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ASYMPTOTICS OF MEIXNER POLYNOMIALS AND CHRISTOFFEL–DARBOUX KERNELS

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Dedicated to A. A. Gonchar on the occasion of his eightieth birthday

Abstract. We obtain the asymptotics of the classical Meixner polynomials (orthogonal with respect to a discrete measure supported at the nonnegative integer points) and the corresponding reproducing kernels (Christoffel–Darboux kernels) as the number n of the polynomial and the variable x tend to infinity under various relationships between their growth rates. (These asymptotics are known as the Plancherel–Rotach asymptotics.)

1. Introduction

We study the limit modes as $n \to \infty$ of the Christoffel–Darboux kernels (CD-kernels)

$$K_n(\tilde{x}, \tilde{y}) := \sqrt{W(\tilde{x})W(\tilde{y})} \sum_{i=0}^{n-1} \frac{M_i(\tilde{x})M_i(\tilde{y})}{\|M_i\|^2_{l_2,W}} \tag{1.1}$$

for the sequence of polynomials $\{M_n(\tilde{x})\}$ obtained by the orthogonalization of the sequence of powers $\{\tilde{x}^k\}$ in $l^2(\mathbb{Z}_+, W)$ with the Meixner weight [1]

$$W := W_{\beta,\sigma}(\tilde{x}) := \frac{\Gamma(\beta + \tilde{x})}{\tilde{x}!}\sigma^{\tilde{x}}, \qquad \tilde{x} \in \mathbb{Z}_+, \quad \beta > 0, \quad 0 < \sigma < 1. \tag{1.2}$$

The limit behavior of the CD-kernels (1.1) of orthogonal polynomials possesses universality properties for broad classes of weights. This fact, which permits one to describe fluctuations of stochastic processes from their mean values, is of interest in connection with the distributions of eigenvalues of random matrices and other similar problems (see [17], [19], [20], [18], and [21]). We point out that, while the universality of the limits of (1.1) is nowadays being proved in increasingly general classes for polynomials orthogonal with respect to continuous weights (see [22] and [23]), the corresponding "universality" conjectures have not essentially been stated yet for polynomials orthogonal with respect to discrete weights, and current research is directed at studying the limit modes of CD-kernels for specific systems of discrete orthogonal polynomials with subsequent attempts to reveal some patterns these systems have in common (see [24], [25], [26], and [27]).

The outline of the paper is as follows. The subsequent sections of the introduction present some well-known general facts concerning the applications of discrete orthogonal polynomials to stochastic processes (Section 1.1) and the asymptotics of such polynomials. (The "weak" asymptotics are considered in Section 1.2, and the "strong" asymptotics

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are considered in Section 1.3). Section 1.4, which is the last in the Introduction, deals with the strong asymptotic formulas recently obtained in [36] for the Meixner polynomials. In Section 2, we present and discuss the main results of the paper: the asymptotics of the CD-kernels of the Meixner polynomials for various limit modes. In the closing section, Section 3, we prove these asymptotics. The results of the present paper were partly announced in [38].

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1.1. **Point determinant stochastic processes.** To provide a link to the applications that motivated writing the present paper, let us give the main definitions related to stochastic processes.

A *point stochastic process* is a triple

$$(\mathcal{X}, \mathcal{A}, \mathbf{M}),$$

where $\mathcal{X}$ is a discrete sample space (for example, $\mathcal{X} = \mathbb{Z}_+^{\mathbb{N}}$),

$$\mathcal{A} := \{C(x_1, \dots, x_n)\}_{n\in\mathbb{N}}$$

is the σ-algebra of cylindrical subsets $C(x_1, \dots, x_n) := \big\{\{y_j\}_{j\in\mathbb{N}} \colon y_k = x_k,\ k = 1, \dots, n\big\}$ of $\mathcal{X}$, and $\mathbf{M}$ is a measure on $\mathcal{A}$ satisfying the matching condition

$$\sum_{x_{n+1}\in\mathbb{Z}_+} \mathbf{M}(C(x_1, \dots, x_n, x_{n+1})) = \mathbf{M}(C(x_1, \dots, x_n)).$$

Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of random variables with the following properties. The probability that ξ_1 takes the value x_1 is

$$\mathbb{P}\{\xi_1 = x_1\} = \mathbf{M}(C(x_1));$$

next, for the vector random variable (ξ_1, ξ_2),

$$\mathbb{P}\{\xi_1 = x_1, \xi_2 = x_2\} = \mathbf{M}(C(x_1, x_2)),$$

etc. Thus, $\xi_1, \xi_2, \dots, \xi_n, \dots$ form a process *discrete* in time ($t = n$) and discrete, or *point*, in space ($x_j \in \mathbb{Z}_+$).

Next, the *n-correlation function* of m distinct elements of $\mathbb{Z}_+$ is defined as the probability of the event that these elements are contained in the set of coordinates of a random n-vector,

$$\varrho^{(n)}(x_1, \dots, x_m) := \begin{cases} 0 & \text{if } \exists x_i = x_j,\ i \neq j, \\ \mathbb{P}\{(\xi_1, \dots, \xi_n) \supseteq (x_1, \dots, x_m)\} & \text{otherwise.} \end{cases}$$

A process $(\mathbb{Z}_+^{\mathbb{N}}, \mathcal{A}, \mathbf{M})$ is called a *determinant* process if there exists a kernel $K_n(x, y)$ such that

$$\varrho^{(n)}(x_1, \dots, x_n) = \det[K_n(x_i, x_j)]_{i,j=1}^n.$$

Finally, $(\mathbb{Z}_+^{\mathbb{N}}, \mathcal{A}, \mathbf{M})$ is called an *orthogonal polynomial process* (an OP-*process*) *with weight* $W\big|_{\mathbb{Z}_+}$ if

$$\mathbf{M}(C(x_1, \dots, x_n)) = \frac{1}{Z_n} \prod_{i=1}^n W(x_i) \prod_{1\le i<j\le n} (x_i - x_j)^2, \qquad x_1 > \dots > x_n. \tag{1.3}$$

It is well known (e.g., see [20]) that OP-processes are determinant processes whose kernel is the CD-kernel (1.1) of the polynomials $\{P_n(\tilde{x})\}$ orthogonal with weight W.

Meixner processes, i.e., OP-processes with Meixner weight (1.2), have broad applications. The Meixner weight arises in connection with some degenerations of Z-measures on Young diagrams, which, in turn, arise when dealing with problems of harmonic analysis on infinite symmetric groups. (See [24], [25], and [26] for more details.) Another interesting model of the Meixner processes occurs in percolation problems. We present this model following Johansson [26]. Let $\omega(i,j)$ be independent random variables defined at the points $(i,j) \in \mathbb{Z}_+^2$ of the two-dimensional lattice with probability distributions

$$\mathbb{P}\{\omega(i,j) = k\} = (1-q)q^k, \qquad k \in \mathbb{N}, \quad q \in (0;1).$$

Consider the set of paths $\Pi_{l,n} := \{\pi\colon (1,1) \to (l,n)\}$ with nondecreasing coordinates and define the random variable

$$G(l,n) := \max_{\pi \in \Pi_{l,n}} \sum_{(i,j) \in \pi} \omega(i,j).$$

It turns out (see [26]) that the distribution of this variable obeys the Meixner process (1.3)–(1.2) with $\sigma = q$ and $\beta = l - n + 1$:

$$\mathbb{P}\{G(l,n) \le t\} = \frac{1}{Z_n} \sum_{\max\{x_i\} \le t+n-l} \prod_{i=1}^{n} W_{\beta,\sigma}(x_i) \prod_{1 \le i < j \le n} (x_i - x_j)^2.$$

The leading term of the asymptotics (as $n \to \infty$) of the Meixner polynomials with weight W given by (1.2), which is "variable" (i.e., depends on n) for $l - n \neq \text{const}$, was found in [26] by applying the mountain pass method to an integral representation. This permitted determining the limit mean values of this random variable for all $q \in (0,1)$ and $\gamma \ge 1$:

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\{G([\gamma n], n)\} = \frac{(1+\sqrt{q\gamma})^2}{1-q} =: \Omega(\gamma, q).$$

A finer analysis of the limit CD-kernel in a neighborhood of the maximum zeros of these polynomials gives the well-known Tracy–Widom distribution $F(s)$ for the fluctuations of this process,

$$\lim_{n\to\infty} \mathbb{P}\left\{ \frac{G([\gamma n], n) - n\Omega(\gamma, q)}{\alpha(\gamma, q) n^{1/3}} \le s \right\} = F(s) := \exp\left[-\int_s^\infty (x-s) u(x)^2 \, dx \right],$$

where $u(x)$ is the solution (with the Airy asymptotics as $x \to \infty$) of the Painlevé II equation

$$u'' = 2u^3 + xu$$

and

$$\alpha := \frac{q^{1/6}\gamma^{-1/6}}{1-q} (\sqrt{\gamma} + \sqrt{q})^{2/3} (1 + \sqrt{q\gamma})^{2/3}.$$

(See [26] for more details.) The exponent 1/3 in the fluctuation scale is of universal character.

1.2. **Distribution of zeros of discrete orthogonal polynomials and the exponent in the leading term of the asymptotics.** The limit behavior of point ensembles of orthogonal polynomials is essentially determined by the limit behavior of zeros of the corresponding sequences of orthogonal polynomials. The zeros of orthogonal polynomials lie in the convex hull of the support of their orthogonality measure. The distribution of zeros of polynomials orthogonal with respect to discrete measures has the specific feature that an orthogonal polynomial can have at most one zero between two neighboring masses of the orthogonality measure. Hence, for a given power n of a discrete orthogonal polynomial $P_n(\tilde{x})$, one can coarsely single out ranges of the variable $\tilde{x}$ in which

- There are no zeros of the polynomial (the *free* region).
- There are zeros of the polynomial in some but not all intervals between neighboring masses of the measure (the *equilibrium* region).
- There is a zero of the polynomial in each interval between neighboring masses of the measure (the *saturation* region).

Accordingly, as n varies, one can single out such regions in the $(\tilde{x}, n)$-plane.

The logarithmic potential $\mathcal{P}^{\nu}(z) = -\int \ln|z-t|\,d\nu(t)$ of the unit measure $\nu_{Q_n}(\tilde{x})$ uniformly distributed on the set of zeros of the polynomial P_n (the *zero counting measure*) is obviously given by

$$\mathcal{P}^{\nu_{P_n}} = -\frac{1}{n}\ln|P_n|.$$

Consequently, the potential of the limit zero counting measure of orthogonal polynomials as $n \to \infty$ determines the *exponent of the leading term* of their asymptotics. To describe the limit measure, it is convenient to use minimization problems for the energy of the logarithmic potential. Usually, one scales the problem, $\tilde{x} \to \tilde{t}$: $\tilde{x} = \tilde{x}(\tilde{t}, n)$ (for example, $\tilde{x} = \tilde{t}n$), to ensure that the support of the limit measure is compact. Then the new orthogonality weight depends on n,

$$W_n(\tilde{t}) = W(\tilde{x}(\tilde{t}, n)) =: \exp\{-2nV_1(\tilde{t}) + V_0(\tilde{t}) + \cdots\},$$

and accordingly, the same is true for its discrete support $\{\tilde{t}_{k,n}\} : \tilde{x}_k = \tilde{x}_k(\tilde{t}_{k,n}, n)$, $k \in \mathbb{Z}_+$, for which one can introduce the point counting measure

$$\tau_n(\tilde{t}) := \frac{1}{n}\sum_k \delta(\tilde{t} - \tilde{t}_{k,n})$$

and its weak limit

$$\tau(\tilde{t}) = \lim_n \tau_n(\tilde{t}).$$

Then the limit measure $\lambda(\tilde{t}) = \lim_n \nu_{Q_n}(\tilde{x}(\tilde{t}, n))$ minimizes the energy functional

$$\mathcal{E} := \int (\mathcal{P}^{\mu}(t) + 2V_1(t))\,d\mu(t) \tag{1.4}$$

with external field V_1 in the class of measures μ satisfying

$$|\mu| = 1, \qquad \mu \le \tau. \tag{1.5}$$

The constraint that μ is bounded by τ is due to the above-mentioned specific feature of discrete orthogonality.

The potential of the limit measure λ satisfies the conditions of equilibrium in the external field:

$$\mathcal{P}^{\lambda}(\tilde{t}) + V_1(\tilde{t}) \begin{cases} \le \kappa, & \tilde{t} \in \operatorname{supp}\lambda, \\ = \kappa, & \tilde{t} \in \operatorname{supp}(\tau - \lambda) \cap \operatorname{supp}(\lambda) := \Sigma, \\ \ge \kappa, & \tilde{t} \in \operatorname{supp}(\tau - \lambda). \end{cases} \tag{1.6}$$

Techniques related to the equilibrium of the logarithmic potential in an external field have been used in various fields of mathematics since Gauss. Presently, these techniques have been perfected in detail by Gonchar (see [2] and [3]) and Gonchar together with his students (see [4], [5], [6], [7], [8], [9], and [10]) in connection with problems of rational approximation with free poles. Rakhmanov [11] developed this approach for weak asymptotics of polynomials orthogonal with respect to discrete measures by suggesting describing the limit measure of the distribution of zeros of such discrete orthogonal polynomials in the form of the solution of the minimization problem for the energy of the logarithmic potential in the class of measures bounded by the counting measure of the support of the orthogonality measure. Later, this approach was extended in [12] by including an external field. (Equilibrium problems in external fields and their generalizations are comprehensively studied in the monographs [13] and [14].)

Let us illustrate the notions introduced above by using the Meixner polynomials as an example (see also [15] and [16]). The scaling

$$\tilde{x} =: n\tilde{t}$$

of the variable $\tilde{x}$ in (1.2) results in the stabilization of zeros (they do not leave some compact set as $n \to \infty$) of the polynomial

$$\widetilde{P}_n(\tilde{t}) := P_n(\tilde{x}),$$

which, treated as a polynomial in $\tilde{t}$, is orthogonal to the powers $\{\tilde{t}^k\}_{k=0}^{n-1}$ with respect to the weight

$$\widetilde{W}_n(\tilde{t}) := \frac{\Gamma(\beta + \tilde{t}n)}{(\tilde{t}n)!}\sigma^{\tilde{t}n},$$

depending on n and concentrated at the points $\left\{\frac{k}{n}\right\}_{k\in\mathbb{Z}_+}$. Since

$$\lim_{n\to\infty} |\widetilde{W}_n(\tilde{t})|^{1/n} = \sigma^{\tilde{t}} = \exp\left\{-\tilde{t}\ln\frac{1}{\sigma}\right\},$$

it follows that the external field for the extremal problem (1.4)–(1.5) and the equilibrium problem (1.6) has the form

$$V_1(\tilde{t}) = \ln\frac{1}{\sigma}\operatorname{Re}\tilde{t}. \tag{1.7}$$

It is easily seen that

$$\frac{1}{n}\sum_{k=0}^{\infty}\delta\left(\tilde{t} - \frac{k}{n}\right) \xrightarrow{*} \operatorname{mes}(\tilde{t}) =: \tau(\tilde{t}), \qquad \tilde{t} \in \mathbb{R}_+, \tag{1.8}$$

where $\operatorname{mes}(\tilde{t})$ is the Lebesgue measure with unit density ($d\tilde{t}$) on the half-line.

Thus, the limit measure $\lambda(\tilde{t})$ of the distribution of zeros of the polynomial $\widetilde{P}_n(\tilde{t})$,

$$\nu_{\widetilde{P}_n(\tilde{t})} \xrightarrow{*} \lambda(\tilde{t}), \tag{1.9}$$

satisfies the extremal problem (1.4)–(1.5) and the equilibrium conditions (1.6) with the external field (1.7) and the constraint (1.8).

Let us find the measure $\lambda(\tilde{t})$ in closed form. Usually, one solves equilibrium problems with an external field by finding the Cauchy transform

$$h(\tilde{z}) := \int \frac{d\lambda(\tilde{t})}{\tilde{z} - \tilde{t}}, \qquad \tilde{z} \in \overline{\mathbb{C}} \setminus \operatorname{supp}\lambda, \tag{1.10}$$

of the measure λ in the form of an analytic function on some two-sheeted Riemann surface $\mathcal{R} := (\mathcal{R}^{(0)}, \mathcal{R}^{(1)})$, $\pi(\mathcal{R}^{(j)}) = \overline{\mathbb{C}}$, $j = 1, 2$. The analytic continuation of the function $h(\tilde{z})$ to the other sheet of the Riemann surface through the compact set Σ is possible, which can be seen by applying the differentiation ∂_z to the second relation in (1.6). Thus, the function $h := (h_0, h_1)$ on $\mathcal{R}$ is subjected to some natural conditions on the values taken at the point ∞. For the principal branch h_0, in view of (1.10) (and under the normalization $|\lambda| = 1$ of the measure), one has

$$h(\tilde{z}) := \frac{1}{\tilde{z}} + \cdots, \qquad \tilde{z} \to \infty^{(0)}, \tag{1.11}$$

while for its continuation h_1 (with the derivative ∂_z applied to the field (1.7) taken into account) one has

$$h(\tilde{z}) := \ln\frac{1}{\sigma} - \frac{1}{\tilde{z}} + \cdots, \qquad \tilde{z} \to \infty^{(1)}. \tag{1.12}$$

The density $\lambda'(\tilde{t})$ of the desired measure is proportional to the jump of the imaginary part of the function $h(\tilde{z})$ along $\mathbb{R}_+$. Moreover (as is clear from electrostatic considerations for problem (1.4)–(1.5)), this jump should be constant in a neighborhood of the left endpoint of $\operatorname{supp}\lambda$ (i.e., to the right of zero), because here the measure λ attains the constraining Lebesgue measure $\tau(\tilde{t})$. This behavior can be modeled if $h(\tilde{z})$ is the logarithm of some function $E(\tilde z)$ taking negative (real) values on that part of the support of λ (in the saturation region). Thus, we seek $h(\tilde z)$ in the form

$$h(\tilde z) = \ln E(\tilde z), \tag{1.13}$$

where $E(\tilde z)$ is a rational function on $\mathcal{R}$ with one zero and one pole (at the points over the point 0 on distinct sheets of $\mathcal{R}$),

$$E(\tilde z) = \begin{cases} 0, & \tilde z = 0^{(0)}, \\ \infty, & \tilde z = 0^{(1)}, \end{cases} \tag{1.14}$$

whose position has been chosen to ensure that the branches of h (see (1.13)) be holomorphic in $\mathbb{C}\backslash\mathbb{R}_+$. Conditions (1.11) and (1.12) give the following behavior of $E(\tilde z)$ in a neighborhood of infinity on both sheets:

$$E(\tilde z) = \begin{cases} 1 + \frac{1}{\tilde z}, & \tilde z \to \infty^{(0)}, \\ \frac{1}{\sigma}\left(1 - \frac{1}{\tilde z}\right), & \tilde z \to \infty^{(1)}. \end{cases} \tag{1.15}$$

It turns out that conditions (1.14) and (1.15) are sufficient for determining $E(\tilde z)$ and the equation of its Riemann surface. Indeed, Viète's relations give

$$E^2 + \left[\frac{1}{\tilde z}\left(\frac{1}{\sigma} - 1\right) - \left(\frac{1}{\sigma} + 1\right)\right] E + \frac{1}{\sigma} = 0. \tag{1.16}$$

