ON PROBLEMS CONCERNING MOMENT-ANGLE COMPLEXES
AND POLYHEDRAL PRODUCTS

A. BAHRI, M. BENDERSKY, F. R. COHEN, AND S. GITLER

It is a pleasure for the authors
to congratulate Vitya Buchstaber
on this occasion of his 70th birthday

ABSTRACT. The main goal of this paper is to give a list of problems closely connected to moment-angle complexes, polyhedral products, and toric varieties. Another purpose is to exhibit the ubiquity and utility of these spaces which have been the subject of seminal work of Buchstaber and Panov as well as many others.

§ 1. INTRODUCTION

The main goal of this paper is to give a list of problems closely connected to moment-angle complexes, polyhedral products, and toric varieties. Another purpose is to exhibit the ubiquity and utility of these spaces which have been the subject of seminal work of Buchstaber and Panov [15], [16] as well as many others.

These topological spaces have origins dating back to work of Poincaré [60], C. L. Siegel [66], Ganea [39], Porter [61], as well as Camacho, Kuiper, and Palis [19]. This last setting was explained in a beautiful work of Santiago López de Medrano [55], who gave methods to understand topological invariants of these spaces. H. S. M. Coxeter also developed some basic examples in 1938 [23] as explained by A. Suciu [70].

On the other hand, these spaces experienced deep, independent developments in different directions of combinatorics, geometry, and topology. Some of this includes work of Hochster [45], Goresky and MacPherson [41], Ziegler and Zivaljević [77], Davis and Januszkiewicz [25], and Buchstaber and Panov [15], and [16].

The point of view of this paper is that basic properties of polyhedral products inform many questions which touch on several parts of mathematics. This paper describes a few of these connections as well as related problems.

The authors thank the referee for useful suggestions.

§ 2. DEFINITIONS

The basic constructions addressed in this article are defined in this section. First recall the definition of an abstract simplicial complex.
Definition 2.1.

(1) Let $K$ be an abstract simplicial complex with $m$ vertices labeled by the set $[m] = \{1, 2, \ldots, m\}$. Thus, $K$ is a subset of the power set of $[m]$ such that an element given by a $(k-1)$ simplex $\sigma$ of $K$ is given by an ordered sequence $\sigma = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \ldots < i_k \leq m$ such that if $\tau \subset \sigma$, then $\tau$ is a simplex of $K$. In particular the empty set $\emptyset$ is a subset of $\sigma$ and so it is in $K$. The set $[m]$ is minimal in the sense that every $i \in [m]$ belongs to at least one simplex of $K$. The length $k$ of $\sigma$ is denoted $|\sigma|$.

(2) Given a sequence $I = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \ldots < i_k \leq m$, define $K_I \subseteq K$ to be the full sub-complex of $K$ consisting of all simplices of $K$ which have all of their vertices in $I$; that is $K_I = \{ \sigma \cap I \mid \sigma \in K \}$.

(3) In case $I = (i_1, \ldots, i_k)$, define $X_I = X_{i_1} \times X_{i_2} \times \ldots \times X_{i_k}$.

(4) Let $\partial_\sigma(K)$ denote the geometric realization of the link of $\sigma$ in $K$.

(5) Let $\Delta[m-1]$ denote the abstract simplicial complex given by the power set of $[m] = \{1, 2, \ldots, m\}$. Let $\Delta[m-1]_q$ denote the subset of the power set of $[m]$ given by all subsets of cardinality at most $q+1$. Thus $\Delta[m-1]_q$ is the $q$ skeleton of $\Delta[m-1]$.

(6) A polyhedron is defined to be the geometric realization of a simplicial complex.

Let $(X, A)$ denote the collection of spaces $\{(X_i, A_i, x_i)\}_{i=1}^m$.

Definition 2.2.

(1) The generalized moment-angle complex or polyhedral product determined by $(X, A)$ and $K$ denoted $Z(K; (X, A))$ is defined using the functor $D: K \to CW_*$ as follows: For every $\sigma$ in $K$, let

$$D(\sigma) = \prod_{i=1}^m Y_i, \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \in [m] - \sigma, \end{cases}$$

with $D(\emptyset) = A_1 \times \ldots \times A_m$.

(2) The polyhedral product is

$$Z(K; (X, A)) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim} D(\sigma),$$

where the colimit is defined by the inclusions, $d_{\sigma, \tau}$ with $\sigma \subset \tau$ and $D(\sigma)$ is topologized as a subspace of the product $X_1 \times \ldots \times X_m$. The polyhedral product is the underlying space $Z(K; (X, A))$ with base-point $x = (x_1, \ldots, x_m) \in Z(K; (X, A))$.

