Труды Москов. Матем. Общ.
Том 74 (2013), вып. 2

Trans. Moscow Math. Soc.
2013, Pages 261–279
S 0077-1554(2014)00219-3
Article electronically published on April 9, 2014

DISCRETIZATION OF BAKER–AKHIEZER MODULES AND COMMUTING DIFFERENCE OPERATORS IN SEVERAL DISCRETE VARIABLES

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Dedicated to Viktor Matveevich Buchstaber on his seventieth birthday

Abstract. We introduce the notion of discrete Baker-Akhiezer (DBA) modules, which are modules over the ring of difference operators, as a discretization of Baker-Akhiezer modules, which are modules over the ring of differential operators. We use it to construct commuting difference operators with matrix coefficients in several discrete variables.

1. Introduction

In this paper we introduce the notion of discrete Baker–Akhiezer modules and, with the help of them, construct commutative rings of difference operators with matrix coefficients in several discrete variables and with spectral parameter from certain algebraic varieties.

We first recall some basic facts on commuting difference operators in one variable. Common eigenfunctions of two commuting difference operators,

$$L_1 = \sum_{i=N_-}^{N_+} v_i(n)T^i, \quad L_2 = \sum_{i=M_-}^{M_+} u_i(n)T^i, \tag{1}$$

are parametrized by points of some algebraic curve Γ:

$$L_1\psi(n,P) = \lambda(P)\psi(n,P), \ L_2\psi(n,P) = \mu(P)\psi(n,P).$$

Krichever and Novikov [1] proved that there are points $P_1,\dots,P_k$ on Γ such that the whole commutative ring of difference operators, containing L_1 and L_2, is isomorphic to the ring of meromorphic functions with poles only at $P_1,\dots,P_k$. In the case $k=2$ (two-point construction) explicit forms of operators were found in [2, 3]. The theory of n-point operators was developed in [1]. Krichever and Novikov classified one-point operators of rank l and found operators of rank two corresponding to the spectral curve of genus one. The theory of such operators is connected with the theory of higher rank algebro-geometric solutions of the 2D-Toda chain [1]. In [4] Krichever–Novikov operators with polynomial coefficients are found.

In the case of operators, either differential or difference, of several variables, there has been no classification theorem up to now (for some results in this direction see [5, 6, 7]). The main difficulty is as follows. If ordinary difference operators (1) have a family of common eigenfunctions parametrized by an algebraic curve with λ and μ being functions on it, then they commute. On the other hand, in the case of operators of several variables, the existence of a big family of common eigenfunctions is not enough for commutativity. For example, it is not difficult to construct operators possessing a family of common eigenfunctions parametrized by points of an algebraic variety which do not commute. This is a major difference between one and higher dimensional cases.

Then the main question is how many common eigenfunctions are enough for the commutativity in the multi-dimensional case. An answer to this question is partially given in the papers of the second author [8, 9]. In these papers the notion of a Baker–Akhiezer (BA) module over the ring of differential operators is introduced. It allows one to obtain commuting differential operators in several variables with matrix coefficients.

In this paper we introduce a discrete analogue of BA modules. This makes it possible to construct commuting partial difference operators with matrix coefficients as an analogue of the construction of commuting differential operators.

Definition 1. Let X be an algebraic variety and Y a subvariety of X. Then the set of $\mathbb{C}$-valued functions $\hat{M} = \{\psi(n,P)\,|\,n \in \mathbb{Z}^g, P \in X\}$ is called a DBA module if the following conditions are satisfied:

1. $T_i\psi(n,P) \in \hat{M}$, where T_i is a shift operator on the i-th discrete variable of $n = (n_1,\dots,n_g)$.

2. $f(n)\psi(n,P) \in \hat{M}$, for an arbitrary function $f(n)$ from a certain class.

3. $\lambda(P)\psi(n,P) \in \hat{M}$ for each meromorphic function $\lambda(P)$ on X having poles only on Y.

