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JUSTIFICATION OF THE ADIABATIC PRINCIPLE IN THE ABELIAN HIGGS MODEL

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ABSTRACT. A justification of the adiabatic principle for the $(2 + 1)$-dimensional Abelian Higgs model is given. It is shown that near any geodesic on the space of static solutions there exists a solution of the dynamical Euler–Lagrange equations.

§ 1. INTRODUCTION

The (2+1)-dimensional Abelian Higgs model appears in the theory of superconductivity. It is given by a hyperbolic action functional defined on pairs (A, Φ), where A is an electromagnetic gauge vector potential and Φ is a complex scalar Higgs field on the plane $\mathbb{C}$. The action functional has the standard form of an integral with respect to time of the difference between the kinetic energy (depending on the derivatives of the components of A and Φ with respect to time) and the potential energy (depending only on a point in the configuration space).

We shall only study the so-called critical case, when λ, the scale parameter of the model, is equal to 1. If we fix time in the model under consideration, confining ourselves to stationary solutions, then we arrive at the (static) two-dimensional Abelian Higgs model, which has been completely investigated by Taubes [1]. All its solutions are divided into classes parametrized by an integer topological invariant, which is called the vortex number. Solutions with vortex number N (they are called N-vortex for $N > 0$, and $|N|$-antivortex for $N < 0$) are uniquely determined (up to gauge equivalence) by the zeros of the field Φ; taking account of multiplicity, these number exactly $|N|$. Thus, the moduli space of N-vortex solutions for $N > 0$ can be identified with $\mathbb{C}^N$ (and similarly for $N < 0$).

From a physical viewpoint, the zeros of the field Φ can be interpreted as the positions of the centres of the vortices, and studying their trajectories is an important physical problem. However, although the static solutions of the model have been described completely, it does not seem possible to obtain any explicit description of the motion of the vortices. Therefore, a current problem is to give an approximate description of this motion.

To do this, the so-called adiabatic principle is used. Originally, Manton [2] proposed this principle from heuristic considerations for the problem of the dynamics of magnetic monopoles, which is similar to our problem. This principle was applied to the problem of the dynamics of vortices by Ruback [3]. The essence of the principle is that the motion of a system of N slowly moving vortices can be approximately described by geodesics on the moduli space of static N-vortex solutions in the metric defined by the kinetic

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energy (for short, the kinetic metric). Even in the case $N = 2$ no one has yet obtained explicit formulae for this metric, but its properties have been studied in detail using both analytic and numerical methods. This made it possible (under the assumption that the adiabatic principle holds) to draw conclusions about the dynamics of vortices too. In particular, the problem of scattering of two vortices has been investigated by many authors. Among these, apart from [3], we mention Sergeev and Chechin [5], where the kinetic metric in the case $N = 2$ is described, and the papers of Samols [4] and Stuart [6]. In [7], the smoothness of the kinetic metric in symmetric coordinates was established for an arbitrary N. This made it possible to prove that after a head-on symmetric collision of N vortices they scatter at an angle π/N.

This paper is devoted to a rigorous justification of the adiabatic principle. (The precise result is given in Theorem 2.) The first attempt to justify the adiabatic principle was made in [6] for a similar Higgs model with a scale parameter $\lambda \neq 1$. In spite of the fact that we found some gaps in the paper on careful reading, it proved to be useful to us, and on the whole we follow the approach proposed in it. The principal difference between our paper and [6] is a new proof of the central result — the theorem on the existence of a solution on a sufficiently long time interval. The proof of this theorem is given in §§ 7, 8.

The paper is organized as follows. In § 2 we describe the static two-dimensional model and vortex solutions. In § 3 we introduce the dynamical $(2 + 1)$-dimensional model and state the main result of the paper. In § 4 we clarify the structure of the tangent bundle of the moduli space of N-vortex solutions and define the kinetic metric on this space, using the linearized vortex equation to do this. In § 5 we propose an ansatz for solution of the Euler–Lagrange equations in the case of a dynamical model (the dynamical equations, for short). Substituting the ansatz into the dynamical equations gives rise to a hyperbolic system with infinite-dimensional degeneration. We propose that an auxiliary system with finite-dimensional degeneration be used to solve it, and we study this system in §§ 5, 6. In § 7 we prove the local existence theorem, which guarantees the existence of a solution on a sufficiently small time interval. Finally, in § 8 we obtain a priori estimates, and using them we succeed in proving the existence of a bounded solution on a sufficiently long time interval. In the Appendix (§ 9) we collect the proofs of some technical results.

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§ 2. The static two-dimensional Abelian Higgs model. Vortex solutions

The *two-dimensional Abelian Higgs model* is defined by the following *action functional*:

$$V(A,\Phi) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|d_A\Phi|^2 + F_{12}^2 + \frac{\lambda}{4}(|\Phi|^2-1)^2\right)dx\,dy, \tag{1}$$

where $A = -iA_1\,dx - iA_2\,dy$ is a gauge potential with smooth real-valued coefficients A_1, A_2 on $\mathbb{R}^2$, $\Phi = \Phi_1 + i\Phi_2$ is a Higgs field given by a smooth complex-valued function on the plane $\mathbb{R}^2$, and $\lambda > 0$ is a constant. We denote the gauge field generated by the potential (A_1, A_2) by $F_{12} := \partial_1 A_2 - \partial_2 A_1$. Henceforth, $\partial_1 := \partial_x$ and $\partial_2 := \partial_y$.

The action functional V is invariant under *gauge transformations* of the following form:

$$A \mapsto \widetilde{A} = A - id\chi, \quad \Phi \mapsto \widetilde{\Phi} = e^{i\chi}\Phi,$$

where χ is a smooth real-valued function on $\mathbb{R}^2$.

Integrating by parts, we can rewrite the action functional V in the following form (found in 1976 by Bogomol'nyĭ):

$$
\begin{aligned}
V = \frac{1}{2}\int_{\mathbb{R}^2} \Big\{ &\big((\partial_1\Phi_1 + A_1\Phi_2) \mp (\partial_2\Phi_2 - A_2\Phi_1)\big)^2 \\
&+ \big((\partial_2\Phi_1 + A_2\Phi_2) \pm (\partial_1\Phi_2 - A_1\Phi_1)\big)^2 + \Big(F_{12} \pm \frac{1}{2}(|\Phi|^2 - 1)\Big)^2 \Big\}\, dx\, dy \\
&\pm \frac{1}{2}\int_{\mathbb{R}^2} F_{12}\, dx\, dy + \frac{\lambda - 1}{4}\int_{\mathbb{R}^2} (|\Phi|^2 - 1)^2 dx\, dy.
\end{aligned} \tag{2}
$$

In what follows we consider only the critical case $\lambda = 1$ (see [1]). Then the right-hand side of the last equation is the sum of nonnegative terms with $\frac{1}{2}\int_{\mathbb{R}^2} F_{12}\, dx\, dy$. Under certain additional conditions on the behaviour of the components of the field (A, Φ) at infinity we can show that the latter term is a topological invariant of the field (A, Φ).

We now give more detail. Suppose that the function Φ has no zeros outside a disc of sufficiently large radius R_0. Then the degree N of the map $\Phi/|\Phi|\colon S_R \to S_1$ of the circle S_R of radius $R > R_0$ is independent of the choice of R and is called the *vortex number* of the field (A, Φ).

Suppose that the following conditions hold:

(1) $F_{12} \in L^1(\mathbb{R}^2)$;

(2) $|\Phi| \to 1$ as $r := \sqrt{x^2 + y^2} \to \infty$;

(3) $|d_A\Phi| \le C/r^{1+\gamma}$ for some $\gamma > 0$.

Then the vortex number N can be calculated using the following formula (see [1]):

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} F_{12}\, dx\, dy = N.$$

Obviously, this number is invariant under gauge transformations.

We fix a vortex number $N \ge 0$. Then it follows from Bogomol'nyĭ's formula that $V(A, \Phi) \ge \pi N$, and the minimal value of V (equal to πN) is attained on solutions of the system of equations

$$
\begin{cases}
\partial_1\Phi_1 + A_1\Phi_2 = \partial_2\Phi_2 - A_2\Phi_1, \\
\partial_2\Phi_1 + A_2\Phi_2 = -\partial_1\Phi_2 + A_1\Phi_1, \\
\qquad\qquad F_{12} = -\dfrac{1}{2}(|\Phi|^2 - 1),
\end{cases} \tag{3}
$$

which are called *vortex equations*. These equations are invariant under gauge transformations.

We introduce the complex coordinate $z = x + iy$ on the plane (x, y) and denote, as usual,

$$\partial := \partial_z := \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \bar\partial := \partial_{\bar z} := \frac{1}{2}(\partial_1 + i\partial_2).$$

Setting $\mathscr{A} := \frac{1}{2}(A_1 - iA_2)$ we can write the first two equations in the system (3) in the more compact form

$$\bar\partial\Phi = i\bar{\mathscr{A}}\Phi\,. \tag{4}$$

For completeness we observe that in the case $N < 0$ the minimal value of V, which is equal to $\pi|N|$, is attained on solutions of a similar system of equations, which are called *antivortex equations*. The change of variables $\widetilde{\mathscr{A}}(z) = -\bar{\mathscr{A}}(-\bar z)$, $\widetilde\Phi(z) = \Phi(-\bar z)$ associates a solution $(\mathscr{A}, \Phi)$ of the vortex equations with the solution $(\widetilde{\mathscr{A}}, \widetilde\Phi)$ of the antivortex equations, and conversely. Therefore in what follows we confine ourselves to the case $N \ge 0$.

The following theorem on the existence and uniqueness of solutions of the vortex equations was proved in [1].

Theorem 1 (Taubes)**.** *Let $N \geq 0$. Let $Z_1, Z_2, \ldots, Z_N$ be arbitrary (not necessarily distinct) points on the complex plane. Then there exists a (smooth) solution (A_1, A_2, Φ) of the vortex equations such that the zeros of Φ coincide with the points $Z_1, \ldots, Z_N$ and*

$$\Phi(z, \bar{z}) \sim c_j (z - Z_j)^{n_j}$$

in a neighbourhood of each of the points Z_j. Here, n_j is the multiplicity of Z_j in the set $\{Z_1, \ldots, Z_N\}$ and c_j is a nonzero constant.

For this solution, $|\Phi|$ tends exponentially to 1 as $|z| \to \infty$, while $|(\partial_1 - iA_1)\Phi|$ and $|(\partial_2 - iA_2)\Phi|$ are exponentially decreasing. More precisely,

$$|d_A \Phi| \leq C(1 - |\Phi|) \quad \textit{for some } C > 0,$$

and for any $\gamma > 0$ there exists $C(\gamma) > 0$ such that

$$1 - |\Phi| \leq C(\gamma) e^{-(1-\gamma)|z|}.$$

The vortex number of this solution is equal to N. A solution with these properties is unique up to gauge equivalence.

The solution whose existence is established by this theorem is called the *N-vortex solution.*

The moduli space of N-vortex solutions, denoted by $\mathscr{M}_N$, is by definition the set of gauge equivalence classes of N-vortex solutions. Due to Taubes's theorem this space can be identified with the Nth symmetric power $S^N\mathbb{C}$, that is, with the set of unordered sets of N complex numbers (coinciding with the zeros of Φ). The set $S^N\mathbb{C}$, in turn, can be identified with the space $\mathbb{C}^N$ by associating with every set $\{Z_1, \ldots, Z_N\}$ the polynomial $p(z)$ with zeros $Z_1, \ldots, Z_N$ and leading coefficient 1:

$$p(z) = (z - Z_1) \cdots (z - Z_N) = z^N + S_1 z^{N-1} + \cdots + S_{N-1} z + S_N.$$

The numbers $S_1, \ldots, S_N$ can be used as coordinates on the moduli space $\mathscr{M}_N$. In addition we also introduce real coordinates q^μ, $\mu = 1, 2, \ldots, 2N$, by setting $q^{2j-1} = \operatorname{Re} S_j$ and $q^{2j} = \operatorname{Im} S_j$ for $j = 1, \ldots, N$.

§ 3. The dynamical problem. Statement of the result

The *dynamical* $(2+1)$*-dimensional Higgs model* is defined by the *action functional*

$$(5) \quad \mathscr{S}(A, \Phi) = \frac{1}{2} \int_{t_1}^{t_2} dt \int_{\mathbb{R}^2} \Big\{ \big(|D_0\Phi|^2 + F_{01}^2 + F_{02}^2\big) - \Big(|D_1\Phi|^2 + |D_2\Phi|^2 + F_{12}^2 + \frac{\lambda}{4}(|\Phi|^2 - 1)^2\Big) \Big\}\, dx\, dy.$$

Here,

1) $\Phi(t, x, y)$ is a smooth complex-valued function;

2) the components $A_j(t, x, y)$ are smooth real-valued functions, $j = 0, 1, 2$;

3) $D_j \Phi := \partial_j \Phi - iA_j \Phi$ is the covariant derivative, $j = 0, 1, 2$;

4) $F_{jk} := \partial_j A_k - \partial_k A_j$ are components of the curvature form, $j, k = 0, 1, 2$, $j \neq k$, with $\partial_0 = \partial_t$, $\partial_1 = \partial_x$, $\partial_2 = \partial_y$;

5) $\lambda > 0$ is a constant (the parameter of the model).

The action functional can be represented in the standard form

$$\mathscr{S} = \int (T - V)\, dt,$$

where the *potential energy* V is given by formula (1), and the *kinetic energy* T is equal to

$$T = \frac{1}{2}\int_{\mathbb{R}^2} (|D_0\Phi|^2 + F_{01}^2 + F_{02}^2)\,dx\,dy.$$

This functional is invariant under *dynamical gauge transformations* of the form

$$\widetilde{A}_j = A_j + \partial_j\chi, \quad \widetilde{\Phi} = e^{i\chi}\Phi,$$

where $\chi(t,x,y)$ is a smooth real-valued function.

