Труды Москов. Матем. Общ.
Том 73 (2012)

Trans. Moscow Math. Soc.
2012, Pages 167–182
S 0077-1554(2013)00201-0
Article electronically published on January 24, 2013

TOPOLOGICAL APPLICATIONS OF GRADED FROBENIUS n-HOMOMORPHISMS, II

D. V. GUGNIN

Abstract. This paper strengthens Theorems 3.1.7 and 3.2.4 in *Topological applications of graded Frobenius n-homomorphisms*, by D. V, Gugnin, Tr. Mosk. Mat. Obs. **72** (2011), no. 1, 127–188; English transl., Trans. Moscow Math. Soc. **72** (2011), no. 1, 97–142. The improved version of Theorem 3.1.7 allows us to use integral techniques when working with rational cohomology algebras of nH-spaces. We introduce a rather broad class of even-dimensional manifolds $\mathcal{M}$ and, using integrality conditions, we show that those manifolds do not admit a 2-valued multiplication with identity. In particular, we show that complex projective spaces $\mathbb{C}P^m, m \geq 2$, are not $2H$-spaces. This fact has only been known for $\mathbb{C}P^2$.

§ 1. Introduction

In Chapter 3 of [4], we investigated n-valued topological groups X and, more generally, nH-spaces, nH-monoids, and nH-groups. Those spaces X were subject to the following restrictions: locally contractible, paracompact, finite-dimension of rational cohomology, and a paracompact X^{n^2}. One of the goals of the present paper is to extend Theorems 3.1.7 and 3.2.4 of [4] to a less exotic, from the point of view of algebraic topology, class of spaces.

In fact, we consider Hausdorff spaces X which are homotopy equivalent to a CW-complex and such that $\dim H^q(X;\mathbb{Q}) < \infty\ \forall q \geq 0$. It turns out that the existence of a cellular structure (for some space homotopy equivalent to X) and, accordingly, cellular cohomology, allows for a substantial strengthening of Theorem 3.1.7.

More precisely, we show that, for an nH-space X, the diagonal $\Delta\colon H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ of the graded n-Hopf algebra sends integral rational cohomology classes $\alpha \in H^*(X;\mathbb{Q})$ to integral elements of the tensor product $H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$.

The obtained integrality condition allows us to show that manifolds in a rather wide class $\mathcal{M}$, introduced in this paper, do not admit a 2-valued multiplication with identity. In particular, we show that projective spaces $\mathbb{C}P^m$, $m \geq 2$, do not admit a 2-valued multiplication with identity. Previously, this fact had been known only for $\mathbb{C}P^2$ (see [2, p. 340]), even though it was announced for all $m \geq 2$. Recall, however, that $\mathbb{C}P^m$, $m \geq 1$, carries a structure of $(m+1)!$-valued topological groups (see [1, p. 64]).

§ 2. Main integrality lemma

Consider an arbitrary *compact polyhedron* X. Let $\mathbb{K}$ be the field of coefficients, $\operatorname{char}\mathbb{K} = 0$ or p, $p > n$. In this case, one can easily compute the cohomology ring $H^*(\operatorname{Sym}^n X;\mathbb{K})$ of the nth symmetric power of X. The answer is given by

2010 *Mathematics Subject Classification.* Primary 13A02, 16T05, 55P45, 57N65.

Key words and phrases. Symmetric power, graded algebra, n-valued topological group, graded Frobenius n-homomorphism, n-Hopf algebra.

Supported by the RFFI grants 11-01-00694-a and 12-01-92104-YaF-a, and the RF Government Grant RF no. 2010-220-01-077, Contract 11.G34.31.0005.

Proposition α. *Let X be a compact polyhedron, $\mathbb{K}$ a field,* $\operatorname{char}\mathbb{K} = 0$ *or p, $p > n$. Then the canonical projection $\pi_n\colon X^n \to \operatorname{Sym}^n X = X^n/S_n$ induces an isomorphism*

$$\pi_n^*\colon H^*(\operatorname{Sym}^n X;\mathbb{K}) \to H^*(X^n;\mathbb{K})^{S_n} = (H^*(X;\mathbb{K})^{\otimes n})^{S_n},$$

where $H^*(X^n;\mathbb{K})^{S_n} \subset H^*(X^n;\mathbb{K})$ is the subalgebra of S_n-invariant cohomology.

Proposition α is a consequence of the following simple result (the Transfer Theorem).

Proposition β. *Let W be an arbitrary simplicial complex on which a finite group G acts simplicially. Let $\mathbb{K}$ be a field,* $\operatorname{char}\mathbb{K} = 0$ *or p, and $(p, |G|) = 1$. Then the projection map $\pi_G : W \to W/G$ induces an isomorphism $\pi_G^*\colon H^*(W/G;\mathbb{K}) \to H^*(W;\mathbb{K})^G$.*

It follows from the general results of A. Grothendieck in Ch. 5 of [6] that Propositon β remains true for arbitrary actions of finite groups G on a paracompact W if singular cohomology H^* is replaced by Čech cohomology $\check{H}^*$ (for a proof, see [8, p. 319] and [7, p. 567]). For locally contractible paracompact spaces W, singular cohomology $H^*(W)$ is canonically isomorphic to Čech cohomology $\check{H}^*(W)$. Because of this and of the aforementioned consequence of [6], for locally contractible paracompact spaces X with a paracompact nth degree X^n, we have an isomorphism between singular cohomology

$$(*) \qquad \pi_n^*\colon H^*(\operatorname{Sym}^n X;\mathbb{K}) \to H^*(X^n;\mathbb{K})^{S_n},$$

where $\mathbb{K}$ is a field, $\operatorname{char}\mathbb{K} = 0$ or p, $p > n$.

In this paper, we want to work with Hausdorff spaces X which are homotopy equivalent to CW-complexes, with an additional condition $\dim H^q(X;\mathbb{K}) < \infty\ \forall q \geq 0$. The proof of the existence of isomorphism $(*)$ for this class of spaces is not difficult, but the author could not find it in the literature. For this reason, it is given here. Moreover, Lemma 1 below establishes an additional integrality condition which holds for the isomorphism $\pi_n^*\colon H^*(\operatorname{Sym}^n X;\mathbb{Q}) \to H^*(X^n;\mathbb{Q})^{S_n}$.

Lemma 1. *Let X be a Hausdorff space homotopy equivalent to a CW-complex and such that* $\dim H^q(X;\mathbb{K}) < \infty\ \forall q \geq 0$, *where $\mathbb{K}$ is the field of coefficients. Let $n \geq 2$ be a natural number and* $\operatorname{char}\mathbb{K} = 0$ *or $p, p > n$. Then:*

(1) *The canonical projection $\pi_n\colon X^n \to \operatorname{Sym}^n X$ induces an isomorphism*

$$\pi_n^*\colon H^*(\operatorname{Sym}^n X;\mathbb{K}) \to S^n H^*(X;\mathbb{K}) = (H^*(X;\mathbb{K})^{\otimes n})^{S_n} = H^*(X^n;\mathbb{K})^{S_n}.$$

(2) *Let $\mathbb{K} = \mathbb{Q}$. Then for any integral rational cohomology class $\alpha \in H^q(X;\mathbb{Q})$, $q \geq 0$, the class*

$$\alpha \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes \alpha \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \alpha \in S^n H^*(X;\mathbb{Q})$$

is mapped by the isomorphism $(\pi_n^*)^{-1}\colon S^n H^*(X;\mathbb{Q}) \to H^*(\operatorname{Sym}^n X;\mathbb{Q})$ to an integral rational cohomology class in $H^q(\operatorname{Sym}^n X;\mathbb{Q})$.

Proof. We begin with the first part of the lemma. Since Sym^n is a homotopy functor, we may assume that X is a CW-complex or even a simplicial complex, because CW-complexes can be approximated simplicially. The space X^n with the usual topology of the direct product decomposes into cells, which are direct products of open simplices of X, but in the case of an uncountable X, it need not be a cellular space. X^n becomes a cellular space $\widehat{X}^n$ after passing to the standard completion of the topology of X^n over all compacts.

Direct verification shows that S_n also acts continuously on $\widehat{X}^n$ and that the topology of $\widehat{\operatorname{Sym}}^n X = \widehat{X}^n/S_n$ is obtained from that of $\operatorname{Sym}^n X$ by completion over compacts. The identity map $\widehat{\operatorname{Id}}_{X^n}\colon \widehat{X}^n \to X^n$ is trivially S_n-equivariant.

For any Hausdorff space Y, let $\widehat{Y}$ denote the space Y with the topology completed by the compacts. The identity map $\widehat{\mathrm{Id}}_Y \colon \widehat{Y} \to Y$ is a homeomorphism on compacts and therefore induces an isomorphism on (co)homology with arbitrary coefficients.

This fact and the obvious commutative diagram

$$\begin{array}{ccc} \widehat{X}^n & \xrightarrow{\widehat{\mathrm{Id}}_{X^n}} & X^n \\ {\scriptstyle /S_n}\downarrow & & \downarrow{\scriptstyle /S_n} \\ \widehat{\mathrm{Sym}}^n X & \xrightarrow{\widehat{\mathrm{Id}}_{\mathrm{Sym}^n X}} & \mathrm{Sym}^n X, \end{array}$$

show that it suffices to prove the lemma for $\widehat{X}^n$ and $\widehat{\mathrm{Sym}}^n X$. Throughout the proof, we shall write X^n instead of $\widehat{X}^n$, and $\mathrm{Sym}^n X$ instead of $\widehat{\mathrm{Sym}}^n X$, and always consider these spaces with the completed topology.

We may assume (by a theorem of Zermelo) that the set of vertices $\mathrm{Vert}(X)$ of the simplicial complex X is linearly ordered. Define a partial S_n-invariant order on the vertices of X^n by setting

$$(u^1, \ldots, u^n) \le (v^1, \ldots, v^n) \Leftrightarrow u^1 \le v^1, \ldots, u^n \le v^n,$$
$$\forall u^1, \ldots, u^n, v^1, \ldots, v^n \in \mathrm{Vert}(X).$$

As a CW-complex, X^n admits a standard triangulation arising from the subdivision of the closed cells $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$ into simplices according to the following rule. The product $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$ is naturally a convex polyhedron in the affine space $\mathbb{R}^{q_1+\cdots+q_n}$. The vertices of any closed simplex $\Delta^q \subset \Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$ of the desired triangulation are arbitrary linearly ordered sets of vertices

$$\big((u_0^1, \ldots, u_0^n) < (u_1^1, \ldots, u_1^n) < \ldots < (u_q^1, \ldots, u_q^n)\big)$$

of $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$. The closed simplex Δ^q is the convex hull of its vertices, and a linear order on its vertices defines a canonical orientation of it.