(3) Note that the definition of $Z(K; (X, A))$ did not require spaces to be either based or $CW$ complexes.

(4) In the special case where $X_i = X$ and $A_i = A$ for all $1 \leq i \leq m$, it is convenient to denote the polyhedral product by $Z(K; (X, A))$ to coincide with the notation in [29].

A direct variation of the structure of the polyhedral product follows next. Products of pointed spaces admit natural quotients called smash products. Spaces analogous to polyhedral products are given next where products of spaces are replaced by smash products, a setting in which base-points are required.

Definition 2.3. The smash product $X_1 \wedge X_2 \wedge \ldots \wedge X_m$ is given by the quotient space $(X_1 \times \ldots \times X_m) / S(X_1 \times \ldots \times X_m)$ where $S(X_1 \times \ldots \times X_m)$ is the subspace of the product with at least one coordinate given by the base-point $x_j \in X_j$.

The (reduced) suspension of a (pointed) space $(X, *)$

$$\Sigma(X)$$

is the smash product

$$S^1 \wedge X.$$
Definition 2.4. Given a polyhedral product $Z(K; (X, A))$ obtained from $(X, A, z)$, the polyhedral smash product $\hat{Z}(K; (X, A))$ is defined to be the image of $Z(K; (X, A))$ in the smash product $X_1 \wedge X_2 \wedge \ldots \wedge X_m$.

The image of $D(\sigma)$ in $\hat{Z}(K; (X, A))$ is denoted by $\hat{D}(\sigma)$ and is $Y_1 \wedge Y_2 \wedge \ldots \wedge Y_m$ where

$$Y_i = \begin{cases} X_i &\text{if } i \in \sigma, \\ A_i &\text{if } i \in [m] - \sigma. \end{cases}$$

As in the case of $Z(K; (X, A))$, note that $\hat{Z}(K; (X, A))$ is the colimit obtained from the spaces $\hat{D}(\sigma)$ with $\hat{D}(\sigma) \cap \hat{D}(\tau) = \hat{D}(\sigma \cap \tau)$.

One theorem proven in [5] is as follows.

Theorem 2.5. Let $K$ be an abstract simplicial complex with $m$ vertices. Given $(X, A) = \{(X_i, A_i)\}_{i=1}^m$ where $(X_i, A_i, x_i)$ are pointed triples of CW-complexes, there is a natural pointed homotopy equivalence

$$H: \Sigma Z(K; (X, A)) \to \Sigma \left( \bigvee_{I \subseteq [m]} \hat{Z}(K_I; (X_I, A_I)) \right).$$

A second theorem in [5] is as follows.

Theorem 2.6. Let $K$ be an abstract simplicial complex with $m$ vertices and let $\hat{K}$ be its associated poset. Let $(X, A)$ have the property that the inclusion $A_i \subset X_i$ is null-homotopic for all $i$. Then there is a homotopy equivalence

$$\hat{Z}(K; (X, A)) \to \bigvee_{\sigma \in K} |lk_\sigma(K)| * \hat{D}(\sigma),$$

where $|lk_\sigma(K)|$ is the link of $\sigma$ in $K$.

§ 3. COHOMOLOGY

The questions in this section address homological properties of the spaces $Z(K; (X, A))$. A brief list of topics is given next with fuller details below.

1. Identify related cohomology rings.
2. Identify whether these cohomology groups are given in terms of the Tor functor via the Eilenberg–Moore spectral sequence.

One immediate consequence of the stable decomposition of Theorem 2.5 of [5] is stated next where the classical evaluation map is given by

$$ev: \Sigma \Omega(X) \to X.$$

Corollary 3.1. Let $K$ be an abstract simplicial complex with $m$ vertices, given that $(X, A) = \{(X_i, A_i)\}_{i=1}^m$ where $(X_i, A_i, x_i)$ are pointed triples of CW-complexes, the adjoint of the natural pointed homotopy equivalence

$$H: \Sigma Z(K; (X, A)) \to \Sigma \left( \bigvee_{I \subseteq [m]} \hat{Z}(K_I; (X_I, A_I)) \right)$$

is a map

$$h: Z(K; (X, A)) \to \Omega \Sigma \left( \bigvee_{I \subseteq [m]} \hat{Z}(K_I; (X_I, A_I)) \right).$$