It follows that the Riemann surface of $E(\tilde z)$ has the branching points

$$e_1 = \frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}, \qquad e_2 = \frac{1}{e_1}.$$

Moreover, both branches of $E(\tilde z)$ are negative on $[0, e_1]$ and positive on $[e_2, \infty]$. Hence the jump of the imaginary part of $h(\tilde z)$ is constant on $[0, e_1]$ and zero on $[e_2, \infty]$. Consequently, $\operatorname{supp}\lambda = [0, e_2]$, $[0, e_1]$ is a saturation region, $[e_1, e_2] = \Sigma$ is an equilibrium region, and $[e_2, \infty]$ is a free region. The resulting measure λ satisfies the equilibrium equation (1.6), and after integration, as a by-product, we can obtain the equilibrium constant in (1.6) in closed form,

$$\kappa = 1 + \ln\frac{1-\sigma}{\sigma}.$$

Let us write the function h in closed form. In what follows, it is convenient to pass to the variable x:

$$\tilde x \to x := (c-1)\tilde x - \beta, \qquad c := \frac{1}{\sigma}.$$

Since $nz = (c-1)n\tilde z - \beta$ after scaling, it follows that $z = (c-1)\tilde z$, and for the solution of (1.16) in the new variables we have

$$E = \frac{(c-1)^2 - (c+1)z \pm (c-1)\sqrt{(z-c-1)^2 - 4c}}{2z}.$$

From this (see (1.13)), we obtain

$$h(\tilde z(z)) = \ln\sqrt{c} \pm \cosh^{-1}\frac{(c-1)^2 - (c+1)z}{2z\sqrt{c}}. \tag{1.17}$$

1.3. **Global asymptotics of Plancherel–Rotach type of solutions of recursion relations.** Modern applications such as describing eigenvalue distributions for random matrices or obtaining limit CD-kernels in closed form convenient for subsequent analysis necessitate knowing not only the leading term of the asymptotics but also the subsequent terms whereby one speaks of the *strong asymptotics*. Moreover, one needs the asymptotics of $Q_n(x)$ as the number n and the variable x tend to infinity under various relationships between their growth rates. Such asymptotics are referred to as *asymptotics of Plancherel–Rotach type* (e.g., see [20]). They originally arose (see [30]) in the asymptotic description of the Hermite polynomials H_n as $n \to \infty$ and

$$\begin{aligned} &\text{(a)}\quad x = (2n+1)^{1/2}\tau, && 1+\varepsilon \le \tau \le C,\\ &\text{(b)}\quad x = (2n+1)^{1/2} - n^{-1/6}\, t, && t \in K \Subset \mathbb{C},\\ &\text{(c)}\quad x = (2n+1)^{1/2}\theta, && -1+\varepsilon \le \theta \le 1-\varepsilon, \end{aligned} \tag{1.18}$$

for given positive ε and C and complex t. Recall that the Hermite polynomials can be defined by the recursion relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad H_0 = 1, \quad H_{-1} = 0, \quad n \in \mathbb{N}.$$

The asymptotics of the Hermite polynomials in the regions (1.18) of the (x, n)-plane were obtained by Plancherel and Rotach by the mountain pass method from integral representations of $H_n(x)$. We point out that the regions (1.18) in the classical result due to Plancherel and Rotach do not overlap.

The *matrix Riemann–Hilbert problem method* is a universal, widespread method for finding the asymptotics of orthogonal polynomials (including asymptotics of Plancherel–Rotach type). The main advantage of the method is that it provides asymptotic expansions of an orthogonal polynomial in domains covering the (x, n)-plane. Thus, one can speak of a *global asymptotic representation*. The method is based on a restatement (originally suggested in [31]) of the definition of orthogonal polynomials in the form of a boundary value problem for a matrix analytic function with a jump depending on the orthogonality weight W; for this problem, one finds the solution for large n. (This was first shown for analytic weights W in [29].) Later, the method was extended to nonanalytic weights (see [32]) and even to discrete orthogonality measures (see [27]). In the latter case, the original matrix boundary value problem in [31] is replaced by some interpolation problem suggested in [33]. However, one should admit that the matrix Riemann–Hilbert problem method has not become widespread in the case of nonanalytic and discrete weights. (Recently there have been new results in this direction; see [34] and [35].)

The present paper uses global asymptotics obtained for the Meixner polynomials in a different way. Namely, one uses a general method for constructing the asymptotic expansions of *basis solutions* of the recursion relations

$$Q_{n+1}(x) = \sum_{j=0}^{p-1} a_j(x, n) Q_{n-j}(x), \qquad n \in \mathbb{N}, \tag{1.19}$$

in various overlapping (extending to infinity) regions covering the (x, n)-plane; then the particular solution to be studied is expanded in the basis solutions in one of the regions, and the corresponding basis expansions in the neighboring regions are matched with this particular solution. This method was suggested in [36] and justified for a sufficiently general class of recursions (1.19) with coefficients $\{a_j(x, n)\}_{j=0}^{p-1}$ polynomial in x and rational in n.

Let us briefly explain what the method is about. The recursion relations (1.19) are written out in the matrix form

$$F_{n+1} = A_n(x)F_n, \qquad n \in \mathbb{N}, \tag{1.20}$$

where F_n is the vector $(Q_n, \dots, Q_{n-p+1})^T$ of successive solutions of (1.19) and $A_n(x)$ is the $p \times p$ matrix of appropriately arranged coefficients $\{a_j(x,n)\}$. To construct basis solutions of (1.20), one seeks "diagonalizing" transformations V_n such that the new transition matrix $(V_{n+1}^{-1}A_nV_n)$,

$$U_{n+1} = (V_{n+1}^{-1}A_nV_n)U_n, \qquad F_n = V_nU_n,$$

is in a sense close to a diagonal matrix. The procedures for constructing V_n depend on the values of the parameters (x,n).

The (x,n)-plane is divided into two regions: the *region of good separation of eigenvalues* of $A_n(x)$, which comprises the domains that are connected components of the set

$$\Omega := \{(x,n) \in \mathbb{C}^2 \colon |n| > C, \quad |\mathcal{D}(x,n)| > 1\}, \tag{1.21}$$

where $\mathcal{D}(x,n)$ is the discriminant of the characteristic equation of the matrix $A_n(x)$ and C is a given constant, and the *transient region*

$$\Xi^c := \left\{(x,n) \colon \max_{i \neq j} \frac{|\lambda_i| + |\lambda_j|}{|\lambda_i - \lambda_j|} \geq |n|^\alpha, \quad 1 \leq i,j \leq k\right\}, \tag{1.22}$$

in whose connected components some eigenvalues $\{\lambda_j\}$ of the matrix $A_n(x)$ are close to each other. It is important that, by appropriately choosing the constant $\alpha > 0$, one can ensure that a neighborhood of infinity in $\mathbb{C}^2$ is completely covered:

$$\Omega \cup \Xi^c \supset (\mathbb{C} \setminus O_{R})^2. \tag{1.23}$$

Under fairly general conditions (see [36] for details), one can construct a diagonalizing transformation in Ω by an iterative procedure starting from the eigenvectors of $A_n(x)$. For the basis solutions of the recursion relations (1.19)–(1.20), this permits one to construct formal series expansions majorized by asymptotic series that can be truncated at an arbitrary term with the following remainder estimate:

$$\Pi_j(x,n) = \pi_j(x,n)\Bigl(1 + \sum_{m=1}^{s} \psi_m^{(j)}(x,n) + o\bigl(\widetilde{\varphi}(x,n)\varphi^s(x,n)\bigr)\Bigr), \qquad j = 1, \dots, p. \tag{1.24}$$

The majorizing estimate has the form $\psi_m^{(j)} = O(\widetilde{\varphi}\varphi^m)$. The paper [36] also provides a description of the procedure for finding the terms of the series (1.24), estimates of these terms, and estimates of how close the partial sums of the series (1.24) are to the corresponding basis solution. (Here φ depends on the choice of the region in the (x,n)-space.)

At the same time, the construction of expansions of the basis solutions in the transient regions Ξ^c is rather special. Here the difference problem with respect to the variable n is transformed (see [36] for details) into a differential problem with respect to a new variable $z := z(x,n)$ related to the scale of the transient region. The subsequent analysis depends on the type of the differential problem obtained. The paper [36] gives a more detailed consideration only for the cases in which $p = 2$ and the differential problem can be reduced to the Airy equation. Then the recursion relation (1.19) in the transient regions has two basis solutions of the form

$$\exp(E(z,x))\,\mathrm{Ai}(h(z,x)) \quad \text{and} \quad \exp(E(z,x))\,\mathrm{Bi}(h(z,x)), \tag{1.25}$$

where Ai and Bi are the Airy functions (linearly independent solutions of the Airy equation; see [37] for the definition). The paper [36] describes a procedure for finding closed-form expressions for the terms of the series

$$E(z,x) = E_0(z,x)\biggl(1+\sum_{m=1}^{\infty}\gamma_m(z,x)\biggr), \qquad h(z,x)=\sum_{m=0}^{\infty}\eta_m(z,x), \tag{1.26}$$

which are majorized by asymptotic series in the same sense as in (1.24). However, φ and $\widetilde{\varphi}$ in the remainder in (1.24) have the simpler form $|z|^{r_1}|x|^{r_2}$ in these cases, with r_1 and r_2 for φ and $\widetilde{\varphi}$ depending only on the problem.

Recall that the ultimate goal of the method is to find the global asymptotics of a particular solution of the recursion relations (1.19)–(1.20), for example, a sequence of orthogonal polynomials. To achieve this, one should first find the expansion of the desired particular solution in the basis solutions in some part of the (x,n)-plane. To this end, along with the asymptotics of the coefficients of the recursion relations, one should know some additional information on the particular solution. For example, if the particular solution is known to be polynomials and if several leading coefficients of these polynomials are known (this kind of additional information can be obtained by several summation operations and hence may be available for orthogonal polynomials), then the expansion of the particular solution can be found in the subdomain $\{(x,n)\colon x\gg n\}\subset\Omega$. To continue the asymptotics of the particular solution into other subdomains of (x,n), one should match the solutions constructed in various regions of the (x,n)-space. This is possible because both representations (1.24) and (1.25) of basis solutions hold simultaneously in the intersections of the regions (1.23) and hence there exists a function

$$K(x)=\sum_{m=0}^{\infty}\kappa_m(x), \tag{1.27}$$

independent of n (in view of the homogeneity of (1.20) in F) but depending on x, such that the multiplication by $K(x)$ of one of the solutions not only forces the two solutions to have the same growth but also ensures that the respective subsequent terms in their asymptotic series coincide. Being successful in finding such a function (in examples where this method is applied) may serve as an additional confirmation that the expansions obtained for the solutions of (1.20) in various regions of the (x,n)-plane are correct.

1.4. **Global asymptotics of Meixner polynomials for large n and x.** Let us make a change of variable and introduce a different notation for the parameters of the Meixner polynomials orthogonal with respect to the weight (1.2):

$$c:=1/\sigma, \qquad b:=\beta-1, \qquad x:=\Bigl(\frac{1}{\sigma}-1\Bigr)\tilde{x}-\beta. \tag{1.28}$$

We do so to make the recursion relations for the Meixner polynomials $Q_n(x)$ as simple as possible. In the new notation, we have

$$\begin{pmatrix}Q_{n+1}\\ Q_n\end{pmatrix}=\begin{pmatrix}x-(c+1)n & -cn(n+b)\\ 1 & 0\end{pmatrix}\begin{pmatrix}Q_n\\ Q_{n-1}\end{pmatrix}, \qquad Q_n(x)=x^n+\cdots. \tag{1.29}$$

We also introduce the notation

$$s:=\sqrt{c}, \qquad D:=(x-(c+1)n)^2-4cn(n+b). \tag{1.30}$$

An analysis of the eigenvalues of the transition matrix in (1.29) permits one to give an accurate qualitative description of regions on the (x,n)-plane with various asymptotic behavior of the orthogonal polynomial $Q_n(x)$. Let $\varepsilon>0$ be given, and let $0\le\operatorname{Im}(x)\lesssim|x|^{1/3+\varepsilon}$. (We use the $\lesssim$ symbol to indicate bounds on the order of growth.) Then we obtain the following results for the Meixner polynomials:

I. A free region for $n < \frac{|x|}{(s+1)^2} - |x|^{1/3+\varepsilon}$.

III. An equilibrium region for $\frac{|x|}{(s+1)^2} + |x|^{1/3+\varepsilon} < n < \frac{|x|}{(s-1)^2} - |x|^{1/3+\varepsilon}$.

V. A saturation region for $n > \frac{|x|}{(s-1)^2} + |x|^{1/3+\varepsilon}$.

These regions are formed by the intersection of the *region of good separation of eigenvalues* (see (1.21)) of the transition matrix in (1.29) with a neighborhood of the real axis. One also singles out *transient regions* (see (1.22)), where the corresponding eigenvalues are close to each other and where the asymptotic behavior of the polynomial changes:

II. Transient (I $\leftrightarrow$ III) region for $n = \frac{x}{(s+1)^2} + zx^{1/3}$, $|z| \lesssim |x|^{2/3-\varepsilon}$.

IV. Transient (III $\leftrightarrow$ V) region for $n = \frac{x}{(s-1)^2} + zx^{1/3}$, $|z| \lesssim |x|^{2/3-\varepsilon}$.

We see that these regions overlap to cover the (x, n)-plane.

The asymptotic expansions in these regions of both basis solutions of the second-order recursion relation for the Meixner polynomials were found in [36]. In regions I, III, and V, the leading terms of the asymptotic expansions (of the basis solutions $\Pi_1(x, n)$ and $\Pi_2(x, n)$) can be written in closed form via algebraic functions. Here

$$\Pi_m(x, n) = \pi_m(x, n)(1 + \Upsilon_m(x, n)), \qquad m = 1, 2, \tag{1.31}$$

where

$$\pi_1 = \left(\frac{x - (c+1)n + \sqrt{D}}{2}\right)^{n-1/2} \left[\frac{2c(x+b)}{c-1} - x - (c-1)n - \sqrt{D}\right]^{\frac{2x+(b+1)(c+1)}{2(c-1)}} \times \frac{[2bc(n+b) + (x - (c+1)n + \sqrt{D})(x + (c+1)b)]^{b/2}}{\exp(n)\sqrt[4]{D}} \tag{1.32}$$

and the expression for π_2 can be obtained from (1.32) by changing the sign of $\sqrt{D}$. Moreover, the branches of multivalued functions in the expressions for π_1 and π_2 are chosen in such a way that all powers and roots are arithmetic for real $x \gg n \gg 1$. The estimates for $\Upsilon_{1,2}$ have the form

$$\Upsilon_1(x, n) = O\left(\max\left(\frac{|x|^2}{n^3}, \frac{|x|}{|n - n_1(x)|^2}, \frac{|x|}{|n - n_2(x)|^2}\right)\right),$$
$$\Upsilon_2(x, n) = O\left(\max\left(\frac{1}{|x|}, \frac{|x|}{n - n_1(x)|^2}, \frac{|x|}{|n - n_2(x)|^2}\right)\right).$$

Here $n_1(x)$ and $n_2(x)$ stand for the roots of D and have the asymptotics $\frac{x}{(\sqrt{c}\pm 1)^2} + O(1)$.

In transient regions II and IV, the strong asymptotics of the basis solutions can be expressed by two independent solutions of the Airy equation (see (1.25)–(1.26)) with the use of the asymptotic expansions

$$E_1(z) = \frac{(s+1)^2 z^2}{2x^{1/3}} + \frac{\left(\frac{b(s+1)^2}{2} + \frac{(s+1)^2(s^2 - 10s + 1)}{15s}\right)z}{x^{2/3}} + \cdots + O\left(\frac{|z|_+^{(k+3)/2}}{|x|^{k/3}}\right),$$
$$h_1(z) = -\sqrt[3]{\frac{(s+1)^4}{s}}\left(z + \frac{sb - (s^2+s+1)}{(s+1)^2 x^{1/3}} + \cdots\right) + O\left(\frac{|z|_+^{(k+2)/2}}{|x|^{k/3}}\right), \tag{1.33}$$

(1.34)

$$E_2(z) = \frac{(s-1)^2 z^2}{2x^{1/3}} + \frac{\left(\frac{b(s-1)^2}{2} - \frac{(s-1)^2(s^2+10s+1)}{15s}\right)z}{x^{2/3}} + \cdots + O\left(\frac{|z|_+^{(k+3)/2}}{|x|^{k/3}}\right),$$

$$h_2(z) = \sqrt[3]{\frac{(s-1)^4}{s}}\left(z - \frac{sb + (s^2-s+1)}{(s-1)^2 x^{1/3}} + \cdots\right) + O\left(\frac{|z|_+^{(k+2)/2}}{|x|^{k/3}}\right).$$

The series in (1.33) and (1.34) are majorized by the asymptotic series as $n, x \to \infty$.

The asymptotic expansion of the specific particular solution (of the recursion relations) corresponding to the Meixner polynomials $Q_{n-1}(x)$ is first found in region I for $x \gg n$, where the desired particular solution can be expressed via the basis solutions $\Pi_1(x,n)$ and $\Pi_2(x,n)$ with the use of known closed-form expressions for the leading coefficient of the Meixner polynomial. As a result, the orthogonal polynomial has the asymptotic expansion

$$Q_{n-1}(x) = K_0(x)\Pi_1(x,n)$$

in this region; the second basis solution $\Pi_2(x,n)$ is exponentially small in this region compared with the first. For the function that provides the match with the Meixner polynomial, one obtains the expansion

$$\ln K_0(x) := -\frac{2x + (b+1)(c+1)}{2(c-1)} \ln\left(\frac{2x}{c-1}\right) - \frac{b}{2}\ln(2x^2) - \frac{bc}{c-1} + O(x^{-1}). \tag{1.35}$$

Further, the expansions of the particular solution in regions II, III, IV, and V are successively found by matching the basis expansions with the asymptotics already obtained in the preceding regions. As a result, the asymptotics in the first transient region II has the form

$$Q_{n-1}(x) = \frac{K_0(x)}{K_1(x)} \exp(E_1(z)) \operatorname{Ai}\big(h_1(z)\big) \tag{1.36}$$

with the function

$$K_1(x) := \sqrt{\frac{2x}{\pi}} \left[\frac{c-1}{2sx}\right]^{\frac{2x+(b+1)(c+1)}{2(c-1)}} \left[\frac{(s+1)^2}{sx^2}\right]^{\frac{b-1}{2}} e^{\frac{x}{(s+1)^2} - \frac{bc}{c-1}} \sqrt[3]{\frac{sx}{s+1}} e^{O(x^{-1})}$$

providing the match with the asymptotics in region I. In region III, one obtains the oscillatory asymptotics

$$Q_{n-1}(x) = K_0(x)(\Pi_1(x,n) + i\Pi_2(x,n)); \tag{1.37}$$

here the two basis solutions are equal in absolute value, and hence the asymptotics has the form of a cosine with a variable phase. In the second transient region IV, the asymptotics has the form

(1.38)

$$Q_{n-1}(x) = K_0(x)\exp(E_2(z))\left(\frac{\operatorname{Ai}(h_2(z)) - i\operatorname{Bi}(h_2(z))}{2\overline{K}_2(x)} + \frac{\operatorname{Ai}(h_2(z)) + i\operatorname{Bi}(h_2(z))}{2K_2(x)}\right).$$

Here the basis solutions are matched with the oscillatory asymptotics with the use of the functions

$$\begin{aligned} K_2(x) &:= i^{b-1-\frac{2x+(b+1)(c+1)}{c-1}} K(x), \qquad \overline{K}_2(x) := (-i)^{b-1-\frac{2x+(b+1)(c+1)}{c-1}} K(x), \\ K(x) &:= \sqrt{\frac{2x}{\pi}} \left[\frac{c-1}{2sx}\right]^{\frac{2x+(b+1)(c+1)}{2(c-1)}} \left[\frac{(s-1)^2}{sx^2}\right]^{\frac{b-1}{2}} e^{\frac{x}{(s-1)^2} - \frac{bc}{c-1}} \sqrt[3]{\frac{sx}{s-1}} e^{O(x^{-1})}. \end{aligned} \tag{1.39}$$

Finally, the most interesting result emerges in saturation region V. Here the asymptotics has the form

$$Q_{n-1}(x) = K_0(x)\bigg(\Big(1 - \frac{\overline{K}_2(x)}{K_2(x)}\Big)\Pi_1(x,n) + i\Pi_2(x,n)\bigg). \tag{1.40}$$

The basis solution $\Pi_1(x,n)$ grows exponentially more rapidly than $\Pi_2(x,n)$ but is multiplied by a coefficient obtained by matching with the particular solution in region IV. This coefficient has the following special form for $\mathrm{Im}(x) = 0$:

$$\frac{\overline{K}_2}{K_2} = \exp\Big\{ i\pi \frac{x+b+c}{c-1} \Big\} + \varepsilon(x), \tag{1.41}$$

where the remainder $\varepsilon(x)$ decays more rapidly than any power of $|x|$ as $|x| \to \infty$. (This is the quotient of two asymptotic series in powers of x^{-1}, and hence, even though terms of all orders cancel out, we cannot claim that we have obtained an exact value and should add a remainder term.)