By a choice of the gauge we can make sure that the condition $A_0 = 0$ holds (the so-called time gauge). In this case,

$$T = \frac{1}{2}(\|\partial_0\Phi\|_{L^2}^2 + \|\partial_0 A_1\|_{L^2}^2 + \|\partial_0 A_2\|_{L^2}^2).$$

Thus, the functional T defines a metric on the space $\mathscr{M}_N$, which is called the *kinetic* metric. (A precise description of this metric is given in § 4.) In [7] it was shown that the kinetic metric on $\mathscr{M}_N$ is smooth in the coordinates $S_1,\dots,S_N$.

As mentioned in the Introduction, we assume that $\lambda = 1$. In this case the Euler–Lagrange equations for the action $\mathscr{S}(A,\Phi)$ have the form

$$\begin{aligned} \partial_0\partial_1 A_1 + \partial_0\partial_2 A_2 - \Delta A_0 &= \operatorname{Im}(\bar{\Phi} D_0\Phi),\\ \partial_0^2 A_1 - \partial_2^2 A_1 + \partial_1\partial_2 A_2 - \partial_1\partial_0 A_0 &= \operatorname{Im}(\bar{\Phi} D_1\Phi),\\ \partial_0^2 A_2 - \partial_1^2 A_2 + \partial_2\partial_1 A_1 - \partial_2\partial_0 A_0 &= \operatorname{Im}(\bar{\Phi} D_2\Phi),\\ (D_0^2 - D_1^2 - D_2^2)\Phi &= \frac{1}{2}\Phi(1-|\Phi|^2). \end{aligned} \tag{6}$$

They are invariant under dynamical gauge transformations.

The problem is to describe the moduli space of solutions of these equations, which are called for brevity the *dynamical equations*. Taubes's theorem gives a description in the case of vortex solutions that are static solutions of these equations. However, we cannot expect anything like this in the general case. Nevertheless, we can try to obtain an approximate description of the moduli space of dynamical solutions by following Manton's idea (see [2] and [3]). This idea, which can be called the *adiabatic principle*, consists in considering geodesics on the moduli space of static solutions $\mathscr{M}_N$ with respect to the kinetic metric as an approximation to dynamical solutions of equations (6) describing the trajectories of a system of N slowly moving vortices. Even though Manton's idea is based on heuristic considerations, it has led to a whole host of papers devoted to describing the geodesics on spaces of static solutions. Our paper is devoted to justifying the adiabatic principle; more precisely, we prove the following result.

Theorem 2. *Let $Q = Q(\tau)$ be a parametrization of a segment of an arbitrary geodesic $Q\colon [0;\tau_0] \to \mathscr{M}_N$ on the moduli space $\mathscr{M}_N$ with respect to the kinetic metric, with a natural parameter τ chosen on it.*

Then there exist

- *positive numbers $\tau_1 \le \tau_0$, ε_0, M;*
- *a smooth curve $(\alpha_1(\tau),\alpha_2(\tau),\phi(\tau))$ defined on the interval $[0;\tau_0]$ in the space of static N-vortex solutions whose gauge class $[\alpha_1(\tau),\alpha_2(\tau),\phi(\tau)]$ coincides with the class $Q(\tau)\in\mathscr{M}_N$ for every fixed $\tau\in[0,\tau_0]$*

that have the following properties.

For any $\varepsilon\in(0;\varepsilon_0)$ there exists a solution $(A_0^\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t),\ \Phi^\varepsilon(t))$ of the Euler–Lagrange equations (6) *defined on the interval $t\in[0;\tau_1/\varepsilon]$ whose deviation from the curve*

$(\alpha_1(\varepsilon t), \alpha_2(\varepsilon t), \phi(\varepsilon t))$ is of order ε^2. More precisely, this solution admits a representation

$$
\begin{aligned}
&A_0^\varepsilon(t,x,y) = \varepsilon^3 a_0^\varepsilon(t,x,y),\\
&A_1^\varepsilon(t,x,y) = \alpha_1(\varepsilon t; x,y) + \varepsilon^2 a_1^\varepsilon(t,x,y),\\
&A_2^\varepsilon(t,x,y) = \alpha_2(\varepsilon t; x,y) + \varepsilon^2 a_2^\varepsilon(t,x,y),\\
&\Phi^\varepsilon(t,x,y) = \phi(\varepsilon t; x,y) + \varepsilon^2 \varphi^\varepsilon(t,x,y)
\end{aligned}
\tag{7}
$$

in which the remainder terms a_0^ε, a_1^ε, a_2^ε, $\varphi^\varepsilon = \varphi_1^\varepsilon + i\varphi_2^\varepsilon$ are uniformly bounded with respect to ε; more precisely, the estimate

$$\max\{\|a_0^\varepsilon(t)\|_3, \|a_1^\varepsilon(t)\|_3, \|a_2^\varepsilon(t)\|_3, \|\varphi_1^\varepsilon(t)\|_3, \|\varphi_2^\varepsilon(t)\|_3\} \le M$$

holds for all $t \in [0;\tau_1/\varepsilon]$, where $\|\cdot\|_3$ denotes the norm in the Sobolev space $H^3(\mathbb{R}^2)$.

The solutions found in this theorem have the following properties:

- the functions α_1, α_2, ϕ are smooth and bounded on the set $[0;\tau_1] \times \mathbb{R}^2$;
- the function a_0^ε belongs to the class $C^1\bigl([0;\tau_1/\varepsilon], H^3(\mathbb{R}^2)\bigr)$, and the functions $a_1^\varepsilon, a_2^\varepsilon$, φ_1^ε, φ_2^ε belong to the class

$$C\bigl([0;\tau_1/\varepsilon], H^3(\mathbb{R}^2)\bigr) \cap C^1\bigl([0;\tau_1/\varepsilon], H^2(\mathbb{R}^2)\bigr) \cap C^2\bigl([0;\tau_1/\varepsilon], H^1(\mathbb{R}^2)\bigr). \tag{8}$$

Sections 4–8 of this paper are devoted to proving Theorem 2. The strategy of the proof consists in substituting the ansatz (7) into the dynamical Euler–Lagrange equations and obtaining a system of equations for the correction terms $a_0^\varepsilon, a_1^\varepsilon, a_2^\varepsilon, \varphi^\varepsilon$. To solve this system we introduce another, auxiliary system. For this auxiliary system we prove the local existence theorem and derive a priori estimates, which make it possible to prove the existence of a solution on a time interval of order $1/\varepsilon$. Hence the original Euler–Lagrange system of equations also has a solution defined on time intervals of order $1/\varepsilon$; that is, the assertion of Theorem 2 follows.

§ 4. The vortex operator. Tangent structure of the moduli space $\mathscr{M}_N$. The kinetic metric

If we substitute the ansatz (7) into the equations of the system (6) and separate the real and imaginary parts in the last equation, then for a given curve $(\alpha_1(\tau), \alpha_2(\tau), \phi(\tau))$ we obtain a system of five partial differential equations with five unknown real functions $a_0^\varepsilon, a_1^\varepsilon, a_2^\varepsilon, \varphi_1^\varepsilon, \varphi_2^\varepsilon$ (in what follows we omit the index ε). The linear part of this system is expressed in terms of infinitesimal vortex and gauge operators, which we will now describe.

Let $A_1 = \alpha_1$, $A_2 = \alpha_2$, $\Phi = \phi$ be a solution of the vortex equations (3) (in what follows, we denote it by (α, ϕ) for brevity). Linearizing the system (3) in a neighbourhood of this solution, we obtain the following equations:

$$
\begin{cases}
\partial_1 a_2 - \partial_2 a_1 = -\phi_1\varphi_1 - \phi_2\varphi_2,\\
\partial_1\varphi_1 + a_1\phi_2 + \alpha_1\varphi_2 = \partial_2\varphi_2 - a_2\phi_1 - \alpha_2\varphi_1,\\
\partial_2\varphi_1 + a_2\phi_2 + \alpha_2\varphi_2 = -\partial_1\varphi_2 + a_1\phi_1 + \alpha_1\varphi_1,
\end{cases}
\tag{9}
$$

where the functions (a_1, a_2, φ), $\varphi = \varphi_1 + i\varphi_2$, define a perturbation of the original solution.

We introduce the *linearized vortex operator* defined by equations (9):

$$D_{\alpha,\phi} = \begin{pmatrix} -\partial_2 & \partial_1 & \phi_1 & \phi_2 \\ \phi_2 & \phi_1 & \partial_1 + \alpha_2 & -\partial_2 + \alpha_1 \\ \phi_1 & \phi_2 & \partial_2 - \alpha_1 & \partial_1 + \alpha_2 \end{pmatrix}.$$

The operator $D_{\alpha,\phi}$ acts from the space $(L^2)^4$ into $(L^2)^3$. We denote the operator that is formally conjugate to $D_{\alpha,\phi}$ by $D^*_{\alpha,\phi}$; it acts from the space $(L^2)^3$ into $(L^2)^4$.

For what follows we need a description of the kernel of the operator $D_{\alpha,\phi}$. This kernel is infinite-dimensional, since the linearized vortex equations (9) are invariant under *infinitesimal gauge transformations* of the form

$$a_1 \mapsto a_1 + \partial_1\chi, \quad a_2 \mapsto a_2 + \partial_2\chi, \quad \varphi \mapsto \varphi + i\phi\chi, \tag{10}$$

where χ is a smooth real-valued function on the plane. This invariance is a consequence of the invariance of the original vortex equations under static gauge transformations. More concretely, infinitesimal gauge transformations are obtained by differentiation from the gauge transformations introduced in § 3:

$$\frac{d}{dt}\,(\alpha + td\chi, \phi e^{it\chi})\big|_{t=0} = (d\chi, i\phi\chi).$$

Because the linearized vortex equations are invariant under infinitesimal gauge transformations, for sufficiently smooth functions χ the vectors $(\partial_1\chi, \partial_2\chi, -\phi_2\chi, \phi_1\chi)$ belong to the kernel of $D_{\alpha,\phi}$. (One of the consequences of the strong degeneration of the operator $D_{\alpha,\phi}$ is the fact that the norm $(D^*_{\alpha,\phi}D_{\alpha,\phi}\psi, \psi)_{L^2} = \|D_{\alpha,\phi}\psi\|^2$ is not equivalent to the Sobolev norm $\|\psi\|^2_{H^1}$.)

Now suppose that (α, ϕ) is some smooth bounded N-vortex solution. It is easy to see that in this case the vortex operator $D_{\alpha,\phi}$ defines a bounded operator from the Sobolev space $(H^1)^4$ into $(L^2)^3$.

We introduce the *tangent gauge operator* defined by formula (10):

$$G_\phi\chi = (\partial_1\chi, \partial_2\chi, -\phi_2\chi, \phi_1\chi).$$

This is a differential operator acting from the space $L^2(\mathbb{R}^2)$ into $(L^2(\mathbb{R}^2))^4$. With any sufficiently smooth function χ (belonging, for example, to the Sobolev space $H^2(\mathbb{R}^2)$) it associates the solution $G_\phi\chi$ of the linearized vortex equations (9); in other words, the vector function $G_\phi\chi$ belongs to the kernel of the operator $D_{\alpha,\phi}$. One can show that the image of the operator G_ϕ regarded as a bounded operator from $H^2(\mathbb{R}^2)$ into $(H^1(\mathbb{R}^2))^4$ is closed.

Apart from the infinite-dimensional image of G_ϕ, the kernel of $D_{\alpha,\phi}$ also contains a certain finite-dimensional subspace. We let $\mathscr{N}_{\alpha,\phi}$ denote the subspace of the kernel $\ker D_{\alpha,\phi}$ of the operator $D_{\alpha,\phi}$ consisting of vectors that are L^2-orthogonal to the image $\operatorname{im} G_\phi$ of the operator G_ϕ. We claim that the subspace $\mathscr{N}_{\alpha,\phi}$ has dimension $2N$.

The condition

$$(\psi, G_\phi\chi)_{(L^2)^4} = 0 \quad \text{for all } \chi \in H^2$$

on a function $\psi \in (H^1)^4$ is equivalent to the equation $G^*_\phi\psi = 0$, where $G^*_\phi : (L^2)^4 \to L^2$ is the operator formally conjugate to G_ϕ:

$$G^*_\phi(a_1, a_2, \varphi_1, \varphi_2) = -\partial_1 a_1 - \partial_2 a_2 - \phi_2\varphi_1 + \phi_1\varphi_2. \tag{11}$$

(Note that the operators G_ϕ and G^*_ϕ are defined for any continuous bounded function ϕ on the plane.)

The subspace $\mathscr{N}_{\alpha,\phi}$ coincides with the kernel of the operator $\mathscr{D}_{\alpha,\phi}$ obtained by 'combining' the operators $D_{\alpha,\phi}$ and G^*_ϕ:

$$\mathscr{D}_{\alpha,\phi} = \begin{pmatrix} -\partial_1 & -\partial_2 & -\phi_2 & \phi_1 \\ -\partial_2 & \partial_1 & \phi_1 & \phi_2 \\ \phi_2 & \phi_1 & \partial_1 + \alpha_2 & -\partial_2 + \alpha_1 \\ \phi_1 & \phi_2 & \partial_2 - \alpha_1 & \partial_1 + \alpha_2 \end{pmatrix}.$$

The operator $\mathscr{D}_{\alpha,\phi}$ is a bounded operator from the space $(H^1)^4$ into $(L^2)^4$. (A similar operator was introduced in [6] to study the tangent bundle of the moduli space $\mathscr{M}_N$.) We now describe the kernel of the operator $\mathscr{D}_{\alpha,\phi}$ following [6].

Let $Q \in \mathscr{M}_N$ be a point of the moduli space with coordinates $q = (q^1, \dots, q^{2N})$. Then for any N-vortex solution (A, Φ) in the gauge class $Q = [A, \Phi]$, the function $\Phi(z)$ has zeros coinciding (taking account of multiplicity) with the zeros of the polynomial

$$P(z) = z^N + (q^1 + iq^2)z^{N-1} + \cdots + (q^{2N-1} + iq^{2N}).$$

As shown in [1], the gauge class Q contains a unique solution for which the function Φ has the form

$$\Phi(z) = P(z)f(z), \quad \text{where } f(z) > 0 \text{ everywhere in } \mathbb{C}.$$

We will call such a solution *canonical* and denote its components by $(A(q), \Phi(q)) = (A_1(q), A_2(q), \Phi(q))$. An arbitrary N-vortex solution (α, ϕ) in the class Q can be written in the form

$$\alpha_1 = A_1(q) + \partial_1\chi, \quad \alpha_2 = A_2(q) + \partial_2\chi, \quad \phi = \Phi(q)e^{i\chi}$$

with some smooth real-valued function χ.