Furthermore, the map $h$ satisfies the property that the following composite is a homotopy equivalence

$$\Sigma Z(K; (X, A)) \xrightarrow{\Sigma(h)} \Sigma \Omega \Sigma \left( \bigvee_{I \subseteq [m]} \hat{Z}(K_I; (X_I, A_I)) \right) \xrightarrow{ev} \Sigma \left( \bigvee_{I \subseteq [m]} \hat{Z}(K_I; (X_I, A_I)) \right).$$
Corollary 3.2. Let $K$ be an abstract simplicial complex with $m$ vertices. Given that $(X, A) = \{(X_i, A_i)\}_{i=1}^n$ where $(X_i, A_i, x_i)$ are pointed triples of CW-complexes. For any cohomology theory $E^*(-)$, the map

$$h: Z(K; (X, A)) \to \Omega \bigg( \bigvee_{I \subseteq [m]} \tilde{Z}(K_I; (X_I, A_I)) \bigg)$$

induces an epimorphism in cohomology

$$E^*(h): E^*\left( \Omega \bigg( \bigvee_{I \subseteq [m]} \tilde{Z}(K_I; (X_I, A_I)) \bigg) \right) \to E^*\left( Z(K; (X, A)) \right),$$

which is additively (not necessarily preserving products) a split epimorphism.

Thus a determination of the cohomology algebra of the polyhedral product $Z(K; (X, A))$ follows from the structure of the cohomology algebra of $\tilde{Z}(K_I; (X_I, A_I))$ with possibly non-trivial new relations for products arising as the $K_I$ vary. That is the approach of [17] where the language is described in terms of adjoints of the above maps.

In addition, classical work of Bott and Samelson [14] gave the additive structure of the cohomology algebra of $\Omega \Sigma(X)$ for any path-connected CW-complex $X$ in terms of the homology of $X$ with $k$ coefficients as long as all modules in question are free $k$ modules. Furthermore, they showed that if $X$ is a suspension, then the homology of $\Omega \Sigma(X)$ is a primitively generated Hopf algebra in case the homology groups $H_*(X; k)$ are free over a ring $k$. This information determines the Hopf algebra structure for the singular homology of $\Omega \Sigma\left( \bigvee_{I \subseteq [m]} \tilde{Z}(K_I; (X_I, A_I)) \right)$ for many spaces.

3.1. Singular cohomology. The structure of the singular homology groups of $Z(K; (X, A))$ has a long, solid history. Some of the important basic developments were due to Davis and Januszkiewicz [23], and Buchstaber and Panov [15]. In the special cases of

$$DJ(K) = Z(K; (\mathbb{C}P^\infty, *)) \quad \text{and} \quad Z(K; (D^2, S^1)).$$

The stable decomposition of $Z(K; (X, A))$ exhibited in [5], [6] stated as Theorem 2.5 here implied the structure of these cohomology algebras for many of these spaces as a consequence of basic properties of that stable decomposition. Thus it is natural to ask about the structure of the cohomology algebras of $H^*(Z(K; (X, A)); \mathbb{F})$ for fields $\mathbb{F}$, and other pairs of spaces $(X, A)$.

In case each $A_i$ is contractible, then the cohomology algebra was determined completely as a property of the stable decomposition [5]. The resulting answer is given by a natural extension of the definition of the Stanley–Reisner ring of a simplicial complex as described next.

A finite set of path-connected spaces $X_1, \ldots, X_m$ is said to satisfy the strong form of the Künneth Theorem over $R$ provided that the natural map

$$\bigotimes_{1 \leq j \leq k} \tilde{H}^*(X_{i_j}; R) \to \tilde{H}^*(X_{i_1} \wedge \ldots \wedge X_{i_k}; R)$$

is an isomorphism for every sequence of integers $1 \leq i_1 < i_2 < \ldots, i_k \leq m$.

Notice that if each $A_i$ is contractible, then the natural inclusion

$$j: Z(K; (X, A)) \subset \prod_{i=1}^m X_i$$

induces an epimorphism in cohomology [5]. Furthermore, the kernel of $j^*$ is given by the generalized Stanley–Reisner ideal $I(K)$ which is generated by all elements $x_{j_1} \otimes x_{j_2} \otimes \ldots \otimes x_{j_k}$.
... \otimes x_j$, for which $x_j \in \bar{H}^*(X_j; R)$ and the sequence $J = (j_1, \ldots, j_l)$ is not a simplex of $K$ [5].

