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EXCELLENT AFFINE SPHERICAL HOMOGENEOUS SPACES OF SEMISIMPLE ALGEBRAIC GROUPS

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Abstract. A spherical homogeneous space G/H of a connected semisimple algebraic group G is called excellent if it is quasi-affine and its weight semigroup is generated by disjoint linear combinations of the fundamental weights of the group G. All the excellent affine spherical homogeneous spaces are classified up to isomorphism.

1. Introduction

1.1. In this paper the ground field is the field $\mathbb{C}$ of complex numbers, all topological terms relate to the Zariski topology, all groups are assumed to be algebraic, and their subgroups closed. The tangent algebras of groups denoted by capital Latin letters are denoted by the corresponding small German letters. The term "variety" always means an algebraic variety. Irreducible representations of semisimple algebraic groups are assumed to be finite-dimensional. The fundamental weights of simple algebraic groups are numbered as in [1] and are denoted by the symbols π_i. The fundamental weight of the group SL_2 is denoted by the symbol π.

Throughout the paper (apart from in § 1.3) the symbol G denotes a connected semisimple algebraic group. Unless otherwise stipulated, we assume a Borel subgroup B in G to be fixed, together with a maximal torus T and the maximal unipotent subgroup U contained in B. The highest weight vectors and highest weights of irreducible representations of the group G are considered with respect to the Borel subgroup B. The set of dominant weights of the group G with respect to the group B (with the structure of a semigroup with respect to addition) is denoted by $\Lambda_+(G)$. The symbol $\widetilde{G}$ denotes the simply connected covering group of the group G and φ denotes the covering homomorphism $\widetilde{G} \to G$. For every subgroup $H \subset G$, the symbol $\widetilde{H}$ denotes its inverse image in $\widetilde{G}$, that is, $\widetilde{H} = \varphi^{-1}(H) \subset \widetilde{G}$.

The actions of the group G (as well as of any of its subgroups) on the space $\mathbb{C}[G]$ of regular functions on G given by the formulae $(gf)(x) = f(g^{-1}x)$ and $(gf)(x) = f(xg)$ ($g, x \in G$, $f \in \mathbb{C}[G]$) are called the actions *on the left* and *on the right*, respectively. For every subgroup $H \subset G$ the algebra of functions in $\mathbb{C}[G]$ that are invariant under the action of the group H on the left (on the right) is denoted by ${}^H\mathbb{C}[G]$ ($\mathbb{C}[G]^H$).

For an arbitrary subgroup $H \subset G$ there is a natural representation of the group G in the space $\mathbb{C}[G/H] = \mathbb{C}[G]^H$ (where G acts on the left). The highest weights of the irreducible G-modules that occur in the spectrum of this representation form a semigroup called the *weight semigroup* of the homogeneous space G/H. We denote it by $\Lambda_+(G/H)$.

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Let H be a spherical subgroup of G (equivalent formulations are: G/H is a spherical homogeneous space; (G,H) is a spherical pair). By definition this means that under the natural action $G : G/H$ the group B has an open orbit. It is known (see [2, Corollary 2]) that for G/H quasi-affine, the sphericity of the subgroup H is equivalent to the condition that the representation of G in $\mathbb{C}[G/H]$ is multiplicity-free. Suppose that a decomposition of the space $\mathbb{C}[G/H]$ into the direct sum of irreducible G-modules is fixed, and fix a highest weight vector in each of these G-modules. Then all these highest weight vectors form a basis of the algebra ${}^U\mathbb{C}[G]^H$ (as a vector space over $\mathbb{C}$). As the representation $G : \mathbb{C}[G/H]$ is multiplicity-free, for every weight $\omega \in \Lambda_+(G/H)$ in the algebra ${}^U\mathbb{C}[G/H] = {}^U\mathbb{C}[G]^H$, the subspace of functions of weight ω is one-dimensional. We fix some nonzero function in this subspace and denote it by $f_H[\omega]$. Clearly, this function is a highest weight vector of the G-module generated by it. If $\omega_1, \omega_2 \in \Lambda_+(G/H)$, then $f_H[\omega_1]f_H[\omega_2]$ is a highest weight vector of weight $\omega_1 + \omega_2$; therefore $f_H[\omega_1]f_H[\omega_2] = cf_H[\omega_1 + \omega_2]$ for some $c \in \mathbb{C}^\times$.

It is known (see [3, Corollary 1 on p. 19]) that the sphericity of the homogeneous space G/H is a local property, that is, it depends only on the tangent algebras $\mathfrak{g}$ and $\mathfrak{h}$.

Recall that subgroups H for which the variety G/H is quasi-affine are called *observable*. Furthermore, the variety G/H is affine if and only if the subgroup H is reductive.

The following well-known characterization of observable spherical subgroups (including reductive) follows from the Frobenius reciprocity theorem.

Theorem 1. *An observable subgroup $H \subset G$ is spherical if and only if the inequality $\dim V^H \le 1$ holds for every irreducible G-module V. Furthermore, $\dim V^H = 1$ if and only if the highest weight of the module V^* (if H is reductive, then also of the module V) belongs to $\Lambda_+(G/H)$.*

Let $\omega_1, \dots, \omega_m$ be all fundamental weights of the group G. We consider an arbitrary dominant weight ω of the group G. It can be represented in the form $\omega = k_1\omega_1 + \dots + k_m\omega_m$, where $k_i \in \mathbb{Z}$, $k_i \ge 0$. We define the *support of the weight* ω to be the set $\operatorname{Supp}\omega = \{\omega_i : k_i > 0\}$. Next, for every set Ω of dominant weights of G, we define the *support of the set* Ω to be the set $\operatorname{Supp}\Omega = \bigcup_{\omega\in\Omega}\operatorname{Supp}\omega$. In particular, the support $\operatorname{Supp}\Lambda_+(G/H)$ of the weight semigroup $\Lambda_+(G/H)$ is defined for any subgroup $H \subset G$.

We now introduce the key notion of this paper.

Definition 1. A spherical subgroup $H \subset G$ is called *excellent* (the spherical homogeneous space G/H is called *excellent*, the spherical pair (G,H) is called *excellent*) if

1) the variety G/H is quasi-affine;
2) the semigroup $\Lambda_+(G/H)$ is generated by weights $\lambda_1, \dots, \lambda_n$ satisfying the condition $\operatorname{Supp}\lambda_i \cap \operatorname{Supp}\lambda_j = \varnothing$ for $i \ne j$.

We also note that if λ is an indecomposable element of $\Lambda_+(G/H)$, then it is present in any system of generating elements of this semigroup.

From § 2.2 to the end of the paper we shall only deal with reductive spherical subgroups H. Further, according to what we have said above, condition 1) of Definition 1 holds automatically for them.

We also point out that condition 2) of Definition 1 implies that the semigroup $\Lambda_+(G/H)$ is free.

The class of excellent spherical homogeneous spaces includes all simply connected (complex) symmetric spaces, as well as most of the simply connected affine spherical homogeneous spaces G/H, where G is simple (see Table 1). The space G/U, where G is simply connected, gives an example of a nonaffine excellent spherical homogeneous space. It is well known that $\Lambda_+(G/U) = \Lambda_+(G)$.

In this paper, up to isomorphism, we classify all the excellent *affine* spherical homogeneous spaces, that is, spaces G/H, where G is a connected semisimple algebraic group and $H \subset G$ is an excellent reductive spherical subgroup. See the precise formulations in § 1.2.

Remark 1. For every subgroup $H \subset G$, the natural action of the group G on G/H can be extended to an action of the group $\widetilde{G}$ on G/H in such a way that $\operatorname{Ker}\varphi$ acts trivially. Hence there is an isomorphism $G/H \simeq \widetilde{G}/\widetilde{H}$ of homogeneous spaces of the group $\widetilde{G}$. For this reason we identify the homogeneous spaces G/H and $\widetilde{G}/\widetilde{H}$.

The motivation for classifying excellent spherical homogeneous spaces is given by the following theorem, which was communicated to the author by Vinberg. The key part of this theorem was proved in [4, Theorem 5.5].

Theorem 2. *Let G be a simply connected group, and $H \subset G$ an observable spherical subgroup. Let $X = \operatorname{Spec}\mathbb{C}[G/H]$ (so that $\mathbb{C}[X] \simeq \mathbb{C}[G/H]$). Suppose that the semigroup $\Lambda_+(G/H)$ is freely generated by elements $\mu_1, \dots, \mu_r$.*

1) *If the subgroup H is excellent, then the morphism*
$$X \to \mathbb{C}^r, \quad x \mapsto (f_H[\mu_1](x), \dots, f_H[\mu_r](x)),$$
is surjective and equidimensional.
2) *If H contains some maximal unipotent subgroup of G, then the assertion converse to 1) holds.*

In what follows we shall find the notion of an almost excellent spherical subgroup useful, so we introduce it now.

Definition 2. A spherical subgroup $H \subset G$ is called *almost excellent* (the spherical homogeneous space G/H is called *almost excellent*, the spherical pair (G, H) is called *almost excellent*) if the convex cone $\mathbb{Q}_+\Lambda_+(G/H)$ (that is, the set of finite linear combinations of elements of $\Lambda_+(G/H)$ with nonnegative rational coefficients) is generated by weights $\lambda_1, \dots, \lambda_n$ with the condition $\operatorname{Supp}\lambda_i \cap \operatorname{Supp}\lambda_j = \varnothing$ for $i \neq j$.

Clearly, any excellent spherical subgroup $H \subset G$ is almost excellent. It is easy to show that if a spherical subgroup $H \subset G$ is almost excellent and the semigroup $\Lambda_+(G/H)$ is free, then the subgroup H is excellent. Moreover, in § 2.1 below, we shall prove (see Corollary 1) that, just like the property that a homogeneous space G/H be spherical, the property that a spherical homogeneous space be almost excellent is a local property, that is, it depends only on the tangent algebras $\mathfrak{g}$ and $\mathfrak{h}$.

1.2. In this subsection we state the main results obtained in this paper.

Theorem 3. *Let G/H be an almost excellent spherical homogeneous space. Then the spherical homogeneous space $\widetilde{G}/(\widetilde{H})^0$, which is the simply connected covering space for G/H, is excellent.*

This theorem will be proved in § 2.1. Here we note that in the statement of the theorem we have not required that the homogeneous space G/H be affine.

Theorem 3 establishes a close connection between excellent and almost excellent spherical homogeneous spaces. First, as we mentioned above, every excellent spherical homogeneous space generates a local isomorphism class of almost excellent spherical homogeneous spaces. Second, by Theorem 3, for every local isomorphism class of almost excellent spherical homogeneous spaces, a simply connected space contained in it is excellent.

Further results relate to the classification, up to isomorphism, of all excellent affine spherical homogeneous spaces. Before stating these results, we introduce some additional concepts.

If $H \subset G$ is a spherical subgroup, then any subgroup $K \supset H$ is also spherical in G. If the subgroup H is connected, then the spherical homogeneous space G/H is called *saturated* (the spherical subgroup H is called *saturated*) if the group $N_G(H)/H$ is finite (see [3, § 1.3.4]). For any spherical homogeneous space G/H the space $G/N_G(H)^0$ is saturated; it is called the *saturation* of the space G/H ([3, § 1.3.4]). Next, a direct product of spherical homogeneous spaces $(G_1/H_1) \times (G_2/H_2) = (G_1 \times G_2)/(H_1 \times H_2)$ is a spherical homogeneous space. Spaces of this form, as well as spaces locally isomorphic to them, are called *reducible*; others are called *irreducible*. A spherical homogeneous space is called *strictly irreducible* if its saturation is irreducible ([3, § 1.3.6]).

Theorem 4. *Suppose that G/H is an excellent affine spherical homogeneous space. Then there exist strictly irreducible excellent affine spherical homogeneous spaces $G_1/H_1, \dots, G_m/H_m$ such that $G/H = G_1/H_1 \times \cdots \times G_m/H_m$.*

This theorem reduces the above-mentioned classification to the classification, up to isomorphism, of all strictly irreducible excellent affine spherical homogeneous spaces. The proof of Theorem 4 is contained § 2.2.

In fact, Theorem 4 means that every irreducible excellent affine spherical homogeneous space is strictly irreducible. Generally speaking, this is false for arbitrary affine spherical homogeneous spaces.

Theorem 5. *A simply connected strictly irreducible affine spherical homogeneous space of a semisimple group is excellent if and only if it is locally isomorphic to one of the spaces G/H in Tables* 1 *and* 2 *whose number is not in a square.*

This result follows easily from the data contained in Tables 1 and 2. The explanations for the tables are given in §1.3.

The following theorem completes the classification.

Theorem 6. *The set of nonsimply connected strictly irreducible excellent affine spherical homogeneous spaces is exhausted up to isomorphism by the spaces G/H in Table* 3.

The proof of this theorem is contained in Sections 3 and 4.

1.3. This paper is based on the known classification, up to local isomorphism, of all strictly irreducible affine spherical homogeneous spaces G/H of connected semisimple algebraic groups G (for simple groups G this classification coincides with the classification of all irreducible spherical homogeneous spaces of the type indicated). For simple groups G this classification was obtained in [5], and for nonsimple semisimple groups G in [6] and [7] (see also the refinements in [8]).

In [5] all pairs (G, H), where G is a simple compact Lie group and H is a connected spherical subgroup of it, were found up to local isomorphism. The spherical subgroup was defined by the condition $\dim V^H \leq 1$ for any finite-dimensional irreducible G-module. This result can be extended to the case of complex Lie groups, since a pair of compact groups (G, H) is spherical if and only if the pair $(G^{\mathbb{C}}, H^{\mathbb{C}})$ is spherical, where $G^{\mathbb{C}}$ and $H^{\mathbb{C}}$ are the complexifications of the groups G and H, respectively (see [2, Remark 5]).

Furthermore, in the same paper, the semigroup $\Lambda_+(\widetilde{G}/(\widetilde{H})^0)$ was indicated for each of the pairs (G, H) found. The list of spherical pairs (G, H) (up to local isomorphism), where G is simple and H is reductive, is given in Table 1. If we compare this table with Table 1 in [5], we have corrected some inaccuracies, and eliminated some repeating cases, as well as cases corresponding to the nonsimple group $G = \mathrm{SO}_4$.

TABLE 1

No.	G	H	Embedding	$\Lambda_+\big(\widetilde{G}/(\widetilde{H})^0\big)$	Notes
1	SL_n	SO_n	β_n	$2\pi_1,\ 2\pi_2,\ \ldots,\ 2\pi_{n-1}$	$n \geq 2$
2	SL_{n+m}	$\mathrm{S}(\mathrm{L}_n \times \mathrm{L}_m)$	$(\alpha_n + \alpha'_m)\big\vert_H$	$\pi_1 + \pi_{n+m-1},\ \pi_2 + \pi_{n+m-2}, \ldots, \pi_m + \pi_n$	$n > m \geq 1$
3	SL_{2n}	$\mathrm{S}(\mathrm{L}_n \times \mathrm{L}_n)$	$(\alpha_n + \alpha'_n)\big\vert_H$	$\pi_1 + \pi_{2n-1},\ \pi_2 + \pi_{2n-2},\ \ldots,\ 2\pi_n$	$n \geq 1$
4^*	SL_{n+m}	$\mathrm{SL}_n \times \mathrm{SL}_m$	$\alpha_n + \alpha'_m$	$\pi_1 + \pi_{n+m-1}, \pi_2 + \pi_{n+m-2},\ \ldots,\ \pi_{m-1} + \pi_{n+1},$ $\pi_m,\ \pi_n$	$n > m \geq 1$
5	SL_{2n}	Sp_{2n}	γ_n	$\pi_2,\ \pi_4,\ \ldots,\ \pi_{2k},\ \ldots,\ \pi_{2n-2}$	$n \geq 2$
6^*	SL_{2n+1}	Sp_{2n}	$\gamma_n + 1$	$\pi_1,\ \pi_2,\ \ldots,\ \pi_{2n}$	$n \geq 1$
$\boxed{7^*}$	SL_{2n+1}	$\mathbb{C}^\times \times \mathrm{Sp}_{2n}$	$\lambda \cdot \gamma'_n + \lambda^{-2n}$	$k_1\pi_1 + k_2\pi_2 + \cdots + k_{2n}\pi_{2n}$, where $\sum_{i=0}^{n-1}(n-i)k_{2i+1} - \sum_{i=1}^{n} ik_{2i} = 0$	$n \geq 1$
8	SO_{2n}	GL_n	$\alpha_n + \alpha^*_n$	$\pi_2,\ \pi_4,\ \ldots,\ \pi_{n-2},\ 2\pi_n$ for even n; $\pi_2,\ \pi_4,\ \ldots,\ \pi_{n-3},\ \pi_{n-1} + \pi_n$ for odd n	$n \geq 3$
9^*	SO_{2n}	SL_n	$\alpha_n + \alpha^*_n$	$\pi_2,\ \pi_4,\ \ldots,\ \pi_{n-3},\ \pi_{n-1},\ \pi_n$	$n \geq 3$, n is odd
10^*	SO_{2n+1}	GL_n	$\alpha_n + \alpha^*_n + 1$	$\pi_1,\ \pi_2,\ \ldots,\ \pi_{n-1},\ 2\pi_n$	$n \geq 2$

