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SIMPLE SPECTRUM OF THE TENSOR PRODUCT OF POWERS OF A MIXING AUTOMORPHISM

V. V. RYZHIKOV

Abstract. It is proved that there exists a mixing automorphism of a Lebesgue space for which the tensor product of all its positive powers has simple spectrum.

§ 1. Introduction

A measure-preserving invertible transformation T of a probability Lebesgue space (X,μ), often called an automorphism, induces a unitary operator U_T, $U_T f(x) = f(Tx)$, which acts in the space $L_2(X,\mu)$. In what follows we denote a transformation and its corresponding operator in the same way .

The aim of the paper is to describe the construction of a mixing automorphism T such that the tensor product $T \otimes T^2 \otimes T^3 \otimes \dots$ has simple spectrum. Recall that mixing means weak convergence $T^i \to \Theta$, where Θ is the orthoprojection onto the space of constants in $L_2(X,\mu)$. This result was announced in [9], and recently Tikhonov [12] used it in proving the existence of a mixing automorphism with homogeneous spectrum of multiplicity $m > 2$. We point out that for nonmixing transformations Rokhlin's problem on homogeneous spectrum for $m > 2$ was solved in [2]. In the same paper, Ageev proved that the spectrum of tensor products of the powers of a typical nonmixing automorphism is simple. To find out about Rokhlin's problem and problems concerning realizing sets of multiplicities of the spectrum of a dynamical system we recommend Anosov's book [3] to the reader, and also the detailed survey by Danilenko [5], where not only are the results discussed but also the methods used to obtain them.

An infinite product $T_1 \otimes T_2 \otimes T_3 \otimes \dots$ has simple spectrum only in the case where the spectra of all the finite products $T_1 \otimes T_2 \otimes \dots \otimes T_n$ are simple. This property implies the mutual singularity not only of the spectral measures σ_i of the automorphisms T_i but also of various convolution products of them: for example, $\sigma_1 * \sigma_2$ and $\sigma_2 * \sigma_3 * \sigma_5$.

One of the ways of proving that the spectrum of an operator is simple consists in producing a cyclic vector. In the paper we prove the existence of a cyclic vector for all finite products of the form $T \otimes T^2 \otimes \dots \otimes T^n$ for some nonmixing automorphisms T. Also, the nonmixing property, and what is more, even the existence of nontrivial polynomials in the weak closure of powers will play an important role in proving that the spectrum is simple. Although this method does not work directly for mixing systems, it can be applied to nonmixing transformations that approximate some mixing transformation and ensure that the property of the spectrum being simple holds for the products $T \otimes T^2 \otimes \dots \otimes T^n$.

The plan of the paper is as follows. Constructions of rank 1 are described in § 2. In § 3 stochastic and staircase constructions are discussed, nonmixing modifications of which are considered in §§ 4, 5. That the spectrum of tensor products of powers of special nonmixing

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transformations is simple is proved in § 6. Then, preserving the spectral property, we diminish the effect of nonmixing and obtain the required result by a limiting procedure.

§ 2. Constructions of rank 1

We now describe an inductive construction of an automorphism which is defined by the following parameters: h_1 (the height of the tower at stage 1), a sequence r_j (where r_j is the number of columns into which the tower is cut at stage j), and a sequence of integer vectors

$$\bar{s}_j = (s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j))$$

(where $s_j(i)$ is the height of the spacer over the column with number i at stage j).

At stage j, the partially defined transformation T is a permutation of disjoint intervals

$$E_j, TE_j, T^2E_j, \dots, T^{h_j-1}E_j.$$

We now pass to step $j+1$. We represent the interval E_j as the union of r_j subintervals of equal measures, that is, in the form

$$E_j = E_j^1 \sqcup E_j^2 \sqcup E_j^3 \sqcup \dots \sqcup E_j^{r_j}.$$

For $i = 1, 2, \dots, r_j$ we consider the columns

$$E_j^i, TE_j^i, T^2E_j^i, \dots, T^{h_j-1}E_j^i$$

(the ith column at stage j). Over every column with number i we add $s_j(i)$ new intervals and obtain the tuple

$$E_j^i, TE_j^i, T^2E_j^i, \dots, T^{h_j-1}E_j^i, T^{h_j}E_j^i, T^{h_j+1}E_j^i, \dots, T^{h_j+s_j(i)-1}E_j^i$$

(all the sets above are disjoint). For every $i < r_j$ we set

$$T^{h_j+s_j(i)}E_j^i = E_j^{i+1}.$$

We thus glue the columns together into one tower,

$$E_{j+1}, TE_{j+1}, T^2E_{j+1}, \dots, T^{h_{j+1}}E_{j+1},$$

where

$$E_{j+1} = E_j^1,$$
$$h_j = h_j r_j + \sum_{i=1}^{r_j} s_j(i).$$

Thus, the automorphism T is (partially) defined at step $j+1$.