**Theorem 3.3.** Let $K$ be an abstract simplicial complex with $m$ vertices and let

$$\langle X, A \rangle = \{(X_i, A_i, x_i)\}_{i=1}^m$$

be $m$ pointed, connected CW-pairs. If all of the $A_i$ are contractible and coefficients are taken in a ring $R$ for which the spaces $X_1, \ldots, X_m$ satisfy the strong form of the K"unneth Theorem over $R$, then there is an isomorphism of algebras

$$\bigotimes_{i=1}^m H^*(X_i; R)/I(K) \to H^*(Z(K; \langle X, A \rangle); R).$$

Furthermore, there are isomorphisms of underlying abelian groups given by

$$E^*(Z(K; \langle X, A \rangle)) \to \bigoplus_{I \in K} E^*(\hat{X}^I)$$

for any reduced cohomology theory $E^*$.

The ring structure of $H^*(Z(K; \langle X, A \rangle))$ is also determined in the following cases.

1. The map $A_i \to X_i$ induces a split epimorphism in integer cohomology for all $i$.
2. All spaces $X_i$ and $A_i$ are of finite type.
3. The strong form of the K"unneth theorem holds.

The natural analogue of the Stanley–Reisner ideal in this setting is the ideal $I(K)$ generated by monomials in $\bigotimes_{i=1}^m H^*(X_i)$ given by

$$u_{i_1} \otimes u_{i_2} \otimes \ldots \otimes u_{i_k} \in H^*(X_{i_1}/A_{i_1}) \otimes H^*(X_{i_2}/A_{i_2}) \otimes \ldots \otimes H^*(X_{i_k}/A_{i_k})$$

regarded as a sub-algebra of $\bigotimes_{1 \leq i \leq m} H^*(X_i)$ corresponding to non-simplices $\tau = (i_1, i_2, \ldots, i_k) \subset [m]$.

The next theorem is given in [9].

**Theorem 3.4.** Let $K$ be an abstract simplicial complex with $m$ vertices. Assume that $(X, A)$ is a family of CW pairs where $A_i \to X_i$ induces a split epimorphism in integer cohomology for all $i$. Suppose also that the strong form of the K"unneth theorem holds; then there is an isomorphism of rings

$$H^*(Z(K; \langle X, A \rangle)) \to \bigotimes_{i=1}^m H^*(X_i)/I(K).$$

A spectral sequence abutting to the cohomology of $Z(K; \langle X, A \rangle)$ is given in [9]. Furthermore, in case the maps $H_*(A_i; \mathbb{F}) \to H_*(X_i; \mathbb{F})$ with field coefficients induce split monomorphisms, the spectral sequence collapses at $E_2$ for the cohomology algebra of $Z(K; \langle X, A \rangle)$ as shown in an unpublished paper [9]. As mentioned above, the precise algebra structure is still not known. In the same paper, the authors proved that with field coefficients $\mathbb{F}$, the homology of $Z(K; \langle X, A \rangle)$ depends only on the full sub-complexes of $K$ and the maps $H_*(A_i; \mathbb{F}) \to H_*(X_i; \mathbb{F})$ with field coefficients [9].

Subsequently, Wang and Zheng [74] developed further algebraic techniques arising from the Taylor resolution which gave some additional cohomology algebras. Related ring calculations over a field have been done by Zheng [76].

One natural question is to determine the cohomology ring structure of

$$H^*(Z(K; \langle X, A \rangle); \mathbb{F}).$$

The stable structure theorem forces restrictions on the ring structure. For example, the following result follows from [7].
Theorem 3.5. Let $K$ be an abstract simplicial complex with $m$ vertices and let

$$(X, A) = \{(X_i, A_i, x_i)\}_{i=1}^m$$

be $m$ pointed, CW-pairs. The following cohomology rings regarded as ungraded rings taken with coefficients in $\mathbb{F}_2$ for $m > 0$,

$$H^*(Z(K; (\Sigma^m(X), \Sigma^m(A))); \mathbb{F}_2),$$

are isomorphic [7].

Question 1. Determine the structure of the cohomology algebra

$$H^*(Z(K; (X, A)); \mathbb{F}).$$

Is the ring structure determined by properties of $K$ and the induced maps

$$H^*(X_i; \mathbb{F}) \to H^*(A_i; \mathbb{F})$$
on the level of cohomology?

Question 2. What information determines the action of the Steenrod algebra on the mod-$p$ cohomology of $H^*(Z(K; (X, A)); \mathbb{F}_p)$? How does $H^*(Z(K; (X, A)); \mathbb{F}_p)$ decompose as a module over the mod-$p$ Steenrod algebra? What is the action of the Steenrod algebra on the mod-$p$ cohomology of the polyhedral smash product $\widehat{Z}(K_i; (X_i, A_i))$?