Then the operators $\mathscr{D}_{\alpha,\phi}$ and $\mathscr{D}_{A(q),\Phi(q)}$ are connected by the relation

$$\mathscr{D}_{\alpha,\phi} = U_\chi \mathscr{D}_{A(q),\Phi(q)} U_\chi^{-1}, \tag{12}$$

where

$$U_\chi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\chi & -\sin\chi \\ 0 & 0 & \sin\chi & \cos\chi \end{pmatrix}. \tag{13}$$

Obviously, U_χ defines a bounded invertible operator from the space $(H^1)^4$ into itself; therefore,

$$\ker \mathscr{D}_{\alpha,\phi} = U_\chi \ker \mathscr{D}_{A(q),\Phi(q)}$$

and the problem of describing the kernel of the operator $\mathscr{D}_{\alpha,\phi}$ reduces to giving a description of the kernel of $\mathscr{D}_{A(q),\Phi(q)}$.

It was shown in [6] that the kernel of the operator $\mathscr{D}_{A(q),\Phi(q)}$ has dimension $2N$. A basis of this space was constructed in [7]. To be explicit, in [7] it was proved that the components of the canonical solution $A_1(q), A_2(q), \Phi(q)$ depend smoothly on the coordinates q. Their derivatives $\frac{\partial}{\partial q^\mu} A_1(q)$, $\frac{\partial}{\partial q^\mu} A_2(q)$, $\frac{\partial}{\partial q^\mu} \Phi(q)$, as derivatives of vortex solutions with respect to the parameters, satisfy the linearized vortex equations. However, they may violate the condition of fixing the gauge $G^*_{\Phi(q)}\psi = 0$. In order to make sure that this condition holds, we choose the gauge functions χ_μ in such a way that this condition holds for the set of gauge-corrected vector functions

$$n_\mu = \Big(\frac{\partial A_1}{\partial q_\mu} + \partial_1\chi_\mu, \frac{\partial A_2}{\partial q_\mu} + \partial_2\chi_\mu, \frac{\partial \Phi_1}{\partial q_\mu} - \chi_\mu\Phi_2, \frac{\partial \Phi_2}{\partial q_\mu} + \chi_\mu\Phi_1\Big). \tag{14}$$

As was shown in [7], the $2N$ vector functions n_μ, $\mu = 1, \dots, 2N$, constructed above belong to the space $(H^1)^4$, are linearly independent at every point $Q = (q^\mu)$, and depend smoothly on the coordinates q^μ; that is, they define smooth maps from $\mathbb{R}^{2N}$ into $(H^1)^4$.

Consequently, for every Q with coordinates $q = (q^\mu)$ the vectors $n_\mu(q)$ form a basis of the space $\ker \mathscr{D}_{A(q),\Phi(q)}$, and the vectors $\tilde{n}_\mu := U_\chi n_\mu$ form a basis of the space $\ker \mathscr{D}_{\alpha,\phi}$.

We define the kinetic metric on the space $\mathscr{M}_N$ using the basis constructed above. To do this we identify the tangent space to $\mathscr{M}_N$ at a point Q with the kernel of the operator $\mathscr{D}_{A(q),\Phi(q)}$ (see [6]) and consider the decomposition

$$\ker D_{A(q),\Phi(q)} = \operatorname{im} G_{\Phi(q)} \oplus \ker \mathscr{D}_{A(q),\Phi(q)}.$$

In other words, this decomposition means that the tangent space to the space of N-vortex solutions is the direct sum of the tangent space to the orbit of the gauge group passing through this point and the tangent space to the moduli space. Here the tangent vector

∂_{q^μ} in the space $T_Q\mathscr{M}_N$ is identified with the vector $n_\mu \in \ker \mathscr{D}_{A(q),\Phi(q)}$. The *kinetic metric on $\mathscr{M}_N$* is defined by the formula

$$g_{\mu\nu}(q) = \langle n_\mu(q), n_\nu(q)\rangle_{(L^2)^4},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in the space $(L^2)^4$. Note that $\langle \tilde{n}_\mu, \tilde{n}_\nu\rangle_{(L^2)^4} = \langle n_\mu, n_\nu\rangle_{(L^2)^4}$.

§ 5. The auxiliary system

As a rule, henceforth we shall omit the arguments of functions, but bear in mind that the argument $\tau = \varepsilon t$ is substituted for functions of τ: for example, $\partial_\tau \phi_2$ is the abbreviated notation for the expression $\partial_\tau\phi_2(\tau,x,y)|_{\tau=\varepsilon t}$.

When we substitute the ansatz (7) into (6) we obtain a system of equations, which conveniently can be written in the form of one equation for the function a_0 and one equation for the vector function $\psi = (a_1, a_2, \varphi_1, \varphi_2)$. After dividing by ε^3, using the notation introduced in the preceding section, the first equation can be written in the form

$$-\Delta a_0 + |\phi|^2 a_0 = \frac{1}{\varepsilon^2} G_\phi^* \partial_\tau(\alpha,\phi) + \frac{1}{\varepsilon}\partial_t G_\phi^*\psi + 2(\partial_\tau\phi_2\cdot\varphi_1 - \partial_\tau\phi_1\cdot\varphi_2) + \varepsilon J_0, \tag{15}$$

where $J_0(\alpha,\phi,a_0,\psi)$ is the sum of nonlinear terms of the following form:

$$J_0 = \varphi_2(\varphi_1)_t - \varphi_1(\varphi_2)_t - 2\varepsilon(\phi_1\varphi_1 + \phi_2\varphi_2)a_0 - \varepsilon^3(\varphi_1^2+\varphi_2^2)a_0.$$

Dividing by ε^2, the second equation can be written in the form

$$\psi_{tt} + D^*_{\alpha,\phi}D_{\alpha,\phi}\psi = -\partial_\tau^2(\alpha_1,\alpha_2,\phi_1,\phi_2) + \varepsilon G_\phi(a_0)_t + \varepsilon^2 J(\alpha,\phi,a_0,\psi), \tag{16}$$

where the components of the vector $J(\alpha,\phi,a_0,\psi)$ are the sums of the terms that are nonlinear in ψ and depend only on ψ, a_0 and the derivatives of ψ of the first order with respect to the space variables. For completeness we give their explicit form:

$$\begin{aligned}
J_1 &= \varphi_1\partial_1\varphi_2 - \varphi_2\partial_1\varphi_1 - \alpha_2(\varphi_1^2+\varphi_2^2) - 2(\phi_1\varphi_1+\phi_2\varphi_2)a_1\\
&\quad - \varepsilon^2 a_1(\varphi_1^2+\varphi_2^2);\\
J_2 &= \varphi_1\partial_2\varphi_2 - \varphi_2\partial_2\varphi_1 - \alpha_2(\varphi_1^2+\varphi_2^2) - 2(\phi_1\varphi_1+\phi_2\varphi_2)a_2\\
&\quad - \varepsilon^2 a_2(\varphi_1^2+\varphi_2^2);\\
J_3 &= -2(\partial_\tau\phi_2)a_0 + 2a_1\partial_1\varphi_2 + \varphi_2\partial_1 a_1 - 2\alpha_1 a_1\varphi_1 - \phi_1 a_1^2 + 2a_2\partial_2\varphi_2\\
&\quad + \varphi_2\partial_2 a_2 - 2\alpha_2 a_2\varphi_1 - \phi_1 a_2^2 - (\phi_1\varphi_1+\phi_2\varphi_2)\varphi_1 - \frac{1}{2}\phi_1(\varphi_1^2+\varphi_2^2)\\
&\quad - \varepsilon^2\Bigl(a_1^2\varphi_1 + a_2^2\varphi_1 + \frac{1}{2}\varphi_1(\varphi_1^2+\varphi_2^2) - \phi_1 a_0^2\Bigr) + \varepsilon^4\varphi_1 a_0^2;\\
J_4 &= 2(\partial_\tau\phi_1)a_0 - 2a_1\partial_1\varphi_1 - \varphi_1\partial_1 a_1 - 2\alpha_1 a_1\varphi_2 - \phi_2 a_1^2 - 2a_2\partial_2\varphi_1\\
&\quad - \varphi_1\partial_2 a_2 - 2\alpha_2 a_2\varphi_2 - \phi_2 a_2^2 - (\phi_1\varphi_1+\phi_2\varphi_2)\varphi_2 - \frac{1}{2}\phi_2(\varphi_1^2+\varphi_2^2)\\
&\quad - \varepsilon^2\Bigl(a_1^2\varphi_1 + a_2^2\varphi_1 + \frac{1}{2}\varphi_2(\varphi_1^2+\varphi_2^2) - \phi_2 a_0^2\Bigr) + \varepsilon^4\varphi_2 a_0^2.
\end{aligned}$$

The term of order $1/\varepsilon^2$ on the right-hand side of equation (15) can be reduced to zero, as we show in the next section, by using a suitable choice of the gauge of the curve of N-vortex solutions (α,ϕ). Assuming that such a gauge is already chosen, we omit this term in what follows.

On the left-hand side of the system (16) there is a hyperbolic operator, which is strongly degenerate, as mentioned in the preceding section. Therefore it is not convenient to work with this system. However, it is possible to construct a solution of the original system by solving a certain auxiliary system. This auxiliary system has the same form as the original one, but instead of the operator $D_{\alpha,\phi}$, which has an infinite-dimensional degeneration, it involves an operator $\mathscr{D}_{\alpha,\phi}$ with finite-dimensional kernel, and the terms of order $1/\varepsilon$ are eliminated from the first equation. The operator $\mathscr{D}_{\alpha,\phi}$ in the new system is connected with the original operator $D_{\alpha,\phi}$ by the relation

$$\mathscr{D}^*_{\alpha,\phi}\mathscr{D}_{\alpha,\phi} = D^*_{\alpha,\phi}D_{\alpha,\phi} + G_\phi G^*_\phi.$$

Suppose that the solution of the system (15), (16) satisfies the condition

$$G^*_{\phi(t)}\psi(t) = 0 \quad \text{for every} \quad t. \tag{17}$$

Remark. The relation $G^*_\phi\psi = 0$ plays the role of 'condition of least squares'; namely, a vortex solution (α, ϕ) satisfying this condition is the closest in the L^2-norm to $(\alpha, \phi)+\varepsilon^2\psi$ among all the solutions that are gauge equivalent to (α, ϕ).

If (a_0, ψ) is a solution of the system (15), (16) which satisfies condition (17), then it also satisfies the system

$$-\Delta a_0 + |\phi|^2 a_0 = 2(\partial_\tau\phi_2 \cdot \varphi_1 - \partial_\tau\phi_1 \cdot \varphi_2) + \varepsilon J_0,$$

$$\psi_{tt} + \mathscr{D}^*_{\alpha,\phi}\mathscr{D}_{\alpha,\phi}\psi = -\partial^2_\tau(\alpha, \phi) + \varepsilon G_\phi(a_0)_t + \varepsilon^2 J.$$

This system already satisfies both requirements that we impose on the auxiliary system. However, it is convenient to 'improve it' by adding the term $G_{\phi+\varepsilon^2\varphi}G^*_\phi\psi$ on the left-hand side of (16) instead of the term $G_\phi G^*_\phi\psi$. Since

$$G_{\phi+\varepsilon^2\varphi}G^*_\phi\psi = G_\phi G^*_\phi\psi + \varepsilon^2(0, 0, -\varphi_2 G^*_\phi\psi, \varphi_1 G^*_\phi\psi),$$

the system thus obtained has the same form as the previous one. Namely, if we introduce the notation

$$J' := J - (0, 0, -\varphi_2 G^*_\phi\psi, \varphi_1 G^*_\phi\psi),$$

then the new system can be written in the following form:

$$-\Delta a_0 + |\phi|^2 a_0 = 2(\partial_\tau\phi_2 \cdot \varphi_1 - \partial_\tau\phi_1 \cdot \varphi_2) + \varepsilon J_0, \tag{18}$$

$$\psi_{tt} + \mathscr{D}^*_{\alpha,\phi}\mathscr{D}_{\alpha,\phi}\psi = -\partial^2_\tau(\alpha, \phi) + \varepsilon G_\phi(a_0)_t + \varepsilon^2 J'. \tag{19}$$

From here on the system (18), (19) is the main object of our study. In §§ 6–8 we shall prove that this system has a solution defined on a sufficiently long time interval, but first we explain how this system can be used to obtain a solution of the original system. As we noted above, for a solution of the system (18), (19) to satisfy the original system (15), (16), condition (17) must hold. However, this condition holds automatically for solutions of the system (18), (19) provided the initial conditions are chosen appropriately (in particular, zero initial conditions are suitable, and the solution constructed in §§ 6–8 will satisfy these). This is a consequence of the following theorem.

Theorem 3. *Suppose that $(\alpha(t), \phi(t))$, $t \in [t_0; t_1]$, is a smooth curve in the space of N-vortex solutions such that its projection $[\alpha(t), \phi(t)]$ onto the moduli space $\mathscr{M}_N$ is a smooth curve in $\mathscr{M}_N$, and suppose that the components α_1, α_2, ϕ_1, ϕ_2 are bounded on the set $[t_0; t_1] \times \mathbb{R}^2$. Suppose that a function $a_0 \in C^1([t_0; t_1], H^3)$ and a vector function $\psi = (a_1, a_2, \varphi_1, \varphi_2)$ with components of class* (8) *define a solution of the system* (18), (19) *on the interval $[t_0; t_1]$, with the given α and ϕ. Suppose that the following conditions hold:*

$$G^*_{\phi(t_0)}\psi(t_0) = 0, \tag{20}$$

$$\frac{d}{dt}\bigg|_{t=t_0} G^*_{\phi(t)}\psi(t) = 0. \tag{21}$$

Then the vector function ψ satisfies condition (17) *and, consequently, the pair (a_0, ψ) is a solution of the system* (15), (16).