The original cellular structure on X^n is invariant under the right action of S_n, $\tau(x_1, \ldots, x_n) = (x_{\tau(1)}, \ldots, x_{\tau(n)})$ $\forall \tau \in S_n$, $\forall x_1, \ldots, x_n \in X$. This action also preserves the partial order on the set $\mathrm{Vert}(X^n)$ of vertices, and the map

$$\tau \colon \Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n} \to \Delta_{\tau(1)}^{q_{\tau(1)}} \times \ldots \times \Delta_{\tau(n)}^{q_{\tau(n)}}, \quad \forall \tau \in S_n,$$

is an affine isomorphism of polyhedra.

This implies that any permutation $\tau \in S_n$ linearly maps an arbitrary simplex

$$\Delta^q = \big[(u_0^1, \ldots, u_0^n) < \ldots < (u_q^1, \ldots, u_q^n)\big] \subset \Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$$

of the defined triangulation to the simplex

$$\tau(\Delta^q) = [(u_0^{\tau(1)}, \ldots, u_0^{\tau(n)}) < \ldots < (u_q^{\tau(1)}, \ldots, u_q^{\tau(n)})] \subset \Delta_{\tau(1)}^{q_{\tau(1)}} \times \ldots \times \Delta_{\tau(n)}^{q_{\tau(n)}}.$$

Thus, the action of S_n on X^n is simplicial.

Let us show that, in any closed simplex $\Delta^q = \big[(u_0^1, \ldots, u_0^n) < \ldots < (u_q^1, \ldots, u_q^n)\big]$, there are no distinct points from the same orbit. Since the action of S_n is simplicial, it suffices to show that no two vertices $(u_i^1, \ldots, u_i^n)$ and $(u_j^1, \ldots, u_j^n)$, $i < j$, belong to the same orbit. This can be deduced from the following result: if $\tau(u^1, \ldots, u^n) \ne (u^1, \ldots, u^n)$, then the vertices $\tau(u^1, \ldots, u^n)$ and $(u^1, \ldots, u^n)$ are not comparable.

Indeed, suppose $\tau(u^1, \ldots, u^n) > (u^1, \ldots, u^n)$, i.e., $u^{\tau(i)} \ge u^i$, $1 \le i \le n$, and for some i_0, $1 \le i_0 \le n$, $u^{\tau(i_0)} > u^{i_0}$. Then

$$u^{\tau^m(i)} = u^{\tau(\tau^{m-1}(i))} \ge u^{\tau^{m-1}(i)} \ge u^{\tau^{m-2}(i)} \ge \ldots \ge u^{\tau(i)} \quad \forall m \ge 1, \forall 1 \le i \le n.$$

Therefore, $\tau^m(u^1,\dots,u^n) \geq \tau(u^1,\dots,u^n) > (u^1,\dots,u^n)\ \forall m \geq 1$. This contradicts the fact that $\tau \in S_n$ is of finite order. Similar arguments lead to a contradiction in the case $\tau(u^1,\dots,u^n) < (u^1,\dots,u^n)$ as well. Thus, we have shown that the intersection of the closed simplex $\Delta^q = [(u_0^1,\dots,u_0^n) < \dots < (u_q^1,\dots,u_q^n)]$ with the orbit $\{S_n x\}$ of any point $x \in X^n$ is either empty or consists of a single point.

This implies that the images $\pi_n(\Delta^q)$ of the simplices of X^n under the projection $\pi_n\colon X^n \to \operatorname{Sym}^n X$ form simplices of a simplicially cellular decomposition of $\operatorname{Sym}^n X$, with a preimage-induced orientation.

The triangulation of X^n can be viewed as either a simplicially cellular structure or just as a cellular structure. The canonical projection $\pi_n\colon X^n \to \operatorname{Sym}^n X$ is simplicially cellular and, moreover, the image $\pi_n(\Delta^q)$ of any simplex Δ^q is a simplex of $\operatorname{Sym}^n X$, and the restriction $\pi_n|_{\Delta^q}\colon \Delta^q \to \pi_n(\Delta^q)$ is an orientation-preserving linear homeomorphism. Furthermore, for the open simplices, we have $\pi_n^{-1}(\pi_n(\operatorname{int}\Delta^q)) = \bigsqcup_{i=1}^m \tau_i(\operatorname{int}\Delta^q)$, where $S_n = \{\operatorname{Stab}_{\operatorname{int}\Delta^q}\tau_1\} \sqcup \dots \sqcup \{\operatorname{Stab}_{\operatorname{int}\Delta^q}\tau_m\}$.

Consider cellular cochain complexes $C^*_{\mathrm{simpl}}(X^n)$ and $C^*_{\mathrm{cell}}(\operatorname{Sym}^n X)$ with coefficients in an arbitrary field $\mathbb{K}$ (the subscript "simpl" in $C^*_{\mathrm{simpl}}(X^n)$ indicates that X^n is a simplicial complex). It now follows that the cochain map $\pi_n^*\colon C^*_{\mathrm{cell}}(\operatorname{Sym}^n X) \to C^*_{\mathrm{simpl}}(X^n)$ is a monomorphism, and the image $\operatorname{Im}\pi_n^* = C^*_{\mathrm{simpl}}(X^n)^{S_n}$ is the subcomplex of S_n-invariants. If we identify $C^*_{\mathrm{cell}}(\operatorname{Sym}^n X)$ with the subcomplex $C^*_{\mathrm{simpl}}(X^n)^{S_n}$ via the monomorphism π_n^*, then π_n^* becomes the identity inclusion $i_n\colon C^*_{\mathrm{simpl}}(X^n)^{S_n} \hookrightarrow C^*_{\mathrm{simpl}}(X^n)$.

As is well known, in the case of a (co)chain complex C^* with an action of a finite group G, taking homology commutes with taking the subcomplex of G-invariants, $H^*(C^G) \cong H^*(C^*)^G$, whenever $\operatorname{char}\mathbb{K} = 0$ or p, $(p,|G|) = 1$. This is a simple consequence of the averaging operation $\Sigma_G = \frac{1}{|G|}\sum_{g\in G} g$ over G, defined on both C^* and $H^*(C^*)$.

Returning to the space X^n, let $\operatorname{char}\mathbb{K} = 0$ or $p, p > n$. As we said above, i_n induces an isomorphism $i_n^*\colon H^*(C^*_{\mathrm{simpl}}(X^n)^{S_n}) \cong H^*(X^n)^{S_n}$. This yields an isomorphism $\pi_n^*\colon H^*_{\mathrm{cell}}(\operatorname{Sym}^n X) \cong H^*_{\mathrm{simpl}}(X^n)^{S_n}$. Since cellular (co)homology of CW-complexes is canonically isomorphic to singular (co)homology, we have the desired isomorphism $\pi_n^*\colon H^*(\operatorname{Sym}^n X) \to H^*(X^n)^{S_n}$.

By assumption, $\dim H^q(X) < \infty\ \forall q \geq 0$. Therefore the Künneth formula for cohomology $H^*(X^n) = H^*(X)^{\otimes n}$ applies, and therefore, $H^*(X^n)^{S_n} = (H^*(X)^{\otimes n})^{S_n} = S^n H^*(X)$. Thus, we have an isomorphism $\pi_n^*\colon H^*(\operatorname{Sym}^n X) \to S^n H^*(X)$ and the first part of the lemma is proved.

Let us prove the second part. For any topological space Y, the *integral rational* cohomology classes $\alpha \in H^*(Y;\mathbb{Q})$ are, by definition, the elements in the image of the homomorphism $H^*(Y;\mathbb{Z}) \to H^*(Y;\mathbb{Q})$ induced by the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$. At the level of cochains, such classes are represented by the cocycles $a \in C^*(Y;\mathbb{Q})$ taking on integer values on all singular simplices $\sigma\colon \Delta^q \to Y$. In the case of a CW-complex Y, one needs to consider the cocycles taking on integer values on all cells of Y.

Let $\alpha \in H^q_{\mathrm{simpl}}(X;\mathbb{Q})$, $q \geq 1$, be an integral rational class, $\alpha = [a]$, where $a \in C^q_{\mathrm{simpl}}(X;\mathbb{Q})$ is a representing cocycle taking on integer values on all q-dimensional cells (simplices) X. We want to show that the class

$$\alpha\otimes 1\otimes\dots\otimes 1 + 1\otimes\alpha\otimes\dots\otimes 1 + \dots + 1\otimes\dots\otimes 1\otimes\alpha \in S^n H^*(X;\mathbb{Q}) \cong H^*(\operatorname{Sym}^n X;\mathbb{Q})$$

is integral in $H^*(\operatorname{Sym}^n X;\mathbb{Q})$ (the case $q = 0$ being trivial).

Let $C^*_{\mathrm{cell}}(X^n;\mathbb{Q})$ denote the cochain complex on the CW-complex X^n whose cells are products of open simplices of X. As before, let $C^*_{\mathrm{simpl}}(X^n;\mathbb{Q})$ denote the cochain complex of the simplicial complex X^n with the above triangulation. For cell cochains, one has the $\times$-product. Let $a_1 \in C^{q_1}_{\mathrm{simpl}}(X;\mathbb{Q}), \ldots, a_n \in C^{q_n}_{\mathrm{simpl}}(X;\mathbb{Q})$. Then

$$a_1 \times \ldots \times a_n(\Delta_1^{p_1} \times \ldots \times \Delta_n^{p_n}) = (-1)^{q_2 p_1 + q_3(p_1+p_2)+\ldots+q_n(p_1+p_2+\ldots+p_{n-1})} \times \\ a_1(\Delta_1^{p_1}) a_2(\Delta_2^{p_2}) \ldots a_n(\Delta_n^{p_n}), \quad a_1 \times \ldots \times a_n \in C^{q_1+\ldots+q_n}_{\mathrm{cell}}(X^n;\mathbb{Q}).$$

The element $\alpha_{(n)} := \alpha \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \alpha \in H^q_{\mathrm{cell}}(X^n;\mathbb{Q})$ is represented by the cocycle $a_{(n)} := a \times 1 \times \ldots \times 1 + \ldots + 1 \times \ldots \times 1 \times a \in C^q_{\mathrm{cell}}(X^n;\mathbb{Q})$. Now we shall produce an S_n-invariant integral rational cocycle $\widetilde{a}_{(n)} \in C^q_{\mathrm{simpl}}(X^n;\mathbb{Q})^{S_n}$ such that its cohomology class $[\widetilde{a}_{(n)}] = \widetilde{\alpha}_{(n)} \in H^q_{\mathrm{simpl}}(X^n;\mathbb{Q})$ corresponds to $\alpha_{(n)} \in H^q_{\mathrm{cell}}(X^n;\mathbb{Q})$ under the canonical isomorphism $\psi_{\text{simpl-cell}} \colon H^*_{\mathrm{simpl}}(X^n;\mathbb{Q}) \to H^*_{\mathrm{cell}}(X^n;\mathbb{Q})$.

The smaller simplicial structure on X^n is a refinement of the cell structure. As is known, this implies that $\psi_{\text{simpl-cell}}(\widetilde{\alpha}_{(n)}) = \alpha_{(n)}$ if, for any q-dimensional cell $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}, q_1 + \ldots + q_n = q$, of the cellular structure on X^n, the value $a_{(n)}(\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n})$ coincides with the sum of the values of the cocycle $\widetilde{a}_{(n)}$ on the q-dimensional simplices comprising the cell $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$.

