrams $\mathcal{L}_1^5$, $\mathcal{L}_4^5$, or $\mathcal{L}_5^5$ from Table 2.

Suppose that $L_1 = \mathcal{L}_5^5$. Let $w_1 \in S_1$ be a node joined with a. If w_1a is a triple edge, then Σ has either a Lannér subdiagram of order 3 or a diagram of type H_3 with at least three bad neighbors, which is impossible by Corollary 3.1. If w_1a is a double or a simple edge, then we have either a parabolic subdiagram or a Lannér subdiagram of order 3. Thus the multiplicity of w_1a should be greater than three, which is impossible by Lemma 5.2.

Hence both S_1 and S_2 are of type H_4. For each of the four possible pairs of diagrams L_1 and L_2, there is a unique label for the dashed edge v_1v_2 such that the determinant of $\Sigma \setminus \langle a, b\rangle$ vanishes. Those labels are: $(1+\sqrt{5})/2$ for $L_1 = L_2 = \mathcal{L}_1^5$, $4+2\sqrt{5}$ for $L_1 = L_2 = \mathcal{L}_4^5$, and $(3+\sqrt{5})/\sqrt{2}$ for $L_1 = \mathcal{L}_1^5$, $L_2 = \mathcal{L}_4^5$ (or $L_2 = \mathcal{L}_1^5$, $L_1 = \mathcal{L}_4^5$).

Let $u_1 \in S_1$ and $u_2 \in S_2$ be the leaves of $\Sigma \setminus \langle a, b\rangle$ incident to the triple edges. Suppose that both a and b are joined with $S_1 \setminus u_1$ and $S_2 \setminus u_2$, all edges joining a and b with $S_i \setminus u_i$ are simple, and a is not joined with b. Then $\Sigma \setminus \langle v_1, v_2\rangle$ contains a parabolic subdiagram of type $\widetilde{A}_m$ for some m, $2 \le m \le 7$. If in addition there is a multiple edge joining a or b with $S_1 \setminus u_1$ and $S_2 \setminus u_2$, or a and b are joined, then $\Sigma \setminus v_1$ or $\Sigma \setminus v_2$ contains either a diagram of type H_3 with at least three neighbors, or a Lannér diagram of order 3, or a parabolic diagram of type B_3. Hence at least one of the nodes a and b, say a, is joined with one of the diagrams $S_i \setminus u_i$, say $S_1 \setminus u_1$, by multiple edges only. Thus, by Lemma 5.3, there are two possibilities: either a is joined with $S_1 \setminus u_1$ by a single multiple edge or it is not joined with $S_1 \setminus u_1$ (in the latter case a is joined with u_1).

Case 1. The node a is joined with $S_1 \setminus u_1$ by a single multiple edge au. It is not difficult to see that if $L_1 = \mathcal{L}_4^5$, then u is a leaf of L_1 different from v_1, and a is not joined with v_1 (otherwise we would have either a diagram of type H_3 with at least three bad neighbors, or a Lannér subdiagram of order 3, or a parabolic subdiagram). If au is a double edge, then $\langle a, L_1 \setminus u_1\rangle$ is a parabolic subdiagram of type $\widetilde{B}_4$. Suppose that au is a triple edge. The node a is not joined with u_1; otherwise $\langle u, a, u_1\rangle$ would be of type H_3 with at least three bad neighbors, which is impossible by Corollary 3.1. Consider now the diagram $\Sigma_1 = \langle L_1, a, v_2\rangle$. The node a can be joined only with v_2 (and only by a simple edge, according to Lemma 5.3). A computation shows that Σ_1 is superhyperbolic, whether or not the nodes a and v_2 are joined.

Case 2. The node a is not joined with $L_1 \setminus u_1$. Then a is joined with u_1 by a simple edge. As before, consider the diagram $\Sigma_1 = \langle L_1, a, v_2\rangle$. The node a can be joined only with v_1 and v_2 and only by simple edges. For each of the four pairs of diagrams L_1 and L_2, the diagram Σ_1 is superhyperbolic whether or not a is joined with v_1 and v_2. □

Lemma 5.7. *Any node of Σ is incident to at most one dashed edge.*

Proof. Suppose that $u \in \Sigma$ is a node incident to two or more dashed edges. By Lemma 1.3, if u is incident to more than two dashed edges, then $P(u)$ is a 7-face of a Coxeter polytope with at most eight facets, which is impossible by Corollary 3.1. Thus, u is incident to exactly two dashed edges, and $P(u)$ is a 7-polytope with nine facets, i.e., a product of two simplices whose diagram of missing faces is $\lfloor k, 9-k \rfloor_1$ ($k < 9$). Since the diagram of missing faces of this polytope cannot have large missing faces, it is of the form $\lfloor 4, 5 \rfloor_1$. By Lemma 3.4, this diagram has only one lifting, Θ_1 in Figure 1. Denote the nodes of Θ_1 as shown in Figure 6, and let v and w be the remaining nodes of Σ. Each of them is joined with u by a dashed edge and they are also joined with the subdiagram $\langle v_5, \dots, v_9 \rangle \subset \Sigma$.

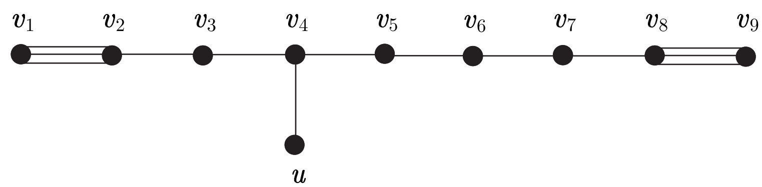


FIGURE 6. Unique lifting of $\lfloor 4, 5 \rfloor_1$

Case 1. The nodes v and w are not joined with the subdiagram $\Sigma \setminus \langle v, w, u \rangle = \langle v_1, \dots, v_9 \rangle$ by dashed edges.

Let $S_0 = \langle v_2, \dots, v_8 \rangle$. Then $P(S_0)$ is a 1-face, and therefore S_0 belongs to exactly two elliptic diagrams of order 8. This means that one of the nodes v and w, say v, either is not joined with S_0 or is a good neighbor of S_0. In particular, any double edge joining v with S_0 may end in v_2 or v_8 only. By Lemma 5.3, no double edge joins v and $\Sigma \setminus \langle v, w, u \rangle$. Hence v can be joined only with v_1, v_9, and one of the nodes of S_0 and only by simple edges. A straightforward calculation shows that the positive inertia index of any diagram $\Sigma \setminus \langle w, u \rangle$ thus obtained is different from 8, contrary to the assumption that Σ is the diagram of an 8-polytope.

Case 2. At least one of the nodes v and w, say v, is joined by a dashed edge with some node $v_x \in \Sigma \setminus \langle v, w, u \rangle$.

In this case, $P(v)$ is a 7-face of P bounded by nine facets, i.e., a face with diagram of missing faces of type $\lfloor 5, 4 \rfloor_1$. The only lifting of this diagram is the diagram Θ_1 (Lemma 3.4). In other words, $\Sigma \setminus \langle u, v_x \rangle$ looks like the diagram of Figure 6, where v takes the place of u, w takes the place of v_x, and, possibly, the node v is not joined with v_4 but is joined with v_6 (when $v_x \neq v_6$) or with w (when $v_x = v_6$).

If v_x and w are not joined by a dashed edge, we obtain a parabolic diagram $\langle v_{x-1}, v_x, v_{x+1}, w \rangle$ (when $v_x \neq v_1$ or v_9) or a Lannér diagram of order 3 (when $v_x = v_1$ or v_9). If v_x and w are joined by a dashed edge, consider the face $P(v_x)$. It is again a 7-polytope with nine facets, but in the lifting of its diagram of missing faces, the node v_x is joined with at least two nodes, contrary to Lemma 3.2. □

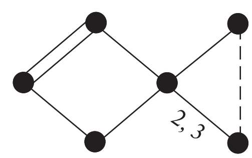
Lemma 5.8. *Σ contains at least two dashed edges.*

Proof. By Lemma 5.4, Σ has a dashed edge. Suppose it is unique. Then by Lemmas 5.2 and 5.3, Σ satisfies assertions (1)–(3) of Lemma 4.1. We shall show that this implies that Σ does not contain Lannér subdiagrams of order 5. By Lemmas 4.3, 5.3, and 1.2 this would prove the lemma.

Suppose that there is a Lannér diagram $L_0 \subset \Sigma$ of order 5. Let $S_0 \subset L_0$ be a subdiagram of type H_4 or F_4. Then $P(S_0)$ is a 4-polytope with at most seven facets and $\Sigma_{S_0} = \bar{S}_0$ (Corollary 1.1). The Coxeter diagram of any 4-polytope with seven facets contains either a Lannér subdiagram of order 3 or at least two dashed edges. The Coxeter diagram of any 4-polytope with six facets which is not a prism contains a Lannér subdiagram of order 3. Hence $P(S_0)$ is a either a Coxeter 4-prism or a Coxeter 4-simplex.

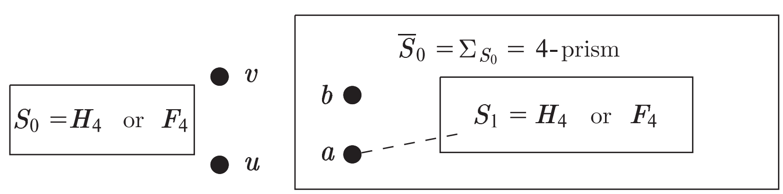
By Lemma 5.6, we may assume that $P(S_0)$ is a prism. Consider two cases.

Case 1. $P(S_0)$ is a prism with Coxeter diagram Σ_{S_0}, different from



where the label "2, 3" indicates that the corresponding nodes are either joined by a simple edge or are not joined.

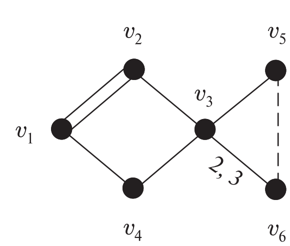
Examining the list of Coxeter diagrams of 4-prisms [13], we see that $\Sigma_{S_0} = \bar{S}_0$ contains a subdiagram S_1 of type H_4 or F_4 with an adjoined dashed edge. Since there is only one dashed edge in Σ, the diagram $\Sigma_{S_1} = \bar{S}_1$ of $P(S_1)$ contains no dashed edges, so $P(S_1)$ is a simplex. Notice that $S_0 \subset \bar{S}_1$. Let u and v be the neighbors of S_0, a the end of the dashed edge contained in S_1, and b the remaining node of Σ not contained in $\langle S_0, S_1\rangle$. Thus, Σ consists of the following parts:



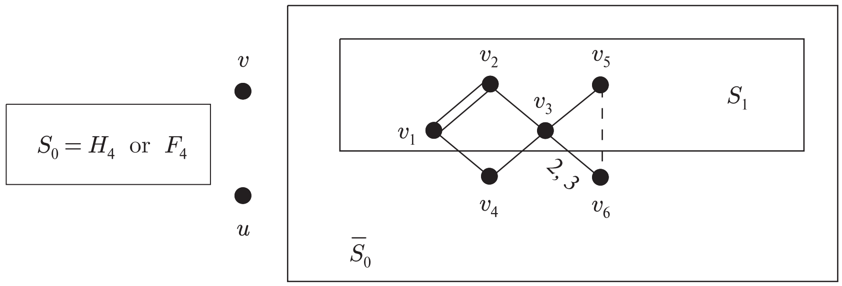
where u and v are neighbors of S_0, and S_0 is disjoint from $\bar{S}_0$. Since $P(S_1)$ is a simplex, b and one of the nodes u and v, say v, are joined with S_1, and $\langle S_0, u\rangle = \bar{S}_1$ is a Lannér diagram.

Consider the diagram $\Sigma \setminus \langle v, a\rangle$ consisting of the Lannér subdiagram $\langle S_0, u\rangle = \bar{S}_1$ and the indefinite subdiagram $\langle S_1, b\rangle$. These subdiagrams can be joined only by the edge $\langle u, b\rangle$, which cannot be dashed. It is not difficult to see that $\Sigma \setminus \langle v, a\rangle$ is superhyperbolic.

Case 2. $P(S_0)$ is a prism with Coxeter diagram Σ_{S_0} of the form



Let $S_1 \subset \Sigma_{S_0}$ be a subdiagram of type B_4, for example, $S_1 = \langle v_1, v_2, v_3, v_5 \rangle$. Then Σ consists of the following parts:



where u and v are neighbors of S_0, and S_0 is disjoint from $\bar{S}_0$. Notice that S_1 has at least two neighbors, namely, v_4 and v_6, and therefore $P(S_1)$ is either a prism, or an Esselmann polytope, or a simplex. Consider these three cases.

Case 2.1: $P(S_1)$ is an Esselmann polytope.

Then Σ_{S_1} contains two disjoint Lannér subdiagrams of order 3, whereas $\bar{S}_1$ contains no such diagrams. Hence Σ_{S_1} contains two nodes, which are good neighbors, in Σ, of the subdiagram S_1 of type B_4. By Corollary 1.2, this is impossible (see the list of Esselmann diagrams in [9]).

Case 2.2: $P(S_1)$ is a prism.

As before, u and v are neighbors of S_0. By assumption, there is only one dashed edge in Σ, and therefore one of the nodes u and v, say v, is a good neighbor of S_1, i.e., the diagram $\langle S_1, v \rangle$ is of type B_5. If u is also a good neighbor of S_1, we obtain either a parabolic subdiagram of Σ or a Lannér subdiagram of order 3. Thus, v is the only good neighbor of S_1.