In other words, the function $G^*_{\phi(t)}\psi(t)$ vanishes on the entire time interval if this function itself and its first derivative with respect to time are both zero at the initial moment.

Proof. We denote the expression under the integral in formula (5) for the action $\mathscr{S}$, in other words the *density of the Lagrangian*, by $\mathscr{L}$. We rewrite the system of equations (6) in Lagrangian form.

To do this we introduce the following notation:

$$l_j := \frac{\partial \mathscr{L}}{\partial A_j} - \sum_{k=0}^{2} \partial_k \Big(\frac{\partial \mathscr{L}}{\partial(\partial_k A_j)} \Big) \quad \text{for } j = 0,1,2;$$
$$l_3 := \frac{\partial \mathscr{L}}{\partial \Phi_1} - \sum_{k=0}^{2} \partial_k \Big(\frac{\partial \mathscr{L}}{\partial(\partial_k \Phi_1)} \Big);$$
$$l_4 := \frac{\partial \mathscr{L}}{\partial \Phi_2} - \sum_{k=0}^{2} \partial_k \Big(\frac{\partial \mathscr{L}}{\partial(\partial_k \Phi_2)} \Big).$$

The Euler–Lagrange system of equations (6) in the Lagrangian form takes the form

$$l_j = 0, \quad j = 0,1,\dots,4.$$

These equations coincide with the first, second, and third equations and the real and imaginary parts of the fourth equation in the system (6), respectively.

The expressions $l_0,\dots,l_4$ satisfy the identity

$$\partial_0 l_0 + \partial_1 l_1 + \partial_2 l_2 + \Phi_2 l_3 - \Phi_1 l_4 = 0. \tag{22}$$

This identity can be verified directly, or it can be proved as follows. Since the functional $\mathscr{S}$ is invariant under dynamical gauge transformations, any smooth compactly supported function $\chi(t,x,y)$ (that is, having support inside the set $(t_0,t_1)\times\mathbb{R}^2$) satisfies the relation

$$\frac{d}{dt}\bigg|_{t=0} \mathscr{S}(A_j + t\partial_j \chi, e^{it\chi}\Phi) = 0,$$

which implies that

$$\int_{t_0}^{t_1} dt \int_{\mathbb{R}^2} (l_0\partial_0\chi + l_1\partial_1\chi + l_2\partial_2\chi - l_3\Phi_2\chi + l_4\Phi_1\chi)\,dxdy = 0. \tag{23}$$

Integrating by parts and noting that the function χ is arbitrary, we obtain the required identity (22).

Remark. Identity (22) means, in particular, that the first equation $l_0 = 0$ plays the role of a connection condition: this equation holds automatically for any t if it holds for some $t = t_0$, under the condition that the other four equations hold.

In the notation of the preceding section, we can write identity (22) in the form

$$\partial_t l_0 - G^*_\Phi(l_1,l_2,l_3,l_4) = 0. \tag{24}$$

To prove the theorem we observe that the system (18), (19) can be rewritten as follows (here, $A_0 = \varepsilon^3 a_0$, $A_j = \alpha_j + \varepsilon^2 a_j$ for $j = 1,2$ and $\Phi = \phi + \varepsilon^2\varphi$):

$$\begin{pmatrix} l_0(A_0,A_1,A_2,\Phi) \\ l_1(A_0,A_1,A_2,\Phi) \\ l_2(A_0,A_1,A_2,\Phi) \\ l_3(A_0,A_1,A_2,\Phi) \\ l_4(A_0,A_1,A_2,\Phi) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -\partial_t \\ \\ G_{\Phi(t)} \\ \\ \end{pmatrix} G^*_{\phi(t)}\psi = 0. \tag{25}$$

In other words, for the given functions A_0, A_1, A_2, Φ, the following equations hold:

$$l_0 = \varepsilon^2 \partial_t G^*_{\phi(t)} \psi, \quad (l_1, l_2, l_3, l_4) = -\varepsilon^2 G_{\Phi(t)} G^*_{\phi(t)} \psi.$$

Substituting these expressions into (24) we obtain the relation

$$\partial_t^2 G^*_{\phi(t)} \psi(t) + G^*_{\Phi(t)} G_{\Phi(t)} G^*_{\phi(t)} \psi(t) = 0.$$

Using the relation $G^*_\Phi G_\Phi = -\Delta + |\Phi|^2$ (which can be proved by an explicit calculation), we can conclude that the function $F(t,x,y) = G^*_{\phi(t,x,y)} \psi(t,x,y)$ satisfies a hyperbolic equation of the form

$$(\partial_t^2 + |\Phi(t)|^2 - \Delta) F = 0. \tag{26}$$

Furthermore, by the hypotheses of the theorem, $F(t)$ satisfies the zero initial conditions for $t = t_0$, that is, $F(t_0) = 0$ and $F_t(t_0) = 0$. In order to prove the theorem, we need to show that $F(t) \equiv 0$ on $[t_0; t_1]$.

Note that the transformations carried out above are well defined for functions in the classes indicated in the hypotheses of the theorem in view of the Sobolev embedding theorems, according to which the embeddings

$$H^2(\mathbb{R}^2) \subset BC(\mathbb{R}^2), \quad H^3(\mathbb{R}^2) \subset BC^1(\mathbb{R}^2) \tag{27}$$

are continuous. Here $BC(\mathbb{R}^2)$ denotes the space of continuous bounded functions on $\mathbb{R}^2$, and $BC^1(\mathbb{R}^2)$ the space of continuous bounded functions with continuous bounded first derivatives on $\mathbb{R}^2$.

It is easy to verify that the function $F(t)$ belongs to the class

$$C\big([t_0; t_1], H^2(\mathbb{R}^2)\big) \cap C^1\big([t_0; t_1], H^1(\mathbb{R}^2)\big) \cap C^2\big([t_0; t_1], L^2(\mathbb{R}^2)\big).$$

We now prove that the solution of equation (26) with zero initial conditions is identically equal to zero. By taking the inner product of each side of (26) with F_t, we obtain the relation

$$\langle F_{tt}, F_t \rangle + \langle \Phi^2(t) F, F_t \rangle - \langle \Delta F, F_t \rangle = 0$$

or

$$\frac{d}{dt} \big(\|F_t\|^2 + \langle \Phi^2(t) F, F \rangle + \|\nabla F\|^2 \big) = 2 \langle (\Phi^2)_t F, F_t \rangle.$$

(Throughout the proof of this theorem, the inner products and norms are taken in $L^2(\mathbb{R}^2)$.)

We fix some time moment $T \in [t_0; t_1]$. Integrating the last inequality with respect to time, taking the initial conditions into account, and only leaving the first summand on the left-hand side, we obtain the inequality

$$\|F_t(T)\|^2 \le 2 \int_{t_0}^T \langle (\Phi^2)_t(s) F(s), F_t(s) \rangle ds. \tag{28}$$

We set

$$C = \max_{t \in [t_0, t_1],\, (x,y) \in \mathbb{R}^2} |(\Phi^2)_t(t,x,y)|.$$

Since $F(t_0) = 0$, the estimate

$$\|F(s)\| \le (s - t_0) \max_{t \in [t_0, s]} \|F_t(t)\| \le (T - t_0) \max_{t \in [t_0, T]} \|F_t(t)\|$$

holds for any $s \in [t_0; T]$.

Using this estimate to estimate the right-hand side of (28), we obtain the inequality

$$\|F_t(T)\|^2 \le 2C(T - t_0)^2 \Big(\max_{t \in [t_0, T]} \|F_t(t)\| \Big)^2. \tag{29}$$

Suppose that the maximum of the right-hand side of (29) for the given T is attained at some point $m(T) \in [t_0, T]$. Then

$$\max_{t\in[t_0,T]} \|F_t(t)\| = \|F_t(m(T))\| = \max_{t\in[t_0,m(T)]} \|F_t(t)\|.$$

Writing out (29) at the point $m(T)$, we obtain the inequality

$$\big(1 - 2C(m(T)-t_0)^2\big)\|F_t(m(T))\|^2 \le 0. \tag{30}$$

In particular, if $T = T_0 = t_0 + \frac{1}{2\sqrt{2C}}$, then

$$2C(m(T_0)-t_0)^2 \le 2C(T_0-t_0)^2 = \frac{1}{2} < 1.$$

Therefore, substituting $T = T_0$ into (30) we see that $\|F_t(m(T_0))\| = 0$ and therefore $F_t(t) \equiv 0$ and $F(t) \equiv 0$ on the interval $[t_0; T_0]$. Taking the point T_0 as the new initial moment, we now conduct the same arguments for the next small interval $[T_0; T_1]$, where $T_1 = T_0 + \frac{1}{2\sqrt{2C}}$, and so on. Proceeding in this way we prove that $F(t) \equiv 0$ on the entire interval $[t_0; t_1]$.

Thus, (17) holds on the entire time interval $[t_0; t_1]$. The theorem is proved. □

Remark. The proof of the theorem is independent of the choice of the gauge of the curve (α, ϕ).

§ 6. The existence theorem for the auxiliary system: preliminary observations

To prove Theorem 2 it remains to show that *the assertion of this theorem holds for the system* (18), (19). More precisely, we need to prove that for sufficiently small ε this system with zero initial conditions has a solution on the interval $[0; \tau_1/\varepsilon]$ whose norm is uniformly bounded with respect to ε. To prove this assertion first, in § 7, we prove a local existence theorem for the system (18), (19), that is, we establish the existence of a solution of this system on a small time interval and for sufficiently small initial conditions $\psi(t_0), \psi_t(t_0)$, and then (in § 8) we use a priori estimates to show that the solution with zero initial conditions can be extended to a time interval of order $1/\varepsilon$.

Before we carry out this plan, we fix a gauge for a solution (α, ϕ). To do this we fix a segment of the geodesic $Q\colon [0, \tau_0] \to \mathscr{M}_N$ in the moduli space of N-vortex solutions given in coordinates by a function $q(\tau)$, and write the solution (α, ϕ) in the form

$$\alpha(\tau) = A(q(\tau)) + d\chi(\tau), \quad \phi(\tau) = \Phi(q(\tau))e^{i\chi(\tau)},$$

where the function χ is defined by the condition

$$\chi_\tau(\tau, x, y) = \sum_{\mu=1}^{2N} \dot q^\mu(\tau)\chi_\mu\big(q(\tau), x, y\big)$$

(henceforth a dot denotes differentiation with respect to τ). Here, for the gauge functions χ_μ we take the functions introduced in § 4 in the definition of the basis $\{n_\mu\}$ (see formula (14)). An explicit form of these functions is given in [7], where, in particular, it is shown that they are smooth bounded functions on the plane in the variables x, y and depend smoothly on the parameter q. We add for definiteness the normalization condition $\chi(0, x, y) \equiv 0$; then the function $\chi(\tau, x, y)$ is uniquely determined by the curve $q(\tau)$.

The gauge chosen above ensures that the derivatives $\partial_\tau^2\alpha_1$, $\partial_\tau^2\alpha_2$, $\partial_\tau^2\phi$ belong to the requisite Sobolev space. Namely, one can verify that

$$\partial_\tau(\alpha, \phi)(\tau) = \sum_{\mu=1}^{2N} \dot q^\mu(\tau) U_{\chi(\tau)} n_\mu(q(\tau)),$$

where the operator U_χ is defined by formula (13) in § 4. It was shown in [7] that the maps $q \mapsto n_\mu(q)$ are smooth maps from $\mathbb{R}^{2N}$ into $(L^2)^4$ with values in $(H^1)^4$. In fact, the arguments in [7] can be used to show that the maps $q \mapsto n_\mu(q)$ are smooth maps from $\mathbb{R}^{2N}$ into $(H^3)^4$. Consequently, for the chosen gauge, the map

$$\tau \mapsto \partial_\tau(\alpha, \phi)$$

is a smooth map of the half-line $[0, \infty)$ into the Sobolev space $(H^3(\mathbb{R}^2))^4$. The functions α_1, α_2, ϕ themselves depend smoothly on the three variables τ, x, y and are bounded on the plane for every fixed value of τ. Moreover, it follows from the results in [7] that on any finite time interval $[0, \tau_0]$ the functions $\alpha_1(\tau, x, y)$, $\alpha_2(\tau, x, y)$, and $\phi(\tau, x, y)$ are bounded on the set $[0, \tau_0] \times \mathbb{R}^2$, together with their derivatives with respect to all variables.

Next, since $\partial_\tau(\alpha, \phi) = \sum \dot{q}^\mu U_\chi n_\mu$, we have $\partial_\tau(\alpha, \phi) \in \ker \mathscr{D}_{\alpha,\phi}$ for every τ (see §4). In particular,

$$G_\phi^* \partial_\tau(\alpha, \phi) = 0;$$

that is, the term of order $1/\varepsilon^2$ in equation (15) is indeed equal to zero.

§ 7. Local existence theorem

In this section we prove the following assertion.

Theorem 4. *There exist positive constants B_1, B_2, B_3 and a number $\delta_0 > 0$ such that*

- *for every time instant $t_0 \in [0; \tau_0/\varepsilon)$,*
- *for any initial conditions*

$$\psi(t_0) = \psi^0 \in (H^3(\mathbb{R}^2))^4, \quad \psi_t(t_0) = \psi^1 \in (H^2(\mathbb{R}^2))^4$$

- *and for every $\varepsilon \in \left(0; \frac{B_1}{1+\left(\|\psi^0\|_{H^3}+\|\psi^1\|_{H^2}\right)^2}\right)$,*

the system (18), (19) has a solution (a_0, ψ) on the interval $\big[t_0; \min\{t_0+\delta_0, \tau_0/\varepsilon\}\big]$ with the given initial conditions $\psi(t_0), \psi_t(t_0)$ such that a_0 belongs to the class $E_0 := C^1([t_0; T], H^3)$, and the components of the vector ψ belong to the class

$$E_1 := C([t_0; T], H^3) \cap C^1([t_0; T], H^2) \cap C^2([t_0; T], H^1).$$

Furthermore, this solution satisfies the estimate

$$\|a_0(t)\|_{H^3} + \|(a_0)_t(t)\|_{H^3} + \|\psi(t)\|_{(H^3)^4} + \|\psi_t(t)\|_{(H^2)^4} \\ \le B_2(\|\psi^0\|_{H^3} + \|\psi^1\|_{H^2}) + B_3 \tag{31}$$

for every t.

