BYPASSES FOR RECTANGULAR DIAGRAMS.
A PROOF OF THE JONES CONJECTURE
AND RELATED QUESTIONS

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Abstract. We give a criterion, in terms of Legendrian knots, for a rectangular diagram to admit a simplification and show that simplifications of two different types are, in a sense, independent of each other. We show that a minimal rectangular diagram maximizes the Thurston-Bennequin number for the corresponding Legendrian links. We prove the Jones conjecture on the invariance of the algebraic number of crossings of a minimal braid representing a given link. We also give a new proof of the monotonic simplification theorem for the unknot.

Introduction

As was shown in [8], any rectangular diagram of the unknot admits a monotonic simplification to the trivial diagram. The starting point for this paper was the question of when a rectangular diagram of an arbitrary link admits a simplification, i.e., when the complexity can be lowered by at least one.

By the complexity of a rectangular diagram we understand one-half of the number of its edges. Any two diagrams representing equivalent links can be transformed into one another by analogs of Reidemeister moves, called here elementary moves. Some of them can change the complexity of the diagram. The complexity-increasing elementary moves are called stabilizations, and their inverses are called destabilizations. By a monotonic simplification we understand a sequence of elementary moves that contains no stabilizations.

Legendrian knots attracted attention in the 1980s due to the developments in contact topology [2]. Each class of topologically equivalent links contains infinitely many classes of Legendrian classes of Legendrian links. There is an integral invariant of Legendrian isotopy, called the Thurston-Bennequin number, which can be made less than any given number for any topological link type. For Legendrian links, one can also define stabilizations (respectively, destabilizations), which preserve the topological type but decrease (respectively, increase) the Thurston-Bennequin number by one.

In 2003, W. Menasco mentioned to the first author a connection between rectangular diagrams and Legendrian links, which can be described as follows ([18], [20]).

With each rectangular diagram $R$ one naturally associates a Legendrian link $L_R$, whose Legendrian type does not change under complexity-preserving elementary moves of $R$. Stabilizations and destabilizations break into two types: the moves of the first type preserve the Legendrian type of $L_R$, and those of the second type preserve the Legendrian type of $L_{R^\leftarrow}$, where $R^\leftarrow$ denotes the rectangular diagram obtained from $R$ by a clockwise rotation through $\pi/2$ (for the topological type of the link, this corresponds to passing...
Moreover, the change in $L_R$ ($L_R\sim$) under (de)stabilization of the second (respectively, first) type is a Legendrian (de)stabilization.

As it turned out, the above question of the ability to simplify a rectangular diagram admits a natural answer in terms of the corresponding Legendrian links. Namely, any Legendrian destabilization $L_R \mapsto L' \leadsto$ can be realized by a monotonic simplification of $R$ which preserves the Legendrian type of $L_R\sim$, and the existence of a Legendrian destabilization for $L_R$ or $L_R\sim$ is equivalent to the ability to simplify $R$.

A notion of a bypass, introduced later in this paper, plays a key role in the proofs of this and other results. We define it both in the language of rectangular diagrams and in the language of Legendrian links. This notion is related to, although differs from, what was called a bypass in [12] and [13].

The idea behind this concept is to represent a Legendrian destabilization as a replacement of an arc in the link by another arc, which we call a bypass. Combinatorially, such a replacement is no longer a destabilization, and it does not necessarily lower the complexity of the diagram, but a rectangular diagram with a bypass can always be transformed by elementary moves so as to preserve the complexity of the diagram and shorten the bypass. This is the main technical result of this paper.

To prove it, we use a technique developed by J. Birman and W. Menasco in a series of papers, where they studied links represented by closed braids [4], [5]. The most important elements of this technique appeared earlier in the paper [2] by D. Bennequin. P. Cromwell [7] noticed that the Birman-Menasco method carries over to arc representations of links (from the combinatorial point of view, they are just rectangular diagrams).

The proof of the main result of [8] on monotonic simplification of the unknot was obtained by further development of this method. That proof can be somewhat simplified by using the aforementioned connection with Legendrian knots. Below, we show how to reduce the monotonic simplification theorem to a theorem of Eliashberg and Fraser [9], [10] on classification of Legendrian unknots or to the theorem of Erlandsson [11] on the negativity of the Thurston-Bennequin number of the unknot.

Some of the consequences of the aforementioned ability to simplify criterion for rectangular diagrams had been formulated earlier as questions or conjectures. For example, below we prove (in a generalized form) a conjecture of Jones that two braids on a minimum number of strands representing the same oriented link have the same algebraic crossing number.

**Convention 1.** Henceforth, except in Section 5 a link is understood as a link in $\mathbb{R}^3$ (i.e., a compact one-dimensional submanifold) endowed with an optional orientation and a coloring of its connected components. This means that each component may or may not be oriented and that a color—a natural number—is assigned to each component. In particular, the link need not be oriented, and the colors of all of its components could be the same, say 1, or, on the contrary, the link could be oriented and the components can have pairwise distinct colors.

When speaking of link isotopy, we assume that the orientation and the component colors are preserved. This additional structure—the optional orientation and the coloring—is extended in an obvious way to the link diagrams considered below (rectangular diagrams and front projections). For brevity, we do not mention this explicitly.

The exception to this convention in Section 5 concerns only the orientation—we only deal with oriented links there.

In this paper, the choice of the coloring plays no role; we only mention that our results remain true for colored links.
§1. Rectangular diagrams

Definition 1. A rectangular link diagram is a finite union of closed broken lines in the plane consisting of horizontal and vertical edges (called edges of the diagram) such that no two of them lie on the same line. Any such diagram is interpreted as a planar link diagram such that, at each crossing, the vertical edge is an overpass. The endpoints of the edges are called vertices of the diagram. The number of vertical edges of a rectangular diagram $R$ is called the complexity of $R$ and is denoted by $c(R)$.

Figure 1 shows an example of a rectangular diagram.

As shown in [7], [8], two rectangular diagrams represent equivalent links if and only if they are related by a finite sequence of the following elementary moves:

(i) The cyclic permutation of vertical or horizontal edges consists of moving one of the extreme (upper, lower, right, or left) edges to the opposite extreme position with simultaneous replacement of the two adjacent edges; see Figure 2.

(ii) The commutation of vertical or horizontal edges consists of exchanging two neighboring parallel edges in such a way that the projections of the pairs of their endpoints to a line of the corresponding direction have no common points and do not interleave. Two parallel edges are considered neighboring if there are no vertices of the diagram between the parallel lines containing those edges; see Figure 3.

(iii) Stabilization consists of replacing a vertex by three new vertices such that all four form the vertices of a small square, addition of two short edges which are sides of that square, and a suitable extension or shortening of the edges adjacent to the deleted vertex; the inverse operation is called destabilization (Figure 4).

We shall distinguish two types of stabilizations and destabilizations.
Figure 3. Commutations.

Figure 4. Stabilizations and destabilizations (the “long” edges may run in the opposite directions, which yields twelve more diagrams of similar kind).
Definition 2. If two short edges run from a stabilization vertex down and left or up and right, then the stabilization is of type I, and otherwise it is of type II; see Figure 5. Notice that in this definition the directions of the edges leaving the stabilization vertex do not matter.

![Types of stabilizations and destabilizations.](image)

**Figure 5.** Types of stabilizations and destabilizations.

The type of a destabilization is defined as the type of the inverse stabilization.

By an elementary simplification of a rectangular diagram we understand a sequence of elementary moves in which the last move is a destabilization and the preceding ones are cyclic permutations and commutations. The type of an elementary simplification is defined as the type of its destabilization.

In combinatorial terms, the central result of this paper can be stated as follows.

**Theorem 1.** Suppose a rectangular diagram $R$ admits $k$ consecutive elementary simplifications $R \mapsto R'_1 \mapsto R'_2 \mapsto \cdots \mapsto R'_k$ of type I, as well as $\ell$ consecutive elementary simplifications $R \mapsto R''_1 \mapsto \cdots \mapsto R''_{\ell}$ of type II.

Then $R'_k$ admits $\ell$ consecutive simplifications of type II, and the resulting diagram is related to $R''_{\ell}$ by a sequence of cyclic permutations, commutations, and stabilizations/destabilizations of type I.

Similarly, $R''_{\ell}$ admits $k$ consecutive simplifications of type I, and the resulting diagram is related to $R'_k$ by a sequence of cyclic permutations, commutations, and stabilizations/destabilizations of type II.

This result will be proved in Sections 3 and 4.

§2. **Legendrian links**

2.1. **Front projections.**

**Definition 3.** A link $L$ in $\mathbb{R}^3$ is said to be Legendrian if it is a smooth curve everywhere tangent to the plane distribution given by the 1-form

$$\omega = x \, dy + dz,$$

and it is called the standard contact structure. Legendrian links are said to be equivalent if they are isotopic within the class of Legendrian links.

The most convenient way to specify Legendrian links is to draw their front projections, i.e., the projections to the plane $Oyz$. A front projection is a piecewise smooth curve with cusp singularities which is nowhere vertical (see Figure 6). For the convenience of the reader, we mark, at each crossing and cusp of front projections, the overcrossing branch (i.e., the branch with a larger $x$-coordinate) and the undercrossing branch. A generic front projection uniquely determines the corresponding Legendrian curve in $\mathbb{R}^3$ since the $x$-coordinate of any point on the curve is given by

$$x = -\frac{dz}{dy}.$$
Figure 6. Front projection of a Legendrian link.

Figure 7 shows schematically (up to central symmetry) which front projection moves yield equivalent Legendrian links; see [23].

Some front projection moves preserving the topological type of the link are not allowed for Legendrian knots because they change the Legendrian type; see Figure 8.

Figure 8. Forbidden front projection moves.

A piecewise smooth link consisting of finitely many arcs which are everywhere tangent to the standard contact structure can be smoothed out at the break points without changing the combinatorial structure of its projection to the Oyz-plane (including information about the cusps). For this reason, such piecewise smooth links also uniquely determine a Legendrian type, and thus they can be considered as Legendrian links.
2.2. **Presentation by rectangular diagrams.** The connection between Legendrian links and rectangular diagrams mentioned in the introduction can be described as follows.

Let $R$ be a rectangular diagram. Rotate it through $\pi/4$ counterclockwise, smooth out the corners pointing up and down, and turn the corners pointing left and right into cusps; see Figure 9.

![Figure 9. Front projection of a Legendrian link corresponding to a rectangular diagram.](image)

The Legendrian link determined by the front projection obtained in this way will be denoted by $L_R$.

**Theorem 2** ([22, Section 4]). Any Legendrian link is equivalent to some link $L_R$.

Legendrian links $L_R$ and $L_{R'}$ are equivalent if and only if $R$ and $R'$ are connected by a sequence of elementary moves not containing type II stabilizations and destabilizations.

By the **Legendrian type** of a rectangular diagram we shall understand the Legendrian type of the corresponding Legendrian link $L_R$. It follows from Theorem [2] that the Legendrian type of a rectangular diagram is its equivalence class with respect to cyclic permutations, commutations, and type I stabilizations/destabilizations.

2.3. **Thurston-Bennequin number.** Notice that the vector $e_z = (0,0,1)$ is nowhere tangent to the contact structure, and therefore a small enough shift of any Legendrian link $L$ along this vector yields a link $L^+$ which is disjoint from $L$. A link obtained by a small shift in the opposite direction will be denoted $L^-$. The **Thurston-Bennequin number** of a Legendrian link $L$, denoted by $tb(L)$, is the linking number $lk(L,L^+) = lk(L,L^-)$. This definition presupposes that $L$ is oriented. Reversing the orientation of all components does not change $tb(L)$. In particular, if $L$ is a knot, then this number does not depend on its orientation.

The Thurston-Bennequin number is a Legendrian isotopy invariant and can be computed by the following simple formula

\[ tb(L) = w(F) - \frac{1}{2} c(F), \]

where $w(F)$ is the writhe number and $c(F)$ is the number of cusps of a front projection $F$ representing $L$.

The rectangular diagram obtained from a rectangular diagram $R$ by a clockwise rotation through $\pi/2$ (with simultaneous flipping of all crossings) will be denoted by $R^\frown$.