Consider the diagram $\bar{S}_1 = \langle S_0, u, v \rangle$. Since v is the only node of $\bar{S}_1$ joined with S_1, the diagram $\bar{S}_1$ differs from Σ_{S_1} only by the multiplicities of the edges incident to v. By Proposition 1.11, any simple edge of $\bar{S}_1$ incident to v becomes a double edge in Σ_{S_1}, and any other edge of $\bar{S}_1$ incident to v becomes a dashed edge. Since $P(S_1)$ is a prism, Σ_{S_1} contains only one dashed edge. At the same time, no Coxeter diagram of a compact Coxeter 4-prism contains nodes incident to both a multiple edge and a dashed edge. Hence v is joined with exactly one node of the diagram $\bar{S}_1 \setminus v = \langle S_0, u \rangle$ (call it w), and $\bar{S}_1$ can be obtained from Σ_{S_1} by replacing the dashed edge by a double or a triple edge.

Recall that v is the only neighbor of S_1 contained in $\bar{S}_1$, and that it is joined with v_5 by a simple edge. Thus, we have either the parabolic diagram $\langle v_1, v_2, v_3, v_5, v, w \rangle$ or the subdiagram $\langle v_3, v_5, v, w \rangle$ of type H_4 with at least four neighbors.

Case 2.3: $P(S_1)$ is a simplex.

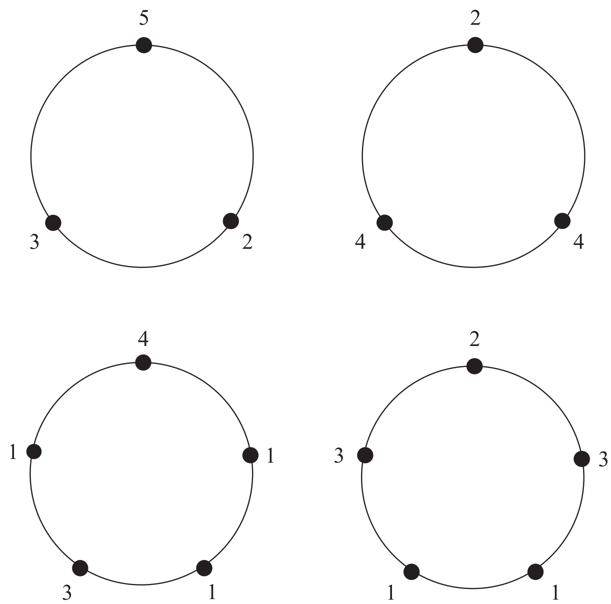
We use the same notation as in Case 1. The difference is that now one of the nodes u and v, say v, is a bad neighbor of S_1. There is only one way to attach a bad neighbor to S_1 such that no parabolic subdiagrams or Lannér subdiagrams of order 3 would be formed: join v with v_5 by a triple edge. But in that case we would have the subdiagram $\langle v_2, v_3, v_5, v \rangle$ of type H_4 with at least four neighbors.

This exhausts all the cases and proves the lemma. □

Theorem 2. *There are no compact hyperbolic Coxeter* 8*-polytopes with* 12 *facets.*

Proof. Suppose P is a compact hyperbolic Coxeter 8-polytope with 12 facets and Σ is its Coxeter diagram. By Lemma 5.4, Σ has a dashed edge $\langle v, w \rangle$. It follows from Lemma 5.7 that $P(v)$ is a 7-polytope with ten facets. By Lemma 5.8, $P(v)$ has a pair of disjoint facets.

The combinatorial structure of $P(v)$ is encoded by a 2-dimensional Gale diagram on ten nodes with a missing face of order 2 and without large missing faces. It is not difficult to check that it must be one of the following diagrams:



By Lemma 3.2, Σ contains either a 0- or a 1-lifting of one of these diagrams. By Lemma 5.3, Σ has no Lannér subdiagrams of order 3. Since P is an 8-polytope, the positive inertia index of Σ is 8. Thus Lemmas 3.7, 3.8, 3.10, and 3.11 apply, and we have that Σ contains $\Sigma(P_8)$. This contradicts Lemma 5.1 and completes the proof of the theorem. □

6. Polytopes in dimension 7

In this section we assume that Σ is the Coxeter diagram of a compact Coxeter 7-polytope with 11 facets, and prove that Σ coincides with Σ_{P_7}, where Σ_{P_7} is the diagram found by Bugaenko [5] and shown in Figure 7.

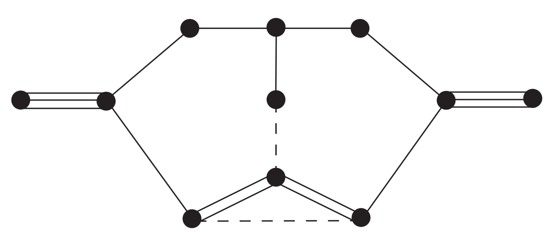


Figure 7. The only compact Coxeter 7-polytope with 11 facets

Theorem 3. *If Σ is the Coxeter diagram of a compact Coxeter 7-polytope with 11 facets, then $\Sigma = \Sigma_{P_7}$.*

The proof in general is similar to the proofs in larger dimensions but it is rather long and involved, with many cases to examine. We show that Σ contains at least one subdiagram of type F_4 or H_4, and finish the proof by presenting Lemmas 6.6–6.9, which deal with subdiagrams of type F_4 and H_4 and take up most of the proof.

We say that a Coxeter diagram satisfies the *signature condition* if it is admissible and its positive inertia index is at most 7.

Recall that if $u, v \in \Sigma$, then $\langle u, v\rangle = m$ (∞ or 2) means that u and v are joined by an edge of multiplicity $m-2$ (respectively, dashed or empty edge).

6.1. **Existence of subdiagrams of type F_4 or H_4.** Here we establish the following properties of Σ:

- any node of Σ is incident to at most one dashed edge (Lemma 6.2);
- Σ contains no subdiagrams of type $G_2^{(k)}$ with $k > 5$ (Lemma 6.4);
- Σ contains at least one subdiagram of type F_4 or H_4 (Lemma 6.5), and any such subdiagram has at least two bad neighbors (Lemma 6.3).

Recall that an elliptic subdiagram of Σ has at most three bad neighbors (Lemma 1.3).

Lemma 6.1. *A subdiagram of type $G_2^{(k)}$ with $k > 3$ has at most two bad neighbors.*

Proof. Suppose that $S_0 \subset \Sigma$ is a subdiagram of type $G_2^{(k)}$, $k > 3$, with three bad neighbors. Then $P(S_0)$ is a 5-simplex, which is impossible. □

Lemma 6.2. *Any node of Σ is incident to at most one dashed edge.*

Proof. Suppose that a node v is incident to two or more dashed edges. Let f be the facet of P corresponding to v. Then f is a (possibly non-Coxeter) 6-polytope with at most $6+2$ facets. By Corollary 3.1, f cannot be a simplex. Therefore, by Proposition 1.7, f is a product of two simplices, i.e., $\Delta^5 \times \Delta^1$, or $\Delta^4 \times \Delta^2$, or $\Delta^3 \times \Delta^3$. The first case is impossible, since Σ has no large missing faces. The remaining cases are impossible, since the diagrams $\lfloor 5,3\rfloor_1$ and $\lfloor 4,4\rfloor_1$ have no 0- and 1-liftings with positive inertia index smaller than 8 (see Lemma 3.6 and 3.11). □

Lemma 6.3. *Any subdiagram of Σ of type H_4 or F_4 has at least two neighbors.*

Proof. Suppose $S_0 \subset \Sigma$ is of type H_4 or F_4. Since Σ is connected, S_0 has at least one neighbor. Suppose that S_0 has exactly one neighbor, and call it a. Then $P(S_0)$ is a Coxeter 3-polytope with $3+3$ facets. A simple Coxeter 3-polytope with six facets can be either a cube or a frustum of a tetrahedron, i.e., a polytope with two triangular, two quadrilateral, and two pentagonal faces. The latter case is impossible for $P(S_0)$, since a triangular facet does not intersect the two other facets, which is impossible by Lemma 6.2 (we use the fact that $\Sigma_{S_0} = \bar{S}_0 \subset \Sigma$, since S_0 is of type H_4 or F_4). Thus, $P(S_0)$ is a cube. Let b_1 and b_2, c_1 and c_2, d_1 and d_2 be the ends of the dashed edges in $\bar{S}_0 = \Sigma_{S_0}$. By Lemma 1.4, a is joined with each of the dashed edges b_1b_2, c_1c_2, and d_1d_2. Assume that a is joined with b_1, c_1, and d_1.

Suppose that $[b_1, c_1] \geq 4$. Then the subdiagram $\langle b_1, c_1\rangle$ has at least three bad neighbors (b_2, c_2 and a), which is impossible by Lemma 6.1. Furthermore, $[b_1, c_1] \neq \infty$, since $P(S_0)$ is a cube. Hence $[b_1, c_1] = 2$ or 3. Similarly, $[b_1, d_1] \leq 3$ and $[c_1, d_1] \leq 3$.

Now suppose that $[a, b_1] \geq 4$. Let $S_1 = \langle a, b_1\rangle$. If $[a, b_1] \geq 6$, then S_1 has at least three bad neighbors (b_2, c_1, and d_1), contrary to Lemma 6.1. Hence $[a, b_1] = 4$ or 5. By Lemma 6.1, S_1 has at most two bad neighbors, and therefore one of the nodes c_1 and d_1 is a good neighbor of S_1 (recall that both c_1 and d_1 are joined with a). Assuming now that c_1 is a good neighbor of S_1, consider the diagram $S_2 = \langle S_1, c_1\rangle$ of type H_3 or B_3. The diagram S_2 has at least four bad neighbors, namely b_2, c_2, d_1 and one of the nodes of S_0 (since a is a neighbor of S_0), which is impossible by Lemma 1.3. The obtained contradiction shows that $[a, b_1] = 3$ ($[a, b_1] \neq \infty$ by Lemma 6.2). Similarly, $[a, c_1] = [a, d_1] = 3$.

Since Σ contains no parabolic subdiagrams and $[b_1, c_1] \le 3$, we have $[b_1, c_1] = 2$. Similarly, $[b_1, d_1] = [c_1, d_1] = 2$, and therefore $\langle a, b_1, c_1, d_1 \rangle$ is of type D_4. This diagram has at least four bad neighbors: b_2, c_2, d_2 and one of the nodes x_i, $1 \le i \le 4$. The obtained contradiction proves the lemma. □

Lemma 6.4. *Σ contains no subdiagrams of type $G_2^{(k)}$ with $k > 5$.*

Proof. Suppose that $S_0 \subset \Sigma$ is a subdiagram of type $G_2^{(k)}$, $k > 5$. Without loss of generality, we may assume that the edge of the subdiagram S_0 is of maximum multiplicity in Σ. Since Σ is connected, S_0 has at least one (obviously, bad) neighbor. By Corollary 1.3, S_0 has at most two neighbors. Consider two cases.

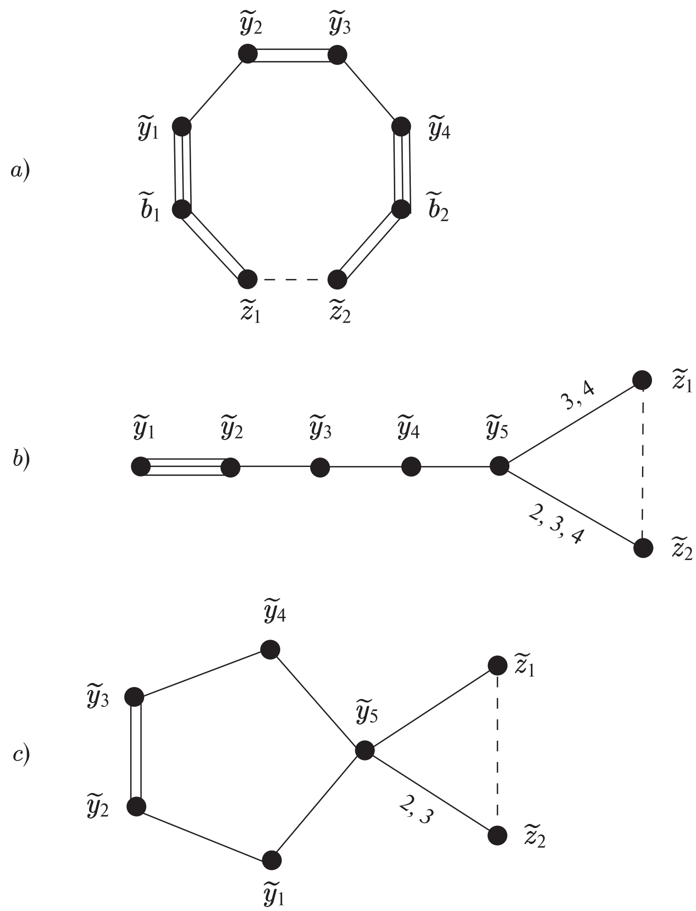


FIGURE 8. To the proofs of Lemmas 6.4 and 6.6

Case 1. Suppose that $S_0 = \langle x_1, x_2 \rangle$ has only one neighbor, and call it a. Then $P(S_0)$ is a 5-polytope with $5 + 3$ facets. By Corollary 1.1, $\Sigma_{S_0} = \bar{S}_0$. Hence, it follows from Lemma 6.2 that any node of Σ_{S_0} is incident to at most one dashed edge. The list of 5-polytopes with eight facets has only one diagram satisfying this condition. We reproduce that diagram in Figure 8, *a*) together with the notation for its nodes. By Lemma 1.4, a is joined with at least one of the nodes z_1 and z_2, say, z_1. Let $S_1 = \langle x_1, z_1 \rangle$. If a is a bad neighbor of S_1, then S_1 has three bad neighbors (y_1, z_2, a), which is impossible by Lemma 6.1. Thus, a is a good neighbor of S_1, which implies that $[a, z_1] = 3$

and that $S_2 = \langle a, S_1 \rangle$ is of type B_3. As a is a neighbor of S_0, we may assume that x_1 is joined with a. If x_1 is a good neighbor of $S_2 = \langle a, S_1 \rangle$, then $\langle x_1, S_2 \rangle$ has at least four bad neighbors (namely, x_2, z_2, y_1 and some node of the Lannér diagram $\langle b_2, y_4, y_3, y_2 \rangle$ joined with a. The last of the above neighbors is actually bad, since a is not a leaf of $\langle x_1, S_2 \rangle$). This is impossible by Lemma 1.3, so x_1 is a bad neighbor of S_2, and S_2 has three bad neighbors (x_1, y_1, z_2). Hence $P(S_2)$ is a 4-simplex and Σ_{S_2} is a Lannér diagram of order 5. By Corollary 1.2, $\bar{S}_2$ is also a Lannér diagram of order 5. At the same time, $\bar{S}_2 = \langle x_2, b_2, y_4, y_3, y_2 \rangle$ cannot be a Lannér diagram, since it contains the Lannér subdiagram $\langle b_2, y_4, y_3, y_2 \rangle$. The obtained contradiction shows that S_0 has exactly two bad neighbors.