Hence the coefficient of the dominating term Π_1 vanishes for

$$x \approx k(c-1) - b - c, \qquad k \in \mathbb{N}, \quad k \gg 0, \tag{1.42}$$

and the polynomials become small in neighborhoods of these points (for any n), which expresses the fact that the zeros of orthogonal polynomials in this region are attracted at an exponential rate (as the ratio Π_2/Π_1) to the points on the right-hand side in (1.42), where the masses of the orthogonality measure are concentrated. These mass points themselves are found from the recursion relations with accuracy $\varepsilon(x)$. (The inverse of the transformation (1.28) takes the points (1.42) to positive integers.)

2. CD-kernels for the Meixner polynomials

In this section, we state and discuss the results of the present paper on the asymptotics of the Christoffel–Darboux kernels for the Meixner polynomials.

The main results are related to the Plancherel–Rotach asymptotic modes, i.e., to modes with large n and x. These results are dealt with in Section 2.1.

Then, to make the picture complete, we consider the mode with bounded x (Section 2.2). We obtain the strong asymptotics of the Meixner polynomials for this mode, which permits us to obtain the asymptotics of the CD-kernels in a neighborhood of the left endpoint of the support of the orthogonality measure.

Our analysis has the specific feature that first we consider the CD-kernels not only at the points where the masses of the orthogonality measure lie (i.e., at the integer points) but also at all points of the real line $\mathbb{R}$. Then the asymptotics obtained are studied on the support. For some modes, the leading terms of the asymptotics degenerate at the points of growth of the discrete orthogonality measure, and we have to study remainders in more detail and resort to additional scaling.

Let us extend the orthogonality measure

$$\mu(x) := \sum_{x_m \in (c-1)\cdot\mathbb{Z}_+ - (b+1)} \rho(x_m)\delta(x - x_m) \tag{2.1}$$

of the Meixner polynomials from the discrete set $(c-1)\cdot\mathbb{Z}_+ - (b+1)$ to the entire $\mathbb{R}$. To this end, we replace the factorial in (1.2) by the corresponding gamma function,

$$\rho(x) := \mathrm{Const}(c,b)\frac{\Big(\dfrac{x+cb+1}{c-1}\Big)!}{\Big(\dfrac{x+b+1}{c-1}\Big)!} c^{-\frac{x}{c-1}} =: e^{-V(x)}, \qquad x \in \mathbb{R}. \tag{2.2}$$

Thus, the CD-kernel for the sequence of orthogonal polynomials Q_n acquires the form

$$K_n(x,y) := \exp\biggl(-\frac{V(x)+V(y)}{2}\biggr)\sum_{k=0}^{n}\frac{1}{\gamma_k}Q_k(x)Q_k(y), \tag{2.3}$$

where γ_n is the squared norm of the monic polynomial Q_n,

$$\gamma_n := n!(n+b)!c^n\left(\frac{c}{c-1}\right)^{b+1}. \tag{2.4}$$

2.1. **Asymptotics of CD-kernels of the Meixner polynomials for large n and x (Plancherel–Rotach modes).** Here we present results on the limit CD-kernels (2.3) following from the asymptotics of the Meixner polynomials described in Section 1.4. Naturally, each of the zero distribution regions (II–V) enjoys its own asymptotics of the CD-kernels. We note in advance that the limit kernels in regions III (the equilibrium region) and II (the free-to-equilibrium transient region) have the traditional form (the sine kernel and the Airy kernel), while the saturation-related regions IV and V produce new kernels.

2.1.1. *Sine mode on the equilibrium part of the spectrum.* The equilibrium region is characterized by the following scale of the variable x:

$$x := nt, \qquad t\in\Delta\Subset((\sqrt{c}-1)^2,(\sqrt{c}+1)^2), \tag{2.5}$$

where Δ is a compact set. In turn, in a neighborhood of x we choose its own scale related to the global characteristic of the distribution of zeros,

$$\lim_{n\to\infty}K_{n-1}(nt,nt) =: \delta(t).$$

Theorem 2.1. *The* CD-*kernel* (2.3) *of the Meixner polynomials satisfies the following relation in the equilibrium region* (2.5) *uniformly with respect to* $t\in\Delta\Subset((\sqrt{c}-1)^2,$ $(\sqrt{c}+1)^2)$:

$$\lim_{n\to\infty}\frac{K_{n-1}\Bigl(nt+\dfrac{\xi}{K_{n-1}(nt,nt)},nt+\dfrac{\eta}{K_{n-1}(nt,nt)}\Bigr)}{K_{n-1}(nt,nt)} = \frac{\sin(\pi(\xi-\eta))}{\pi(\xi-\eta)}. \tag{2.6}$$

Moreover,

$$nK_{n-1}(nt,nt) = \frac{n}{\pi(c-1)}\arccos\biggl(\frac{(c+1)t-(c-1)^2}{2t\sqrt{c}}\biggr)+O(1). \tag{2.7}$$

Thus, Theorem 2.1 establishes the universality of the sine kernel even for polynomials orthogonal with respect to a discrete measure (for the chosen continuation of the discrete weight to $\mathbb{R}$). Note the relationship between the leading term $\delta(t)$ of the asymptotics of the diagonal kernel (2.7) with the weak asymptotics (1.17) of the distribution of zeros of the Meixner polynomials.

The asymptotic formula remains meaningful if one substitutes the points where the discrete masses are concentrated as the arguments of the kernel. Under condition (2.5), let the point nt belong to the support $(c-1)\cdot\mathbb{Z}_+-(b+1)$ of the Meixner measure (see (2.1)), and let the parameters ξ and η satisfy the condition that the points $x_\xi(t,n) := nt+\xi/\delta(t)$ and $x_\eta(t,n) := nt+\eta/\delta(t)$ belong to the support of the Meixner measure as well. Then

$$\lim_{n\to\infty}\frac{1}{\delta(t)}K_{n-1}(x_\xi(t,n),x_\eta(t,n)) = \frac{\sin(\pi(\xi-\eta))}{\pi(\xi-\eta)}.$$

2.1.2. *Modes in transient regions.* We start from transient region II (between the free region and the equilibrium region). This region (see Section 1.4) lies in a neighborhood of the point

$$x_0^{(II)} := n(\sqrt{c}+1)^2,$$

and the scale in this region is determined by the factor

$$s_n^{(II)} := (n\sqrt{c}(\sqrt{c}+1)^4)^{1/3}.$$

Thus, the variables (x, y) in this region are related to the local parameters (u, v) as follows:

$$x := x_0^{(II)} + us_n^{(II)}, \qquad y := x_0^{(II)} + vs_n^{(II)}. \tag{2.8}$$

Theorem 2.2. *The* CD-*kernel* (2.3) *of the Meixner polynomials satisfies the following relation in the transient region* II (*see* (2.8)) *uniformly with respect to* $u, v \in \Delta \Subset \mathbb{C}$ *as* $n \to \infty$:

$$K_n(x,y) = \frac{1}{s_n^{(II)}}\left[\frac{\operatorname{Ai}'(u)\operatorname{Ai}(v) - \operatorname{Ai}'(v)\operatorname{Ai}(u)}{u-v} + O(n^{-1/3})\right].$$

Since the last formula is true for all values in region II, it follows that the passage to the limit is well defined, and we obtain the diagonal values

$$K_n(x,x) = \frac{1}{s_n^{(II)}}\left[\operatorname{Ai}''(u)\operatorname{Ai}(u) - (\operatorname{Ai}'(u))^2 + O(n^{-1/3})\right].$$

Thus, Theorem 2.2 establishes the universality of the Airy kernel for the Meixner polynomials orthogonal with respect to a discrete measure (for the chosen continuation of the discrete weight to $\mathbb{R}$). A straightforward substitution

$$X_k := x_0^{(II)} + us_n^{(II)}, \quad X_m := x_0^{(II)} - vs_n^{(II)}, \qquad k \neq m,$$

of the support points gives

$$K_n(X_k, X_m) = \frac{1}{s_n^{(II)}}\left[\frac{\operatorname{Ai}'(u)\operatorname{Ai}(v) - \operatorname{Ai}'(v)\operatorname{Ai}(u)}{u-v} + O(n^{-1/3})\right]$$

and

$$K_n(X_k, X_k) = \frac{1}{s_n^{(II)}}\left[\operatorname{Ai}''(u)\operatorname{Ai}(u) - (\operatorname{Ai}'(u))^2 + O(n^{-1/3})\right].$$

This result was obtained earlier by Johansson [26].

Now let us state a new result on the limit kernel in the transient region IV (between the saturation and equilibrium regions). This region (see Section 1.4) lies in a neighborhood of the point

$$x_0^{(IV)} := n(\sqrt{c}-1)^2,$$

and the scale in this region is given by the factor

$$s_n^{(IV)} := \left(n\sqrt{c}(\sqrt{c}-1)^4\right)^{1/3}.$$

Thus, the variables (x, y) in this region are related to the local parameters (u, v) as follows:

$$x := x_0^{(IV)} - us_n^{(IV)}, \qquad y := x_0^{(IV)} - vs_n^{(IV)}. \tag{2.9}$$

Theorem 2.3. *The* CD-*kernel* (2.3) *of the Meixner polynomials satisfies the following relation in the transient region* IV (*see* (2.9)) *uniformly with respect to* $u, v \in \Delta \Subset \mathbb{C}$ *as* $n \to \infty$ *:*

$$K_n(x,y) = \frac{1}{s_n}\frac{1}{u-v}\Bigg[\cos\pi\frac{x+b+c}{c-1}\cos\pi\frac{y+b+c}{c-1}(\mathrm{Ai}'(u)\,\mathrm{Ai}(v)-\mathrm{Ai}'(v)\,\mathrm{Ai}(u)) \\ +\cos\pi\frac{x+b+c}{c-1}\sin\pi\frac{y+b+c}{c-1}(-\mathrm{Ai}'(u)\,\mathrm{Bi}(v)+\mathrm{Ai}(u)\,\mathrm{Bi}'(v)) \\ +\sin\pi\frac{x+b+c}{c-1}\cos\pi\frac{y+b+c}{c-1}(-\mathrm{Bi}'(u)\,\mathrm{Ai}(v)+\mathrm{Bi}(u)\,\mathrm{Ai}'(v)) \\ +\sin\pi\frac{x+b+c}{c-1}\sin\pi\frac{y+b+c}{c-1}(\mathrm{Bi}'(u)\,\mathrm{Bi}(v)-\mathrm{Bi}'(v)\,\mathrm{Bi}(u))+O(n^{-1/3})\Bigg]. \tag{2.10}$$

Furthermore, on the diagonal $x=y \;\Rightarrow\; u=v$ *one has*

$$K_n(x,x) = \frac{1}{c-1} + o(1).$$

The trigonometric factors in the variables (u,v) are high-frequency functions, which results in the diagonal dominance of the kernel K_n in these variables, $|K_n| \sim n^{-1/3}$ for $u \neq v$ and $|K_n| \sim 1$ for $u = v$.

Note how the parameters b and c of the weight function of the Meixner polynomials occur on the right-hand side in (2.10). Their occurrence in the limit kernel is caused by our change of variable (1.28), and these parameters disappear from the limit kernel when passing to the standard variable of the Meixner polynomials (for which the masses of the orthogonality measure lie at the positive integer points).

In region IV, the restriction of the kernel to the support of the measure leads to a dramatic simplification. As a result, the asymptotic behavior of the kernel will be similar to that in region II, with the exception for the diagonal entries.

Thus, if X_k and X_m are two distinct mass points in region IV,

$$X_k := x_0^{(IV)} - us_n^{(IV)}, \quad X_m := x_0^{(IV)} - vs_n^{(IV)}, \qquad k \neq m,$$

and (u,v) are the corresponding local variables, then the sines in (2.10) vanish and the cosines are $(-1)^k$ and $(-1)^m$, respectively (modulo ε in (1.41)). We obtain

$$K_n(X_k, X_m) = \frac{(-1)^{k-m}}{s_n^{(IV)}}\left[\frac{\mathrm{Ai}'(u)\,\mathrm{Ai}(v)-\mathrm{Ai}'(v)\,\mathrm{Ai}(u)}{u-v} + O(n^{-1/3})\right].$$

One cannot pass to the limit in this formula; this should be done in formula (2.10), which holds for all values in region IV. As a result,

$$K_n(X_k, X_k) = \frac{1}{c-1} + O(n^{-1/3}).$$

Thus, the restriction of the kernel to the support in region IV has power-law diagonal dominance.

2.1.3. *Modes in the saturation region.* The saturation region for the Meixner polynomials (region V; see Section 1.4) is characterized by the scale

$$x = nt, \qquad t \in \Delta \Subset (0, (\sqrt{c}-1)^2), \quad n \to \infty. \tag{2.11}$$

In the saturation region, the zeros of the orthogonal polynomials are attracted at an exponential rate to the mass points of the discrete orthogonality measure as $n \to \infty$. Hence it is expedient to consider various asymptotic modes for the CD-kernel in this region:

- The mode (2.11) in a neighborhood of an arbitrary point of the saturation region.
- Modes in a neighborhood of the masses

$$X_k = k(c-1) - b - c, \qquad k \in \mathbb{Z}_+, \tag{2.12}$$

with the exponential scaling

(2.13)
$$e^{-2\Sigma(t,n)} := \left(\frac{c+1-t+\sqrt{\widetilde{D}}}{2\sqrt{c}}\right)^{-2n-b+1}\left(\frac{(c-1)^2-(c+1)t+(c-1)\sqrt{\widetilde{D}}}{2t\sqrt{c}}\right)^{2k+b-1},$$

for
$$t := t_k := \frac{X_k}{n} \in \Delta \Subset (0, (\sqrt{c}-1)^2), \quad \widetilde{D}(t) := (c+1-t)^2 - 4c \qquad \text{as } n \to \infty.$$

Note that the scaling function $\Sigma(t,n)$ is related to the solution of the equilibrium problem (1.6). Specifically,
$$\Sigma(t,n) = n\Psi_1(t) + O(1),$$
where
$$\Psi_1(t) := \mathcal{P}^{\lambda}(t) - \kappa + V_1(\tilde{t}(t)), \qquad t := \frac{x}{n}, \quad V_1(\tilde{t}(t)) = \lim_{n\to\infty}\frac{V(nt)}{n},$$
and $\lambda(t)$ is the limit measure of the distribution (1.9) of the "compressed" Meixner polynomials. Since we now consider the function Ψ_1 in the saturation region, it follows that it is positive here according to (1.6). Let us present a closed-form expression for this function (cf. (1.17)):

(2.14)
$$\Psi_1(t) := \cosh^{-1}\left(\frac{c+1-t}{2\sqrt{c}}\right) - \frac{t}{c-1}\cosh^{-1}\left(\frac{(c-1)^2-(c+1)t}{2t\sqrt{c}}\right).$$

Let us state results on the limit kernels for the saturation region modes.

Theorem 2.4. *The* CD-*kernel* (2.3) *of the Meixner polynomials satisfies the following relation in the mode* (2.11) *uniformly with respect to* $\xi, \eta \in \Delta \Subset \mathbb{R}$ *as* $n \to \infty$:
$$nK_{n-1}(nt+\xi, nt+\eta) = \left(F(t) + O\left(\frac{1}{n}\right)\right)\exp\left(n\Psi_1\left(t+\frac{\xi}{n}\right) + n\Psi_1\left(t+\frac{\eta}{n}\right)\right)$$
$$\times \sin\left(\pi\frac{nt+\xi+b+c}{c-1}\right)\sin\left(\pi\frac{nt+\eta+b+c}{c-1}\right) + O(e^{|(\xi-\eta)\Psi_1'(t)|}),$$
where
$$F(t) = \frac{2\sqrt{c}}{\pi\widetilde{D}}\left(\frac{c+1-t+\sqrt{\widetilde{D}}}{2\sqrt{c}}\right)^{b}\left(\frac{(c-1)^2-(c+1)t+(c-1)\sqrt{\widetilde{D}}}{2t\sqrt{c}}\right)^{-\frac{(b+1)(c+1)}{c-1}}.$$

Note that the limit kernel describing the leading term of the asymptotics is degenerate. Moreover, it vanishes at the points where the masses of the discrete measure lie, and lower-order terms play the main role there. Hence, to study the limit kernels in neighborhoods of the masses of the orthogonality measure, one needs an additional scaling in these neighborhoods (the mode (2.12)–(2.13)).

The limit kernel for the mixed mode, where the arguments of the CD-kernel in a neighborhood of the mass x_k are considered in distinct scales (the linear magnification (2.11) and the exponential magnification (2.12)), is as follows.