4. The sum of any two elements of $\hat{M}$ belongs to $\hat{M}$.

Let A_Y be the ring of meromorphic functions on X with poles only on Y and $\mathcal{T}_g = \hat{\mathcal{K}}[T_1,\dots,T_g]$ be the ring of difference operators, where $\hat{\mathcal{K}}$ is a ring of certain functions on $\mathbb{Z}^g$. Properties 1-3 imply that $\hat{M}$ is a module over $\mathcal{T}_g$ and, at the same time, over A_Y. We call $\hat{M}$ a discrete Baker–Akhiezer (DBA) module.

Suppose that $\hat{M}$ is a free $\mathcal{T}_g$-module of finite rank. Then the DBA module allows us to construct commuting difference operators in several variables. Indeed, let us choose a free basis $\psi_1,\dots,\psi_N$ in $\hat{M}$ and consider the vector-valued function $\Psi = {}^t(\psi_1,\dots,\psi_N)$. Then for $\lambda \in A_Y$ there exists a uniquely defined difference operator $D(\lambda)$ with matrix coefficients such that

$$D(\lambda)\Psi = \lambda\Psi,$$

since $\hat{M}$ is a free $\mathcal{T}_g$-module. Similarly, for $\mu \in A_Y$, we have

$$D(\mu)\Psi = \mu\Psi.$$

Operators $D(\lambda)$ and $D(\mu)$ commute, since $\hat{M}$ is free and λ, μ do not depend on the discrete variable n. This means that the family $\{\Psi(n,P)\}$ of common eigenvector-valued functions parametrized by points of X is large enough, and hence the commutativity of these operators on the whole set of vector-valued functions follows from the commutativity on $\{\Psi(n,P)\}$.

We construct examples of free DBA modules of finite rank and commuting difference operators from abelian varieties with non-singular theta divisors and certain rational varieties as discretizations of the corresponding BA modules. We show that a basis of a BA module gives a basis of the corresponding DBA module. The situation when solutions of a continuous system directly give solutions of the corresponding discrete system is well known in soliton equations [10, 11].

The present paper is organized as follows. In section 2 we construct DBA modules explicitly and give main theorems. The DBA modules are defined as certain discretizations of Baker-Akhiezer $\mathcal{D}$ modules. Proofs of theorems are given in section 3. In section 4 we give examples of explicit forms of commuting operators.

2. Construction of free DBA modules

In this section we give two examples of free DBA modules which are constructed from Abelian varieties and certain rational varieties. In the first case elements of DBA modules and coefficients of difference operators are expressed in terms of theta functions, and in the second case the corresponding objects are expressed by elementary functions. All theorems in this section can be proved using the results on their differential analogues. The proofs themselves are given in section 3.

2.1. **DBA modules on Abelian varieties.** Let τ be a point of the Siegel upper half space, $\theta_{a,b}(z,\tau)$ the Riemann's theta function with the characteristic ${}^t({}^ta, {}^tb)$, $a, b \in \mathbb{R}^g$, $X = \mathbb{C}/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$, $\Theta \subset X$ the theta divisor specified by the zero set of $\theta(z) := \theta_{0,0}(z,\tau)$ and $\mathcal{L}_c$, $c \in \mathbb{C}^g$, the flat line bundle on X for which $\theta(z+c)/\theta(z)$ is a meromorphic section. A meromorphic section of $\mathcal{L}_c$ is identified with a meromorphic function $f(z)$ on $\mathbb{C}^g$ satisfying the condition

$$f(z + m + \tau n) = exp(-2\pi i {}^t nc) f(z), \tag{2}$$

for any $m, n \in \mathbb{Z}^g + \tau\mathbb{Z}^g$.

Let $L_c(m)$ be the space of meromorphic sections of $\mathcal{L}_c$ with poles only on Θ of order at most m and $L_c = \bigcup_{m=0}^{\infty} L_c(m)$. A basis of $L_c(m)$ is given quite explicitly. Namely, for a nonnegative integer m and $a \in \mathbb{Z}^g/m\mathbb{Z}^g$ we set

$$F_{m,a}(z,c) = \theta_{a/m,0}(mz + c, m\tau)/\theta(z)^m.$$

Then the set of functions $\{F_{m,a}(z,c)\}$ is a basis of $L_c(m)$.