11	SO_{n+m}	$\mathrm{SO}_n \times \mathrm{SO}_m$	$\beta_n + \beta'_m$	$2\pi_1,\, 2\pi_2,\, \ldots,\, 2\pi_{m-1},\, \pi_m$ if $m < \left[\frac{n+m-1}{2}\right]$ $2\pi_1,\, 2\pi_2,\, \ldots,\, 2\pi_{r-2},\, \pi_{r-1} + \pi_r$ if $m+n = 2r$ and $m = r-1$ $2\pi_1,\, 2\pi_2,\, \ldots,\, 2\pi_r$ if $m+n = 2r+1$ and $m = r$	$n > m \geq 1$, $n + m \neq 4$
12	SO_{2n}	$\mathrm{SO}_n \times \mathrm{SO}_n$	$\beta_n + \beta'_n$	$2\pi_1,\, 2\pi_2,\, \ldots,\, 2\pi_{n-1},\, 2\pi_n$	$n \geq 3$
13^*	SO_9	Spin_7	$R(\pi_3) + 1$	$\pi_1,\, \pi_4$	
14^*	SO_7	G_2	$R(\pi_1)$	π_3	
15^*	SO_8	G_2	$R(\pi_1) + 1$	$\pi_1,\, \pi_3,\, \pi_4$	
$\boxed{16^*}$	SO_{10}	$\mathrm{SO}_2 \times \mathrm{Spin}_7$	$\beta_2 + R(\pi'_3)$	$k_1\pi_1 + k_2\pi_2 + k_4\pi_4 + k_5\pi_5$, where $2k_1 \geq \lvert k_4 - k_5 \rvert$ and $2k_1 - (k_4 - k_5) \equiv 0 \bmod 4$	
17	Sp_{2n}	GL_n	$\alpha_n + \alpha^*_n$	$2\pi_1,\, 2\pi_2,\, \ldots,\, 2\pi_n$	$n \geq 2$
18	Sp_{2n+2m}	$\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$	$\gamma_n + \gamma'_m$	$\pi_2,\, \pi_4,\, \ldots,\, \pi_{2m}$	$n \geq m \geq 1$
19^*	Sp_{2n}	$\mathbb{C}^\times \times \mathrm{Sp}_{2n-2}$	$\lambda + \lambda^{-1} + \gamma'_{n-1}$	$2\pi_1,\, \pi_2$	$n \geq 2$
20^*	G_2	SL_3	$R(\pi_1) + R(\pi_2) + 1$	π_1	
21	G_2	$\mathrm{SL}_2 \times \mathrm{SL}_2$	$R(2\pi) + R(\pi + \pi')$	$2\pi_1,\, 2\pi_2$	
22	F_4	Spin_9	$1 + R(\pi_1) + R(\pi_4)$	π_1	
23	F_4	$\mathrm{Sp}_6 \times \mathrm{SL}_2$	$R(\pi_2) + R(\pi_1 + \pi')$	$2\pi_1,\, 2\pi_2,\, 2\pi_3,\, 2\pi_4$	

24	E_6	Sp_8	$R(\pi_2)$	$2\pi_1,\ 2\pi_2,\ \ldots,\ 2\pi_6$	
25	E_6	F_4	$1+R(\pi_1)$	$\pi_1,\ \pi_5$	
26^*	E_6	Spin_{10}	$1+R(\pi_1)+R(\pi_5)$	$\pi_1,\ \pi_5,\ \pi_6$	
27	E_6	$\mathbb{C}^\times\times\mathrm{Spin}_{10}$	$\lambda^{-4}+\lambda^2R(\pi_1)+\lambda^{-1}R(\pi_5)$	$\pi_1+\pi_5,\ \pi_6$	
28	E_6	$\mathrm{SL}_6\times\mathrm{SL}_2$	$R(\pi_4)+R(\pi_1+\pi')$	$\pi_1+\pi_5,\ \pi_2+\pi_4,\ 2\pi_3,\ 2\pi_6$	
29	E_7	$\mathbb{C}^\times\times\mathrm{E}_6$	$\lambda^3+\lambda^{-3}+\lambda R(\pi_1)+\lambda^{-1}R(\pi_5)$	$2\pi_1,\ \pi_2,\ \pi_6$	
30	E_7	SL_8	$R(\pi_2)+R(\pi_6)$	$2\pi_1,\ 2\pi_2,\ \ldots,\ 2\pi_7$	
31	E_7	$\mathrm{Spin}_{12}\times\mathrm{SL}_2$	$R(\pi_6)+R(\pi_1+\pi')$	$\pi_2,\ \pi_4,\ 2\pi_5,\ 2\pi_6$	
32	E_8	Spin_{16}	$R(\pi_2)+R(\pi_8)$	$2\pi_1,\ 2\pi_2,\ \ldots,\ 2\pi_8$	
33	E_8	$\mathrm{SL}_2\times\mathrm{E}_7$	$R(\pi+\pi_1')+R(2\pi)+R(\pi_6')$	$2\pi_1,\ 2\pi_2,\ \pi_3,\ \pi_7$	

TABLE 2

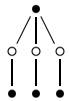
No.	$G \supset H$	Embedding diagram	$\Lambda_+(\widetilde{G}/(\widetilde{H})^0)$	Notes
$\boxed{1}$	$\mathrm{SL}_n \times \mathrm{SL}_{n+1} \supset \mathrm{SL}_n \times \mathbb{C}^\times$		$k_1\varphi_1 + k_2(\pi_{n-1}+\varphi_2) + \dots + k_n(\pi_1+\varphi_n) + l_1(\pi_{n-1}+\varphi_1) + \dots$ $\dots + l_{n-1}(\pi_1+\varphi_{n-1}) + l_n\varphi_n$, where $\sum_{i=1}^n (n+1-i)k_i = \sum_{i=1}^n i l_i$	$n \geq 2$
$\boxed{2}$	$\mathrm{SL}_n \times \mathrm{Sp}_{2m}$ $\supset \mathbb{C}^\times \times \mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$		$\pi_2+\pi_{n-2}$, $2\pi_1+\pi_{n-2}+2\varphi_1$, $\pi_2+2\pi_{n-1}+\varphi_1$, $\pi_1+\pi_{n-1}+2\varphi_1$, $\pi_1+\pi_{n-1}$, φ_2 $(m\geq 2)$ if $n\geq 4$; $\pi_1+\pi_2, 3\pi_1+2\varphi_1, 3\pi_2+2\varphi_1, \pi_1+\pi_2+2\varphi_1, \varphi_2$ $(m\geq 2)$ if $n=3$	$n\geq 3, m\geq 1$
$\boxed{3}$	$\mathrm{SL}_n \times \mathrm{Sp}_{2m} \supset \mathrm{SL}_{n-2} \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m-2}$		π_{n-2}, φ_2 $(m\geq 2)$, $\pi_{n-1}+\varphi_1$, π_2, $\pi_1+\pi_{n-1}$, $\pi_1+\varphi_1$	$n\geq 5$, $m\geq 1$
4	$\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \supset \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2m-2}$		π_2 $(n\geq 2)$, φ_2 $(m\geq 2)$, $\pi_1+\varphi_1$	$n\geq 1$, $m\geq 1$
$\boxed{5}$	$\mathrm{Sp}_{2n} \times \mathrm{Sp}_4 \supset \mathrm{Sp}_{2n-4} \times \mathrm{Sp}_4$		$\pi_1+\varphi_1$, $\pi_2+\varphi_2$, $\pi_3+\varphi_1$, π_4 $(n\geq 4)$, π_2, $\pi_1+\pi_3+\varphi_2$	$n\geq 3$
$\boxed{6}$	$\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \times \mathrm{Sp}_{2l}$ $\supset \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_{2m-2} \times \mathrm{Sp}_{2l-2} \times \mathrm{Sp}_2$		π_2 $(n\geq 2)$, φ_2 $(m\geq 2)$, ψ_2 $(l\geq 2)$, $\pi_1+\varphi_1$, $\varphi_1+\psi_1$, $\pi_1+\psi_1$	$n\geq 1$, $m\geq 1$, $l\geq 1$
$\boxed{7}$	$\mathrm{Sp}_{2n} \times \mathrm{Sp}_4 \times \mathrm{Sp}_{2m}$ $\supset \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2m-2}$		π_2 $(n\geq 2)$, φ_2, ψ_2 $(m\geq 2)$, $\pi_1+\varphi_1$, $\varphi_1+\psi_1$, $\pi_1+\varphi_2+\psi_1$	$n\geq 1$, $m\geq 1$
$\boxed{8}$	$\mathrm{SO}_n \times \mathrm{SO}_{n+1} \supset \mathrm{SO}_n$		$\varphi_1+\varphi_2$, $\pi_1+\varphi_1$, $\pi_1+\varphi_2$ if $n=3$; φ_1, $\pi_1+\varphi_1$, $\pi_1+\varphi_2$, $\pi_2+\varphi_2, \dots$, $\pi_{k-2}+\varphi_{k-1}$, $\pi_{k-1}+\pi_k+\varphi_{k-1}$, $\pi_{k-1}+\varphi_k$, $\pi_k+\varphi_k$ if $n=2k$; φ_1, $\pi_1+\varphi_1$, $\pi_1+\varphi_2$, $\pi_2+\varphi_2$, $\dots$, $\pi_{k-1}+\varphi_{k-1}$, $\pi_{k-1}+\varphi_k+\varphi_{k+1}$, $\pi_k+\varphi_k$, $\pi_k+\varphi_{k+1}$ if $n=2k+1\geq 5$	$n\geq 3$
9	$H \times H \supset H$, where H is simple		$\pi_i+\varphi_i^*$, $i=1,\dots,\mathrm{rk}\,H$	

TABLE 3

No.	Local isomorphism class	$G \supset H_0$	H	$\Lambda_+(G/H)$	Notes
1	1.8	$\mathrm{SO}_8 \supset \mathrm{GL}_4$	$sH_0 \cup H_0$	$2\pi_2, 2\pi_4$	
2	1.11	$\mathrm{SO}_{n+m} \supset \mathrm{SO}_n \times \mathrm{SO}_m$	$\mathrm{S}(\mathrm{O}_n \times \mathrm{O}_m)$	$2\pi_1,\ 2\pi_2,\ \ldots,\ 2\pi_m$ if $m < \left[\frac{n+m-1}{2}\right]$ $2\pi_1,\ 2\pi_2,\ \ldots,\ 2\pi_{r-2},\ 2\pi_{r-1} + 2\pi_r$ if $m+n = 2r$ and $m = r-1$ $2\pi_1,\ 2\pi_2, \ldots, 2\pi_{r-1}, 4\pi_r$ if $m+n = 2r+1$ and $m = r$	$n > m \geq 1$
3	1.14	$\mathrm{SO}_9 \supset \mathrm{Spin}_7$	H_0	$\pi_1, 2\pi_4$	
4	1.15	$\mathrm{SO}_7 \supset \mathrm{G}_2$	H_0	$2\pi_3$	
5	1.20	$\mathrm{Sp}_8 \supset \mathrm{Sp}_4 \times \mathrm{Sp}_4$	$sH_0 \cup H_0$	$2\pi_2, \pi_4$	
6	1.21	$\mathrm{Sp}_{2n} \supset \mathbb{C}^\times \times \mathrm{Sp}_{2n-2}$	$sH_0 \cup H_0$	$4\pi_1, \pi_2$	$n \geq 2$
7	1.22	$\mathrm{G}_2 \supset \mathrm{SL}_3$	$sH_0 \cup H_0$	$2\pi_1$	
8	2.4	$\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \supset \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2m-2}$	$wH_0 \cup H_0$	π_2 $(n \geq 2)$, φ_2 $(m \geq 2)$, $2\pi_1 + 2\varphi_1$	$n \geq 1$, $m \geq 1$
9	2.9	$\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1} \supset \mathrm{SO}_{2n+1}$	H_0	$\pi_1 + \varphi_1, \ldots, \pi_{2n-1} + \varphi_{2n-1}, 2\pi_{2n} + 2\varphi_{2n}$	$n \geq 1$
10	2.9	$\mathrm{Sp}_4 \times \mathrm{Sp}_4 \supset \mathrm{Sp}_4$	$(E, -E)H_0 \cup H_0$	$2\pi_1 + 2\varphi_1, \pi_2 + \varphi_2$	

Up to local isomorphism, all the strictly irreducible spherical homogeneous spaces G/H, where G is a nonsimple semisimple complex algebraic group and H is a reductive subgroup, were found in [6] and [7]. The corresponding list of spherical pairs is presented in Table 2. The semigroups $\Lambda_+(\widetilde{G}/(\widetilde{H})^0)$ for the corresponding simply connected spaces can easily be calculated using the results of [9] (or are calculated in that paper).

In Table 3 we present all the nonsimply connected excellent affine spherical homogeneous spaces G/H, up to isomorphism.

We will now explain the notation in Tables 1, 2, and 3. First of all we point out that in Tables 1 and 2 the map of the group H into G admits a nontrivial finite kernel.

In this paragraph we will only explain the notation in Table 1. If a number is starred, this means that the corresponding spherical homogeneous space is not locally isomorphic to any symmetric space. The column "Embedding" describes the map of the group H into the group G (in most cases this is an embedding). The symbols α_n, β_n, γ_n, λ denote the tautology representations of the groups GL_n (or SL_n), SO_n, Sp_{2n}, $\mathbb{C}^\times$, respectively. If one of these symbols is starred it denotes the dual representation. The symbol 1 denotes the trivial one-dimensional representation. The symbol $R(\omega)$ denotes the irreducible representation of the corresponding group with highest weight ω. If the subgroup H consists of two factors, then the representation of the second factor is marked by a prime. The sign + in the column "Embedding" means the direct sum of the corresponding irreducible H-modules.

In Table 2 the diagrams of embedding of the group H into the group G are deciphered as follows (the description is taken almost word-for-word from [3, § 1.3.6]). The white nodes of a diagram correspond to factors of the group G, and black to factors of the group H. The order of the nodes is the same as the order of the corresponding factors, with the exception of row 6, where the top black node corresponds to the last factor of the group H. The factor of the group H corresponding to a black node v is embedded diagonally into the product of those factors of the group G that correspond to the white nodes connected with v.

Now we clarify the notation in Table 3. The symbol $a.b$ in the column "Local isomorphism class" means that this space is locally isomorphic to the space (or to one of the spaces of the family) in row b of the table with number a. In the column "$G \supset H_0$", the subgroups H_0 are connected subgroups of the group G satisfying $H^0 = H_0$. The embedding of the subgroup H_0 into G is shown in the corresponding place in Table 1 or 2. In the column "H", the subgroups $H \subset G$ are indicated explicitly. In this column (in rows 1, 5, 6, 7), $s \in G$ is an element permuting invariant subspaces of the same dimension of the subgroup H_0 in the space of the tautology (for the group G_2 the simplest) representation of the group G (an explicit expression for s can be found in § 4). See § 4.2 for a description of the element $w \in G$ in row 8. In rows 3, 4 and 9 we have $H = H_0$; here, $G/H \simeq \widetilde{G}/\varphi^{-1}(H_0)$ and the subgroup $\widetilde{H}_0 = \varphi^{-1}(H_0) \subset \widetilde{G}$ consists of two connected components. The symbol E in row 10 denotes the identity of the group G.

In Tables 1 and 2, in the column "$\Lambda_+(\widetilde{G}/(\widetilde{H})^0)$", in all the rows, apart from rows 7 and 16 in Table 1, and row 1 in Table 2, the indecomposable elements of the semigroup $\Lambda_+(\widetilde{G}/(\widetilde{H})^0)$ are indicated. In the three rows singled out, all the elements of this semigroup are indicated. We point out that in Tables 1 and 2 the semigroup $\Lambda_+(\widetilde{G}/(\widetilde{H})^0)$ coincides with the semigroup $\Lambda_+(G/H)$ only if the group G does not contain factors of SO type. In Table 3 in the column "$\Lambda_+(G/H)$" the indecomposable elements of the semigroup $\Lambda_+(G/H)$ are indicated. If the group G is not simple, then the symbols π_i, φ_i, and ψ_i denote the ith fundamental weight of the first, second, and third factors of the group G, respectively. The symbol φ_i^* denotes the highest weight of the representation dual to the representation with highest weight φ_i. If a number in the first column of

Table 1 or Table 2 is in a square, this means that the corresponding space is not almost excellent.