Case 2. Suppose that $S_0 = \langle x_1, x_2 \rangle$ has two bad neighbors, a_1 and a_2. Then $P(S_0)$ is a 5-prism. Such a prism can be one of the two types shown in Figure 8, *b*), *c*). We denote the nodes of $\bar{S}_0$ as is shown in that figure. Let S_1 denote the subdiagram $\langle y_1, y_2, y_3, y_4 \rangle$ of type F_4 or H_4. Since $\bar{S}_1 = \Sigma_{S_1}$ contains the subdiagram S_0 of type $G_2^{(k)}$, $k > 5$, the diagram $\bar{S}_1$ cannot be a Lannér diagram of order 4. Therefore, the face $P(S_1)$ cannot be a 3-simplex. Hence S_1 has at most two neighbors, i.e., at least one of the nodes a_1 and a_2 is not joined with S_1. We may assume that a_1 is not joined with S_1. By Lemma 1.4, we then conclude that a_1 is joined with y_5 (since y_5 belongs to the Lannér diagram $\langle y_1, y_2, y_3, y_4, y_5 \rangle \subset \bar{S}_0$). Now we consider the cases $S_1 = F_4$ and $S_1 = H_4$ separately.

In the case $S_1 = F_4$, consider the subdiagrams $\langle y_2, y_3, y_4, y_5, z_1 \rangle$ and $\langle y_3, y_2, y_1, y_5, z_1 \rangle$ of type B_5. Each of these subdiagrams has three bad neighbors (a_1, y_1, z_2 and a_1, y_4, z_2, respectively), and therefore a_2 is not a bad neighbor of either of these diagrams. Hence a_2 cannot be joined with the Lannér diagram $\langle S_1, y_5 \rangle$, contrary to Lemma 1.4.

Consider now the case $S_1 = H_4$. Since a_1 is not a neighbor of S_1, a_2 is a neighbor of S_1; otherwise S_1 would have just one neighbor, which is impossible by Lemma 6.3. Thus, S_1 has two bad neighbors, y_5 and a_2, and $P(S_1)$ is a 3-prism. This means that the diagram $\bar{S}_1 = \Sigma_{S_1}$ consists of the dashed edge $\langle z_1, z_2 \rangle$ and the Lannér diagram $\langle S_0, a_1 \rangle$ of order 3. Hence the diagram $X = \Sigma \setminus \langle z_1, z_2 \rangle = \langle S_0, a_1, S_1, y_5 \rangle$ consists of the Lannér diagrams $\langle S_0, a_1 \rangle$ and $\langle S_1, y_5 \rangle$ joined only by the edge $a_1 y_5$. If this edge is not dashed and $[a_1, y_5] \neq 5$, then X is superhyperbolic. Consider the cases $[a_1, y_5] = 5$ and $[a_1, y_5] = \infty$.

Case 2.1. Suppose that $[a_1, y_5] = 5$. Then $\langle a_1, y_5, y_4, y_3 \rangle$ is of type H_4 and has three bad neighbors, namely z_1, y_2 and one of the nodes x_1 and x_2, say, x_1. Hence z_2 is not a neighbor of $\langle a_1, y_5, y_4, y_3 \rangle$. In particular, $[z_2, a_1] = 2$, whence (Lemma 1.4) $[a_1, z_1] \neq 2$. Hence z_1 is a bad neighbor of $S_2 = \langle a_1, y_5 \rangle$, and therefore Σ_{S_2} is the diagram of a 5-polytope with at most eight facets. Consider Σ_{S_2}. By Proposition 1.11, the subdiagram $S_1 = \langle y_1, y_2, y_3, y_4 \rangle$ of Σ becomes, in Σ_{S_2}, a linear diagram of order 4 with a triple edge $\widetilde{y}_1 \widetilde{y}_2$, simple edge $\widetilde{y}_2 \widetilde{y}_3$ and an edge $\widetilde{y}_3 \widetilde{y}_4$ labeled by 10. But no diagram of a compact 5-polytope with eight facets contains such a subdiagram. Thus, the case $[a_1, y_5] = 5$ is impossible.

Case 2.2. Now suppose that $[a_1, y_5] = \infty$. Consider the subdiagram $S_2 = \langle y_1, y_2, y_3 \rangle$ of type H_3. If a_2 is a bad neighbor of S_2, then $P(S_2)$ is a 4-polytope with $4 + 3 = 7$ facets; moreover, Σ_{S_2} contains a subdiagram $\langle \widetilde{x}_1, \widetilde{x}_2 \rangle$ of type $G_2^{(k)}$ with $k > 5$ and at least two dashed edges ($\widetilde{z}_1 \widetilde{z}_2$ and $\widetilde{y}_4 \widetilde{y}_5$). But no 4-polytope with seven facets has these properties. Therefore, a_2 is not a bad neighbor of S_2. If a_2 is a good neighbor of S_2, then the diagram $\langle y_1, y_2, y_3, a_2 \rangle$ of type H_4 has at least four bad neighbors (namely, y_4, at least one of the nodes x_1 and x_2, at least one of the nodes z_1 and z_2, and at least one of the nodes a_1 and y_5). Hence a_2 is not a neighbor of S_2, and, by Lemma 6.3, a_2 is joined with y_4. Consider the subdiagram $S_3 = \langle y_2, y_3, y_4, y_5, z_1 \rangle$ of type A_5 or B_5.

It has three bad neighbors (y_1, a_1, z_2), and therefore a_2 is a good neighbor of S_3. Thus $\langle a_2, S_3\rangle$ is of type E_6. But then $\langle a_2, S_3\rangle$ has at least four bad neighbors (y_1, z_2, a_1, and one of the nodes x_1 and x_2). This contradicts Lemma 1.3 and completes the proof of the lemma. □

Lemma 6.5. Σ *contains at least one subdiagram of type* F_4 *or* H_4.

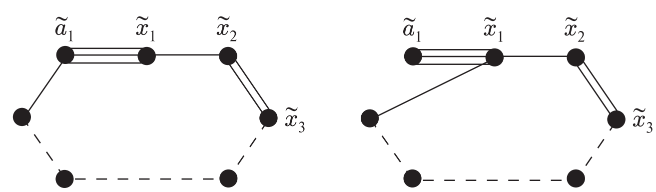
Proof. Suppose that Σ contains no subdiagrams of type F_4 or H_4.

Suppose that Σ has a subdiagram $S_0 = \langle x_1, x_2\rangle$ of type $G_2^{(4)}$ or $G_2^{(5)}$ with a bad neighbor. Then $P(S_0)$ is a 5-polytope with at most eight facets. Hence Σ_{S_0} contains a subdiagram of type F_4 or H_4. Corollary 1.2 implies that $\bar{S}_0$ also contains a subdiagram of that type, which is impossible by assumption.

It now follows (in view of Lemma 6.4) that Σ has no Lannér diagrams of order 3. Since any Lannér diagram of order 5 contains a subdiagram of type F_4 or H_4, Σ does not contain Lannér subdiagrams of order 5. By [8, Prop. 6.9], any simple polytope of dimension $d > 4$ with $d + 4$ facets has at least one missing face of order greater than two. Thus, Σ contains a Lannér subdiagram L of order 4. Let $S_0 \subset L$ be a subdiagram of type H_3 or B_3. Set $a_1 = L \setminus S_0$ and $S_0 = \langle x_1, x_2, x_3\rangle$.

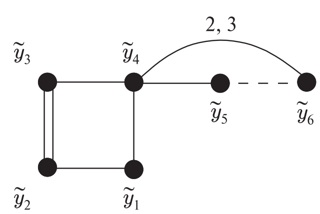
Suppose that S_0 has three bad neighbors. Then $P(S_0)$ is a 4-simplex, and Σ_{S_0} (and therefore $\bar{S}_0$) contains a subdiagram of type F_4 or H_4, which is impossible by assumption.

Now suppose that S_0 has only one bad neighbor, a_1. Then $P(S_0)$ is a 4-polytope with $4 + 3$ facets. Corollary 1.2 implies that its diagram contains no subdiagrams of type F_4 and H_4. Any diagram of a 4-polytope with $4 + 3$ facets that does not contain subdiagrams of type F_4, H_4, and $G_2^{(k)}$ $(k > 5)$ contains a subdiagram $\langle \widetilde{y}_1, \widetilde{y}_2\rangle$ of type $G_2^{(4)}$ or $G_2^{(5)}$ with a bad neighbor. If S_0 is of type H_3, then it has no good neighbors (since Σ has no subdiagrams of type H_4), and therefore $\bar{S}_0 = \Sigma_{S_0}$ contains a subdiagram $S_1 = \langle y_1, y_2\rangle$ of type $G_2^{(4)}$ or $G_2^{(5)}$ with a bad neighbor. As we have seen before, this is impossible. Therefore, S_0 is of type B_3. Using Corollary 1.1, we have that either $S_1 \subset \bar{S}_0$ is a multiple edge with a bad neighbor (which is impossible) or Σ_{S_0} is one of the following diagrams:



In this case the double edge $\langle \widetilde{y}_1, \widetilde{y}_2\rangle$ may become a simple edge of $\bar{S}_0$, which would yield a subdiagram $\langle y_1, y_2, y_3, y_4\rangle$ of type H_4 in $\bar{S}_0 \subset \Sigma$, which is also impossible. Hence the multiple edge $\langle \widetilde{y}_1, \widetilde{y}_2\rangle$ becomes the multiple edge $\langle y_1, y_2\rangle \subset \bar{S}_0$, and Corollary 1.1 implies that the bad neighbor of that edge remains bad in $\bar{S}_0$, which is also impossible. Hence S_0 cannot have three bad neighbors.

Thus, S_0 has exactly two bad neighbors, a_1 and a_2, and $P(S_0)$ is an Esselmann polytope or a 4-prism. Since Σ contains no subdiagrams of type F_4 and H_4, we see, using Corollary 1.2, that Σ_{S_0} coincides with the following diagram:



By Corollary 1.2, the nodes of Σ_{S_0} (with a possible exception of $\widetilde{y}_6$ in the case $[\widetilde{y}_4, \widetilde{y}_6] = 2$) cannot be good neighbors of S_0. In particular, $\bar{S}_0$ contains a cyclic Lannér diagram of order 4 with exactly one double edge and without subdiagrams of type H_3. Furthermore, it is not difficult to check that the dashed edge $\langle \widetilde{y}_5, \widetilde{y}_6 \rangle$ of Σ_{S_0} becomes the dashed edge $\langle y_5, y_6 \rangle$ in $\bar{S}_0$ (since otherwise $[y_4, y_6] = 2$ and if $y_5 y_6$ is a triple edge, then $[y_4, y_6] = 2$ and $\langle y_3, y_4, y_5, y_6 \rangle \subset \bar{S}_0$ is a subdiagram of type H_4, which is impossible by assumption; if $y_5 y_6$ is a double edge, then $\langle y_2, y_3, y_4, y_5, y_6 \rangle \subset \Sigma$ is a parabolic subdiagram of type $\widetilde{C}_4$, which is also impossible; if $y_5 y_6$ is a simple edge, then $\langle y_5, y_6, S_0 \rangle$ is of type B_5 and, by Proposition 1.11, $\widetilde{y}_5 \widetilde{y}_6$ must be a double edge, not dashed, in Σ_{S_0}). Thus, $\bar{S}_0$ consists of a dashed edge and a cyclic Lannér subdiagram of order 4, and $\bar{S}_0 = \Sigma_{S_0}$.

Consider the diagram $S_1 = \langle y_1, y_2, y_3 \rangle$, which is a subdiagram of type B_3 in the Lannér diagram $\langle y_1, y_2, y_3, y_4 \rangle$ of order 4. Arguing as for $\bar{S}_0$, we have that S_1 has exactly two bad neighbors (y_4 and one of the nodes a_1 and a_2) and that $\bar{S}_1$ consists of a dashed edge $y_5 y_6$ and a cyclic Lannér diagram ($\langle S_0, a_2 \rangle$ or $\langle S_0, a_1 \rangle$, respectively). Without loss of generality, we may assume that a_2 is a bad neighbor of S_1 (and $\langle a_1, S_0 \rangle$ is a cyclic Lannér diagram). By Corollary 1.2, a_1 cannot be a good neighbor of S_1 and, by Lemma 1.4, a_1 is joined with y_4. If $[a_1, y_4] = 3$ or 4, then $\langle x_1, x_2, a_1, y_2, y_3, y_4 \rangle$ contains a parabolic subdiagram of type $\widetilde{B}_5$ or $\widetilde{B}_3$. If $[a_1, y_4] = 5$, then $\langle x_2, x_3, a_1, y_4 \rangle$ is of type H_4. Therefore, $[a_1, y_4] = \infty$.