Continuing the construction we obtain an invertible transformation T on the union X of all the intervals. Here two cases can arise. In the first case the measure of X is infinite, and in the second finite. We are interested in the latter case, and for its realization it is sufficient (and necessary) to require that the series

$$\sum_j \frac{s_j(1) + s_j(2) + \dots + s_j(r_j)}{h_j r_j}$$

converges. Below we consider stochastic constructions for which the r_j increase sufficiently rapidly, and the $s_j(i)$ slowly, so that the corresponding series are majorised by the series $\sum_j \frac{\mathrm{Const}}{j!}$, and it only remains for us to normalize the finite measure of the phase space X.

It is convenient to represent the sets that we called intervals in the form of real semi-closed intervals situated in the plane (or on a straight line) with their Lebesgue measure. The transformation T maps the interval T^nE_j to $T^{n+1}E_j$ by translation (for $n < h_j$). It is obvious that the measure is preserved and that the transformation is invertible. It is important for us that any measurable set is approximated by finite unions of intervals (or, as they say, by floors of the tower) in the set $E_j, TE_j, \dots, T^{h_j}E_j$. A transformation of rank 1 is ergodic and has simple spectrum, as is well known.

§ 3. Stochastic and staircase constructions

Ornstein proved the mixing property for almost all special stochastic constructions [7]. Recall their definition. We fix h_1 and sequences $r_j \to \infty$ and $H_j \to \infty$ such that $H_j \ll r_j$. We consider all possible sequences of the form $a_j(i) \in \{0, 1, \dots, H_j\}$, that is, points of the space

$$\{0, 1, \dots, H_1\} \times \{0, 1, \dots, H_2\} \times \dots$$

equipped with the natural probability measure (the product of uniform distributions) and set

$$s_j(i) = H_j + a_j(i) - a_j(i+1).$$

To each random sequence there corresponds a random transformation of rank 1.

Of course, it is possible to take the quantities $a_j(i)$ directly for the $s_j(i)$. But if we do this it becomes significantly harder to control the mixing; nontrivial problems arise related to large deviations. This work was accomplished by Prikhod'ko [8], who considered the corresponding constructions of flows of rank 1. Ornstein's idea to use the so-called coboundary $a_j(i) - a_j(i+1)$ simplifies the situation. Looking a little bit ahead, we should explain that in this case for large n the operator T^n is approximated in the weak topology by the operators of averaging with respect to powers, and it is not hard to show that in this case these are close to Θ. Consequently, the T^n tend to Θ, and this means that T is mixing. We now explain formally how the mixing property is established for almost all transformations.

Below, instead of expressions of the form $A(m) - B(m) \underset{\mathrm{w}}{\to} 0$ (convergence in the weak operator topology) as $m \to \infty$, we shall also write $A(m) \underset{\mathrm{w}}{\approx} B(m)$ for convenience. For $m \in [h_j, h_{j+1})$ the specific character of transformations of rank 1 ensures we have an approximation of the form

$$T^m \underset{\mathrm{w}}{\approx} \widehat{D}_1 P_1 + \widehat{D}_2 P_2 + \widehat{D}_3 P_3, \tag{1}$$

where the operators $\widehat{D}_i$ and P_i depend on m. The operators $\widehat{D}_i$ are multiplications by the indicators of certain sets (domains) D_i. These domains (see [10] for a description) are asymptotically invariant with respect to T, and for almost all the constructions we consider the operators P_i have the following form:

$$P_1 = \sum_{-H_{j+1} \le n \le H_{j+1}} c_{j+1}(n) T^{k_1+n}, \quad c_{j+1}(n) = \frac{H_{j+1} + 1 - |n|}{(H_{j+1}+1)^2},$$

$$P_i = \sum_{-H_j \le n \le H_j} c_j(n) T^{k_i+n}, \quad c_j(n) = \frac{H_j + 1 - |n|}{(H_j+1)^2}, \quad i = 2, 3,$$

where k_1 and k_2 depend on m.