We denote by $\mathcal{K}$ the ring of meromorphic functions on $\mathbb{C}^g$. We denote the variable of a function of $\mathcal{K}$ by $x = (x_1, \ldots, x_g)$. Define the space M_c by

$$M_c = \bigcup_{m=0}^{\infty} M_c(m), \quad M_c(m) = \sum_a \mathcal{K} F_{m,a}(z, c + x).$$

This is nothing but the underlying space of the Baker-Akhiezer module of (X, Θ) [8]. We shall discretize it as follows.

For a function $F(z,x)$ define the operator T_i by

$$T_i F(z,x) = F(z, x + h_i e_i) \frac{\theta(z - h_i e_i)}{\theta(z)}, \quad F(z,x) \in M_c,$$

where e_i is the i-th unit vector of $\mathbb{C}^g$ and $h_i \in \mathbb{C}$ is a parameter. It is easy to see that T_i acts on M_c, since it preserves the relation (2) for L_{c+x}.

For $f(x) \in \mathcal{K}$ we associate the map $\hat{f} : \mathbb{Z}^g \to \mathcal{K}$ by

$$\hat{f}(n) = f(x + nh),$$

where $n = (n_1, \ldots, n_g)$ and $nh = (n_1 h_1, \ldots, n_g h_g)$. We identify the map $\hat{f}$ with its value $\hat{f}(n)$. Let

$$\hat{\mathcal{K}} = \{\hat{f}(n) | f \in \mathcal{K}\}.$$

The space $\mathcal{K}$ naturally becomes a ring which we consider the ring of discrete functions with the discrete variable $n \in \mathbb{Z}^g$.

For a nonnegative integer m and $a \in \mathbb{Z}^g/m\mathbb{Z}^g$ we define the map $\hat{F}_{m,a} : \mathbb{Z}^g \to M_c$ by

$$\hat{F}_{m,a}(n) = T^n F_{m,a}(z, c + x),$$

where $T^n = T_1^{n_1} \cdots T_g^{n_g}$. We identify the map $\hat{F}_{m,a}$ and its value $\hat{F}_{m,a}(n)$. We write $\hat{F}_{m,a}(n, z)$ if it is necessary to indicate the dependence on the variable z.

Now we define the discrete Baker-Akhiezer module $\hat{M}_c$ by

$$\hat{M}_c = \bigcup_{m=0}^{\infty} \hat{M}_c(m), \qquad \hat{M}_c(m) = \sum_{a\in\mathbb{Z}^g/m\mathbb{Z}^g} \hat{\mathcal{K}}\hat{F}_{m,a}(n).$$

Explicitly,

$$\hat{M}_c(m) = \sum_a \hat{\mathcal{K}} \frac{\theta_{a/m,0}(mz+c+x+nh, m\tau)}{\theta(z)^m} \prod_{j=1}^{g} \left(\frac{\theta(z-he_j)}{\theta(z)}\right)^{n_j}. \tag{3}$$

We give an example of the elements in $\hat{M}_c(m)$ with $m = 1, 2$.

Example.

$$\frac{\theta(z+c+x+nh)}{\theta(z)} \prod_{j=1}^{g} \left(\frac{\theta(z-he_j)}{\theta(z)}\right)^{n_j} \in \hat{M}_c(1),$$

$$\frac{\theta(z+c+x+nh+\beta)\theta(z-\beta)}{\theta^2(z)} \prod_{j=1}^{g} \left(\frac{\theta(z-he_j)}{\theta(z)}\right)^{n_j} \in \hat{M}_c(2),$$

where β is an arbitrary constant from $\mathbb{C}^g$. The first example corresponds to $m = 1$, $a = 0$ in (3).