Consider the diagram $S_2 = \langle y_2, y_3, y_4 \rangle$. Since it has two bad neighbors (a_1 and y_1), the node a_2 cannot be a bad neighbor of S_2 (we are applying the results proved for the diagram S_0 to the diagram S_2). Hence a_2 is joined with y_1, and, repeating the foregoing arguments, we have $[a_2, y_1] = \infty$. But then the diagram $\langle y_1, y_3, y_4, y_5 \rangle$ of type D_4 has four bad neighbors, namely a_1, a_2, y_2, y_6, which is impossible. □

6.2. **Subdiagrams of type F_4 and H_4.** Thus, Σ contains at least one subdiagram of type F_4 or H_4. In Lemmas 6.6 and 6.7 we shall prove that such subdiagrams always have three neighbors. Next, in Lemma 6.8 we shall show that Σ contains no subdiagrams of type F_4. Thus Σ contains a subdiagram of type H_4, and we shall finish the proof of Theorem 3 by Lemma 6.9, which shows that $\Sigma = \Sigma_{P_7}$. We remark that the proofs of Lemmas 6.7 and 6.9, dealing with subdiagrams of type H_4, are much longer than the proofs of the similar Lemmas 6.6 and 6.8 dealing with subdiagrams of type F_4. A possible reason for that is the fact that H_4 is contained in many more diagrams of d-polytopes with at most $d+3$ facets than F_4.

Lemma 6.6. *Any subdiagram of type F_4 has exactly three neighbors.*

Proof. Assume the contrary. By Lemma 6.3, this means that Σ contains a subdiagram S_0 of type F_4 with two neighbors. Then $P(S_0)$ is a 3-prism, and $\Sigma_{S_0} = \bar{S}_0$ consists of a dashed edge (denote it $z_1 z_2$) and a Lannér subdiagrams of order 3 (denote it $L = \langle x_1, x_2, x_3 \rangle$) containing a multiple edge. Assuming that an edge $x_1 x_2$ is of maximal multiplicity in L, let $S_1 = \langle x_1, x_2 \rangle$. This diagram has at least one bad neighbor, x_3. By Lemma 6.1, S_1 has either one or two bad neighbors.

Suppose that S_1 has only one bad neighbor x_3. Then $P(S_1)$ is a 5-polytope with $5+3 = 8$ facets. The diagram S_0 is of type F_4 and is not joined with S_1. Therefore, Σ_{S_1} contains a subdiagram of type F_4, and Σ_{S_1} is the diagram shown in Figure 8, *a*). Let y_1, y_2, y_3, y_4 be the nodes of $S_0 \subset \bar{S}_1$, and b_1, b_2 the neighbors of S_0. By Corollary 1.2, the nodes b_1 and b_2 cannot be good neighbors of S_1. Since the Lannér diagram $\langle S_1, x_3 \rangle$ must be joined with the Lannér diagram $\langle b_1, y_1, y_2, y_3 \rangle$, the node x_3 is joined with b_1. Similarly, x_3 is joined with b_2. Consider the diagram $S_2 = \langle b_1, y_1, y_2 \rangle$ of type H_3. It has three bad neighbors (y_3, x_3, and z_1) and no good ones. Therefore, $P(S_2)$ is a 4-simplex and $\bar{S}_2 = \Sigma_{S_2} = \langle y_4, b_2, z_2, x_1, x_2 \rangle$ is a Lannér diagram of order 5. Since $y_4 b_2$ and $x_1 x_2$ are disjoint multiple edges, $\bar{S}_2$ is a linear Lannér diagram (with edges labeled by 5, 3, 3, 4 or 5, 3, 3, 5). Since $[y_4, b_2] = 5$, the subdiagram $\langle y_4, b_2, z_2, x_1\rangle$ (or $\langle y_4, b_2, z_2, x_2\rangle$) is of type H_4. Thus, the simple edge b_2z_2 of Σ becomes a double edge $\widetilde{b_2}\widetilde{z_2}$ in Σ_{S_1}. By Corollary 1.1, this means that $[x_1, x_2] = 4$. Since x_1x_2 is of maximum multiplicity in the Lannér diagram $\langle x_1, x_2, x_3\rangle$, we see that x_3 is joined with both x_1 and x_2. Hence the diagram $\langle y_4, b_2, z_2, x_1\rangle$ (or $\langle y_4, b_2, z_2, x_2\rangle$) is of type H_4 with at least four neighbors: y_3, x_2 (or x_1), x_3, and z_1, which is impossible.

The obtained contradiction shows that S_1 has two bad neighbors, namely x_3 and some other node a_1, i.e., $P(S_1)$ is a 5-prism, containing a subdiagram of type F_4 (see Figure 8, *c*) for the notation). Recall that $a_1 \notin \bar{S}_1$, and therefore a_1 is a bad neighbor of $S_0 = \langle y_1, y_2, y_3, y_4\rangle$. Consider the diagrams $\langle y_2, y_3, y_4, y_5, z_1\rangle$ and $\langle y_3, y_2, y_1, y_5, z_1\rangle$ of type B_5. The node a_1 is a bad neighbor of at least one of these diagrams, say $\langle y_2, y_3, y_4, y_5, z_1\rangle$. Furthermore, by Lemma 1.4, the node y_5 is joined with the Lannér subdiagram $\langle x_1, x_2, x_3\rangle \subset \bar{S}_0$. This means that $\langle y_2, y_3, y_4, y_5, z_1\rangle$ has at least four bad neighbors (y_1, a_1, z_2, and one of the nodes x_1, x_2, x_3). The obtained contradiction completes the proof of the lemma. □

Lemma 6.7. *Any subdiagram of type H_4 has exactly three neighbors.*

Proof. Assume the contrary. By Lemma 6.3, this means that Σ has a subdiagram S_0 of type H_4 with two neighbors. Then $P(S_0)$ is a 3-prism, and $\Sigma_{S_0} = \bar{S}_0$ consists of a dashed edge (call it z_1z_2) and a Lannér subdiagram of order 3 (denote it $L = \langle y_1, y_2, y_3\rangle$) containing a multiple edge. Assuming that the edge y_1y_2 is of maximum multiplicity in L, let $S_1 = \langle y_1, y_2\rangle$. By Lemma 6.1, S_1 can have one or two neighbors.

Case 1. Suppose that S_1 has only one bad neighbor, y_3. Then $P(S_1)$ is a 5-polytope with $5 + 3 = 8$ facets. Since Σ_{S_1} contains a subdiagram S_0 of type H_4, the diagram Σ_{S_1} has one of the three types *a*), *b*), and *c*) shown in Figure 9. Let x_1, x_2, x_3, x_4 be the nodes of $S_0 \subset \bar{S}_1$, and a_1, a_2 the neighbors of S_0 (see Figure 9). We now examine the types *a*), *b*), and *c*) of Σ_{S_1} separately.

Case 1.1. Suppose that Σ_{S_1} is the diagram shown in Figure 9, *a*). Recall that z_1z_2 is a dashed edge of Σ, and therefore Lemma 6.2 implies that a_1 is a good neighbor of S_1. Without loss of generality, we may assume that $[a_1, y_1] = 3$ and $[a_1, y_2] = 2$. On the other hand, a_2 is not a good neighbor of $S_1 = \langle y_1, y_2\rangle$, since $[a_2, z_2] = 5$ (see Corollary 1.2). Therefore, a_2 is joined with y_3 (otherwise the Lannér diagrams $\langle x_1, x_2, x_3, x_4, a_2\rangle$ and $L = \langle S_1, y_3\rangle$ would not be joined). Notice that the diagram $S_2 = \langle z_2, a_2, x_4, x_3\rangle$ of type H_4 has three neighbors (x_2, z_1, y_3), and therefore $P(S_2)$ is a 3-simplex and $\bar{S}_2 = \langle x_1, a_1, y_1, y_2\rangle$ is a Lannér diagram. Furthermore, $\bar{S}_2$ is a linear Lannér diagram, since x_1 is not joined with $\langle y_1, y_2\rangle$, and a_1 is not joined with y_2 and is joined with y_1 by a simple edge. Hence $[x_1, a_1] = 4$ or 5. Consider the diagram $S_3 = \langle x_1, a_1, y_1\rangle$ of type H_3 or B_3. It has three bad neighbors (x_2, z_1, y_2), and therefore $P(S_3)$ is a 4-simplex and Σ_{S_3} is a Lannér diagram of order 5. Thus, by Corollary 1.2, $\bar{S}_0$ is also a Lannér diagram. However, in $\bar{S}_3 = \langle x_3, x_4, a_2, z_2, y_3\rangle$ the valency of the end a_2 of the triple edge z_2a_2 cannot be less than 3, which is impossible in a Lannér diagram of order 5.

Case 1.2. Suppose that Σ_{S_1} is one of the diagrams shown in Figure 9, *b*) or *c*). Then neither a_1 nor a_2 can be a good neighbor of S_1 (Corollary 1.2). Hence, by Lemma 6.2, both z_1 and z_2 are good neighbors of S_1 (recall that z_1z_2 is a dashed edge of Σ). By Lemma 6.4, $[a_i, z_i] = 3$, 4, or 5 ($i = 1, 2$). If $[a_1, z_1] = 5$, then the diagram $\langle z_1, a_1, x_4, x_3\rangle$ is of type H_4 and has at least four neighbors (x_2, z_2, a_2, and at least one of the nodes y_1 and y_2). If $[a_1, z_1] = 4$, then the diagram $\langle z_1, a_1, x_4, x_3, x_2\rangle$ is of type B_5 and has at least four bad neighbors (x_1, z_2, a_2, and at least one of the nodes y_1 and y_2). Finally, if

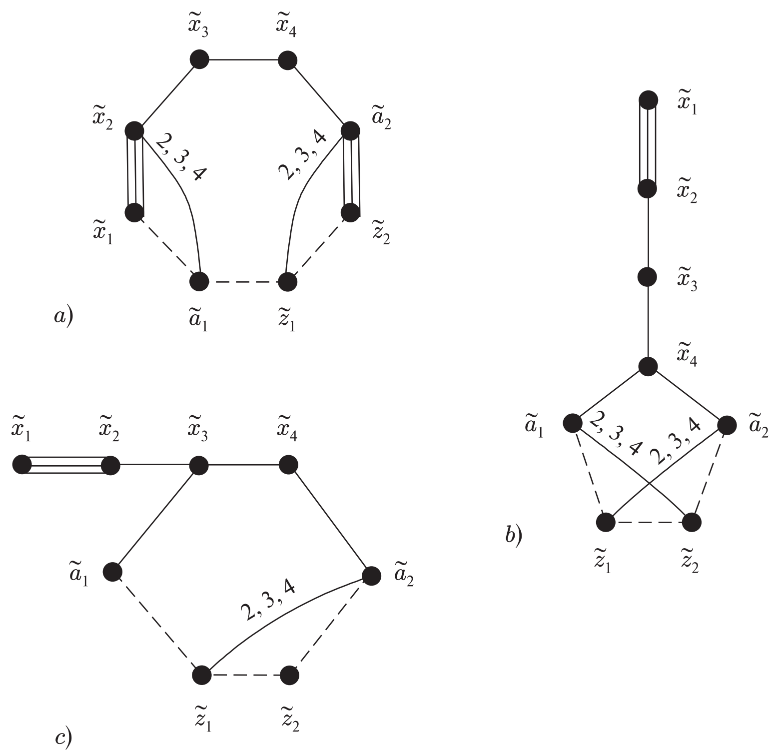
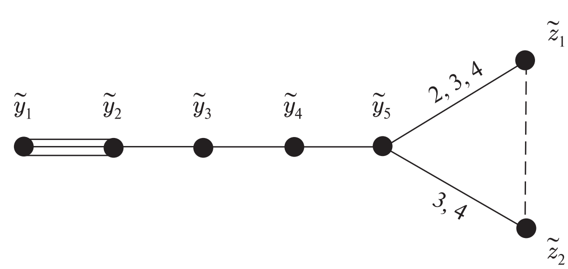


FIGURE 9. Possibilities for Σ_{S_1} (Case 1)

$[a_1, z_1] = 3$, then the edge $\widetilde{a}_1\widetilde{z}_1$ of Σ_{S_1} should be either a double edge or an edge labeled by 10, but not a dashed edge.

Case 2. Suppose that S_1 has exactly two bad neighbors: y_3 and one of the nodes a_1 and a_2, say a_1. Therefore, $P(S_1)$ is a 5-prism, whose diagram contains a subdiagram of type H_4, i.e., Σ_{S_1} is of the form



By Corollary 1.2, a_2 cannot be a good neighbor of S_1, and therefore (Lemma 1.4) a_2 is joined with y_3. If $[a_2, y_3] \neq \infty$, then the diagram $\langle S_0, a_2, y_3, S_1\rangle$ is superhyperbolic (since Σ contains no subdiagrams of type $G_2^{(k)}$ with $k > 5$ and y_1y_2 has maximum multiplicity in $L = \langle y_1, y_2, y_3\rangle$). Hence $[a_2, y_3] = \infty$. Set $S_4 = \langle x_1, x_2, x_3\rangle$ and consider three cases: either the node a_1 is a bad neighbor of S_4, or a_1 is a good neighbor of S_4, or a_1 is not joined with S_4.