It is straightforward to verify that in this situation

$$\widehat{D}_i P_i T - \widehat{D}_i P_i \underset{\mathrm{w}}{\to} 0$$

for $i = 1, 2, 3$. Consequently, as $m \to \infty$ we have

$$T^{m+1} - T^m \underset{\mathrm{w}}{\to} 0$$

and, as is well known, for an ergodic automorphism this implies mixing:

$$T^m \underset{\mathrm{w}}{\to} \Theta.$$

We now explain how averaging operators appear. We begin with the following observation:

$$\mu(T^{h_j}A\cap B)\approx \frac{1}{r_j-1}\sum_{i=1}^{r_j-1}\mu(T^{-s_j(i)}A\cap B). \tag{2}$$

This formula uses the specific characteristics of transformations of rank 1. The sets A and B under consideration are approximated by the unions of floors of the tower that appear at stage j. Moreover, we can assume that they are precisely such unions, since this property is preserved at subsequent stages (a floor in the j-tower is the union of some floors of the j'-tower for $j'>j$). Under the action of T^{h_j} every column with number $i<r_j$ goes to the next one with delay $s_j(i)$, since it passes through a spacer of height $s_j(i)$. Hence on the column with number $i+1$ the conditional measure of the set $T^{h_j}A\cap B$ is equal to $\mu(T^{-s_j(i)}A\cap B)$. This is what gives rise to the approximation (2).

For a fixed $p>1$ we have

$$\mu(T^{ph_j}A\cap B)\approx \frac{1}{r_j-p}\sum_{i=1}^{r_j-p}\mu(T^{-S_j(i,p)}A\cap B),$$

where $S_j(i,p)$ is defined as the sum of the delays $s_j(i)$:

$$S_j(i,p)=s_j(i)+s_j(i+1)+\ldots+s_j(i+p-1)=pH_j+a_j(i)-a_j(i+p).$$

Recall that $a_j(i)$ takes values from 0 to H_j, $H_j\ll r_j$, randomly; therefore for $|n|\le 2H_j$ the probability that $S_j(i,p)-pH_j=n$ is close to $c_j(n)=\frac{H_j+1-|n|}{(H_j+1)^2}$.

In the general case for $m_j\in[h_j,h_{j+1})$, by representing

$$m_j=qh_j+k$$

we can decode formula (1) as follows:

$$\begin{aligned}\mu(T^{m_j}A\cap B)\approx \frac{1}{r_{j-1}}&\sum_{i=1}^{r_{j+1}-1}\mu(T^{m_j-h_{j+1}}T^{-s_{j+1}(i)}A\cap B\cap D_1)\\ &+\frac{1}{r_j-q}\sum_{i=1}^{r_j-q}\mu(T^{k}T^{-S_j(i,q)}A\cap B\cap D_2)\\ &\quad+\frac{1}{r_j-q-1}\sum_{i=1}^{r_j-q-1}\mu(T^{-h_j+k}T^{-S_j(i,q+1)}A\cap B\cap D_3),\end{aligned}$$

where the domain D_1 is the union of columns (of stage j) with numbers from 1 to q, the domain D_2 is the union of floors of height greater than $k-qH_j$ for the remaining columns, and D_3 is the corresponding union of floors of height less than $k-qH_j$. Note that when $qH_j\ll h_j$, correction by the quantity $-qH_j$ is not important.

The staircase construction is defined by the explicit sequence of spacers $s(i)=i$, $i=1,2,\ldots,r_j$, $r_j\to\infty$. Adams [1] found a nontrivial method of proving the mixing property in this case. As we pointed out above, the mixing problem reduces to analysing a sequence of certain operators averaging over powers of the transformation under consideration. For the staircase construction, it is easy to establish mixing for a sequence $\{m_j\}$ when $m_j\in[h_j,Ch_j]$ for a fixed number $C>1$. For example, using the ergodicity of the power T^p we obtain

$$\mu(T^{ph_j}A\cap B)\approx\frac{1}{r_j-p}\sum_{i=0}^{r_j-p-1}\mu(T^{-pi-(1+2+\ldots+p)}A\cap B)\approx\mu(A)\mu(B).$$

In the general case averagings of the form

$$\frac{1}{N_j}\sum_{i=0}^{N_j-1}\mu(T^{-d_ji+k_j}A\cap B\cap D(j)),\quad N_j=r_j-d_j,$$

arise, where the d_j may tend to infinity (the domain $D(j)$ is an analogue of the domains D_2 or D_3 described in the stochastic constructions).