Theorem 2.5. *The* CD-*kernel* (2.3) *of the Meixner polynomials satisfies the following relation uniformly with respect to*
$$t := \frac{X_k}{n} \in \Delta \Subset (0, (\sqrt{c}-1)^2)$$
and $\eta, \xi \in \widetilde{\Delta} \Subset \mathbb{R}$ *as* $n \to \infty$:

(2.15)
$$K_n\big(X_k + e^{-2\Sigma(t,n)}n\xi, X_k + \eta\big) = \frac{\sin\left(\dfrac{\pi\eta}{c-1}\right)}{\eta}e^{\eta\Psi_1'(t)}\left(\xi\eta\widetilde{F}(t) + \frac{1}{\pi}\right) + O\left(\frac{1}{n}\right),$$

where X_k *is given in* (2.12), Σ *is given in* (2.13), Ψ_1 *is given in* (2.14), *and the function* $\widetilde{F}$ *has the following closed-form expression depending on the parameter of the Meixner weight and independent of* n, ξ, *and* η:

$$\widetilde{F} = \frac{c+1-t+\sqrt{\widetilde{D}}}{2(c-1)\widetilde{D}}.$$

Note that if we distribute the exponential scaling (2.13) between both arguments of the kernel (2.3),

$$\Sigma_k = \widetilde{\Sigma}_k + \widetilde{\widetilde{\Sigma}}_k, \tag{2.16}$$

then the limit kernel will have the form

$$K_n\big(X_k + e^{-2\widetilde{\Sigma}_k(t,n)}n\xi, X_k + e^{-2\widetilde{\widetilde{\Sigma}}_k(t,n)}n\eta\big) = \frac{\pi}{c-1}\bigg(\xi\eta\widetilde{F}(t) + \frac{1}{\pi}\bigg)O\bigg(\frac{1}{n}\bigg). \tag{2.17}$$

Finally, let us present a result in which each argument of the kernel is scaled exponentially but in a neighborhood of its own mass point X_k or X_m, respectively, these points being assumed to be not too far from each other.

Theorem 2.6. *Assume that, for each* $n \in \mathbb{N}$, *we choose a pair* (X_k, X_m) *of mass points such that* $X_j - n\hat{t} \in K \Subset \mathbb{R}$ *for* $j = k, m$ *and for some given* $\hat{t} \in \Delta \Subset (0, (\sqrt{c}-1)^2)$. *Then the* CD-*kernel* (2.3) *of the Meixner polynomials satisfies the following relation uniformly with respect to* $\hat{t}$ *and the choice of the sequence of pairs* (X_k, X_m) *and local coordinates* $\eta, \xi \in \widetilde{\Delta} \Subset \mathbb{R}$ *as* $n \to \infty$:

$$\begin{aligned}\frac{K_n\big(X_k + e^{-2\Sigma(t_k,n)}n\xi, X_m + e^{-2\Sigma(t_m,n)}n\eta\big)}{(-1)^{(k-m)}\exp(-\Sigma(t_k,n) - \Sigma(t_m,n))} &= \frac{\pi(c+1-\hat{t}+\sqrt{\widetilde{D}})n + O(1)}{2(c-1)^2\widetilde{D}}\xi\eta \\ &+ \frac{n+O(1)}{(k-m)(c-1)^2}\xi - \frac{n+O(1)}{(k-m)(c-1)^2}\eta - \frac{c+1-\hat{t}-\sqrt{\widetilde{D}}}{2\pi\widetilde{D}\cdot n} + O\bigg(\frac{1}{n^2}\bigg),\end{aligned}$$

where $\widetilde{D} = \widetilde{D}(\hat{t})$.

Now we are in a position to state some corollaries of Theorems 2.5 and 2.6 on the behavior of the kernels on the support of the orthogonality measure. By substituting $\xi = \eta = 0$ into (2.15) and into the asymptotic formula in Theorem 2.6 (because the short-range exponential scaling in a neighborhood of a mass point prevents one from preserving the parameters describing neighboring mass points in the local mode), we obtain

$$K_n(X_k, X_k) = \frac{1}{c-1} + O\bigg(\frac{1}{n}\bigg) \tag{2.18}$$

and

$$\frac{K_n(X_k, X_m)}{(-1)^{(k-m)}\exp(-\Sigma(t_k,n) - \Sigma(t_m,n))} = -\frac{c+1-\hat{t}-\sqrt{\widetilde{D}}}{2\pi\widetilde{D}\cdot n} + O\bigg(\frac{1}{n^2}\bigg). \tag{2.19}$$

Thus, we see that the CD-kernel has a diagonal dominance on the support of the orthogonality measure in the saturation region.

2.2. **Asymptotics of the Meixner polynomials and CD-kernels for bounded** x. The Plancherel–Rotach modes (in which x grows with n) do not include the important mode of finite x as $n \to \infty$. It is this mode that describes the transient (I$\leftrightarrow$V) region, i.e., the transition from the zero-free region to the saturation region (see Section 1.4).

2.2.1. *Asymptotics of the Meixner polynomials and the continuous CD-kernel in a neighborhood of zero.* One cannot completely reproduce the approach in Section 1.3 for the case of bounded x, although, at least for Π_1, the formal series is still majorized by a series asymptotic as $n \to \infty$ even if x is bounded rather than tending to infinity. Once one solution of second-order recursion relations is known, one can determine the other, and so Π_2 can be found via Π_1 (rather than directly), which gives a series with asymptotic properties. The remaining problem is to expand the particular solution $Q_n(x)$ in the basis solutions, but the asymptotic data are not sufficient for solving this problem, because the expansion should use the initial conditions, i.e., the case of bounded n and bounded x. Here we again need additional information, and the exact positions of discrete masses serve as such.

Thus, we need some transformations of the asymptotic series (1.24) for Π_1 and Π_2. By adding some functions depending only on x to $\psi_m^{(j)}(x,n)$ (1.24), one can eliminate the singularities that arise for bounded x owing to the method. This changes the factor $K_0(x)$ and destroys the symmetry between the expressions for Π_1 and Π_2; one cannot be obtained from the other any longer by simply changing the sign of $\sqrt{D}$. (This only pertains to the subsequent terms of the asymptotics, which are not given in the paper in closed form anyway.) However, the main thing is that the expressions for Π_1 and Π_2 are now holomorphic in x in a large disk (say, $|x| < \sqrt{n}$). The polynomials $Q_n(x)$ themselves are holomorphic as well. (This is obvious!) Hence the coefficient of Π_1 in the expansion is a holomorphic function of x, and we know it depends on x *alone.* We know its asymptotics as $|x| \to \infty$, and we know that it vanishes only at the mass points. Hence we can determine this holomorphic function completely.

As a result, the basis solutions (1.31)–(1.32) can be reduced to the form

(2.20)
$$\Pi_1 := (-1)^n \left(\frac{(c+1)n - x + \sqrt{D}}{2}\right)^{n-\frac{1}{2}} \left[\sqrt{D} + (c-1)n - \frac{(c+1)x + 2bc}{c-1}\right]^{-\frac{2x+2bc+c+1}{2(c-1)}} \times \frac{\left[(c-1)n - x - 2b + \sqrt{D}\right]^b}{\exp(n)\sqrt[4]{D}} \exp\Bigl(\sum_j \Psi_j^{(1)}\Bigr)$$

and a similar expression for Π_2 with the opposite sign of $\sqrt{D}$. The new expression for the leading term is proportional to the original one. It is crucial for this expression that all roots and powers in it are arithmetic for $n \gg |x|$ and $x \in \mathbb{R}$. By matching with the leading coefficients of the Meixner polynomial, we obtain the following theorem.

Theorem 2.7. *The Meixner polynomial can be represented as follows via the basis solutions* (2.20):

$$Q_{n-1}(x) = -2\sqrt{\pi c}(2ce)^{\tilde{x}} \left[\frac{c}{c-1}\right]^b \frac{\exp\left(\dfrac{-1}{c-1}\right)}{\Gamma(-\tilde{x})} \Pi_1(x,n) + O(\Pi_2(x,n)),$$

where the relationship between the variables x and $\tilde{x}$ is given in (1.28).

Recall that $\tilde{x}$ in (1.28) is the original variable $\tilde{x} = \frac{x+b+1}{c-1}$ of the Meixner polynomials, for which the polynomials $M_n(\tilde{x}) := (-1)^n Q_n(x)$ are orthogonal with respect to a discrete measure concentrated at the nonnegative integer points. Hence we obtain the following corollary.

Corollary 2.1. *The Meixner polynomials* (1.2) *satisfy the relation*

$$\frac{M_{n-1}(\tilde{x})}{\sqrt{\gamma_{n-1}}} = \frac{c^{n/2}}{\sqrt{cn}}\left(\frac{c}{n(c-1)}\right)^{\tilde{x}+\frac{b+1}{2}} \frac{\left(1+O\left(\frac{1}{n}\right)\right)}{\Gamma(-\tilde{x})}, \qquad x \in \Delta,$$

uniformly on compact sets $\Delta \Subset \mathbb{R}$ *as* $n \to \infty$, *where* γ_{n-1} *is the norm of the Meixner polynomials* (*see* (2.4)).

Now we are in a position to state the corresponding claim on the limit CD-kernel for this mode. Recall that, owing to the dilation with ratio $c-1$ and the additional factor $(-1)^n$, there will be a notational difference (which does not occur in the expanded formulas (1.1) and (2.3)) between the standard Meixner polynomials $M_n(\tilde{x})$ and the monic polynomials $Q_n(x)$ after contracting the sum in the Christoffel–Darboux formulas; namely, an extra factor will occur:

$$\frac{Q_{n+1}(x)Q_n(y) - Q_{n+1}(y)Q_n(x)}{x-y} = \frac{-1}{c-1}\,\frac{M_{n+1}(\tilde{x})M_n(\tilde{y}) - M_{n+1}(\tilde{y})M_n(\tilde{x})}{\tilde{x}-\tilde{y}}.$$

To make the comparison with the asymptotics of the kernel for the preceding modes more convenient, we, as before, state the asymptotic results for the kernel (2.3), i.e., for the left-hand side of the last relation.

Theorem 2.8. *The* CD-*kernel* (2.3) *of the Meixner polynomials satisfies the following relation uniformly with respect to* $x \in \Delta \Subset \mathbb{R}$ *as* $n \to \infty$:

$$K_{n-1}(x,y) = c^{n-1}\left(\frac{c}{n(c-1)}\right)^{\tilde{x}+\tilde{y}+b+2} \frac{\sqrt{\rho(x)\rho(y)}}{\Gamma(-\tilde{x})\Gamma(-\tilde{y})}\left(1+O\left(\frac{1}{n}\right)\right) + O(nc^{-n}),$$

where ρ *is the continuous extension* (2.2) *of the Meixner weight.*

2.2.2. *Asymptotics of the Meixner polynomials and the discrete CD-kernel in a neighborhood of zero.* In this section, we present an expression for the Christoffel–Darboux kernel of the Meixner polynomials at the mass points and in their neighborhoods for the case of bounded x. Unfortunately, in contrast to what has been done in other sections, for this case we do not have a general approach based on general properties of the orthogonality measure and the knowledge of recursion coefficients. Hence we derive the desired formulas from the relationship between the Meixner polynomials and special functions, namely, the hypergeometric function ${}_2F_1$ (see [37]).

Specifically, the Meixner polynomials can be expressed as

$$M_n(\tilde{x}) = (\beta)_n\, {}_2F_1\left(-n, -\tilde{x}; \beta; 1-\frac{1}{\sigma}\right) = \frac{(b+n)!}{b!}\, {}_2F_1(-n, -\tilde{x}; b+1; 1-c).$$

By using Euler's representation for the hypergeometric functions (see Section 3.3.5 for details), we obtain the following integral representation of the Meixner polynomials.

Theorem 2.9. *The Meixner polynomial has the representation*

$$M_n(\tilde{x}) = \frac{(b+n)!}{(b+\tilde{x})!}\cdot \tilde{x}! \cdot \frac{\sin\pi\tilde{x}}{\pi} I_1 + \frac{(b+n)!}{(b+\tilde{x})!}\cdot\tilde{x}!\cdot\cos\pi\tilde{x} I_2, \tag{2.21}$$

where

$$I_1 = \frac{1}{2}\left(\oint_{\widetilde{\theta}_+}\mathcal{F}(t)\,dt + \oint_{\widetilde{\theta}_-}\mathcal{F}(t)\,dt\right), \qquad I_2 = \frac{1}{2\pi i}\oint_{\theta}\mathcal{F}(t)\,dt, \tag{2.22}$$

the contours θ_+ *and* θ_- *join the points* 1 *and* $-\frac{1}{c-1}$ *in the upper and lower half-planes, respectively, and the contour* θ *issues from the point* $-\frac{1}{c-1}$ *into the lower half-plane, goes* *around zero counterclockwise, and returns to the point* $-\frac{1}{c-1}$ *through the upper half-plane. The branch of the function*

$$\mathcal{F}(t) := t^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n$$

is taken in the domain with the cut $(-\infty;0)\cup(1;+\infty)$.

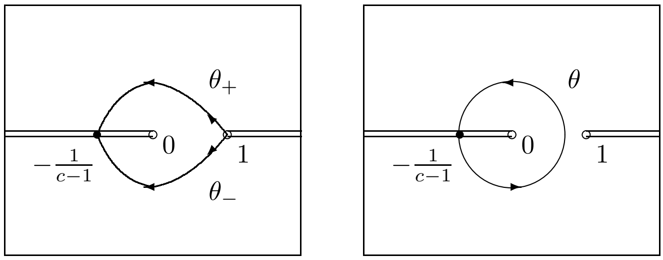

FIGURE 1. The contours θ_+, θ_-, and θ

By using the asymptotics

$$I_1 \sim -(b+\tilde{x})!\left(\frac{c}{n(c-1)}\right)^{b+\tilde{x}+1}c^n, \qquad I_2 \sim \frac{1}{\tilde{x}!}(n(c-1))^{\tilde{x}} \tag{2.23}$$

of these integrals, we see that we again have a two-term asymptotic representation of Meixner polynomials, where one of the terms grows exponentially more rapidly than the other as $n\to\infty$ but is multiplied by a factor that vanishes at the mass points. Note also a recursion relation satisfied by these integrals and the value of their determinant, which follows from this relation.

Proposition 2.1. *For* $j=1,2$ *and* $n\in\mathbb{N}$, *one has*

$$(n+b+1)I_2(\tilde{x},n+1)+[(c-1)\tilde{x}-(c+1)n-b-1]I_2(\tilde{x},n)+cnI_2(\tilde{x},n-1)=0$$

and also

$$\Delta_I(\tilde{x}) := I_1(\tilde{x},n+1)I_2(\tilde{x},n)-I_1(\tilde{x},n)I_2(\tilde{x},n+1)=\mathrm{Const}\,\frac{c^n n!}{(n+b+1)!}, \tag{2.24}$$

where

$$\mathrm{Const} := -\frac{(b+\tilde{x})!}{\tilde{x}!}\frac{c^{b+\tilde{x}+1}}{(c-1)^b}.$$

Let us write out the representation of the discrete CD-kernel of the Meixner polynomials via the integrals I_1 and I_2.

Theorem 2.10. *The* CD*-kernel* (2.3) *of the Meixner polynomials satisfies*

$$K_n(x,y) = (-1)^{\tilde{x}+\tilde{y}+1}\varpi_n\frac{\Delta_2(\tilde{x},\tilde{y})}{(\tilde{x}-\tilde{y})}, \qquad \tilde{x},\tilde{y}\in\mathbb{Z}_+, \quad \tilde{x}\neq\tilde{y}, \tag{2.25}$$

where

$$\varpi_n := \frac{(n+b+1)!c^{-n-1}}{n!(c-1)}\left(\frac{c-1}{c}\right)^b\sqrt{\frac{\tilde{x}!\tilde{y}!c^{-\tilde{x}-\tilde{y}}}{(b+\tilde{x})!(b+\tilde{y})!}} \tag{2.26}$$

and

$$\Delta_2(\tilde{x},\tilde{y}) := I_2(\tilde{x},n+1)I_2(\tilde{y},n)-I_2(\tilde{x},n)I_2(\tilde{y},n+1).$$

On the diagonal, one has

$$K_n(x,x) = -\varpi_n\left(I_2^2(\tilde{x},n)\cdot\frac{d}{d\tilde{x}}\left(\frac{I_2(\tilde{x},n+1)}{I_2(\tilde{x},n)}\right) + \Delta_I(\tilde{x})\right), \qquad \tilde{x}\in\mathbb{Z}_+, \tag{2.27}$$

where $\Delta_I(\tilde{x})$ *is defined in* (2.24).

This theorem readily implies a corollary concerning the asymptotics of the discrete CD-kernel of the Meixner polynomials. First, note the order of growth in n of the terms on the right-hand side in (2.25),

$$\varpi_n \sim \mathrm{const}(\tilde{x},\tilde{y})n^{b+1}c^{-n},$$

and also

$$\frac{\Delta_2(\tilde{x},\tilde{y})}{(x-y)} \sim \frac{(n(c-1))^{\tilde{x}+\tilde{y}}}{\tilde{x}!\tilde{y}!}\left[\frac{(1+\frac{1}{n})^{\tilde{x}}-(1+\frac{1}{n})^{\tilde{y}}}{(c-1)(\tilde{x}-\tilde{y})}\right] \sim \frac{(n(c-1))^{\tilde{x}+\tilde{y}-1}}{\tilde{x}!\tilde{y}!}.$$

Corollary 2.2. *The* CD*-kernel* (2.3) *of the Meixner polynomials satisfies the following relation uniformly with respect to* $\tilde{x},\tilde{y}\Subset\mathbb{Z}_+$ *as* $n\to\infty$:

$$K_n(x,y) \sim (-1)^{\tilde{x}+\tilde{y}+1}\frac{(n+b+1)!}{n!}\frac{c^{-n-b-1}(c-1)^{\tilde{x}+\tilde{y}+b}n^{\tilde{x}+\tilde{y}-1}}{\sqrt{\tilde{x}!\tilde{y}!(\tilde{x}+b)!(\tilde{y}+b)!c^{\tilde{x}+\tilde{y}}}}, \qquad \tilde{x}\neq\tilde{y}. \tag{2.28}$$

On the diagonal $\tilde{x}=\tilde{y}$, *one has*

$$K_n(x,x) = \frac{1}{c-1} + o(1), \qquad \tilde{x}\in\mathbb{Z}_+.$$

3. Derivation of the limit CD-kernels of Meixner polynomials

3.1. **Proof of Theorem 2.1.**

3.1.1. *Asymptotics of the Meixner polynomials in the equilibrium region.* First, we transform the asymptotic formulas (1.37) to a form more convenient for the analysis of CD-kernels. For the continualization of the orthogonality weight (2.2), we make the following refinement:

$$\mathrm{Const}(c,b) := \frac{1}{c-1}c^{-\frac{b+1}{c-1}}$$

and

$$V(x) = \frac{\ln c}{c-1}x - b\ln\frac{x}{c-1} + \frac{b+1}{c-1}\ln c + \ln(c-1) + O\left(\frac{1}{x}\right).$$

Recall the expression for the norm of the monic polynomials orthogonal with respect to the measure (2.1),

$$\int Q_n^2\,d\mu = \Gamma(\beta)n!(\beta)_n\sigma^{-n}(1-\sigma)^{-\beta} = n!(n+b)!c^n\left(\frac{c}{c-1}\right)^{b+1} =: \gamma_n, \tag{3.1}$$

and the relationship between the norm and the recursion coefficient (1.29),

$$\frac{\gamma_n}{\gamma_{n-1}} = cn(n+b).$$

Now for the CD-kernel (by the Christoffel–Darboux formula) we have

$$K_n(x,y) = \frac{1}{\gamma_n}\exp\left(-\frac{V(x)+V(y)}{2}\right)\frac{Q_{n+1}(x)Q_n(y)-Q_{n+1}(y)Q_n(x)}{x-y}.$$