Case 2.1. If a_1 is a bad neighbor of S_4, then $P(S_4)$ is a 4-polytope with $4+3$ facets, and Σ_{S_4} contains at least three dashed edges, namely $\widetilde{z}_1\widetilde{z}_2$, $\widetilde{a}_2\widetilde{y}_3$, and $\widetilde{x}_4\widetilde{a}_2$. But no diagram of a 4-polytope with $4+3$ facets has more than three dashed edges; moreover, in any diagram with exactly three dashed edges, each dashed edge has a node incident to some other dashed edge. However, the edge $\widetilde{z}_1\widetilde{z}_2$ has no nodes incident to $\widetilde{a}_2\widetilde{y}_3$ and $\widetilde{x}_4\widetilde{a}_2$, a contradiction.

Case 2.2. Let a_1 be a good neighbor of S_4. Consider the diagram $S_5 = \langle x_1, x_2, x_3, a_2\rangle$ of type H_4. It has three bad neighbors: x_4, one of y_1 and y_2 (say, y_1), and one of z_1 and z_2. Hence y_2 cannot be a neighbor of S_5 and, in particular, $[a_1, y_2] = 2$ and $[a_1, x_1] = 2$. On the other hand, a_1 is a bad neighbor of y_1y_2, and therefore $[a_1, y_1] \in \{4, 5, \infty\}$.

Consider the diagram $S_6 = \langle x_2, x_3, x_4, a_2, z_1\rangle$ of type A_5 or B_5. Since a_1 is a neighbor of $S_0 = \langle x_1, x_2, x_3, x_4\rangle$ and, as we have shown before, $[a_1, x_1] = 2$, we see that a_1 is a neighbor of S_6. Since S_6 already has three bad neighbors (x_1, z_2, y_3), the node a_1 is a good neighbor of S_6. Hence $\langle a_1, S_6\rangle$ is a diagram of type D_6 with three bad neighbors x_1, z_2, y_3. However, $[a_1, y_1] = 4$, 5, or ∞, and therefore y_1 is also a bad neighbor of $\langle a_1, S_6\rangle$, which is impossible.

Case 2.3. If a_1 is not a neighbor of $S_4 = \langle x_1, x_2, x_3\rangle$, then a_1 is joined with x_4 (as a neighbor of $S_0 = \langle x_1, x_2, x_3, x_4\rangle$). Consider the diagram $S_7 = \langle x_2, x_3, x_4, a_2, z_1\rangle$ of type A_5 or B_5. It has three bad neighbors, x_1, z_2, y_3. Hence a_1 is a good neighbor of S_7, and $\langle S_7, a_1\rangle$ is of type E_6. However, $\langle S_7, a_1\rangle$ has four bad neighbors: x_1, z_2, y_3, and one of y_1 and y_2 (since, by assumption, a_1 is a bad neighbor of S_1).

Thus S_1 cannot have two bad neighbors, and the lemma is proved. □

Lemma 6.8. *Σ has no subdiagrams of type F_4.*

Proof. Let $S_0 = \langle x_1, x_2, x_3, x_4\rangle \subset \Sigma$ be a subdiagram of type F_4. By Lemma 6.6, S_0 has exactly three neighbors; call them a_1, a_2, and a_3. Then $P(S_0)$ is a 3-simplex, and $\bar{S}_0 = \Sigma_{S_0} = \langle y_1, y_2, y_3, y_4\rangle$ is a Lannér diagram of order 4. Let $S_1 = \langle y_1, y_2, y_3\rangle \subset \bar{S}_0$ be a subdiagram of type H_3 or B_3, and $\langle S_2, \bar{S}_0\rangle$ another such subdiagram (it exists by Lemma 1.5); we may assume that $S_2 = \langle y_2, y_3, y_4\rangle$.

The diagram S_1 has at least one bad neighbor, y_4. Consider the three cases when S_1 has one, two, or three bad neighbors, respectively.

Case 1. Suppose that S_1 has only one bad neighbor y_4. Then $P(S_1)$ is a 4-polytope with $4 + 3$ facets.

Assume in addition that each of the diagrams $\langle a_i, S_0, \bar{S}_0\rangle$, $i \in \{1, 2, 3\}$, contains a dashed edge. Then each a_i $(i = 1, 2, 3)$ is incident to a dashed edge. Consider the other ends of those dashed edges. If S_0 contains three ends of dashed edges, then Σ_{S_1} contains three dashed edges with pairwise distinct ends (since $S_0 \subset \Sigma_{S_1}$ and $a_i \in \Sigma_{S_1}$ for all $i = 1, 2, 3$). However, there are no such diagrams of 4-polytopes with seven facets. Therefore, at least one of the ends of the dashed edges belongs to $\bar{S}_0$. Since $a_i \in \Sigma_{S_1}$ $(i = 1, 2, 3)$, the diagram S_1 contains no end of a dashed edge. Hence y_4 is the only end of a dashed edge in $\bar{S}_0$. Then $S_2 = \langle y_2, y_3, y_4\rangle$ has at least two bad neighbors (namely, y_1 and some node a_j such that $[a_j, y_4] = \infty$). If S_2 has exactly two bad neighbors, then Σ_{S_2} is either the diagram of a 4-prism or the diagram of an Esselmann polytope. However, Σ_{S_2} contains two dashed edges, which is impossible. If S_2 has three bad neighbors, then Σ_{S_2} is the diagram of a 4-simplex, but Σ_{S_2} contains at least one dashed edge.

Thus, we may assume that the diagram $\langle a_1, S_0, \bar{S}_0\rangle$ contains no dashed edges. By Lemma 6.4, we have only finitely many possibilities for the diagram $\langle a_1, S_0, \bar{S}_0\rangle$. The only two diagrams satisfying the signature condition and the assumption that S_1 has only one bad neighbor are shown in Figure 10.

For the diagram shown in Figure 10, a), consider the subdiagram $S_3 = \langle a_1, y_4\rangle$ with two bad neighbors. Then Σ_{S_3} is the diagram of a 5-prism. However, Σ_{S_3} contains the subdiagram $\langle\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{y}_1, \widetilde{y}_2\rangle$ of type $B_3 + B_2$, which is not a subdiagram of the diagram of any 5-prism.

Consider now the diagram shown in Figure 10, b). The subdiagram $S_4 = \langle x_4, a_1, y_3, y_2\rangle$ is of type H_4 and has three bad neighbors, x_3, y_4, and y_1. Hence a_2 and a_3 are not neighbors of S_4.

Suppose that $[a_2, y_4] \neq 0$. Then $S_5 = \langle a_1, y_3, y_4\rangle$ is of type H_3 and has three bad neighbors (x_4, y_2, and a_2). Therefore $[a_3, y_4] = 2$ and $[a_3, y_1] \neq 0$ (apply Lemma 1.4 to the Lannér diagram $L = \langle y_1, y_2, y_3, y_4\rangle$). Furthermore, $P(S_5)$ is a simplex and Σ_{S_5} is a Lannér diagram of order 5. By Corollary 1.2, the diagram $\bar{S}_5$ is also Lannér. Since $\bar{S}_5$ contains a subdiagram S_0 of type F_4, we have that $\bar{S}_0$ is a cyclic Lannér subdiagram and $[a_3, x_1] = [a_3, x_4] = 3$. However, in this case the diagram $\langle x_1, a_3, x_4, a_1\rangle$ is of type H_4 and has at least four neighbors (x_2, x_3, y_3, and y_1, which is joined with a_3). The obtained contradiction shows that $[a_2, y_4] = 0$. Similarly, $[a_3, y_4] = 0$.

By Lemma 1.4, the nodes a_2 and a_3 are joined with the Lannér diagram $\langle y_1, y_2, y_3, y_4\rangle$, and therefore a_2 and a_3 are joined with y_1. However, $\langle a_1, y_3, y_2, y_1\rangle$ is a diagram of type B_4 and has four bad neighbors, (a_2, a_3, x_4 and y_4), which is impossible.

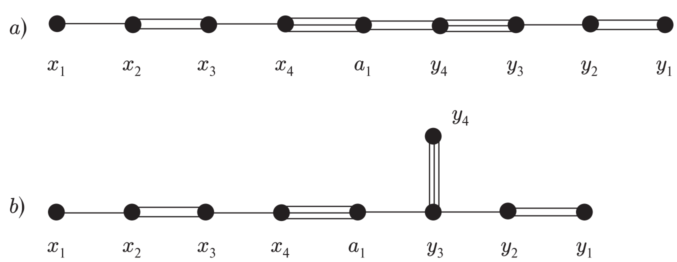


FIGURE 10. Case 1: two possibilities for $\langle a_1, S_0, \bar{S}_0\rangle$

Case 2. Suppose that S_1 has two bad neighbors, y_4 and a_1. Then $P(S_1)$ is a either an Esselmann polytope or a 4-prism. Notice that Σ_{S_1} contains a subdiagram S_0 of type F_4 and contains no subdiagrams of type $G_2^{(k)}$ with $k > 5$ (see Corollary 1.1). There are only two prisms and one Esselmann polytope (see Figure 11) satisfying these conditions. We consider these polytopes separately.

Case 2.1. Let Σ_{S_1} be the diagram shown in Figure 11, a). Then, by Corollary 1.2, $\Sigma_{S_1} = \bar{S}_1$. The node a_1 is a neighbor of $S_0 = \langle x_1, x_2, x_3, x_4\rangle$. Without loss of generality, we may assume that a_1 is a neighbor of $\langle x_1, x_2, x_3\rangle$. Then the subdiagram $\langle x_1, x_2, x_3, a_3\rangle$ is of type F_4 and has four bad neighbors (x_4, a_1, a_2, and some node of $\bar{S}_0$ joined with a_3), which is impossible.

Case 2.2. Let Σ_{S_1} be the subdiagram shown in Figure 11, b). By Corollary 1.2, we have $\Sigma_{S_1} = \bar{S}_1$. In particular, the nodes a_2 and a_3 cannot be neighbors of S_1, and therefore (by Corollary 1.2) they are joined with y_4. Furthermore, the diagram $\langle x_1, x_2, x_3, a_3\rangle$ is of type F_4 and has three neighbors, x_4, a_2, and y_4. Hence a_1 is not a neighbor of this diagram, and therefore $[a_1, x_4] \neq 2$ (since a_1 is a neighbor of $S_0 = \langle x_1, x_2, x_3, x_4\rangle$). Consider the diagram $S_6 = \langle a_2, x_1, x_2\rangle$ of type H_3. It has no good neighbors and $\bar{S}_6 = \Sigma_{S_6}$. Furthermore, $\bar{S}_6$ contains the dashed edge a_3x_4, so S_6 has at most two bad neighbors. Then S_6 has exactly two neighbors, x_3 and y_4. Hence, $\langle a_3, x_4, a_1\rangle \subset \Sigma_{S_6}$, and Σ_{S_6} is the diagram of a 4-prism. Thus, in the diagram of the 4-prism, the edge $\widetilde{x}_4, \widetilde{a}_1$ has a common node with a dashed edge, hence $[x_4, a_1] = 3$. It follows that $\langle x_1, x_2, x_3, x_4, a_1\rangle$ is a parabolic diagram of type $\widetilde{F}_4$, which is impossible.

Case 2.3. Let Σ_{S_1} be the diagram shown in Figure 11, c). Consider two cases: either $[a_3, x_4] = \infty$ or $[a_3, x_4] \neq \infty$ in Σ.

Case 2.3.1. Suppose that $[a_3, x_4] = \infty$. Since a_2 is not a good neighbor of S_1, y_4 is the only node of $\bar{S}_0$ joined with a_2.

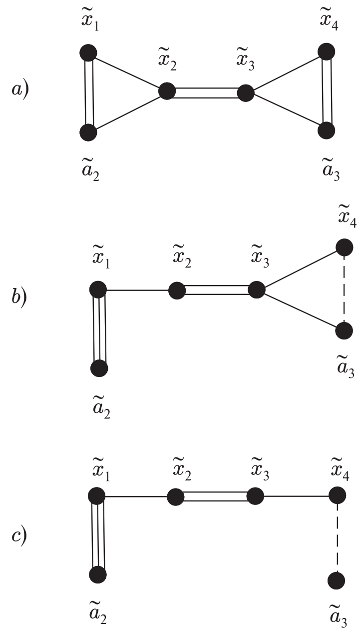


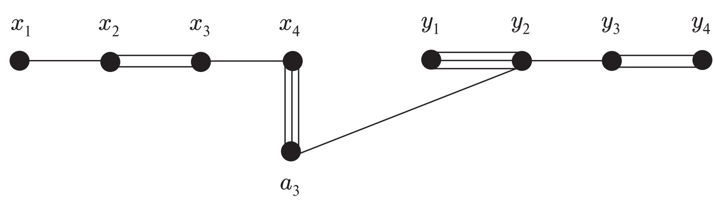
FIGURE 11. Diagrams of 4-polytopes with six facets containing F_4 and containing no $G_2^{(k)}$ with $k > 5$

Consider the diagram $S_7 = \langle a_2, x_1 \rangle$ of type $G_2^{(5)}$. If S_7 has a bad neighbor, then $P(S_7)$ is a 5-polytope with at most $5+3$ facets. Then Σ_{S_7} contains the subdiagram $\langle \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{a}_3 \rangle$, and $\widetilde{x}_2\widetilde{x}_3$ is a dashed edge in Σ_{S_7}. However, no diagram of a 5-polytope with at most $5+3$ facets contains two dashed edges ($\widetilde{x}_2\widetilde{x}_3$ and $\widetilde{x}_4\widetilde{a}_3$), joined by a simple edge ($\widetilde{x}_3\widetilde{x}_4$). Hence S_7 has no bad neighbors. In particular, $[a_2, y_4] = 3$.

Consider the diagram $\langle a_2, S_0, \bar{S}_0 \rangle$. It contains no dashed edges, and we know the multiplicities of all edges in this diagram except for the edges in the Lannér subdiagram $\bar{S}_0$. Since the number of Lannér diagrams of order 4 is finite, for $\langle a_2, S_0, \bar{S}_0 \rangle$ there are only finitely many possibilities. None of these satisfies the signature condition. Therefore the case $[a_3, x_4] = \infty$ is impossible.