Adams' approach (as we interpret it) consists in the following: for the operators

$$P_j=\frac{1}{N_j}\sum_{i=0}^{N_j-1}T^{-d_ji}$$

there exists a sequence l_j such that for any L the following conditions hold:

$$T^{l_j}T^{2l_j}\dots T^{Ll_j}\underset{\mathrm{w}}{\approx}\Theta,$$
$$\|P_j-P_jT^{l_j}\|\to 0,\ \dots,\ \|P_j-P_jT^{Ll_j}\|\to 0.$$

But if the sequence L_j is slowly increasing, the first condition implies the strong convergence,

$$\frac{1}{L_j}\sum_{i=0}^{L_j-1}T^{il_j}\to_s\Theta.$$

Then from the second condition we see that the operators P_j converge strongly to Θ. But when $\mu(B\cap D(j))\approx\mu(B)\mu(D(j))$ (which is true in this case) this implies mixing:

$$\frac{1}{N_j}\sum_{i=0}^{N_j-1}\mu(T^{-d_ji+k_j}A\cap B\cap D(j))\approx\mu(A)\mu(B)\mu(D(j)).$$

For a more detailed account of constructions of rank 1 and methods for proving the mixing property, see [1], [4], [8]–[10].

§ 4. The convergence $T^{sm_j}\underset{\mathrm{w}}{\to}\Theta$ for a particular s

Weak limits of the form $aI+(1-a)\Theta$ are well known in ergodic theory. They were used in the solution of Kolmogorov's problem on the group property of the spectrum of an automorphism [11] and played an important role in the construction of examples of transformations with unusual properties [6]. If all limit operators for powers of an automorphism contain the component $a\Theta$, then such an automorphism is said to be a-mixing. Since we do not require a to be maximal, it follows from the definition that a mixing transformation is a-mixing for any $a\in(0,1]$.

Lemma 1. *For any $\varepsilon>0$ and positive integers $n>1$ and $s\in[1,n]$ there exists a $(1-\varepsilon)$-mixing construction T with the following property: for some sequence m_j and a positive number a,*

$$(\mathbf{s},\ \mathbf{n})\qquad\begin{cases}T^{sm_j}\underset{\mathrm{w}}{\to}\Theta,\\ T^{km_j}\underset{\mathrm{w}}{\to}(1-a_k)\Theta+a_kI,\quad a_k>a,\ k\neq s,\ 1\le k\le n.\end{cases}$$

Proof. For definiteness let $s=3$, $n>3$. We define a sequence of spacers as follows. We define the vector $\bar s_j$ in the form of a successive union of arrays,

$$S_1,S_1,A_1,S_2,S_2,A_2,S_4,S_4,A_4,S_5,S_5,A_5,\dots,S_n,S_n,A_n,$$

where the S_k,A_k are independent random vectors satisfying the same conditions as those described in Ornstein's construction. We require that the size of the array S_k be equal to kL_j ($L_j\to\infty$), and the average value of its elements be equal to H_j. Let the array

A_k have length $[2\varepsilon^{-1}kL_j]$. Because of the large arrays A_k we always have mixing on a part of the phase space with measure greater than $1-\varepsilon$.

Let $m_j = (h_j + H_j)L_j$; then for almost all such realizations for $k \neq s$, $1 \le k \le n$, we obtain

$$T^{km_j} \underset{\mathrm{w}}{\approx} a_k I + (1-a_k)\Theta, \quad a_k > \frac{1}{2n^2(1+\varepsilon^{-1})}.$$

Indeed, we have mixing on the whole tower, apart from the part D_k situated under the second array of spacers S_k. On D_k we obtain

$$\mu(T^{km_j}A \cap B \cap D_k) \approx \mu(D_k)\mu(A\cap B),$$

which is what gives rise to the component $a_k I$. (Since the ratio of the length of the array S_k to the length of the array A_k is less than ε, we obtain $a_k < \varepsilon$.) However,

$$T^{3m_j} \underset{\mathrm{w}}{\approx} \Theta,$$

since we deliberately did not duplicate the random array of length $3L_j$. □

§ 5. The convergence $T^{nm_j} \underset{\mathrm{w}}{\to} aI + bT + c\Theta$ for a particular n

Constructions with parameters (n, a, b)**.** Let $r_j \to \infty$. We fix positive numbers a, b, c such that $a+b+c = 1$, and a positive integer $n > 1$.