The operator T_i acts on $\hat{M}_c$ as the shift operator:

$$T_i\left(\hat{f}(n)\hat{F}_{m,a}(n)\right) = \hat{f}(n+e_i)\hat{F}_{m,a}(n+e_i).$$

Let $\mathcal{T}_g = \hat{\mathcal{K}}[T_1, \ldots, T_g]$ be the ring of difference operators with the coefficients in $\hat{\mathcal{K}}$. Then $\hat{M}_c$ becomes a $\mathcal{T}_g$-module.

Let $A = L_0$ be the ring of meromorphic functions on X having poles only on Θ. Obviously the space L_{c+x} is an A-module. It follows that the ring A also acts on $\hat{M}_c$. In fact, for $f(z) \in A$, we have

$$f(z)F_{m,a}(z, c+x) = \sum_{m',a'} f_{m',a'}(x)F_{m',a'}(z, c+x), \tag{4}$$

for some $f_{m',a'}(x) \in \mathcal{K}$, since L_{c+x} is an A-module. Notice that the multiplication by $f(z)$ commutes with the action of T_i. Therefore, applying T^n to (4), we have

$$f(z)\hat{F}_{m,a}(n) = \sum_{m',a'} \hat{f}_{m',a'}(n)\hat{F}_{m',a'}(n),$$

which shows that $f(z)\hat{M}_c \subset \hat{M}_c$. Consequently, $\hat{M}_c$ is a $(\mathcal{T}_g, A)$ bi-module.

In the following we assume that Θ is nonsingular. Then our first theorem is

Theorem 1. *For an uncountable number of $h \in (\mathbb{C}^*)^g$ the module $\hat{M}_c$ is a free $\mathcal{T}_g$-module of rank $g!$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.*

Corollary 1. *For values of h specified in Theorem 1 there exists a ring mono-morphism*

$$A \to \mathrm{Mat}(g!, \mathcal{T}_g).$$

2.2. **DBA-modules on rational varieties.** We construct a rational spectral variety Γ from $\mathbb{C}P^1 \times \mathbb{C}P^{g-1}$ by identifying two hypersurfaces. In general we denote a point of the projective space $\mathbb{C}P^{m-1}$ by $[t_1, \ldots, t_m]$, while a point of the m dimensional affine space is denoted by $(t_1 \ldots, t_m)$.

Let us fix a_1, a_2, b_1, $b_2 \in \mathbb{C}$ such that $(a_i, b_i) \neq (0,0)$ and $[a_1, b_1] \neq [a_2, b_2]$. Let $\mathcal{P}$ be a nondegenerate linear map $\mathcal{P} : \mathbb{C}^g \to \mathbb{C}^g$, and λ_j and $v_j, j = 1 \ldots, g$, be the eigenvalues and the eigenvectors of $\mathcal{P}$ respectively. We assume that $\lambda_i \neq \lambda_j$ for $i \neq j$. Denote the induced map $\mathbb{C}P^{g-1} \to \mathbb{C}P^{g-1}$ by the same symbol $\mathcal{P}$.

We set

$$\Gamma = \mathbb{C}P^1 \times \mathbb{C}P^{g-1} / \{([a_1, b_1], t) \sim ([a_2, b_2], \mathcal{P}(t)),\ t \in \mathbb{C}P^{g-1}\}.$$

Then on Γ there is a structure of an algebraic variety [12].

Let $f(P)$, $f_i(P)$, $i = 1, \ldots, g$, be the function of $P = (z_1, z_2, t_1, ..., t_g) \in \mathbb{C}^{g+2}$ of the form

$$f(z_1, z_2, t_1, \ldots, t_n) = \sum_{k=1}^{g} (\alpha_k z_1 t_k + \beta_k z_2 t_k),\ \alpha_k, \beta_k \in \mathbb{C}, \tag{5}$$

$$f_i(z_1, z_2, t_1, \ldots, t_g) = \sum_{k=1}^{g} \left(\alpha_{ik} z_1 t_k + \beta_{ik} z_2 t_k\right),\ \alpha_{ik}, \beta_{ik} \in \mathbb{C}. \tag{6}$$