Proof. We represent the operator $\mathscr{D}^*_{\alpha,\phi}\mathscr{D}_{\alpha,\phi}$ as a sum of two operators, the first of which includes all the second order derivatives of the vector function ψ, and the second operator includes the derivatives of at most first order.

We set

$$L_0 := \operatorname{diag}(1 - \Delta, 1 - \Delta, 1 - \Delta, 1 - \Delta).$$

It is easy to see that the difference $W = \mathscr{D}^*_{\alpha,\phi}\mathscr{D}_{\alpha,\phi} - L_0$ does not contain derivatives of the second order, and the coefficients of the first derivatives and the free term are smooth bounded functions that can be expressed in terms of α_1, α_2, ϕ and their first derivatives with respect to x and y.

For brevity we denote

$$K := -\partial_\tau^2(\alpha_1, \alpha_2, \phi_1, \phi_2) \quad \text{and} \quad W_0\psi := 2(\varphi_1\partial_\tau\phi_2 - \varphi_2\partial_\tau\phi_1)$$

and write (18), (19) in the form

$$-\Delta a_0 + |\phi|^2 a_0 = W_0\psi + \varepsilon J_0(a_0, \psi, \psi_t), \tag{32}$$

$$\psi_{tt} + L_0\psi + W\psi = K + \varepsilon G_\phi(a_0)_t + \varepsilon^2 J'(a_0, \psi). \tag{33}$$

We seek a solution of the system (32), (33) with the initial conditions $\psi(t_0) = \psi^0$, $\psi_t(t_0) = \psi^1$ on some small interval $[t_0, T]$ (the choice of T will be made later).

To do this as in [6], we use the following iteration process. As our zeroth approximation we take $a_{0,0} = 0$, $\psi_0 = 0$ and seek the nth approximation $(a_{0,n}, \psi_n)$, $n \in \mathbb{N}$, from the equations

$$-\Delta a_{0,n} + |\phi|^2 a_{0,n} = W_0\psi_{n-1} + \varepsilon J_0\bigl(a_{0,n-1}, \psi_{n-1}, (\psi_{n-1})_t\bigr), \tag{34}$$

$$(\psi_n)_{tt} + L_0\psi_n = -W\psi_{n-1} + K + \varepsilon G_\phi(a_{0,n-1})_t + \varepsilon^2 J'(a_{0,n-1}, \psi_{n-1}) \tag{35}$$

with the initial conditions $\psi_n(t_0) = \psi^0$, $(\psi_n)_t(t_0) = \psi^1$.

This process is justified by the following lemma.

Lemma 1. *If $\psi^0 \in (H^3)^4$ and $\psi^1 \in (H^2)^4$, then at every step we can uniquely define functions $a_{0,n}$ and ψ_n in such a way that the function $a_{0,n}$ belongs to the class E_0, and the components of the vector ψ_n to the class E_1.*

The proof of Lemma 1 is given in § 9.

Now we look at whether or not this iteration process converges. We introduce the notation $\|f\|_k := \max_{t_0\le t\le T} \|f(t)\|_{H^k}$ and

$$M_n := \max\{\|\psi_n\|_3, \|(\psi_n)_t\|_2, \|a_{0,n}\|_3, \|(a_{0,n})_t\|_3\}.$$

Furthermore, we set

$$\begin{aligned}\sigma_k := \|\psi_{k+1} - \psi_k\|_3 + \|(\psi_{k+1})_t - (\psi_k)_t\|_2 \\ + \|a_{0,k+1} - a_{0,k}\|_3 + \|(a_{0,k+1})_t - (a_{0,k})_t\|_3\end{aligned} \tag{36}$$

and

$$\Sigma_n = \sum_{k=0}^{n-1} \sigma_k.$$

Since $a_{0,0} = \psi_0 = 0$, it is clear that $M_n \le \Sigma_n$.

The proof is conducted in three steps. First we obtain an estimate for the quantity σ_n in terms of σ_{n-1} and σ_{n-2} *under the assumption that $\varepsilon < 1/M_k^2$ for $k = n$, $n-1$, $n-2$*. (The significance of these assumptions will become clear in the course of the proof.) Then we derive an estimate for M_n which is *uniform in n*, under the condition that the quantities ε and $T - t_0$ are small. Finally, this estimate enables us to choose the range of variation of ε. Finally, we prove that, for the chosen ε and T, the sequences ψ_n and $a_{0,n}$ converge to a solution of the system (32), (33).

In what follows it is assumed that $\varepsilon < 1$.

Lemma 2. *There exists a positive constant C such that the inequality*

$$\sigma_n \le C(\sqrt{\varepsilon} + T - t_0)(\sigma_{n-1} + \sigma_{n-2}) \tag{37}$$

holds for any $n \ge 2$ and any ε satisfying the condition $\varepsilon < 1/M_k^2$ for $k = n$, $n-1$, $n-2$.

Using this lemma we can show that for sufficiently small ε and $T - t_0$ the quantities Σ_n can be estimated from above by the partial sums of a series whose terms form a convergent geometric progression. Therefore Σ_n can be estimated by a quantity (namely, by the sum of this series) that is independent of n. Hence the same assertion is also valid for M_n. We use this scheme to prove Lemma 3, which follows.

Lemma 3. *There exist constants $C_0 > 0$, $C_1 > 1$, and $K_0 > 0$ for which the following assertion holds. Let $n \geq 2$. If $\varepsilon < M_k^{-2}$ for $k = 0, 1, \dots, n-1$ and the quantities ε and $T - t_0$ satisfy the condition $C_0(\sqrt{\varepsilon} + T - t_0) < 1$, then*

$$\Sigma_n \leq \frac{C_1\big(\|\psi^0\|_{(H^3)^4} + \|\psi^1\|_{(H^2)^4} + (\sqrt{\varepsilon} + T - t_0)K_0\big)}{1 - C_0(\sqrt{\varepsilon} + T - t_0)}. \tag{38}$$

The proofs of Lemmas 2 and 3 are given in § 9.

We now choose $\delta_0 = T - t_0 = \frac{1}{4C_0}$ and suppose that $\sqrt{\varepsilon} < \frac{1}{4C_0}$. Then we can conclude from inequality (38) that

$$M_n \leq \Sigma_n \leq 8C_1\big(\|\psi^0\|_{(H^3)^4} + \|\psi^1\|_{(H^2)^4} + K_0/(2C_0)\big) \tag{39}$$

for all $n = 0, 1, 2, \dots$.

Using the estimate (39) we obtain the final bound for ε. Namely, suppose that

$$\varepsilon < \min\left\{1,\ \frac{1}{16C_0^2},\ \frac{1}{64C_1^2(\|\psi^0\|_{(H^3)^4} + \|\psi^1\|_{(H^2)^4} + K_0/(2C_0))^2}\right\}. \tag{40}$$

For such ε, a uniform estimate for M_n can be obtained using induction on n. Indeed, (39) obviously holds for $n = 0$ and $n = 1$. Then $\varepsilon < M_k^{-2}$ for $k = 0, 1$. Consequently, by Lemma 3, the estimate (39) holds for $n = 2$. Therefore, $\varepsilon < M_2^{-2}$. Then again by Lemma 3, inequality (39) holds for $n = 3$. But then $\varepsilon < M_3^{-2}$, and so on. In the general case we have the estimate (39) for M_k for $k = 0, 1, \dots, N$, which in combination with the estimate (40) for ε enables us to conclude that $\varepsilon < M_k^{-2}$ for $k = 0, 1, \dots, N$. Therefore, applying Lemma 3 we obtain that (39) holds for $n = N + 1$.

To complete the proof of Theorem 4, we observe that for the chosen values of δ_0 (or T) and ε, the series $\sum_{n=0}^{\infty} \sigma_n$ converges, since its partial sums Σ_n are bounded. Therefore the sequences ψ_n, $(\psi_n)_t$, $a_{0,n}$, $(a_{0,n})_t$ converge in the corresponding normed spaces (for example, the series $\sum_{n=0}^{\infty}(\psi_{n+1} - \psi_n)$ is majorized in norm by the series $\sum_{n=0}^{\infty} \sigma_n$). The limit functions

$$(a_0, \psi) = \lim_{n\to\infty} (a_{0,n}, \psi_n)$$

belong to the required function classes, and the maximum of the sum of their norms can be estimated from above by the right-hand side of (39). As a consequence of equation (35), the functions $(\psi_n)_{tt}$ converge uniformly with respect to t in the space $(H^1)^4$. Therefore the limit functions a_0 and ψ give solutions of the system (18), (19) that belong to the classes E_0 and E_1 and satisfy (for the corresponding choice of the constants B_2, B_3, which can be easily expressed in terms of the constants C_0, C_1, and K_0) the estimates given in the statement of the theorem. The constant B_1 can be chosen in terms of the constants C_0, C_1, and K_0 in such a way that the estimate $\varepsilon < \frac{B_1}{1+(\|\psi^0\|_{H^3}+\|\psi^1\|_{H^2})^2}$ implies (40). □

§ 8. Extending a solution with respect to time

The goal of this section is to prove that the system (18), (19) has a solution that can be extended to a time interval of order $1/\varepsilon$. In order to achieve this, we firstly prove a conditional a priori estimate for a solution.

For a pair (a_0, ψ) in the classes E_0 and E_1 on some interval (see the statement of Theorem 4), we introduce the notation

$$\|(a_0(t), \psi(t))\| := \|a_0(t)\|_{H^3} + \|(a_0)_t(t)\|_{H^3} + \|\psi(t)\|_{(H^3)^4} + \|\psi_t(t)\|_{(H^2)^4}.$$

Theorem 5. *Let (a_0, ψ) be a solution of the system (18), (19) in the class $E_0 \times E_1$ on an interval $[0, T_0]$ with zero initial conditions: $\psi(0) = 0$, $\psi_t(0) = 0$. Suppose that for* *some constant M the inequality*

$$\|(a_0(t), \psi(t))\| \leq M$$

holds for all $t \in [0; T_0]$, and let $\varepsilon < M^{-1}$.

Then for some positive constant B_4 independent of M the estimate

$$\|\psi(t)\|_{(H^3)^4} + \|\psi_t(t)\|_{(H^2)^4} \leq B_4(1 + \varepsilon t(1 + M^4)) \tag{41}$$

holds for any $t \in [0; T_0]$.

Proof. We use the method proposed in [6], which is based on estimating the operator $\mathscr{D}_{\alpha,\phi}$ from below. We consider the operator $\mathscr{D}_{\alpha,\phi}$ in which (α, ϕ) is the canonical N-vortex solution, that is, $(\alpha, \phi) = (A(q), \Phi(q))$. It follows from Theorem 3.1 in [6] that vectors $\psi \in (H^1)^4$ that are L^2-orthogonal to the kernel of the operator $\mathscr{D}_{A(q),\Phi(q)}$ satisfy the estimate

$$\|\mathscr{D}_{A(q),\Phi(q)}\psi\|_{L^2} \geq \eta(q)\|\psi\|_{H^1}, \tag{42}$$

where $\eta(q)$ is a continuous strictly positive function of q.

The minimum of the function $\eta(q(\tau))$ on the interval $[0, \tau_0]$ is positive and the expression (12) for the operator $\mathscr{D}_{\alpha,\phi}$ implies that there exists a constant $\gamma > 0$ such that the estimate

$$\|\mathscr{D}_{\alpha(\tau),\phi(\tau)}\psi\|_{L^2} \geq \gamma\|\psi\|_{H^1}$$

holds uniformly in $\tau \in [0, \tau_0]$ for vectors ψ that are orthogonal to the kernel of the operator $\mathscr{D}_{\alpha(\tau),\phi(\tau)}$.

We now estimate the component of the vector ψ orthogonal to the kernel of $\mathscr{D}_{\alpha,\phi}$. To do this we introduce the operator $L(\tau) := \mathscr{D}^*_{\alpha(\tau),\phi(\tau)}\mathscr{D}_{\alpha(\tau),\phi(\tau)}$. Equation (19) can be written in the short form

$$\psi_{tt} + L(\varepsilon t)\psi = K + \varepsilon G_\phi(a_0)_t + \varepsilon^2 J'(a_0, \psi). \tag{43}$$

(The argument εt of the operator $L(\varepsilon t)$ is omitted in what follows.)

We represent the vector function ψ in the form

$$\psi(t) = \psi_1(t) + \psi_2(t),$$

where $\psi_1 \perp \ker \mathscr{D}_{\alpha,\phi}$ and $\psi_2 \in \ker \mathscr{D}_{\alpha,\phi}$.

In order to obtain an estimate for ψ_1 in the norm of $(H^3)^4$, we firstly find an estimate for the norm $\|L\psi\|_{H^1}$. We introduce the function

$$\begin{aligned} Q_2(t) :=& \frac{1}{2}\big(\langle (L\psi)_t, (L\psi)_t\rangle + \langle \mathscr{D}_{\alpha,\phi}L\psi, \mathscr{D}_{\alpha,\phi}L\psi\rangle\big) \\ =& \frac{1}{2}\big(\langle (L\psi)_t, (L\psi)_t\rangle + \langle L^2\psi, L\psi\rangle\big). \end{aligned} \tag{44}$$

(The angle brackets henceforth denote the inner product in the space $(L^2)^4$.) If we formally differentiate Q_2 with respect to time, then we obtain the equation

$$\frac{d}{dt}Q_2 = \langle (L\psi)_t, (L\psi)_{tt}\rangle + \langle L^2\psi, (L\psi)_t\rangle + \frac{1}{2}\varepsilon\langle L_\tau L\psi, L\psi\rangle.$$

Note that all the terms on the right-hand side of (43) belong to the space $(H^2)^4$ for every t. Therefore the left-hand side of the equation also belongs to this space for every t. Applying the operator L to both sides of equation (43) we obtain

$$L(\psi_{tt} + L\psi) = LK + \varepsilon LG_\phi(a_0)_t + \varepsilon^2 LJ'(a_0, \psi). \tag{45}$$

(To justify these calculations we are using the fact that $\psi_{tt} + L\psi \in (H^2)^4$, and that the operator $\mathscr{D}_{\alpha,\phi}$ is the sum of a first order differential operator with constant coefficients and an operator of order zero.)