For some sequence of even *indices* j, the vectors of spacers will consist of three parts: a flat a-part (which will give the summand aI), a special b-part (summand bT), and a mixing c-part (summand $c\Theta$). For a change, we define the mixing part explicitly in the so-called staircase form[1] (following [1], [9]).

We set $H_j = h_{j-1}$.

a-Part: for $i = 1, 2, \dots, [ar_j]$,

$$s_j(i) = H_j.$$

b-Part: for $i \in ([ar_j], [(a+b)r_j])$, for $i = ni'$,

$$s_j(i) = nH_j - 1,$$

otherwise

$$s_j(i) = 0.$$

c-Part: $s_j(i) = i - [(a+b)r_j]$ for $i > [(a+b)r_j]$.

Thus, the vectors of spacers for $n = 3$ will have the form

$$H_j, H_j \dots, H_j,\ 0, 0, 3H_j - 1,\ 0, 0, 3H_j - 1, \dots, 0, 0, 3H_j - 1,\ 1, 2, 3, 4, \dots.$$

For the sequence of odd *indices* $j-1$, the vectors of spacers will have the purely staircase form:

$$s_j(i) = i \text{ for } i = 1, 2, \dots, r_j.$$

Weak limits. Consider the sequence $m_j = h_j + H_j$. The following convergence properties hold for this sequence (property (**n**)):

$$(\mathbf{n}) \qquad \begin{cases} T^{m_j} \underset{\mathrm{w}}{\to} aI + (b+c)\Theta, \\ T^{2m_j} \underset{\mathrm{w}}{\to} aI + (b+c)\Theta, \\ \dots\dots\dots\dots\dots\dots\dots \\ T^{(n-1)m_j} \underset{\mathrm{w}}{\to} aI + (b+c)\Theta, \\ T^{nm_j} \underset{\mathrm{w}}{\to} aI + bT + c\Theta. \end{cases}$$

[1]We could equally well use stochastic or so-called algebraic spacers for this mixing part; see [10].

Indeed, using the formula

$$\mu(T^{ph_j}A\cap B)\underset{\mathrm{w}}{\approx}\frac{1}{r_j-p}\sum_{i=1}^{r_j-p}\mu(T^{-S_j(i,p)}A\cap B),$$

where $S_j(i,p)=s_j(i)+s_j(i+1)+\ldots+s_j(i+p-1)$, for $S(p)=1+2+\ldots+p$ we obtain

$$T^{pm_j}\underset{\mathrm{w}}{\approx}aT^0+b\left(\frac{n-p}{n}T^{pH_j}+\frac{p}{n}T^{(p-n)H_j+1}\right)+\frac{1}{r_j}\sum_{i=1}^{r_j-[(a+b)r_j]}T^{-pi-S(p-1)}$$
$$\underset{\mathrm{w}}{\approx}aI+b\left(\frac{n-p}{n}T^{pH_j}+\frac{p}{n}T^{(p-n)H_j+1}\right)+c\Theta.$$

We use the fact that $T^h\underset{\mathrm{w}}{\approx}\Theta$ for $h\in[H_j,CH_j]$ (see [1]). For $p=1,2,\ldots,n-1$ we have

$$b\left(\frac{n-p}{n}T^{pH_j}+\frac{p}{n}T^{(p-n)H_j+1}\right)\underset{\mathrm{w}}{\approx}b\Theta.$$

For $p=n$ we obtain

$$b\left(\frac{n-p}{n}T^{pH_j}+\frac{p}{n}T^{(p-n)H_j+1}\right)=bT.$$

Thus property (**n**) is established.

§ 6. The main result

Assertion 1. *If a transformation T with simple spectrum has properties* (**m**), (**s, n**), *for any m,s such that $1<m\le n$, $1\le s\le n$, then the product $T\otimes T^2\otimes\ldots\otimes T^n$ has simple spectrum.*

Proof. The following short argument will serve as a compensation for the complexity of the constructions. Let f be a cyclic vector for the operator T restricted to the space $H=L_2^0(\mu)$ (where $L_2^0(\mu)$ is the space orthogonal to the constants). We claim that $H^{\otimes n}=C_F$ for the operator $T\otimes T^2\otimes\ldots\otimes T^n$, where C_F is the cyclic space generated by the vector $F=f^{\otimes n}$. By induction we assume that the operator $S=T\otimes T^2\otimes\ldots\otimes T^{n-1}$ has simple spectrum.