Define a cochain $\widetilde{a}_{(n)} \in C^q_{\mathrm{simpl}}(X^n;\mathbb{Q})$ by the formula

$$\widetilde{a}_{(n)}\big[(u_0^1,\ldots,u_0^n) < (u_1^1,\ldots,u_1^n) < \ldots < (u_q^1,\ldots,u_q^n)\big] = \\ a(\Delta[u_0^1,u_1^1,\ldots,u_q^1]) + a(\Delta[u_0^2,u_1^2,\ldots,u_q^2]) + \ldots + a(\Delta[u_0^n,u_1^n,\ldots,u_q^n]),$$

where we set $a(\Delta[u_0^j,u_1^j,\ldots,u_q^j]) = 0$ if there are repetitions among the vertices $u_0^j \le u_1^j \le \ldots \le u_q^j$, $1 \le j \le n$. In the case $u_0^j < u_1^j < \ldots < u_q^j$, the expression $\Delta[u_0^j,u_1^j,\ldots,u_q^j]$ denotes the q-dimensional simplex of X on the vertices $u_0^j,\ldots,u_q^j$, $1 \le j \le n$, with the (canonical) orientation determined by the ordering $u_0^j < \ldots < u_q^j$.

Clearly, the cochain $\widetilde{a}_{(n)}$ is integral and S_n-invariant. Let us check the cocycle condition $\delta\widetilde{a}_{(n)} = 0$. We have $\delta\widetilde{a}_{(n)}(\Delta^{q+1}) = \pm\widetilde{a}_{(n)}(\partial\Delta^{q+1})$ for any $(q+1)$-dimensional simplex $\Delta^{(q+1)} = \big[(u_0^1,\ldots,u_0^n) < \ldots < (u_{q+1}^1,\ldots,u_{q+1}^n)\big]$ of the simplicial complex X^n.

The boundary of $\Delta^{(q+1)}$ is of the form

$$\partial\Delta^{q+1} = \sum_{s=0}^{q+1} (-1)^s \big[(u_0^1,\ldots,u_0^n) < \ldots < \widehat{(u_s^1,\ldots,u_s^n)} < \ldots < (u_{q+1}^1,\ldots,u_{q+1}^n)\big].$$

We then have

$$\widetilde{a}_{(n)}(\partial\Delta^{q+1}) = \\ \sum_{s=0}^{q+1} (-1)^s \big(a(\Delta[u_0^1,\ldots,\widehat{u_s^1},\ldots,u_{q+1}^1]) + \ldots + a(\Delta[u_0^n,\ldots,\widehat{u_s^n},\ldots,u_{q+1}^n])\big) = \\ \sum_{k=1}^{n} a\left(\sum_{s=0}^{q+1} (-1)^s \Delta[u_0^k,\ldots,\widehat{u_s^k},\ldots,u_{q+1}^k]\right) = \sum_{k=1}^{n} a(\partial\Delta[u_0^k,\ldots,u_{q+1}^k]) = 0.$$

Therefore, $\widetilde{a}_{(n)} \in C^q_{\mathrm{simpl}}(X^n;\mathbb{Q})$ is a cocycle. Now we want to show that the cohomology class $[\widetilde{a}_{(n)}] = \widetilde{\alpha}_{(n)} \in H^q_{\mathrm{simpl}}(X^n;\mathbb{Q})$ is mapped to the class $\alpha_{(n)} \in H^q_{\mathrm{cell}}(X^n;\mathbb{Q})$ under the canonical isomorphism $H^q_{\mathrm{simpl}}(X^n;\mathbb{Q}) \cong H^q_{\mathrm{cell}}(X^n;\mathbb{Q})$. As before, it suffices to show that for any cell $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$, $q_1 + \ldots + q_n = q$, of X^n the value $a_{(n)}(\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n})$ coincides with the sum of the values of the cocycle $\widetilde{a}_{(n)}$ on the q-dimensional simplices comprising the cell $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$. We shall refer to this condition as the *coherence condition*.

Fix an arbitrary q-dimensional cell $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$, $q_1 + \ldots + q_n = q$, of X^n. We have two cases: (A) $q_k < q$ $\forall 1 \le k \le n$, and (B) $\exists!\, i$, $1 \le i \le n$, $q_i = q$, $q_j = 0$ $\forall j \ne i$.

Case (A). Since $a_{(n)} = a \times 1 \times \ldots \times 1 + \ldots + 1 \times \ldots \times 1 \times a$, we have $a_{(n)}(\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}) = 0$. Consider an arbitrary simplex $\Delta^q[(u_0^1, \ldots, u_0^n) < \ldots < (u_q^1, \ldots, u_q^n)]$ in the cell $\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n}$. Since $q_k < q$ $\forall 1 \le k \le n$, among the vertices $u_0^k \le u_1^k \le \ldots \le u_q^k$ of $\Delta_k^{q_k}$ there are always repetitions for any k, $1 \le k \le n$. Therefore,

$$\widetilde{a}_{(n)}\big(\Delta^q[(u_0^1, \ldots, u_0^n) < \ldots < (u_q^1, \ldots, u_q^n)]\big) = \sum_{k=1}^{n} a(\Delta[u_0^k, \ldots, u_q^k]) = \sum_{k=1}^{n} 0 = 0,$$

and, in this case, the coherence condition is trivially true.

Case (B). Let $q_i = q$ for some i, $1 \le i \le n$. Then $q_j = 0$ for any $j \ne i$, $1 \le j \le n$. Therefore, the cell

$$\Delta_1^{q_1} \times \ldots \times \Delta_n^{q_n} = \mathrm{pt}_1 \times \ldots \times \Delta_i^q[u_0^i, u_1^i, \ldots, u_q^i] \times \ldots \times \mathrm{pt}_n$$

is a q-dimensional simplex parallel to the i-factor of X^n. It is easy to see that

$$a_{(n)}\big(\mathrm{pt}_1 \times \ldots \times \Delta_i^q[u_0^i, u_1^i, \ldots, u_q^i] \times \ldots \times \mathrm{pt}_n\big) = a(\Delta_i^q[u_0^i, u_1^i, \ldots, u_q^i]).$$

Since the simplex $\mathrm{pt}_1 \times \ldots \times \Delta_i^q \times \ldots \times \mathrm{pt}_n$ does not break up when passing to the simplicial complex X^n, it remains to compute

$$\widetilde{a}_{(n)}(\mathrm{pt}_1 \times \ldots \times \Delta_i^q \times \ldots \times \mathrm{pt}_n) = \sum_{k=1}^{n} a(\Delta[u_0^k, u_1^k, \ldots, u_q^k]) = a(\Delta_i^q[u_0^i, u_1^i, \ldots, u_q^i]).$$

Therefore, the coherence condition

$$a_{(n)}(\mathrm{pt}_1 \times \ldots \times \Delta_i^q \times \ldots \times \mathrm{pt}_n) = \widetilde{a}_{(n)}(\mathrm{pt}_1 \times \ldots \times \Delta_i^q \times \ldots \times \mathrm{pt}_n)$$

is true in this case as well. Thus, the coherence condition for the cocycles $\widetilde{a}_{(n)}$ and $a_{(n)}$ is always true.

In summary, we have the following. We have constructed an S_n-invariant integral rational cocycle $\widetilde{a}_{(n)} \in C^q_{\mathrm{simpl}}(X^n;\mathbb{Q})^{S_n}$ whose cohomology class $[\widetilde{a}_{(n)}] \in H^q_{\mathrm{simpl}}(X^n;\mathbb{Q})^{S_n}$ corresponds to the class $\alpha \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \alpha \in S^n H^*(X;\mathbb{Q})$. But because we have, at the level of cochain complexes, the canonical isomorphism $C^q_{\mathrm{simpl}}(X^n;\mathbb{Q})^{S_n} \cong C^q_{\mathrm{cell}}(\mathrm{Sym}^n X;\mathbb{Q})$ (sending integral cochains to integral cochains), we may assume that the integral rational cocycle $\widetilde{a}_{(n)}$ lies in $C^q_{\mathrm{cell}}(\mathrm{Sym}^n X;\mathbb{Q})$. Moreover, its cohomology class $[\widetilde{a}_{(n)}] \in H^q_{\mathrm{cell}}(\mathrm{Sym}^n X;\mathbb{Q})$ is integral and is mapped to the class $\alpha \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \alpha \in S^n H^*(X;\mathbb{Q})$ under the isomorphism $H^*(\mathrm{Sym}^n X;\mathbb{Q}) \cong S^n H^*(X;\mathbb{Q})$ constructed above. This finishes the proof of the lemma. □

§ 3. A strengthening of Theorems 3.1.7 and 3.2.4

We want to prove the following strengthening of Theorem 3.1.7 of [4].

Theorem 1. (1) *Let X be a connected Hausdorff space homotopy equivalent to a CW-complex, and such that* $\dim H^q(X;\mathbb{Q}) < \infty$ $\forall q \ge 0$. *If X has a structure of nH-space, then the algebra $H^*(X;\mathbb{Q})$ has a structure of graded n-Hopf prealgebras.*

(2) *Let X be a connected Hausdorff space homotopy equivalent to a countable CW-complex, and such that* $\dim H^q(X;\mathbb{Q}) < \infty$ $\forall q \ge 0$. *If X has a structure of a nH-monoid (nH-group), then the algebra $H^*(X;\mathbb{Q})$ has a structure of a graded n-bialgebra (n-Hopf algebra).*

In both cases, if the groups $H_q(X;\mathbb{Z})$ are finitely generated for all $q \geq 0$, then the diagonal $\Delta\colon H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ of the graded n-Hopf prealgebra (n-bialgebra, n-Hopf algebra) $H^*(X;\mathbb{Q})$ maps integral rational classes $\alpha \in H^*(X;\mathbb{Q})$ to integral elements of $H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$.

Proof. (1) The proof of Theorem 3.1.7 (which is an obvious modification of that of Theorem 3.1.5) is completely applicable in this case because of the isomorphism

$$\pi_n^*\colon H^*(\operatorname{Sym}^n X;\mathbb{Q}) \to S^n H^*(X;\mathbb{Q}) = H^*(X^n;\mathbb{Q})^{S_n},$$

where $\pi_n\colon X^n \to \operatorname{Sym}^n X$ is the canonical projection. The isomorphism π_n^* holds because of Lemma 1.

(2) The only thing needed to use the proof of Theorem 3.1.7 again is an isomorphism $\pi_{n(Y)}^*\colon H^*(\operatorname{Sym}^n Y;\mathbb{Q}) \to S^n H^*(Y;\mathbb{Q})$ for $Y = X \times X$ and $Y = \operatorname{Sym}^n X$. By Lemma 1, it suffices that Y be a Hausdorff space homotopy equivalent to a CW-complex and have finite-dimensional rational cohomology in each dimension: $\dim H^q(Y;\mathbb{Q}) < \infty\ \forall q \geq 0$. The first and the third conditions are obviously satisfied for both $X \times X$ and $\operatorname{Sym}^n X$.