Equation (45) can be rewritten in the form

$$(L\psi)_{tt} + L^2\psi = LK + \varepsilon LG_\phi(a_0)_t + \varepsilon^2 LJ' + 2\varepsilon L_\tau \psi_t + \varepsilon^2 L_{\tau\tau}\psi.$$

We take the inner product of each side of this equation with $(L\psi)_t$ and integrate from 0 to t taking into account that $\psi(0) = \psi_t(0) = 0$. Then we apply integration by parts to the first summand:

$$\int_0^t \langle LK, (L\psi)_t\rangle ds = \langle LK(t), L\psi(t)\rangle - \varepsilon \int_0^t \langle (LK)_\tau, L\psi\rangle ds.$$

As a result we obtain the following equation:

$$\begin{aligned}(46)\quad Q_2(t) &= \langle LK(t), L\psi(t)\rangle \\ &\quad + \varepsilon \int_0^t \Big(-\langle L\psi, (LK)_\tau\rangle - \frac{1}{2}\langle L_\tau(L\psi), L\psi\rangle + \langle LG_\phi(a_0)_t, (L\psi)_t\rangle \\ &\qquad + 2\langle L_\tau\psi_t, (L\psi)_t\rangle + \varepsilon\big(\langle L_{\tau\tau}\psi, (L\psi)_t\rangle + \langle LJ'(a_0,\psi), (L\psi)_t\rangle\big)\Big)\, ds.\end{aligned}$$

We observe that $L\psi \perp \ker \mathscr{D}_{\alpha,\phi}$, since $L = \mathscr{D}^*_{\alpha,\phi}\mathscr{D}_{\alpha,\phi}$; therefore by the Cauchy–Bunyakovskiĭ inequality we have

$$|\langle LK(t), L\psi(t)\rangle| \le \frac{1}{\gamma}\|LK(t)\|^2_{L^2} + \frac{\gamma}{4}\|L\psi\|^2_{L^2}.$$

Since

$$\frac{\gamma}{4}\|L\psi\|^2_{L^2} \le \frac{\gamma}{4}\|L\psi\|^2_{H^1} \le \frac{1}{4}\langle L^2\psi, L\psi\rangle \le \frac{1}{2}Q_2(t),$$

we obtain an estimate for Q_2 of the form

$$Q_2(t) \le c(\|LK(t)\|^2_{L^2} + \varepsilon t(1 + M^4)). \tag{47}$$

(Henceforth we will use c to denote various positive constants.) Since $\|L\psi\|^2_{H^1} \le 2Q_2$, we have

$$\|\psi_1\|_{H^3} \le c\sqrt{1 + \varepsilon t(1 + M^4)}. \tag{48}$$

We now estimate the component of ψ contained in the kernel of the operator $\mathscr{D}_{\alpha,\phi}$. In doing this we use the fact that $Q(\tau)$ is a geodesic on the space $\mathscr{M}_N$.

We take the inner product of each side of equation (43) with $\tilde{n}_\mu$. Since $\langle L\psi, \tilde{n}_\mu\rangle = \langle \psi, L\tilde{n}_\mu\rangle = 0$ and $\langle G_\phi(a_0)_t, \tilde{n}_\mu\rangle = \langle (a_0)_t, G^*_\phi \tilde{n}_\mu\rangle = 0$, we obtain

$$\langle \psi_{tt}, \tilde{n}_\mu\rangle = \langle K, \tilde{n}_\mu\rangle + \varepsilon^2\langle J'(a_0,\psi), \tilde{n}_\mu\rangle, \tag{49}$$

where $K := -\partial^2_\tau(\alpha_1, \alpha_2, \phi_1, \phi_2)$.

The following assertion holds: for the choice of the function χ defined by the curve $Q\colon q = q(\tau)$ we made in § 6, the condition that the curve Q is a geodesic of the kinetic metric is equivalent to the following condition:

$$\langle \partial^2_\tau(\alpha,\phi), \tilde{n}_\mu\rangle = 0 \quad \text{for all } \mu = 1,\dots,2N.$$

Indeed, one can verify that this condition is equivalent to the condition that the functional

$$\int \langle n_\mu(q), n_\nu(q)\rangle \dot{q}^\mu \dot{q}^\nu\, d\tau$$

is extremal, which coincides with the condition that the curve $Q\colon q = q(\tau)$ is a geodesic.

Hence the first summand on the right-hand side of equation (49) is equal to zero. In view of the formula

$$\langle \psi, \tilde{n}_\mu\rangle_{tt} = \langle \psi_{tt}, \tilde{n}_\mu\rangle + 2\varepsilon\langle \psi_t, (\tilde{n}_\mu)_\tau\rangle + \varepsilon^2\langle \psi, (\tilde{n}_\mu)_{\tau\tau}\rangle,$$

it follows from equation (49) that

$$\langle \psi, \tilde{n}_\mu \rangle_{tt} = \varepsilon^2 \langle J'(a_0, \psi), \tilde{n}_\mu \rangle + 2\varepsilon \langle \psi_t, (\tilde{n}_\mu)_\tau \rangle + \varepsilon^2 \langle \psi, (\tilde{n}_\mu)_{\tau\tau} \rangle. \tag{50}$$

Integrating by parts we obtain

$$\begin{aligned}\langle \psi, \tilde{n}_\mu \rangle_t(t) &= 2\varepsilon \int_0^t \langle \psi_t, (\tilde{n}_\mu)_\tau \rangle \, ds + \varepsilon^2 \int_0^t \big(\langle J'(a_0, \psi), \tilde{n}_\mu \rangle + \langle \psi, (\tilde{n}_\mu)_{\tau\tau} \rangle \big) ds \\ &= 2\varepsilon \langle \psi(t), (\tilde{n}_\mu)_\tau(\varepsilon t) \rangle + \varepsilon^2 \int_0^t \big(\langle J'(a_0, \psi), \tilde{n}_\mu \rangle - \langle \psi, (\tilde{n}_\mu)_{\tau\tau} \rangle \big) ds.\end{aligned} \tag{51}$$

Since $\varepsilon t \le \tau_0$, this implies that

$$|\langle \psi, \tilde{n}_\mu \rangle_t| \le c(2\varepsilon M + \varepsilon \tau_0 (1 + M^3)) \le \varepsilon c (1 + M^3).$$

Therefore,

$$|\langle \psi, \tilde{n}_\mu \rangle| \le c \varepsilon t (1 + M^3).$$

Since the vectors $\tilde{n}_\mu$ form a basis in the space $\ker \mathscr{D}_{\alpha,\phi}$ and $\langle \tilde{n}_\mu, \tilde{n}_\nu \rangle = g_{\mu\nu}$, the projection of ψ onto $\ker \mathscr{D}_{\alpha,\phi}$, denoted by ψ_2, is equal to

$$\psi_2 = \sum_{\mu,\nu} g^{\mu\nu} \langle \psi, \tilde{n}_\mu \rangle \tilde{n}_\mu.$$

Since the elements of the inverse matrix $g^{\mu\nu}(q(\tau))$ are continuous and bounded on the interval $[0, \tau_0]$, and the $\tilde{n}_\mu$ are smooth maps from $[0, \tau_0]$ into $(H^3)^4$ (see § 6), we have

$$\|\psi_2(t)\|_{(H^3)^4} \le c \varepsilon t (1 + M^3). \tag{52}$$

Combining the estimates (48) and (52), we obtain the following estimate for $\|\psi(t)\|_{(H^3)^4}$:

$$\|\psi(t)\|_{(H^3)^4} \le c\big(1 + \sqrt{\varepsilon t (1 + M^4)} + \varepsilon t (1 + M^4)\big).$$

We now obtain an estimate for $\|\psi_t\|_{(H^2)^4}$. We observe that inequality (47) implies an estimate for $\|(L\psi)_t\|_{L^2}$ and therefore also an estimate for the norm $\|\psi_t\|_{(H^2)^4}$, since $L\psi_t = (L\psi)_t - \varepsilon L_\tau \psi$. The component of ψ_t contained in the kernel of the operator $\mathscr{D}_{\alpha,\phi}$ can be estimated using the relation

$$\langle \psi_t, \tilde{n}_\mu \rangle = \langle \psi, \tilde{n}_\mu \rangle_t - \varepsilon \langle \psi, (\tilde{n}_\mu)_\tau \rangle.$$

Since we assume that $\varepsilon M < 1$, the additional terms will just change the constant (independent of M) in the estimates:

$$\|\varepsilon L_\tau \psi\| \le c \varepsilon M \le c, \quad |\varepsilon \langle \psi, (\tilde{n}_\mu)_\tau \rangle| \le c \varepsilon M \le c.$$

Therefore we can indeed obtain the estimate (41) for some new constant B_4. □

To complete the proof of the main theorem, we apply a method which consists of using the local existence theorem (Theorem 4) with the estimates (31) and (41) repeatedly. Recall that the first of these estimates has the form

$$\begin{aligned}\|a_0(t), \psi(t)\| &= \|a_0(t)\|_{H^3} + \|(a_0)_t(t)\|_{H^3} + \|\psi(t)\|_{(H^3)^4} + \|\psi_t(t)\|_{(H^2)^4} \\ &\le B_2(\|\psi^0\|_{H^3} + \|\psi^1\|_{H^2}) + B_3,\end{aligned}$$

and the second, $\|\psi(t)\|_{(H^3)^4} + \|\psi_t(t)\|_{(H^2)^4} \le B_4(1 + \varepsilon t(1 + M^4))$.

We choose some positive number M greater than $B_2 B_4 + B_3$. After that we determine a number τ_1 from the equation

$$B_2 B_4 (1 + \tau_1 (1 + M^4)) + B_3 = M.$$

We claim that for sufficiently small ε we can find a solution of the system (18), (19) on the interval $[0; \tau_1/\varepsilon]$.

By Theorem 4, for $\varepsilon < B_1$, we can find a solution of the system (18), (19) on a sufficiently small interval $[0;\delta_0]$ with zero initial conditions $\psi(0)=0$, $\psi_t(0)=0$. If $\delta_0 \ge \tau_1/\varepsilon$, the solution has been found. If this is not the case, we observe that by the local estimate (31) the inequality $\|a_0(t),\psi(t)\| \le B_3 < M$ holds on the entire interval $[0;\delta_0]$. Therefore we can apply the global estimate (41) and obtain the estimate

$$\|\psi(t)\|_{(H^3)^4} + \|\psi_t(t)\|_{(H^2)^4} \le B_4(1+\varepsilon t(1+M^4)) \le B_4(1+\tau_1(1+M^4))$$

on the entire interval $[0;\delta_0]$. In particular, this estimate holds at the right-hand endpoint of the interval, that is, at the point δ_0. We again apply the local existence theorem and find a solution of the system on the interval $[\delta_0;2\delta_0]$ with the initial conditions $\psi(\delta_0)$, $\psi_t(\delta_0)$, which are equal to the values of ψ and ψ_t at the endpoint δ_0 for the solution already found on the interval $[0;\delta_0]$. Since the initial conditions satisfy the estimate

$$\|\psi(\delta_0)\|_{(H^3)^4} + \|\psi_t(\delta_0)\|_{(H^2)^4} \le B_4(1+\tau_1(1+M^4)), \tag{53}$$

the solution will exist for $\varepsilon < \frac{B_1}{1+(B_4(1+\tau_1(1+M^4)))^2}$. By the local estimate, the inequality

$$\begin{aligned}\|a_0(t),\psi(t)\| &\le B_2(\|\psi(\delta_0)\|_{(H^3)^4} + \|\psi_t(\delta_0)\|_{(H^2)^4}) + B_3 \\ &\le B_2B_4(1+\tau_1(1+M^4)) + B_3 = M\end{aligned} \tag{54}$$

holds on the entire interval $[\delta_0;2\delta_0]$. Therefore, by combining the solutions on the intervals $[0;\delta_0]$ and $[\delta_0;2\delta_0]$, we obtain a solution of the system on the interval $[0;2\delta_0]$, which satisfies the condition $\|a_0(t),\psi(t)\| \le M$ on this entire interval. Then we can apply the global estimate to it and obtain that

$$\|\psi(t)\|_{(H^3)^4} + \|\psi_t(t)\|_{(H^2)^4} \le B_4(1+\varepsilon t(1+M^4)) \le B_4(1+\tau_1(1+M^4))$$

for every $t \in [0;2\delta_0]$. In particular, it holds at the point $2\delta_0$. We now seek a solution of the system on the interval $[2\delta_0;3\delta_0]$, and take the values $\psi(2\delta_0)$ and $\psi_t(2\delta_0)$ already obtained as initial conditions. Since these initial conditions again satisfy an estimate of the form (53), and so a solution exists for $\varepsilon < \frac{B_1}{1+(B_4(1+\tau_1(1+M^4)))^2}$ and satisfies an estimate of the type of (54), from which we conclude that the norm of the solution is bounded above by M everywhere on the new interval. Therefore, combining this solution with the one already obtained, we find a solution on the interval $[0;3\delta_0]$ with zero initial conditions and again apply the global estimate to it.