Notice that if $R_1 \rightarrow R_2$ is a type I stabilization, then $R_1^\frown \rightarrow R_2^\frown$ is a type II stabilization and vice versa.
For front projections $F$ and $F^\sim$ corresponding to the rectangular diagrams $R$ and $R^\sim$ one has $w(F) = -w(F^\sim)$, and the total number of their cusps equals the number of vertices of $R$. Therefore

$$c(R) = \text{tb}(L_R) + \text{tb}(L_R^\sim).$$

It follows from formula (1) that the forbidden moves shown in Figure 8 change the Thurston-Bennequin number by one. These moves for front projections of Legendrian links will also be called stabilizations and destabilizations. Those of the first type make the Thurston-Bennequin number smaller, whereas the latter make it bigger.

Stabilizations, unlike destabilizations, can be applied to any front projection. If the Legendrian equivalence class of a given Legendrian link $L_R$ contains a representative whose front projection admits a destabilization, then we say that the Legendrian link $L$ does also.

Clearly, type II stabilizations and destabilizations of a rectangular diagram $R$ result, respectively, in stabilizations and destabilizations of the corresponding Legendrian link $L_R$.

The following result is due to T. Erlandsson [11] (see also [2, Theorem 2]).

**Theorem 3.** The Thurston-Bennequin number of a Legendrian unknot is always negative.

Y. Eliashberg and M. Fraser [9], [10] proved a stronger result, a particular case of which can be stated as follows.

**Theorem 4.** Let $K$ be an oriented Legendrian knot with a topological type of the unknot. Then it is Legendrian equivalent to one of the following knots:

\[\text{in which the numbers of left and right “zigzags” independently run through the set of nonnegative integers.}\]

§3. **Bypasses**

For the proofs of our main results, we only need a combinatorial version of bypasses. To make the definition less artificial, we first describe its geometric prototype.

**3.1. Legendrian description.**

**Definition 4.** Let $L$ be a Legendrian link. A pair $(\alpha, \beta)$, where $\alpha \subset \mathbb{R}^3$ is a smooth simple Legendrian (i.e., everywhere tangent to the standard contact structure) arc with endpoints on $L$ and $\beta \subset L$ is an arc with the same endpoints, is called a bypass for $L$ if there is a two-dimensional disk $D$ in $\mathbb{R}^3$ such that
(A0) $D$ is the image of the semidisk \[
\{(x,y) \in \mathbb{R}^2; \ x^2 + y^2 \leq 1, \ x \leq 0\}\]
under a smooth embedding in $\mathbb{R}^3$;
(A1) the disk boundary $\partial D$ coincides with $\alpha \cup \beta$;
(A2) the intersection $D \cap L$ coincides with $\beta$;
(A3) $D$ is tangent to the standard contact structure along $\alpha$.

We want to discuss condition (A3) in more detail because later on we shall replace it with another condition which can be formulated in combinatorial terms. Suppose we have a pair $(\alpha, \beta)$ and a disk $D$ satisfying (A0)–(A2).

Since $\alpha$ and $\beta$ are Legendrian, $D$ is tangent to the standard contact structure at the endpoints $\partial \alpha = \partial \beta$, but not necessarily along the entire arc $\alpha$. Therefore, when traversing $\alpha$ from one endpoint to the other, the plane tangent to $D$ makes an integer or half-integer algebraic number of turns about the tangent vector to $\alpha$ with respect to the standard contact structure. Denote this number by $r$. If $r = 0$, then $D$ can be deformed near $\alpha$ so as to satisfy (A3), and therefore $(\alpha, \beta)$ is a bypass.

Denote the link $(L \setminus \beta) \cup \alpha$ by $L_{\beta \to \alpha}$. It is not difficult to see that the pairs of linking numbers \[
\{\text{lk}(L_{\beta \to \alpha}, (\alpha \cup \beta)^+), \text{lk}(L_{\beta \to \alpha}, (\alpha \cup \beta)^-)\}
\]
and \[
\{\text{lk}(L_{\beta \to \alpha}^+, \alpha \cup \beta), \text{lk}(L_{\beta \to \alpha}^-, \alpha \cup \beta)\}
\]
will coincide, after a suitable choice of orientations, with $\{[r], [r+1/2]\}$, where $[x]$ denotes the integer part of $x$. If all these numbers are zero, then the knot $\alpha \cup \beta$ shifted in either way is actually unlinked with $L_{\beta \to \alpha}$, i.e., separated from $L_{\beta \to \alpha}$ by an embedded 2-sphere. Thus, condition (A3) in the definition of bypass can be replaced by any of the following three conditions:

(A3′) $L_{\beta \to \alpha}$ is unlinked with $(\alpha \cup \beta)^+ \cup (\alpha \cup \beta)^-$;
(A3″) $L_{\beta \to \alpha}^+ \cup L_{\beta \to \alpha}^-$ is unlinked with $\alpha \cup \beta$;
(A3‴) $\text{lk}(L_{\beta \to \alpha}, (\alpha \cup \beta)^+) = \text{lk}(L_{\beta \to \alpha}, (\alpha \cup \beta)^-) = 0$.

Notice also that condition (A0) then becomes redundant.

With (A3) replaced by one of (A3′), (A3″), or (A3‴), it becomes convenient to use front projections to check whether or not an arc is a bypass. For example, in Figure 10 the dashed line on the left is not a bypass, whereas the one on the right is.

It follows from (A3‴) and Theorem 3 that the link $L_{\beta \to \alpha}$ is topologically equivalent to $L$ but has a larger Thurston-Bennequin number, namely \[
\text{tb}(L_{\beta \to \alpha}) = \text{tb}(L) - \text{tb}(\alpha \cup \beta).
\]
As we shall see below, (in combinatorial terms) $L_{\beta \to \alpha}$ can be obtained from $L$ by a sequence of $(-\text{tb}(\alpha \cup \beta))$ destabilizations.

3.2. Description in terms of rectangular diagrams. We shall now translate all of this into combinatorial language. For ease of exposition, we introduce the following terminological convention.

**Convention 2.** Since a rectangular diagram is uniquely determined by its set of vertices, henceforth we shall not distinguish the diagram and the set of its vertices, but we shall use the same notation for both. Thus, a finite set of points in the plane is considered a rectangular diagram if each vertical or horizontal line either contains no points from this set or contains exactly two of them.
Definition 5. The Thurston-Bennequin number $\text{tb}(R)$ of a rectangular diagram of a knot $R$ is the Thurston-Bennequin number of the corresponding Legendrian knot:
\[ \text{tb}(R) = \text{tb}(L_R). \]

For a rectangular diagram $R$ we denote by $R^{\uparrow}$ a diagram obtained from $R$ by a small positive shift along the vector $(1, 1)$, where the meaning of being small will be clear from the context. Likewise, $R^{\downarrow}$ denotes the result of a small shift in the opposite direction.

If $R$ is a rectangular diagram of a knot, then $\text{tb}(R)$ is by definition equal to the linking number $\text{lk}(R, R^{\uparrow})$, provided the components of the link determined by $R \cup R^{\uparrow}$ are oriented coherently (in place of $\text{lk}(R, R^{\uparrow})$ one may also take $\text{lk}(R, R^{\downarrow})$ or $\text{lk}(R^{\downarrow}, R^{\uparrow})$).

Definition 6. A rectangular path is a set of points in the plane that can be made a rectangular diagram of a knot by adding one or two points belonging to a vertical or a horizontal line. Furthermore, a single point is also regarded as a rectangular path.

Horizontal and vertical line segments connecting points of a rectangular path are called edges, and the points themselves are called vertices of the rectangular path. Horizontal and vertical lines containing exactly one point of a path are called the ends of the path.

Figure 10. Bypass on front projection.

Figure 11. Examples of rectangular paths. Solid lines are edges and dashed lines contain the endpoints of the paths.
and the corresponding vertices are called the *endpoints*. All other vertices are said to be *internal*.

Figure 11 shows examples of rectangular paths; for clarity, the edges and the ends of these paths are also shown.

**Convention 3.** Whenever we discuss the union of several rectangular diagrams of links or of rectangular paths we shall assume that they are in general position, namely, no two vertices belong to the same vertical or horizontal line unless they are required to do so by construction.

For example, under these conventions, two distinct rectangular paths with a common end form a rectangular diagram of a knot. We shall express the fact that an edge of a rectangular diagram or of a rectangular path is contained in an end of a rectangular path in the opposite (but colloquially more common) way, i.e., we shall say that the end of the path lies on the edge. Notice however that the endpoint of the path is not required to actually belong to the edge as it can belong to its extension; see Figure 12. Our terminology is also justified by the following construction.

For a finite subset $X \subset \mathbb{R}^2$ let $\tilde{X}$ denote the following union of line segments in $\mathbb{R}^3$:

$$\tilde{X} = \bigcup_{(i,j) \in X} \left[ (2i,0,1), (0,2j,-1) \right].$$

The broken line $\tilde{X}$ is characterized as the union of all line segments with endpoints on the lines $\ell_1 = \mathbb{R} \times \{0\} \times \{1\}$ and $\ell_2 = \{0\} \times \mathbb{R} \times \{-1\}$ and passing through the plane $\mathbb{R}^2 \times \{0\}$ at the points of $X$; see Figure 13.

It is not difficult to see that if $R$ is a rectangular diagram, then $\tilde{R}$ is a link in $\mathbb{R}^3$ isotopic to the one determined by $R$. Indeed, the intersection of $\tilde{R}$ with the upper half-space $\mathbb{R}^2 \times [0, +\infty)$ consists of piecewise linear arcs

$$[(i,j_1,0), (2i,0,1)] \cup [(2i,0,1), (i,j_2,0)],$$

for which $[(i,j_1), (i,j_2)]$ is a vertical edge of $R$. This family is isotopic in $\mathbb{R}^2 \times [0, +\infty)$ to the union of the vertical edges of $R$, with the arc endpoints being fixed by the isotopy.

Likewise, the intersection of $\tilde{R}$ with the lower half-space $\mathbb{R}^2 \times (-\infty, 0]$ consists of arcs isotopic in this half-space to the union of the horizontal edges of $R$.

For a rectangular path $\alpha$ with endpoints on edges of a rectangular diagram $R$, the endpoints of $\tilde{\alpha}$ lie in $\tilde{R}$. Moreover, by Convention 3, the interior of the broken arc $\tilde{\alpha}$ does not intersect $\tilde{R}$. If two rectangular paths $\alpha$ and $\beta$ have common ends, then $\tilde{\alpha}$ and $\tilde{\beta}$ have common endpoints.

We are now ready to define our key object.
**Definition 7.** A bypass for a rectangular diagram $R$ is a pair $(\alpha, \beta)$ of rectangular paths with the same endpoints, where $\beta$ is a subset of $R$, if there is an embedded 2-disk $D \subset \mathbb{R}^3$ such that

- (B1) the disk boundary $\partial D$ coincides with $\tilde{\alpha} \cup \tilde{\beta}$;
- (B2) the intersection $D \cap \tilde{R}$ coincides with $\tilde{\beta}$;
- (B3) in the link determined by the rectangular diagram $$(R \setminus \beta) \cup \alpha \cup (\alpha \cup \beta)'^c \cup (\alpha \cup \beta)'^c,$$

the components represented by $(R \setminus \beta) \cup \alpha$ are unlinked with the other two.

The bypass $(\alpha, \beta)$ is said to be elementary if $\text{tb}(\alpha \cup \beta) = -1$. The number $- \text{tb}(\alpha \cup \beta)$ will be called the weight of the bypass $\alpha$, and the path $\beta$ will be called the bypassed path.

In most cases, the path $\beta$ is uniquely determined by $\alpha$. For example, this is true whenever the component of $\tilde{R}$ containing the endpoints of $\tilde{\alpha}$ is knotted or linked with the rest of the link. In any case, given the path $\alpha$, there are at most two choices for $\beta$. For this reason, we shall often refer to $\alpha$ as a bypass, with the convention that in the case when there are two options for $\beta$, one of them is chosen and fixed.

It is not difficult to see that any bypass for a rectangular diagram $R$ can be made into a bypass for the Legendrian link $L_R$ using a procedure similar to the one turning a rectangular diagram into a front projection. Namely, we add edges (if an endpoint of the bypass is outside of an edge of $R$ but is collinear with it, we also connect it with the nearest endpoint of that edge), thus obtaining a projection of a knotted graph isotopic to $\tilde{R} \cup \tilde{\alpha}$ (at self-intersections, vertical edges are always overpasses). Then we rotate the whole configuration counterclockwise, smooth out some corners, and turn the others into cusps, according the general principles; see Figure 14.

Also, it is not difficult to show that any bypass for the link $L_R$ considered up to Legendrian isotopy can be obtained this way. We shall not use these facts in the proofs; we leave them to the reader as an exercise.