3.2. Other cohomology theories. The following theorem is proven in [9].

Theorem 3.6. Let $K$ be an abstract simplicial complex with $m$ vertices, given that $(X, A) = \{(X_i, A_i)\}_{i=1}^m$ where $(X_i, A_i, x_i)$ are pointed triples of CW-complexes. Then the homology of $Z(K; (X, A))$ with field coefficients $\mathbb{F}$ depends only on (i) the full subcomplexes of $K$ and (ii) the maps $H_*(A_i; \mathbb{F}) \to H_*(X_i; \mathbb{F})$.

Thus it is natural to ask what information determines the additive homology of $E_*(Z(K; (X, A)))$ for other theories $E_*(\cdot)$. The reason for asking this question is technical as the proof of the result for singular homology, Theorem 3.6, depends heavily on properties of the Dold–Thom construction and it is unclear whether an analogous result holds more generally. In other words, the stable decomposition gives some incomplete information about the additive as well as multiplicative structure of cohomology. It was necessary to use the Dold theorem to prove Theorem 3.6.

Question 3. What works more generally? The answer for a general theory $E_*(\cdot)$ is a functor. Describe precise properties of this functor. Some crude information is given in [7]. In addition, $KO^*(DJ(K))$ was worked out in [4]. What is the structure of the cohomology algebra $E^*(Z(K; (X, A)))$? Is the resulting algebra determined by properties of $K$ and the map $E^*(X_i) \to E^*(A_i)$?

3.3. Eilenberg–Moore spectral sequence. There has been substantial work concerning the Eilenberg–Moore spectral sequence and the cohomology of polyhedral products. Elegant general theorems are given in work of Buchstaber and Panov [15, 16, 17] and M. Franz [37, 38, 39]. Preprints of Wang and Zheng give further results about these cohomology rings in [74, 75], which do not address the Eilenberg and Moore spectral sequence directly, but which identify cohomology groups in terms of an appropriate $\text{Tor}$.

Consider the Denham–Suciu fibrations

$$Z(K; (\overline{BG_i, *})) \to BG_1 \times BG_2 \times \ldots \times BG_m$$

with fibre $Z(K; (EG_i, G_i))$ where each topological group $G_i$ is assumed to be path-connected so that $\overline{BG_i}$ is simply-connected. Denham and Suciu pose the question for which Lie groups $G_i$ do the Eilenberg–Moore spectral sequence collapse in this case [29].
A preprint of Luo, Matsumura, and Moore gives additional results in this direction. Some collapse results for the associated Eilenberg–Moore spectral sequence are given in the thesis of A. Al-Raisi.

**Question 4.** Identify conditions which guarantee the collapse at $E_2$ of the Eilenberg–Moore spectral sequence for the fibration
$$Z(K; (BG_i, *)) \to BG_1 \times BG_2 \times \ldots \times BG_m.$$

**Question 5.** Suppose that $f : X^m \to B$ is a fibration where $B$ is simply-connected and the Eilenberg–Moore spectral sequence collapses at $E_2$ for this fibration. Does the Eilenberg–Moore spectral sequence for the fibration $Z(K; (X, *)) \to B$ collapse at $E_2$?

### § 4. Local systems

The questions in this section address homological properties of the spaces $Z(K; (X, A))$ arising from natural covering spaces. A brief list of topics is given next with fuller details below.

1. Develop the properties of natural local coefficient systems arising from fibrations for $Z(K; (X, A))$.
2. Identify the cohomology groups of automorphism groups as well as their combinatorial features arising from $Z(K; (X, A))$.

#### 4.1. Local systems for fibrations.

Given a fibration $F \to E \to B$ with all spaces path-connected, there is the natural monodromy representation
$$\zeta : \pi_1(B) \to Aut(H_jF)$$

obtained by lifting a loop in the base to a map $h : [0, 1] \times F \to E$ such that $h(0, f) = f$, and $h(1, f) \in F$. The map $\zeta$ is that induced by the map $h(1, -) : F \to E$.

The analysis of monodromy for various natural elementary fibrations has proven to be fruitful as well as interesting. Some examples are Drinfel’d, Kohno, S. Suciu, and Shimura. Questions about monodromy arising from fibrations involving polyhedral products turn out to be interesting with some further information as follows.

One example arises where $X_i = BG_i$, and $A_i = *$ where $G_i$ is a discrete group. In this case, there is a fibration due to Denham and Suciu:

$$Z(K; (BG_i, *)) \to BG_1 \times BG_2 \times \ldots \times BG_m$$

with fibre $Z(K; (EG_i, G_i))$.