Let us check the homotopy equivalence. By assumption, X is homotopy equivalent to a countable CW-complex Z. This gives rise to a homotopy equivalence $X \times X \sim Z \times Z$, where $Z \times Z$ is a CW-complex as a finite product of countable CW-complexes.

It remains to consider the case $Y = \operatorname{Sym}^n X$. Since a countable CW-complex has a countable simplicial approximation, we may assume that Z is a countable simplicial complex. Because of that, the space Z^n is a CW-complex with products of open simplices of Z as cells.

Linearly ordering the set $\operatorname{Vert}(Z)$ of vertices of Z and applying the construction from the proof of Lemma 1, we can refine the original cellular structure on Z^n to obtain a simplicial decomposition invariant under the action of S_n. The "weak topology" axiom, which holds for the cellular structure on Z^n, automatically holds for the simplicial decomposition. Thus, Z^n has an S_n-invariant triangulation.

It is known that, for simplicial actions of an arbitrary finite group G on an arbitrary simplicial complex W, the quotient W/G is a simplicial complex. The simplices of W/G are the images of the simplices of the second barycentric subdivision of W under the projection $\pi_G\colon W \to W/G$. As we observed above, the space $\operatorname{Sym}^n Z = Z^n/S_n$ can be triangulated. Since Sym^n is a homotopy functor, the space $\operatorname{Sym}^n X$ is homotopy equivalent to the simplicial complex $\operatorname{Sym}^n Z$.

Thus, in both cases $Y = X \times X$ and $Y = \operatorname{Sym}^n X$, we have checked all three conditions guaranteeing an isomorphism $\pi_{n(Y)}^*\colon H^*(\operatorname{Sym}^n Y;\mathbb{Q}) \to S^n H^*(Y;\mathbb{Q})$. The isomorphisms $\pi_{n(X \times X)}^*$ and $\pi_{n(\operatorname{Sym}^n X)}^*$, needed for the proof of the coassociativity of the diagonal $\Delta\colon H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$, allow us to reduce the proof of the second part of Theorem 1 to a verbatim repetition of the proof of Theorem 3.1.7.

In both parts of Theorem 1, the diagonal $\Delta\colon H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ is constructed as $\Delta = \mu^* \circ (\pi_n^*)^{-1} \circ \chi_n$, where $\chi_n\colon H^*(X;\mathbb{Q}) \to S^n H^*(X;\mathbb{Q})$, $\chi_n(\alpha) = \alpha \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \alpha$ is a natural n-homomorphism of the algebra $H^*(X;\mathbb{Q})$, $\pi_n^*\colon H^*(\operatorname{Sym}^n X;\mathbb{Q}) \to S^n H^*(X;\mathbb{Q})$ is the canonical isomorphism, and $\mu^*\colon H^*(\operatorname{Sym}^n X;\mathbb{Q}) \to H^*(X^2;\mathbb{Q}) = H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ is the algebra homomorphism induced by the n-valued multiplication $\mu\colon X^2 \to \operatorname{Sym}^n X$.

By Lemma 1, if $\alpha \in H^q(X;\mathbb{Q})$, $q \geq 0$, is an integral element, then $(\pi_n^*)^{-1} \circ \chi_n(\alpha) \in H^*(\operatorname{Sym}^n X;\mathbb{Q})$ is also integral. Since the homomorphism $\mu^*\colon H^*(\operatorname{Sym}^n X;\mathbb{Q}) \to H^*(X^2;\mathbb{Q})$ maps integral elements to integral elements, $\mu^* \circ (\pi_n^*)^{-1} \circ \chi_n(\alpha) = \Delta(\alpha) \in H^*(X^2;\mathbb{Q})$ is integral.

Suppose now that the groups $H_q(X;\mathbb{Z})$ are finitely generated for all $q \geq 0$. Then the cohomology Künneth formula

$$H^q(X \times X;\mathbb{Z}) = \Big(\bigoplus_{i+j=q} H^i(X;\mathbb{Z}) \otimes H^j(X;\mathbb{Z})\Big)$$
$$1 \oplus \Big(\bigoplus_{i+j=q+1} \operatorname{Tor}(H^i(X;\mathbb{Z}), H^j(X;\mathbb{Z}))\Big),$$

applies, where the first summand is natural and is the image of the $\times$-product

$$\nu_{\mathbb{Z}}\colon H^*(X;\mathbb{Z}) \otimes H^*(X;\mathbb{Z}) \to H^*(X \times X;\mathbb{Z}).$$

Since the integral $\times$-product $\nu_{\mathbb{Z}}$ becomes the rational $\times$-product

$$\nu_{\mathbb{Q}}\colon H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q}) \to H^*(X \times X;\mathbb{Q})$$

under the natural homomorphism of the coefficient rings $\mathbb{Z} \hookrightarrow \mathbb{Q}$, the integral elements of $H^*(X \times X;\mathbb{Q})$ are exactly the integral elements of the tensor square $H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$. Thus, the element $\Delta(\alpha) \in H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ is integral for any integral $\alpha \in H^*(X;\mathbb{Q})$. The theorem is proved. ☐

Now we want to strengthen Theorem 3.2.4.

Theorem 2. *Let X be a connected Hausdorff space homotopy equivalent to a CW-complex and such that* $\dim H^q(X;\mathbb{Q}) < \infty$ $\forall q \geq 0$. *If X admits a structure of a $2H$-space, then its fundamental group $\pi_1(X)$ does not belong to the class $\mathcal{C}$.*

Proof. Assume the opposite. Let X be a $2H$-space and $G := \pi_1(X) \in \mathcal{C}$. By Theorem 1, $H^*(X;\mathbb{Q})$ is a graded 2-Hopf prealgebra. This means that there is a 2-homomorphism $\Delta\colon H^*(X;\mathbb{Q}) \to H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$ satisfying the counit condition: $\Delta(u) = 2u \otimes 1 + 1 \otimes 2u + \sum_{i=1}^N u_i' \otimes u_i''$, $|u_i'| \geq 1$, $|u_i''| \geq 1$, $1 \leq i \leq N$ for any $u \in H^*(X;\mathbb{Q}), |u| \geq 1$. Suppose a CW-complex Y is homotopy equivalent to X and $f\colon X \to Y$ is a homotopy equivalence.

Consider a cellular realization of $BG = K(G,1)$. Since $G \in \mathcal{C}$, there are a basis $\alpha_1, \dots, \alpha_{2g}, \beta_1, \dots, \beta_s \in H^1(BG;\mathbb{Q})$, $g \geq 2$, $s \geq 0$, and a nonzero element $\gamma \in H^2(BG;\mathbb{Q})$ such that both conditions of Definition 3.2.1 of [4] of the class $\mathcal{C}$ are satisfied. Fix base points $y_0 \in Y$ and $b_0 \in BG$. It is well known from homotopy theory of cellular spaces that there is a unique (up to homotopy of pointed spaces) continuous map $h_\varphi\colon (Y, y_0) \to (BG, b_0)$ inducing an arbitrary isomorphism $\varphi\colon \pi_1(Y) = \pi_1(X) \to G = \pi_1(BG)$.

Fix an isomorphism $\varphi_0\colon \pi_1(Y) \to G$ and let $h = h_{\varphi_0}\colon (Y, y_0) \to (BG, b_0)$. Since $h_* = \varphi_0\colon \pi_1(Y) \to \pi_1(BG)$ is an isomorphism, $h_*\colon H_1(Y;\mathbb{Z}) \to H_1(BG;\mathbb{Z})$ is also an isomorphism. Therefore, we also have isomorphisms $h_*\colon H_1(Y;\mathbb{Q}) \to H_1(BG;\mathbb{Q})$ and $h^*\colon H^1(BG;\mathbb{Q}) \to H^1(Y;\mathbb{Q})$.

By a theorem of Hopf, the map $h_*\colon H_2(Y;\mathbb{Z}) \to H_2(BG;\mathbb{Z})$ is an epimorphism whose kernel coincides with the image of the Hurewicz homomorphism $\pi_2(Y) \to H_2(Y;\mathbb{Z})$. This implies that $h_*\colon H_2(Y;\mathbb{Q}) \to H_2(BG;\mathbb{Q})$ is an epimorphism, and hence $h^*\colon H^2(BG;\mathbb{Q}) \to H^2(Y;\mathbb{Q})$ is a monomorphism.

Since $h^*\colon H^*(BG;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is an algebra homomorphism inducing an isomorphism on H^1 and a monomorphism on H^2, both conditions of Definition 3.2.1 are satisfied for the basis $h^*(\alpha_1), \dots, h^*(\alpha_{2g}), h^*(\beta_1), \dots, h^*(\beta_s) \in H^1(Y;\mathbb{Q})$ and the nonzero element $h^*(\gamma) \in H^2(Y;\mathbb{Q})$. Since the homotopy equivalence $f\colon X \to Y$ induces an algebra isomorphism $f^*\colon H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$, both conditions of Definition 3.2.1 are also satisfied for the basis $\bar\alpha_1 = f^* \circ h^*(\alpha_1), \dots, \bar\alpha_{2g} = f^* \circ h^*(\alpha_{2g})$, $\bar\beta_1 = f^* \circ h^*(\beta_1), \dots, \bar\beta_s = f^* \circ h^*(\beta_s) \in H^1(X;\mathbb{Q})$ and the nonzero element $\bar\gamma = f^* \circ h^*(\gamma) \in H^2(X;\mathbb{Q})$. Denote the algebra $H^*(X;\mathbb{Q})$ by A^*.

In summary, we have an algebra A^*, a basis $\bar{\alpha}_1, \ldots, \bar{\alpha}_{2g}, \bar{\beta}_1, \ldots, \bar{\beta}_s \in A^1$, $g \geq 2$, $s \geq 0$, and a nonzero element $\bar{\gamma} \in A^2$ satisfying the conditions of Definition 3.2.1, and we also have a 2-homomorphism $\Delta\colon A^* \to A^* \otimes A^*$ satisfying the counit axiom. Now, using the same arguments as in the proof of Theorem 3.2.4, we have $g = -3$, a contradiction. The theorem is proved. □

§ 4. A class $\mathcal{M}$ of even-dimensional manifolds which are not $2H$-spaces

Definition 1. We say that a connected closed orientable topological $2n$-dimensional manifold M^{2n}, $n \neq 1, 2, 4$, belongs to the class $\mathcal{M}$, if the first betti number of M^{2n} is zero, $b_1(M^{2n}) = 0$, and there is an element $u \in H^2(M^{2n}; \mathbb{Z})$ such that $u^n = k[M^{2n}]$ and $n \nmid 4k^2$.

The following result shows that the class $\mathcal{M}$ is rather wide.