Case 2.3.2. Suppose that $[a_3, x_4] \neq \infty$. Then a_3 is a good neighbor of S_1 and therefore $[a_3, x_4] = 5$ (otherwise either $[a_3, x_4] = 3$ and $\langle a_3, S_0 \rangle$ is a parabolic diagram of type $\widetilde{F}_4$ or $[a_3, x_4] = 4$ and $\langle a_3, x_4, x_3, x_2 \rangle$ is a parabolic diagram of type $\widetilde{C}_4$).

Consider the subdiagram $X = \langle S_0, \bar{S}_0, a_3 \rangle$, which we know completely with the exception of finitely many possibilities for $\bar{S}_0$ (S_0 is of type F_4, $\bar{S}_0$ is a Lannér diagram of order 4, $S_1 \subset \bar{S}_0$ is of type H_3 or B_3, a_3 is a good neighbor of S_1, and a_3 is joined with S_0 as shown in Figure 11, *c*)). The only diagram satisfying these conditions and the signature condition is



It is clear that a_2 is not a good neighbor of S_1, and therefore a_2 is not a neighbor of $S_1 = \langle y_1, y_2, y_3 \rangle$ and $[a_2, y_4] \neq 2$ (Lemma 1.4). Hence $S_2 = \langle y_2, y_3, y_4 \rangle$ has three bad neighbors: a_2, a_3, and y_1. Therefore Σ_{S_2} is a Lannér diagram of order 5 containing a subdiagram of type F_4. It follows that a_1 is joined with x_1 and x_4, but then the diagram $\langle x_4, a_3, y_3, y_2\rangle$ of type H_4 has four bad neighbors a_1, x_3, y_1, y_4.

Thus, we have examined two prisms and an Esselmann polytope and in each case obtained a contradiction.

Case 3. Suppose that S_1 has three bad neighbors, y_4, a_1, and a_2. Then $\bar{S}_1$ is a cyclic Lannér diagram of order 5 (i.e., $\bar{S}_1 = \mathcal{L}_5^5$), $[a_3, x_1] = [a_3, x_4] = 3$, and $[a_3, x_2] = [a_3, x_3] = 2$. By Corollary 1.2, a_3 is not a good neighbor of S_1, and therefore, y_4 is the only node joined with a_3.

Recall that $S_2 \subset \bar{S}_0$ is a subdiagram of type B_3 or H_3, $S_2 = \langle y_2, y_3, y_4\rangle$. As was shown in Cases 1 and 2, we may assume that S_2 also has three bad neighbors. Arguing as before (with S_2 in place of S_1), we have that the nodes a_i, $i = 1, 2, 3$, cannot be good neighbors of S_2, and exactly one of these three nodes is not joined with a_i. Hence a_3 and one of the nodes a_1 and a_2 (say, a_1) are bad neighbors of S_2, and a_2 is not joined with this diagram. Moreover, $\langle a_1, S_0\rangle$ is a cyclic Lannér diagram and y_1 is the only node of $\bar{S}_0$ joined with a_1. Without loss of generality, we may assume that a_2 is joined with $\langle x_1, x_2, x_3\rangle$. Then the diagram $\langle a_1, x_1, x_2, x_3\rangle$ is of type B_4 and has three bad neighbors, x_1, a_2, and a_3. Therefore y_1 is a good neighbor of $\langle a_1, x_1, x_2, x_3\rangle$, i.e., $[y_1, a_1] = 3$. Recall that a_1 is a bad neighbor of S_1. This means that either $[y_1, y_2] = 4, 5$ or $[y_1, y_3] = 4, 5$. Hence one of the nodes y_2 and y_3 is a bad neighbor of $\langle y_1, a_1, x_1, x_2, x_3\rangle$, which is impossible (since there are three other bad neighbors, x_1, a_2, and a_3). □

Lemma 6.9. *If Σ contains a subdiagram of type H_4, then $\Sigma = \Sigma_{P_7}$.*

Proof. Suppose that $S_0 = \langle x_1, x_2, x_3, x_4\rangle$ is a subdiagram of Σ of type H_4. By Lemmas 6.3 and 6.7, S_0 has exactly three neighbors; call them a_1, a_2, and a_3. Hence $P(S_0)$ is a 3-simplex, and $\bar{S}_0 = \Sigma_{S_0} = \langle y_1, y_2, y_3, y_4\rangle$ is a Lannér diagram of order 4. Let $S_1 = \langle y_1, y_2, y_3\rangle \subset \bar{S}_0$ be a subdiagram of type H_3 or B_3. Then S_1 has at least one bad neighbor, y_4. Consider the three cases where S_1 has one, two, or three bad neighbors, respectively.

Case 1. Suppose S_1 has only one bad neighbor, y_4. Then $P(S_1)$ is a 4-polytope with 4+3 facets. Arguing as in Case 1 of Lemma 6.8, we see that for some $i \in \{1, 2, 3\}$ the diagram $\langle a_i, S_0, \bar{S}_0\rangle$, say $\langle a_1, S_0, \bar{S}_0\rangle$, has no dashed edges. By Lemma 6.4, there are finitely many possibilities for the diagram $\langle a_1, S_0, \bar{S}_0\rangle$. Recall that $S_1 \subset \bar{S}_0$ is a subdiagram of type H_3 or B_3 with only one bad neighbor. There are only three possibilities for the diagram $\langle a_1, S_0, \bar{S}_0\rangle$ satisfying this condition and the signature condition. We list those diagrams in Figure 12 and consider them separately.

Consider the diagrams shown in Figure 12, *a*) and *b*). It is clear that $S_1 = \langle y_2, y_3, y_4\rangle$. Suppose that a_2 is joined with y_3 or y_4. Since a_2 is not a bad neighbor of S_1, the diagram $\langle a_2, S_1\rangle$ is of type F_4, contrary to Lemma 6.8. Hence both a_2 and a_3 are joined with either y_1 or y_2 (Lemma 1.4), and therefore the subdiagram $\langle y_1, y_2, y_3\rangle$ is of type H_3 and has four bad neighbors (y_4, a_1, a_2, a_3), which is impossible.

Consider the diagram shown in Figure 12, *c*). Without loss of generality, we may assume that $S_1 = \langle y_1, y_2, y_3\rangle$. Each of the diagrams $S_2 = \langle a_1, y_1, y_2, y_3\rangle$ and $S_3 = \langle a_1, y_4, y_3, y_2\rangle$ is of type H_4 and has two bad neighbors, x_4 and y_1 (y_4 in the latter case). Hence each of these diagrams has at most one extra neighbor (Lemma 6.7). On the other hand, by Lemma 1.4, each of the nodes a_2 and a_3 is joined with $\langle a_1, y_1, y_2, y_3, y_4\rangle$. Hence we may assume that a_2 is not joined with S_2 and is joined with y_4, and a_3 is not joined with S_3 and is joined with y_1. Since S_1 has no other bad neighbors besides y_4, we have $[a_4, y_1] = 3$. Furthermore, $[a_3, a_1] = 2$ (otherwise the diagram $\langle S_1, a_3\rangle$ is of type H_4 with

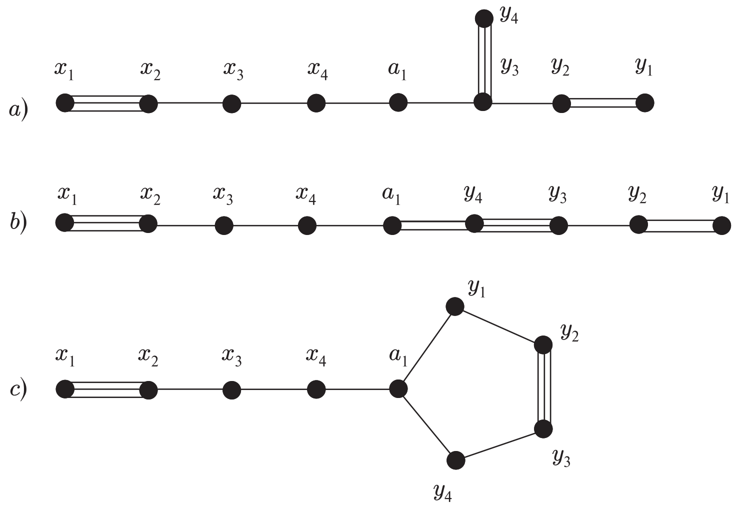


FIGURE 12. Three possibilities for $\langle a_1, S_0, \bar{S}_0\rangle$

four bad neighbors, y_4, a_2, a_1, and some x_i, $i \in \{1,2,3,4\}$, joined with a_3). If $[a_1, x_4] = 2$, then $\langle x_4, a_1, y_4, y_1, y_2, a_3\rangle$ is a parabolic diagram of type $\widetilde{D}_5$, and therefore, $[a_3, x_4] \neq 2$. Moreover, $[a_1, x_4] \neq 3$ (otherwise $\langle x_4, a_1, y_1, a_3\rangle$ is a parabolic diagram of type $\widetilde{A}_3$). Hence $S_4 = \langle x_2, x_3, x_4, a_1, y_4, y_3\rangle$ is of type A_6 and has three bad neighbors, x_1, y_2, and a_3. Thus, a_2 is a good neighbor of S_4, $[a_2, y_4] = 3$, and $[a_2, x_4] = [a_2, x_3] = [a_2, x_2] = 2$. It follows that $\langle x_3, x_4, a_1, y_4, a_2, y_1, y_2\rangle$ is a parabolic diagram of type $\widetilde{E}_6$, which is impossible.

Case 2. Suppose that S_1 has two bad neighbors, y_4 and a_1. Then $P(S_1)$ is either an Esselmann polytope or a 4-prism, and

$$\Sigma_{S_1} = \langle \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{a}_2, \widetilde{a}_3\rangle.$$

The Coxeter diagram of an Esselmann polytope not containing subdiagrams of type F_4 and $G_2^{(k)}$, $k \geq 6$, is one of the two diagrams shown in Figure 13. The Coxeter diagram of a 4-prism not containing subdiagrams of type F_4 or $G_2^{(k)}$, $k \geq 6$, is one of the diagrams shown in Figure 14. Diagram *a*) is shown three times, since there are three different ways of embedding S_0 in this diagram (in any other diagram containing more than one subdiagram of type H_4, those subdiagrams are permuted by automorphisms of the diagram).

We may assume that $\langle \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3\rangle$ is a subdiagram of Σ_{S_1} of type H_3. In any possible case for $\bar{S}_1$, that subdiagram has exactly one bad neighbor (in Σ_{S_1}). Assume that this is $\widetilde{a}_2$.

We first consider the Esselmann polytopes and then the prisms.

Case 2.1. Suppose that Σ_{S_1} is the diagram of an Esselmann polytope (see Figure 13). The subdiagram $S_5 = \langle x_1, x_2, x_3, a_3\rangle$ of type H_4 has three bad neighbors in Σ (a_2, x_4, and some node of $\bar{S}_0$ joined with a_3). Therefore, a_1 is not a neighbor of $S_5 = \langle x_1, x_2, x_3, a_3\rangle$, and a_1 is joined with x_4 (since a_1 is a neighbor of S_0). Furthermore, the diagram $\langle S_0, a_3, \bar{S}_0\rangle$ consists of the subdiagrams $\langle S_0, a_3\rangle$ and $\bar{S}_0$ joined by the edge a_3y_4 only (notice that this edge is not empty by Lemma 1.4). Moreover, this is a dashed edge; otherwise the diagram $\langle S_0, a_3, \bar{S}_0\rangle$ would be superhyperbolic. It follows that the three bad neighbors of $S_5 = \langle x_1, x_2, x_3, a_3\rangle$ are a_2, x_4, y_4, and we conclude that $\langle S_1, a_1\rangle = \bar{S}_5 = \Sigma_{S_5}$ is a Lannér diagram of order 4.

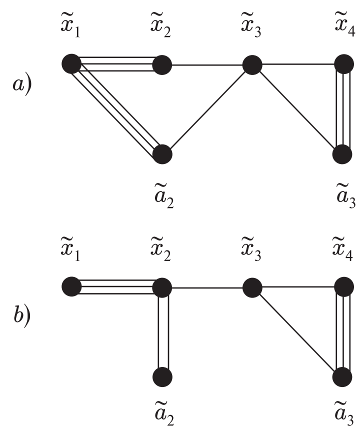


FIGURE 13. Diagrams of Esselmann polytopes containing H_4 and containing no $G_2^{(k)}$ with $k > 5$

Consider the diagram $S_6 = \langle a_3, x_4 \rangle$ of type $G_2^{(5)}$. It has at least two bad neighbors, y_4 and x_4, and therefore $P(S_6)$ is a 5-polytope with at most $5 + 2$ facets. Hence $P(S_6)$ is a 5-prism. Notice that the diagram $X = \langle x_1, x_2, a_2, y_1, y_2, y_3 \rangle$ is not joined with S_6 in Σ, so it does not differ from the subdiagram $\langle \widetilde{x}_1, \widetilde{x}_2, \widetilde{a}_2, \widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \rangle$ in Σ_{S_6}. It is clear that X contains no dashed edges (a_2 cannot be joined with S_1 by a dashed edge, since a_2 is not a bad neighbor of S_1). Since the diagram of a 5-prism does not contain Lannér subdiagrams of order 3, we have that a_2 is a good neighbor of S_1, Σ_{S_1} is the diagram shown in Figure 13, b), $[a_2, x_2] = 3$, and S_1 is a diagram of type B_3 (if S_1 was of type H_3, then $\widetilde{a}_2\widetilde{x}_2$ would be a dashed edge of Σ_{S_6}, and so it could not have a common node with the triple edge $\widetilde{x}_1\widetilde{x}_2$ in the diagram of a 5-prism). Without loss of generality, we may assume that $[y_1, y_2] = 3$ and $[y_2, y_3] = 4$. Thus, we have a linear diagram $\langle x_1, x_2, a_2, y_1, y_2, y_3 \rangle$ with edges labeled by 5, 3, 3, 3, 4. It follows that $[\widetilde{a}_1, \widetilde{y}_3] = \infty$ in Σ_{S_6}, and therefore $[a_1, y_3] \neq 2, 3$ in Σ (if $[a_1, y_3] = 3$, then the edge $\widetilde{a}_1\widetilde{y}_3$ of Σ_{S_6} would be labeled by 10, but it should be a dashed edge). However, in the Lannér diagram $\langle S_1, a_1 \rangle$ of order 4, the multiple edge a_1y_3 cannot have a common node with the multiple edge y_3y_2.