This iteration process can be continued until the right-hand endpoint of the next interval in turn, $[0,k\delta_0]$, is situated to the right of the point τ_1/ε. When this happens, we obtain a solution on the interval $[0;\tau_1/\varepsilon]$ with the estimate $\|a_0(t),\psi(t)\| \le M$, as required. Note that the range of variation of ε stops changing after the second step. Therefore after this step we can set

$$\varepsilon_0 = \frac{B_1}{1+(B_4(1+\tau_1(1+M^4)))^2}.$$

When solutions on different intervals are combined, the question arises of whether the gluing at the endpoints of adjacent intervals gives continuity. The functions ψ and ψ_t are obtained to be continuous by construction. We now consider the functions a_0, $(a_0)_t$, and ψ_{tt}.

Equation (18) can be rewritten in the following form:

$$(-\Delta + |\phi+\varepsilon^2\varphi|^2)a_0 = (\phi_2)_\tau\varphi_1 - (\phi_2)_\tau\varphi_1 + \varphi_1(\varphi_2)_t - \varphi_2(\varphi_1)_t. \tag{55}$$

The operator $-\Delta + |\phi+\varepsilon^2\varphi|^2$ is a bounded invertible operator from the space H^3 into H^1. This can be proved in the same fashion as for the operator $-\Delta+|\phi|^2$ (see (61) and the arguments after this formula). Therefore the value of a_0 at the beginning of the interval is uniquely determined by the values of ψ and ψ_t at the same point, whence the function a_0 is also continuous at the gluing points.

A more complicated situation arises when the functions $(a_0)_t$ and ψ_{tt} are considered. An equation for $(a_0)_t$ can be obtained by differentiating equation (55) with respect to time. However, then the second derivatives $(\varphi_1)_{tt}$ and $(\varphi_2)_{tt}$ appear on the right-hand side. (This difficulty has already arisen in the proof of the local existence theorem.)

Equation (55) makes it possible to eliminate a_0 and $(a_0)_t$ from equation (19). As a result we obtain an equation of the form

$$\psi_{tt} + \mathscr{D}^*_{\alpha,\phi}\mathscr{D}_{\alpha,\phi}\psi = \varepsilon G_\phi(-\Delta + |\phi + \varepsilon^2\varphi|^2)^{-1}(\varphi_1(\varphi_2)_{tt} - \varphi_2(\varphi_1)_{tt}) + F_1,$$

where F_1 depends on α, ϕ, ψ, $\nabla\psi$, and ψ_t. Now suppose that $\psi(t_0)$ and $\psi_t(t_0)$ are known. We let H denote the following linear operator acting on vector functions $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$:

$$H\Psi = G_\phi(-\Delta + |\phi(\varepsilon t_0) + \varepsilon^2\varphi(t_0)|^2)^{-1}(\varphi_1(t_0)\Psi_4 - \varphi_2(t_0)\Psi_3).$$

Then $\psi_{tt}(t_0)$ satisfies the linear equation

$$(Id - \varepsilon H)\psi_{tt}(t_0) = -\mathscr{D}^*_{\alpha,\phi}\mathscr{D}_{\alpha,\phi}\psi(t_0) + F_1.$$

This equation is uniquely solvable for sufficiently small ε. Hence for such an ε the values $\psi_{tt}(t_0)$ and $(a_0)_t(t_0)$ are uniquely determined by $\psi(t_0)$ and $\psi_t(t_0)$. Therefore the functions $(a_0)_t$ and ψ_{tt} are also glued together continuously at the endpoints of the intervals.

Thus, for sufficiently small ε we have constructed solutions of the system (18), (19) with zero initial conditions on the interval $[0;\tau_1/\varepsilon]$. By Theorem 3 the same functions a_0, ψ are also solutions of the system (15), (16), and the expressions (7) constructed from them are solutions of the Euler–Lagrange equations (6). Theorem 2 is proved.

§ 9. Appendix

In this section we give the proofs of the technical results stated as lemmas in § 7 in the proof of the local existence theorem.

Proof of Lemma 1. We apply induction on n. Obviously, the zeroth approximations $a_{0,0}$ and ψ_0 belong to the classes E_0 and E_1, respectively. Suppose that the approximations $a_{0,n-1} \in E_0$ and $\psi_{n-1} \in E_1$ have already been constructed. We need to show that the next approximations $a_{0,n} \in E_0$ and $\psi_n \in E_1$ are uniquely determined by equations (34), (35) with the initial conditions $\psi_n(t_0) = \psi^0$, $(\psi_n)_t(t_0) = \psi^1$.

Using the continuity of the Sobolev embeddings

$$H^2(\mathbb{R}^2) \subset BC(\mathbb{R}^2), \quad H^3(\mathbb{R}^2) \subset BC^1(\mathbb{R}^2)$$

already mentioned in § 5 (formula (27)), we can show that the right-hand side of equation (34) belongs to the class $C^1([t_0;T], H^1(\mathbb{R}^2))$, and the components of the right-hand side of equation (35) to the class $C([t_0;T], H^2(\mathbb{R}^2))$, under the condition that $a_{0,n-1} \in E_0$ and $\psi_{n-1} \in E_1$.

Thus to find ψ_n we have to solve four scalar equations of the form

$$u_{tt} + (1 - \Delta)u = f(t) \tag{56}$$

in the unknown function u satisfying the initial conditions $u(t_0) = u_0 \in H^3$, $u_t(t_0) = u_1 \in H^2$ with the right-hand side satisfying $f \in C([t_0;T], H^2(\mathbb{R}^2))$.

To solve equation (56) we take the Fourier transform with respect to the space variables. As a result we obtain the equation

$$\hat{u}_{tt} + (1 + x^2 + y^2)\hat{u} = \hat{f}$$

with the initial conditions $\hat{u}(t_0) = \hat{u}_0$, $\hat{u}_t(t_0) = \hat{u}_1$. The solution of the latter problem can be written out explicitly:

$$(57)\quad \hat{u} = \left(\hat{u}_0 - \frac{1}{R}\int_{t_0}^{t} \hat{f}(s)\sin(R(s-t_0))\,ds\right)\cos(R(t-t_0)) + \left(\frac{\hat{u}_1}{R} + \frac{1}{R}\int_{t_0}^{t} \hat{f}(s)\cos(R(s-t_0))\,ds\right)\sin(R(t-t_0)),$$

where $R = \sqrt{1+x^2+y^2}$.

Note that if $\hat{u}$ is defined by formula (57), then

$$(58)\quad \hat{u}_t = \left(-R\hat{u}_0 + \int_{t_0}^{t} \hat{f}(s)\sin(R(s-t_0))\,ds\right)\sin(R(t-t_0)) + \left(\hat{u}_1 + \int_{t_0}^{t} \hat{f}(s)\cos(R(s-t_0))\,ds\right)\cos(R(t-t_0)).$$

A solution of equation (56) is obtained from formula (57) by applying the inverse Fourier transform. It follows from formulae (57), (58) and equation (56) that the solution we obtain does indeed belong to the class E_1. Moreover, it is easy to obtain estimates for the norm of the solution and its derivative in the Sobolev spaces H^3 and H^2, respectively from formulae (57), (58):

$$(59)\quad \|u(t)\|_{H^3} \le c_0\big(\|u_0\|_{H^3} + \|u_1\|_{H^2} + 2(t-t_0)\max_{t_0\le s\le t}\|f(s)\|_{H^2}\big),$$

$$(60)\quad \|u_t(t)\|_{H^2} \le c_0\big(\|u_0\|_{H^3} + \|u_1\|_{H^2} + 2(t-t_0)\max_{t_0\le s\le t}\|f(s)\|_{H^2}\big),$$

where $c_0 > 0$ is some absolute constant. (We shall need these estimates in the proof of convergence of the iteration process.)

We now consider equation (34) defining the approximation $a_{0,n}$. As shown in [7], the operator $-\Delta + |\phi|^2$ on the left-hand side of this equation is invertible as an operator from $H^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$. One can show that it is also invertible as an operator from $H^3(\mathbb{R}^2)$ into $H^1(\mathbb{R}^2)$. This is clear from the following sequence of inequalities (here $v \in H^3(\mathbb{R}^2)$ is an arbitrary function, and the letter c denotes various positive constants):

$$(61)\quad \begin{aligned}\|v\|_{H^3} &\le c\big(\|v\|_{H^2} + \|\nabla v\|_{H^2}\big) \le c\big(\|(-\Delta+|\phi|^2)v\|_{L^2} + \|(-\Delta+|\phi|^2)\nabla v\|_{L^2}\big)\\ &\le c\big(\|(-\Delta+|\phi|^2)v\|_{L^2} + \|\nabla((-\Delta+|\phi|^2)v)\|_{L^2} + \|(\nabla|\phi|^2)v)\|_{L^2}\big)\\ &\le c\|(-\Delta+|\phi|^2)v\|_{H^1} + c\|v\|_{L^2} \le c\|(-\Delta+|\phi|^2)v\|_{H^1}.\end{aligned}$$

We observe that the operator-valued function $\tau \mapsto -\Delta+|\phi(\tau)|^2$ is continuous and even C^1-smooth on the interval $[0;\tau_0]$, and the operator $-\Delta+|\phi(\tau)|^2$ is invertible for every τ; furthermore, the function $\tau \mapsto (-\Delta+|\phi(\tau)|^2)^{-1}$ is also C^1-smooth. As we have already mentioned, the right-hand side of equation (34) belongs to the class $C^1([t_0;T],H^1)$. Therefore, applying the inverse operator to it for every t, that is, setting

$$a_{0,n}(t) := (-\Delta+|\phi(\varepsilon t)|^2)^{-1}\big[W_0(\varepsilon t)\psi_{n-1}(t) + \varepsilon J_0\big(a_{0,n-1}(t),\psi_{n-1}(t),(\psi_{n-1})_t(t)\big)\big],$$

we obtain the approximation $a_{0,n}$ in the class E_0. □

Proof of Lemma 2. We write equation (35) for the nth and $(n+1)$st approximations and subtract the first equation (with index n) from the second (with index $n+1$). Then the nonhomogeneous term K cancels out, and the difference $\psi_{n+1} - \psi_n$ will satisfy the equation

$$
\begin{aligned}
(62)\quad & (\psi_{n+1}-\psi_n)_{tt}+L_0(\psi_{n+1}-\psi_n)\\
&= -W(\psi_n-\psi_{n-1})+\varepsilon G_\phi(a_{0,n}-a_{0,n-1})_t+\varepsilon^2(J'(a_{0,n},\psi_n)-J'(a_{0,n-1},\psi_{n-1}))
\end{aligned}
$$

with zero initial conditions. Using estimates of the type of (59), (60), one can show that the following estimates hold:

$$
\begin{aligned}
(63)\quad & \|\psi_{n+1}-\psi_n\|_3 \le c_1(T-t_0)\\
&\times\Big(\big(V+\varepsilon^2(M_n+M_{n-1})\max\{1,\varepsilon^2(M_n+M_{n-1})\}\big)\|\psi_n-\psi_{n-1}\|_3\\
&\qquad + V\|a_{0,n}-a_{0,n-1}\|_3+\varepsilon V_1\|(a_{0,n})_t-(a_{0,n-1})_t\|_3\Big),\\
(64)\quad & \|(\psi_{n+1})_t-(\psi_n)_t\|_2 \le c_1(T-t_0)\\
&\times\Big(\big(V+\varepsilon^2(M_n+M_{n-1})\max\{1,\varepsilon^2(M_n+M_{n-1})\}\big)\|\psi_n-\psi_{n-1}\|_3\\
&\qquad + V\|a_{0,n}-a_{0,n-1}\|_3+\varepsilon V_1\|(a_{0,n})_t-(a_{0,n-1})_t\|_3\Big),
\end{aligned}
$$

where c_1 is some absolute constant and the constants V, V_1 depend only on the maxima of the functions α_1,α_2,ϕ and their derivatives on the set $[0,\tau_0]\times\mathbb{R}^2$.

Suppose that $\varepsilon\le\min\{1/M_n,1/M_{n-1}\}$. Then the coefficient

$$
V+\varepsilon^2(M_n+M_{n-1})\max\{1,\varepsilon^2(M_n+M_{n-1})\}
$$

on the right-hand sides of inequalities (63) and (64) can be estimated by the quantity $1+V$. We set $\widetilde{C}_1 := c_1(1+V)$. Then from the estimates (63) and (64) we obtain the following (henceforth $\|\cdot\|$ denotes $\|\cdot\|_3$):

$$
\begin{aligned}
(65)\quad & \|\psi_{n+1}-\psi_n\| \le \widetilde{C}_1(T-t_0)\big(\|\psi_n-\psi_{n-1}\|+\|a_{0,n}-a_{0,n-1}\|\\
&\qquad + \varepsilon V_1\|(a_{0,n}-a_{0,n-1})_t\|\big),\\
(66)\quad & \|(\psi_{n+1}-\psi_n)_t\|_2 \le \widetilde{C}_1(T-t_0)\big(\|\psi_n-\psi_{n-1}\|+\|a_{0,n}-a_{0,n-1}\|\\
&\qquad + \varepsilon V_1\|(a_{0,n}-a_{0,n-1})_t\|\big).
\end{aligned}
$$

To obtain an estimate for $a_{0,n+1}-a_{0,n}$, we recall that the operator-valued function $\tau\mapsto(-\Delta+|\phi(\tau)|^2)^{-1}$ is continuous on $[0,\tau_0]$; therefore the norm of the inverse operator on this interval is bounded by some constant. Writing out equation (34) for the indices n and $n+1$ and subtracting the first equation from the second, we obtain

$$
(67)\qquad (-\Delta+|\phi|^2)(a_{0,n+1}-a_{0,n}) = W_0(\psi_n-\psi_{n-1})+\varepsilon(J_{0,n}-J_{0,n-1}),
$$

where $J_{0,n}$ denotes $J_0\big(a_{0,n},\psi_n,(\psi_n)_t\big)$. Since we assumed that $\varepsilon\le\min\{1/M_n,1/M_{n-1}\}$, this implies the inequality

$$
(68)\qquad \|a_{0,n+1}-a_{0,n}\|\le c\|\psi_n-\psi_{n-1}\|.
$$

(Henceforth the letter c again denotes various positive constants, the numerical value of which is unimportant for us.)