Proposition 1. (1) *Suppose $M^{2n} \in \mathcal{M}$ and N^{2n} is a connected closed orientable topological $2n$-dimensional manifold such that $b_1(N^{2n}) = 0$, and there is a continuous map $f\colon N^{2n} \to M^{2n}$ with $(\deg f, n) = 1$. Then N^{2n} belongs to $\mathcal{M}$.*

(2) *Suppose $M^{2n} \in \mathcal{M}$ and N^{2n} is a connected closed orientable topological $2n$-dimensional manifold with $b_1(N^{2n}) = 0$. Then $M^{2n} \# N^{2n} \in \mathcal{M}$.*

(3) *$\mathbb{C}P^n \in \mathcal{M}$ for any $n \neq 1, 2, 4$.*

(4) *Suppose $M^{2n} \subset \mathbb{C}P^N, n \neq 1, 2, 4$, is a nonsingular connected projective algebraic variety of dimension n and degree m. If $n \nmid 4m^2$ and $b_1(M^{2n}) = 0$, then $M^{2n} \in \mathcal{M}$.*

Proof. (1) Let $u \in H^2(M^{2n}; \mathbb{Z})$ be an element such that $u^n = k[M^{2n}]$ and $n \nmid 4k^2$. By assumption, there is a continuous map $f\colon N^{2n} \to M^{2n}$ with $(\deg f, n) = 1$. Let $v = f^* u \in H^2(N^{2n}; \mathbb{Z})$. Then

$$v^n = f^*(u^n) = f^*(k[M^{2n}]) = k f^*[M^{2n}] = k \deg f [N^{2n}].$$

Since $n \nmid 4k^2$ and the integer $\deg f$ is relatively prime with n, we have that $n \nmid 4(k \deg f)^2 = 4k^2 \deg^2 f$. Thus, N^{2n} belongs to $\mathcal{M}$. This proves the first assertion.

(2) We have $b_1(M^{2n} \# N^{2n}) = b_1(M^{2n}) + b_1(N^{2n}) = 0$. Now the desired assertion follows from (1) because there is always a natural continuous map $f\colon M^{2n} \# N^{2n} \to M^{2n}$ of degree 1 (collapsing N^{2n} to a point). This proves the second assertion.

(3) This assertion is obvious. Taking $u \in H^2(\mathbb{C}P^n; \mathbb{Z})$ as a canonical generator, we have $u^n = 1 \cdot [\mathbb{C}P^n]$ and $n \nmid 4 = 4 \cdot 1^2$, if $n \neq 1, 2, 4$.

(4) As is well known, if the degree of a nonsingular connected algebraic n-dimensional subvariety $i\colon M^{2n} \hookrightarrow \mathbb{C}P^N$ is m, then $v^n = m[M^{2n}]$, where $v = i^* u \in H^2(M^{2n}; \mathbb{Z})$ and $u \in H^2(\mathbb{C}P^N; \mathbb{Z})$ is the canonical generator. This immediately implies that if $n \nmid 4m^2$ and $b_1(M^{2n}) = 0$, then M^{2n} belongs to $\mathcal{M}$. The proposition is proved. □

Theorem 3. *If M^{2n} belongs to $\mathcal{M}$, then it does not admit a structure of $2H$-space.*

Proof. Assume the opposite: a connected closed orientable topological $2n$-dimensional manifold M^{2n} belongs to $\mathcal{M}$ and admits a structure of $2H$-space. By the Borsuk criterion, any topological manifold (possibly, with boundary) is an ENR-space, and any compact ENR, by a result of F. Quinn, is homotopy equivalent to a finite CW-complex. Thus, Theorem 1 applies to the $2H$-space M^{2n}. This means that there is a 2-homomorphism $\Delta\colon H^*(M^{2n}; \mathbb{Q}) \to H^*(M^{2n}; \mathbb{Q}) \otimes H^*(M^{2n}; \mathbb{Q})$ satisfying the counit axiom and mapping integral elements $\alpha \in H^*(M^{2n}; \mathbb{Q})$ to integral elements of $H^*(M^{2n}; \mathbb{Q}) \otimes H^*(M^{2n}; \mathbb{Q})$.

Since integral homology $H_q(M^{2n}; \mathbb{Z})$ is finitely generated for all $q \geq 0$, we have a natural isomorphism $H^*(M^{2n}; \mathbb{Z}) \otimes \mathbb{Q} = H^*(M^{2n}; \mathbb{Q})$. Denote the cohomology algebra $H^*(M^{2n}; \mathbb{Q})$ by A^*.

Let $A^*_{\mathbb{Z}} \subset A^*$ denote the subring of integral elements of A^*. It is easy to see that $A^*_{\mathbb{Z}} = H^*(M^{2n};\mathbb{Z})/\mathrm{Tor}$. Denote by γ the fundamental cohomology class $[M^{2n}] \in A^{2n}_{\mathbb{Z}}$. Since Poincaré duality holds for closed orientable topological manifolds, we have $A^i_{\mathbb{Z}} \cong A^{2n-i}_{\mathbb{Z}} \cong \mathbb{Z}^{b_i(M^{2n})} = \mathbb{Z}^{b_{2n-i}(M^{2n})}$, $0 \le i \le 2n$. Moreover, multiplication in cohomology $A^i_{\mathbb{Z}} \otimes A^{2n-i}_{\mathbb{Z}} \to A^{2n}_{\mathbb{Z}} = \mathbb{Z}\langle\gamma\rangle$ defines a nondegenerate unimodular pairing $A^i_{\mathbb{Z}} \otimes A^{2n-i}_{\mathbb{Z}} \to \mathbb{Z}$, $A^{2n-i}_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(A^i_{\mathbb{Z}}, \mathbb{Z})$, $0 \le i \le 2n$.

Since $M^{2n} \in \mathcal{M}$, we have $A^1 = \{0\}$ and there is an element $u \in H^2(M^{2n};\mathbb{Z})$ such that $u^n = k[M^{2n}]$ and $n \nmid 4k^2$. Denote by the same symbol u the image of $u \in H^2(M^{2n};\mathbb{Z})$ under the epimorphism $H^*(M^{2n};\mathbb{Z}) \to H^*(M^{2n};\mathbb{Z})/\mathrm{Tor} = A^*_{\mathbb{Z}}$. Thus, we have an element $u \in A^2_{\mathbb{Z}}$ such that $u^n = k\gamma$ and $n \nmid 4k^2$.

Consider $u^{n-1} \in A^{2n-2}_{\mathbb{Z}}$. Suppose a homomorphism $\varphi_{u^{n-1}} \in \mathrm{Hom}_{\mathbb{Z}}(A^2_{\mathbb{Z}}, \mathbb{Z})$ corresponds to $u^{n-1} \in A^{2n-2}_{\mathbb{Z}}$ under the canonical isomorphism $A^{2n-2}_{\mathbb{Z}} \cong \mathrm{Hom}_{\mathbb{Z}}(A^2_{\mathbb{Z}}, \mathbb{Z})$. The subgroup $\mathrm{Ker}\varphi_{u^{n-1}} \subset A^2_{\mathbb{Z}}$ is free abelian as a subgroup of the free abelian group $A^2_{\mathbb{Z}}$. It is easy to see that $\mathrm{rk}(\mathrm{Ker}\,\varphi_{u^{n-1}}) = \mathrm{rk}(A^2_{\mathbb{Z}}) - 1 = b_2(M^{2n}) - 1$. Let $b_2(M^{2n}) = N$.

By the structure theorem for finitely generated abelian groups, for any free finitely generated abelian group F and its (free abelian) subgroup $G \subset F$ there is a basis $f_1, \dots, f_l$ of F and numbers $\mu_1, \dots, \mu_m \in \mathbb{N}$, $m = \mathrm{rk}(G) \le \mathrm{rk}(F) = l$ such that $\mu_1 f_1, \dots, \mu_m f_m \in F$ is a basis of G. Therefore, there is a basis $v_1, v_2, \dots, v_N$ of $A^2_{\mathbb{Z}}$ such that $\mu_2 v_2, \dots, \mu_N v_N$ is a basis of $\mathrm{Ker}\,\varphi_{u^{n-1}}$ for some $\mu_2, \dots, \mu_N \in \mathbb{N}$. But since $\varphi_{u^{n-1}}(\mu_j v_j) = \mu_j \varphi_{u^{n-1}}(v_j) = 0$, $2 \le j \le N$, we also have $\varphi_{u^{n-1}}(v_j) = 0$, $2 \le j \le N$, and therefore, $\mu_2 = \ldots = \mu_N = 1$. Thus, we have a basis $v_1, v_2, \dots, v_N$ of $A^2_{\mathbb{Z}}$ such that $v_2, \dots, v_N$ is a basis of $\mathrm{Ker}\,\varphi_{u^{n-1}}$, i.e., $u^{n-1}v_2 = \ldots = u^{n-1}v_N = 0$ and $u^{n-1}v_1 \ne 0$.

Write the element $u \in A^2_{\mathbb{Z}}$ as a linear combination of the basis vectors $v_1, v_2, \dots, v_N$: $u = k_1 v_1 + k_2 v_2 + \ldots + k_N v_N$, $k_i \in \mathbb{Z}$, $1 \le i \le N$. Clearly, $k_1 \ne 0$. It is also clear that $u, v_2, \dots, v_N$ is a basis of $A^2 = A^2_{\mathbb{Z}} \otimes \mathbb{Q}$. By Theorem 1, applied to the $2H$-space M^{2n}, there is a $\mathbb{Z}$-linear map $\Delta_{\mathbb{Z}}\colon A^*_{\mathbb{Z}} \to A^*_{\mathbb{Z}} \otimes A^*_{\mathbb{Z}}$ such that its $\mathbb{Q}$-linear extension $\Delta = \Delta_{\mathbb{Z}} \otimes \mathrm{Id}\colon A^* = A^*_{\mathbb{Z}} \otimes \mathbb{Q} \to A^* \otimes A^* = (A^*_{\mathbb{Z}} \otimes \mathbb{Q}) \otimes (A^*_{\mathbb{Z}} \otimes \mathbb{Q}) = (A^*_{\mathbb{Z}} \otimes A^*_{\mathbb{Z}}) \otimes \mathbb{Q}$ is a 2-homomorphism and satisfies the counit axiom.

Let $f = \frac{1}{2}\Delta$. The condition $\Phi_3(\Delta)(a, b, c) = 0$ $\forall a, b, c \in A^{2*}$, rewritten in terms of f, yields

$$f(abc) = -2f(a)f(b)f(c) + f(ab)f(c) + f(ac)f(b) + f(bc)f(a) \quad \forall a, b, c \in A^{2*}. \tag{1}$$

It follows from the counit axiom that

$$\begin{aligned} &f(u) = u \otimes 1 + 1 \otimes u, \\ &f(u^2) = u^2 \otimes 1 + \alpha u \otimes u + \sum_{j=2}^{N} \varepsilon_j v_j \otimes u + \sum_{j=2}^{N} \delta_j u \otimes v_j + \sum_{i,j=2}^{N} \omega_{ij} v_i \otimes v_j + 1 \otimes u^2, \end{aligned} \tag{2}$$

for some constants $\alpha, \varepsilon_*, \delta_*, \omega_{**} \in \mathbb{Q}$.