3.3. $\Theta$-diagram. Let us take a closer look at the object formed by a rectangular diagram with a bypass.

![Figure 13. A rectangular path X and its three-dimensional realization $\tilde{X}$ by a broken line.](image-url)
Definition 8. A rectangular Θ-diagram is the union $\alpha \cup \beta \cup \gamma \cup \delta$ of three rectangular paths $\alpha, \beta, \gamma$ with common pairs of ends and a rectangular diagram of a link $\delta$. We distinguish these paths; so formally a Θ-diagram is a quadruple $(\alpha, \beta, \gamma, \delta)$, where $\alpha, \beta,$ and $\gamma$ are rectangular paths with common pairs of endpoints and $\delta$ is a rectangular diagram. In accordance with Convention 3, we assume by default that all edges are noncollinear with each other and with the ends of $\alpha, \beta,$ and $\gamma$.

Two rectangular Θ-diagrams are said to be Legendrian equivalent if one of them can be obtained from the other by a finite sequence of admissible moves, which include cyclic permutations, commutations, type I stabilizations/destabilizations, and end shifts, as defined below. The corresponding equivalence class of a rectangular Θ-diagram will be called its Legendrian type.

Cyclic permutations, commutations, and (de)stabilizations for rectangular Θ-diagrams are defined similarly to those for rectangular diagrams of links. Let us however discuss the difference between the two notions.

At an extreme left, right, upper, or lower position, a rectangular Θ-diagram may have an edge as well as an end of three rectangular paths. A shift of it to the opposite side is also considered a cyclic permutation (see Figure 15).

Commutations for edges of a rectangular Θ-diagram are defined the same way as for rectangular diagrams of links: one may interchange two parallel neighboring edges of
α, β, γ, or δ, provided their endpoints do not interleave. Common ends of α, β, and γ may also be exchanged with edges parallel to them and with each other, provided that no pair of endpoints on a common end interleave with the vertices of the other edge being commuted or with a pair of endpoints on the other end, respectively; see Figure 16.

Type I stabilizations and destabilizations can be performed at any vertex of a rectangular Θ-diagram, including the endpoints of α, β, and γ.

Finally, we shall need one more type of moves—an end shift—which we now define. Let P₁, P₂, P₃ be three endpoints of the rectangular paths α, β, γ (not necessarily in this order) lying on the same horizontal line and listed from left to right. Let P′₁ and P″₁ be the points obtained by shifting P₁ by the vector (0, ε) or (0, −ε), respectively, where ε > 0 is a sufficiently small real number (it should be less than the vertical distance between any two vertices of the diagram not lying on the same horizontal line). An end shift consists of one of the following six replacements in α, β, and γ:

\[
P₁ \mapsto \{ P'₁, P₂ \}, \quad P₂ \mapsto \emptyset, \quad P₃ \mapsto \{ P₂, P₃ \};
\]

\[
P₁ \mapsto \{ P''₁, P₃' \}, \quad P₂ \mapsto \{ P₂, P₃ \}, \quad P₃ \mapsto \emptyset;
\]

\[
P₁ \mapsto \{ P₁, P₃ \}, \quad P₂ \mapsto \{ P₂', P₃' \}, \quad P₃ \mapsto \emptyset;
\]

\[
P₁ \mapsto \emptyset, \quad P₂ \mapsto \{ P₂'', P₁' \}, \quad P₃ \mapsto \{ P₁, P₃ \};
\]

\[
P₁ \mapsto \emptyset, \quad P₂ \mapsto \{ P₁, P₂ \}, \quad P₃ \mapsto \{ P₃'', P₃'' \};
\]

\[
P₁ \mapsto \{ P₁, P₂ \}, \quad P₂ \mapsto \emptyset, \quad P₃ \mapsto \{ P₃', P₂'' \},
\]

provided that the result is again a rectangular Θ-diagram. Namely, the first and sixth replacements are allowed if P₂ is not the unique vertex of the corresponding rectangular path, the second and third are allowed if P₃ has the same property, and the fourth and fifth are allowed if P₁ has that property.

Notice that, from the combinatorial point of view, the above six replacements break into three pairs of coinciding moves.

A similar operation is defined for three endpoints of rectangular paths in a rectangular Θ-diagram if they lie on the same vertical line; see Figure 17. Namely, the above construction should be reflected in a diagonal (any of the two will work).

Thus, in all cases, an end shift consists of deleting one of the vertices P₁, P₂, P₃ and adding two vertices near the remaining ones. If the deleted vertex is an endpoint of the rectangular path α of the rectangular Θ-diagram α ∪ β ∪ γ ∪ δ, then the rectangular diagram β ∪ γ ∪ δ does not change under the end shift, and the path α acquires an extra
edge one of whose ends slides along $\beta \cup \gamma$ to an adjacent edge. Moreover, one of the paths $\beta$ and $\gamma$ gives the other path one of its vertices.

We do not define the inverse of an end shift because one can come back by moving the end backward and then perform several commutations and a type I destabilization (see Figure 18).

**Proposition 1.** Let $\alpha \cup \beta \cup \gamma \cup \delta$ and $\alpha' \cup \beta' \cup \gamma' \cup \delta'$ be Legendrian equivalent rectangular $\Theta$-diagrams such that $(\alpha, \beta)$ is a weight $b$ bypass for $\beta \cup \gamma \cup \delta$. Then:

(i) $(\alpha', \beta')$ is a weight $b$ bypass for $\beta' \cup \gamma' \cup \delta'$;

(ii) the rectangular diagrams $\alpha \cup \gamma \cup \delta$ and $\alpha' \cup \gamma' \cup \delta'$ are Legendrian equivalent.

**Proof.** It suffices to prove the statement in the case when $\alpha \cup \beta \cup \gamma \cup \delta \rightsquigarrow \alpha' \cup \beta' \cup \gamma' \cup \delta'$ is a single admissible move. This reduces to a direct verification that the move preserves the isotopy classes of $\tilde{\alpha} \cup \tilde{\beta} \cup \tilde{\gamma} \cup \tilde{\delta}$ and of $\tilde{D}$, where $D = (\alpha \cup \beta) \cup (\alpha \cup \beta) \cup (\alpha \cup \gamma \cup \delta)$,
Figure 19. In all cases, the diagram $D' = (\alpha' \cup \beta' \cup (\alpha' \cup \beta')) \cup \alpha' \cup \gamma' \cup \delta'$ is equivalent to $D = (\alpha \cup \beta) \cup (\alpha \cup \beta) \cup \alpha \cup \gamma \cup \delta$, and the diagram $\alpha \cup \gamma \cup \delta$ is Legendrian equivalent to $\alpha' \cup \gamma' \cup \delta'$. 

\[\begin{align*}
\alpha' & \cup \beta' \\
\alpha' & \cup \gamma' \\
\alpha' & \cup \delta'
\end{align*}\]
as well as the Legendrian type of the link $\alpha \cup \gamma \cup \delta$. The cases of a commutation, an admissible destabilization, or a cyclic permutation are trivial and are left to the reader.

The least obvious part here is to see that the isotopy class of $\tilde{D}$ does not change under an end shift. This is illustrated in Figure 19 for one of the possible relative positions of the end vertices of $\alpha$, $\beta$, and $\gamma$. Since for cyclic permutations our assertion is true, only the cyclic order of the end vertices matters, and this, in turn, can be reversed using central symmetry.

The case of a horizontal end shift is dealt with by a diagonal reflection. Notice that the desired Legendrian equivalences of rectangular diagrams are established by commutations and stabilizations which can be done by moving only vertices from a small neighborhood of a common end of $\alpha$, $\beta$, $\gamma$, and therefore the positions of the other vertices connected by edges with the end vertices involved in the moves play no role. It also does not matter if all three paths $\alpha$, $\beta$, and $\gamma$ have more than one vertex. Thus, all remaining cases reduce to the one shown in Figure 19.

It should be clear (although we do not formally use this fact) that there is a natural correspondence between Legendrian types of rectangular $\Theta$-diagrams and Legendrian types of one-dimensional complexes in $\mathbb{R}^3$ consisting of several circles and one $\Theta$–component, which is a graph with two vertices connected by three edges such that all the curves in the complex are Legendrian. For completeness, we briefly mention the additional moves of front projections of such graphs that involve the trivalent vertex and, together with the moves from Figure 7, yield the equivalence of Legendrian graphs which are of interest to us. These moves are shown in Figure 20 (one should also add their mirror images).

![Figure 20. Moves of a Legendrian graph involving the trivalent vertex.](image)

**Proposition 2.** Let $R$ and $R'$ be rectangular diagrams of links of the same Legendrian type, and let $\alpha$ be a rectangular path with ends on edges of $R$. Then there is a rectangular path $\alpha'$ with ends on edges of $R'$ such that the rectangular $\Theta$-diagrams $R \cup \alpha$ and $R' \cup \alpha'$ are Legendrian equivalent.

**Proof.** It suffices to prove the desired assertion in the case when $R'$ is obtained from $R$ by a single elementary move preserving the Legendrian type. For a stabilization this is
obvious: just take $\alpha' = \alpha$ (we assume that the new edges resulting from the stabilization are sufficiently short).

Let $R \mapsto R'$ be a cyclic permutation. For definiteness, we assume that the leftmost edge moves to the right. The remaining cases are completely analogous.

If there are edges of $\alpha$ to the left of $R$, we move them to the right by a cyclic permutation and then freely apply the desired cyclic permutation, regardless of whether or not one of the ends of $\alpha$ lies on the affected edge.

Let $R \mapsto R'$ be a commutation. Because our construction is symmetric, it suffices to consider a commutation of horizontal edges. Moreover, because for cyclic permutations the desired assertion is already established, we may assume that the horizontal projections of the exchanged edges do not overlap and that the lower of the two edges has its left endpoint on the left edge of the $\Theta$-diagram $R \cup \alpha$, i.e., there are no vertices of $R$ or of $\alpha$ to the left of it.

In the presence of $\alpha$, there could only be two obstacles to proceeding with the desired commutation: 1) the endpoints of $\alpha$ lie on the commuted edges or on the edges adjacent to them; 2) there may be edges of $\alpha$ in the horizontal strip between the commuted edges.

In the former case, we move the ends of $\alpha$ to other horizontal edges of $R$, which can be done by applying several end shifts. We now consider the other obstacle.

Denote the left of the commuted edges of $R$ by $e$ and the right edge by $e'$. Let $P$ denote the right vertex of $e$. In the absence of $\alpha$, the desired commutation could have been made by shifting $e$ upward. In our case, we may be prevented by the edges of $\alpha$ lying above $P$ but below $e'$; see Figure 21(a).

Let $\ell$ be the vertical line passing through $P$. Let us “break” every obstructing edge of $\alpha$ near its intersection with $\ell$. More precisely, apply a suitable admissible stabilization near one of the vertices of each obstructing edge and then shift, using commutations, the new short vertical edge toward $\ell$. At the same time, arrange the new short vertical edges to the right of $\ell$ so that the closer an edge is to $\ell$ the lower it is; see Figure 21(b).
Now we can freely shift upward all edges lying above $e$ but below $e'$, thus removing the obstacle to performing the desired commutation; see Figure 21(c).

It remains to consider the case when $R \mapsto R'$ is an admissible destabilization. There are two types of such destabilizations, but since they are symmetric with each other, we only consider one of them, when the edges being shortened come out of their common vertex upward and to the right. Using cyclic permutations and shifting the ends of $\alpha$, we can make the common vertex of the edges being shortened the lowest and the leftmost vertex of $R \cup \alpha$ and, at the same time, the endpoints of $\alpha$ not lie on the edges being shortened and on their adjacent edges.

Let $e$ be the horizontal edge to be shortened, and let $e'$ be the horizontal edge of $R$ immediately above it. An obstruction to shifting $e$ upward to make it neighboring with $e'$ in $R \cup \alpha$ may come from the edges of $\alpha$ located above the right endpoint of $e$ but below $e'$.

We proceed exactly as in the case of commutations; see Figure 22.

![Figure 22. Destabilization in the presence of a path. There may be vertical edges of $\alpha$ (not shown) coming out of the boxes $X$ and $Y$.](image)

Now $e$ can be shifted upward, next to $e'$, and then the vertical edge being shortened can be freely shifted to the right, thus making the desired destabilization possible. □

3.4. Key lemma and its consequences. By a connected component of a rectangular diagram $R$ of a link we understand any subset $K \subset R$ which is a rectangular diagram of a knot.