Some notation and conventions:

$\mathbb{N}$ is the set of positive integers, $\mathbb{N} = \{1, 2, \dots\}$;
$\mathbb{Z}_+$ is the set of nonnegative integers;
$\mathbb{C}^\times$ is the multiplicative group of the field $\mathbb{C}$;
$Z_L(K)$ is the centralizer of a subgroup K in a group L;
$N_L(K)$ is the normalizer of a subgroup K in a group L;
$Z(L)$ is the centre of a group L;
L' is the derived subgroup of a group L;
L^0 is the connected component of the identity of a group L;
$\mathfrak{X}(L)$ is the character group (in the additive notation) of a group L;
$\operatorname{Aut} L$ is the automorphism group of a group L;
$\operatorname{Int} L$ is the group of inner automorphisms of a group L;
$\operatorname{rk} L$ is the rank of a reductive group L, that is, the dimension of a maximal torus in L;
X^L is the set of elements of a set X that are fixed under the action of a group L;
X^l is the set of elements of a set X that are fixed under an element $l \in L$ under the action of a group L;
$a(g)$ is the automorphism of a group G that takes an arbitrary element $x \in G$ to gxg^{-1} $(g \in G)$;
$V[\lambda]$ is an irreducible G-module with highest weight λ;
V^* is the dual space of a vector space V, that is, the space of linear functions on V;
E_n is the identity matrix of order n;
F_n is the matrix of order n with ones on the skew diagonal and zeros elsewhere;
$\operatorname{diag}(a_1, \dots, a_n)$ is the diagonal matrix of order n with elements $a_1, \dots, a_n$ on the diagonal;
id is the identity map;
the identity of a group G is denoted by E;
if L is a group, when we write $L = L_1 \leftthreetimes L_2$ (or $L = L_2 \rtimes L_1$) we mean that L is a *semidirect product* of subgroups L_1, L_2, that is, $L = L_1L_2$, the subgroup L_1 is normal in L and the intersection $L_1 \cap L_2$ consists of the identity of the group L;
if L is a group, when we write $L = L_1 \cdot L_2$ we mean that L is an *almost direct product* of subgroups L_1, L_2, that is, $L = L_1L_2$, the subgroups L_1, L_2 commute with each other, and the intersection $L_1 \cap L_2$ is finite;
if λ is a dominant weight of a group G, then the highest weight of the irreducible G-module $V[\lambda]^*$ is called the *dual weight of the weight* λ and is denoted by λ^*; if furthermore $\lambda^* = \lambda$, then the weight λ is called *self-dual*;
if $v_1, \dots, v_k$ are elements of a vector space V, then $\langle v_1, \dots, v_k \rangle$ denotes their linear span in V;
if $l_1, \dots, l_k$ are elements of a group L, then $\langle l_1, \dots, l_k \rangle$ denotes the subgroup of L generated by these elements;
the groups $\mathfrak{X}(B)$ and $\mathfrak{X}(T)$ are identified by restricting characters from B to T;
automorphisms of algebraic groups and their differentials, which are automorphisms of the corresponding Lie algebras, are denoted by the same symbols.

2. Reduction of the classification to the case of strictly irreducible spaces

2.1. First we point out several simple properties of spherical homogeneous spaces.

Lemma 1. *Let $H_1 \subset H_2 \subset G$ be a chain of groups. Then $\Lambda_+(G/H_2) \subset \Lambda_+(G/H_1)$. If H_1, H_2 are reductive spherical subgroups of G, then this inclusion is strict if and only if $H_1 \neq H_2$.*

Proof. As the morphism $\xi : G/H_1 \to G/H_2$, $gH_1 \mapsto gH_2$, is surjective, there is an embedding of algebras and G-modules $\xi^* : \mathbb{C}[G/H_2] \hookrightarrow \mathbb{C}[G/H_1]$. Therefore, $\Lambda_+(G/H_2) \subset \Lambda_+(G/H_1)$.

Now suppose that the groups H_1, H_2 are reductive and spherical. Then both varieties G/H_1 and G/H_2 are affine and the spectra of the representations of the group G in the spaces $\mathbb{C}[G/H_1]$ and $\mathbb{C}[G/H_2]$ are simple. If $H_1 \neq H_2$, then ξ is not an isomorphism, so neither is ξ^*, and hence $\Lambda_+(G/H_2) \neq \Lambda_+(G/H_1)$. □

We now consider a situation we encounter in this paper when we study homogeneous spaces which are locally isomorphic to a given simply connected homogeneous space. Suppose that $H_0 \subset H \subset G$ is a chain of groups, where H_0 is connected and spherical, and $H^0 = H_0$. Since the group H_0 is normal in H, the action of the group H on the right on the spaces $\mathbb{C}[G/H_0]$ and ${}^U\mathbb{C}[G/H_0]$ is well defined. Since the subgroup H_0 acts trivially here, in fact we are dealing with the action on the right of the (finite) quotient group H/H_0. This action commutes with the action on the left of the torus T; therefore the group H/H_0 preserves all the weight subspaces with respect to T. Consider an arbitrary element $\omega \in \Lambda_+(G/H_0)$. Since the subgroup H_0 is spherical, the weight subspace of weight ω with respect to T in the space ${}^U\mathbb{C}[G/H_0]$ is one-dimensional and coincides with $\langle f_{H_0}[\omega]\rangle$. Therefore it is invariant under H/H_0; that is, under the action of the group H/H_0 the function $f_{H_0}[\omega]$ is multiplied by some character χ_ω of this group. Furthermore, $\omega \in \Lambda_+(G/H)$ if and only if the function $f_{H_0}[\omega]$ is invariant under H/H_0; that is, it is contained in the algebra ${}^U\mathbb{C}[G/H]$. If d_ω is the order of the character χ_ω in the group $\mathfrak{X}(H/H_0)$, then the function $(f_{H_0}[\omega])^{d_\omega}$, which is proportional to the function $f_{H_0}[d_\omega\omega]$, is invariant under H/H_0; therefore $d_\omega\omega \in \Lambda_+(G/H)$. Since the number d_ω divides the order of the group H/H_0, we have proved the following.

Lemma 2. *Suppose that $H_0 \subset H \subset G$ is a chain of subgroups in which H_0 is connected and spherical, and $H^0 = H_0$. Let d denote the order of the finite group H/H_0. Then $\Lambda_+(G/H_0) \supset \Lambda_+(G/H) \supset d\Lambda_+(G/H_0)$.*

Corollary 1. *The property that a spherical subgroup $H \subset G$ be almost excellent is local; that is, it depends only on the Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$.*

Proof. This follows from Lemma 2 and the equation

$$\mathbb{Q}_+\Lambda_+(G/H_0) = \mathbb{Q}_+(d\Lambda_+(G/H_0))$$

for any $d \in \mathbb{N}$. □

Corollary 2. *Under the hypotheses of Lemma* 2, $\operatorname{Supp}\Lambda_+(G/H_0) = \operatorname{Supp}\Lambda_+(G/H)$.

This corollary obviously follows from Lemma 2.

Now our immediate aim is to prove Theorem 3. Before we do this, we recall one more concept. For every character $\chi \in \mathfrak{X}(H)$, consider the subspace of the algebra $\mathbb{C}[G]$,

$$V_\chi = \{f \in \mathbb{C}[G] : \ f(gh) = \chi(h)f(g) \text{ for any } g \in G,\ h \in H\}.$$

It is easy to see that the action of the group G on the left on the space $\mathbb{C}[G]$ preserves the subspace V_χ for any $\chi \in \mathfrak{X}(H)$. If we take all pairs of the form (λ, χ), where $\lambda \in \Lambda_+(G)$ and $\chi \in \mathfrak{X}(H)$, for which the space V_χ contains an irreducible G-module with highest weight λ, they form a semigroup with respect to addition. This semigroup is called the *extended weight semigroup* of the homogeneous space G/H (see [9] for a more detailed

description). We denote this semigroup by $\widehat{\Lambda}_+(G/H)$. We have $\Lambda_+(G/H) = \{(\lambda, \chi) \in \widehat{\Lambda}_+(G/H) : \chi = 0\}$.

Proof of Theorem 3. Taking Remark 1 into account, we can assume that the group G is simply connected.

By Theorem 1 in [9], the semigroup $\widehat{\Lambda}_+(G/H^0)$ is free. We list all its indecomposable elements as $(\lambda_1, \chi_1), \dots, (\lambda_n, \chi_n)$. Next, let the elements $\mu_1, \dots, \mu_m \in \Lambda_+(G/H^0)$ be such that

1) they generate the convex cone $\mathbb{Q}_+\Lambda_+(G/H^0)$;
2) $\operatorname{Supp} \mu_i \cap \operatorname{Supp} \mu_j = \varnothing$ for any $i \neq j$;
3) for every $i = 1, \dots, m$ we have $\mu_i/z \notin \Lambda_+(G/H^0)$ for any positive integer $z \geq 2$.

Then $\mu_i = \sum_{j=1}^n a_{ij}\lambda_j$ for every $i = 1, \dots, m$, where $a_{i1}, \dots, a_{in}$ are uniquely determined nonnegative integers. Furthermore, $\sum_{j=1}^n a_{ij}\chi_j = 0$ $(i = 1, \dots, m)$. Property 3) means that the numbers $a_{i1}, \dots, a_{in}$ are relatively prime (in totality) for each $i = 1, \dots, m$. Property 2) implies that, for any fixed j, at most one of the numbers $a_{1j}, \dots, a_{mj}$ is nonzero. Consider an arbitrary element $\omega \in \Lambda_+(G/H^0)$. By property 1) for some (uniquely determined) nonnegative rational numbers $b_1, \dots, b_m$ we have $\omega = \sum_{i=1}^m b_i\mu_i = \sum_{i=1}^m \sum_{j=1}^n b_i a_{ij}\lambda_j = \sum_{j=1}^n c_j\lambda_j$, where $c_j = \sum_{i=1}^m b_i a_{ij}$. Furthermore, in the sum defining c_j there is at most one nonzero term. We observe that $c_j \in \mathbb{Z}$ for any $j = 1, \dots, n$, since $\omega \in \widehat{\Lambda}_+(G/H^0)$. Suppose that $b_k \notin \mathbb{Z}$ for some $k \in \{1, \dots, m\}$. Since the numbers $a_{k1}, \dots, a_{kn}$ are relatively prime (in totality), there exists a_{kl} such that $b_k a_{kl} \notin \mathbb{Z}$. Therefore, $c_l \notin \mathbb{Z}$, a contradiction. Thus, all the numbers $b_1, \dots, b_m$ are integers and the semigroup $\Lambda_+(G/H^0)$ is freely generated by the elements $\mu_1, \dots, \mu_m$. $\square$

2.2. In this subsection we prove Theorem 4.

Lemma 3. *Suppose that G/H is a strictly irreducible affine spherical homogeneous space, where the group G is simply connected and the subgroup H is connected and saturated. Then* $\operatorname{Supp} \Lambda_+(G/H) = \operatorname{Supp} \Lambda_+(G/H')$.

Proof. If $H' = H$ there is nothing to prove. Therefore we assume that $H' \neq H$.

First suppose that the group G is simple. Then the (simply connected) space G/H is locally isomorphic to one of the spaces of Table 1, and the semigroup $\Lambda_+(G/H)$ is described in the column "$\Lambda_+(\widetilde{G}/(\widetilde{H})^0)$". If the space G/H' is spherical (that is, G/H is locally isomorphic to the space in one of rows 2, 7, 8 (n odd) or 27 of Table 1), then a space locally isomorphic to it and the description of the semigroup $\Lambda_+(G/H')$ are also contained in Table 1 (G/H' is locally isomorphic to the space in row 4, 6, 9 or 26 of Table 1, respectively). Therefore we can verify the assertion of the lemma directly. If the space G/H' is not spherical (that is, G/H is locally isomorphic to the space in one of rows 3, 8 (n even), 10, 11 ($m = 2$), 16, 17, 19 or 29 in Table 1), then it is a space of complexity 1 and is locally isomorphic to one of the spaces in Table 1 of [10] (and has number 1, 6, 5, 4, 13, 7, 8 or 16, respectively, in the table). The semigroups $\Lambda_+(G/H')$ are also described in the table. For the reader's convenience, we give the corresponding fragment of Table 1 from [10] in Table 4. In this table we have used the same notation as in Table 1. By comparing the descriptions of the semigroups $\Lambda_+(G/H)$ and $\Lambda_+(G/H')$, we see that the assertion of the lemma holds in this case too.

Now suppose that the group G is not simple. Then G/H is the space in either row 1 or 2 of Table 2. For each of these spaces, the extended weight semigroup is known (see Table 1 in [9]). As we know this semigroup, the semigroup $\Lambda_+(G/H')$ can easily be calculated in both cases in view of Remark 2 of [9]. Furthermore, if G/H is the space in row 2 of Table 2, then for $n \geq 5$ the space G/H' is spherical and is listed in row 3 of Table 2, and the semigroup $\Lambda_+(G/H')$ is also described there. By comparing

Table 4

No.	G	H	Embedding	$\Lambda_+(\widetilde{G}/(\widetilde{H})^0)$	Notes
1	SL_{2n}	$\mathrm{SL}_n \times \mathrm{SL}_n$	$\alpha_n + \alpha'_n$	$\pi_1 + \pi_{2n-1},\ \pi_2 + \pi_{2n-2}, \dots,$ $\pi_{n-1} + \pi_{n+1},\ \pi_n$	$n \geq 1$
2	SO_{2n}	SL_n	$\alpha_n + \alpha^*_n$	$\pi_2,\ \pi_4,\ \dots,\ \pi_{n-2},\ \pi_n$	$n \geq 4$, n is even
3	SO_{2n+1}	SL_n	$\alpha_n + \alpha^*_n + 1$	$\pi_1,\ \pi_2,\ \dots,\ \pi_{n-1},\ \pi_n$	$n \geq 2$
4	SO_{n+2}	SO_n	$\beta_n + 1 + 1$	$\pi_1,\ \pi_2$	$n \geq 3$
5	SO_{10}	Spin_7	$R(\pi_3) + 1 + 1$	$\pi_1,\ \pi_2,\ \pi_4,\ \pi_5$	
6	Sp_{2n}	SL_n	$\alpha_n + \alpha^*_n$	$2\pi_1,\ 2\pi_2,\ \dots,\ 2\pi_{n-1},\ \pi_n$	$n \geq 2$
7	Sp_{2n}	Sp_{2n-2}	$\gamma_{n-1} + 1 + 1$	$\pi_1,\ \pi_2$	$n \geq 2$
8	E_7	E_6	$R(\pi_1) + R(\pi_5) + 1 + 1$	$\pi_1,\ \pi_2,\ \pi_6$	

the semigroups $\Lambda_+(G/H)$ and $\Lambda_+(G/H')$ in both cases, we obtain the assertion of the lemma. $\square$

Proof of Theorem 4. We can assume without loss of generality (see Remark 1) that the group G is simply connected. It follows from the classification of affine spherical homogeneous spaces (see [8]) that there exist strictly irreducible spherical pairs $(G_1, H_1), \dots, (G_m, H_m)$ satisfying the following conditions:

1) G_i is a simply connected semisimple group and H_i is a connected saturated reductive subgroup of it ($i = 1, \dots, m$);
2) $G = G_1 \times \dots \times G_m$;
3) $H^0 = Z \cdot (H'_1 \times \dots \times H'_m)$, where Z is some subgroup of $Z(H_1) \times \dots \times Z(H_m)$;
4) $G_i/p_i(H)$ is a spherical homogeneous space, where $p_i : G \to G_i$ is the projection onto the factor G_i, $i = 1, \dots, m$.

We note that, generally speaking, conditions 1)–4) are not sufficient for the subgroup H to be spherical in G (see [8]).

For every $i = 1, \dots, m$, we set $B_i = p_i(B)$, $T_i = p_i(T)$, $U_i = p_i(U)$. Clearly, B_i, T_i, and U_i are a Borel subgroup, a maximal torus, and a maximal unipotent subgroup in G_i, respectively, $i = 1, \dots, m$.

We have the chain of inclusions

$$H'_1 \times \dots \times H'_m \subset H \subset p_1(H) \times \dots \times p_m(H). \tag{1}$$

For every $i = 1, \dots, m$, by Corollary 2 (as $p_i(H)^0 = p_i(H^0)$) we obtain

$$\mathrm{Supp}\ \Lambda_+(G_i/p_i(H)) = \mathrm{Supp}\ \Lambda_+(G_i/p_i(H)^0) = \mathrm{Supp}\ \Lambda_+(G_i/p_i(H^0)).$$

From the classification in Tables 1 and 2, $\dim Z(H_i) \leq 1$, and therefore $p_i(H^0)$ coincides with either H_i or H'_i. By Lemma 3, $\mathrm{Supp}\ \Lambda_+(G_i/H'_i) = \mathrm{Supp}\ \Lambda_+(G_i/H_i)$, and so $\mathrm{Supp}\ \Lambda_+(G_i/p_i(H)) = \mathrm{Supp}\ \Lambda_+(G_i/H'_i)$.

We now show that $\Lambda_+(G/H) \cap \Lambda_+(G_i) = \Lambda_+(G_i/p_i(H))$ for any $i = 1, \dots, m$, where $\Lambda_+(G_i)$ is the set of dominant weights of the group G_i with respect to B_i. The inclusion "$\supset$" is obvious. We will prove the reverse inclusion. Let $\omega \in \Lambda_+(G/H) \cap \Lambda_+(G_i)$ be an arbitrary element. Consider the function $f_H[\omega] \in {}^U\mathbb{C}[G]^H$ depending on the argument $g = (g_1, \dots, g_m) \in G$, where $g_k \in G_k$ for $k = 1, \dots, m$. Consider any $j \neq i$ and fix

arbitrary elements g_k, $k \neq j$, in the argument g. Then the function $f_H[\omega](g)$ becomes a regular function on G_j. The latter function is invariant under B_j (since $\omega \in \Lambda_+(G_i)$) and therefore is constant. Therefore, the function $f_H[\omega]$ is independent of g_j for $j \neq i$ and can be regarded as a regular function on G_i. This function, obviously, lies in the subspace ${}^{U_i}\mathbb{C}[G_i]^{p_i(H)}$ and is B_i-semi-invariant of weight ω; therefore, $\omega \in \Lambda_+(G_i/p_i(H))$, which proves the inclusion "$\subset$".