**Question 6.** What is the induced representation
$$G_1 \times G_2 \times \ldots \times G_m \to Aut \left( H_1(Z(K; (EG_i, G_i))) \right) ?$$

If $G_i$ is a finite, discrete group, the fibre has the homotopy type of $Z(K; ([0, 1], S_i))$ where $S_i$ is a finite discrete set with cardinality $|G_i|$ elements. Thus the question of whether $Z(K; (BG_1, *))$ is a $K(\pi, 1)$ depends on whether $Z(K; ([0, 1], S_i))$ is a $K(\pi, 1)$ where $S_i$ is a finite discrete set.

If $K$ has a minimal non-face of dimension $q > 2$, then the Hurewicz homomorphism is non-trivial in dimension $q - 1$, and thus $Z(K; (BG_1, *))$ is not a $K(\pi, 1)$.

The action of the fundamental group of $BG_1 \times BG_2 \times \ldots \times BG_m$ on the homology of $Z(K; ([0, 1], S_i))$ seems to be a delicate problem. General features of this monodromy are essentially equivalent to the classical Feit Thompson theorem in case the $G_i$ are all of odd order. Indeed, this turns out to be exactly the case where the $G_i$ run over the $p$-Sylow subgroups of a transitive commutative finite group with trivial center. This
point is developed in [1] in which polyhedral products were not used but where the space $B(2, G)$ is a bouquet, and thus homotopy equivalent to a polyhedral product where $K$ is precisely a set of vertices.

**Question 7.** What is the structure of the monodromy for the fibrations

$$Z(K; (BG_i, *)) \to BG_1 \times BG_2 \times \ldots \times BG_m$$

with fibre $Z(K; (EG_i, G_i))$?

**Question 8.** Can the fibrations be used to reprove special cases of the Feit–Thompson theorem? What are the essential features of the geometry needed here? The fibrations of use here are $\lambda: Z(K; (BG_i, *)) \to BG$ where the map $\lambda$ is a maximal extension over $K$ of the natural map $BG_1 \vee BG_2 \vee \ldots \vee BG_m \to BG$ where $G_i$ is a subgroup of $G$.

Some information is given in work of M.Stafa [67].

### 4.2. Loop spaces of polyhedral products.

Identify the structure of the pointed loop space as well as free loop spaces of $Z(K; (X, A))$. The first question arises from some computations for free loop spaces of the spaces $DJ(K) = Z(K; (\mathbb{C}P^\infty, *))$.

**Question 9.** It seems likely that the Poincaré series for the rational homology of the free loop space of $DJ(K)$ is a rational function if and only if the rational homotopy groups of $Z(K; (D^2, S^1))$ are totally finite. This property holds for many cases (in unpublished work of the authors).

Next, consider the pointed loop spaces $\Omega Z(K; (X, A))$. The following result follows immediately from work of Félix–Tanré who analyzed the rational homotopy of $Z(K; (X, A))$ [35] although they do not state this result directly.

**Proposition 4.1** (Félix–Tanré). Let $K$ be an abstract simplicial complex with $m$ vertices, given that $(X, A) = \{(X_i, A_i)\}_{i=1}^m$ where $(X_i, A_i, x_i)$ are pointed triples of simply-connected CW-complexes. Then $\Omega(Z(K; (X, A)))$ is homotopy equivalent to a finite product

$$\Omega(X_1) \times \Omega(X_2) \times \ldots \times \Omega(X_m) \times \prod_{1 \leq \alpha \leq m} \Omega \Sigma(Y_\alpha)$$

for some choice of spaces $Y_\alpha$.

**Question 10.** Identify the spaces $Y_\alpha$. Work out the Pontrjagin rings of $\Omega(Z(K; (X, A)))$.

Some examples have been worked out in work of N.Dobrinskaya [30], [31], Grbić–Panov, Theriault, and Wu [43] as well as unpublished work of the authors. The precise algebra extension given for the Pontrjagin ring of $\Omega(Z(K; (X, A)))$ may admit interesting consequences.

Question [10] above is to identify the “twisting” in the Pontrjagin ring. Identifying this algebra extension seems like a natural question. The case for which configuration spaces arising from complements of diagonals in Euclidean space are used in place of polyhedral products gives an extension which is in fact the enveloping algebra for the $E^0$-term of the integral analogue of the classical unstable Adams spectral sequence abutting to the homotopy of $\Omega(S^2)$ [22].