Key lemma. Let $R$ be a rectangular diagram of a link and let $(\alpha, \beta)$ be a bypass whose weight is smaller than the number of vertical edges of the component of $R$ whose edges contain the ends of $\alpha$.

Then there is a rectangular diagram $R'$ which is Legendrian equivalent to $(R \setminus \beta) \cup \alpha$ and can be obtained from $R$ by $b$ successive type II elementary simplifications, where $b$ is the weight of the bypass $(\alpha, \beta)$.

The next section is entirely devoted to the proof of this assertion. Here, we want to deduce our main results from it.
Theorem 5. Suppose rectangular diagrams $R_1$ and $R_2$ are Legendrian equivalent and $R_1$ admits an elementary type II simplification $R_1 \mapsto R'_1$. Then $R_2$ admits an elementary type II simplification $R_2 \mapsto R'_2$ such that $R'_2$ is Legendrian equivalent to $R'_1$.

Proof. Since commutations and cyclic permutations of edges preserve the Legendrian type of a rectangular diagram, it suffices to consider the case when $R_1 \mapsto R'_1$ is a type II stabilization. Then, as is seen from Figure 23, there is an elementary bypass $\alpha_1$ for $R_1$ consisting of a single vertex such that $R'_1$ is obtained from $R_1$ by replacing the corresponding bypassed path $\beta_1$ (having three vertices) by $\alpha_1$.

![Figure 23. Elementary one-vertex bypass.](image)

It follows from Proposition 2 that for $R_2$ there is also a rectangular path $\alpha_2$ such that the rectangular $\Theta$-diagrams $R_1 \cup \alpha_1$ and $R_2 \cup \alpha_2$ are Legendrian equivalent. By Proposition 1 this implies that $\alpha_2$ is an elementary bypass for $R_2$. Let $\beta_2$ be the corresponding bypassed path. Then $(R_2 \setminus \beta_2) \cup \alpha_2$ is Legendrian equivalent to $(R_1 \setminus \beta_1) \cup \alpha_1 = R'_1$.

Since the bypass $\alpha_2$ is elementary, it satisfies the assumptions of the key lemma. Applying it, we have the desired assertion. □

The next result immediately follows from Theorems 2 and 5.

Corollary 1. Let $R$ be a rectangular diagram of a link such that the corresponding Legendrian link $L_R \ (L_R \rt)$ admits a destabilization $L_R \mapsto L'$ (respectively, $L_R \rt \mapsto L'$). Then $R$ admits a type II (respectively, type I) elementary simplification $R \mapsto R'$ such that the links $L'$ and $L_R'$ (respectively, $L'$ and $L_R'$) are Legendrian equivalent.

Moreover, we recover the main result of [8]:

Corollary 2 (Monotonic simplification theorem for the unknot). Any nontrivial diagram of the unknot admits a sequence of elementary simplifications ending with a trivial diagram.

Proof. The fact that $R$ is nontrivial means that its complexity is greater than 2. Together with formula (2), this means that at least one of the Legendrian knots $L_R$ and $L_R \rt$ has Thurston-Bennequin number less than $-1$. The Eliashberg-Fraser theorem on the classification of topologically trivial Legendrian knots (Theorem 4 above) implies that this knot admits a destabilization. Hence, $R$ admits an elementary simplification. Now, an obvious induction on the complexity of $R$ completes the proof. □

At the end of Section 4 we shall mention another proof of the monotonic simplification theorem which avoids the use of the Eliashberg-Fraser theorem.

The result mentioned at the very beginning of the paper is yet another consequence of Theorem 5.
Proof of Theorem 1. By assumption, the diagrams $R$ and $R'_k$ are Legendrian equivalent. Hence, applying Theorem $5$ $\ell$ times, we have that there is a sequence $R'_0 \mapsto R''_0 \mapsto R''''_0 \mapsto \cdots \mapsto R''''_{\ell}$ of elementary simplifications in which $R''''_i$ is Legendrian equivalent to $R''''_i$ for all $i = 1, \ldots, \ell$, which is a reformulation of the first assertion of the theorem. The second assertion is symmetric to the first one. \qed

Now we want to describe the connections between bypasses and simplifications in more general terms.

**Theorem 6.** Let $R$ be a rectangular diagram of a link, $K \subset R$ one of its connected components, and $L$ a Legendrian type. The following conditions are equivalent:

(C1) $R$ admits $b > 0$ consecutive type II elementary simplifications on $K$ resulting in a diagram with Legendrian type $L$;

(C2) $R$ admits a bypass $\alpha$ of weight $b$ with ends on the edges of $K$ such that the replacement of the bypassed path by the bypass yields a diagram with Legendrian type $L$.

**Proof.** (C2) $\Rightarrow$ (C1) Subjecting $R$ to type I stabilizations on $K$ outside the bypassed path, we can make the number of vertical edges of that component larger than $b$. Denote the obtained diagram by $\tilde{R}$.

By the key lemma, there is a sequence of elementary simplifications $\tilde{R} \mapsto \tilde{R}_1 \mapsto \tilde{R}_2 \mapsto \cdots \mapsto \tilde{R}_b$ such that the last diagram in it is Legendrian equivalent to $(\tilde{R}\setminus \beta) \cup \alpha$, which in turn is Legendrian equivalent to $(R\setminus \beta) \cup \alpha$, where $\beta$ is the bypassed path.

Applying Theorem $5$ $b$ times, we have that there is a sequence of elementary simplifications $R \mapsto R_1 \mapsto R_2 \mapsto \cdots \mapsto R_b$ with each $R_i$ Legendrian equivalent to the corresponding diagram $\tilde{R}_i$.

(C1) $\Rightarrow$ (C2) We need the following two lemmas, which will also be used later on.

**Lemma 1.** Let $R_0 \mapsto R_1 \mapsto \cdots \mapsto R_m$ be an arbitrary sequence of elementary moves in which there are only $k$ stabilizations and $\ell$ destabilizations. Then there is another sequence of elementary moves $R_0 \mapsto R'_1 \mapsto \cdots \mapsto R'_m = R_m$ in which the first $k$ moves are stabilizations, the last $\ell$ moves are destabilizations, and there are no stabilizations and destabilizations in between those groups. Moreover, in the new sequence, the number of stabilizations and destabilizations of each type is the same as in the original one.

**Proof.** We induct on the quadruple $(k, \ell, p, s)$, where $k$ is the total number of stabilizations, $\ell$ is the total number of destabilizations, $p + 1$ is the ordinal number of the first stabilization (in the absence of such, we set $p = 0$), and $m − s$ is the ordinal number of the last destabilization (in the absence of such, we set $s = 0$). The quadruples $(k, \ell, p, s)$ are ordered lexicographically.

The induction base $k = \ell = 0$ is obvious. For the induction step, it suffices to prove the claim for $m = 2$. Indeed, if the first move is a stabilization, then, removing it, we have a sequence with a smaller $k$ and the same $\ell$, the validity of the claim for which implies that for the original sequence. Likewise, if the last move is a destabilization, then, removing it, we have the same $k$ but a smaller $\ell$. If none of these holds but $k > 0$, then applying the assertion to the subsequence consisting of the $p$th and the $(p + 1)$st moves, we have a sequence with the same $k$ and $\ell$, but a smaller $p$. Finally, if $k = 0$ and $\ell > 0$, we similarly apply the assertion to the subsequence consisting of the $(m − s)$th and the $(m − s + 1)$st moves.

Thus, it remains to check our assertion for sequences consisting of two moves in the following two cases: 1) the second move is a stabilization, whereas the first one is not; 2) the first move is a destabilization, whereas the second one is not. Each of these cases can be obtained from the other by reversing the entire sequence, and therefore it suffices to consider only one of them. Figure $24$ shows how to move a stabilization to the
beginning of the sequence when it is preceded by a cyclic permutation, a commutation, or a destabilization in some of those cases when these two moves cannot be simply interchanged. The remaining cases are dealt with in a similar way, and we leave them to the reader.

\[ \square \]

\[ \text{Figure 24. Moving a stabilization to the beginning of the sequence.} \]

**Lemma 2.** Let \( R \mapsto R' \) be a stabilization, let \( P \) be any vertex of the component of \( R \) where this stabilization is performed. Then there is a stabilization \( R \mapsto R'' \) of the same type as \( R \mapsto R' \) in a neighborhood of \( P \) such that \( R' \) and \( R'' \) are connected by a sequence of commutations and cyclic permutations.

**Proof.** It suffices to consider the case when \( P \) is connected by an edge with the vertex in whose neighborhood the stabilization \( R \mapsto R' \) occurs. Denote that edge by \( e \). As a result of the stabilization \( R \mapsto R' \), we have two short edges. The one perpendicular to \( e \) can, using commutations and, possibly, one cyclic permutation, be shifted toward \( P \). For the diagram \( R'' \) obtained in this way, the move \( R \mapsto R'' \) is a stabilization; see Figure 25. \( \square \)

Returning to the proof of Theorem 6, let \( R = R_0 \mapsto R_1 \mapsto \cdots \mapsto R_b \) be a sequence of elementary simplifications. It follows from Lemma 1 that \( R_b \) can be obtained from \( R \) by a sequence of elementary moves in which the last \( b \) are destabilizations and all the preceding ones are cyclic permutations and commutations. Moreover, by Lemma 2 we may assume that all \( b \) destabilizations take place in a small neighborhood of the same vertex \( P \) of \( R_b \).
Thus, we can find a sequence of complexity-preserving moves from $R$ to some diagram $R'$ such that $R_b$ is obtained from $R'$ by a sequence of $b$ type II destabilizations all of which take place in a small neighborhood of some vertex $P$ of $R_b$. It is not difficult to see that this vertex $P$ is then a bypass of weight $b$ and that the transition from $R'$ to $R_b$ is a replacement of the bypassed path by $P$; see Figure 26. It now follows from Propositions 2 and 1 that for $R$ there is a bypass of weight $b$ such that replacing the bypassed path with it results in a diagram Legendrian equivalent to $R_b$. \[\square\]

**Remark 1.** It follows from Theorem 6 that the restriction imposed on the bypass in the statement of the key lemma is in fact redundant—the weight of a bypass is always less than the number of vertical edges of the component containing its endpoints.

**Remark 2.** Theorem 6 can be strengthened by allowing simplifications to take place in different components and by introducing an appropriate notion of independent bypasses.
Namely, bypasses are independent if their corresponding disks, mentioned in the definition of a bypass, do not intersect. Given a set of independent bypasses, each one of them gives rise to as many elementary simplifications of the component to which it is attached as is its weight. The proof of this claim is not harder than the proof of Theorem 6, but for the sake of clarity we want to consider the case of just one bypass.

**Theorem 7.** Let $L_1$ and $L_2$ be Legendrian types of Legendrian links with mirror-symmetric topological types. Then there is a rectangular diagram $R$ such that the Legendrian links $L_R$ and $L_{R'}$ have types $L_1$ and $L_2$, respectively.

**Proof.** Let $R_1$ and $R_2$ be rectangular diagrams whose corresponding Legendrian links $L_{R_1}$ and $L_{R_2'}$ have types $L_1$ and $L_2$, respectively. Then the topological link types determined by these diagrams are the same and each diagram can be obtained from the other by a sequence of elementary moves.

By Lemma 1, we may assume that that sequence starts with stabilizations, ends with destabilizations, and in between it has only commutations and cyclic permutations. It follows from Lemma 2 that we can have stabilization of different types occur far from each other, i.e., in neighborhoods of distinct vertices. In that case, they commute and without loss of generality we may assume that type I stabilizations occur first and are followed by all type II stabilizations. But type I stabilizations do not change the Legendrian type of $L_{R_1}$, and we may assume that they are not present in the sequence.

Likewise, without loss of generality we may assume that our sequence of elementary moves has no type II destabilizations.

Starting with $R_1$ and performing all (type II) stabilizations, we obtain a diagram $R'$ such that $R_1$ can be obtained from it by type II elementary simplifications and $R_2$ can be obtained from it by type I elementary simplifications. It now follows from Theorem 1 that there is a diagram $R$ which can be obtained from $R_2$ by type II elementary simplifications and which is Legendrian equivalent to $R_1$. This is the desired diagram $R$. $\square$

Theorem 7 and equality (2) yield a positive answer to a question of J. Green; see [19, Question 1]:

**Corollary 3.** If $R$ is the simplest rectangular diagram representing a given topological link type, then $L_R$ has the largest Thurston-Bennequin number among all Legendrian links of the same topological type.