Since the space G/H is excellent, by the equation $\Lambda_+(G/H) \cap \Lambda_+(G_i) = \Lambda_+(G_i/p_i(H))$ $(i = 1, \dots, m)$ every indecomposable element of the semigroup $\Lambda_+(G_i/p_i(H))$ is an indecomposable element of the semigroup $\Lambda_+(G/H)$. Therefore the space $G_i/p_i(H)$ is also excellent.

It follows from the chain (1) that

$$\bigoplus_{i=1}^{m} \Lambda_+(G_i/H_i') \supset \Lambda_+(G/H) \supset \bigoplus_{i=1}^{m} \Lambda_+(G_i/p_i(H)).$$

Let ω be an arbitrary indecomposable element of the semigroup $\Lambda_+(G/H)$. Then

$$\operatorname{Supp} \omega \subset \bigcup_{i=1}^{m} \operatorname{Supp} \Lambda_+(G_i/H_i') = \bigcup_{i=1}^{m} \operatorname{Supp} \Lambda_+(G_i/p_i(H)).$$

Hence for some $j = 1, \dots, m$ and some indecomposable element ω' of the semigroup $\Lambda_+(G_j/p_j(H))$ we obtain $\operatorname{Supp} \omega \cap \operatorname{Supp} \omega' \neq \varnothing$, which implies that $\omega = \omega'$ and also $\omega \in \Lambda_+(G_j/p_j(H))$. Therefore, $\Lambda_+(G/H) = \bigoplus_{i=1}^{m} \Lambda_+(G_i/p_i(H))$, so by Lemma 1 we obtain $H = p_1(H) \times \dots \times p_m(H)$ and $G/H = G_1/p_1(H) \times \dots \times G_m/p_m(H)$. □

3. Preliminaries for the final stage of the classification

3.1. By Theorem 4 the problem of classifying all excellent affine spherical homogeneous spaces, up to isomorphism, is reduced to classifying the *strictly irreducible* spaces of this type. Let G/H be a strictly irreducible excellent affine spherical homogeneous space. Then by Theorem 3 its simply connected covering space $\widetilde{G}/(\widetilde{H})^0$ is also excellent. Hence the space G/H is locally isomorphic to one of the spaces in Tables 1 or 2 whose number is not inside a square. We can now reformulate our problem as follows: *for every known simply connected strictly irreducible excellent affine spherical homogeneous space G/H_0, find all the excellent spherical homogeneous spaces that are locally isomorphic to it.*

In the case of a simply connected group G, this problem is equivalent to finding all the excellent spherical subgroups $H \subset G$ such that $H^0 = H_0$. At that, G is a spinor group, then we find it preferable to do all the calculations not for G but for the corresponding orthogonal group. Because of this, generally speaking, the stated problem becomes more difficult. But in a number of cases, Lemma 5 and its Corollary 3, which we present below, turn out to be useful to us. Before stating and proving Lemma 5, we prove one more lemma.

Lemma 4. *Let $\omega \in \Lambda_+(G/H_0)$. Then $f_{H_0}[\omega](xz) = \omega(z)^{-1} f_{H_0}[\omega](x)$ for any element $z \in Z(G)$.*

Proof. The result follows from the chain of equalities:

$$f_{H_0}[\omega](xz) = f_{H_0}[\omega](zx) = (z^{-1} f_{H_0}[\omega])(x) = \omega(z)^{-1} f_{H_0}[\omega](x). \qquad \square$$

Lemma 5. *Suppose that $K \subset \widetilde{G}$ is a spherical subgroup and all weights in $\Lambda_+(\widetilde{G}/K^0)$ are equal to 1 on $\operatorname{Ker} \varphi$ (that is, all these weights are weights of the group G). Then $\widetilde{G}/K \simeq G/\varphi(K)$.*

Proof. We have $\mathbb{C}[G/\varphi(K^0)] = \mathbb{C}[\widetilde{G}/(K^0 \cdot \operatorname{Ker}\varphi)] = \mathbb{C}[\widetilde{G}/K^0]^{\operatorname{Ker}\varphi} \subset \mathbb{C}[\widetilde{G}/K^0]$. Consider an arbitrary weight $\omega \in \Lambda_+(\widetilde{G}/K^0)$ and the function $f_{K^0}[\omega] \in {}^U\mathbb{C}[\widetilde{G}/K^0]$ corresponding to it. Since $\operatorname{Ker}\varphi \subset Z(\widetilde{G})$, by Lemma 4 the action on the right of an arbitrary element $z \in \operatorname{Ker}\varphi$ on the function $f_{K^0}[\omega]$ multiplies it by $w(z)^{-1}$. By hypothesis we have $w(z) = 1$, whence $f_{K^0}[\omega] \in \mathbb{C}[\widetilde{G}/K^0]^{\operatorname{Ker}\varphi}$. Consequently, $\mathbb{C}[\widetilde{G}/K^0]^{\operatorname{Ker}\varphi} = \mathbb{C}[\widetilde{G}/K^0]$ and therefore, $\operatorname{Ker}\varphi \subset K^0$. It follows from the last inclusion that the map $\widetilde{G}/K \to G/\varphi(K)$, $gK \mapsto \varphi(g)\varphi(K)$, is an isomorphism. □

Corollary 3. *Suppose that $H_0 \subset G$ is a connected spherical subgroup and all the weights in $\Lambda_+(\widetilde{G}/(\widetilde{H}_0)^0)$ are equal to 1 on $\operatorname{Ker}\varphi$. Then the homogeneous space G/H_0 is simply connected and every homogeneous space locally isomorphic to it has the form G/H, where $H^0 = H_0$.*

If, for an orthogonal group G with a connected spherical subgroup H_0, the space $\widetilde{G}/(\widetilde{H}_0)^0$ is excellent while the space G/H_0 itself is not simply connected, then we need additional results to solve our problem. We will return to this later, in § 3.5.

3.2. As we saw in § 3.1, to solve the stated problem we need to know how to describe all the finite extensions of a given connected subgroup $H_0 \subset G$. We devote this subsection to achieving this.

Lemma 6. *Suppose that $H_0 \subset H \subset G$ is a chain of groups such that H_0 is connected and $H^0 = H_0$. Then $H \subset N_G(H_0)$.*

Proof. Consider an arbitrary element $h \in H$. It defines conjugation in the group G which leaves the group H, and consequently also its connected component H_0, invariant. Therefore, $h \in N_G(H_0)$. □

Thus, we need to be able to calculate the group $N_G(H_0)$. In this matter we shall be helped by Lemma 7, below. In order to state this lemma, we introduce some additional concepts. Let $\tau \in \operatorname{Aut} H_0/\operatorname{Int} H_0$ be a nontrivial outer automorphism of the group H_0. We say that τ *is realized in* G by an element $g \in G$ (an element g *realizes in* G the automorphism τ) if the automorphism $a(g) \in \operatorname{Int} G$ leaves the group H_0 invariant and defines an automorphism of it that belongs to the coset of the outer automorphism τ modulo $\operatorname{Int} H_0$.

Lemma 7. *Let $\tau_1, \dots, \tau_k$ be all different nontrivial outer automorphisms of the group H_0 realized in G, and let them be realized by elements $g_1, \dots, g_k \in G$, respectively. Then the group $N_G(H_0)$ is generated by the elements $g_1, \dots, g_k$, the subgroup $Z_G(H_0) \subset G$, and the group H_0.*

Proof. To prove this, we will consider all the automorphisms of the group G under the restriction to H_0. Consider an arbitrary element $g \in N_G(H_0)$. Then $a(g)$ is an automorphism of the group H_0. If g realizes one of the outer automorphisms τ_i, then $a(g_i^{-1}g) \in \operatorname{Int} H_0$; therefore it is sufficient to confine ourselves to the case $a(g) \in \operatorname{Int} H_0$. In this case there exists an element $h \in H_0$ such that $a(g) = a(h)$. Hence $a(h^{-1}g)$ is the identity automorphism of the group H_0, and therefore $h^{-1}g \in Z_G(H_0)$. □

In the case of a simple group G, we apply the arguments described below to find the outer automorphisms of the connected reductive spherical subgroup H_0 that are realized in G. Let K be the derived subgroup of the group H_0. Then either $K = \{e\}$ or K is a semisimple group. To make our arguments uniform, in the case $K = \{e\}$ we will assume that the empty root system and empty Dynkin diagram correspond to the group K (and to the group H_0). We assume that $G \subset \mathrm{GL}(V)$ for some finite-dimensional vector space V.

Let $\tau \in \operatorname{Aut} H_0/\operatorname{Int} H_0$ be some outer automorphism (possibly the identity), and $\sigma \in \operatorname{Aut} H_0$ some representative of it. Multiplying the automorphism σ by the automorphism $a(h)$ for a suitable choice of $h \in H_0$, we can assume that σ preserves some fixed maximal torus S in K and a fixed simple root system $\Pi \subset \mathfrak{X}(S)$ with respect to S. Thus σ defines some automorphism θ of the system Π. We identify the automorphism θ with the corresponding automorphism of the Dynkin diagram of the subgroup K (and of the subgroup H_0). Next, we let ν denote the restriction of the automorphism σ to the group $Z(H_0)^0$. Clearly, ν is an automorphism of this group. Since $\dim Z(H_0) \leq 1$ (see Table 1), ν is either the identity map or the inversion (the latter is possible only for $\dim Z(H_0) = 1$). Thus, we have associated the pair (θ, ν) with the outer automorphism $\tau \in \operatorname{Aut} H_0/\operatorname{Int} H_0$. It is easy to verify that this pair is independent of the choice of the representative σ of the outer automorphism $\tau \in \operatorname{Aut} H_0/\operatorname{Int} H_0$.

It is well known that the automorphism $\overline{\theta} = \sigma|_K \in \operatorname{Aut} K$ which preserves a maximal torus $S \subset K$ can be uniquely reconstructed from the corresponding automorphism θ of the Dynkin diagram of the group H_0, up to multiplication by $a(s)$, $s \in S$. Since the subgroup H_0 decomposes into an almost direct product of the form $H_0 = Z(H_0)^0 \cdot K$ (see Problem 4.1.22 in [1]), the outer automorphism τ of the group H_0 is uniquely determined by the corresponding pair (θ, ν). We point out that if the group $Z(H_0)^0 \cap K$ has elements of order greater than 2, then the automorphism ν is uniquely determined by the action of the automorphism σ on K (that is, by the automorphism θ).

Let θ be an automorphism of the Dynkin diagram of the subgroup H_0, and ν an automorphism of the group $Z(H_0)^0$. We say that the pair (θ, ν) is *compatible* if

1) there exists an automorphism $\overline{\theta} \in \operatorname{Aut} K$ that preserves a maximal torus $S \subset K$ and acts on the simple root system Π in the same way as θ;

2) the restrictions of the automorphisms $\overline{\theta}$ and ν to the subgroup $Z(H_0)^0 \cap K$ coincide.

For a pair (θ, ν), compatibility is well defined, since the automorphism $\overline{\theta}\big|_S$ is independent of the choice of $\overline{\theta}$. Obviously, every pair (θ, ν) corresponding to an outer automorphism of the group H_0 is compatible.

Now suppose that some element $g \in G$ realizes an automorphism $\tau \in \operatorname{Aut} H_0/\operatorname{Int} H_0$ in G, to which the pair (θ, ν) described above corresponds. We assume without loss of generality that the automorphism $\sigma = a(g)$ preserves the torus S and the simple root system Π with respect to S. We introduce the notation $W(\lambda, \chi)$ for an irreducible H_0-module that has the highest weight λ as a K-module and on which the group $Z(H_0)^0$ acts by multiplication by the character χ. Let $W \subset V$ be an arbitrary irreducible H_0-submodule such that $W \simeq W(\lambda, \chi)$. Then the subspace $gW \subset V$ is also an irreducible H_0-submodule. Let $w \in W$ be a weight vector of some weight $\mu \in \mathfrak{X}(S)$ with respect to S. Then for any element $s \in S$ we have $s(gw) = g(g^{-1}sg)w = g\mu(g^{-1}sg)w = \mu(\sigma^{-1}(s))gw = (\sigma(\mu))(s)gw$; that is, the vector $gw \in gW$ is a weight vector with respect to S of weight $\sigma(\mu)$. Therefore, the irreducible K-module gW has the highest weight $\sigma(\lambda)$. In a similar way, we see that the group $Z(H_0)^0$ acts on gW by multiplication by the character $\sigma(\chi)$. Thus, $gW \simeq W(\theta(\lambda), \nu(\chi))$.

By combining the results of the two preceding paragraphs, we obtain the following necessary condition for an outer automorphism of the group H_0 to be realized in G.

Theorem 7. *Let $G \subset \mathrm{GL}(V)$ be a simple group, and H_0 a connected reductive spherical subgroup of it. Let θ be an automorphism of the Dynkin diagram of the subgroup H_0, and ν an automorphism of the group $Z(H_0)^0$. Suppose that there exists an outer automorphism σ of the group H_0 to which the pair (θ, ν) corresponds and that σ is realized in G. Then*

1) *the pair (θ, ν) is compatible;*

2) *the multiplicity of occurrence in V of any irreducible H_0-module $W(\lambda,\chi)$ coincides with the multiplicity of occurrence in V of the irreducible H_0-module $W(\theta(\lambda),\nu(\chi))$.*

In a number of cases the following lemma provides information about the centralizer of a connected reductive subgroup in a connected reductive group.

Lemma 8. *Suppose that L is a connected reductive group, with a connected reductive subgroup $K \subset L$, and $\operatorname{rk} L = \operatorname{rk} K$. Then $Z_L(K) \subset K$.*

Proof. Consider a maximal torus S in the group K. Since $\operatorname{rk} L = \operatorname{rk} K$, it follows that S is also a maximal torus of the group L. Then we have the chain of inclusions $Z_L(K) \subset Z_L(S) = S \subset K$. □

Remark 2. In fact, under the hypotheses of Lemma 8, we have $Z_L(K) = Z(K)$, since $Z(K)$ coincides with the intersection of all maximal tori of the group K.

3.3. Once we have described all the subgroups $H \subset G$ such that $H^0 = H_0$ for a simply connected excellent affine spherical homogeneous space G/H_0 (the subgroup H_0 is automatically connected), to solve our problem it remains to determine which of these subgroups H are excellent in G. We note that in the cases that we consider, it is relatively rare for a nonconnected subgroup H to turn out to be excellent (see Table 3); therefore for our purpose we do not have to calculate the semigroups $\Lambda_+(G/H)$ for all subgroups H satisfying $H^0 = H_0$. In order to prove that a given subgroup H is nonexcellent, it is sufficient to verify that it does not satisfy condition 2) of Definition 1. In many cases it is much easier to prove the latter than to calculate $\Lambda_+(G/H)$. In this subsection we prove several sufficient conditions for nonexcellence of subgroups H with the condition $H^0 = H_0$. Here we will apply the material expounded in § 2.1 without further explanation.