From property (**n**) we obtain

$$b^{n-1}f^{\otimes n-1}\otimes(aI+bT)f\in C_F;$$

consequently, $f^{\otimes n-1}\otimes Tf\in C_F$. Thus,

$$f^{\otimes n-1}\otimes T^mf\in C_F$$

for all m. We obtain

$$f^{\otimes n-1}\otimes H\subset C_F,\quad S^mf^{\otimes n-1}\otimes H\subset C_F,\quad H^{\otimes n-1}\otimes H\subset C_F.$$

We now verify that the operator $T\otimes T^2\otimes\ldots\otimes T^n$ has simple spectrum when its action is considered on the whole space $L_2(\mu^{\otimes n})$. To do this it is sufficient to know that different products of the form $T^{n_1}\otimes\ldots\otimes T^{n_k}$ acting on the spaces $H^{\otimes k}$ are spectrally disjunct. But this is a consequence of property (**s, n**) (Lemma 1). For example, let $s=2$; then on the space $H^{\otimes 3}$ we have

$$(T\otimes T^2\otimes T^5)^{m_j}\underset{\mathrm{w}}{\to}0,$$

but at the same time,

$$(T\otimes T^3\otimes T^5)^{m_j}\underset{\mathrm{w}}{\to}a_1a_3a_5I.$$

Hence it is evident that the operators $(T \otimes T^2 \otimes T^5)$ and $(T \otimes T^3 \otimes T^5)$ cannot have nonzero interlacing. Indeed, from

$$(T \otimes T^2 \otimes T^5)J = J(T \otimes T^3 \otimes T^5),$$

by passing to the weak limit of powers, we obtain

$$0 = a_1 a_3 a_5 J, \quad J = 0.$$

The assertion is proved. □

We point out that a consequence of the assertion is that the spectrum is simple for all powers T^n, $n > 1$.

Theorem 1. *There exists a mixing transformation T such that the product $T \otimes T^2 \otimes T^3 \dots$ has simple spectrum.*

Proof. We consider a sequence T_p of constructions of rank 1 that have properties (**n**) and (**s, n**). Note that to do this we use sparse subsequences of indices in such a way that the required weak limits are obtained independently of one another. Simultaneously we ensure that the spectrum for $T_p \otimes T_p^2 \otimes T_p^3 \dots$ is simple and that the automorphism T has the c_p-mixing property (Assertion 1). Properties (**n**) and (**s, n**) are defined in such a way that it is possible to choose c_p to be arbitrarily close to 1. By making c_p tend to 1 we obtain a limit mixing construction T with the required spectral property. The paper [9] contains a detailed description of the passage to the limit $T_p \to T$ (as $p \to \infty$) which ensures that the spectrum is simple for all symmetric powers $T^{\odot n}$. Our problem is solved in a similar fashion: using the same method we find that the spectrum of the products $T \otimes T^2 \otimes \dots \otimes T^n$ are simple.

In general outline, the essence of the method is as follows. Under iteration, for a long time the limit construction T behaves in the same way as T_p. When large powers of the construction T start to differ substantially from the powers of the construction T_p, then they differ (extremely) little from the powers of the construction T_{p+1}, and so on. The product $T_p \otimes T_p^2 \otimes \dots \otimes T_p^n$ has a cyclic vector v_p. Replacing T_p by T we obtain a good approximation of a fixed set of vectors by elements of the cyclic space generated by the vector v_p under the action of the operator $T \otimes T^2 \otimes \dots \otimes T^n$. Then we work with the vector v_{p+1}, and so on. Enlarging the set of vectors that we are approximating in such a way that it is dense in the product $L_2(\mu) \otimes L_2(\mu) \otimes \dots$, we find that the spectrum of the product $T \otimes T^2 \otimes T^3 \otimes \dots$ is simple. □

Remark. The following assertion, which we state as a conjecture, is an analogue of our result for flows. *There exists a mixing flow such that for any set of different numbers $t_i > 0$ the product $T_{t_1} \otimes T_{t_2} \otimes T_{t_3} \dots$ has simple spectrum.*

If we only require that the spectrum be simple for a fixed sequence of numbers $t_i > 0$, then the problem becomes much easier and the construction of the flow is adapted to this sequence. In the case $t_i = i$ the construction of the flow is similar to the construction we have looked at for an automorphism, but we need additional properties to ensure that the spectrum of T_1 is simple. Theorem 1 cannot be applied directly here, since a mixing transformation of rank 1 does not embed in a flow; moreover, such a transformation commutes only with its powers [7].

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Moscow State University
E-mail address: `vryzh@mail.ru`

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