Let

$$\widetilde{q}_n(x) := \frac{1}{\sqrt{\gamma_n}}\exp\left(-\frac{V(x)}{2}\right)Q_n(x); \tag{3.2}$$

now we see that we need the asymptotics of

$$K_n(x,y) = \sqrt{\frac{\gamma_{n+1}}{\gamma_n}}\,\frac{\widetilde{q}_{n+1}(x)\widetilde{q}_n(y) - \widetilde{q}_{n+1}(y)\widetilde{q}_n(x)}{x-y}.$$

Let us transform the expressions for the asymptotics of $\widetilde{q}_n$. We have (see (1.37))

$$\widetilde{q}_{n-1}(x) = \frac{K_0(nt)}{\sqrt{\gamma_{n-1}}}\exp\left(-\frac{V(x)}{2}\right)(\Pi_1(x,n) + i\Pi_2(x,n)),$$

where (see (1.35))

$$\ln K_0(x) := -\frac{2x+(b+1)(c+1)}{2(c-1)}\ln\left(\frac{2x}{c-1}\right) - \frac{b}{2}\ln(2x^2) - \frac{bc}{c-1} + O\left(\frac{1}{x}\right)$$

and

$$\begin{aligned}\Pi_1 := \left(\frac{x-(c+1)n+\sqrt{D}}{2}\right)^{n-1/2}&\left[\frac{2c(x+b)}{c-1} - x - (c-1)n - \sqrt{D}\right]^{\frac{2x+(b+1)(c+1)}{2(c-1)}}\\ &\times\frac{[2bc(n+b) + (x-(c+1)n+\sqrt{D})(x+(c+1)b)]^{b/2}}{\exp(n)\sqrt[4]{D}}\\ &\qquad\times\left[1 + O\left(\max\left(\frac{|x|^2}{n^3}, \frac{|x|}{|n-n_i(x)|^2}\right)\right)\right].\end{aligned}$$

Recall (see (1.30)) that

$$D := (x-(c+1)n)^2 - 4cn(n+b)$$

and the $n_j(x)$, $j=1,2$, are the roots of D. Moreover, $\Pi_2(x,n)$ differs from $\Pi_1(x,n)$ by the opposite sign of $\sqrt{D}$, and the factor containing the remainder acquires the form

$$\left[1 + O\left(\max\left(\frac{1}{|x|}, \frac{|x|}{|n-n_i(x)|^2}\right)\right)\right].$$

Thus,

$$\widetilde{q}_{n-1}(nt) =: e^{\Psi(t,n)}\cos\Phi(t,n), \tag{3.3}$$

where

$$\begin{aligned}\Psi(t,n) = &\left(-\frac{2nt+(b+1)(c+1)}{2(c-1)}\ln\frac{2nt}{c-1} - \frac{b}{2}\ln(2n^2t^2) - \frac{bc}{c-1}\right) - \frac{1}{2}\ln\gamma_{n-1}\\ &- \frac{nt\ln c}{2(c-1)} + \frac{b}{2}\ln\frac{nt}{c-1} - \frac{(b+1)\ln c}{2(c-1)} - \frac{\ln(c-1)}{2} + \left(n-\frac{1}{2}\right)\ln\sqrt{cn(n+b)}\\ &- n + \frac{2nt+(b+1)(c+1)}{2(c-1)}\ln\frac{2\sqrt{c(nt+b)(nt+bc)}}{c-1}\\ &+ \frac{b}{2}\ln 2(n+b)\sqrt{c(nt+b)(nt+bc)} - \frac{1}{2}\ln\sqrt{-D} + \ln 2 + O\left(\frac{1}{nt}\right)\\ = &-\frac{1}{2}\ln\sqrt{-D} - \frac{1}{2}\ln\gamma_{n-1} + \left(n+\frac{b-1}{2}\right)\ln n - n\\ &+ \frac{n+b}{2}\ln c - \frac{b+1}{2}\ln(c-1) + \ln 2 + O\left(\frac{1}{n}\right).\end{aligned}$$

Here the last equation has been obtained by expanding in n in a neighborhood of the point at infinity. Finally, by applying the Stirling formula to (3.1), we obtain

$$\Psi(t,n) = \frac{1}{2}\ln\frac{2}{\pi} - \frac{1}{2}\ln\sqrt{-D} + O\left(\frac{1}{n}\right). \tag{3.4}$$

For the phase function $\Phi(t,n)$, we have

$$\Phi(t,n) = -\frac{2nt+(b+1)(c+1)}{2(c-1)}\arccos\frac{(c+1)nt+2bc-(c-1)^2n}{2\sqrt{c(nt+b)(nt+bc)}} - \frac{\pi}{4}$$
$$+\left(n-\frac{1}{2}\right)\arccos\frac{nt-(c+1)n}{2\sqrt{cn(n+b)}} + \frac{b}{2}\arccos\frac{(nt+(c+1)b)n(t-c-1)+2bc(n+b)}{2(n+b)\sqrt{c(nt+b)(nt+bc)}}.$$

Let

$$\Phi(t,n) =: n\varphi_1(t)+\varphi_0(t)+O\left(\frac{1}{n}\right).$$

Then

$$\varphi_1(t) = \arccos\frac{t-c-1}{2\sqrt{c}} - \frac{t}{c-1}\arccos\frac{(c+1)t-1}{2t\sqrt{c}}$$

and

$$\varphi_0(t) = \frac{b-1}{2}\arccos\frac{t-c-1}{2\sqrt{c}} - \frac{(b+1)(c+1)}{2(c-1)}\arccos\frac{(c+1)t-(c-1)^2}{2t\sqrt{c}} - \frac{\pi}{4}.$$

Note that the terms growing with n in the expression for $\Psi(t,n)$ have the form $-\frac{1}{2}\ln n$:

$$\Psi(t,n) = \frac{1}{2}\ln\frac{2}{\pi\sqrt{-\widetilde{D}}} - \frac{1}{2}\ln n + O\left(\frac{1}{n}\right).$$

Here we have used the notation

$$\widetilde{D} := \lim_{n\to\infty}\frac{D|_{x=nt}}{n^2}. \tag{3.5}$$

In view of the Christoffel–Darboux formula, for the kernel (2.3) we also need the asymptotics of

$$\widetilde{q}_n(nt) = \widetilde{q}_n\left((n+1)\left(t-\frac{t}{n+1}\right)\right),$$

which, in turn, can be determined with the help of

$$\Phi\left(t-\frac{t}{n+1}, n+1\right) = (n+1)\left(\varphi_1(t) - \frac{t}{n+1}\varphi_1'(t)+\cdots\right) + \varphi_0(t)+\cdots$$
$$=: \Phi(t,n)+\delta\Phi(t,n),$$

$$\delta\Phi(t,n) = \varphi_1(t) - t\varphi_1'(t) + O\left(\frac{1}{n}\right) = \arccos\frac{t-c-1}{2\sqrt{c}} + O\left(\frac{1}{n}\right),$$

and

$$\Psi\left(t-\frac{t}{n+1}, n+1\right) = \Psi_0(t) - \frac{1}{2}\ln n + O\left(\frac{1}{n}\right) =: \Psi(t,n)+\delta\Psi, \qquad \delta\Psi = O\left(\frac{1}{n}\right).$$

3.1.2. *Derivation of the sine kernel.* First, let us compute

$$nK_{n-1}(nt,nt) = \lim_{t_1\to t}\sqrt{\frac{\gamma_n}{\gamma_{n-1}}}\frac{\widetilde{q}_n(nt)\widetilde{q}_{n-1}(nt_1)-\widetilde{q}_n(nt_1)\widetilde{q}_{n-1}(nt)}{t-t_1}.$$

We have (recall notation (3.3))

$$nK_{n-1}(nt,nt) = \sqrt{\frac{\gamma_n}{\gamma_{n-1}}}\lim_{t_1\to t}\frac{1}{t_1-t}\det\begin{pmatrix} e^{\Psi}e^{\delta\Psi}\cos(\Phi+\delta\Phi)[t_1] & e^{\Psi}\cos\Phi[t_1] \\ e^{\Psi}e^{\delta\Psi}\cos(\Phi+\delta\Phi)[t] & e^{\Psi}\cos\Phi[t]\end{pmatrix}.$$

We continue the computation by subtracting the rows and by passing to the limit:

$$nK_{n-1}(nt,nt) = \sqrt{cn(n+b)}\exp(2\Psi+\delta\Psi)\biggl(\Psi'\cdot 0 + \frac{d}{dt}(\delta\Psi)\cos(\Phi+\delta\Phi)\cos\Phi + \frac{d}{dt}(\delta\Phi)(-\sin(\Phi+\delta\Phi))\cos\Phi - \Phi'\det\begin{pmatrix}\sin(\Phi+\delta\Phi) & \sin\Phi\\ \cos(\Phi+\delta\Phi) & \cos\Phi\end{pmatrix}\biggr).$$

Since

$$e^{2\Psi} = \frac{2+O\left(\frac{1}{n}\right)}{\pi\sqrt{-D}}, \qquad \delta\Phi \asymp 1, \qquad \delta\Psi = O\left(\frac{1}{n}\right),$$
$$\sin(\Phi+\delta\Phi)\cos\Phi = \frac{1}{2}(\sin(2\Phi+\delta\Phi)+\sin(\delta\Phi))$$

(where the symbol $\asymp$ stands for the asymptotic boundedness above and below) and also

$$\sin(\delta\Phi) = \sin\left[\arccos\frac{t-(c-1)}{2\sqrt{c}} + O\left(\frac{1}{n}\right)\right] = \frac{\sqrt{-\widetilde{D}}}{2\sqrt{c}} + O\left(\frac{1}{n}\right) = \frac{\sqrt{-D}}{2n\sqrt{c}} + O\left(\frac{1}{n}\right),$$

we eventually obtain

$$\begin{aligned}
&nK_{n-1}(nt,nt)\\
&\quad= \sqrt{cn(n+b)}e^{2\Psi}\left[1+O\left(\frac{1}{n}\right)\right]\left(O\left(\frac{1}{n}\right) - (\delta\Phi)'\sin(\Phi+\delta\Phi)\cos\Phi - \Phi'\sin(\delta\Phi)\right)\\
&\quad= -e^{2\Psi}[n\sqrt{c}+O(1)]\sin(\delta\Phi)\left[\Phi' + (\delta\Phi)'\frac{\sin(\Phi+\delta\Phi)\cos\Phi}{\sin(\delta\Phi)} + O\left(\frac{1}{n}\right)\right]\\
&\quad= -\frac{2n\sqrt{c}+O(1)}{\pi\sqrt{-D}}\sin(\delta\Phi)(\Phi'+O(1)) = \left[-\frac{1}{\pi}+O\left(\frac{1}{n}\right)\right](\Phi'+O(1))\\
&\quad= -\frac{1}{\pi}n\varphi_1' + O(1) = \frac{n}{\pi(c-1)}\arccos\frac{(c+1)t-(c-1)^2}{2t\sqrt{c}} + O(1).
\end{aligned}$$

For brevity, we write

$$t_1 = t + \frac{\xi}{nK_{n-1}(nt,nt)}, \qquad t_2 = t + \frac{\eta}{nK_{n-1}(nt,nt)}.$$

Now consider

$$\begin{aligned}
n\mathcal{K}_{n-1}(\xi,\eta) &:= nK_{n-1}\left(nt + \frac{\xi}{K_{n-1}(nt,nt)}, nt + \frac{\eta}{K_{n-1}(nt,nt)}\right) = nK_{n-1}(nt_1,nt_2)\\
&= \frac{1}{t_1-t_2}\sqrt{\frac{\gamma_n}{\gamma_{n-1}}}\det\begin{pmatrix} e^{\Psi+\delta\Psi}\cos(\Phi+\delta\Phi)(t_1) & e^{\Psi}\cos(\Phi)(t_1)\\ e^{\Psi+\delta\Psi}\cos(\Phi+\delta\Phi)(t_2) & e^{\Psi}\cos(\Phi)(t_2)\end{pmatrix}.
\end{aligned}$$

We single out common factors and transform the determinant:

$$\begin{aligned}
&nK_{n-1}(nt_1,nt_2)\\
&\quad= \sqrt{\frac{\gamma_n}{\gamma_{n-1}}}\frac{\exp(\Psi(t_1)+\Psi(t_2))}{t_1-t_2}\det\begin{pmatrix} e^{\delta\Psi(t_1)}\cos(\Phi(t_1)+\delta\Phi(t_1)) & \cos\Phi(t_1)\\ e^{\delta\Psi(t_2)}\cos(\Phi(t_2)+\delta\Phi(t_2)) & \cos\Phi(t_2)\end{pmatrix}\\
&= e^{\Psi(t_1)+\Psi(t_2)+\delta\Psi(t_2)}\sqrt{\frac{\gamma_n}{\gamma_{n-1}}}\biggl(\frac{e^{\delta\Psi(t_1)-\delta\Psi(t_2)}-1}{t_1-t_2}\det\begin{pmatrix}\cos(\Phi(t_1)+\delta\Phi(t_1)) & \cos\Phi(t_1)\\ 0 & \cos\Phi(t_2)\end{pmatrix}\\
&\qquad + \frac{1}{t_1-t_2}\det\begin{pmatrix}\cos(\Phi(t_1)+\delta\Phi(t_1)) & \cos\Phi(t_1)\\ \cos(\Phi(t_2)+\delta\Phi(t_2)) & \cos\Phi(t_2)\end{pmatrix}\biggr).
\end{aligned}$$

In the first term, we have

$$\frac{e^{\delta\Psi(t_1)-\delta\Psi(t_2)}-1}{t_1-t_2}=(\delta\Psi)'(t)+O\big(\max(|t_1-t|,|t_2-t|)\cdot(\delta\Psi)''(t)\big)=O\Big(\frac{1}{n}\Big).$$

The second term can be reduced to the form

$$\begin{aligned}&\frac{\cos(\Phi(t_1)+\delta\Phi(t_1))\cos\Phi(t_2)-\cos\Phi(t_1)\cos(\Phi(t_2)+\delta\Phi(t_2))}{t_1-t_2}\\&=\frac{1}{2(t_1-t_2)}\big[\cos(\Phi(t_1)+\Phi(t_2)+\delta\Phi(t_1))+\cos(\Phi(t_1)-\Phi(t_2)+\delta\Phi(t_1))\\&\qquad-\cos(\Phi(t_1)+\Phi(t_2)+\delta\Phi(t_2))-\cos(\Phi(t_1)-\Phi(t_2)-\delta\Phi(t_2))\big]\\&=\frac{\sin\Big(\dfrac{\delta\Phi(t_2)-\delta\Phi(t_1)}{2}\Big)}{t_1-t_2}\sin\Big(\Phi(t_1)+\Phi(t_2)+\frac{\delta\Phi(t_2)+\delta\Phi(t_1)}{2}\Big)\\&\qquad-\sin\Big(\frac{\delta\Phi(t_1)+\delta\Phi(t_2)}{2}\Big)\frac{\sin\Big(\Phi(t_1)-\Phi(t_2)+\dfrac{\delta\Phi(t_1)-\delta\Phi(t_2)}{2}\Big)}{t_1-t_2}.\end{aligned}$$

Next, we carry out the estimates

$$\begin{aligned}\frac{\sin\Big(\dfrac{\delta\Phi(t_2)-\delta\Phi(t_1)}{2}\Big)}{t_1-t_2}&=\frac{\delta\Phi(t_2)-\delta\Phi(t_1)}{2(t_1-t_2)}\Big(1+O([\delta\Phi(t_2)-\delta\Phi(t_1)]^2)\Big)\\&=\frac{1}{2}(\delta\Phi)'(t)\big[1+O\big((t_1-t_2)^2[(\delta\Phi)'(t)]^2\big)\big]\\&\qquad+O\big(\max(|t_1-t|,|t_2-t|)\cdot(\delta\Phi)''(t)\big)=O(1),\\ \Phi(t_1)-\Phi(t_2)+\frac{\delta\Phi(t_1)-\delta\Phi(t_2)}{2}&=(t_1-t_2)\big[\Phi'(t)+O\big(\max(|t_1-t|,|t_2-t|)\cdot\Phi''(t)\big)\\&\quad+O((\delta\Phi)'(t))\big]=\frac{\xi-\eta}{nK_{n-1}(nt,nt)}\big(n{\varphi'}_1(t)+O(1)\big)=(\xi-\eta)\Big[-\pi+O\Big(\frac{1}{n}\Big)\Big].\end{aligned}$$

Finally, we obtain

$$\begin{aligned}n\mathcal{K}_{n-1}(\xi,\eta)&=\Big[1+O\Big(\frac{1}{n}\Big)\Big]e^{2\Psi(t)}\sqrt{cn(n+b)}\\&\quad\times\Big[O(1)+\sin\Big(\delta\Phi(t)+O\Big(\frac{1}{n}\Big)\Big)\frac{nK_{n-1}(nt,nt)}{\xi-\eta}\sin\Big((\xi-\eta)\Big[\pi+O\Big(\frac{1}{n}\Big)\Big]\Big)\Big]\\&=-\frac{1}{\pi}\Big(n{\varphi'}_1(t)\frac{\sin(\pi(\xi-\eta))}{\pi(\xi-\eta)}+O(1)\Big).\end{aligned}$$

The proof of Theorem 2.1 is complete.

3.2. **Derivation of the limit CD-kernel in transient regions.** Let us prove Theorem 2.3. The asymptotic formula (1.38) provides asymptotics of the functions $\widetilde{q}_{n-1}(x)$ (see (3.3)) convenient for the analysis of the CD-kernel. We have

$$\widetilde{q}_{n-1}(x)\simeq\Xi(x,n)\cdot\exp(E_2(z))\Big[\mathrm{Ai}(h_2(z))\cos\pi\frac{x+b+c}{c-1}-\mathrm{Bi}(h_2(z))\sin\pi\frac{x+b+c}{c-1}\Big],$$

where (recall notation (1.30), (1.35), and (1.39))

$$\Xi(x,n)=2\frac{K_0(x)}{K(x)}\Big(-\frac{sx}{(s-1)^2}\Big)^n\exp\Big(-\frac{V(x)}{2}\Big)\frac{1}{\sqrt{\gamma_{n-1}}}.$$

Here the first terms of the asymptotics have the form

$$\ln \Xi(x,n) = \pi i n - \frac{1}{3}\ln\frac{sx}{s-1} + o(1), \qquad z \in K \Subset \mathbb{C}.$$

Recall that h_2 and E_2 in (1.34) depend on x and n indirectly via z. Hence $h_2(x,n)$ stands for $h_2(z(x,n),x)$. Note also that

$$E_2(z) = \frac{(s-1)^2 z^2}{2\sqrt[3]{x}} + \cdots = o(1).$$

Summarizing, we obtain (having in mind (1.41))

$$\begin{aligned}\widetilde{q}_{n-1}(x) = ((-1)^n + o(1))\sqrt[3]{\frac{s-1}{sx}}\biggl[\operatorname{Ai}(h_2(z))\cos\biggl(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\biggr) \\ - \operatorname{Bi}(h_2(z))\sin\biggl(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\biggr)\biggr].\end{aligned}$$

Let us fix n. Set $\hat{x} := n(s-1)^2$. Let $x, y \in \{x\colon |x-\hat{x}| = O(n^{1/3})\}$, and let u and v satisfy

$$\begin{aligned} x &=: \hat{x} - u\sqrt[3]{s(s-1)^2\hat{x}} = \hat{x} - u\sqrt[3]{s(s-1)^4 n}, \\ y &=: \hat{x} - v\sqrt[3]{s(s-1)^2\hat{x}} = \hat{x} - v\sqrt[3]{s(s-1)^4 n}. \end{aligned}$$

Then

$$h_2(x,n) = u + O(n^{-1/3}), \qquad h_2(y,n) = v + O(n^{-1/3}).$$

It is important to us that all O- and o-terms occurring in the estimates can be expanded in powers of n and can be differentiated, with exponent in n being reduced by 1.