Lemma 9. *Suppose that $H_0 \subset H \subset G$ is a chain of groups, where $H \neq H_0$, $H^0 = H_0$, and H_0 is an excellent spherical subgroup. Let ω_1, ω_2 be different indecomposable elements of the semigroup $\Lambda_+(G/H_0)$. Suppose that χ is a nontrivial character of the group H/H_0 such that $f_{H_0}[\omega_1](xh) = \chi(h) f_{H_0}[\omega_1](x)$ and $f_{H_0}[\omega_2](xh) = \chi(h)^{-1} f_{H_0}[\omega_2](x)$ for any $h \in H/H_0$. Then the subgroup H is not excellent.*

Proof. Since the character χ is nontrivial, the functions $f_{H_0}[\omega_1]$ and $f_{H_0}[\omega_2]$ are not H/H_0-invariant. Therefore, $\omega_1, \omega_2 \notin \Lambda_+(G/H)$. On the other hand, the function $f_{H_0}[\omega_1] f_{H_0}[\omega_2]$ is invariant under H/H_0; therefore, $\omega_1 + \omega_2 \in \Lambda_+(G/H)$, and the element $\omega_1 + \omega_2$ is indecomposable in the semigroup $\Lambda_+(G/H)$. The elements $d\omega_1$ and $d\omega_2$ are also indecomposable in this semigroup, where $d > 1$ is the order of the character χ in the group $\mathfrak{X}(H/H_0)$. Thus, the semigroup $\Lambda_+(G/H)$ is not free; therefore the subgroup H is not excellent. □

Lemma 10. *Suppose that $H_0 \subset G$ is a connected reductive spherical subgroup and a subgroup $H \subset G$ is such that $H^0 = H_0$. For every $\lambda \in \Lambda_+(G/H_0)$ we denote by χ_λ the character by which the group H/H_0 acts on the right on the function $f_{H_0}[\lambda]$. Then*

1) *χ_λ coincides with the character by which H/H_0 acts on the one-dimensional space $V[\lambda^*]^{H_0} \subset V[\lambda^*]$ by multiplication;*

2) *$\chi_{\lambda^*} = -\chi_\lambda$ for any $\lambda \in \Lambda_+(G/H_0)$.*

Proof. It is well known that the algebra $\mathbb{C}[G]$ as a $(G \times G)$-module (where the left factor acts on the left, and the right one on the right) is isomorphic to the direct sum $\bigoplus_\lambda V[\lambda] \otimes V[\lambda]^*$, where λ runs over all the dominant weights of the group G and the group $G \times G$ acts on every term of the form $V[\lambda] \otimes V[\lambda]^*$ component-wise. For every λ

we denote the highest weight vector of the irreducible G-module $V[\lambda]$ by v_λ. We have the following chain of isomorphisms of H/H_0-modules:

$$\begin{aligned} {}^U\mathbb{C}[G/H_0] = {}^U\mathbb{C}[G]^{H_0} &\simeq \bigoplus_\lambda V[\lambda]^U \otimes (V[\lambda]^*)^{H_0} \\ &\simeq \bigoplus_\lambda \langle v_\lambda\rangle \otimes (V[\lambda]^*)^{H_0} \simeq \bigoplus_\lambda (V[\lambda]^*)^{H_0}, \end{aligned} \tag{2}$$

where, as before, λ runs over the set of dominant weights of the group G.

Consider an arbitrary weight $\lambda \in \Lambda_+(G/H_0)$. Using (2) we see at once that Assertion 1) holds for it. Since the group H_0 is reductive, we have $V[\lambda]^* = (V[\lambda]^*)^{H_0} \oplus W_\lambda$ for some H_0-invariant subspace $W_\lambda \subset V[\lambda]^*$. We now consider a linear function $\xi \in (V[\lambda]^*)^* \simeq V[\lambda]$ such that $\xi|_{W_\lambda} \equiv 0$ and $\xi|_{(V[\lambda]^*)^{H_0}} \not\equiv 0$. It is easy to see that this linear function is H_0-invariant and under the action of the group H/H_0 it is multiplied by the opposite character to the character χ_λ. On the other hand, the condition $\xi \in V[\lambda]^{H_0}$ and assertion 1) imply that the group H/H_0 acts on ξ by multiplication by the character χ_{λ^*} (here, we have taken into account the fact that $V[\lambda]^* \simeq V[\lambda^*]$). Thus, $\chi_{\lambda^*} = -\chi_\lambda$ for any $\lambda \in \Lambda_+(G/H_0)$ and assertion 2) is proved. □

Lemma 11. *Let G/H_0 be a simply connected excellent affine spherical homogeneous space. Suppose that there are no self-dual elements among the indecomposable elements of the semigroup $\Lambda_+(G/H_0)$. Then none of the subgroups $H \subset G$ with $H \neq H_0$ and $H^0 = H_0$ is excellent.*

Proof. Suppose that $H \subset G$ is an arbitrary subgroup such that $H \neq H_0$ and $H^0 = H_0$. By Lemma 1 there exists an indecomposable element λ of the semigroup $\Lambda_+(G/H_0)$ that is not contained in $\Lambda_+(G/H)$. Then by Lemma 10 the group H/H_0 acts on the function $f_{H_0}[\lambda]$ by the character χ_λ, and on the function $f_{H_0}[\lambda^*]$ by the character $-\chi_\lambda$. By hypothesis we have $\lambda \neq \lambda^*$ and therefore Lemma 9 can be applied, which gives the required result. □

Lemma 12. *Suppose that G/H_0 is a simply connected excellent affine spherical homogeneous space and let $\omega_1, \dots, \omega_n$ be all the indecomposable elements of the semigroup $\Lambda_+(G/H_0)$. Also suppose that $\omega_1, \dots, \omega_{n-2}, \omega_{n-1} + \omega_n$ are all the indecomposable elements of the semigroup $\Lambda_+(G/N_G(H_0))$. Then no subgroup $H \subset G$ with $H \neq H_0$ and $H^0 = H_0$ is excellent.*

Proof. Let a subgroup $H \subset G$ be such that $H \neq H_0$ and $H^0 = H_0$. From Lemma 6 we obtain $H \subset N_G(H_0)$. Then $\Lambda_+(G/H_0) \supset \Lambda_+(G/H) \supset \Lambda_+(G/N_G(H_0))$ by Lemma 1; furthermore, in view of the hypothesis, the elements $\omega_1, \dots, \omega_{n-2}$ are indecomposable in the semigroup $\Lambda_+(G/H)$. If the element $\omega_{n-1} + \omega_n$ is decomposable in $\Lambda_+(G/H)$, then $\omega_{n-1}, \omega_n \in \Lambda_+(G/H)$. Hence, $\Lambda_+(G/H) = \Lambda_+(G/H_0)$, which is impossible by the condition $H \neq H_0$ and Lemma 1. Therefore the element $\omega_{n-1} + \omega_n$ is indecomposable in $\Lambda_+(G/H)$. It follows from Lemma 2 that for some positive integer $p \geq 1$ the semigroup $\Lambda_+(G/H)$ contains the indecomposable element $p\omega_n$. Thus condition 2) of Definition 1 does not hold for the subgroup H; therefore H is not excellent. □

3.4. In this subsection we describe a general method for calculating the weight semigroups for (algebraic) symmetric spaces (we follow the survey [11]; see also [12]). Furthermore, we discuss a situation where this method can be applied in § 4.1. The proofs of Theorems 8–10 are contained in Section 26 of [11], although these theorems are not stated in [11] in the form in which they are presented here. For Satake diagrams, see also [1, § 5.4.3].

Let σ be a nontrivial involutive automorphism of a group G (that is, $\sigma^2 = id$), and let $K = G^\sigma \subset G$ be the fixed point subgroup with respect to σ (note that K is reductive). In this situation the homogeneous space G/K is called an (algebraic) symmetric space. All symmetric spaces are classified up to local isomorphism; for a fixed group G the symmetric space G/K is uniquely defined by its *Satake diagram* $S(G/K)$, up to isomorphism. In this subsection we describe how the semigroup $\Lambda_+(G/K)$ can be calculated from the diagram $S(G/K)$.

Let G/K be a symmetric space, and $S(G/K)$ its Satake diagram. Let $T \subset G$ be an arbitrary σ-invariant maximal torus. We consider the root system $\Delta \subset \mathfrak{X}(T)$ of the group G with respect to the torus T and choose some simple root system Π in it. In the space $\mathfrak{X}(T) \otimes \mathbb{Q}$ we fix an inner product that is invariant with respect to the Weyl group $W = N_G(T)/T$ and the automorphism σ. We establish the natural one-to-one correspondence between the set Π and the nodes of the diagram $S(G/K)$. We denote by Π_0 the set of simple roots in Π which have corresponding black nodes in $S(G/K)$. Let $\Delta_0 \subset \Delta$ be the root system with the simple root system Π_0, and let $w_0 \in W$ be an element of greatest length in the Weyl group of the root system Δ_0. Finally, we introduce an involution ι of the simple root system Π as follows. If there are no arrows in the diagram $S(G/K)$, then we set $\iota = id$. If there are arrows in the diagram $S(G/K)$, then we take as ι the unique involution of the set Π which transposes any two nodes connected by arrows. Clearly, ι can be extended to an involutive automorphism of the group $\mathfrak{X}(T)$. Using this notation, the following theorem holds.

Theorem 8. *Let G/K be a symmetric space, and $S(G/K)$ its Satake diagram. Then there exist a σ-invariant maximal torus $T \subset G$ and a simple root system $\Pi \subset \Delta$ such that $\sigma|_{\mathfrak{X}(T)} = -\iota w_0$.*

For every symmetric space G/K we fix a σ-invariant maximal torus T and a simple root system $\Pi \subset \Delta$ satisfying the conditions of Theorem 8. We also fix the Borel subgroup $B \supset T$ that is uniquely determined by the simple root system Π. We denote by Q and P, respectively, the root lattice and the weight lattice of the root system Δ; $Q, P \subset \mathfrak{X}(T) \otimes \mathbb{Q}$. Consider the endomorphism ν of the space $\mathfrak{X}(T) \otimes \mathbb{Q}$ given by the formula $\nu(\lambda) = (\lambda - \sigma(\lambda))/2$. We set $\Delta_{G/K} = \nu(\Delta)\backslash\{0\}$, $\Pi(G/K) = \nu(\Pi)\backslash\{0\}$, $Q_{G/K} = \nu(Q)$, $P_{G/K} = \nu(P)$. Next, for every simple root $\alpha_i \in \Pi$ we denote the corresponding fundamental weight by ω_i. Now, with every orbit of the group $\{id, \iota\}$ on the set of white nodes of the diagram $S(G/K)$ we associate a dominant weight of the group G as follows. With a pair of different nodes $\{k, \iota(k)\}$ we associate the weight $\omega_k + \omega_{\iota(k)}$. If the ith node is not connected by arrows with other white nodes and is connected by an edge with a black node, then we associate with it the weight ω_i. If the jth node is not connected by arrows with other white nodes and is not connected by an edge with any black node, then we associate the weight $2\omega_j$ with it. We combine all the weights obtained, $\omega_k + \omega_{\iota(k)}$, ω_i, $2\omega_j$, in the set Ω_0.

Theorem 9. *In the notation introduced in the preceding paragraph,*

1) *$\Delta_{G/K}$ is a root system (possibly, nonreduced);*
2) *$\Pi_{G/K}$ is a simple root system of the root system $\Delta_{G/K}$;*
3) *$Q_{G/K}$ is the root lattice corresponding to the root system $\Delta_{G/K}$;*
4) *$P_{G/K}$ is the weight lattice corresponding to the root system $\Delta_{G/K}$;*
5) *Ω_0 is contained in $2P_{G/K}$ and consists of the fundamental weights corresponding to the simple root system $2\Pi_{G/K}$.*

The root system $\Delta_{G/K}$ is called the *(small) root system of the symmetric space G/K*.

The list of all symmetric spaces, up to local isomorphism, as well as of the corresponding Satake diagrams and small root systems is contained in [11, Table 5.9].

Table 5

No.	Number in Table 1	$G \supset H$	$\Delta_{\widetilde{G}/(\widetilde{H})^0}$
1	3	$\mathrm{SL}_{2n} \supset \mathrm{S}(\mathrm{L}_n \times \mathrm{L}_n)$	A_{n-1}
2	8	$\mathrm{SO}_{4n} \supset \mathrm{GL}_{2n}$	C_n
3	11	$\mathrm{SO}_{n+m} \supset \mathrm{SO}_n \times \mathrm{SO}_m$	B_m
4	12	$\mathrm{SO}_{2n} \supset \mathrm{SO}_n \times \mathrm{SO}_n$	D_n
5	17	$\mathrm{Sp}_{2n} \supset \mathrm{GL}_n$	C_n
6	18	$\mathrm{Sp}_{4n} \supset \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}$	C_n
7	29	$\mathrm{E}_7 \supset \mathbb{C}^{\cdot} \times \mathrm{E}_6$	C_3
8	30	$\mathrm{E}_7 \supset \mathrm{SL}_8$	E_7

It follows from part 5) of Theorem 9 that the elements of the set Ω_0 freely generate the semigroup of dominant weights corresponding to the simple root system $2\Pi_{G/K}$ of the root system $2\Delta_{G/K}$. We denote this semigroup by $\Omega(G/K)$.

Theorem 10. $\Lambda_+(G/K) = \Omega(G/K) \cap 2\nu(\mathfrak{X}(T))$.

Theorem 10 provides a practical method for calculating the semigroup $\Lambda_+(G/K)$. Namely, one can calculate the set $2\nu(\mathfrak{X}(T))$ knowing the action of σ on $\mathfrak{X}(T)$, and this action can be found from Theorem 8.

We now consider the situation in which we shall apply Theorem 10 (more precisely, Corollary 4 given below) in § 4.1. We denote the adjoint group of $G/Z(G)$ by $\widehat{G}$. Since $\sigma(Z(G)) = Z(G)$, the automorphism $\widehat{\sigma} \in \mathrm{Aut}(\widehat{G})$ given by the rule $\widehat{\sigma}(gZ(G)) = \sigma(g)Z(G)$ is well defined. The results of [13] (see 2.2, Lemma 1) imply that $\widehat{G}^{\widehat{\sigma}} = N_G(K)/Z(G)$. Therefore the homogeneous space $G/N_G(K) \simeq \widehat{G}/\widehat{G}^{\widehat{\sigma}}$ is a symmetric space of the adjoint group $\widehat{G}$. For this space we have $Q = \mathfrak{X}(T)$, so, taking part 3) of Theorem 9 into account, we obtain the following corollary to Theorem 10.

Corollary 4. *The semigroup $\Lambda_+(\widehat{G}/\widehat{G}^{\widehat{\sigma}}) \simeq \Lambda_+(G/N_G(H_0))$ coincides with the intersection of the semigroup $\Omega(G/K)$ of dominant weights corresponding to the simple root system $2\Pi_{G/K}$ with the root lattice of the root system $2\Delta_{G/K}$.*

The method for calculating the intersection mentioned in the corollary will now be described in the general situation.

Let Φ be an arbitrary reduced root system, $\Psi = \{\gamma_1, \dots, \gamma_n\} \subset \Phi$ a simple root system, and $\omega_1, \dots, \omega_n$ the fundamental weights corresponding to this simple root system. We have $(\gamma_1, \dots, \gamma_n) = (\omega_1, \dots, \omega_n)A^\top$, where A is the Cartan matrix of the simple root system Ψ, where the symbol $\top$ denotes taking the transpose. Let ω be an arbitrary dominant weight. It has the form $\omega = y_1\omega_1 + \dots + y_n\omega_n = (\omega_1, \dots, \omega_n)Y^\top$, where $Y = (y_1, \dots, y_n)$ is a tuple of nonnegative integers. Hence, $\omega = (\gamma_1, \dots, \gamma_n)(A^\top)^{-1}Y^\top$. Therefore, the *weight ω is contained in the root lattice of the root system Φ if and only if the column $(A^\top)^{-1}Y^\top$ consists of integers.*

In Table 5 we present the types of small root systems of some symmetric spaces, which we shall need in § 4.1. For every space G/H in Table 5 the fundamental weights corresponding to the simple roots in $2\Pi_{\widetilde{G}/(\widetilde{H})^0}$ can be found in Table 1 at the intersection

of the corresponding row with the column "$\Lambda_+(G/H)$". In all cases, apart from space 7, the fundamental weights are written out in Table 1 in the order corresponding to the numbering of the simple roots we have adopted; for space 7 the fundamental weights are written out in the reverse order.

3.5. In this subsection we discuss the situation where, for a connected spherical subgroup H_0 of an orthogonal group G, the space $\widetilde{G}/(\widetilde{H}_0)^0$ is excellent, while the space G/H_0 itself is not simply connected. In this situation the following lemma can be applied.

Lemma 13. *For an arbitrary semisimple group G and an arbitrary connected subgroup H_0 of it we have $N_{\widetilde{G}}((\widetilde{H}_0)^0) = \varphi^{-1}(N_G(H_0))$.*

Proof. Let $g \in N_{\widetilde{G}}((\widetilde{H}_0)^0)$. This means that $g(\widetilde{H}_0)^0 g^{-1} \subset (\widetilde{H}_0)^0$. Then $\varphi(g)H_0\varphi(g)^{-1} \subset H_0$ and therefore $\varphi(g) \in N_G(H_0)$. Hence, $g \in \varphi^{-1}(N_G(H_0))$.

Conversely, let $g \in \varphi^{-1}(N_G(H_0))$. Then $\varphi(g)H_0\varphi(g)^{-1} \subset H_0$. For any element $h \in (\widetilde{H}_0)^0$ we have $\varphi(ghg^{-1}) \in H_0$, whence $ghg^{-1} \in \varphi^{-1}(H_0) = \widetilde{H}_0$. Since the group $(\widetilde{H}_0)^0$ is connected, we obtain $ghg^{-1} \in (\widetilde{H}_0)^0$. □

In view of Lemma 13, in § 4.1 when we are in the situation we consider there, we shall always act as follows to describe all the homogeneous spaces locally isomorphic to G/H_0. First we will calculate the subgroup $N_G(H_0)$ (for example, by using the methods of § 3.2), and only then apply additional arguments.

4. The final stage of the classification

We now move on to the part of the paper where we do the calculations. In this section, for every almost excellent spherical homogeneous space G/H in Tables 1 and 2, we find all the nonsimply connected excellent spherical homogeneous spaces locally isomorphic to it. The methods described in § 3 provide a theoretical basis for the calculations. We consider the spaces in Table 1 in § 4.1, and the spaces in Table 2 in § 4.2.

4.1. In this subsection almost excellent spherical pairs are considered consecutively, following the way they are numbered in Table 1. Unless otherwise stipulated, the term "outer automorphism" means a nontrivial outer automorphism of the group H_0.