It is natural to ask what are the analogous universal relations for the loop space of the polyhedral product. The case of configuration spaces of points in Euclidean spaces arises as the universal setting for the horizontal 4T relations sometimes known as the Yang–Baxter–Lie relations. Applications of these relations are given in [20], [32], [31], [52], [53]. The analogous relations for the homology of $\Omega(Z(K; (D^2, S^1)))$ look compelling.
The cohomology of the free loop space of a complex projective space has been computed by several authors. Seeliger gives an elegant method in [64]. The following is a natural question.

**Question 11.** Extend the calculation of [64] from complex projective spaces to all toric manifolds. Observe that elliptic toric manifolds (i.e., toric manifolds which are rationally elliptic) are a special case as elliptic toric manifolds have the property that their 2-connected covers are homotopy equivalent to a product of odd spheres. Thus in this special case, elliptic toric manifolds have the property that (1) the universal cover of their pointed loop space is homotopy equivalent to a product of pointed loop spaces of odd spheres, and (2) the free loop space of the 2-connected cover of $M$ is homotopy equivalent to a product of free loop spaces of odd spheres.

A natural conjecture analogous to Question 9 above is as follows: The Poincaré series for the free loop space of a toric manifold is a rational function if and only if the toric manifold has totally finite rational homotopy groups.

A closely related question is as follows.

**Question 12.** Let $F \to E \to B$ be a fibration of path-connected spaces where (i) the base $B$ is a finite product of circles and complex projective spaces $\mathbb{CP}^\infty$, and (ii) the fibration has a trivial local coefficient system. Does the rational homology of $F$ have rational Poincaré series if and only if the Poincaré series for the rational homology of $E$ has rational Poincaré series?

4.3. **Automorphisms and cohomology.** Let $\pi(K)$ ambiguously denote either

(1) the automorphism group of the simplicial complex $K$ or

(2) the automorphism group of the cohomology ring for $\mathbb{Z}(K; \langle X, A \rangle)$.

Then $\pi(K)$ acts naturally on $H^*(Z(K; \langle X, A \rangle))$.

**Question 13.** Describe the cohomology of the discrete group $\pi(K)$ with coefficients in the natural representation:

$$H^*\left(\pi(K); H^*(Z(K; \langle X, A \rangle)); \mathbb{F}\right).$$

In the first example where $K = \Delta[1]$ the 1-simplex dates back to G. Shimura. [65]. In this case, the cohomology ring of $Z(K; \langle \mathbb{C}P^\infty, * \rangle)$ is $\mathbb{Z}[x_1, x_2]$ with $x_i$ of degree two where $\pi(K) = GL(2, \mathbb{Z})$. Shimura [65] gives the real cohomology of $H^*(SL(2, \mathbb{Z}); \mathbb{R}[x_1, x_2])$ as the classical ring of modular forms. In subsequent work of F. Callegaro, M. Salvetti, and one of the authors [18], it is shown that the $p$-torsion for $p > 3$ is given in terms of the cohomology of certain fibrations due to D. Anick, and which appear in classical homotopy theory [3]. Thus this question seems to be a source of interesting examples.

§ 5. **Topology and geometry**

The problems in this section address naive properties of the polyhedral product at the interface of elementary geometry, and related applications:

1. The J-construction is an operation on simplicial complexes which gives new moment-angle complexes from old. The combinatorial input is the simplicial wedge construction together with resulting moment-angle complexes, and toric manifolds [8]. Related elegant work is given in [40], [71], [72], [34], [21], [33], and [69].

2. Flows on polyhedral products have been used in concrete engineering questions as a practical language for motions of certain robotic legs [44]. We develop methods for describing flows on natural subspaces of polyhedral products.
5.1. The J-Construction. The J-construction is an operation on simplicial complexes which gives new moment-angle complexes from old. The combinatorial input is the “simplicial wedge construction” together with resulting moment-angle complexes, and toric manifolds. The initial data is a simplicial complex $K$ which is a polytopal sphere with $m$ vertices together with an ordered sequence of strictly positive integers $J = (j_1, j_2, \ldots, j_m)$. The output is a new simplicial complex $K(J)$, the “simplicial wedge construction”, with $j_1 + j_2 + \ldots + j_m$ vertices determined by the minimal non-faces of $K$.

Furthermore, given a torus $T^{m-n}$ of maximal rank which acts freely on $Z(K; (D^2, S^1))$, then there is a torus $T^{m-n}$ of maximal rank which acts freely on both $Z(K; (D^{2J}, S^{2J-1}))$, and $Z(K(J); (D^2, S^1))$ to give diffeomorphic toric manifolds

$$Z(K; (D^{2J}, S^{2J-1}))/T^{m-n} \rightarrow M(J), \quad \text{and} \quad Z(K(J); (D^2, S^1))/T^{m-n} \rightarrow M(J)$$

with a new simplicial complex $K(J)$ with $j_1 + j_2 + \ldots + j_m$ vertices determined by the minimal non-faces of $K$.