It follows easily from the theory of nongraded k-homomorphisms that, for any natural numbers $k \ge 1$ and $m \ge k+1$, any $\mathbb{Q}$-algebra k-homomorphism $g\colon A^* \to B^*$, and any homogeneous element $a \in A^{2*}$, we have $g(a^m) = P_{m,k}(g(a), g(a^2), \dots, g(a^k))$, where $P_{m,k}(z_1, \dots, z_k) \in \mathbb{Q}[z_1, \dots, z_k]$ is a universal homogeneous polynomial of degree m (the degree of z_s equals $s, 1 \le s \le k$) whose coefficients depend only on the pair (m, k). More precisely, $P_{m,k}(z_1, \dots, z_k) \in \mathbb{Q}[z_1, \dots, z_k]$ can be determined from the identity

$$t_1^m + t_2^m + \ldots + t_k^m = P_{m,k}(t_1 + \ldots + t_k, t_1^2 + \ldots + t_k^2, \dots, t_1^k + \ldots + t_k^k),$$

where $t_1, \dots, t_k$ are algebraically independent over $\mathbb{Q}$ formal variables.

Consider now the case $k = 2$, $m = n + 1$, $a = u \in A^2$, and the universal polynomial $P_{n+1,2}(z_1, z_2) = \sum_{s_1+2s_2=n+1} c_{s_1 s_2} z_1^{s_1} z_2^{s_2}$, $c_{**} \in \mathbb{Q}$. We then have

$$(3)\quad 0 = f(u^{n+1}) = \frac{1}{2}\Delta(u^{n+1}) = \frac{1}{2}P_{n+1,2}(2f(u), 2f(u^2)) = \sum_{s_1+2s_2=n+1} 2^{s_1+s_2-1} c_{s_1 s_2} f^{s_1}(u) f^{s_2}(u^2).$$

Since $f(u) \in A^2 \otimes A^0 \oplus A^0 \otimes A^2 = (A^* \otimes A^*)^2$ and $f(u^2) \in A^4 \otimes A^0 \oplus A^2 \otimes A^2 \oplus A^0 \otimes A^4 = (A^* \otimes A^*)^4$, we have, for any $s_1, s_2 \geq 0$, $s_1 + 2s_2 = n + 1$, that

$$f^{s_1}(u) f^{s_2}(u^2) \in (A^{2*} \otimes A^{2*})^{2n+2} = A^{2n} \otimes A^2 \oplus A^{2n-2} \otimes A^4 \oplus \ldots \oplus A^2 \otimes A^{2n}.$$

Let $[\cdot]_{2n,2}\colon (A^{2*} \otimes A^{2*})^{2n+2} \to A^{2n} \otimes A^2$ denote the canonical projector to the direct summand $A^{2n} \otimes A^2 \subset (A^{2*} \otimes A^{2*})^{2n+2}$. By (3),

$$(4)\qquad \sum_{s_1+2s_2=n+1} 2^{s_1+s_2-1} c_{s_1 s_2} [f^{s_1}(u) f^{s_2}(u^2)]_{2n,2} \;=\; 0.$$

We know that $A^{2n} = \mathbb{Q}\langle \gamma \rangle = \mathbb{Q}\langle u^n \rangle$ and $A^2 = \mathbb{Q}\langle u, v_2, \ldots, v_N \rangle$. Therefore, the elements $u^n \otimes u, u^n \otimes v_2, \ldots, u^n \otimes v_N$ form a $\mathbb{Q}$-basis of $A^{2n} \otimes A^2$. Taking into account (2), we rewrite (4) as

$$\sum_{s_1+2s_2=n+1} 2^{s_1+s_2-1} c_{s_1 s_2} [f^{s_1}(u) f^{s_2}(u^2)]_{2n,2} = Q_{n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**}) u^n \otimes u + R_{2,n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**}) u^n \otimes v_2 + \ldots + R_{N,n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**}) u^n \otimes v_N = 0.$$

It now follows that

$$(5)\qquad Q_{n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**}) = 0.$$

We want to show that the rational constant $Q_{n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**})$ depends only on α and does not depend on ε_*, δ_* and ω_{**}.

Indeed, $Q_{n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**})$ is a sum of the first coordinates of the elements $[f^{s_1}(u) f^{s_2}(u^2)]_{2n,2} \in A^{2n} \otimes A^2, s_1 + 2s_2 = n + 1$ with certain universal coefficients (in the basis $u^n \otimes u, u^n \otimes v_2, \ldots, u^n \otimes v_N$ of $A^{2n} \otimes A^2$). We now examine in more detail the expression for the first coordinate of $[f^{s_1}(u) f^{s_2}(u^2)]_{2n,2}, s_1 + 2s_2 = n + 1$. If $s_2 = 0$ and $s_1 = n + 1$, then the first coordinate of $[f^{n+1}(u)]_{2n,2}$ depends only on n.

Suppose $s_2 \geq 1$. Distributing all products in $f^{s_1}(u) f^{s_2}(u^2)$, we can see that, in the resulting sum of the monomials of the form $c_* u^q v_{j_1} \ldots v_{j_s} \otimes u^{q'} v_{j'_1} \ldots v_{j'_{s'}}$, the monomials in $A^{2n} \otimes A^2$ are either products of only *one monomial* of the form $\alpha u \otimes u$, $\varepsilon_j v_j \otimes u$, $2 \leq j \leq N$, $\delta_j u \otimes v_j$, $2 \leq j \leq N$, $\omega_{ij} v_i \otimes v_j$, $2 \leq i, j \leq N$, and the remaining s_1 monomials $u \otimes 1$ and $(s_2 - 1)$ monomials $u^2 \otimes 1$, or products of $(s_1 - 1)$ monomials $u \otimes 1$, *one monomial* $1 \otimes u$, and s_2 monomials $u^2 \otimes 1$.

The summands appearing in the second case depend only on n. In the first case, any summand belongs to one of the following four types:

(A) $\alpha u^n \otimes u$;
(B) $\varepsilon_j u^{n-1} v_j \otimes u$, $2 \leq j \leq N$;
(C) $\delta_j u^n \otimes v_j$, $2 \leq j \leq N$;
(D) $\omega_{ij} u^{n-1} v_i \otimes v_j$, $2 \leq i, j \leq N$.

Since $u^{n-1} v_2 = \ldots = u^{n-1} v_N = 0$, the summands of types (B) and (D) are zero. Thus, we only have summands of types (A) and (C). But it is easy to see that only elements of type (A) contribute to the first coordinate of $[f^{s_1}(u) f^{s_2}(u^2)]_{2n,2}$. Such elements depend only on α. Thus, we have shown that $Q_{n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**})$ depends only on α.

Now let us compute this constant

$$Q_{n+1}(\alpha) = Q_{n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**}) = Q_{n+1}(\alpha, 0, 0, 0).$$

Consider the algebra $\mathbb{Q}[\tilde{u}]$ with $\deg \tilde{u} = 2$.

By the so-called completeness property of Frobenius m-homomorphisms, we have that, for any even-graded commutative $\mathbb{Q}$-algebra B^{2*} and any homogeneous elements $b_1 \in B^2$, $b_2 \in B^4$, $\dots$, $b_m \in B^{2m}$, there is a unique m-homomorphism $g_{b_1,b_2,\dots,b_m} \colon \mathbb{Q}[\tilde{u}] \to B^{2*}$ such that $g_{b_1,b_2,\dots,b_m}(\tilde{u}^k) = b_k$, $1 \le k \le m$ (see [5, Theorem 4.1]). Now take the algebra $A^{2*} \otimes A^{2*}$ as B^{2*}. Then there is a unique 2-homomorphism $\widetilde{\Delta} \colon \mathbb{Q}[\tilde{u}] \to A^{2*} \otimes A^{2*}$ such that $\widetilde{\Delta}(\tilde{u}) = 2u \otimes 1 + 1 \otimes 2u$ and $\widetilde{\Delta}(\tilde{u}^2) = 2u^2 \otimes 1 + 2\alpha u \otimes u + 1 \otimes 2u^2$.

Consider the map $\tilde{f} = \frac{1}{2}\widetilde{\Delta} \colon \mathbb{Q}[\tilde{u}] \to A^{2*} \otimes A^{2*}$. Using the same arguments as for f, we have

$$(6) \quad [\tilde{f}(\tilde{u}^{n+1})]_{2n,2} = \sum_{s_1+2s_2=n+1} 2^{s_1+s_2-1} c_{s_1 s_2} [\tilde{f}^{s_1}(\tilde{u}) \tilde{f}^{s_2}(\tilde{u}^2)]_{2n,2} = Q_{n+1}(\alpha) u^n \otimes u + R_{2,n+1}(\alpha, 0,0,0) u^n \otimes v_2 + \ldots + R_{N,n+1}(\alpha,0,0,0) u^n \otimes v_N.$$

Moreover, it is easy to see that

$$\tilde{f}^{s_1}(\tilde{u}) \tilde{f}^{s_2}(\tilde{u}^2) = \sum_{k=0}^{s_1+2s_2} d_{s_1 s_2;k}(\alpha) u^{s_1+2s_2-k} \otimes u^k \quad \forall s_1, s_2 \ge 0.$$

Thus, we have

$$\tilde{f}(\tilde{u}) = u \otimes 1 + 1 \otimes u, \quad \tilde{f}(\tilde{u}^2) = u^2 \otimes 1 + \alpha u \otimes u + 1 \otimes u^2;$$
$$\tilde{f}(\tilde{u}^m) = u^m \otimes 1 + Q_m(\alpha) u^{m-1} \otimes u + R_m(\alpha) \cdot 1 \otimes u^2, \quad m \ge 2,$$

where $Q_m(\alpha) \in \mathbb{Q}$ and $R_m(\alpha) \in A^{2*} \otimes A^{2*}$, $m \ge 2$.