To obtain an estimate for the norm of the derivative $(a_{0,n+1}-a_{0,n})_t$, we differentiate equation (67) with respect to time:

$$
\begin{aligned}
(69)\quad (-\Delta+|\phi|^2)[(a_{0,n+1}-a_{0,n})_t] &= -\varepsilon|\phi|^2_\tau(a_{0,n+1}-a_{0,n})+\varepsilon(W_0)_\tau(\psi_n-\psi_{n-1})\\
&\quad + W_0[(\psi_n-\psi_{n-1})_t]+\varepsilon(J_{0,n}-J_{0,n-1})_t.
\end{aligned}
$$

Because the norm of the operator $(-\Delta+|\phi(\tau)|^2)^{-1}$ is bounded, we obtain the inequality

$$\begin{aligned}(70)\quad \|(a_{0,n+1}-a_{0,n})_t\| &\le \varepsilon c\|a_{0,n+1}-a_{0,n}\|+\varepsilon c\|\psi_n-\psi_{n-1}\| \\ &\quad + c\|(\psi_n-\psi_{n-1})_t\|_2+\varepsilon c\|(J_{0,n}-J_{0,n-1})_t\|_1.\end{aligned}$$

Estimating the last term is made more difficult by the fact that the expression for $(J_0)_t$ contains terms depending on the second derivatives of ψ (namely, terms of the form $\varphi_2(\varphi_1)_{tt}$ and $\varphi_1(\varphi_2)_{tt}$).

Based on the explicit form of J_0 and using the assumption about ε, we can conclude that the following inequality holds:

$$\begin{aligned}(71)\quad \|(J_{0,n})_t-(J_{0,n-1})_t\|_1 &\le c(M_n+M_{n-1})\|(\psi_n-\psi_{n-1})_{tt}\|_1 \\ &\quad +\varepsilon c\|\psi_n-\psi_{n-1}\|_1+\varepsilon c\|(\psi_n-\psi_{n-1})_t\|_1.\end{aligned}$$

Now suppose that $\varepsilon\le\min\{1/M_n^2,1/M_{n-1}^2\}$. (Since $\varepsilon<1$, this assumption is stronger compared with the preceding assumption $\varepsilon\le\min\{1/M_n,1/M_{n-1}\}$.) Using this new assumption and substituting the estimate (71) into (70), we obtain the inequality

$$\begin{aligned}(72)\quad \|(a_{0,n+1}-a_{0,n})_t\| &\le \varepsilon c\|a_{0,n+1}-a_{0,n}\|+\varepsilon c\|\psi_n-\psi_{n-1}\| \\ &\quad + c\|(\psi_n-\psi_{n-1})_t\|_2+c\sqrt{\varepsilon}\,\|(\psi_n-\psi_{n-1})_{tt}\|_1.\end{aligned}$$

The norm of the derivative $(\psi_n-\psi_{n-1})_{tt}$ can be estimated based on equation (62). We write it out for $n-1$ and transfer the term $L_0(\psi_n-\psi_{n-1})$ from the left-hand side to the right-hand side. Under the assumption that $\varepsilon\le 1/M_{n-2}$ we can obtain the estimate

$$\begin{aligned}(73)\quad \|(\psi_n-\psi_{n-1})_{tt}\|_1 &\le c(\|\psi_n-\psi_{n-1}\|+\|\psi_{n-1}-\psi_{n-2}\|) \\ &\quad +\varepsilon c(\|(a_{0,n-1}-a_{0,n-2})_t\|+\|a_{0,n-1}-a_{0,n-2}\|).\end{aligned}$$

Substituting (73) into (72) we obtain that

$$\begin{aligned}(74)\quad \|(a_{0,n+1}-a_{0,n})_t\| &\le \varepsilon c\|a_{0,n+1}-a_{0,n}\|+c\sqrt{\varepsilon}\,\|\psi_n-\psi_{n-1}\|+c\|(\psi_n-\psi_{n-1})_t\|_2 \\ &\quad + c\sqrt{\varepsilon}\big(\|\psi_{n-1}-\psi_{n-2}\|\varepsilon\|(a_{0,n-1}-a_{0,n-2})_t\|+\varepsilon\|a_{0,n-1}-a_{0,n-2}\|\big).\end{aligned}$$

To simplify the formulae we introduce the following notation: $p_n:=\|\psi_{n+1}-\psi_n\|$, $q_n:=\|(\psi_{n+1}-\psi_n)_t\|_2$, $r_n:=\|a_{0,n+1}-a_{0,n}\|$, $s_n:=\|(a_{0,n+1}-a_{0,n})_t\|$. Bringing together the estimates (65), (66), (67), (74), we see that for some positive constant, which we denote by c_2, the following inequalities hold:

$$(75)\qquad p_n\le c_2(T-t_0)(p_{n-1}+\varepsilon r_{n-1}+\varepsilon s_{n-1}),$$

$$(76)\qquad q_n\le c_2(T-t_0)(p_{n-1}+\varepsilon r_{n-1}+\varepsilon s_{n-1}),$$

$$(77)\qquad r_n\le c_2p_{n-1},$$

$$(78)\qquad s_n\le \varepsilon c_2r_n+c_2p_{n-1}+c_2q_{n-1}+c_2\sqrt{\varepsilon}(p_{n-2}+\varepsilon c_2r_{n-2}+\varepsilon c_2s_{n-2}).$$

Because of the first inequality, the third implies that

$$r_n\le c_2^2(T-t_0)(p_{n-2}+\varepsilon r_{n-2}+\varepsilon s_{n-2}),$$

and, using this, together with the first and second inequalites, the fourth implies that

$$s_n\le(\varepsilon c_2^3(T-t_0)+2c_2^2(T-t_0)+c_2\sqrt{\varepsilon})(p_{n-2}+\varepsilon r_{n-2}+\varepsilon s_{n-2}).$$

We again use the notation $\sigma_n=p_n+q_n+r_n+s_n$ (see (36)). From the inequalities obtained above we can conclude that

$$(79)\qquad \sigma_n\le C(\sqrt{\varepsilon}+T-t_0)(\sigma_{n-1}+\sigma_{n-2})$$

for some new constant C, and this is precisely the required inequality (37). $\square$

Proof of Lemma 3. Recall that we wish to obtain an estimate for the quantity Σ_n that is independent of n.

Suppose that $\varepsilon < M_k^{-2}$ for $k = 0, 1, \ldots, n-1$. Then by Lemma 2 for some constant C, inequality (79) holds for the numbers $2, 3, \ldots, n-1$.

For brevity we introduce the notation $\kappa := C(\sqrt{\varepsilon} + T - t_0)$. First suppose that $n \geq 4$. We set $\tilde{\sigma}_l = \sigma_{l+1} + \sigma_l$. Then for $l = 2, 3, \ldots, n-2$ we can write

$$\begin{aligned}\tilde{\sigma}_l = \sigma_{l+1} + \sigma_l &\leq \kappa(\sigma_l + \sigma_{l-1}) + \sigma_l = (\kappa+1)\sigma_l + \kappa\sigma_{l-1} \\ &\leq (\kappa+1)\kappa(\sigma_{l-1} + \sigma_{l-2}) + \kappa\sigma_{l-1} \\ &\leq (\kappa^2 + 2\kappa)(\sigma_{l-1} + \sigma_{l-2}) = (\kappa^2 + 2\kappa)\tilde{\sigma}_{l-2}.\end{aligned} \tag{80}$$

Assuming that $T - t_0 < 1$ we obtain that

$$\tilde{\sigma}_l \leq C_0(\sqrt{\varepsilon} + T - t_0)\tilde{\sigma}_{l-2}$$

for some positive constant $C_0 > C$. We denote by κ_0 the quantity $\kappa_0 = C_0(\sqrt{\varepsilon} + T - t_0)$.

If n is even, $n = 2m$, then we can obtain the following estimate:

$$\begin{aligned}\Sigma_n = \sigma_0 + \sigma_1 + \sigma_2 + \cdots + \sigma_{2m-1} &= \tilde{\sigma}_0 + \tilde{\sigma}_2 + \tilde{\sigma}_4 + \cdots + \tilde{\sigma}_{2m-2} \\ &\leq \tilde{\sigma}_0(1 + \kappa_0 + \kappa_0^2 + \cdots + \kappa_0^{m-1}) \leq \frac{\tilde{\sigma}_0}{1-\kappa_0}.\end{aligned} \tag{81}$$

If, however, $n = 2m+1$, then

$$\begin{aligned}\Sigma_n = \sigma_0 + \sigma_1 + \sigma_2 + \cdots + \sigma_{2m} &= \tilde{\sigma}_0 + \tilde{\sigma}_2 + \tilde{\sigma}_4 + \cdots + \tilde{\sigma}_{2m-2} + \sigma_{2m} \\ &\leq \tilde{\sigma}_0(1 + \kappa_0 + \kappa_0^2 + \cdots + \kappa_0^{m-1}) + \sigma_{2m}.\end{aligned} \tag{82}$$

Since $\varepsilon < M_{2m}^{-2}$, it follows from the estimate (79) that

$$\sigma_{2m} \leq C(\sqrt{\varepsilon} + T - t_0)(\sigma_{2m-1} + \sigma_{2m-2}) \leq \kappa_0\tilde{\sigma}_{2m-2} \leq \kappa_0^m\tilde{\sigma}_0.$$

Therefore the estimate

$$\Sigma_n \leq \frac{\tilde{\sigma}_0}{1-\kappa_0}$$

also holds in this case.

We consider the cases $n = 3$ and $n = 2$ separately. For $n = 3$ we have $\Sigma_3 = \sigma_2 + \sigma_1 + \sigma_0$. Applying Lemma 2 for σ_2 we obtain

$$\sigma_2 \leq \kappa(\sigma_1 + \sigma_0) \leq \kappa_0(\sigma_1 + \sigma_0).$$

Consequently, $\Sigma_3 \leq (1 + \kappa_0)\tilde{\sigma}_0$. It is also obvious that $\Sigma_2 = \sigma_1 + \sigma_0 = \tilde{\sigma}_0$. Therefore the inequality $\Sigma_n \leq \frac{\tilde{\sigma}_0}{1-\kappa_0}$ holds for all $n \geq 2$.

We now estimate $\tilde{\sigma}_0 = \sigma_0 + \sigma_1$. (This has to be done, since this quantity depends on the choice of ε and T.) We know that

$$\sigma_0 = p_0 + q_0 + r_0 + s_0 = \|\psi_1\| + \|(\psi_1)_t\|_2 + \|a_{0,1}\| + \|(a_{0,1})_t\|.$$

Recall that the vector function ψ_1 is a solution of the equation

$$(\psi_1)_{tt} + (1 - \Delta)\psi_1 = K$$

with the initial conditions $\psi_1(t_0) = \psi^0$, $(\psi_1)_t(t_0) = \psi^1$. We now set

$$K_0 = \max_{0 \leq \tau \leq \tau_0} \|K(\tau)\|_{(H^2)^4}.$$

By the estimates (59), (60) we have the inequalities

$$\|\psi_1\| \leq \|\psi^0\|_{(H^3)^4} + \|\psi^1\|_{(H^2)^4} + 2(T - t_0)K_0, \tag{83}$$

$$\|(\psi_1)_t\|_2 \leq \|\psi^0\|_{(H^3)^4} + \|\psi^1\|_{(H^2)^4} + 2(T - t_0)K_0. \tag{84}$$

The function a_1 is identically equal to zero, since it satisfies equation (34) for $n = 1$. Consequently,

$$\sigma_0 \leq 2(\|\psi^0\|_{(H^3)^4} + \|\psi^1\|_{(H^2)^4} + 2(T - t_0)K_0).$$

Next, $\sigma_1 = p_1 + q_1 + r_1 + s_1$. The first three summands satisfy inequalities (75)–(77):

$$p_1 \leq c_2(T - t_0)p_0, \quad q_1 \leq c_2(T - t_0)p_0, \quad r_1 \leq c_2 p_0.$$

In order to estimate $s_1 = \|(a_{0,2})_t\|$, we observe that $(a_{0,2})_t$ is a solution of equation (69) for $n = 2$. Therefore,

$$\|(a_{0,2})_t\| \leq \varepsilon c\big(\|a_{0,2}\| + \|\psi_1\| + \|(J_{0,1})_t\|_1\big) + c\|(\psi_1)_t\|_2.$$

To estimate the term $\|(J_{0,1})_t\|_1$ we use inequality (71) (recall that $J_{0,0} = 0$):

$$\|(J_{0,1})_t\|_1 \leq cM_1\|(\psi_1)_{tt}\|_1 + \varepsilon c\|\psi_1\|_1 + \varepsilon c\|(\psi_1)_t\|_1.$$

Finally, we observe that $(\psi_1)_{tt} = -L_0\psi_1 + K$. Therefore,

$$\|(\psi_1)_{tt}\|_1 \leq c(\|\psi_1\| + K_0).$$

Assuming that $\varepsilon < M_1^{-2}$, we can estimate the expression $\varepsilon c\|(J_{0,1})_t\|_1$ in the estimate for $\|(a_{0,2})_t\|$ by the quantity $c(K_0\sqrt{\varepsilon} + \|\psi_1\| + \|(\psi_1)_t\|_2)$.

Collecting together the estimates for p_1, q_1, r_1, s_1, we obtain the inequality

$$\sigma_1 = p_1 + q_1 + r_1 + s_1 \leq c(\sigma_0 + \sqrt{\varepsilon}K_0).$$

Consequently, we have the inequality

$$\tilde{\sigma}_0 = \sigma_0 + \sigma_1 \leq C_1\big(\|\psi^0\|_{(H^3)^4} + \|\psi^1\|_{(H^2)^4} + (\sqrt{\varepsilon} + T - t_0)K_0\big), \tag{85}$$

where C_1 is some positive constant.

By combining this last inequality with the estimates (81) and (82), we obtain the required estimate:

$$\Sigma_n \leq \frac{C_1\big(\|\psi^0\|_{(H^3)^4} + \|\psi^1\|_{(H^2)^4} + (\sqrt{\varepsilon} + T - t_0)K_0\big)}{1 - C_0(\sqrt{\varepsilon} + T - t_0)}. \qquad \square$$

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