Let us proceed to the asymptotic analysis of the kernel. We have

$$\begin{aligned} K_{n-1}(x,y) &= K_{n-1}\bigl(\hat{x} - u\sqrt[3]{s(s-1)^4 n}, \hat{x} - v\sqrt[3]{s(s-1)^4 n}\bigr) \\ &= \frac{s\sqrt{n(n+b)}}{v-u}\frac{1}{\sqrt[3]{s(s-1)^4 n}}(1+o(1))(-1)^{2n+1}\sqrt[3]{\frac{s-1}{sx}}\sqrt[3]{\frac{s-1}{sy}}\det\begin{pmatrix} A(x) & B(x) \\ A(y) & B(y)\end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} A(x) &:= \operatorname{Ai}(h_2(x,n+1))\cos\biggl(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\biggr) \\ &\qquad - \operatorname{Bi}(h_2(x,n+1))\sin\biggl(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\biggr) \\ &= \operatorname{Ai}\biggl(h_2(x,n) + \sqrt[3]{\frac{(s-1)^2}{sn}} + O\biggl(\frac{1}{n}\biggr)\biggr)\cos\biggl(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\biggr) \\ &\qquad - \operatorname{Bi}\biggl(h_2(x,n) + \sqrt[3]{\frac{(s-1)^2}{sn}} + O\biggl(\frac{1}{n}\biggr)\biggr)\sin\biggl(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\biggr) \end{aligned}$$

and

$$B(x) := \operatorname{Ai}(h_2(x,n))\cos\biggl(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\biggr) - \operatorname{Bi}(h_2(x,n))\sin\biggl(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\biggr).$$

Let us transform the determinant by expanding the rows into separate sums over sines and cosines. We obtain a sum of four determinants and estimate each of these. Thus,

$$\begin{aligned}\det\begin{pmatrix}A(x) & B(x)\\ A(y) & B(y)\end{pmatrix} &= \cos\Big(\pi\frac{x+b+c}{c-1}+\varepsilon(x)\Big)\cos\Big(\pi\frac{y+b+c}{c-1}+\varepsilon(y)\Big)\Delta_1\\ &\quad-\cos\Big(\pi\frac{x+b+c}{c-1}+\varepsilon(x)\Big)\sin\Big(\pi\frac{y+b+c}{c-1}+\varepsilon(y)\Big)\Delta_2\\ &\quad-\sin\Big(\pi\frac{x+b+c}{c-1}+\varepsilon(x)\Big)\cos\Big(\pi\frac{y+b+c}{c-1}+\varepsilon(y)\Big)\Delta_3\\ &\quad+\sin\Big(\pi\frac{x+b+c}{c-1}+\varepsilon(x)\Big)\sin\Big(\pi\frac{y+b+c}{c-1}+\varepsilon(y)\Big)\Delta_4.\end{aligned}\tag{3.6}$$

Here

$$\begin{aligned}\Delta_1 &= \det\begin{pmatrix}\mathrm{Ai}(h_2(x,n+1)) & \mathrm{Ai}(h_2(x,n))\\ \mathrm{Ai}(h_2(y,n+1)) & \mathrm{Ai}(h_2(y,n))\end{pmatrix}\\ &= \sqrt[3]{\frac{(s-1)^2}{sn}}\det\begin{pmatrix}\mathrm{Ai}'(u+O(n^{-1/3})) & \mathrm{Ai}(u+O(n^{-1/3}))\\ \mathrm{Ai}'(v+O(n^{-1/3})) & \mathrm{Ai}(v+O(n^{-1/3}))\end{pmatrix}+O\Big(\frac{1}{n}\Big)\\ &= \sqrt[3]{\frac{(s-1)^2}{sn}}(\mathrm{Ai}'(u)\,\mathrm{Ai}(v)-\mathrm{Ai}(u)\,\mathrm{Ai}'(v))+O(n^{-2/3}).\end{aligned}$$

Likewise,

$$\begin{aligned}\Delta_2 &= \sqrt[3]{\frac{(s-1)^2}{sn}}(\mathrm{Ai}'(u)\,\mathrm{Bi}(v)-\mathrm{Ai}(u)\,\mathrm{Bi}'(v))+O(n^{-2/3}),\\ \Delta_3 &= \sqrt[3]{\frac{(s-1)^2}{sn}}(\mathrm{Bi}'(u)\,\mathrm{Ai}(v)-\mathrm{Bi}(u)\,\mathrm{Ai}'(v))+O(n^{-2/3}),\\ \Delta_4 &= \sqrt[3]{\frac{(s-1)^2}{sn}}(\mathrm{Bi}'(u)\,\mathrm{Bi}(v)-\mathrm{Bi}(u)\,\mathrm{Bi}'(v))+O(n^{-2/3}).\end{aligned}$$

This gives the asymptotic formula (2.10) in Theorem 2.3.

To derive the formula on the diagonal ($x=y$), note that $u=v$ in this case. When passing to the limit, we take into account the fact that $\frac{x-y}{u-v}=-\sqrt[3]{s(s-1)^4n}$, and here we need the differentiability of the remainders. In (3.6), we transform the sum of the second and third terms by trigonometric formulas, thus obtaining

$$\begin{aligned}K_{n-1}(x,x) &= \frac{(1+o(1))}{(s-1)^2}\Big[\cos^2\Big(\pi\frac{x+b+c}{c-1}+\varepsilon(x)\Big)\lim_{v\to u}\frac{\Delta_1}{u-v}\\ &\quad+\sin^2\Big(\pi\frac{x+b+c}{c-1}+\varepsilon(x)\Big)\lim_{v\to u}\frac{\Delta_4}{u-v}-\sin\Big(2\pi\frac{x+b+c}{c-1}+\varepsilon(x)\Big)\lim_{v\to u}\frac{\Delta_2+\Delta_3}{2(u-v)}\\ &\quad-\lim_{v\to u}\frac{\sin\Big(\frac{\pi(x-y)}{c-1}+\varepsilon(x)-\varepsilon(y)\Big)}{u-v}\lim_{v\to u}\frac{\Delta_3-\Delta_2}{2}\Big]\\ &= \frac{\pi}{c-1}[\mathrm{Ai}(u)\,\mathrm{Bi}'(u)-\mathrm{Bi}(u)\,\mathrm{Ai}'(u)]+o(1)=\frac{1}{c-1}+o(1).\end{aligned}$$

The proof of Theorem 2.3 is complete.

Note that the terms neglected in the last formula because of the assumption that $u,v\in K\Subset\mathbb{C}$ include the term $(\mathrm{Bi}'^2(u)-\mathrm{Bi}(u)\,\mathrm{Bi}''(u))O(n^{-1/3})$, which becomes comparable to the leading term if

$$u = \sqrt[3]{4}\frac{\frac{1}{4}\ln n + \ln\ln n + O(1)}{\sqrt[3]{\ln n}}, \text{ i.e., } x = n(s-1)^2 - \sqrt[3]{\frac{4n}{\ln n}}\left(\frac{1}{4}\ln n + \ln\ln n + O(1)\right).$$

3.3. **Derivation of the limit CD-kernels in the saturation region.**

3.3.1. *Transformation of the asymptotics of $\widetilde{q}_{n-1}(x)$ in the saturation region.* We have

$$\widetilde{q}_{n-1}(x) = \frac{1}{\sqrt{\gamma_{n-1}}}K_0(x)\exp\left(-\frac{V(x)}{2}\right)\left[\left(1 - \frac{\overline{K}_2}{K_2}(x)\right)\Pi_1 + i\Pi_2\right],$$

where, with ε used to denote a different thing for convenience,

$$\frac{\overline{K}_2}{K_2}(x) = \exp\left(2\pi i\left[\frac{2x+(b+1)(c+1)}{2(c-1)} - \frac{b-1}{2}\right] + 2\varepsilon(x)\right)$$
$$= \exp\left(2\pi i\frac{x+b+c}{c-1} + 2i\varepsilon(x)\right).$$

Thus,

$$1 - \frac{\overline{K}_2}{K_2} = -2i\exp\left(\pi i\frac{x+b+c}{c-1} + i\varepsilon(x)\right)\sin\left(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\right).$$

Let us represent $\widetilde{q}_{n-1}(x)$ in the form

$$\widetilde{q}_{n-1}(x) =: \exp\{\Psi(x,n)\}\sin\left(\pi\frac{x+b+c}{c-1} + \varepsilon(x)\right) + \exp\{\Xi(x,n)\}. \tag{3.7}$$

Then

$$\begin{aligned}\Psi(x,n) = &-\frac{1}{2}\ln\gamma_{n-1} + \ln K_0(x) - \frac{V(x)}{2} + \pi i\frac{x+b+c}{c-1} + i\varepsilon(x) - \frac{\pi i}{2} + \ln 2\\ &- n - \frac{1}{4}\ln D - \frac{\pi i}{2} + \left(n - \frac{1}{2}\right)\left(\pi i + \ln\frac{(c+1)n - x + \sqrt{D}}{2}\right)\\ &+ \frac{2x+(c+1)(b+1)}{2(c-1)}\left(-\pi i + \ln\left[x + (c-1)n - \frac{2c(x+b)}{c-1} - \sqrt{D}\right]\right) + \pi i\frac{b}{2}\\ &+ \frac{b}{2}\ln[-2bc(n+b) + ((c+1)n - x + \sqrt{D})(x + (c+1)b)] + O(x^{-1}).\end{aligned}$$

Let us continue the transformations:

$$\begin{aligned}\Psi(x,n) = &\,\pi i(n-1) + \frac{1}{2}\ln\frac{2}{\pi} - \frac{n+b}{2}\ln c + \frac{b+1}{2}\ln(c-1) - \frac{bc}{c-1} - \frac{V(x)}{2}\\ &+ \left(n - \frac{1}{2}\right)\ln\frac{(c+1)n - x + \sqrt{D}}{2n} - \frac{1}{2}\ln n - \frac{1}{4}\ln\frac{D}{n^2}\\ &+ \frac{2x+(c+1)(b+1)}{2(c-1)}\ln\frac{(c-1)^2 n - (c+1)x - 2bc - (c-1)\sqrt{D}}{2x}\\ &+ \frac{b}{2}\ln\frac{-2bc(n+b) + ((c+1)n - x + \sqrt{D})(x + (c+1)b)}{2nx^2} + O(x^{-1}) + O(n^{-1}).\end{aligned}$$

Finally, we obtain

$$\Psi(x,n) = \pi i(n-1) - \frac{1}{2}\ln n + \widetilde{\Psi}(x,n),$$

where

$$\widetilde{\Psi} := \left(n-\frac{1}{2}\right)\ln\frac{(c+1)n-x+\sqrt{D}}{2n\sqrt{c}} - \frac{1}{4}\ln\frac{D}{n^2} + \frac{1}{2}\ln\frac{2}{\pi} - \frac{bc}{c-1} \\ + \frac{2x+(b+1)(c+1)}{2(c-1)}\ln\frac{(c-1)^2n-(c+1)x-2bc-(c-1)\sqrt{D}}{2x\sqrt{c}} \\ + \frac{b}{2}\ln\frac{-2bc(n+b)+((c+1)n-x+\sqrt{D})(x+(c+1)b)}{2\sqrt{c}nx} + O(x^{-1}).$$

For Ξ in (3.7), we have

$$\Xi(x,n) = -\frac{1}{2}\ln\gamma_{n-1} - \frac{V(x)}{2} + \ln K_0(x) + \left(n-\frac{1}{2}\right)\left(-\pi i + \ln\frac{(c+1)n-x-\sqrt{D}}{2}\right) \\ + O(x^{-1}) + \frac{\pi i}{2} + \frac{2x+(c+1)(b+1)}{2(c-1)}\left(\pi i + \ln\frac{(c-1)^2n-(c+1)x-2bc+(c-1)\sqrt{D}}{c-1}\right) \\ - \frac{\pi i b}{2} + \frac{b}{2}\ln(-2bc(n+b) + \big((c+1)n-x-\sqrt{D}\big)(x+(c+1)b)) - n - \frac{\pi i}{2} - \frac{1}{4}\ln D \\ = -\pi i n + \pi i\frac{x+b+c}{c-1} + \widetilde{\Xi}(x,n) - \frac{1}{2}\ln n + O(x^{-1}) + O(n^{-1}),$$

where

$$\widetilde{\Xi}(x,n) := \left(n-\frac{1}{2}\right)\ln\frac{(c+1)n-x-\sqrt{D}}{2n\sqrt{c}} - \frac{1}{4}\ln\frac{D}{n^2} + \frac{1}{2}\ln\frac{1}{2\pi} \\ + \frac{2x+(c+1)(b+1)}{2(c-1)}\ln\frac{(c-1)^2n-(c+1)x-2bc+(c-1)\sqrt{D}}{2x\sqrt{c}} \\ + \frac{b}{2}\ln\frac{-2bc(n+b)+\big((c+1)n-x-\sqrt{D}\big)(x+(c+1)b)}{2\sqrt{c}nx} - \frac{bc}{c-1}.$$

We point out that the asymptotics of $\widetilde{\Psi}(nt,n)$ and $\widetilde{\Xi}(nt,n)$ do not contain terms logarithmic in n and can be expanded in powers of n with coefficients depending on t. To obtain the asymptotics of $\widetilde{q}_{n-1}(x)$ in the mode $x = nt$ as $n \to \infty$, we introduce the function

$$\text{(3.8)}\quad \Sigma(t,n) := \frac{1}{2}\left[\Psi(nt,n) + \pi i\left(\frac{nt+b+c}{c-1} + 1 - 2n\right) - \Xi(nt,n)\right] \\ = \frac{1}{2}\left[\widetilde{\Psi}(nt,n) - \widetilde{\Xi}(nt,n)\right] = \frac{1}{2}\ln 2 + \left(n+\frac{b-1}{2}\right)\cosh^{-1}\frac{c+1-t}{2\sqrt{c}} \\ - \frac{2nt+(b+1)(c+1)}{2(c-1)}\cosh^{-1}\frac{(c-1)^2-(c+1)t}{2t\sqrt{c}} + O\left(\frac{1}{n}\right).$$

This function specifies the scale in a neighborhood of mass points. We also compute the expression

$$\text{(3.9)}\quad \frac{1}{2}\left[\pi i + \Psi(nt,n) - \pi i\frac{nt+b+c}{c-1} + \Xi(nt,n) + \ln n\right] = -\frac{1}{2}\ln\pi - \frac{1}{4}\ln\widetilde{D}(t) + O\left(\frac{1}{n}\right).$$

As a result, since $n \in \mathbb{N}$, we have obtained the representation

$$
\begin{aligned}
\widetilde{q}_{n-1}(nt) &= \frac{(-1)^{n-1}}{\sqrt{n}}\bigg[e^{\widetilde{\Psi}(nt,n)}\sin\bigg(\pi\frac{nt+b+c}{c-1}+\varepsilon(nt)\bigg) \\
&\qquad\qquad -e^{\widetilde{\Xi}(nt,n)}\exp\bigg(i\pi\frac{nt+b+c}{c-1}+i\varepsilon(nt)\bigg)\bigg], \\
\widetilde{q}_{n-1}(nt) &= (-1)^{n-1}\sqrt{\frac{1}{\pi n\sqrt{\widetilde{D}}}}\bigg[\exp\{\Sigma\}\sin\bigg(\pi\frac{nt+b+c}{c-1}+\varepsilon(nt)\bigg) \\
&\qquad - \exp\{-\Sigma\}\exp\bigg(i\pi\frac{nt+b+c}{c-1}+i\varepsilon(nt)\bigg)\bigg]\bigg(1+O\Big(\frac{1}{n}\Big)\bigg).
\end{aligned}
\tag{3.10}
$$

3.3.2. *Proof of Theorem* 2.4. For the mode in a neighborhood of an arbitrary point of the saturation region, we have

$$
\begin{aligned}
K_{n-1}(nt+\xi,nt+\eta) &= \frac{1}{\xi-\eta}\det\begin{pmatrix}\widetilde{q}_{n-1}(nt+\eta) & \widetilde{q}_n(nt+\eta)\\ \widetilde{q}_{n-1}(nt+\xi) & \widetilde{q}_n(nt+\xi)\end{pmatrix}\sqrt{\frac{\gamma_n}{\gamma_{n-1}}} \\
&= -\frac{1}{n}\sqrt{\frac{\gamma_n}{\gamma_{n-1}}}\sqrt{\frac{n}{n+1}}\frac{1}{\xi-\eta}\det\begin{pmatrix}\widetilde{A}(\eta) & \widetilde{B}(\eta)\\ \widetilde{A}(\xi) & \widetilde{B}(\xi)\end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{A}(\eta) := \exp\{\widetilde{\Psi}(nt+\eta,n)\}\sin\bigg(\pi\frac{nt+\eta+b+c}{c-1}+\varepsilon(nt+\eta)\bigg) \\
- \exp\{\widetilde{\Xi}(nt+\eta,n)\}\exp\bigg\{\pi i\frac{nt+\eta+b+c}{c-1}+i\varepsilon(nt+\eta)\bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{B}(\eta) := \exp\bigg\{\widetilde{\Psi}(nt+\eta,n)+\delta\widetilde{\Psi}\Big(t+\frac{\eta}{n},n\Big)\bigg\}\sin\bigg(\pi\frac{nt+\eta+b+c}{c-1}+\varepsilon(nt+\eta)\bigg) \\
- \exp\bigg\{\widetilde{\Xi}(nt+\eta,n)+\delta\widetilde{\Xi}\Big(t+\frac{\eta}{n},n\Big)\bigg\}\exp\bigg\{\pi i\frac{nt+\eta+b+c}{c-1}+i\varepsilon(nt+\eta)\bigg\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\det\begin{pmatrix}\widetilde{A}(\eta) & \widetilde{B}(\eta)\\ \widetilde{A}(\xi) & \widetilde{B}(\xi)\end{pmatrix} &= \exp\{\widetilde{\Psi}(nt+\xi,n)+\widetilde{\Psi}(nt+\eta,n)+\delta\widetilde{\Psi}(t,n)\} \\
&\times\sin\bigg(\pi\frac{nt+\eta+b+c}{c-1}+\varepsilon(nt+\eta)\bigg)\sin\bigg(\pi\frac{nt+\xi+b+c}{c-1}+\varepsilon(nt+\xi)\bigg) \\
&\times\frac{\exp\Big[\delta\widetilde{\Psi}\Big(t+\frac{\xi}{n},n\Big)\Big]-\exp\Big[\delta\widetilde{\Psi}\Big(t+\frac{\eta}{n},n\Big)\Big]}{\exp(\delta\widetilde{\Psi}(t,n))(\xi-\eta)} \\
&\qquad + O\big(\exp\{\widetilde{\Psi}(nt,n)+\widetilde{\Xi}(nt,n)+O(|\xi|+|\eta|)\}\big).
\end{aligned}
$$

Consider the expansions

$$
\begin{aligned}
\widetilde{\Psi}(nt,n) &= \Psi_1(t)n+\Psi_0(t)+\Psi_{-1}(n)n^{-1}+\cdots, \\
\delta\widetilde{\Psi}(t,n) &= [\Psi_1(t)-t\Psi_1'(t)]+\frac{\frac{t^2}{2}\Psi_1''-t\Psi_0'}{n}+\cdots.
\end{aligned}
$$

Here

$$\begin{aligned}\Psi_1 &= \cosh^{-1}\frac{c+1-t}{2\sqrt{c}} - \frac{t}{c-1}\cosh^{-1}\frac{(c-1)^2-(c+1)t}{2t\sqrt{c}},\\ \Psi_0 &= \frac{b-1}{2}\cosh^{-1}\frac{c+1-t}{2\sqrt{c}}\\ &\quad - \frac{(b+1)(c+1)}{2(c-1)}\cosh^{-1}\frac{(c-1)^2-(c+1)t}{2t\sqrt{c}} + \frac{1}{2}\ln\frac{2}{\pi} - \frac{1}{4}\ln\widetilde{D}.\end{aligned}$$

As a result, we obtain

$$\begin{aligned}K_{n-1}(nt+\xi,nt+\eta) &= \frac{\sqrt{c}}{n}t\Psi_1''(t)\cdot\exp\{2n\Psi_1(t)+2\Psi_0(t)+\Psi_1(t)-t\Psi_1'(t)\}\\ &\times e^{(\xi+\eta)\Psi_1'(t)}\sin\Bigl(\pi\frac{nt+\eta+b+c}{c-1}+\varepsilon(nt+\eta)\Bigr)\sin\Bigl(\pi\frac{nt+\xi+b+c}{c-1}+\varepsilon(nt+\xi)\Bigr)\\ &\times\Bigl[1+O\Bigl(\frac{1}{n}\Bigr)\Bigr]+O\Bigl(\frac{\exp\{(|x-y|-t)\Psi_1'(t)+\Psi_1(t)\}}{\sqrt{\widetilde{D}}}\Bigr), \qquad t\in\mathbb{R}_+.\end{aligned}$$

Note that the last $O(\cdot)$ is $O(1)$. The proof of Theorem 2.4 is complete.