### §4. Proof of the key lemma

The plan of the proof is as follows.

- First, with the rectangular diagram $R$ and the paths $\alpha$ and $\beta$ we associate geometric objects $\hat{R}$, $\hat{\alpha}$, and $\hat{\beta}$ in $\mathbb{R}^3$, called arc presentations.
- We span the unknot $\hat{\alpha} \cup \hat{\beta}$ by a disk $D$, subject to some restrictions. Using an “open book” foliation, we define a special combinatorial structure on that disk.
- Next we induct. For the induction step, we modify $\hat{R} \cup D$ to make $D$ simpler. Those modifications will lead to type II destabilizations on $\beta$ and type I destabilizations on $\alpha$, as well as to commutations and cyclic permutations for the $\Theta$-diagram $R \cup \alpha$.
- Rearrangement of saddles is an operation that in some cases shortens $\alpha$ or $\beta$ and in other cases leads to a situation where a wrinkle can be smoothed out inside $D$.
- Smoothing out a wrinkle simplifies $D$. If the wrinkle is at the boundary, then $\alpha$ or $\beta$ will become shorter.
- If further simplification of $D$ is impossible, then the disk has a simple standard form, and the transition from $R$ to $(R \setminus \beta) \cup \alpha$ is a type II elementary simplification. This is our induction base.
4.1. **Arc presentations.** We fix the standard cylindrical coordinate system \((\rho, \theta, z)\) in \(\mathbb{R}^3\). The axis of it will be denoted by \(\ell\) and called the *binding line*. The half-planes of the form \(\{\theta = \text{const}\}\) will be called *pages*. The page \(\{\theta = \theta_0\}\) will be denoted by \(P_{\theta_0}\).

**Definition 9.** An *arc presentation* of a topological link type \(L\) is a link \(L \in \mathcal{L}\) consisting of a finite number of smooth arcs with endpoints on \(\ell\), each of which lies entirely on a separate page and has its interior disjoint from the binding line. The intersection points \(L \cap \ell\) are called *vertices* of the arc presentation \(L\).

As was mentioned in [8], from the combinatorial point of view, arc presentations are the same as rectangular diagrams. Namely, with each arc presentation \(L\) we can associate a rectangular diagram \(R\) whose vertices are those points \((\theta_0, z_0) \in \mathbb{R}^2, \theta_0 \in [0, 2\pi)\) for which \(L\) has an arc on the page \(P_{\theta_0}\) with an endpoint \(z = z_0\); see Figure 27.

Given a rectangular diagram, the inverse construction of an arc presentation is obvious, provided the rectangular diagram lies in the strip \([0, 2\pi) \times \mathbb{R}\), which imposes no restriction on its combinatorics and which will henceforth be tacitly assumed. The arc presentation corresponding to a rectangular diagram \(R\) will be denoted by \(\hat{R}\).

For a rectangular path \(\gamma\), the corresponding *book-like path* \(\hat{\gamma}\) is defined similarly, provided the ends of the path are horizontal. Namely, to any vertical edge \([((\theta_0, z_1), (\theta_0, z_2))\] there corresponds an arc on the page \(P_{\theta_0}\) with endpoints \(z = z_1\) and \(z = z_2\); see Figure 28. Only such rectangular paths will be considered from now on. By the *length* of the path \(\gamma\) we understand the number of its vertical edges, which coincides with the number of pages occupied by \(\hat{\gamma}\).

Notice that the geometric realization \(\hat{X}\) of a rectangular diagram or of a rectangular path \(X\) differs only in a minor way from the construction of \(\tilde{X}\) in Subsection 3.2. To obtain \(\hat{X}\) from \(\tilde{X}\), one only needs to designate \(\ell_2\) to be the binding line, smooth out the arcs, and rotate them about the line keeping their cyclic order.

By the hypothesis of the key lemma, for the rectangular diagram \(R\) we have a bypass \(\alpha\) of weight \(b\), and, in addition, the component of the diagram containing \(\partial \alpha\) has complexity larger than \(b\). The lemma assumes nothing on the positions of the ends of the bypass, but using end shift moves we can place the ends on any edges within the same component. Thus, without loss of generality, we may adopt the following.

**Convention 4.** Henceforth we shall assume that the ends of the bypass \(\alpha\) lie on horizontal edges of \(R\) and that the length of the bypassed path \(\beta\) is exactly \(b\).
4.2. Suitable disk $D$. Now we want to construct an embedded cooriented two-dimensional disk $D \hookrightarrow \mathbb{R}^3$ such that

1. the boundary $\partial D$ coincides with $\alpha \cup \beta$, and the interior of $D$ is disjoint from $\widehat{R} \cup \widehat{\alpha}$;
2. $D$ is the image of an $(a+b)$-gon under a regular smooth map, sending the vertices of the polygon to the vertices of the knot $\alpha \cup \beta$, where $a$ is the length of $\alpha$;
3. the interior of $D$ intersects the binding line $\ell$ transversally and only finitely many times;
4. $D$ is orthogonal to $\ell$ at the points of $\partial D \cap \ell$; moreover, in a small neighborhood $U$ of any of the ends of $\widehat{\alpha}$ and $\widehat{\beta}$, the arc of $\widehat{R}$ not belonging to $\widehat{\beta}$ lies outside the $\theta$-interval occupied by $D \cap U$ (see Figure 29);
5. the foliation with singularities $F$, defined on $D \setminus \ell$ by the restriction of the form $d\theta$, has only simple saddle singularities inside $D$ and has no closed regular leaves;
6. at the points of $D \cap \ell$, the coorientation of $D$ coincides with the positive direction of $\ell$;
7. on each arc of the form $\alpha \cap P_t$ (or $\beta \cap P_t$) the foliation $F$ has exactly one positive (respectively, negative) half-saddle (see definitions below);
8. all saddles and half-saddles of $F$ lie on different pages.

Most of these conditions are quite common for the technique we are going to use (see [4], [5], [7], [8]), and so we mention them only briefly. The main point here is the consistency of condition (D7) both when constructing the disk $D$ and during its subsequent simplification.

A disk $D$ satisfying (D1)–(D8) is said to be suitable. Let us construct it.

Recall that by the definition of a bypass in the link defined by the diagram $(R \setminus \beta) \cup \alpha \cup (\alpha \cup \beta)^{\wedge} \cup (\alpha \cup \beta)^{\vee}$, the components represented by $(\alpha \cup \beta)^{\wedge} \cup (\alpha \cup \beta)^{\vee}$ are unknotted and are not linked with the rest of the link, and their linking number equals $-b$. Here $X^{\wedge}$ denotes the result of the shift of $X$ by the vector $(\varepsilon, \varepsilon)$, and $X^{\vee}$ denotes the result of the shift by the vector $(-\varepsilon, -\varepsilon)$ with a sufficiently small $\varepsilon$.

Let $X^{\wedge}$ and $X^{\vee}$ be the shifts of $X$ by the vectors $(-\varepsilon, \varepsilon)$ and $(\varepsilon, -\varepsilon)$, respectively; let $X^{\wedge \vee}$ be $X^{\wedge} \cup X^{\vee}$ and let $X^{\vee \wedge}$ be $X^{\vee} \cup X^{\wedge}$.

We claim that in the link $(R \setminus \beta) \cup \alpha \cup \alpha^{\wedge} \cup \beta^{\vee}$ the two components represented by $\alpha^{\wedge} \cup \beta^{\vee}$ are not linked with the rest of the link and with each other.

Indeed, $\alpha^{\wedge} \cup \beta^{\vee}$ is obtained from $\alpha^{\vee} \cup \beta^{\wedge}$ by the forbidden permutations of $b$ pairs of vertical edges originating from the edges of $\beta$, each of which increases the linking number by one; see Figure 30.
Figure 29. Relative positions of a suitable disk and the link at the endpoints of $\hat{\alpha}$ and $\hat{\beta}$.

Figure 30. Under a “forbidden commutation”, the linking number goes up by one.

These moves do not exchange the edges of the components being altered with the edges of $(R \setminus \beta) \cup \alpha$, and therefore each of the modified components remains unlinked with $(R \setminus \beta) \cup \alpha$.

Now we proceed from rectangular diagrams to arc presentations. The path $\hat{\alpha} \nearrow$ is obtained from $\hat{\alpha}$ by a positive (i.e., in the increasing direction for $\theta$) rotation about the binding line $\ell$, followed by an upward shift; see Figure 31.

Likewise, $\searrow$, $\nearrow$, and $\swarrow$ correspond to, respectively, a negative rotation with an upward shift, a negative rotation with a downward shift, and a positive rotation with a downward shift.

Let $S$ denote a narrow band spanning the link $\hat{\alpha} \cup \hat{\beta}$ whose core line is the trivial knot $\hat{\alpha} \cup \hat{\beta}$. It consists of strips each of which binds to the binding line at the ends and is twisted a half turn as shown in Figure 32.

By construction, the boundary of $S$ forms a trivial link of two components, each of which is unlinked. We shall see in a moment that their union is also unlinked with $(\hat{R} \setminus \beta) \cup \hat{\alpha}$ and that its core line $\hat{\alpha} \cup \hat{\beta}$ is spanned by an embedded disk whose interior is disjoint from $\hat{R}$. Topologically, the situation is described by Figure 33.

This means that we can span the knot $\hat{\alpha} \cup \hat{\beta}$ by a disk $D$ whose interior is still disjoint from $\hat{R}$ and which is orthogonal to $S$ along its boundary. Moreover, at the endpoints of $\hat{\alpha}$
we can make $D$ and the ends of $R \setminus \beta$ approach $S$ from the opposite sides. Also, we can have the general position conditions (D2) and (D3) satisfied. Notice that conditions (D1) and (D4) are satisfied by construction.

Consider the foliation with singularities $\mathcal{F}$ on the disk $D$ given by the 1-form $d\theta$. It is undefined at the intersection points of $D$ and the binding line $\ell$, and it has singularities at the tangency points of $D$ and the pages $P_t$. By a general position argument, we may assume that those singularities are of Morse type.

Since the strip $S$ is orthogonal to the disk along its boundary $\partial D$, the foliation $\mathcal{F}$ has singularities on $\partial D$. Namely, since each such strip making up $S$ and having ends on the binding line is twisted a half turn, there is a half-saddle on each corresponding segment of $\partial D$; see Figure 34.

With each saddle or half-saddle we associate a sign, “+” or “−”, depending on whether $\theta$ increases or decreases in the direction of coorientation of $D$ on the (half-) saddle.

The intersection points of $D$ and the binding line will be called vertices of $D$, and we will distinguish the boundary and the interior vertices, depending on whether or not they lie on $\partial D$. With each vertex we associate a sign, “+” or “−”, indicating whether or not the coorientation of $D$ at the vertex coincides with the orientation of $\ell$.

Since the boundary of the disk twists a half turn between any two boundary vertices, it is not difficult to see that all boundary vertices have the same sign. We choose the coorientation of $D$ in a way that makes them all positive. Thus condition (D6) is satisfied.

Moreover, it is not difficult to see that, with our choice of the coorientation, condition (D7) is also satisfied.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{paths.png}
\caption{Paths $\hat{\alpha}$ and $\tilde{\alpha}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{strip.png}
\caption{Strips forming the band $S$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{topology.png}
\caption{The band $S$ viewed topologically.}
\end{figure}
Condition (D8) can be satisfied by performing, if necessary, a small perturbation of $D$. It remains to check condition (D5) that is, to verify that $\mathcal{F}$ has no singularities of “pole” type and no closed regular leaves. This is done in a standard way.

To that end, we remove from $D$ the union of all disks bounded by the closed leaves of $\mathcal{F}$. As a result, $D$ will have holes, around each of which the foliation is of the form shown in Figure 35.

The boundary of each hole consists of a single separatrix forming a loop. It lies entirely on one page and bounds a disk in it. Plugging the hole by that disk and slightly perturbing the obtained surface, we remove that singularity. Now the only type of singularity which $\mathcal{F}$ can have inside $D$ is a simple saddle, and no separatrix forms a loop.

Each regular leaf of $\mathcal{F}$ is an arc joining two vertices of opposite signs. There are four (respectively, three) separatrices attached to every saddle (respectively, half-saddle) with their other ends approaching vertices. Thus, the entire foliation $\mathcal{F}$ can be cut along regular leaves, as shown in Figure 36.
The separatrices of $\mathcal{F}$ partition $D$ into what we call cells of $\mathcal{F}$. Each cell is filled by regular leaves and has, on the boundary, two vertices of opposite signs and two (half-)saddles whose signs could be arbitrary; see Figure 37.