As a rule, the way we calculate the normalizers is based on Lemma 7: first, we find elements of the group G that realize nontrivial outer automorphisms of the subgroup H_0 (or we prove that certain outer automorphisms are not realized in G), and then we calculate the group $Z_G(H_0)$. We also use the data in Table 3 from [1].

In all cases V denotes the space of the tautology representation (for exceptional groups, the simplest) of a simple group G , so that $V \simeq V[\pi_1]$. In the space V we consider a fixed basis $e_1, \dots, e_p$, where $p = \dim V$. We consider all elements of the group G with respect to this basis. For the group $G = \mathrm{SO}_n$ (Sp_{2n}) we denote by Ω^O (Ω^S, respectively) a nondegenerate symmetric (skew-symmetric, respectively) bilinear form in V preserved by the group G. In the case $G = \mathrm{SO}_n$, unless otherwise stipulated, the basis $e_1, \dots, e_n$ is chosen so that the matrix $\Omega^O_n = F_n$ serves as the matrix of the form Ω^O in it. In the case $G = \mathrm{Sp}_{2n}$ the basis $e_1, \dots, e_{2n}$ is always chosen so that the matrix

$$\Omega^S_{2n} = \begin{pmatrix} 0 & F_n \\ -F_n & 0 \end{pmatrix}$$

is the matrix of the form Ω^S.

For a group G of type SL or Sp we assume by default that the subgroups B, U, and T consist, respectively, of all upper-triangular, upper-unitriangular and diagonal matrices contained in G.

1°. $G = \mathrm{SL}_n$, $H_0 = \mathrm{SO}_n$. If n is odd, then we are under the hypotheses of Lemma 11; therefore none of the nontrivial finite extensions of the subgroup H_0 is an excellent subgroup.

Now suppose that n is even. In this case it is more convenient for us to assume that in the basis $e_1, \dots, e_n$, the group SO_n preserves the symmetric bilinear form with matrix E_n. There is an outer automorphism realized in G by the element $s = \varepsilon \operatorname{diag}(-1, 1, \dots, 1)$, where $\varepsilon = e^{\pi i/n}$. We have $Z_G(H_0) = Z(G) \simeq \mathbb{Z}_n$ and $Z_G(H_0) = \langle s^2 \rangle$. Thus, $N_G(H_0) = \langle s \rangle \cdot H_0$.

Consider an arbitrary matrix $A \in G$. Let $A = (x_{ij})$ and $A^{-1} = (X_{ij})$. Then we have $f_{H_0}[2\pi_1](A) = X_{11}^2 + X_{21}^2 + \dots + X_{n1}^2$ and $f_{H_0}[2\pi_{n-1}](A) = x_{n1}^2 + x_{n2}^2 + \dots + x_{nn}^2$, up to a nonzero factor. As is easy to verify, the element s acts on the right on $f_{H_0}[2\pi_1]$ and $f_{H_0}[2\pi_{n-1}]$ by multiplication by ε^{-2} and ε^2, respectively. Let H be a nontrivial finite extension of the group H_0. It has the form $H = Z \cdot H_0$, where Z is the subgroup of $\langle s \rangle$ generated by the element $z = s^d$, where d is some divisor of the number $2n$. Since $s^n \in H_0$, we can assume that $d < n$. The element z acts on the right on $f_{H_0}[2\pi_1]$ and $f_{H_0}[2\pi_{n-1}]$ by multiplication by ε^{-2d} and ε^{2d}, respectively. Now $\varepsilon^{2d} \neq 1$ for $d < n$, so the subgroup H/H_0 acts on the right on the functions $f_{H_0}[2\pi_1]$ and $f_{H_0}[2\pi_{n-1}]$ by multiplication by nontrivial characters, and these characters are mutually inverse. For $n > 2$ by Lemma 9 we obtain that the subgroup H is not excellent.

For $n = 2$ we have $N_G(H_0) = sH_0 \cup H_0$, the space $G/N_G(H_0)$ is isomorphic to the space $\mathrm{SO}_3/\mathrm{S}(\mathrm{O}_2 \times \mathrm{O}_1)$ (see case 11°) and is excellent.

2°. $G = \mathrm{SL}_{n+m}$, $H_0 = \mathrm{S}(\mathrm{L}_n \times \mathrm{L}_m)$, $n > m \geq 1$. By Theorem 7 there are no outer automorphisms realized in G. Since $\operatorname{rk} G = \operatorname{rk} H_0$, by Lemma 8 we obtain $Z_G(H_0) \subset H_0$. Therefore, $N_G(H_0) = H_0$.

3°. $G = \mathrm{SL}_{2n}$, $H_0 = \mathrm{S}(\mathrm{L}_n \times \mathrm{L}_n)$, $n \geq 1$. By Theorem 7 there is only one outer automorphism; it is realized in G by the element $s = F_{2n}$ for even n, and by the element $s = e^{\pi i/(2n)} F_{2n}$ for odd n. From the condition $\operatorname{rk} G = \operatorname{rk} H_0$ we obtain $Z_G(H_0) \subset H_0$ by Lemma 8. Therefore, $N_G(H_0) = sH_0 \cup H_0$.

Clearly, the unique finite extension of the subgroup H_0 is the group $H = N_G(H_0)$. Since G/H_0 is a symmetric space, using Corollary 4 we can calculate the semigroup $\Lambda_+(G/H)$. We obtain

$$\Lambda_+(G/H) = \{k_1(\pi_1 + \pi_{2n-1}) + k_2(\pi_2 + \pi_{2n-2}) + \dots + 2k_n\pi_n \mid \\ k_i \in \mathbb{Z}_+, k_1 + k_3 + \dots + k_{2m-1} \in 2\mathbb{Z}\},$$

where $m = \left[\frac{n+1}{2}\right]$. For $n \geq 3$ the elements $2(\pi_1 + \pi_{2n-1})$ and $\pi_1 + \pi_3 + \pi_{2n-3} + \pi_{2n-1}$ are indecomposable in $\Lambda_+(G/H)$; therefore condition 2) of Definition 1 does not hold and the subgroup H is not excellent.

For $n = 2$ and $n = 1$ the space G/H is isomorphic to the space $\mathrm{SO}_6/\mathrm{S}(\mathrm{O}_4 \times \mathrm{O}_2)$ and $\mathrm{SO}_3/\mathrm{S}(\mathrm{O}_2 \times \mathrm{O}_1)$, respectively (see case 11°); both spaces are excellent.

4°. $G = \mathrm{SL}_{n+m}$, $H_0 = \mathrm{SL}_n \times \mathrm{SL}_m$, $n > m \geq 1$. By Theorem 7 there are no outer automorphisms realized in G. We assume without loss of generality that the first and second factors of the group H_0 act irreducibly on the subspaces $\langle e_1, \dots, e_n \rangle$ and $\langle e_{n+1}, \dots, e_{n+m} \rangle$ of the space V, respectively. We have

$$Z_G(H_0) = \{\operatorname{diag}(\underbrace{\lambda, \dots, \lambda}_{n}, \underbrace{\mu, \dots, \mu}_{m}) \mid \lambda, \mu \in \mathbb{C}^\times, \lambda^n \mu^m = 1\}.$$

Hence, $N_G(H_0) = \mathrm{S}(\mathrm{L}_n \times \mathrm{L}_m)$.

We have shown that $(G, N_G(H_0))$ is the spherical pair in row 2 of Table 1. It is easy to verify that the hypotheses of Lemma 12 hold; therefore the group H_0 has no nontrivial finite extensions H that are excellent subgroups in G.

5°. $G = \mathrm{SL}_{2n}$, $H_0 = \mathrm{Sp}_{2n}$. There are no outer automorphisms. Furthermore, $Z_G(H_0) = Z(G) = \langle s\rangle$, where $s = e^{\pi i/n}E_{2n}$. Hence, $N_G(H_0) = \langle s\rangle \cdot H_0$.

Let H be a nontrivial finite extension of the group H_0. Then $H = Z \cdot H_0$, where $Z = \langle s^d\rangle$ for some divisor d of the number $2n$. Since $s^n \in H_0$, we can assume that $d < n$. By Lemma 4, under the action on the right of the element s^d, the functions $f_{H_0}[\pi_2]$ and $f_{H_0}[\pi_{2n-2}]$ are multiplied by $e^{-2\pi i d/n}$ and $e^{2\pi i d/n}$, respectively. This action is nontrivial, since $d < n$. If $n \geq 3$, then the subgroup H is not excellent by Lemma 9.

For $n = 2$ we have $N_G(H_0) = sH_0 \cup H_0$, the space $G/N_G(H_0)$ is isomorphic to the space $\mathrm{SO}_6/\mathrm{S}(\mathrm{O}_5 \times \mathrm{O}_1)$ (see case 11°) and is excellent.

6°. $G = \mathrm{SL}_{2n+1}$, $H_0 = \mathrm{Sp}_{2n}$. There are no outer automorphisms. We assume without loss of generality that the subgroup H_0 acts irreducibly on the subspace $\langle e_1, \dots, e_{2n}\rangle$ of the space V, preserves the skew-symmetric bilinear form with matrix Ω^S_{2n} in it, and preserves the vector $e_{2n+1} \in V$. We have $Z_G(H_0) = \{\mathrm{diag}(t, \dots, t, t^{-2n}) \mid t \in \mathbb{C}^\times\}$, whence $N_G(H_0) = Z \cdot H_0$, where $Z = Z_G(H_0)$.

Let H be an arbitrary finite extension of the subgroup H_0, $H \neq H_0$. We denote by $\nu(t)$ the image of an element $t \in \mathbb{C}^\times$ in Z. We have $V[\pi_2] \simeq \bigwedge^2 V$; furthermore, the vector $w = e_1 \wedge e_{2n} + e_2 \wedge e_{2n-1} + \dots + e_n \wedge e_{n+1} \in \bigwedge^2 V$ is invariant under H_0 and is multiplied by t^2 under the action of the element $\nu(t)$. By Lemma 10 the action of an arbitrary element $\nu(t) \in Z \cap H$ multiplies the functions $f_{H_0}[\pi_2]$ and $f_{H_0}[\pi_{2n-1}]$ by t^2 and t^{-2}, respectively. Since $\nu(-1) \in H_0$, the hypotheses of Lemma 9 apply; therefore the subgroup H is not excellent.

8°. $G = \mathrm{SO}_{2n}$, $H_0 = \mathrm{GL}_n$. In this case the hypotheses of Corollary 3 hold; therefore every homogeneous space locally isomorphic to G/H_0 has the form G/H, where $H^0 = H_0$. The group H_0 has one outer automorphism defined by the conjugation by the matrix $s = F_{2n}$. For even n we have $s \in G$; therefore the element s realizes in G this outer automorphism. For odd n we have $s \in \mathrm{O}_{2n}\backslash\mathrm{SO}_{2n}$, and this outer automorphism is not realized in G. Indeed, if this automorphism were realized in G by an element $s_0 \in G$, then we would have $a(s^{-1}s_0) \in \mathrm{Int}\, H_0$; we can assume without loss of generality that $s^{-1}s_0 \in Z_{\mathrm{O}_{2n}}(H_0)$. But $Z_{\mathrm{O}_{2n}}(H_0) \subset G$, which gives rise to a contradiction. Next, $Z_G(H_0) \subset H_0$; therefore $N_G(H_0) = sH_0 \cup H_0$ for even n, and $N_G(H_0) = H_0$ for odd n.

Let $n = 2m$. Then the group H_0 admits the unique nontrivial finite extension $H = N_G(H_0)$. Since $\widetilde{G}/\widetilde{H}_0$ is a symmetric space, we can calculate the semigroup $\Lambda_+(G/H)$ using Corollary 4. We obtain

$$\Lambda_+(G/H) = \{k_1\pi_2 + k_2\pi_4 + \dots + k_{m-1}\pi_{2m-2} + 2k_m\pi_{2m} \mid k_i \in \mathbb{Z}_+, k_1 + k_3 + \dots + k_{2l-1} \in 2\mathbb{Z}\},$$

where $l = [\frac{m+1}{2}]$. If $m \geq 3$, then the semigroup $\Lambda_+(G/H)$ contains the indecomposable elements $2\pi_2$ and $\pi_2 + \pi_6$ for $m > 3$, and $2\pi_2$ and $\pi_2 + 2\pi_6$ for $m = 3$. Condition 2) of Definition 1 does not hold; therefore the subgroup H is not excellent.

For $m = 2$ the subgroup H is excellent. In this case the indecomposable elements of the semigroup $\Lambda_+(G/H)$ are $2\pi_2, 2\pi_4$.

9°. $G = \mathrm{SO}_{2n}$, $H_0 = \mathrm{SL}_n$, n is odd. We assume without loss of generality that the subgroup H_0 acts irreducibly on the subspaces $\langle e_1, \dots, e_n\rangle, \langle e_{n+1}, \dots, e_{2n}\rangle \subset V$. The subgroup H_0 has one outer automorphism, but it is not realized in G (see case 8°). Next, $Z_G(H_0) = \{\mathrm{diag}(\underbrace{t, \dots, t}_{n}, \underbrace{t^{-1}, \dots, t^{-1}}_{n})\}$. Therefore the subgroup $N_G(H_0)$ coincides with the subgroup GL_n in case 8°. By Lemma 13 the subgroup $N_{\widetilde{G}}(\widetilde{H}_0)$ coincides with the subgroup $N_{\widetilde{G}}(\widetilde{H}_0)$ for case 8°. The hypotheses of Lemma 12 apply; therefore none of the nontrivial finite extensions of the subgroup $\widetilde{H}_0$ is an excellent subgroup in $\widetilde{G}$.

10°. $G = \mathrm{SO}_{2n+1}$, $H_0 = \mathrm{GL}_n$. In this case Corollary 3 can be applied; therefore we have to find finite extensions of the subgroup H_0. We assume without loss of generality

that H_0 acts irreducibly on the subspaces $\langle e_1, \dots, e_n\rangle, \langle e_{n+2}, \dots, e_{2n+1}\rangle \subset V$ and preserves the vector e_{n+1}. There is an outer automorphism, which is realized in G by the element s, where s is the matrix with zeros outside the skew diagonal and ones on the skew diagonal, except for the central element, which is equal to 1 for even n, and to -1 for odd n. Furthermore, we have $Z_G(H_0) \subset H_0$. Thus, $N_G(H_0) = sH_0 \cup H_0$.

There is the unique nontrivial finite extension $H = N_G(H_0)$ of the subgroup H_0. We set $V_1 = \langle e_1, \dots, e_n, e_{n+2}, \dots, e_{2n+1}\rangle$ and $V_2 = \langle e_{n+1}\rangle$. Both spaces V_1, V_2 are H-invariant, and $V = V_1 \oplus V_2$. The irreducible representations of the group SO_{2n+1} with highest weights $\pi_1, \pi_2, \dots, \pi_{n-1}, 2\pi_n$ are realized in the spaces V, $\bigwedge^2 V$, $\dots$, $\bigwedge^{n-1} V$, $\bigwedge^n V$, respectively (see [14, Supplement, §§5, 26]). By the branching rule for the group SO_{2n+1} (see, for example, Theorem 4 in [9] or the original paper [15]), for $j \le n-1$ the irreducible $\mathrm{SO}(V)$-module $\bigwedge^j V$ decomposes into the direct sum of two irreducible $\mathrm{SO}(V_1)$-modules with highest weights π_j and π_{j-1}, respectively, in the case $j \le n-2$, and π_{n-2} and $\pi_{n-1} + \pi_n$ in the case $j = n-1$. Next we consider two cases depending on the parity of n.

If n is even, then the subgroup H is contained in $\mathrm{SO}(V_1)$ and we can use the results obtained in case 8° considering irreducible G-modules as $\mathrm{SO}(V_1)$-modules. For $n \ge 4$ we obtain that in the irreducible G-modules with highest weights π_2, π_3 there are no H-invariant vectors. Therefore, on the weight functions $f_{H_0}[\pi_2]$, $f_{H_0}[\pi_3]$ the element s acts on the right by multiplication by -1. Then by Lemma 9 the subgroup H is not excellent. For $n = 2$ the space G/H is locally isomorphic to the space $\mathrm{Sp}_4/(\mathbb{C}^\times \times \mathrm{Sp}_2)$ (see case 19°) and is excellent.

We now consider the case where n is odd. We note that in the irreducible G-modules $V \simeq V[\pi_1]$ and $\bigwedge^2 V \simeq V[\pi_2]$ the vectors fixed under H_0 are e_{2n+1} and $e_1 \wedge e_{2n+1} + e_2 \wedge e_{2n} + \dots + e_n \wedge e_{n+2}$, respectively. It is easy to see that the action of the element s multiplies each of these vectors by -1. By Lemma 10, under the action of the element s on the right the functions $f_{H_0}[\pi_1]$, $f_{H_0}[\pi_2]$ are multiplied by -1. By Lemma 9 we obtain that the subgroup H is not excellent.