3.3.3. *Proof of Theorem* 2.5. Now consider the modes

$$\begin{aligned}&K_{n-1}(X_k+e^{-2\Sigma(t,n)}n\xi, X_k+\eta),\\ &K_{n-1}(X_k+e^{-2\widetilde{\Sigma}}n\xi, X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta),\end{aligned}\qquad n\to\infty. \tag{3.11}$$

Here the $X_k := k(c-1)-b-c+\varepsilon_k$ are the mass points of the orthogonality measure, and hence

$$\pi\frac{X_k+b+c}{c-1}+\varepsilon(X_k)=\pi k.$$

By using the Schwarz lemma and the Cauchy formula, having surrounded the point X_k with a contour of radius $(c-1)/2$, we find that the sine factor in our formulas has the following representation in a neighborhood of X_k:

$$\sin\Bigl(\pi\frac{x+b+c}{c-1}+\varepsilon(x)\Bigr) = (x-X_k)\left[\frac{\sin\pi\dfrac{x+b+c}{c-1}}{x+b+c-k(c-1)}+\hat{\varepsilon}(x)\right], \tag{3.12}$$

where $\hat{\varepsilon}(x)$, as well as $\varepsilon(x)$, decays more rapidly than any power of x as $|x|\to\infty$.

Owing to this, we can consider exponentially small neighborhoods of the points X_k, even though, without additional information about the orthogonality measure, the position of the points X_k themselves is known from the recursion relations only with power-law accuracy.

In what follows, we write $t := X_k/n$, and the scale of contraction of one variable of the kernel is given by the function

$$e^{-2\Sigma(t,n)} := \frac{1}{2}\Bigl(\frac{c+1-t+\sqrt{\widetilde{D}}}{2\sqrt{c}}\Bigr)^{-2n-b+1}\Bigl(\frac{(c-1)^2-(c+1)t+(c-1)\sqrt{\widetilde{D}}}{2t\sqrt{c}}\Bigr)^{2k+b-1}.$$

In this formula, Σ coincides with (3.8) modulo $O\bigl(\frac{1}{n}\bigr)$. To account for the changes in Σ, one should write formula (3.10) with distinct exponents in each factor (which will be denoted by H and $\widetilde{H}$), each of which has the form (3.9)

$$\widetilde{q}_{n-1}(nt) = \frac{(-1)^{n-1}}{\sqrt{n}}\big[e^{H+\Sigma}\sin\Phi(nt) - e^{\widetilde{H}-\Sigma}\exp(i\Phi(nt))\big],$$
$$\Phi(x) = \pi\frac{x+b+c}{c-1} + \varepsilon(x).$$

Let us find the asymptotics of the functions $\widetilde{q}_n$ forming the kernel (3.11). By (3.12), in a neighborhood of X_k we have (we omit the arguments of ε and $\hat{\varepsilon}$ for brevity)

$$\sin\Phi(nt+z) = (-1)^k z\left(\frac{\sin\dfrac{\pi}{c-1}(z+\varepsilon(\cdot))}{z+\varepsilon(\cdot)} + (-1)^k\hat{\varepsilon}(\cdot)\right)$$
$$= (-1)^k z\left(\frac{\sin\dfrac{\pi z}{c-1}}{z} + O(\|\varepsilon\|)\right);$$

hence

$$\Phi(nt+z) = \pi k + \frac{\pi z}{c-1} + O(z\|\varepsilon\|).$$

We continue the computation for $\widetilde{q}_{n-1}$:

$$\begin{aligned}\widetilde{q}_{n-1}\big(X_k + e^{-2\widetilde{\Sigma}}n\xi\big) = \frac{(-1)^{n+k-1}}{\sqrt{n}}\Bigg[&e^{\widetilde{\Psi}-2\widetilde{\Sigma}}\frac{\pi\xi n}{c-1}\bigg(1+O(\|\varepsilon\|)\\ &+ O\big(|e^{-2\widetilde{\Sigma}}n|^2\big) + e^{-2\widetilde{\Sigma}}O\bigg(\bigg|\frac{d}{dt}\widetilde{\Psi}(nt,n)\bigg|\bigg)\bigg)\\ &- e^{\widetilde{\Xi}}\bigg(1+O(\|\varepsilon\|)+O(|e^{-2\widetilde{\Sigma}}n|) + e^{-2\widetilde{\Sigma}}O\bigg(\bigg|\frac{d}{dt}\widetilde{\Xi}(nt,n)\bigg|\bigg)\bigg)\Bigg].\end{aligned} \tag{3.13}$$

By taking into account the properties of $\widetilde{\Psi}$ and $\widetilde{\Xi}$ and by assuming that $ne^{-\widetilde{\Sigma}} = o(1)$, we obtain

$$\widetilde{q}_{n-1}\big(X_k + e^{-2\widetilde{\Sigma}}n\xi\big) = (-1)^{n+k-1}\frac{e^{\widetilde{\Psi}-2\Sigma}}{\sqrt{n}}\left[e^{2(\Sigma-\widetilde{\Sigma})}\frac{\pi\xi n}{c-1}\left(1+O\Big(\frac{1}{n}\Big)\right) - 1 + O\Big(\frac{1}{n}\Big)\right],$$

$$\widetilde{q}_{n-1}\big(X_k + e^{-2\Sigma}n\xi\big) = (-1)^{n+k-1}\frac{e^{\widetilde{\Psi}-2\Sigma}}{\sqrt{n}}\left[\frac{\pi\xi n}{c-1} - 1 + O\Big(\frac{1}{n}\Big)\right],$$

$$\begin{aligned}\widetilde{q}_{n-1}\big(X_k+\eta\big) = \frac{(-1)^{n+k-1}}{\sqrt{n}}\Bigg[&e^{\widetilde{\Psi}(nt+\eta.n)}\sin\Big(\frac{\pi\eta}{c-1} + \eta O(\|\varepsilon\|)\Big)\\ &- e^{\widetilde{\Xi}(nt+\eta,n)}\exp\Big(i\frac{\pi\eta}{c-1} + i\eta O(\|\varepsilon\|)\Big)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{q}_n\big(X_k + e^{-2\Sigma}n\xi\big) = \frac{(-1)^{n+k}}{\sqrt{n+1}}\Bigg[&e^{\widetilde{\Psi}(nt,n)-2\Sigma}e^{\delta\widetilde{\Psi}(t,n)}\frac{\pi\xi n}{c-1}(1+O(\|\varepsilon\|+ne^{-2\Sigma}))\\ &- e^{\widetilde{\Xi}(nt,n)+\delta\widetilde{\Xi}(t,n)}(1+O(\|\varepsilon\|))\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{q}_n\big(X_k+\eta\big) = \frac{(-1)^{n+k}}{\sqrt{n+1}}\Bigg[&e^{\widetilde{\Psi}(nt+\eta,n)}\sin\Big(\frac{\pi\eta}{c-1} + \eta O(\|\varepsilon\|)\Big)e^{\delta\widetilde{\Psi}(t+\eta/n,n)}\\ &- e^{\widetilde{\Xi}(nt+\eta,n)}\exp\Big(i\frac{\pi\eta}{c-1} + i\eta O(\|\varepsilon\|)\Big)e^{\delta\widetilde{\Xi}(t+\eta/n,n)}\Bigg].\end{aligned}$$

Note that $O(e^{-\Sigma}) = O(\frac{1}{n})$ and

$$\Sigma(t+\eta/n, n) = \Sigma(t,n) + \eta\Psi_1' + O\Big(\frac{1}{n}\Big).$$

Let us proceed to the derivation of the limit kernel. We have

$$\begin{aligned}
&K_{n-1}(X_k + e^{-2\Sigma}n\xi, X_k+\eta) \\
&\quad= \frac{\sqrt{\dfrac{\gamma_n}{\gamma_{n-1}}}}{\eta + O(ne^{-2\Sigma})} \det\begin{pmatrix} \widetilde{q}_n(X_k+\eta) & \widetilde{q}_n(X_k+e^{2\Sigma}n\xi) \\ \widetilde{q}_{n-1}(X_k+\eta) & \widetilde{q}_{n-1}(X_k+e^{2\Sigma}n\xi)\end{pmatrix} \\
&\quad= \frac{\sqrt{\dfrac{c(n+b)}{n+1}}}{\eta + O(ne^{-2\Sigma})} e^{\widetilde{\Psi}(nt+\eta,n)+\widetilde{\Psi}(nt,n)-2\Sigma} \sin\Big(\frac{\pi\eta}{c-1} + \eta O(\|\varepsilon\|)\Big) \\
&\qquad\times\Big[e^{\delta\widetilde{\Psi}(t+\eta/n,n)}\Big(1 - \frac{\pi\xi n}{c-1} + O\Big(\frac{1}{n}\Big)\Big) \\
&\qquad\quad + e^{\delta\widetilde{\Psi}(t,n)}\frac{\pi\xi n}{c-1}\Big(1+\eta O(\|\varepsilon\|)\Big) - e^{\delta\widetilde{\Xi}(t,n)}\Big(1+O\Big(\frac{1}{n}\Big)\Big) + O(e^{\Xi})\Big].
\end{aligned}$$

Next,

(3.14)

$$\begin{aligned}
&K_{n-1}\big(X_k + e^{-2\Sigma}n\xi, X_k+\eta\big) \\
&\quad= \sqrt{c}e^{2\widetilde{\Psi}-2\Sigma}\frac{\sin\dfrac{\pi\eta}{c-1}}{\eta}e^{\eta\Psi_1'} \\
&\qquad\times\Big[2\sinh\delta\Psi(t,n) + O\Big(\frac{1}{n}\Big) - \frac{\pi n\xi}{c-1}\big(e^{\delta\Psi(t+\eta/n,n)} - e^{\delta\Psi(t,n)}\big)\Big]\Big(1+O\Big(\frac{1}{n}\Big)\Big) \\
&\quad= \frac{1}{\pi}\frac{\sqrt{c}}{\sqrt{\widetilde{D}}}\frac{\sin\frac{\pi\eta}{c-1}}{\eta}e^{\eta\Psi_1'} \\
&\qquad\times\Big[2\sinh(\Psi_1 - t\Psi_1') + \frac{\pi\xi\eta}{c-1}t\Psi_1''e^{\Psi_1-t\Psi_1'}\Big] + O\Big(\frac{1}{n}\Big).
\end{aligned}$$

The proof of Theorem 2.5 is complete.

The case in which the scales are distributed between the arguments of the kernel (2.16) requires lengthier transformations. First, one should substitute the arguments in the original form,

$$\begin{aligned}
\widetilde{q}_{n-1}(X_k + e^{-2\widetilde{\Sigma}}n\xi) &= \frac{(-1)^{n-1}}{n}\big[e^{\widetilde{\Psi}(X_k+e^{-2\widetilde{\Sigma}}n\xi,n)}\sin\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi) \\
&\qquad - e^{\widetilde{\Xi}(X_k+e^{-2\widetilde{\Sigma}}n\xi,n)}\exp i\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi)\big], \\
\widetilde{q}_n(X_k + e^{-2\widetilde{\Sigma}}n\xi) &= \frac{(-1)^n}{n+1}\big[e^{\widetilde{\Psi}(X_k+e^{-2\widetilde{\Sigma}}n\xi,n)+\delta\widetilde{\Psi}(t_k+e^{-2\widetilde{\Sigma}}\xi,n)}\sin\Phi\big(X_k+e^{-2\widetilde{\Sigma}}n\xi\big) \\
&\qquad - e^{\widetilde{\Xi}(X_k+e^{-2\widetilde{\Sigma}}n\xi,n)+\delta\widetilde{\Xi}(t_k+e^{-2\widetilde{\Sigma}}\xi,n)}\exp i\Phi\big(X_k+e^{-2\widetilde{\Sigma}}n\xi\big)\big].
\end{aligned}$$

Now we write out the kernel, multiply out, and arrange the terms:

$$K_{n-1}\big(X_k+e^{-2\widetilde{\Sigma}}n\xi, X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta\big) = -\sqrt{\frac{c(n+b)}{n+1}}\bigg[e^{\widetilde{\Psi}(X_k+e^{-2\widetilde{\Sigma}}n\xi,n)+\widetilde{\Psi}(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta,n)}$$
$$\times\frac{e^{\delta\widetilde{\Psi}(t_k+e^{-2\widetilde{\Sigma}}\xi,n)}-e^{\delta\widetilde{\Psi}(t_k+e^{-2\widetilde{\widetilde{\Sigma}}}\eta,n)}}{n(e^{-2\widetilde{\Sigma}}\xi-e^{-2\widetilde{\widetilde{\Sigma}}}\eta)}\sin\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi)\sin\Phi(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta)$$
$$+e^{\widetilde{\Xi}(X_k+e^{-2\widetilde{\Sigma}}n\xi,n)+\widetilde{\Xi}(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}\eta,n)}\frac{e^{\delta\widetilde{\Xi}(t_k+e^{-2\widetilde{\Sigma}}\xi,n)}-e^{\delta\widetilde{\Xi}(t_k+e^{-2\widetilde{\widetilde{\Sigma}}}\eta,n)}}{n(e^{-2\widetilde{\Sigma}}\xi-e^{-2\widetilde{\widetilde{\Sigma}}}\eta)}O(1)$$
$$-\frac{E_1}{n(e^{-2\widetilde{\Sigma}}\xi-e^{-2\widetilde{\widetilde{\Sigma}}}\eta)}\sin\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi)\exp i\Phi(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta)$$
$$+\frac{E_2}{n(e^{-2\widetilde{\Sigma}}\xi-e^{-2\widetilde{\widetilde{\Sigma}}}\eta)}\sin\Phi(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta)\exp i\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi)\bigg],$$

where

$$E_1=e^{\widetilde{\Psi}(X_k+e^{-2\widetilde{\Sigma}}n\xi,n)+\widetilde{\Xi}(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta,n)}\big(e^{\delta\widetilde{\Psi}(t_k+e^{-2\widetilde{\Sigma}}\xi,n)}-e^{\delta\widetilde{\Xi}(t_k+e^{-2\widetilde{\widetilde{\Sigma}}}\eta,n)}\big),$$
$$E_2=e^{\widetilde{\Psi}(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta,n)+\widetilde{\Xi}(X_k+e^{-2\widetilde{\Sigma}}n\xi,n)}\big(e^{\delta\widetilde{\Psi}(t_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta,n)}-e^{\delta\widetilde{\Xi}(t_k+e^{-2\widetilde{\Sigma}}\xi,n)}\big).$$

Let us estimate the first two terms and identically transform the sum of the last two terms. We obtain

$$K_{n-1}\big(X_k+e^{-2\widetilde{\Sigma}}n\xi, X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta\big)$$
$$=-\sqrt{c}\bigg[e^{2\widetilde{\Psi}(X_k,n)+\delta\widetilde{\Psi}(t_k,n)}\frac{(\delta\widetilde{\Psi})'}{n}\big(1+O(ne^{-2\widetilde{\Sigma}})+O(ne^{-2\widetilde{\widetilde{\Sigma}}})\big)$$
$$\times\sin\bigg(\frac{\pi n\xi}{c-1}e^{-2\widetilde{\Sigma}}[1+O(\|\varepsilon\|)]\bigg)\sin\bigg(\frac{\pi n\eta}{c-1}e^{-2\widetilde{\widetilde{\Sigma}}}[1+O(\|\varepsilon\|)]\bigg)$$
$$+O(e^{2\widetilde{\Xi}(X_k,n)})-\frac{E_1-E_2}{2n(e^{-2\widetilde{\Sigma}}\xi-e^{-2\widetilde{\widetilde{\Sigma}}}\eta)}\Big[\sin\big(\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi)+\Phi(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta)\big)$$
$$+2i\sin\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi)\sin\Phi(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta)\Big]$$
$$-(E_1+E_2)\frac{\sin(\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi)-\Phi(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta))}{2n(e^{-2\widetilde{\Sigma}}\xi-e^{-2\widetilde{\widetilde{\Sigma}}}\eta)}\bigg].$$

Now let us estimate the fractions. We obtain the following asymptotics:

$$\frac{\sin\big(\Phi(X_k+e^{-2\widetilde{\Sigma}}n\xi)-\Phi(X_k+e^{-2\widetilde{\widetilde{\Sigma}}}n\eta)\big)}{2n(e^{-2\widetilde{\Sigma}}\xi-e^{-2\widetilde{\widetilde{\Sigma}}}\eta)}$$
$$=\frac{1}{2n}\frac{\sin\bigg(\dfrac{\pi n\xi}{c-1}e^{-2\widetilde{\Sigma}}[1+O(\|\varepsilon\|)]-\dfrac{\pi n\eta}{c-1}e^{-2\widetilde{\widetilde{\Sigma}}}[1+O(\|\varepsilon\|)]\bigg)}{e^{-2\widetilde{\Sigma}}\xi-e^{-2\widetilde{\widetilde{\Sigma}}}\eta}$$
$$=\frac{\pi}{2(c-1)}+O(ne^{-2\widetilde{\Sigma}})+O(ne^{-2\widetilde{\widetilde{\Sigma}}});$$

$$
\begin{aligned}
&\frac{E_1 - E_2}{2n(e^{-2\widetilde{\Sigma}}\xi - e^{-2\widetilde{\widetilde{\Sigma}}}\eta)} \\
&\quad = \frac{1}{2n} e^{\widetilde{\Psi}(X_k,n)+\widetilde{\Xi}(X_k,n)} \big[e^{\delta\widetilde{\Psi}(t_k,n)}\big(1 + O(ne^{-2\widetilde{\Sigma}}) + O(ne^{-2\widetilde{\widetilde{\Sigma}}})\big)(\widetilde{\Psi}' \\
&\qquad - \widetilde{\Xi}' + (\delta\widetilde{\Psi})') - e^{\delta\widetilde{\Xi}(t_k,n)}(1 + O(ne^{-2\widetilde{\Sigma}}) + O(ne^{-2\widetilde{\widetilde{\Sigma}}}))(\widetilde{\Psi}' - \widetilde{\Xi}' - (\delta\widetilde{\Xi})')\big] = O(1).
\end{aligned}
$$

After the substitution and some simplifications, taking into account the relation $\widetilde{\Sigma}+\widetilde{\widetilde{\Sigma}} = \Sigma$ and assuming that $ne^{-\widetilde{\Sigma}} = o(1)$, $ne^{-\widetilde{\widetilde{\Sigma}}} = o(1)$, we obtain the desired formula (2.17).