4.3. The induction. For convenience, both saddles and half-saddles will simply be called saddles, and the saddles inside $D$ will be said to be interior.

Thus, we have a quadruple $(R, \alpha, \beta, D)$, where $\alpha$ is a bypass of weight $b$ and length $a$ with ends on horizontal edges of the rectangular diagram $R$, $\beta$ is a bypassed path of length $b$, and $D$ is a suitable disk for $(\hat{R}, \hat{\alpha})$. Our goal is to show that we can apply $b$ successive elementary simplifications to $R$ and obtain a diagram Legendrian equivalent to $(R \setminus \beta) \cup \alpha$. This will be done by induction on $(a + b, c, d)$, where $c$ is the number of interior vertices of $D$ and $d$ is the number of interior saddles that do not lie on bridges (see below). Triples $(a + b, c, d)$ are ordered lexicographically.

A bridge is an arc in $D$ which connects two boundary vertices and consists of two separatrices; see Figure 38.
The induction base corresponds to the case $a = b = c = 1, d = 0$. For the induction step, in all other cases we modify the quadruple $(R, \alpha, \beta, D)$ and obtain another quadruple $(R', \alpha', \beta', D')$ with the same properties (for which the lengths of $\alpha'$ and $\beta'$, the number of interior vertices of $D'$, and the number of interior saddles lying outside bridges are denoted, respectively, by $a', b', c'$, and $d'$) such that one of the following holds:

(E1) $R \mapsto R'$ is a type II elementary simplification, the transformation $(R \setminus \beta) \cup \alpha \mapsto (R' \setminus \beta') \cup \alpha'$ preserves the Legendrian type of the diagram, and $a' = a, b' = b - 1$;

(E2) $R'$ is obtained from $R$ by commutations and cyclic permutations, $(R \setminus \beta) \cup \alpha \mapsto (R' \setminus \beta') \cup \alpha'$ is a type I elementary simplification, and $a' = a - 1, b' = b$;

(E3) $R' \cup \alpha'$ is obtained from $R \cup \alpha$ by commutations and cyclic permutations, $a' = a, b' = b$, and either $c' = c - 2$ or $c' = c, d' = d - 1$.

The possibility of applying one of those manipulations is completely determined by the presence of certain patterns in $F$ (including the signs of vertices and saddles). The reason for this, as well as what happens with $R, \beta$, and $\alpha$, will be discussed later; here we only want to show how these manipulations affect $F$ and prove that at least one of the patterns enabling the induction step must be present.

By the *star* of a vertex (whether interior or boundary) of $D$ we mean the closure of the union of all regular leaves approaching that vertex. The *valence* of a vertex is the number of separatrices having it as an endpoint.

*Rearrangement of saddles*, which may be regular interior saddles or half-saddles on the boundary $\partial D$, is possible whenever saddles of the same sign appear on the boundary of the same cell. This makes $F$ change, as shown in Figures 39, 40, and 41.

If the two saddles being rearranged are negative half-saddles, we have case (E1), and if they are positive half-saddles, we have case (E2). If at least one of the saddles being rearranged is interior, then the diagram $R \cup \alpha$ undergoes commutations and cyclic permutations, and the numbers $a, b, c$ remain the same. We shall use this trick in a special situation, which will lead to case (E3).

*Smoothing out a wrinkle* is possible whenever there is a 2-valent vertex and one of the following two conditions is satisfied:

(i) the 2-valent vertex is connected by a regular leaf with an interior vertex (Figure 42), and then we have case (E3);

(ii) the star of the 2-valent vertex contains exactly one half-saddle (Figure 43), and then we have case (E1) for a negative half-saddle and case (E2) for a positive one.

Now we show that, whenever $a + b > 2$, at least one of the induction steps (E1), (E2), or (E3) is possible.

If $D$ has no bridges, we set $D_0 = D$. Otherwise, among the parts into which the bridges partition $D$, there are at least two disks whose boundaries contain only one bridge. We say that such disks are *terminal*. Let us mark the endpoints of the bridges and of $\tilde{\alpha}$. The boundary of at least one of the terminal disks will have no more than three marked vertices. Call that disk $D_0$ and consider the restriction of $F$ to it.
Let $V_k$ be the number of $k$-valent interior vertices of $D_0$, and let $B_k$ be the number of $k$-valent vertices on $\partial D_0$ (the valence is defined with respect to $D_0$). By construction, the vertices on $\partial D_0$ alternate with half-saddles (the saddle on the bridge cutting out $D_0$ is considered as a half-saddle for $D_0$). Hence, there are $\sum_k B_k$ half-saddles on $\partial D_0$ and the number of saddles inside $D_0$ is one less than that of the vertices, i.e., $\sum_k V_k - 1$, by an Euler characteristic argument. Notice also that there are no vertices of valence less than 2 on the boundary of or inside the disk $D_0$.

Now let us count the number of separatrices in two ways: by the saddles and by the vertices. There are three separatrices coming out of each half-saddle and four from each interior saddle. Hence

$$3 \sum_k B_k + 4\left(\sum_k V_k - 1\right) = \sum_k kB_k + \sum_k kV_k.$$
Rearranging the summands, we have

\( B_2 + 2V_2 + V_3 = 4 + \sum_{k \geq 3} (k - 3)B_k + \sum_{k \geq 4} (k - 4)V_k \geq 4. \)

If \( V_2 > 0 \), then there is a 2-valent vertex inside \( D_0 \). If at least one of the other vertices in its star is interior, then we can smooth out a wrinkle inside \( D \). If both vertices in its star belong to \( \partial D \), then \( D_0 \) is the whole star. If \( D_0 \neq D \), then a wrinkle on the boundary can be smoothed out. If \( D_0 = D \), then \( a = b = c = 1 \) and \( d = 0 \), which is the induction base.

Now suppose \( V_2 = 0 \) and \( V_3 > 0 \), i.e., there is a 3-valent interior vertex \( P \) of \( D_0 \). Its star contains two saddles of the same sign, which we can rearrange. If they both lie on \( \partial D \), then the rearrangement results in a removal of one boundary vertex. Otherwise, \( a, b, \) and \( c \) are preserved, but \( P \) becomes a 2-valent vertex.

In the latter case, it may happen that both remaining vertices in the star of \( P \) lie on \( \partial D \). This means that the rearrangement results in a new bridge, and \( d' = d - 1 \). Otherwise, we can smooth out a wrinkle inside \( D \). In either case we have (E3).

If \( V_2 = V_3 = 0 \), then it follows from (3) that \( B_2 \geq 4 \). By construction, on the boundary of \( D_0 \) there are at most three vertices which are endpoints of a bridge or of \( \hat{\alpha} \). Hence, among the remaining vertices there is at least one 2-valent vertex. It is an interior vertex either for \( \hat{\alpha} \) or for \( \hat{\beta} \) and is 2-valent not only with respect to \( D_0 \) but also with respect to \( D \). Rearrangement of the saddles in the star of that vertex results in its removal.

Now we describe in detail the moves making the induction step possible in all of the above cases.

**4.4. Rearrangement of saddles.** This trick can be used whenever there is a cell \( \Delta \) in \( D \) with two saddles of the same sign on the boundary. Let \( S_1 \) and \( S_2 \) be those saddles, and suppose they lie on pages \( P_{\theta_1}, P_{\theta_2} \), respectively, with \( 0 < \theta_1 < \theta_2 < 2\pi \) and with \( \Delta \) lying in the sector \( \theta_1 \leq \theta \leq \theta_2 \) (we can always achieve this by rotating the entire construction.
about the binding line, which would make $R \cup \alpha$ undergo only cyclic permutations); see Figure 44.

![Figure 44. Pulling saddles of the same sign toward each other.](image)

Denote this sector by $K$. The disk $\Delta$ cuts it into two parts, which we denote by $K_1$ and $K_2$. Since $S_1$ and $S_2$ have the same sign, the separatrices coming out of these saddles and not lying on the boundary of $\Delta$ lie in $K$ on the opposite side from $\Delta$. With an appropriate numbering of $K_1$ and $K_2$, the separatrices coming out of the saddle $S_i$ lie on the boundary of $\partial K_i$, $i = 1, 2$, which we shall henceforth tacitly assume.

If there are arcs of $\hat{R} \cup \hat{\alpha}$ or saddles of $F$ in $K_1$, then, by rotating those arcs about the binding line in the positive direction and deforming the disk $D$, we can push them out of $K$ into the region $\theta_2 < \theta < \theta_2 + \varepsilon$ with a small enough $\varepsilon > 0$. Likewise, the arcs and the saddles inside $K_2$, can be pushed into the sector $\theta_1 - \varepsilon < \theta < \theta_1$ by a negative rotation and a deformation of $D$. Under these modifications, the corresponding $\Theta$-diagram $R \cup \alpha$ undergoes only commutations of vertical edges, while the combinatorial structure of $F$ remains the same.

As a result, we can remove topological obstructions to pulling the saddles $S_1$ and $S_2$ toward each other. If both saddles are interior, then, by deforming $D$, we can collapse $\Delta$ and thus produce a monkey saddle, which can be resolved into a pair of simple saddles in three different ways. Figure 45 shows consecutive sections of the part of $D$ being altered by the pages from $K$ in all three cases.

If exactly one of the saddles $S_1$ and $S_2$ is a half-saddle, say $S_2$, then only two of the three resolutions of the monkey saddle are left; see Figure 46.
If both $S_1$ and $S_2$ are half-saddles, then, after the above procedure of pushing arcs and saddles out of $K$, there will be exactly two arcs from $\alpha \cup \beta$ in $K$ which contain $S_1$ and $S_2$. The vertical edges of the $\Theta$-diagram $R \cup \alpha$ corresponding to those arcs become neighboring, so, after several commutations, we can apply a destabilization; see Figure 47 where we only show one of the four possible positions of $\Delta$ in 3-space. The remaining cases are obtained from this one by symmetries with respect to either the plane perpendicular to $\ell$ or the bisector plane between $P_{\theta_1}$ and $P_{\theta_2}$.

Moreover, because $S_1$ and $S_2$ have the same sign, they both lie either in $\widehat{\alpha}$ or in $\widehat{\beta}$. An easy direct verification shows that in the former case $\alpha$ undergoes a type I destabilization,
and in the latter case \( \beta \) undergoes a type II destabilization. The corresponding change in the sections of \( D \) by pages from \( K \) is shown in Figure 48. The original sequence is shown in the top of Figure 48 and in the bottom of Figure 48 one can see the sequence after the rearrangement of half-saddles.

In each rearrangement case, one can directly check that the saddle signs are preserved and the combinatorics of \( \mathcal{F} \) changes as in Figures 39, 40, 41. In each rearrangement case,
$R \cup \alpha$ was subjected only to cyclic permutations, commutations, and, when rearranging two boundary saddles, one destabilization. Thus, the corresponding conditions (E1), (E2), or (E3) for the induction step have been satisfied.

4.5. **Smoothing out a wrinkle.** Let $P_0$ be a 2-valent interior vertex of $D$. At the moment, we do not specify whether or not the other two vertices, denoted by $P_1$ and $P_2$, in its star are interior or boundary. The star of $P_0$ consists of two cells, which we denote by $\Delta_1$ and $\Delta_2$, in accordance with the positions of $P_1$ and $P_2$. Suppose the vertices $P_0, P_1, P_2$ have $z$-coordinates $z = z_0, z_1, z_2$, respectively. Without loss of generality, we may assume that these points have the following order on the binding line: $z_1 < z_0 < z_2$. Indeed, if $z_0$ does not lie between $z_1$ and $z_2$, then we can add a point $\infty$ to $R$ (and to the binding line), thus obtaining a 3-sphere, and then remove an arbitrary point from the interval between $z_1$ and $z_2$ disjoint from $D \cup \tilde{R}$. For the corresponding $\Theta$-diagram $\tilde{R} \cup \tilde{\alpha}$ this would yield a cyclic permutation of horizontal edges and, possibly, of the ends of $\alpha$.

If $z_2 < z_0 < z_1$, then we just change the numbering of the vertices $P_1$ and $P_2$.