11°. $G = \mathrm{SO}_{n+m}$, $H_0 = \mathrm{SO}_n \times \mathrm{SO}_m$. If $n+m$ is even (odd), then we assume that $m+n = 2r$ ($m+n = 2r+1$). The hypotheses of Corollary 3 hold; therefore every homogeneous space locally isomorphic to G/H_0 has the form G/H, where $H^0 = H_0$. To find the group $N_G(H_0)$ we proceed as follows. Suppose that $V_1, V_2 \subset V$ are invariant subspaces of the subgroup H_0 such that the first and second factors of H_0 act irreducibly on V_1 and V_2, respectively, so that $\dim V_1 = n$, $\dim V_2 = m$. Any element of the subgroup $N_G(H_0)$ must take H_0-invariant subspaces to H_0-invariant subspaces. Since $m \ne n$, it follows that $N_G(H_0)$ preserves each of the subspaces V_1, V_2. Furthermore, the group $N_G(H_0)$ preserves the symmetric bilinear forms $\Omega^O\big|_{V_1}$ and $\Omega^O\big|_{V_2}$, and so $N_G(H_0) = G \cap (\mathrm{O}_n \times \mathrm{O}_m) = \mathrm{S}(\mathrm{O}_n \times \mathrm{O}_m)$.

The only nontrivial finite extension of the subgroup H_0 is the subgroup $H = N_G(H_0)$. Since $\widetilde{G}/\widetilde{H}_0$ is a symmetric space, we can use Corollary 4 to calculate the semigroup $\Lambda_+(G/H)$. As a result we obtain the following. For $m < [\frac{m+n-1}{2}]$ the semigroup $\Lambda_+(G/H)$ is generated by the elements $2\pi_1, \dots, 2\pi_m$; for $n+m = 2r$, $m = r-1$, by the elements $2\pi_1, \dots, 2\pi_{r-2}, 2\pi_{r-1} + 2\pi_r$; for $n+m = 2r+1$, $m = r$, by the elements $2\pi_1, \dots, 2\pi_{r-1}, 4\pi_r$. Therefore, the subgroup H is excellent for any m, n.

12°. $G = \mathrm{SO}_{2n}$, $H_0 = \mathrm{SO}_n \times \mathrm{SO}_n$, $n \ge 3$. Corollary 3 can be applied; therefore every homogeneous space locally isomorphic to G/H_0 has the form G/H, where $H^0 = H_0$. We assume that the basis $e_1, \dots, e_{2n}$ of the space V is orthonormal with respect to the form Ω^O and that the first and second factors of the group G act irreducibly on the subspaces $V_1 = \langle e_1, \dots, e_n\rangle$ and $V_2 = \langle e_{n+1}, \dots, e_{2n}\rangle$, respectively. We find the group $N_G(H_0)$ in the same way as in case 11°. As in case 11°, the set of elements of $N_G(H_0)$

preserving invariant subspaces of the subgroup H_0 is a subgroup $K = \mathrm{S}(\mathrm{O}_n \times \mathrm{O}_n) \subset G$. Let $s = F_{2n}$ for even n, and $s = \Omega^S_{2n}$ for odd n. In both cases, $s \in G$. Then we have $s \in N_G(H_0)$, and s transposes the spaces V_1 and V_2. Thus, $N_G(H_0) = sK \cup K$. We fix the element $z = \mathrm{diag}(-1, 1, 1, \dots, 1, -1) \in K \backslash H_0$.

Consider the element $w = \mathrm{diag}(\underbrace{1, \dots, 1}_{n}, \underbrace{-1, \dots, -1}_{n})$ and the involutive automorphism $\sigma = a(w) \in \mathrm{Aut}\, G$. We have $G^\sigma = K$. The Satake diagram of the symmetric space G/K coincides with the Dynkin diagram D_n. After choosing a σ-invariant maximal torus T and a simple root system $\Pi \subset \mathfrak{X}(T)$ as in Theorem 8, we have $\sigma(\chi) = -\chi$ for any character $\chi \in \mathfrak{X}(T)$. Using Theorem 10 we obtain the following system of indecomposable elements of the semigroup $\Lambda_+(G/K)$: $2\pi_1, \dots, 2\pi_{n-2}, 4\pi_{r-1}, 2\pi_{r-1} + 2\pi_r, 4\pi_r$. Hence, first, under the action on the right of the element z the functions $f_{H_0}[2\pi_1], \dots, f_{H_0}[2\pi_{n-2}]$ are left invariant, while the functions $f_{H_0}[2\pi_{n-1}], f_{H_0}[2\pi_n]$ are multiplied by -1. Second, we obtained that the subgroup K is not excellent in G, since condition 2) of Definition 1 does not hold for it.

Now suppose that n is even. Then $s^2 \in H_0$; therefore the group H_0 has four different nontrivial finite extensions: K, $sH_0 \cup H_0$, $szH_0 \cup H_0$, and $N_G(H_0)$. Calculating the semigroup $\Lambda_+(G/N_G(H_0))$, by Corollary 4 we have

$$\Lambda_+(G/N_G(H_0)) = \{2k_1\pi_1 + \dots + 2k_n\pi_n \mid \\ k_i \in \mathbb{Z}_+, k_{n-1} + k_n \in 2\mathbb{Z}, k_1 + k_3 + \dots + k_{n-1} \in 2\mathbb{Z}\}.$$

The elements $4\pi_n, 2\pi_1 + 2\pi_{n-1} + 2\pi_n$ are indecomposable in $\Lambda_+(G/N_G(H_0))$; therefore the subgroup $N_G(H_0)$ is not excellent. Comparing the semigroups $\Lambda_+(G/K)$ and $\Lambda_+(G/N_G(H_0))$ we find that under the action on the right of the element s the functions $f_{H_0}[2\pi_2], f_{H_0}[2\pi_4], \dots, f_{H_0}[2\pi_{n-2}], (f_{H_0}[2\pi_{n-1}])^2, (f_{H_0}[2\pi_n])^2$ are invariant, and the functions $f_{H_0}[2\pi_1], f_{H_0}[2\pi_3], \dots, f_{H_0}[2\pi_{n-3}], f_{H_0}[2\pi_{n-1}]f_{H_0}[2\pi_n]$ are multiplied by -1. Therefore, under the action on the right of the element s exactly one of the functions $f_{H_0}[2\pi_{n-1}]$, $f_{H_0}[2\pi_n]$ is invariant, and the other is multiplied by -1. Applying Lemma 9 to the latter function and to the function $f_{H_0}[2\pi_1]$ we find that the subgroup $sH_0 \cup H_0$ is not excellent. It is easy to see that the element sz acts on the right on the functions $f_{H_0}[2\pi_1]$, $\dots$, $f_{H_0}[2\pi_{n-2}]$, $(f_{H_0}[2\pi_{n-1}])^2$, $f_{H_0}[2\pi_{n-1}]f_{H_0}[2\pi_n]$, $(f_{H_0}[2\pi_n])^2$ in the same way as the element s. Therefore the subgroup $szH_0 \cup H_0$ is not excellent for the same reasons as the subgroup $sH_0 \cup H_0$.

We now consider the case of odd n. We have $s^2 \in K \backslash H_0$ and $s^4 = E$; therefore, $N_G(H_0) = \langle s \rangle \leftthreetimes H_0$, where $\langle s \rangle \simeq \mathbb{Z}_4$. Therefore the subgroup H_0 has only two nontrivial finite extensions K and $N_G(H_0)$. By using Corollary 4 we can calculate the semigroup $\Lambda_+(G/N_G(H_0))$. We obtain

$$\Lambda_+(G/N_G(H_0)) = \{2k_1\pi_1 + \dots + 2k_n\pi_n \mid \\ k_i \in \mathbb{Z}_+, k_{n-1} + k_n \in 2\mathbb{Z}, 2k_1 + 2k_3 + 2k_{n-2} + k_{n-1} + 3k_n \in 4\mathbb{Z}\}.$$

In this semigroup the elements $8\pi_n$ and $2\pi_{n-1} + 2\pi_n$ are indecomposable; therefore the subgroup $N_G(H_0)$ is not excellent.

13°. $G = \mathrm{SO}_9$, $H_0 = \mathrm{Spin}_7$. We choose a basis $e_1, \dots, e_9$ of the space V so that in it the form Ω^O has matrix E_9. We assume without loss of generality that the group H_0 acts irreducibly on the subspace $V_1 = \langle e_1, \dots, e_8 \rangle$, and trivially on the subspace $V_2 = \langle e_9 \rangle$. There are no outer automorphisms. The group $Z_G(H_0)$ acts as a scalar on each of the subspaces V_1, V_2. Since the restrictions $\Omega^O\big|_{V_1}$ and $\Omega^O\big|_{V_2}$ of the form Ω^O are nondegenerate, on each of the subspaces V_1 and V_2 the group $Z_G(H_0)$ can act only by multiplication by $+1$ or -1. Clearly, the action on V_2 is trivial; therefore

$Z_G(H_0) = Z(\mathrm{SO}_8) \subset \mathrm{Spin}_7$ and $N_G(H_0) = H_0$. By Lemma 13 we obtain $N_{\widetilde{G}}((\widetilde{H}_0)^0) = \widetilde{H}_0 = Z(\widetilde{G}) \times (\widetilde{H}_0)^0$.

The only nontrivial finite extension of the subgroup $(\widetilde{H}_0)^0$ in the group $\widetilde{G}$ is the group $\widetilde{H}_0 = N_{\widetilde{G}}((\widetilde{H}_0)^0)$; furthermore, $\widetilde{G}/\widetilde{H}_0 \simeq G/H_0$.

Since a nontrivial element of the group $Z(\widetilde{G})$ acts on the irreducible $\widetilde{G}$-modules with highest weights π_1 and π_4 as the identity and by multiplication by -1, respectively, the semigroup $\Lambda_+(\widetilde{G}/\widetilde{H}_0)$ is generated by the elements π_1, $2\pi_4$, and the subgroup $\widetilde{H}_0$ is excellent in $\widetilde{G}$.

14°. $G = \mathrm{SO}_7$, $H_0 = \mathrm{G}_2$. There are no outer automorphisms. The subgroup H_0 acts on V irreducibly; therefore $Z_G(H_0) = Z(G) = \{e\}$. Hence $N_G(H_0) = H_0$ and by Lemma 13 we obtain $N_{\widetilde{G}}((\widetilde{H}_0)^0) = \widetilde{H}_0 = Z(\widetilde{G}) \times (\widetilde{H}_0)^0$.

There is a unique nontrivial finite extension $\widetilde{H}_0 = N_{\widetilde{G}}((\widetilde{H}_0)^0)$ of the subgroup $(\widetilde{H}_0)^0$ in the group $\widetilde{G}$; furthermore, $\widetilde{G}/\widetilde{H}_0 \simeq G/H_0$. A nontrivial element of the group $Z(\widetilde{G})$ acts on the irreducible $\widetilde{G}$-module with highest weight π_3 by multiplication by -1; therefore the semigroup $\Lambda_+(\widetilde{G}/\widetilde{H}_0)$ is generated by the element $2\pi_3$, and the subgroup $\widetilde{H}_0$ is excellent in $\widetilde{G}$.

15°. $G = \mathrm{SO}_8$, $H_0 = \mathrm{G}_2$. We choose a basis $e_1, \dots, e_8$ of the space V so that in it the form Ω^O has matrix E_8. We assume without loss of generality that the group H_0 acts irreducibly on the subspace $V_1 = \langle e_1, \dots, e_7 \rangle$, and trivially on the subspace $V_2 = \langle e_8 \rangle$. There are no outer automorphisms. The group $Z_G(H_0)$ acts as a scalar on each of the subspaces V_1, V_2. The restrictions $\Omega^O\big|_{V_1}$ and $\Omega^O\big|_{V_2}$ of the form Ω^O are nondegenerate; therefore on each of the subspaces V_1 and V_2 the group $Z_G(H_0)$ acts by multiplication by $+1$ or -1. Hence we obtain $Z_G(H_0) = \{\pm E\} = Z(G)$, $N_G(H_0) = Z(G) \times H_0$, and $N_{\widetilde{G}}((\widetilde{H}_0)^0) = Z(\widetilde{G}) \times (\widetilde{H}_0)^0$ by Lemma 13.

Let z_1, z_2 be the elements of the group $\mathrm{Z}(\widetilde{G})$ that correspond to the elements $(h_1 + h_3)/2$ and $(h_3 + h_4)/2$, respectively, of the Lie algebra $\mathfrak{g}$ (see Table 3 in [1]). Then on the irreducible $\widetilde{G}$-modules with highest weights π_1, π_3, π_4, the element z_1 acts by multiplication by -1, -1, 1, respectively; the element z_2 by multiplication by 1, -1, -1, respectively; the element $z_1 z_2$ by multiplication by -1, 1, -1, respectively. The action is also the same on the functions $f_{\widetilde{H}_0}[\pi_1], f_{\widetilde{H}_0}[\pi_3], f_{\widetilde{H}_0}[\pi_4] \in {}^{\widetilde{U}}\mathbb{C}[\widetilde{G}]^{\widetilde{H}_0}$. For each of the elements z_1, z_2, $z_1 z_2$, among the functions $f_{\widetilde{H}_0}[\pi_1]$, $f_{\widetilde{H}_0}[\pi_3]$, $f_{\widetilde{H}_0}[\pi_4]$ there exist two on which this element acts by multiplication by -1; therefore by Lemma 9 none of the subgroups $z_1(\widetilde{H}_0)^0 \cup (\widetilde{H}_0)^0$, $z_2(\widetilde{H}_0)^0 \cup (\widetilde{H}_0)^0 = \widetilde{H}_0$, $z_1 z_2(\widetilde{H}_0)^0 \cup (\widetilde{H}_0)^0$ is excellent in $\widetilde{G}$. Since the semigroup $\Lambda_+(N_{\widetilde{G}}((\widetilde{H}_0)^0))$ contains the indecomposable elements $2\pi_1, \pi_1 + \pi_3 + \pi_4$, the subgroup $N_{\widetilde{G}}((\widetilde{H}_0)^0)$ is not excellent in $\widetilde{G}$ either.

17°. $G = \mathrm{Sp}_{2n}$, $H_0 = \mathrm{GL}_n$, $n \geq 2$. There is one outer automorphism realized in G by the element $s = \Omega^S_{2n}$. We have $Z_G(H_0) \subset H_0$; therefore, $N_G(H_0) = sH_0 \cup H_0$.

The only nontrivial finite extension of the subgroup H_0 is the group $H = N_G(H_0)$. Since G/H_0 is a symmetric space, we can use Corollary 4 to calculate the semigroup $\Lambda_+(G/H)$. We obtain

$$\Lambda_+(G/H) = \{2k_1\pi_1 + \dots + 2k_n\pi_n \mid k_i \in \mathbb{Z}_+, k_1 + k_3 + \dots + k_{2l-1} \in 2\mathbb{Z}\},$$

where $l = [\frac{n+1}{2}]$. Thus, for $n \geq 3$ the semigroup $\Lambda_+(G/H)$ contains the indecomposable elements $4\pi_1$, $2\pi_1 + 2\pi_3$, and the subgroup H is not excellent. For $n = 2$ the space G/H is isomorphic to the space $\mathrm{SO}_5/\mathrm{S}(\mathrm{O}_3 \times \mathrm{O}_2)$ (see case 11°) and is excellent.

18°. $G = \mathrm{Sp}_{2n+2m}$, $H_0 = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$, $n \geq m \geq 1$. We assume without loss of generality that the first and second factors of the subgroup H_0 act irreducibly on the subspaces $\langle e_1, \dots, e_n, e_{2m+n+1}, \dots, e_{2m+2n} \rangle$ and $\langle e_{n+1}, e_{n+2}, \dots, e_{n+2m} \rangle$, respectively. An

outer automorphism exists only in the case $m = n$; it is realized in G by the element $s = \begin{pmatrix} F_{2n} & 0 \\ 0 & F_{2n} \end{pmatrix}$. Next, $Z_G(H_0) \subset H_0$. Therefore we have $N_G(H_0) = H_0$ for $m \neq n$, and $N_G(H_0) = sH_0 \cup H_0$ for $m = n$.

For $n = m$ there is a unique nontrivial finite extension $H = N_G(H_0)$ of the subgroup H_0. Since the space G/H_0 is symmetric, we can calculate the semigroup $\Lambda_+(G/H)$ using Corollary 4. We obtain

$$\Lambda_+(G/H) = \{k_1\pi_2 + k_2\pi_4 + \cdots + k_n\pi_{2n} \mid k_i \in \mathbb{Z}_+, k_1 + k_3 + \cdots + k_{2l-1} \in 2\mathbb{Z}\},$$

where $l = [\frac{n+1}{2}]$. For $n \geq 3$ this semigroup contains the indecomposable elements $2\pi_2$ and $\pi_2 + \pi_6$; therefore the subgroup H is not excellent.

For $n = 2$ the semigroup $\Lambda_+(G/H)$ is generated by the elements $2\pi_2$, π_4; therefore the subgroup H is excellent. For $n = 1$ the space G/H is isomorphic to the space $\mathrm{SO}_5/\mathrm{S}(\mathrm{O}_4 \times \mathrm{O}_1)$ (see case 11°) and is also excellent.