Question 14. What restrictions on $K$ and a torus $T^{m-n}$ acting on $Z(K; (D^2, S^1))$ ensure that $H^*(Z(K; (D^2, S^1))/T^{m-n}; Z)$ is concentrated in even degrees? Every toric variety with a simplicial fan whose rays span the integer lattice has such a description. The torus $T^{m-n}$ does not need to act freely; weighted projective spaces have such a description and their cohomology is concentrated in even degrees only.

Question 15. Suppose that the torus $T^{m-n}$ acts on $Z(K; (D^2, S^1))$ with finite isotropy groups and $Z(K; (D^2, S^1))/T^{m-n}$ has integral cohomology concentrated in even degrees. Does $Z(K(J); (D^2, S^1))/T^{m-n}$ have integral cohomology concentrated in even degrees?

Question 16. Is it the case that if $Z(K; (D^2, S^1))$ is homotopy equivalent to a wedge of spheres then so is $Z(K(J); (D^2, S^1))$?

Question 17. Does the Lusternik–Schnirelmann category and the topological complexity of $Z(K; (D^2, S^1))$ imply information about that for $Z(K(J); (D^2, S^1))$?

5.2. Flows on polyhedral products. The purpose of this section is to state a concrete problem concerning $Z(K; (S^1, E_-))$ where $E_-$ denotes the lower hemisphere in $S^1$. Points in this polyhedral product can be regarded as positions of legs in certain robots.

Flows which reflect the change of “gaits” of robotic legs rarely exist on all of $Z(K; (S^1, E_-))$ as the homology groups are of rank greater than 1 in the case where $K$ has at least two vertices. We remedy this difficulty by developing flows on subspaces of $Z(K; (S^1, E_-))$.

Question 18. Exhibit natural flows on subspaces of $Z(K; (S^1, E_-))$ which reflect the motions of the robot “rhex”. Find maximal subspaces on which the constructed flows exist.

5.3. Spaces of sphere packings. Polyhedral products have been used to obtain information about spaces of sphere packings. The case addressed below concerns packings of spheres in a disk defined as follows.

Fix natural numbers $q$ and $n$ together with a positive real number $0 < r \leq 1$. Define $X_q(r, n)$ as follows: $X_q(r, n) = \{ (z_1, \ldots, z_q) \in (\mathbb{R}^n)^q \mid \text{which satisfy the conditions (1–2) listed next} \}$:

1. $2r \leq |z_i - z_j|$ if $i \neq j$, and
2. $|z_i| \leq 1 - r$ for all $i$. 


The space $X_q(r, n)$ can be regarded as the space of ordered $q$-tuples of standard spheres of radius $r$ in a disk of radius 1 in $\mathbb{R}^n$ which are not allowed to overlap on their interiors. These spaces, described by natural inequalities, are semi-algebraic sets which have appeared in broad contexts.

Polyhedral products have been used to inform the stable structure as well as cohomology of spaces of packing of spheres $X_q(r, n)$. For example, the thesis of Sun Qiang gives subgroups of the fundamental groups of spaces of packings by using configuration spaces of singular spaces and comparisons to polyhedral products. A. Putman has considered polyhedral products in certain “small packings” (to be defined precisely elsewhere) which are given up to homotopy by $\mathbb{Z}(K; (S^1, E_-))$ where $K$ is a certain choice of flag complex. The fundamental group of $\mathbb{Z}(K; (S^1, E_-))$ is sometimes known as a RAAG (right-angled Artin group).

**Question 19.** Identify the stable homotopy type (or possibly the homotopy type) of $X_q(r, n)$ in terms of more tractable spaces. Do polyhedral products “fill up” certain spaces of packings in a way which allows a determination of the homology of $X_q(r, n)$?

**Acknowledgements**

Happy birthday Victor. The authors would like to thank you and your collaborators for producing fruitful, very interesting mathematics.

**References**


[64] Suciu A. Private communication.


Department of Mathematics, Rider University, Lawrenceville, New Jersey
E-mail address: bahri@rider.edu

Department of Mathematics, CUNY, New York, New York
E-mail address: mbenders@xena.hunter.cuny.edu

Department of Mathematics, University of Rochester, Rochester, New York
E-mail address: cohf@math.rochester.edu

Department of Mathematics, Cinvestav, San Pedro Zacatenco, Mexico
E-mail address: sgitler@math.cinvestav.mx

Originally published in English