Moreover, without loss of generality we may assume that the cell $\Delta_1$ is contained in the half-space $\theta \in [0, \pi]$ and that the cell $\Delta_2$ is contained in the half-space $\theta \in [\pi, 2\pi]$. Indeed, this can always be done by applying the map $(\rho, \theta, z) \mapsto (\rho, f(\theta), z)$, where $f$ is an appropriate degree 1 self-diffeomorphism of the circle. Such a map is isotopic to the identity. The corresponding diagram $R \cup \alpha$ may undergo only cyclic permutations of vertical edges.

The position of the cells $\Delta_1$ and $\Delta_2$ in 3-space is shown in Figure 49. Each of $\Delta_1$ and $\Delta_2$ cuts off a three-dimensional half-ball from its half-space, which we denote by $K_1$ and $K_2$, respectively.
Deforming $D$ and rotating the arcs of $\hat{R} \cup \hat{\alpha}$ about the binding line, we can push all saddles of $F$ and all the arcs out of $K_1$ and $K_2$. Under this procedure, the corresponding diagram $R \cup \alpha$ may only undergo commutations and cyclic permutations of vertical edges. As a result, the following rectangles will become free of the vertices of $R \cup \alpha$:

$$[0, \pi] \times [z_1, z_0] \quad \text{and} \quad [\pi, 2\pi] \times [z_0, z_2].$$

Now there are no topological obstructions to moving all vertices of $D$, $\hat{R}$, and $\hat{\alpha}$ from the interval $(z_1, z_0)$ to the interval $(z_2, z_2 + \varepsilon)$, and from $(z_0, z_2)$ to $(z_1 - \varepsilon, z_1)$ provided $\varepsilon$ is sufficiently small; see Figure 50. In each of the two sets of shifted vertices, their relative order on the binding line is preserved.

![Figure 50. Change in vertex positions under smoothing out a wrinkle.](image)

Notice that no section of the form $P_t \cap (D \cup \hat{R})$ contains more than one arc coming out from a vertex of $D$ or of $\hat{R}$. In our construction, we need to know this for $P_0$, $P_1$, and $P_2$, because otherwise there could be obstructions to the vertex exchange mentioned above. The absence of arcs with common ends on the pages $P_t$ is guaranteed by the conditions (D3) and (D4) from the definition of a suitable disk.

Now there are no vertices of $D$ and $\hat{R}$ between $P_1$ and $P_2$, except $P_0$. Our next moves depend on the relative position of the pattern $\Delta_1 \cup \Delta_2$ with respect to the boundary of $D$.

If one of the vertices $P_1$ and $P_2$ is interior, then, deforming the disk, we can remove two vertices; see Figure 51, where we assume that $P_1$ is an interior vertex. This deformation can be chosen such as to effect the change in $F$ from Figure 42.

The boundary of $D$ is not involved in this procedure and therefore the disk remains suitable.

Suppose now that both $P_1$ and $P_2$ are on the boundary of $D$. In Subsection 4.3 above, we did not need to smooth out a wrinkle in the case when the star $\Delta_1 \cup \Delta_2$ of $P_0$ was bounded by two bridges. We excluded that case by requiring that $D_0$ be terminal.

Thus, if $P_1, P_2 \in \partial D$, then at least one of the arcs with endpoints $P_1, P_2$ that bound $\Delta_1 \cup \Delta_2$ is contained in $\hat{\alpha} \cup \beta$. Consider the case when there is just one such arc and it is contained in $\hat{\alpha}$, i.e., it contains a positive half-saddle. For the positions of $\Delta_1$ and $\Delta_2$ that we are considering, the positive half-saddle will be the one on page $P_{\pi}$; see Figure 52.
Smoothing out a wrinkle in this case consists of the following. First, we remove the interior of the disk $\Delta_1 \cup \Delta_2$ together with the part of the boundary lying in $\widehat{\alpha}$. Then we collapse to a point the segment $[P_1, P_2]$ of the binding line and, at the same time, collapse to a line segment the disk cut off from $\mathcal{P}_0$ by the separatrices lying on the boundary $\partial(\Delta_1 \cup \Delta_2)$; see Figure 53.
In the corresponding rectangular diagram, this results in collapsing the vertical edge \( \pi \times [z_1, z_2] \), which is part of \( \alpha \); see Figure 54. There are no other vertices of \( R \cup \alpha \) in the strip \( \mathbb{R} \times [z_1, z_2] \); hence this transformation can be written as the composition of commutations of vertical edges and destabilizations, whenever \( \alpha \) contains at least two edges. More precisely, if the edge \( z = z_1 \) belongs to \( \alpha \), then the short vertical edge \( \pi \times [z_1, z_2] \) should be shifted by commutations to the right until the edge \( z = z_1 \) becomes short, and then a destabilization can be applied. If this edge does not belong to \( \alpha \), then the short vertical edge should be similarly shifted to the left.

Examining the relative positions of the reduced edges, one can see that the destabilization is of type I. One can also see that the corresponding change in \( F \) is of the form shown in Figure 43.

If \( \pi \times [z_1, z_2] \) is the only edge of \( \alpha \), then a destabilization is impossible. Assuming that this is the case, \( \alpha \cup \beta \) still admits a type I simplification, which results in a diagram of the unknot of complexity equal to its Thurston-Bennequin number. Because of (2), this contradicts Theorem 3.

Thus, the situation in which \( \pi \times [z_1, z_2] \) is the only edge of \( \alpha \), which creates an obstruction to smoothing out a wrinkle, cannot occur.

If the boundary of \( \Delta_1 \cup \Delta_2 \) contains exactly one arc of \( \widehat{\alpha} \cup \widehat{\beta} \) and that arc belongs to \( \widehat{\beta} \), then a similar procedure allows us to simplify the path \( \beta \). Its length, which is also the weight of the bypass \( \alpha \), goes down by one. By symmetry arguments, \( \beta \) undergoes a type II simplification.
In that case, to smooth out a wrinkle, \( \beta \) must have more than one edge, which is again proved by contradiction. More precisely, if \( \beta \) consists of a single edge, then \( \text{tb}(\alpha \cup \beta) = -1 \) and \( \alpha \cup \beta \) admits a type II destabilization resulting in an unknot diagram with \( \text{tb} = 0 \), contrary to Theorem 3.

In all the cases, under smoothing out a wrinkle, \( R \cup \alpha \) undergoes only cyclic permutations, commutations, and, in the case of a boundary wrinkle, one destabilization. Hence conditions (E1), (E2), or (E3) for the induction step have been satisfied.

Finally, the boundary of \( \Delta_1 \cup \Delta_2 \) may coincide with \( \hat{\alpha} \cup \hat{\beta} \). This means that each of the paths \( \hat{\alpha} \) and \( \hat{\beta} \) consists of a single arc and \( \Delta_1 \cup \Delta_2 \) coincides with \( D \). This situation is discussed next.

4.6. The induction base. We continue the foregoing discussion. As a result of the manipulations described there, each of \( \alpha \) and \( \beta \) has a single edge, \( \pi \times [z_1, z_2] \) and \( 0 \times [z_1, z_2] \), respectively, and the ends of \( R \setminus \beta \) are inside the segments \( [0, \pi] \times z_2 \) and \( [\pi, 2\pi] \times z_1 \); see Figure 55.

Moreover, there are no vertices of \( R \cup \alpha \) inside the strip \( \mathbb{R} \times [z_1, z_2] \); see Figure 56. Clearly, \( R \) admits a type II simplification, whose resulting diagram can also be obtained from \( (R \setminus \beta) \cup \alpha \) by a type I simplification, and is therefore Legendrian equivalent to \( (R \setminus \beta) \cup \alpha \).

The key lemma is proved.

Now we mention another proof of the monotonic simplification theorem for the unknot. In the proof of Corollary 2 above we used the classification theorem of Eliashberg-Fraser, which proves much more than we would need if we choose not to use the key lemma in its present form but instead re-prove it with some minor simplifications.

Let \( K \) be a rectangular diagram of the unknot. Let \( a \) and \( b \) denote the numbers \( -\text{tb}(K^\sim) \) and \( -\text{tb}(K) \), respectively. By Theorem 3 we have \( a, b > 0 \). Recall that,

\[
\text{Figure 55. Relative position of the wrinkle and the arcs of } \hat{R} \cup \hat{\alpha} \text{ for a completely simplified disk.}
\]
according to (2), the total number of vertical edges of $K$ is $a + b$. Therefore $K$ can be represented as the union of two rectangular paths $\alpha$ and $\beta$, having, respectively, $a$ and $b$ vertical edges.

Now we simply go back to the beginning of Subsection 4.2 and repeat the construction and simplification of $D$, ignoring the rectangular path $R \setminus \beta$ and everything related to it.

\[\square\]

§5. Applications to braids and transversal links

5.1. Birman-Menasco classes. By a well-known theorem of J. Alexander [1], any oriented link can be represented by a closed braid; see Figure 57 for an example. Another well-known theorem, due to Markov [17], [3], asserts that the closures of two braids are equivalent as oriented links if and only if they can be obtained from each other by what are now called Markov moves. These transformations include the usual conjugations in the group-theoretic sense, stabilizations, and destabilizations (see Figure 58). Algebraically, they can be described as follows.

Let $B_n$ denote the braid group on $n$ strands, with standard Artin generators $\sigma_1, \ldots, \sigma_{n-1}$. The embedding $\iota_n: B_n \to B_{n+1}$ is defined by adding a free $(n+1)$st strand and is written tautologically in the terms of the generators: $\sigma_i \mapsto \sigma_i$, $i = 1, \ldots, n-1$.

In this notation, a stabilization of a braid $\beta \in B_n$ is defined as the transformation

$$\beta \mapsto \iota_n(\beta)\sigma_n^{\pm 1}.$$
The stabilization is said to be positive or negative depending on the power of $\sigma_n$. A destabilization (positive or negative) is defined as the inverse operation.

A monotonic simplification theorem for braids, proved in [5], asserts that every braid whose closure is unknotted can be transformed into a trivial braid on one strand using conjugations, destabilizations, and exchange moves introduced by J. Birman and W. Menasco, which are defined algebraically as follows:

$$\beta_1\sigma_n\beta_2\sigma_n^{-1} \mapsto \beta_1\sigma_n^{-1}\beta_2\sigma_n,$$

where $\beta_1, \beta_2 \in \iota_n(B_n)$; see Figure 59. The inverse operation is clearly a composition of conjugations and an exchange move. (Actually, more types of moves are used in [5], but those moves easily reduce to the ones we defined. This was noticed by the authors of [5] later.)

**Definition 10.** The Birman-Menasco class of a braid $\beta \in B_n$ is the set of all braids $\beta' \in B_n$ which can be obtained from $\beta$ by a sequence of conjugations and exchange moves. It will be denoted by $[\beta]_{BM}$, and the set of all Birman-Menasco classes in $B_n$ by $BM_n$.

We shall say that $\mathcal{B} \in BM_n$ admits a positive (negative) destabilization $\mathcal{B} \mapsto \mathcal{B}'$ if there is a positive (respectively, negative) destabilization $\beta \mapsto \beta'$ with $\beta \in \mathcal{B}$, $\beta' \in \mathcal{B}'$. In that case, we shall also say that $\mathcal{B}$ is obtained from $\mathcal{B}'$ by a positive (respectively, negative) stabilization.

**Theorem 8.** Suppose Birman-Menasco classes $\mathcal{B}_1$ and $\mathcal{B}_2$ define equivalent oriented links. Then there is a Birman-Menasco class $\mathcal{B}$ which can be obtained from $\mathcal{B}_1$ by a sequence of positive stabilizations and destabilizations and from $\mathcal{B}_2$ by a sequence of negative stabilizations and destabilizations.

The same is true if the Birman-Menasco classes are replaced by braid conjugacy classes.

![Figure 58. Stabilizations applied to a braid $\beta$.](image)

![Figure 59. Exchange move.](image)
Before proceeding with the proof, we recall connections between rectangular diagrams and braids.

Throughout this section we deal only with *oriented* rectangular diagrams of links. This means that their edges have consistent orientations, i.e., ones resulting in an oriented link diagram. It is convenient to specify the orientation by coloring vertices in black and white, so that the vertical edges run from a black vertex to a white one, whereas the horizontal edges run in the opposite direction. Rectangular diagrams with such a coloring of vertices are also called *grid diagrams*.

For oriented rectangular diagrams one naturally distinguishes four, rather than two, types of stabilizations, which we shall call *oriented types*. Each of the types I and II splits into two oriented types: \( \rightarrow I \), \( \leftarrow I \) and \( \rightarrow II \), \( \leftarrow II \), respectively, where the arrows indicate the direction of the short horizontal edge which appears as a result of a stabilization. In terms of colored vertices, all four types are shown in Figure 60.