3.3.4. *Proof of Theorem* 2.6 *and the corollaries for the kernels at the discrete mass points.* We use formula (3.13) for both neighborhoods. It is important for us to keep track of all terms carefully, because in what follows we need the result for $\xi = \eta = 0$. In the situation of Theorem 2.6, we have $t_k - \hat{t} = O\big(\frac{1}{n}\big)$ and $t_m - \hat{t} = O\big(\frac{1}{n}\big)$. Hence

$$
\begin{aligned}
K_{n-1}(X_k + e^{-2\Sigma_k} n\xi, X_m + e^{-2\Sigma_m} n\eta) &= \sqrt{\frac{\gamma_n}{\gamma_{n-1}}} \frac{(-1)^{k+m-1}}{\sqrt{n(n+1)}(X_k - X_m + O(ne^{-2\Sigma}))} \\
\times \bigg[e^{(\widetilde{\Psi}-2\Sigma)_k + (\widetilde{\Psi}-2\Sigma)_m} &\frac{\pi^2 n^2 \xi\eta}{(c-1)^2} \big(e^{\delta\widetilde{\Psi}_k} - e^{\delta\widetilde{\Psi}_m} + O(\|\varepsilon\| + ne^{-2\Sigma})\big) \\
+ e^{(\widetilde{\Psi}-2\Sigma)_k + \Xi_m} \frac{\pi\xi n}{c-1} &\big(e^{\delta\widetilde{\Xi}_m}\big(1 + O(\|\varepsilon\| + ne^{-2\Sigma})\big) - e^{\delta\widetilde{\Psi}_k}\big(1 + O(\|\varepsilon\| + ne^{-2\Sigma})\big)\big) \\
+ e^{(\widetilde{\Psi}-2\Sigma)_m + \Xi_k} \frac{\pi\eta n}{c-1} &\big(e^{\delta\widetilde{\Psi}_m}\big(1 + O(\|\varepsilon\| + ne^{-2\Sigma})\big) - e^{\delta\widetilde{\Xi}_k}\big(1 + O(\|\varepsilon\| + ne^{-2\Sigma})\big)\big) \\
+ e^{\Xi_k + \Xi_m} &\big(e^{\delta\widetilde{\Xi}_k}\big(1 + O(\|\varepsilon\| + ne^{-2\Sigma})\big) - e^{\delta\widetilde{\Xi}_m}\big(1 + O(\|\varepsilon\| + ne^{-2\Sigma})\big)\big)\bigg].
\end{aligned}
$$

After some simplifications and the omission of $O\big(\frac{1}{n}\big)$ terms, we obtain

$$
\begin{aligned}
K_{n-1}\big(X_k &+ e^{-2\Sigma_k} n\xi, X_m + e^{-2\Sigma_m} n\eta\big) = \sqrt{c}(-1)^{k+m-1} e^{\Xi_k + \Xi_m} \\
&\times \left[\frac{\pi^2 \xi\eta n}{(c-1)^2} \left((\delta\widetilde{\Psi})' e^{\delta\widetilde{\Psi}} + O\left(\frac{1}{n}\right) \right) - \frac{\pi\xi n}{(c-1)^2} \left(\frac{e^{\delta\widetilde{\Psi}} - e^{\delta\widetilde{\Xi}}}{k-m} + O\left(\frac{1}{n}\right) \right) \right. \\
&\left. + \frac{\pi\eta n}{(c-1)^2} \left(\frac{e^{\delta\widetilde{\Psi}} - e^{\delta\widetilde{\Xi}}}{k-m} + O\left(\frac{1}{n}\right) \right) + \frac{1}{n} \left((\delta\widetilde{\Xi})' e^{\delta\widetilde{\Xi}} + O\left(\frac{1}{n}\right) \right) \right] \left(1 + O\left(\frac{1}{n}\right)\right).
\end{aligned}
$$

Next, equivalent transformations of the right-hand side result in the asymptotic formula in Theorem 2.6.

By setting ξ and η to zero in the resulting formula and in (3.14), we obtain

$$
\begin{aligned}
K_{n-1}(X_k, X_m) &= \frac{\sqrt{c}(-1)^{k-m} e^{\Xi_k + \Xi_m}}{n} \left(-(\delta\widetilde{\Xi})' e^{\delta\widetilde{\Xi}} + O\left(\frac{1}{n}\right) \right), \\
K_{n-1}(X_k, X_k) &= \frac{2\sqrt{c}\sinh(\Psi_1 - t\Psi_1')}{\pi\sqrt{\widetilde{D}}} + O\left(\frac{1}{n}\right),
\end{aligned}
$$

whence corollaries (2.18) and (2.19) for the kernels on the support of the discrete orthogonality measure follow.

3.3.5. *Proof of Theorem* 2.9 *and the properties of basis integrals.* Recall that the parameters of the Meixner polynomials satisfy the constraints $b > -1$ and $c > 1$ and that one has the representation

$$M_n(\tilde{x}) = (\beta)_n\, {}_2F_1\left(-n, -\tilde{x}; \beta; 1 - \frac{1}{\sigma}\right) = \frac{(b+n)!}{b!}\, {}_2F_1(-n, -\tilde{x}; b+1; 1-c).$$

If $-b-1 < \mathrm{Re}(\tilde{x}) < 0$, then ${}_2F_1$ can be computed by the Euler formula (see [37]). We obtain

$$M_n(\tilde{x}) = \frac{(b+n)!}{(b+\tilde{x})!\Gamma(-\tilde{x})} \int_0^1 t^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n\, dt.$$

For $t \in (0;1)$, all bases in the integrand are positive, and the standard (arithmetic) values of powers are taken.

The nonintegrable singularity at the point 0 prevents us from applying this formula to $\tilde{x} \geq 0$. Hence we derive a different contour integral formula with contour not passing through 0. To this end, consider the function $(-t)^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n$ analytic in t in the domain with cuts along the rays $\left(-\infty; \frac{-1}{c-1}\right] \cup [0; +\infty)$ and with the arithmetic definition of the powers for $t \in \left(-\frac{1}{c-1}; 0\right)$. Let us integrate this function along a contour γ issuing from the point 1 on the upper coast of the cut, surrounding the point 0, and ending at the point 1 on the lower coast. If $-b-1 < \tilde{x} < 0$, then we can shrink the contour γ and obtain

$$\oint_\gamma (-t)^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n\, dt$$
$$= \left[\exp(-\pi(\tilde{x}+1)i) - \exp(\pi(\tilde{x}+1)i)\right] \int_0^1 t^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n\, dt,$$

whence (we write $x! := \Gamma(x+1)$ for brevity)

$$M_n(\tilde{x}) = \frac{(b+n)!}{(b+\tilde{x})!\Gamma(-\tilde{x})} \cdot \frac{1}{2i\sin\pi\tilde{x}} \oint_\gamma (-t)^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n\, dt.$$

By transforming this with the use of the complement formula, we obtain an expression suitable for $\tilde{x} > 0$:

$$M_n(\tilde{x}) = \frac{(b+n)!}{(b+\tilde{x})!} \cdot \frac{i\tilde{x}!}{2\pi} \oint_\gamma (-t)^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n\, dt. \tag{3.15}$$

Now γ can be deformed into the union of two arcs Γ_+ and Γ_- passing from 1 to $-\frac{1}{c-1}$ in the upper half-plane and from $-\frac{1}{c-1}$ to 1 in the lower half-plane, respectively. Let $\widetilde{\mathcal{F}}(t) = (-t)^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n$ be the integrand in the preceding formula in the domain with the cut $\left(-\infty; \frac{-1}{c-1}\right] \cup [0; +\infty)$. By considering the function

$$\mathcal{F}(t) := t^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n$$

in the domain with the cut $(-\infty; 0) \cup (1; +\infty)$ and by passing to the contours θ_+, θ_-, and θ (see Figure 1 in Section 2.2.2), we obtain

$$\begin{cases} \displaystyle\oint_{\Gamma_+} \widetilde{\mathcal{F}}(t)\, dt = \exp(\pi(\tilde{x}+1)i) \oint_{\theta_+} \mathcal{F}(t)\, dt, \\ \displaystyle\oint_{\Gamma_-} \widetilde{\mathcal{F}}(t)\, dt = -\exp(-\pi(\tilde{x}+1)i) \oint_{\theta_-} \mathcal{F}(t)\, dt, \end{cases}$$

$$\oint_{\theta_-} \mathcal{F}(t)\, dt + \oint_\theta \mathcal{F}(t)\, dt = \oint_{\theta_+} \mathcal{F}(t)\, dt.$$

Then the integral in the representation (3.15) is transformed into

$$\frac{1}{2\pi}\left(\oint_{\theta_+}(\cdot)+\oint_{\theta_-}(\cdot)\right)\sin(\pi\tilde{x})+\frac{i}{2\pi}\left(\oint_{\theta_-}(\cdot)-\oint_{\theta_+}(\cdot)\right)\cos(\pi\tilde{x})$$

and (3.15) becomes (2.21). The proof of Theorem 2.9 is complete.

Now let us prove Proposition 2.1. We start by studying I_2,

$$I_2(n)=\frac{1}{2\pi i}\oint_\theta t^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^n\,dt.$$

Note the relationship between the differentiation of the integrand $\mathcal{F}(t)$ and the shift of the parameter n in the integral:

$$\begin{aligned}\frac{d}{dt}\big[t^{-\tilde{x}}(1-t)^{b+\tilde{x}+1}(1+(c-1)t)^n\big]&=t^{-\tilde{x}-1}(1-t)^{b+\tilde{x}}(1+(c-1)t)^{(n-1)}\\ &\times\big[-\tilde{x}(1-t)(1+(c-1)t)-(b+\tilde{x}+1)t(1+(c-1)t)+(c-1)nt(1-t)\big].\end{aligned}$$

Hence the recursion

$$DI_2(\tilde{x},n+1)+BI_2(\tilde{x},n)+AI_2(\tilde{x},n-1)=0 \tag{3.16}$$

results in the equation

$$\begin{aligned}-\tilde{x}(1-t)(1+(c-1)t)-(b+\tilde{x}+1)t(1+(c-1)t)+(c-1)nt(1-t)&\\ =A+B(1+(c-1)t)+D(1+(c-1)t)^2.&\end{aligned} \tag{3.17}$$

We substitute $t=-\frac{1}{c-1}$, $t=1$, and $t=0$ into this equation and obtain a linear system for the coefficients in (3.16). If the coefficients A,B, and D are such that (3.17) is satisfied identically, then

$$\begin{aligned}DI_2(\tilde{x},n+1)+BI_2(\tilde{x},n)+AI_2(\tilde{x},n-1)&=\oint_\theta\frac{d}{dt}\big[t^{-\tilde{x}}(1-t)^{b+\tilde{x}+1}(1+(c-1)t)^n\big]\,dt\\ &=\big[t^{-\tilde{x}}(1-t)^{b+\tilde{x}+1}(1+(c-1)t)^n\big]\big|_\theta=0.\end{aligned}$$

Consequently, a similar relation holds for the other contours θ_+ and θ_-, and hence I_1 satisfies the same recursion relation as I_2. Finally, the Liouville formula for the resulting recursion gives (2.24), and the constant is obtained from the asymptotics (2.24) with regard for (2.23).

The proof of Proposition 2.1 is complete.

3.3.6. *Derivation of the discrete kernel in the bounded domain.* Now let us prove Theorem 2.10. We recall notation (3.2), substitute the expressions

$$e^{-\frac{V(x)}{2}}=\sqrt{\frac{W(\tilde{x})}{c-1}}=\sqrt{\frac{\Gamma(\beta+\tilde{x})}{\tilde{x}!(c-1)}\sigma^{\tilde{x}}}=\sqrt{\frac{(b+\tilde{x})!}{\tilde{x}!(c-1)}c^{-\tilde{x}}},\quad \gamma_n=n!(n+b)!c^n\left(\frac{c}{c-1}\right)^{b+1}$$

into the expression for $\widetilde{q}_n(x)$, and take into account (2.21), thus obtaining

$$\begin{aligned}\widetilde{q}_n(x)&=\frac{(-1)^n}{\sqrt{\gamma_n}}e^{-\frac{V(x)}{2}}M_n(\tilde{x})\\ &=(-1)^n\sqrt{\frac{(b+\tilde{x})!c^{-\tilde{x}-n-1}}{(b+n)!x!n!}\left(\frac{c}{c-1}\right)^{-b}}\left(\frac{(b+n)!}{(b+\tilde{x})!}\tilde{x}!\frac{\sin\pi\tilde{x}}{\pi}I_1+\frac{(b+n)!}{(b+\tilde{x})!}\tilde{x}!\cos\pi\tilde{x}I_2\right)\\ &=(-1)^n\sqrt{\frac{(b+n)!\tilde{x}!c^{-\tilde{x}-n-1}}{(b+\tilde{x})!n!}\left(\frac{c-1}{c}\right)^{b}}\left(\frac{\sin\pi\tilde{x}}{\pi}I_1+\cos\pi\tilde{x}I_2\right).\end{aligned}$$

Now we have

$$K_n(x,y) = -\varpi_n \mathcal{K}_n(\tilde{x},\tilde{y}),$$

where (recall notation (2.26))

$$\varpi_n := \frac{(n+b+1)!c^{-n-1}}{n!(c-1)}\left(\frac{c-1}{c}\right)^b\sqrt{\frac{\tilde{x}!\tilde{y}!c^{-\tilde{x}-\tilde{y}}}{(b+\tilde{x})!(b+\tilde{y})!}},$$

and for $\mathcal{K}_n(\tilde{x},\tilde{y})$ we have

$$\begin{aligned}\mathcal{K}_n(\tilde{x},\tilde{y}) = \frac{\sin\pi\tilde{x}\sin\pi\tilde{y}}{\pi^2(\tilde{x}-\tilde{y})}I_1(\tilde{x},n)I_1(\tilde{y},n)\left[\frac{I_1(\tilde{x},n+1)}{I_1(\tilde{x},n)} - \frac{I_1(\tilde{y},n+1)}{I_1(\tilde{y},n)}\right]\\ + \frac{\cos\pi\tilde{x}\cos\pi\tilde{y}}{(\tilde{x}-\tilde{y})}I_2(\tilde{x},n)I_2(\tilde{y},n)\left[\frac{I_2(\tilde{x},n+1)}{I_2(\tilde{x},n)} - \frac{I_2(\tilde{y},n+1)}{I_2(\tilde{y},n)}\right]\\ + \frac{\sin\pi\tilde{x}\cos\pi\tilde{y}}{\pi(\tilde{x}-\tilde{y})}\big[I_1(\tilde{x},n+1)I_2(\tilde{y},n) - I_1(\tilde{x},n)I_2(\tilde{y},n+1)\big]\\ + \frac{\cos\pi\tilde{x}\sin\pi\tilde{y}}{\pi(\tilde{x}-\tilde{y})}\big[I_2(\tilde{x},n+1)I_1(\tilde{y},n) - I_2(\tilde{x},n)I_1(\tilde{y},n+1)\big].\end{aligned}$$

Let us transform the last two terms to eliminate the singularity at $\tilde{x}=\tilde{y}$. We obtain

$$\begin{aligned}\sin\pi(\tilde{x}+\tilde{y})\bigg[\frac{I_1(\tilde{x},n+1)I_2(\tilde{y},n) - I_2(\tilde{x},n)I_1(\tilde{y},n+1)}{2\pi(\tilde{x}-\tilde{y})}\\ + \frac{I_2(\tilde{x},n+1)I_1(\tilde{y},n) - I_2(\tilde{y},n+1)I_1(\tilde{x},n)}{2\pi(\tilde{x}-\tilde{y})}\bigg]\\ + \frac{\sin\pi(\tilde{x}-\tilde{y})}{2\pi(\tilde{x}-\tilde{y})}\big[I_1(\tilde{x},n+1)I_1(\tilde{y},n) + I_1(\tilde{y},n+1)I_1(\tilde{x},n)\\ - I_1(\tilde{x},n)I_2(\tilde{y},n+1) - I_1(\tilde{y},n)I_1(\tilde{x},n+1)\big].\end{aligned}$$

By substituting the mass points $\tilde{x}\neq\tilde{y}$, $\tilde{x},\tilde{y}\in\mathbb{Z}_+$ into the resulting expression, we see that all sines are zero, all cosines are ±1, and

$$\mathcal{K}_n(\tilde{x},\tilde{y}) = (-1)^{\tilde{x}+\tilde{y}}\frac{I_2(\tilde{x},n+1)I_2(\tilde{y},n) - I_2(\tilde{x},n)I_2(\tilde{y},n+1)}{(\tilde{x}-\tilde{y})};$$

i.e., we have proved (2.25). Finally, by passing to the limit

$$\begin{aligned}&\lim_{\tilde{y}\to\tilde{x}\in\mathbb{Z}_+}\mathcal{K}_n(\tilde{x},\tilde{y})\\ &= \left(I_2^2(\tilde{x},n)\cdot\frac{d}{d\tilde{x}}\left(\frac{I_2(\tilde{x},n+1)}{I_2(\tilde{x},n)}\right) + I_1(\tilde{x},n+1)I_2(\tilde{x},n) - I_1(\tilde{x},n)I_2(\tilde{x},n+1)\right)\end{aligned}$$

in the resulting expression, we obtain (2.27) with the use of (2.26), (2.23), and (2.24). The proof of Theorem 2.10 is complete.

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