The obtained contradiction shows that Σ_{S_1} cannot be the diagram of an Esselmann polytope.

Case 2.2. Now suppose that Σ_{S_1} is a 4-prism. Notice that in all diagrams of 4-prisms all edges incident to a_2 are simple, and by Corollary 1.2, a_2 cannot be a good neighbor of S_1. Hence a_2 is joined with y_4 (Lemma 1.4).

Furthermore, suppose that $[a_2, y_4] \neq \infty$. Then the diagram $\langle S_0, a_2, \bar{S}_0 \rangle$ satisfies the following conditions: it has no dashed edges, y_4 is the only node of $\bar{S}_0$ joined with a_2, a_2 is a bad neighbor of $H_3 \subset S_0$, and a_2 is joined with S_0 by simple edges only. None of the diagrams satisfies these conditions together with the signature condition, whence $[a_2, y_4] = \infty$.

We now examine the diagrams of 4-prisms case by case.

Case 2.2.1. Consider the diagram shown in Figure 14, $a1$). If a_1 is joined with $\langle x_1, x_2, x_4 \rangle$, then $\langle x_2, x_1, a_2, x_4 \rangle$ is of type H_4 with four neighbors, x_3, a_3, a_1, y_4 (recall that $[a_2, y_4] = \infty$). If a_1 is not a neighbor of $\langle x_1, x_2, x_4 \rangle$, then $[a_1, x_3] \neq 2$ (as a_1 is a neighbor of $S_0 = \langle x_1, x_2, x_3, x_4 \rangle$) and $\langle x_1, x_2, x_3, a_3 \rangle$ is of type H_4 with four neighbors (x_3, a_2, a_1, and some node y_i, $i \in \{1, 2, 3, 4\}$, joined with a_3; see Lemma 1.4).

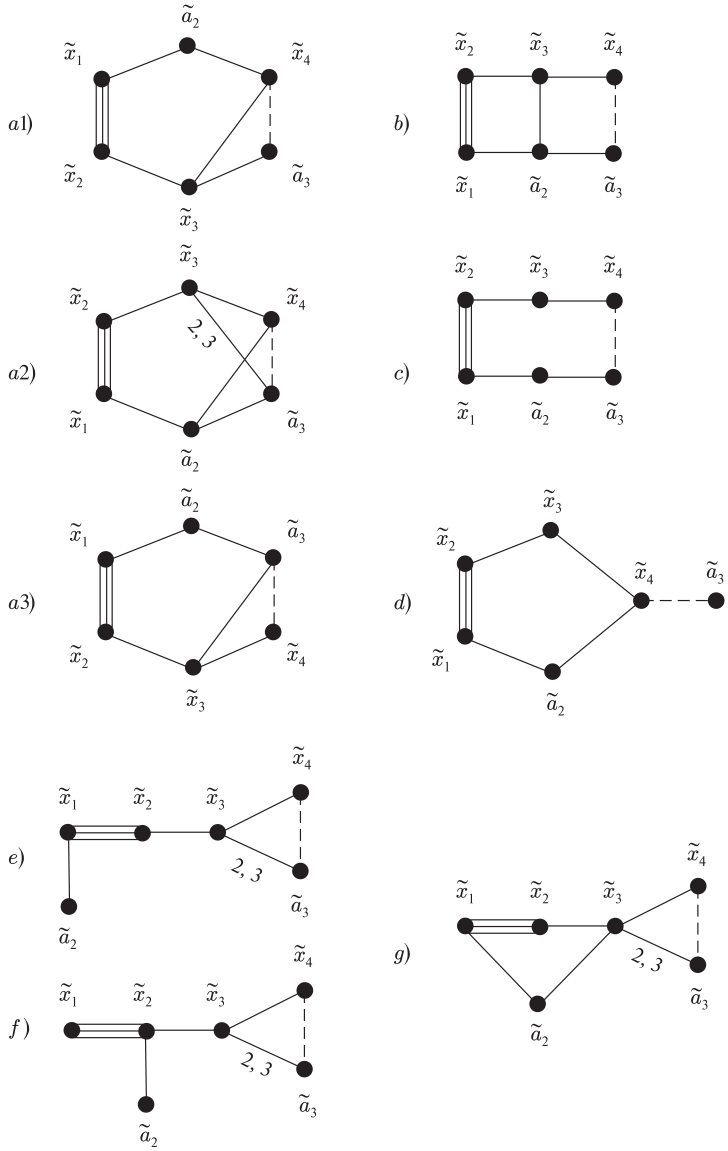


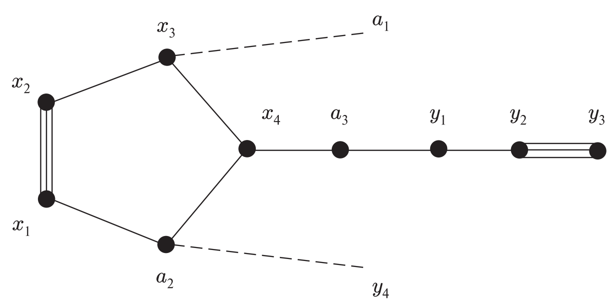
FIGURE 14. Diagrams of 4-prisms containing H_4

Case 2.2.2. Consider the diagrams shown in Figure 14, $a2)$–$c)$. Since $\langle x_2, x_1, a_2, a_3\rangle$ is of type H_4 and has three bad neighbors, x_3, x_4, and y_4 (as $[a_2, y_4] = \infty$), $\langle S_1, a_1\rangle$ is a Lannér diagram. It is clear that the nodes x_1, x_2, a_2 are not joined with that Lannér diagram (otherwise $\langle x_2, x_1, a_2, a_3\rangle$ would have four bad neighbors, x_3, x_4, y_4, and some node of $\langle S_1, a_1\rangle$). Since the Lannér diagrams $\langle a_1, S_1\rangle$ and $\langle x_1, x_2, x_3, a_2\rangle$ must be joined, we have $[x_3, a_1] \neq 2$. The signature condition, applied to the diagram $\langle S_0, a_1, a_2, S_1\rangle$, implies that $[x_3, a_1] = \infty$. This means that $\langle x_2, x_3, x_4, a_2\rangle$ is of type A_4 (in the case of $a2)$), or $A_3 + A_1$ (in the cases of $a3)$ and $c)$), or D_4 (in the case of $b)$) and has four bad neighbors, x_1, a_3, y_4, a_1 (here we again use the fact that $[a_2, y_4] = \infty$).

Case 2.2.3. Consider the diagram shown in Figure 14 (d). Since the subdiagram $\langle x_2, x_1, a_2, x_4\rangle$ is of type H_4 and has three bad neighbors (x_3, a_3 and y_4), a_1 cannot be a neighbor of $\langle x_2, x_1, a_2, x_4\rangle$ and $\langle a_1, S_1\rangle$ is a Lannér diagram. Hence $[a_1, x_3] \neq 2$ (a_1 is a neighbor of $S_0 = \langle x_1, x_2, x_3, x_4\rangle$, but not a neighbor of $\langle x_1, x_2, x_4\rangle$). The signature condition, applied to the diagram $\langle S_0, a_1, a_2, S_1\rangle$, implies that $[x_3, a_1] = \infty$.

Suppose a_3 is not a good neighbor of S_1. Then the dashed edge $\widetilde{x}_4\widetilde{a}_3$ of Σ_{S_1} corresponds to the dashed edge x_4a_3 of $\bar{S}_1$, and the diagram $\langle x_2, x_3, x_4, a_2\rangle$ of type A_4 has four bad neighbors, x_1, a_1, a_3, y_4 (we used the facts that $[x_3, a_1] = \infty$ and $[a_2, y_4] = \infty$), which is impossible.

Suppose that a_3 is a good neighbor of S_1. If the edge $\langle x_4, a_3\rangle \subset \bar{S}_1$ is dashed (as is $\langle \widetilde{x}_4, \widetilde{a}_3\rangle \subset \Sigma_{S_1}$) or multiple, then a_3 is a bad neighbor of $\langle x_2, x_3, x_4, a_2\rangle$, and this diagram still has four bad neighbors. Hence we may assume that the edge x_4a_3 becomes simple in $\bar{S}_1$. This can only happen when S_1 is of type H_3 (otherwise, $\langle a_2, x_3, x_4, a_3, S_1\rangle$ is a parabolic diagram of type $\widetilde{B}_6$). Thus, we have the following diagram:



By Lemma 6.2, $[y_4, a_3] \neq \infty$. If $[y_4, a_3] = 5$, then the diagram $\langle y_4, a_3, x_4, x_3\rangle$ is of type H_4 and has four bad neighbors, a_1, a_2, y_1, x_2. If $[y_4, a_3] = 4$, then the diagram $\langle y_4, a_3, x_4, x_3, x_2\rangle$ is of type B_5 and has four bad neighbors, a_1, a_2, y_1, x_1. Hence $[y_4, a_3] = 2$ or 3. Similarly, $[a_1, a_3] = 2$ or 3.

Consider now the diagram $\langle S_0, a_3, S_1\rangle$. The node y_4 is not joined with S_0. Since $\langle x_2, x_3, x_4, a_3, y_1, y_2\rangle$ has three bad neighbors (x_1, a_1, y_3), y_4 can be joined with a_3 by a simple edge only. Recall that $\langle S_1, y_4\rangle = \bar{S}_0$ is a Lannér diagram, so Lemma 1.4 implies that $[y_4, a_3] \neq 2$. Therefore, $[y_4, a_3] = 3$. Examining the Lannér subdiagrams $\langle y_4, S_1\rangle$ (where S_1 is of type H_3), we find that the diagram $\langle S_0, a_3, y_4, S_1\rangle$ satisfies the signature condition only when $[y_4, y_3] = 3$ and $[y_4, y_1] = [y_4, y_2] = 2$. Similarly, examining the diagram $\langle x_2, x_1, a_2, x_4, a_3, a_1, S_1\rangle$, we have $[a_1, a_3] = [a_1, y_3] = 3$ and $[a_1, y_1] = [a_1, y_2] = 2$. However, in this case the diagram $\langle a_3, y_4, y_3, y_2\rangle$ is of type H_4 and has four neighbors (a_2, x_4, y_1, a_1).

Case 2.2.4. Consider the diagrams shown in Figure 14, e)–g). As before, a_2 cannot be a good neighbor of S_1, i.e., a_2 is not joined with S_1, and, as has been shown above, $[a_2, y_4] = \infty$. However, if $[x_3, a_3] = 2$, then a_3 can be a good neighbor of S_1 and it may happen that $[x_4, a_3] \neq \infty$ in Σ.

Suppose that $[x_4, a_3] \neq \infty$ in Σ and therefore $[x_3, a_3] = 2$. Consider the diagram $Y = \langle S_0, a_2, a_3, S_1\rangle$. If $[x_4, a_3] = 3$ or 5, then the diagram Y does not satisfy the signature condition (more precisely, at least one of its subdiagrams $Y \setminus a_2$ and $Y \setminus y_4$ does not satisfy the signature condition). If $[x_4, a_3] = 4$, then Y contains the parabolic diagram $\langle x_2, x_3, x_4, a_3, y_1\rangle$ of type $\widetilde{F}_4$ (here we assumed that the nodes of S_1 are numbered in such a way that $[y_1, y_2] = 3$ and $[y_2, y_3] = 4$ or 5).

Thus, $[x_4, a_3] = \infty$. Consider the subdiagram $S_7 = \langle x_2, x_1, a_2\rangle$ of type H_3. It has at least two bad neighbors, x_3 and y_4. Furthermore, the edge x_4a_3 is not joined with S_7, so $\langle x_4, a_3\rangle \subset \bar{S}_7$, and therefore S_7 has no bad neighbors besides x_3 and y_4. In particular, a_1 cannot be a bad neighbor of S_7. Suppose that a_1 is a good neighbor of S_7. Then $S_8 = \langle a_1, S_7\rangle$ is of type H_4 with three bad neighbors (x_3, y_4, and the node of S_1 joined with a_1). On the other hand, the diagram $\bar{S}_8 = \Sigma_{S_8}$ contains the dashed edge x_4a_3, contrary to being a Lannér diagram of order 4. The obtained contradiction shows that a_1 is not joined with S_7. Thus, $\bar{S}_7 = \Sigma_{S_7} = \langle x_4, a_3, a_1, S_1\rangle$ is the diagram of a 4-prism, which implies that $\langle a_1, S_1\rangle$ is a Lannér diagram. It must be joined with the Lannér diagram $\langle a_2, x_1, x_2, x_3\rangle$. Therefore $[a_1, x_3] \neq 2$, since a_1 is not joined with S_7. Consider now the diagram $S_9 = \langle a_2, x_2, x_3, x_4\rangle$ (of type $A_1 + A_3$, A_4 or D_4 in the cases e), f), and g), respectively). The diagram S_9 has three bad neighbors, x_1, a_3, y_4, so a_1 is a good neighbor of S_9, $[a_1, x_3] = 3$, and $[a_1, x_4] = 2$.