With each oriented rectangular diagram \( R \) we associate a braid \( \beta_R \) in the following way (see [7], [8], [20]). Suppose all vertices of \( R \) are contained in the strip \([0,1] \times \mathbb{R}\). Each horizontal edge \([x_1,x_2] \times y_0\) oriented from right to left is replaced by a pair of line segments \([0,x_1] \times y_0\) and \([x_2,1] \times y_0\). At each crossing of horizontal and vertical segments, the vertical ones are considered overpasses. Now, tilting the vertical edges and smoothing out the corners, we have a braid diagram of \( \beta_R \); see Figure 61.

**Proposition 3.** Any braid can be represented as \( \beta_R \) for some oriented rectangular diagram \( R \).

For any oriented rectangular diagram \( R \), the corresponding oriented link is equivalent to the closure of the braid \( \beta_R \).

Let \( R \) and \( R' \) be oriented rectangular diagrams. Then the Birman-Menasco classes of \( \beta_R \) and \( \beta_{R'} \) coincide if and only if \( R \) and \( R' \) can be obtained from each other by a sequence of elementary moves avoiding stabilizations and destabilizations of types \( \rightarrow I \) and \( \rightarrow II \).

For a proof, see [20].

It is not difficult to see that a \( \leftarrow I \)-stabilization of \( R \) results in a positive stabilization of \( [\beta_R]_{BM} \) and that a \( \rightarrow II \)-stabilization of \( R \) results in a negative one. Since \( \rightarrow I \) - and \( \rightarrow II \)-stabilizations do not change the class \([\beta_R]_{BM}\), we have that the diagrams obtained from the oriented rectangular diagram \( R \) by stabilizations of different oriented types cannot be obtained from each other by commutations and cyclic permutations.

**Proof of Theorem 8** Let \( R_1 \) and \( R_2 \) be oriented rectangular diagrams with \( \beta_{R_1} \in \mathcal{B}_1 \) and \( \beta_{R_2} \in \mathcal{B}_2 \). By the closure condition, the braids \( \beta_1 \) and \( \beta_2 \) are equivalent; hence, by Proposition 3 the diagrams \( R_1 \) and \( R_2 \) define equivalent oriented links.

By Theorem 7 there is a diagram \( R \) which is Legendrian equivalent to \( R_1 \) and such that \( R^\sim \) is Legendrian equivalent to \( R_2^\sim \). We claim that \([\beta_R]_{BM}\) is the desired class.

![Figure 60. Oriented types of (de)stabilizations.](image-url)
Indeed, $R$ is obtained from $R_1$ by a sequence of cyclic permutations, commutations, and stabilizations/destabilizations of types $\overrightarrow{T}$ and $\overleftarrow{T}$. Of these moves, only stabilizations and destabilizations of type $\overrightarrow{I}$ change the Birman-Menasco class of the corresponding braid, resulting in positive stabilizations and destabilizations.

In the case of $R_2$, the argument is the same, except that the types of stabilizations and destabilizations should be changed to $\overrightarrow{II}$ and $\overleftarrow{II}$ for rectangular diagrams and from positive to negative for the corresponding braids.

The second assertion of the theorem, concerning the braid conjugacy classes, follows from the first one and from an observation by J. Birman and N. Wrinkle [6] that any exchange move can be represented as a sequence of conjugations, one stabilization, and one destabilization, in which the stabilization and the destabilization are of the same sign, which can be chosen arbitrarily. When the stabilization and the destabilization are positive, the decomposition looks as follows:

$$
\beta_1 \sigma_n \beta_2 \sigma_n^{-1}
$$

conjugation $\downarrow$

$$
\sigma_n \beta_1 \sigma_n \beta_2 \sigma_n^{-2}
$$

stabilization $\downarrow$

$$
\sigma_n \beta_1 \sigma_n \beta_2 \sigma_n^{-2} \sigma_n+1
$$

conjugation $\downarrow$

$$
\sigma_n^{-2} \sigma_n+1 \beta_2 \sigma_n^{-2} \sigma_n+1 \sigma_n \beta_1 \sigma_n \sigma_n+1 \sigma_n^2 = \sigma_n^{-2} \beta_2 \sigma_n \beta_1 \sigma_n \sigma_n+1
$$

To obtain a similar decomposition for negative stabilization and destabilization, one should invert all $\sigma_n$ and $\sigma_n+1$, as well as the order of the moves. □

5.2. The Jones conjecture. For any $n \geq 2$, we denote by $c$ the homomorphism from $B_n$ to $\mathbb{Z}$ given by $c(\sigma_1) = 1$. In other words, $c(\beta)$ is the algebraic crossing number of a braid diagram representing the braid $\beta$.

In [14], V. Jones asked the following question: is it true that $c(\beta)$ of a minimal braid is an invariant of the corresponding link? A braid is said to be minimal if it has the smallest possible number of strands among all braids representing the same oriented link.

A positive answer to this question, which is now known as the Jones conjecture, was given for several infinite series of knots; see [15] and the references therein. The following stronger assertion was formulated as a conjecture in [15] and [16] and was called the generalized Jones conjecture.

**Theorem 9.** Suppose braids $\beta_1 \in B_m$ and $\beta_2 \in B_n$ represent the same class of oriented links and $\beta_1$ has the smallest possible number of strands in that class. Then

$$
|c(\beta_2) - c(\beta_1)| \leq n - m.
$$
In particular, when \( n = m \) we have \( c(\beta_1) = c(\beta_2) \).

**Proof.** By Theorem 8, there are a natural number \( p \) and \( \beta \in B_p \) such that the classes \([\beta]_{BM}\) and \([\beta_1]_{BM}\) can be connected by a sequence of positive stabilizations and destabilizations, and the classes \([\beta]_{BM}\) and \([\beta_2]_{BM}\) can be connected by a negative sequence.

Under positive (de)stabilizations, the difference of the algebraic crossing number and the number of strands does not change. Hence

\[
c(\beta_1) - m = c(\beta) - p.
\]

Likewise, under negative (de)stabilizations, the sum of the algebraic crossing number and the number of strands does not change, and

\[
c(\beta_2) + n = c(\beta) + p.
\]

Hence

\[
p = \frac{1}{2}(m + n + c(\beta_2) - c(\beta_1)).
\]

Since \( p \geq m \), we have

\[
c(\beta_2) - c(\beta_1) \geq m - n.
\]

Similarly, choosing \( \beta \) such that \([\beta]_{BM}\) is connected by a sequence of negative (respectively, positive) stabilizations and destabilizations with \([\beta_1]_{BM}\) (respectively, with \([\beta_2]_{BM}\)), we have

\[
c(\beta_2) - c(\beta_1) \leq n - m.
\]

It was noticed in [16] that the generalized Jones conjecture which we have just proved is equivalent to the following statement.

**Theorem 10.** Suppose planar diagrams \( D_1 \) and \( D_2 \) represent equivalent oriented links and have, respectively, \( m \) and \( n \) Seifert circles, where \( m \) is the smallest possible number of Seifert circles for the diagrams representing the same link type. Then

\[
|w(D_2) - w(D_1)| \leq n - m,
\]

where \( w(D) \) denotes the algebraic crossing number of \( D \).

**Remark 3.** Our proof of the generalized Jones conjecture partially follows the approach of K. Kawamuro, proposed in 2008, namely, in the reduction to the “commutativity principle”. However, that formulation was based on conjugacy classes of braids, and not on Birman-Menasco classes, which made it false.

**5.3. Transversal links.**

**Definition 11.** A transversal link is an oriented link such that the restriction to it of the standard contact form \( \omega = x dy + dz \) takes on positive values on the tangent vectors defining the orientation of the link.

Transversal links are said to be transversally equivalent if they are isotopic in the class of transversal links.

With every braid \( \beta \in B_n \) one can canonically associate a transversal link defined up to transversal equivalence. To this end, the closure \( L \) of \( \beta \) should be placed in space so that it transversally intersects each \( P_t \) (see Subsection 4.1) \( n \) times and, moreover, the angular coordinate \( \theta \) is increasing on \( L \) in the direction of the link orientation. For a small enough \( \varepsilon > 0 \), the form \( \varphi_\varepsilon^* \omega \), where \( \varphi_\varepsilon \) is the diffeomorphism \( (x, y, z) \mapsto (2x, y, \varepsilon z - xy) \), is a small perturbation of the form \( \rho d\theta = x dy - y dx \), and therefore the link \( \varphi_\varepsilon(L) \) is transversal. The transversal link obtained in this way will be denoted by \( T_\beta \).
Theorem 11. Each transversal link is transversally equivalent to a link of the form $T_\beta$.

For given braids $\beta_1$ and $\beta_2$, the links $T_{\beta_1}$ and $T_{\beta_2}$ are transversally equivalent if and only if $\beta_1$ and $\beta_2$ can be obtained from each other by a sequence of conjugations and positive stabilizations and destabilizations.

The first part of this theorem was proved by D. Bennequin [2]. The second part was proved by S. Orevkov and V. Shevchishin [21] and independently by N. Wrinkle [24].

The self-linking number $sl(L)$ of a transversal link $L$ is defined as $c(\beta) - n$, where $\beta \in B_n$ is any braid for which $L$ is transversally equivalent to $T_\beta$. As one sees from Theorem 11, this number is well-defined since it does not change under positive stabilizations.

For a braid $\beta$ we denote by $[\beta]_t$ the set of all braids $\beta'$ such that the links $T_{\beta'}$ and $T_\beta$ are transversally equivalent. We call $[\beta]_t$ the transversal class of $\beta$. It follows from (4) that each Birman-Menasco class is a subset of some transversal class.

In the language of transversal links, Theorem 8 means that for any braids $\beta_1$ and $\beta_2$ defining equivalent oriented links there is a braid $\beta$ such that $[\beta]_t = [\beta_1]_t$ and $[\beta^{-1}]_t = [\beta_2^{-1}]_t$.

Theorem 12. Suppose Birman-Menasco classes $B_1$ and $B_2$ are contained in the same transversal class and $B_1$ admits a negative destabilization $B_1 \mapsto B'_1$. Then $B_2$ also admits a negative destabilization $B_2 \mapsto B'_2$ such that $B'_1$ and $B'_2$ are also contained in the same transversal class.

Proof. Let $R_2$ and $R'_2$ be oriented rectangular diagrams such that $\beta_{R_2} \in B_2$ and $\beta_{R'_2} \in B'_2$. Then one can transform $R_2$ into $R'_2$ by a sequence of elementary moves containing exactly one $\overrightarrow{II}$-destabilization and avoiding $\overrightarrow{II}$-stabilizations.

By Lemmas 1 and 2 we can transform $R_2$ into $R'_2$ by a sequence of moves in which all $\overrightarrow{II}$-stabilizations go first and all $\overrightarrow{II}$-destabilizations go last. Since type $\overrightarrow{II}$-stabilizations and destabilizations do not change the corresponding braid, we may assume without loss of generality that they are not present in the sequence.

Applying Lemmas 1 and 2 again, we can move the only $\overrightarrow{II}$-destabilization to the end of the sequence. The obtained sequence of elementary moves with the last destabilization removed contains only type I stabilizations and destabilizations. Let $R$ be the last diagram in it. By construction, it is Legendrian equivalent to $R_2$, and $R \mapsto R'_2$ is a type $\overrightarrow{II}$ destabilization.

It follows from Theorem 5 that $R_2$ admits a type $\overrightarrow{II}$ simplification $R_2 \mapsto R'_2$ with Legendrian equivalents $R'_2$ and $R'_2$. It is not difficult to see from this that $[\beta_{R'_2}]_t = [\beta_{R'_2}]_t$ and that $[R_2]_{BM} \mapsto [R'_2]_{BM}$ is a negative destabilization. \hfill \Box

Finally, we remark that Theorem 9 gives a positive answer to Question 2 of [19]:

Corollary 4. Suppose $\beta$ has the smallest number of strands among all braids representing a given oriented link type. Then the transversal link $T_\beta$ has the largest possible self-linking number among all transversal links of the same topological type.

Proof. Suppose $\beta \in B_m$ and $\beta' \in B_n$ is any other braid representing the same oriented link type. By Theorem 11 we have

$$sl(T_{\beta'}) = c(\beta') - n \leq c(\beta) - m = sl(T_\beta).$$  \hfill \Box

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References


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