19°. $G = \mathrm{Sp}_{2n}$, $H_0 = \mathbb{C}^\times \times \mathrm{Sp}_{2n-2}$, $n \geq 2$. We assume without loss of generality that the factor Sp_{2n-2} of the subgroup H_0 acts irreducibly on the subspace $\langle e_2, e_3, \ldots, e_{2n-1}\rangle$, and the factor $\mathbb{C}^\times \subset H_0$ is embedded in G as the set $\{\mathrm{diag}(t, 1, \ldots, 1, t^{-1}) \mid t \in \mathbb{C}^\times\}$. There is one outer automorphism realized by the element

$$s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & E_{2n-2} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Any element of the group $Z_G(H_0)$ acts scalarly on every H_0-invariant subspace of the space V and is therefore contained in the subgroup H_0 itself. Therefore, $N_G(H_0) = sH_0 \cup H_0$.

The only nontrivial finite extension of the subgroup H_0 is the subgroup $H = N_G(H_0)$. It is easy to verify that, up to proportionality, the functions $f_{H_0}[2\pi_1]$, $f_{H_0}[\pi_2]$ coincide with the functions that at a matrix $A = (a_{k,l}) \in \mathrm{Sp}_{2n}$ take values $a_{2n,1}a_{2n,2n}$ and $a_{2n-1,1}a_{2n,2n} - a_{2n,1}a_{2n-1,2n}$, respectively. The action on the right of the element s multiplies the functions $f_{H_0}[2\pi_1]$ and $f_{H_0}[\pi_2]$ by -1 and 1, respectively. Thus, the semigroup $\Lambda_+(G/H)$ is generated by the elements $4\pi_1, \pi_2$, and the group H is excellent.

20°. $G = \mathrm{G}_2$, $H_0 = \mathrm{SL}_3$. We can assume that $T \subset H_0$. There is an outer automorphism realized in G by an element $s \in N_G(T)$ whose image in the Weyl group $N_G(T)/Z_G(T)$ is -1 (that is, s acts by inversion on T). By Lemma 8 we obtain that $Z_G(H_0) \subset H_0$. Therefore, $N_G(H_0) = sH_0 \cup H_0$.

The only nontrivial finite extension of the subgroup H_0 is the group $H = N_G(H_0)$. Taking Lemma 1 and the fact that the rank of the semigroup $\Lambda_+(G/H_0)$ is equal to 1 into account, we see that the semigroup $\Lambda_+(G/H)$ is generated by the element $2\pi_1$. Therefore the subgroup H is excellent.

21°. $G = \mathrm{G}_2$, $H_0 = \mathrm{SL}_2 \cdot \mathrm{SL}_2$. By Theorem 7 there are no outer automorphisms realized in G. By Lemma 8 we have $Z_G(H_0) \subset H_0$, whence $N_G(H_0) = H_0$.

22°. $G = \mathrm{F}_4$, $H_0 = \mathrm{Spin}_9$. There are no outer automorphisms. By Lemma 8 we have $Z_G(H_0) \subset H_0$; therefore, $N_G(H_0) = H_0$.

23°. $G = \mathrm{F}_4$, $H_0 = \mathrm{Sp}_6 \cdot \mathrm{SL}_2$. There are no outer automorphisms. By Lemma 8 we have $Z_G(H_0) \subset H_0$, whence we obtain $N_G(H_0) = H_0$.

24°. $G = \mathrm{E}_6$, $H_0 = \mathrm{Sp}_8/\{\pm E\}$. There are no outer automorphisms. Since the subgroup H_0 acts on V irreducibly, we have $Z_G(H_0) \subset Z(G)$; therefore, $N_G(H_0) = Z(G) \times H_0$.

The only nontrivial finite extension of the subgroup H_0 is the group $H = N_G(H_0)$. Then the description of the centre of the group G (see [1, Table 3]) implies that the group

$Z(G)$ acts on the right on the functions $f_{H_0}[2\pi_1]$, $f_{H_0}[2\pi_5]$ by multiplication by nontrivial mutually inverse characters. Therefore the subgroup H is not excellent by Lemma 9.

25°. $G = \mathrm{E}_6$, $H_0 = \mathrm{F}_4$. In this case the hypotheses of Lemma 11 hold; therefore none of the nontrivial finite extensions of the subgroup H_0 is an excellent subgroup.

26°. $G = \mathrm{E}_6$, $H_0 = \mathrm{Spin}_{10}$. By Theorem 7 there are no outer automorphisms realized in G. Let $T \subset \mathbb{C}^\times \cdot \mathrm{Spin}_{10}$ (see case 27°) be a maximal torus in G. Then any element of the group $Z_G(H_0)$ acts as a scalar on every H_0-invariant subspace of the space V and therefore commutes with T. Therefore, $Z_G(H_0) \subset Z_G(T) = T$, whence $N_G(H_0)$ coincides with the subgroup H_0 for case 27°.

The hypotheses of Lemma 12 apply; therefore none of the subgroups $H \subset G$ with the conditions $H \neq H_0$ and $H^0 = H_0$ is excellent.

27°. $G = \mathrm{E}_6$, $H_0 = \mathbb{C}^\times \cdot \mathrm{Spin}_{10}$. By Theorem 7 there are no outer automorphisms realized in G. By Lemma 8 we obtain $Z_G(H_0) \subset H_0$. Therefore, $N_G(H_0) = H_0$.

28°. $G = \mathrm{E}_6$, $H_0 = \mathrm{SL}_6 \times \mathrm{SL}_2$. By Theorem 7 there are no outer automorphisms realized in G. By Lemma 8 we obtain $Z_G(H_0) \subset H_0$. Therefore, $N_G(H_0) = H_0$.

29°. $G = \mathrm{E}_7$, $H_0 = \mathbb{C}^\times \cdot \mathrm{E}_6$. We can assume that $T \subset H_0$. There is one outer automorphism realized in G by an element $s \in N_G(T)$ whose image in the Weyl group $N_G(T)/Z_G(T)$ is -1 (that is, s acts on T by inversion). It follows from Lemma 8 that $Z_G(H_0) \subset H_0$. Therefore, $N_G(H_0) = sH_0 \cup H_0$.

The only nontrivial finite extension of the subgroup H_0 is the group $H = N_G(H_0)$. Since G/H_0 is a symmetric space, the semigroup $\Lambda_+(G/H)$ can be calculated using Corollary 4. We obtain

$$\Lambda_+(G/H) = \{2k_1\pi_1 + k_2\pi_2 + k_6\pi_6 \mid k_i \in \mathbb{Z}_+, k_1 + k_6 \in 2\mathbb{Z}\}.$$

Thus, the semigroup $\Lambda_+(G/H)$ contains the indecomposable elements $2\pi_1, \pi_1+\pi_6$; therefore the subgroup H is not excellent.

30°. $G = \mathrm{E}_7$, $H_0 = \mathrm{SL}_8/\{\pm E\}$. We can assume that $T \subset H_0$. There is an outer automorphism realized in G by an element $s \in N_G(T)$ whose image in the Weyl group $N_G(T)/Z_G(T)$ is -1 (that is, s acts on T by inversion). By Lemma 8 we have $Z_G(H_0) \subset H_0$. Thus, $N_G(H_0) = sH_0 \cup H_0$.

There is a unique nontrivial finite extension $H = N_G(H_0)$ of the subgroup H_0. Since G/H_0 is a symmetric space, we can calculate the semigroup $\Lambda_+(G/H)$ by using Corollary 4. We obtain

$$\Lambda_+(G/H) = \{2k_1\pi_1 + \cdots + 2k_7\pi_7 \mid k_i \in \mathbb{Z}_+, k_1 + k_3 + k_7 \in 2\mathbb{Z}\}.$$

Since the semigroup $\Lambda_+(G/H)$ contains the indecomposable elements $4\pi_1, 2\pi_1 + 2\pi_3$, the subgroup H is not excellent.

31°. $G = \mathrm{E}_7$, $H_0 = \mathrm{Spin}_{12} \cdot \mathrm{SL}_2$. By Theorem 7 there are no outer automorphisms realized in G. By Lemma 8 we obtain $Z_G(H_0) \subset H_0$. Therefore, $N_G(H_0) = H_0$.

32°. $G = \mathrm{E}_8$, $H_0 = \mathrm{Spin}_{16}/\{E, z\}$, where $z \in Z(G)$ is the element corresponding to the element $(h_1 + h_3 + h_5 + h_7)/2$ of the Lie algebra $\mathfrak{g}$ (see [1, Table 3]). By Theorem 7 there are no outer automorphisms realized in G. By Lemma 8 we have $Z_G(H_0) \subset H_0$. Therefore, $N_G(H_0) = H_0$.

33°. $G = \mathrm{E}_8$, $H_0 = \mathrm{SL}_2 \cdot \mathrm{E}_7$. There are no outer automorphisms. By Lemma 8 we obtain $Z_G(H_0) \subset H_0$. Therefore, $N_G(H_0) = H_0$.

4.2. In this subsection we work with Table 2. The almost excellent spaces in this table are the spaces in rows 4 and 9. Thus, we only need to consider two cases.

1°. First we consider the series of spaces in row 4. We have $G = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$, $H_0 = \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2m-2}$. Let V be the direct sum of the spaces V_1 and V_2 of the tautology representations of the first and second factors of the group G, respectively. For

$i = 1, 2$ we choose a basis in the space V_i so that, with respect to it, the matrix of the nondegenerate skew-symmetric bilinear form Ω_i preserved by the ith factor of the group G takes the form Ω^S_{2n} for $i = 1$, and Ω^S_{2m} for $i = 2$. Then we can assume that the first and third factors of the subgroup H_0 are embedded in the first and second factors of the group G, respectively, as central blocks of sizes $(2n-2)\times(2n-2)$ and $(2m-2)\times(2m-2)$, while the second factor of the group H_0 is diagonally embedded in each of the factors of the group G as a 2×2 block situated in the first and last rows and columns. We assume that the subgroups B, U, and T of the group G coincide with the sets of all pairs of upper-triangular, upper-unitriangular, and diagonal matrices contained in G, respectively. Let $W_1 \subset V_1$, $W_2 \subset V_2$ be subspaces on which the first and third factors of the group H_0 act irreducibly and nontrivially, respectively, and let $V_1' \subset V_1$, $V_2' \subset V_2$ be subspaces on which the second factor of the group H_0 acts irreducibly and nontrivially.

We now calculate the group $N_G(H_0)$. Let $g \in N_G(H_0)$ be an arbitrary element. We observe that the element g can permute invariant subspaces of the group H_0 only inside V_1 and V_2. Since the subspaces V_1' and V_2' are isomorphic as H_0-modules, gV_1' and gV_2' in V are also isomorphic as H_0-modules. Hence the element g preserves each of the subspaces W_1, V_1', W_2, V_2' and, moreover, $a(g)|_{H_0} \in \operatorname{Int} H_0$. Multiplying the element g by a suitable element of the subgroup H_0 we can assume that g acts as a scalar on W_1, V_1', W_2, V_2'. Since for each $i = 1, 2$ the restriction of the form Ω_i to the subspaces W_i, V_i' is nondegenerate, g acts on each of these subspaces as the identity or by multiplication by -1. Therefore, $N_G(H_0) = wH_0 \cup H_0$, where the element w acts as the identity on the subspaces W_1, W_2, V_1', and by multiplication by -1 on the subspace V_2'.

The only nontrivial finite extension of the subgroup H_0 is the subgroup $H = N_G(H_0)$. We have $G = \{(A, B) \mid A \in \mathrm{Sp}_{2n},\ B \in \mathrm{Sp}_{2m}\}$. Let $A = (a_{ij})$, $B = (b_{ij})$. In this notation, up to proportionality we have $f_{H_0}[\pi_2](A, B) = a_{2n-1,1}a_{2n,2n} - a_{2n,1}a_{2n-1,2n}$ $(n \geq 2)$, $f_{H_0}[\varphi_2](A, B) = b_{2m-1,1}b_{2m,2m} - b_{2m,1}b_{2m-1,2m}$ $(m \geq 2)$, $f_{H_0}[\pi_1 + \varphi_1](A, B) = a_{2n,1}b_{2m,2m} - a_{2n,2n}b_{2m,1}$. The functions $f_{H_0}[\pi_2]$ and $f_{H_0}[\varphi_2]$ are invariant under the action on the right of the element w, and $f_{H_0}[\pi_1 + \varphi_1]$ is multiplied by -1. Therefore the semigroup $\Lambda_+(G/H)$ is generated by the elements π_2, φ_2, and $2\pi_1 + 2\varphi_1$, and the subgroup H is excellent.

2°. We now consider the series of spaces in row 9. We have $G = H \times H$, $H_0 = H$, where H is an arbitrary simply connected simple algebraic group and the subgroup H_0 is embedded in G diagonally. We calculate the group $N_G(H_0)$. Let $h_1, h_2 \in H$ and $(h_1, h_2) \in N_G(H_0)$. Then for any element $h \in H$ we have $h_1hh_1^{-1} = h_2hh_2^{-1}$, so that $h_2^{-1}h_1 \in Z(H)$. Therefore, $N_G(H_0) \simeq Z(H) \times H_0$, and under this isomorphism a pair (z, h), where $z \in Z(H)$, $h \in H_0$, corresponds to the element $(zh, h) \in N_G(H_0)$.

Let K be a nontrivial finite extension of the group H_0. Then $K \simeq Z \times H_0$, where Z is a nontrivial subgroup of $Z(H)$. Next we examine all the possibilities for H and for each of them find out which subgroups K are excellent in G. We use Table 3 from [1] to calculate the action of elements of $Z(H)$ on irreducible H-modules and then Lemma 4 to determine the action of elements of $Z(H)$ on the weight functions $f_{H_0}[\omega]$, $\omega \in \mathfrak{X}(T)$.

For the groups H given in Table 6, none of the subgroups $K \subset G$ is excellent. This follows from Lemma 9 applied to the weights ω_1 and ω_2 of the semigroup $\Lambda_+(G/H_0)$ indicated in Table 6 in the corresponding columns. Next we consider the remaining possibilities for H.

If $H = \mathrm{SL}_2 = \mathrm{Sp}_2$, then the space G/H_0 is a special case of the space in row 4 (see case 1°).

If $H = \mathrm{Sp}_4$, then we have $Z = Z(H)$. The space G/K is excellent and the semigroup $\Lambda_+(G/K)$ is generated by the indecomposable elements $2\pi_1 + 2\varphi_1$ and $\pi_2 + \varphi_2$.

Table 6

H	ω_1	ω_2
SL_n, $n\geq 3$	$\pi_1+\varphi_1^*$	$\pi_{n-1}+\varphi_{n-1}^*$
Spin_{4n+2}, $n\geq 1$	$\pi_{2n}+\varphi_{2n}^*$	$\pi_{2n+1}+\varphi_{2n+1}^*$
Sp_{2n}, $n\geq 3$	$\pi_1+\varphi_1$	$\pi_3+\varphi_3$
E_6	$\pi_1+\varphi_1^*$	$\pi_2+\varphi_2^*$
E_7	$\pi_1+\varphi_1$	$\pi_3+\varphi_3$

If $H=\mathrm{Spin}_{2n+1}$, then we have $Z=Z(H)$. The corresponding space G/K is excellent and the semigroup $\Lambda_+(G/K)$ is generated by the elements $\pi_1+\varphi_1,\ \dots,\ \pi_{n-1}+\varphi_{n-1}$, $2\pi_n+2\varphi_n$. We note that $G/K\simeq(\mathrm{SO}_{2n+1}\times\mathrm{SO}_{2n+1})/\mathrm{SO}_{2n+1}$.

If $H=\mathrm{Spin}_{4n}$, $n\geq 2$, we reason as follows. Let z_1, z_2 be the elements of the centre of the group H corresponding to the elements $(h_1+h_3+\dots+h_{2n-1})/2$, $(h_{2n-1}+h_{2n})/2$ of the Lie algebra $\mathfrak{h}$ (see [1, Table 3]). Then on the functions $f_{H_0}[\pi_1+\varphi_1^*]$, $f_{H_0}[\pi_{2n-1}+\varphi_{2n-1}^*]$, $f_{H_0}[\pi_{2n}+\varphi_{2n}^*]$ the element z_1 acts by multiplication by -1, -1, 1, respectively; the element z_2 acts by multiplication by 1, -1, -1, respectively; the element z_1z_2 acts by multiplication by -1, 1, -1, respectively. Thus, for $|Z|=2$ among these three functions there exist two on which the group Z acts by multiplication by nontrivial mutually inverse characters, whence the subgroup K is not excellent by Lemma 9. The final case is $Z=Z(H)$. Based on the calculations given above, we can conclude that among the indecomposable elements of the semigroup $\Lambda_+(G/\widetilde{H})$ there are the elements $2\pi_1+2\varphi_1^*$ and $\pi_1+\pi_{2n-1}+\pi_{2n}+\varphi_1^*+\varphi_{2n-1}^*+\varphi_{2n}^*$. Therefore, the subgroup K is not excellent in this case either.

If H is a simple group of type E_8, F_4, or G_2, then its centre is trivial; therefore the subgroup H_0 has no nontrivial finite extensions in G.

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