Finally, consider the diagram $Z = \langle S_0, a_2, a_1, S_1\rangle$. Since this diagram has no dashed edges, we have only finitely many possibilities for it. Moreover, Z satisfies the following conditions: $\langle S_0, S_1\rangle$ is of type $H_4 + H_3$ or $H_4 + B_3$, a_2 is not joined with S_1 and is joined with S_0 in one of the ways shown in Figure 14, e)–g), $[a_1, x_3] = 3$, and a_1x_3 is the only edge joining a_1 with $\langle a_2, S_0\rangle$; $\langle a_1, S_1\rangle$ is a Lannér diagram. However, no diagram satisfies all these conditions and the signature condition.

Case 3. Suppose that S_1 has three bad neighbors. Let $S_1' \subset \bar{S}_0$ be a subdiagram of type H_3 or B_3, $S_1' \neq S_1$ (see Lemma 1.5). Set $\langle y_1, y_2, y_3\rangle = S_1$ and $\langle y_2, y_3, y_4\rangle = S_1'$. By Cases 1 and 2, we may assume that both S_1 and S_1' have three bad neighbors. There are two possibilities (up to a permutation of the nodes a_1, a_2, and a_3): either a_1 and a_2 are bad neighbors of both S_1 and S_1' (in addition to the bad neighbors y_4 and y_1 of S_1 and S_1', respectively) or y_4, a_1, a_2 are bad neighbors of S_1, and y_1, a_2, a_3 are bad neighbors of S_1'.

Suppose that a_1 and a_2 are bad neighbors of each of S_1 and S_1'. Then the node a_3 is not a bad neighbor of S_1 and S_1'. Moreover, Σ_{S_1} is a Lannér diagram of order 5. By Corollary 1.2, $\bar{S}_1 = \langle S_0, a_3\rangle$ is also a Lannér diagram. Thus, the diagram $\langle S_0, \bar{S}_0\rangle$ consists of the Lannér diagram $\langle S_0, a_3\rangle$ (where S_0 is a diagram of type H_4) and the Lannér diagram $\bar{S}_0$, the node a_3 being a bad neighbor of both S_1 and S_1'. The only such diagram satisfying the signature condition is shown in Figure 12, b). But in this case, Proposition 1.11 implies that $\widetilde{x}_4\widetilde{a}_3$ is a dashed edge of Σ_{S_1}, contrary to the assumption that Σ_{S_1} is a Lannér diagram of order 5.

Thus we may assume that y_4, a_1, a_2 are bad neighbors of S_1, and y_1, a_2, a_3 are bad neighbors of S_1'. Examining $\bar{S}_1$ and $\bar{S}_1'$, we conclude that $\langle S_0, a_1\rangle$ and $\langle S_0, a_3\rangle$ are Lannér diagrams.

Consider now two cases: either both a_1 and a_3 are joined with $\bar{S}_0$ by dashed edges or at least one of the nodes a_1 and a_3 (say, a_1) is joined with $\bar{S}_0$ by ordinary edges only.

Case 3.1. Suppose that a_1 and a_3 are joined with $\bar{S}_0$ by dashed edges. Since a_1 is a bad neighbor of S_1 but not a bad neighbor of S_1', the dashed edge joining a_1 with $\bar{S}_0$ can only be a_1y_1. Similarly, $[a_3, y_4] = \infty$. Lemma 6.2 implies that $\langle a_1, a_3\rangle \neq \infty$. Hence we have only finitely many possibilities for the diagram $\langle S_0, a_1, a_3\rangle$ (there are four possibilities for each of the Lannér diagrams $\langle S_0, a_1\rangle$ and $\langle S_0, a_3\rangle$ and three possibilities for $[a_1, a_3] \in \{3, 4, 5\}$).

Consider the diagram $S_{10} = \langle x_2, x_3, x_4, a_1\rangle$ of type D_4, A_4, B_4, or H_4. If a_3 is a bad neighbor of S_{10}, then S_{10} has three bad neighbors, x_1, y_1, and a_3, and therefore a_2 cannot be a bad neighbor of it. If a_3 is not a bad neighbor of S_{10}, then $\langle S_{10}, a_3\rangle$ has three bad neighbors x_1, y_1, y_4, and a_2 cannot be a bad neighbor of $\langle S_{10}, a_3\rangle$. In any case, a_2 cannot be a bad neighbor of S_{10}. Similarly, a_2 cannot be a bad neighbor of $S_{11} = \langle x_2, x_3, x_4, a_3\rangle$.

Suppose that a_2 is a bad neighbor of the diagram $S_{12} = \langle x_1, x_2, x_3\rangle$ of type H_3. Then $P(S_{12})$ is a 4-polytope with at most $4+3$ facets. However, $\Sigma_{S_{12}} \neq \bar{S}_{12}$ and $\Sigma_{S_{12}}$ contains four dashed edges $\widetilde{y}_4\widetilde{a}_3, \widetilde{a}_3\widetilde{x}_4, \widetilde{x}_4\widetilde{a}_1, \widetilde{a}_1\widetilde{y}_1$, which is impossible for a 4-polytope with at most seven facets. Hence a_2 cannot be a bad neighbor of S_{12}, which implies that a_2 is not a neighbor of x_1x_2. Therefore, a_2 is a neighbor of x_3x_4 and each of the diagrams S_{10} and S_{11} is of type D_4 or A_4.

Suppose that S_{10} and S_{11} are of the same type. Then $[a_1, a_3] = 2$ (otherwise either $[a_1, a_3] = \infty$, contrary to Lemma 6.2, or the edge a_1a_3 has three bad neighbors, contrary to Lemma 6.1, or one of the diagrams $\langle x_3, a_1, a_3\rangle$ and $\langle x_4, a_1, a_3\rangle$ is parabolic of type $\widetilde{A}_2$). In this case $\langle S_{10}, a_3, a_2\rangle$ contains a parabolic subdiagram. Hence S_{10} and S_{11} are of different types. If $[a_1, a_3] = 2$ or 3, then $\langle S_{10}, a_3, a_2\rangle$ still contains a parabolic subdiagram. If $[a_1, a_3] = 4$ or 5, then $S_{13} = \langle a_1, a_3\rangle$ has two bad neighbors, so $\Sigma_{S_{13}}$ is the diagram of a 5-prism. However, in $\Sigma_{S_{13}}$ the subdiagram $\langle \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4\rangle$ is a linear diagram in which a triple edge and a simple edge are joined by the dashed edge $\widetilde{x}_2\widetilde{x}_3$. No diagram of a 5-prism contains such a subdiagram, which shows that Case 3.1 is impossible.

Case 3.2. Suppose that a_1 is joined with S_0 by ordinary (i.e., non-dashed) edges. Then the subdiagram $T = \langle S_0, a_1, \bar{S}_0\rangle$ contains no dashed edges, and we have finitely many possibilities for the diagram T. Notice that T satisfies the following conditions:

S_0 is of type H_4, and $\langle S_0, a_1\rangle$ is a Lannér diagram;
$\bar{S}_0$ is a Lannér diagram of order 4, and $\bar{S}_0$ is not joined with S_0;
S_1 and S_1' are subdiagrams of $\bar{S}_0$ of type H_3 or B_3;
a_1 is a bad neighbor of S_1 not joined with S_1'.

However, only two subdiagrams satisfy these conditions together with the signature condition; they are shown in Figure 15, *a*) and *b*).

Consider the diagram shown in Figure 15 (*a*). The subdiagram $\langle x_2, x_3, x_4, a_1, y_1\rangle \subset \Sigma$ is of type B_5 and has three bad neighbors, x_1, y_2, and a_3. Hence a_2 cannot be a bad neighbor of $\langle x_2, x_3, x_4, a_1, y_4\rangle$, and so a_2 is joined with x_1x_2. Therefore a_2 is a bad neighbor of $S_{14} = \langle \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3\rangle$, and $P(S_{14})$ is a 4-polytope with $4 + 3$ facets (it is not difficult to check that S_{14} has no other bad neighbors). However, $\Sigma_{S_{14}}$ contains the subdiagram $\langle \widetilde{a}_1, \widetilde{y}_1, \widetilde{y}_2\rangle$ consisting of the double edge $\widetilde{a}_1\widetilde{y}_1$ adjacent to the triple edge $\widetilde{y}_1\widetilde{y}_2$. But no diagram of a 4-polytope with $4 + 3$ facets has such a subdiagram, so the case shown in Figure 15, *a*) is impossible.

In particular, we conclude that for any diagram $S \subset \Sigma$ of type H_4, the diagram $Q = \langle S, \bar{S}, a_1\rangle$ is the diagram shown in Figure 15, *b*), and $\bar{S}$ is a linear Lannér diagram with one double, one simple, and one triple edge.

Consider the diagram $\langle Q, a_3\rangle$. By Lemma 6.2, $\langle Q, a_3\rangle$ contains at most one dashed edge (since any such edge must be incident to a_3 and cannot have endpoints in either S_0 (since $\langle S_0, a_3\rangle$ is a Lannér diagram) or S_1 (since a_3 is not a bad neighbor of S_1). Hence, either $\langle S_0, a_3, y_4, S_1\rangle$ or $\langle S_0, a_3, a_1, S_1\rangle$ contains no dashed edges. The former diagram does not satisfy the signature condition, whereas the latter satisfies it only in the case of the diagram shown in Figure 15, *c*). Moreover, $[a_3, y_4] = \infty$, since otherwise $\det(\langle S_0, a_1, a_3, \bar{S}_0\rangle) \neq 0$.

It remains to determine in how many ways the remaining node a_2 can be adjoined to the diagram $\langle S_0, a_1, a_3, \bar{S}_0\rangle$. Consider the diagram $S_{15} = \langle \widetilde{y}_1, \widetilde{y}_2, \widetilde{a}_1, \widetilde{x}_4\rangle$ of type H_4 with bad neighbors x_3, a_3, y_2. As has been shown before, $\bar{S}_{15} = \langle x_1, x_2, a_2, y_4\rangle$ is a linear Lannér diagram with one double, one simple, and one triple edge, i.e., either $[a_2, x_2] = 2$, $[a_2, x_1] = 3$, $[a_2, y_4] = 4$ or $[a_2, x_2] = 3$, $[a_2, x_1] = 2$, $[a_2, y_4] = 4$. Furthermore, a_2 is not joined with $\langle x_3, x_4, a_3, a_1\rangle$ since the diagram $\langle x_2, x_3, x_4, a_3, a_1, y_2\rangle$ of type E_6 already has three bad neighbors, x_1, y_1, and y_4. Also, a_2 is not joined with the edge y_1y_2, since the diagram $\langle y_1, y_2, a_1, x_4\rangle$ of type H_4 already has three bad neighbors, x_3, a_3, and y_3. Since a_2 is a bad neighbor of S_1, we have $[a_2, y_3] \neq 2$. If $[a_2, y_3] = 3$, then $S_{16} = \langle x_2, x_1, a_2, y_3\rangle$ is a diagram of type H_4 such that $\bar{S}_{16} = \langle y_1, a_1, x_4, a_3\rangle$ is a non-connected diagram, which is impossible. If $[a_2, y_3] = 5$ (or 4), then $\langle a_2, y_3, y_2, a_1\rangle$ (or

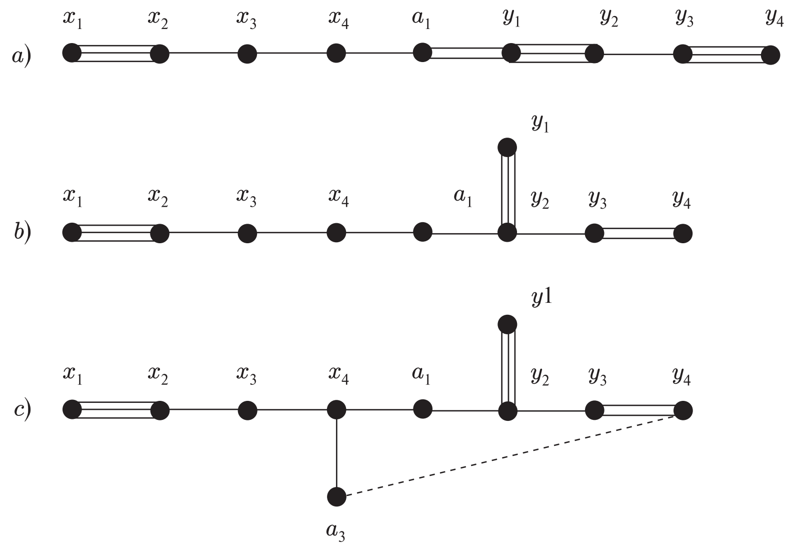


FIGURE 15. To the proof of Lemma 6.9

$\langle a_2, y_3, y_2, a_1, x_4, a_3\rangle$, respectively) is a diagram of type H_4 (or B_6, respectively) with four bad neighbors, y_4, y_1, x_4, and one of the nodes x_1 and x_2. Hence $[a_2, y_3] = \infty$.

Thus, a_2 is incident to three edges: the dashed edge a_2y_3, the double edge a_2y_4, and the simple edge a_2x_1 or a_2x_2. If $[a_2, x_1] = 3$, then the diagram $\langle S_0, a_1, a_2, a_3, y_1, y_2\rangle$ does not satisfy the signature condition. Then $[a_2, x_1] = 2$, $[a_2, x_2] = 3$, and we have the diagram Σ_{P_7}.

Thus, having exhausted all the cases, we found only the polytope P_7, as claimed. □

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