Also, since $2\tilde{f} \colon \mathbb{Q}[\tilde{u}] \to A^{2*} \otimes A^{2*}$ is a 2-homomorphism, the identity $\Phi_3(2\tilde{f})(a,b,c) = 0$ $\forall a,b,c \in \mathbb{Q}[\tilde{u}]$ implies that

$$(7) \quad \tilde{f}(abc) = -2\tilde{f}(a)\tilde{f}(b)\tilde{f}(c) + \tilde{f}(ab)\tilde{f}(c) + \tilde{f}(ac)\tilde{f}(b) + \tilde{f}(bc)\tilde{f}(a) \quad \forall a,b,c \in \mathbb{Q}[\tilde{u}].$$

Substituting $a = b = c = \tilde{u}$ in (7), we have

$$\tilde{f}(\tilde{u}^3) = u^3 \otimes 1 + 3(\alpha - 1) u^2 \otimes u + R_3(\alpha) \cdot 1 \otimes u^2.$$

Substituting $a = b = \tilde{u}$, $c = \tilde{u}^m$, $m \ge 2$ in (7), we have

$$\tilde{f}(\tilde{u}^{m+2}) = -2\tilde{f}^2(\tilde{u})\tilde{f}(\tilde{u}^m) + 2\tilde{f}(\tilde{u})\tilde{f}(\tilde{u}^{m+1}) + \tilde{f}(\tilde{u}^2)\tilde{f}(\tilde{u}^m);$$

$$\tilde{f}(\tilde{u}^{m+2}) = (-u^2 \otimes 1 - 1 \otimes u^2 + (\alpha - 4) u \otimes u) \times (u^m \otimes 1 + Q_m(\alpha) u^{m-1} \otimes u + R_m(\alpha) \cdot 1 \otimes u^2) + 2(u \otimes 1 + 1 \otimes u) \times (u^{m+1} \otimes 1 + Q_{m+1}(\alpha) u^m \otimes u + R_{m+1}(\alpha) \cdot 1 \otimes u^2).$$

Distributing and collecting similar terms, we have a recurrence relation

$$(8) \quad Q_{m+2}(\alpha) = 2Q_{m+1}(\alpha) - Q_m(\alpha) + \alpha - 2, \quad m \ge 2.$$

We also know the initial conditions: $Q_2(\alpha) = \alpha$, $Q_3(\alpha) = 3\alpha - 3$.

Together with (8), this yields $Q_m(\alpha) = a_m \alpha + b_m$ for some $a_m, b_m \in \mathbb{Z}$, $m \ge 2$. For the numbers a_m and $b_m, m \ge 2$, we have, from (8), recurrence relations with constant coefficients

$$\begin{cases} a_{m+2} = 2a_{m+1} - a_m + 1, & m \ge 2; \\ b_{m+2} = 2b_{m+1} - b_m - 2, & m \ge 2. \end{cases}$$

Since the initial conditions $a_2 = 1$, $a_3 = 3$, $b_2 = 0$, $b_3 = -3$ are given, the above system has a unique solution, which can be found by standard methods: $a_m = \frac{1}{2}(m^2 - m)$ and $b_m = -m^2 + 2m$ $\forall m \geq 2$. Thus,

$$Q_m(\alpha) = \frac{1}{2}m(m-1)\alpha - m(m-2) \quad \forall m \geq 2.$$

As a result, we have determined the coefficient $Q_{n+1}(\alpha) = (n+1)\Big(\frac{n}{2}\alpha - (n-1)\Big)$. But earlier in the proof of this theorem we had $Q_{n+1}(\alpha) = Q_{n+1}(\alpha, \varepsilon_*, \delta_*, \omega_{**}) = 0$. Therefore, $\alpha = \frac{2(n-1)}{n} \in \mathbb{Q}$.

Returning to (2), we have

$$f(u^2) = u^2 \otimes 1 + \alpha u \otimes u + \sum_{j=2}^{N} \varepsilon_j v_j \otimes u + \sum_{j=2}^{N} \delta_j u \otimes v_j + \sum_{i,j=2}^{N} \omega_{ij} v_i \otimes v_j + 1 \otimes u^2.$$

We know that $A^2_{\mathbb{Z}} = \mathbb{Z}\langle v_1, v_2, \ldots, v_N\rangle$ and $u = k_1v_1 + k_2v_2 + \ldots + k_Nv_N$, $k_i \in \mathbb{Z}$, $1 \leq i \leq N$, $k_1 \neq 0$.

Since $\Delta(A^*_{\mathbb{Z}}) = 2f(A^*_{\mathbb{Z}}) \subset A^*_{\mathbb{Z}} \otimes A^*_{\mathbb{Z}}$, we have $2f(u^2) \in A^4_{\mathbb{Z}} \otimes A^0_{\mathbb{Z}} \oplus A^2_{\mathbb{Z}} \otimes A^2_{\mathbb{Z}} \oplus A^0_{\mathbb{Z}} \otimes A^4_{\mathbb{Z}}$. Consider the summand $[2f(u^2)]_{2,2} \in A^2_{\mathbb{Z}} \otimes A^2_{\mathbb{Z}}$:

$$\begin{aligned}[2f(u^2)]_{2,2} &= 2\alpha(k_1v_1 + k_2v_2 + \ldots + k_Nv_N) \otimes (k_1v_1 + k_2v_2 + \ldots + k_Nv_N) + \\ &\sum_{j=2}^{N} \varepsilon_j v_j \otimes (k_1v_1 + k_2v_2 + \ldots + k_Nv_N) + \sum_{j=2}^{N} \delta_j (k_1v_1 + k_2v_2 + \ldots + k_Nv_N) \otimes v_j + \\ &\sum_{i,j=2}^{N} \omega_{ij} v_i \otimes v_j \in A^2_{\mathbb{Z}} \otimes A^2_{\mathbb{Z}} = \\ &\mathbb{Z}\langle v_1 \otimes v_1, v_1 \otimes v_j, 2 \leq j \leq N, v_j \otimes v_1, 2 \leq j \leq N, v_i \otimes v_j, 2 \leq i, j \leq N\rangle.\end{aligned}$$

Since $[2f(u^2)]_{2,2} \in A^2_{\mathbb{Z}} \otimes A^2_{\mathbb{Z}}$, we have $2\alpha k_1^2 \in \mathbb{Z}$. Since $\alpha = \frac{2(n-1)}{n}$ and $(n-1, n) = 1$, we have $n \mid 4k_1^2$. Moreover, $u = k_1v_1 + k_2v_2 + \ldots + k_Nv_N$ and $u^n = k\gamma$.

Thus, $u^n = uu^{n-1} = (k_1v_1 + k_2v_2 + \ldots + k_Nv_N)u^{n-1} = k_1v_1u^{n-1} = k_1q\gamma$, where $v_1u^{n-1} = q\gamma, q \in \mathbb{Z}$. Therefore, $k = k_1q$ for some $q \in \mathbb{Z}, q \neq 0$. We then have $n \mid 4k^2 = 4k_1^2q^2$ because $n \mid 4k_1^2$. But, by assumption, $n \nmid 4k^2$. The obtained contradiction proves the theorem. □

The class $\mathcal{M}$ introduced above does not contain 4-dimensional and 8-dimensional manifolds. At the same time, complex projective spaces $\mathbb{C}P^n$ of dimension $n \neq 1, 2, 4$ do belong to $\mathcal{M}$ and therefore do not admit a structure of $2H$-space. The Riemann sphere $\mathbb{C}P^1$ has a structure of 2-valued algebraic groups, introduced by V. M. Buchstaber in [3]. It was shown in [2] that $\mathbb{C}P^2$ is not a $2H$-space. The next theorem proves the same for $\mathbb{C}P^4$ and a number of other spaces.

Theorem 4. *Let X be a connected Hausdorff space homotopy equivalent to a CW-complex and having finitely generated integral homology groups $H_q(X;\mathbb{Z})$ for all $q \geq 0$. Suppose that, for some natural number m, the first m betti numbers of X vanish, $b_1(X) = b_2(X) = \ldots = b_m(X) = 0$. Assume also that $b_{2m}(X) = 1$ and, if $m \geq 2$, then for any k, $1 \leq k \leq m-1$, either $b_{m+k}(X) = 0$ or $b_{3m-k}(X) = 0$. If, for a generator $u \in H^{2m}(X;\mathbb{Q})$, $u^n \neq 0$ and $u^{n+1} = 0$ for some $n \geq 2$, then X is not a $2H$-space.*

Proof. Denote the ring $H^*(X;\mathbb{Z})/\mathrm{Tor}$ by $A^*_{\mathbb{Z}}$. Because the homology $H_q(X;\mathbb{Z}), q \geq 0$ is finitely generated, we have the canonical $\mathbb{Q}$-algebra isomorphism $A^*_{\mathbb{Z}} \otimes \mathbb{Q} := A^* \cong H^*(X;\mathbb{Q})$.

Assume that X is a $2H$-space. According to Theorem 1, there is a $\mathbb{Z}$-linear map $\Delta_{\mathbb{Z}}\colon A^*_{\mathbb{Z}} \to A^*_{\mathbb{Z}} \otimes A^*_{\mathbb{Z}}$ such that the $\mathbb{Q}$-linear extension $\Delta = \Delta_{\mathbb{Z}} \otimes \mathrm{Id}\colon A^* \to A^* \otimes A^*$ is a 2-homomorphism and satisfies the counit axiom. It is easy to see that for the element $u \in A^{2m}$ in the statement of the theorem we can take a generator of $A^{2m}_{\mathbb{Z}} \subset A^{2m}$, $A^{2m}_{\mathbb{Z}} = \mathbb{Z}\langle u\rangle$.

Let $f = \frac{1}{2}\Delta$. By the counit axiom and the assumptions on A^*,

$$f(u) = u \otimes 1 + 1 \otimes u, \quad f(u^2) = u^2 \otimes 1 + 1 \otimes u^2 + \alpha u \otimes u,$$

where $\alpha \in \frac{\mathbb{Z}}{2}$ by the integrality of Δ.

Using the same arguments as in the proof of Theorem 3, we have the implication $u^{n+1} = 0 \Rightarrow f(u^{n+1}) = 0 \Rightarrow \alpha = \frac{2(n-1)}{n}$. Since $\alpha \in \frac{\mathbb{Z}}{2}$, it follows that $n = 2$ or $n = 4$. Thus, for $n \neq 2, 4$ and $n \geq 2$, the theorem is already proved.

Suppose $n = 2$ or 4. Formula (1) from the proof of Theorem 3, which applies to our case, yields a recurrence relation

$$f(u^{m+2}) = -2f^2(u)f(u^m) + 2f(u)f(u^{m+1}) + f(u^2)f(u^m) \quad \forall m \geq 1.$$

The initial conditions ($f(u) = u \otimes 1 + 1 \otimes u$, $f(u^2) = u^2 \otimes 1 + 1 \otimes u^2 + \alpha u \otimes u$) allow us to compute $f(u^m)$ for all $m \geq 3$ whenever α is known.

Let $n = 2$. Then $\alpha = \frac{2(n-1)}{n} = 1$. Direct calculations show that $0 = f(u^4) = -5u^2 \otimes u^2 \neq 0$. The obtained contradiction proves the theorem in the case $n = 2$.

Let $n = 4$. Then $\alpha = \frac{2(n-1)}{n} = \frac{3}{2}$. Calculations show that $0 = f(u^5) = -\frac{15}{4}(u^3 \otimes u^2 + u^2 \otimes u^3) \neq 0$. The obtained contradiction proves the theorem in the remaining case $n = 4$. The theorem is proved. □

§ 5. Retracts of nH-spaces

The next lemma shows that, for CW-complexes X, the homotopy unit axiom for n-valued multiplications $\mu\colon X \times X \to \mathrm{Sym}^n X$ is equivalent to the strong unit axiom.

Lemma 2. *Let X be a connected CW-complex, x_0 and $x_1 \in X$ two arbitrary points of X, and $\mu_0\colon (X, x_0) \times (X, x_0) \to (\mathrm{Sym}^n X, [nx_0])$ an n-valued multiplication for which x_0 is a homotopy unit. Then there is an n-valued multiplication $\mu_1\colon (X, x_1) \times (X, x_1) \to (\mathrm{Sym}^n X, [nx_1])$, freely homotopic to μ_0 and such that the strong unit axiom $\mu_1(x, x_1) = \mu_1(x_1, x) = [nx]$ $\forall x \in X$ holds.*

Proof. The term homotopy will refer to the homotopy of pointed spaces, as opposed to free homotopy (of non-pointed spaces). It is known that any point of an arbitrary CW-complex can be made into a vertex by subdividing a finite number of cells.

Therefore, we may assume that x_0 and $x_1 \in X$ are vertices of the complex X. Since X is connected, the vertices x_0 and x_1 can be connected by a continuous path $h\colon \{x_0\} \times I \to X$, $h(x_0, 0) = x_0$, $h(x_0, 1) = x_1$. The pair $(X, \{x_0\})$ is called a Borsuk pair. Therefore, the free homotopy $h\colon \{x_0\} \times I \to X$ extends to a free homotopy $H\colon X \times I \to X$ such that $H_0 = \mathrm{Id}_X$ and $H_1(x_0) = x_1$. Denote the map $H_1\colon (X, x_0) \to (X, x_1)$ by f.

The map $f\colon (X, x_0) \to (X, x_1)$ is freely homotopic to Id_X and therefore is a homotopy equivalence. Since X is a CW-complex, $f\colon (X, x_0) \to (X, x_1)$ is a pointed homotopy equivalence. Let $g\colon (X, x_1) \to (X, x_0)$ be the homotopy inverse of f.

Let $[F]$ denote the free homotopy class of an arbitrary map $F\colon (Y, y_0) \to (Z, z_0)$, and $[F]^\bullet$ – the pointed homotopy class of F. By definition, $[g]^\bullet = ([f]^\bullet)^{-1}$. This implies that $[g] = [f]^{-1} = \mathrm{Id}_X^{-1} = \mathrm{Id}_X$.

Define a continuous map $\widetilde{\mu}_1\colon (X,x_1)\times(X,x_1)\to(\mathrm{Sym}^n X,[nx_1])$ using the following commutative diagram:

$$\begin{array}{ccc}(X,x_0)\times(X,x_0) & \xrightarrow{\mu_0} & (\mathrm{Sym}^n X,[nx_0])\\ {\scriptstyle g\times g}\uparrow & & \downarrow{\scriptstyle \mathrm{Sym}^n f}\\ (X,x_1)\times(X,x_1) & \xrightarrow{\widetilde{\mu}_1} & (\mathrm{Sym}^n X,[nx_1]).\end{array}$$

Since Sym^n is a homotopy functor, we have $[\mathrm{Sym}^n f]=[\mathrm{Sym}^n \mathrm{Id}_X]=\mathrm{Id}_{\mathrm{Sym}^n X}$. Therefore,

$$[\widetilde{\mu}_1]=[\mathrm{Sym}^n f\circ\mu_0\circ(g\times g)]=[\mathrm{Sym}^n f]\circ[\mu_0]\circ([g]\times[g])=\\ \mathrm{Id}_{\mathrm{Sym}^n X}\circ[\mu_0]\circ\mathrm{Id}_{X^2}=[\mu_0].$$

Define a map $\nu_0\colon (X\times\{x_0\})\cup(\{x_0\}\times X)\to(\mathrm{Sym}^n X,[nx_0])$ by setting $\nu_0(x,x_0)=\nu_0(x_0,x)=[nx]$ $\forall x\in X$. By assumption, $x_0\in X$ is a homotopy unit for the n-valued multiplication $\mu_0\colon (X,x_0)\times(X,x_0)\to(\mathrm{Sym}^n X,[nx_0])$. Since the homotopy here is meant to fix x_0, this condition is equivalent to the equality $[\mu_0|_{(X\times\{x_0\})\cup(\{x_0\}\times X)}]^\bullet=[\nu_0]^\bullet$. Denote $(X\times\{x_i\})\cup(\{x_i\}\times X)$ by $X\vee_{x_i}X$, $i=0,1$.

Consider the restriction $\widetilde{\mu}_1|_{X\vee_{x_1}X}$. Since $g\times g(X\vee_{x_1}X)\subset X\vee_{x_0}X$, we have

$$[\widetilde{\mu}_1|_{X\vee_{x_1}X}]^\bullet=[\mathrm{Sym}^n f]^\bullet\circ[\mu_0|_{X\vee_{x_0}X}]^\bullet\circ[g\times g|_{X\vee_{x_1}X}]^\bullet=\\ [\mathrm{Sym}^n f]^\bullet\circ[\nu_0]^\bullet\circ[g\times g|_{X\vee_{x_1}X}]^\bullet=[\mathrm{Sym}^n f\circ\nu_0\circ(g\times g|_{X\vee_{x_1}X})]^\bullet.$$

It is easy to check that

$$(\mathrm{Sym}^n f\circ\nu_0\circ(g\times g|_{X\vee_{x_1}X}))(x,x_1)=(\mathrm{Sym}^n f\circ\nu_0\circ(g\times g|_{X\vee_{x_1}X}))(x_1,x)=\\ [n(f\circ g)(x)]\in\mathrm{Sym}^n X\quad\forall x\in X.$$

Since $[f\circ g]^\bullet=\mathrm{Id}_{(X,x_1)}$, the last two equalities imply that $[\widetilde{\mu}_1|_{X\vee_{x_1}X}]^\bullet=[\nu_1]^\bullet$, where the map $\nu_1\colon X\vee_{x_1}X\to(\mathrm{Sym}^n X,[nx_1])$ is given by $\nu_1(x,x_1)=\nu_1(x_1,x)=[nx]$ $\forall x\in X$. Thus, we have proved that $x_1\in X$ is a homotopy unit for the multiplication $\widetilde{\mu}_1\colon (X,x_1)\times(X,x_1)\to(\mathrm{Sym}^n X,[nx_1])$.

The pair $(X\times X,X\vee_{x_1}X)$ is the product of the closed Borsuk pair $(X,\{x_1\})$ with itself, and is therefore a closed Borsuk pair (see [9, p. 48]). Hence, the homotopy connecting $\widetilde{\mu}_1|_{X\vee_{x_1}X}$ with ν_1 can be extended from $X\vee_{x_1}X$ to the entire space $X\times X$. Restricting this homotopy to $t=1$, we have the desired n-valued multiplication $\mu_1\colon (X,x_1)\times(X,x_1)\to(\mathrm{Sym}^n X,[nx_1])$ satisfying the strong unit axiom $\mu_1(x,x_1)=\mu_1(x_1,x)=[nx]\forall x\in X$. Moreover, the map μ_1 is homotopic to $\widetilde{\mu}_1$ and, therefore, is freely homotopic to μ_0. The lemma is proved. ☐

Let us show that, for CW-complexes, the property of being an nH-space is invariant under retraction.

Proposition 2. *Let X be a connected CW-complex with an nH-space structure for some $n\geq 1$. Then any retract $Y\subset X$ also has an nH-space structure.*

Proof. Consider an arbitrary point $y_0\in Y$. By Lemma 2, there is an n-valued multiplication $\mu_X\colon X\times X\to\mathrm{Sym}^n X$ with (a strong) unit $y_0\in X$, $\mu_X(y_0,x)=\mu_X(x,y_0)=[nx]$ $\forall x\in X$. Let $r\colon X\to Y$ be a retraction, $r(y)=y$ $\forall y\in Y\subset X$. Define an n-valued multiplication $\mu_Y\colon Y\times Y\to\mathrm{Sym}^n Y$ by the formula

$$\mu_Y(y_1,y_2)=\mathrm{Sym}^n r\circ\mu_X(y_1,y_2)\quad\forall y_1,y_2\in Y.$$

The continuity of μ_Y is obvious. Moreover,

$$\mu_Y(y_0,y)=\mathrm{Sym}^n r\circ\mu_X(y_0,y)=\mathrm{Sym}^n r([ny])=[ny]\in\mathrm{Sym}^n Y\quad\forall y\in Y.$$

Similarly, $\mu_Y(y, y_0) = [ny] \in \operatorname{Sym}^n Y\ \forall y \in Y$. Thus, we have defined an n-valued multiplication $\mu_Y\colon Y \times Y \to \operatorname{Sym}^n Y$ with identity $y_0 \in Y$. The proposition is proved. □

Proposition 2 implies the following

Proposition 3. *Let X be a connected CW-complex having a retract $Y \subset X, r\colon X \to Y$ such that either Y belongs to the class $\mathcal{M}$ introduced above or Y satisfies the conditions of Theorem 4. Then X does not admit a 2-valued multiplication with identity.*

Acknowledgments

The author is deeply grateful to the Corresponding Member of the Russian Academy of Sciences Professor V. M. Buchstaber for posing this problem and for his interest in this work. The author is also grateful to the Lead Researcher A. A. Gaifullin and to the Senior Researcher S. A. Melikhov for valuable discussions.

References

[1] V. M. Buchstaber, *n-valued groups: theory and applications*, Mosc. Math. J. **6** (2006), no. 1, 57-84. MR2265947 (2007i:20102)

[2] V. M. Buchstaber and E. G. Rees, *Multivalued groups, their representations and Hopf algebras*, Transform. Groups **2** (1997), no. 4, 325-349. MR1486035 (99b:20071)

[3] V. M. Buchstaber, *Functional equations that are associated with addition theorems for elliptic functions, and two-valued algebraic groups*, Uspekhi Mat. Nauk **45** (1990), no. 3 (273), 185-186; English transl., Russian Math. Surveys **45** (1990), no. 3, 213-215. MR1071939 (91j:39002)

[4] D. V. Gugnin, *Topological applications of graded Frobenius n-homomorphisms*, Tr. Mosk. Mat. Obs. **72** (2011), no. 1, 127-188; English transl., Trans. Moscow Math. Soc. **72** (2011), no. 1, 97-142.

[5] D. V. Gugnin, *Polynomially dependent homomorphisms and Frobenius n-homomorphisms*, Tr. Mat. Inst. Steklova **266** (2009), Geometriya, Topologiya i Matematicheskaya Fizika. II, 64-96; English transl., Proc. Steklov Inst. Math. **266** (2009), no. 1, 59-90. MR2603261 (2011e:16077)

[6] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119-221. MR0102537 (21:1328)

[7] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Cambridge Philos. Soc. **58** (1962), 563-568. MR0143204 (26:764)

[8] I. G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319-343. MR0151460 (27:1445)

[9] M. M. Postnikov, *Lectures in algebraic topology. Elements of homotopy theory*, Nauka, Moscow, 1984. (Russian) MR776974 (87j:55002)

Chair of Higher Geometry and Topology, Department of Mechanics and Mathematics, Moscow State University, Moscow, Russia

E-mail address: dmitry-gugnin@yandex.ru

Translated by A. MARTSINKOVSKY