Proposition 1 ([12]). *For generic $(\alpha, \beta) \in \mathbb{C}^{2g}$ and generic $(\alpha_i, \beta_i) \in \mathbb{C}^{2g}$, $i = 1, \ldots, g$, there exist A, $c_1, \ldots, c_g \in \mathbb{C}^*$ such that for every $t = (t_1, \ldots, t_g) \in \mathbb{C}^g$ the functions (5), (6) satisfy the following equations:*

$$f(a_1, b_1, v_j) \neq 0, \quad 1 \leq j \leq g, \tag{7}$$

$$f(a_1, b_1, t) - A f(a_2, b_2, \mathcal{P}(t)) = 0, \tag{8}$$

$$f_i(a_1, b_1, t) - c_i f_i(a_2, b_2, \mathcal{P}(t)) = 0, \quad 1 \leq i \leq g. \tag{9}$$

According to (8) the equation

$$f(z_1, z_2, t_1, \ldots, t_g) = 0$$

correctly defines a hypersurface in Γ.

For any $\Lambda \in \mathbb{C}^*$ the discrete Baker-Akhiezer module $\hat{M}_\Lambda$ is similarly defined to the case of Abelian varieties as the discretization of the Baker-Akhiezer module constructed in [12]. It is defined directly by

$$\hat{M}_\Lambda = \bigcup_{k=0}^{\infty} \hat{M}_\Lambda(k), \tag{10}$$

$$\hat{M}_\Lambda(k) = \left\{ \psi(n, P) = \frac{h(n, P)}{f(P)^k} \prod_{j=1}^{g} \left(\frac{f_j(P)}{f(P)} \right)^{n_j} \right\}, \tag{11}$$

where $h(n, P) = h(n, z_1, z_2, t)$ is an arbitrary function of the form

$$h(n, P) = \sum_{0 \leq j \leq k, |\alpha| = k} h_{j\,\alpha}(n)\, z_1^j z_2^{k-j}\, t^\alpha, \tag{12}$$

$\alpha = (\alpha_1, \ldots, \alpha_g)$, $t^\alpha = t_1^{\alpha_1} \cdot \cdots \cdot t_g^{\alpha_g}$, and satisfies the equation

$$h(n, a_1, b_1, t) - \Lambda A^k h(n, a_2, b_2, \mathcal{P}(t)) \prod_{j=1}^{g} \left(\frac{A}{c_j} \right)^{n_j} = 0. \tag{13}$$

This equation is equivalent to a set of linear homogeneous equations for $\{h_{j\alpha}(n)\}$ which has nontrivial solutions under the generic conditions in Proposition 1 [12]. Notice that (13) can be written as

$$\psi(n, a_1, b_1, t) - \Lambda\psi(n, a_2, b_2, \mathcal{P}(t)) = 0. \tag{14}$$

According to (8), (9), (14), if $\psi \in \hat{M}_\Lambda(k)$, then $T_j\psi = \psi(n + e_i, P) \in \hat{M}_\Lambda(k + 1)$. Consequently, we have g mappings

$$T_j : \hat{M}_\Lambda(k) \to \hat{M}_\Lambda(k + 1),\ j = 1, \dots, g.$$

Theorem 2. *For an uncountable number of $h \in (\mathbb{C}^*)^g$ the module $\hat{M}_\Lambda$ is a free $\mathcal{T}_g$-module of rank g generated by g functions from $\hat{M}_\Lambda(1)$.*

Let A be the ring of meromorphic functions on Γ with poles only on the divisor $(f = 0)$.

Corollary 2. *For values of h specified in Theorem 2 there is an embedding of the ring*

$$A \to Mat(g, \mathcal{T}_g).$$

In the case $g = 2$ there is another way of identification of two lines on $\mathbb{C}P^1 \times \mathbb{C}P^1$ which is suitable for our goals.

We set

$$\Omega = \mathbb{C}P^1 \times \mathbb{C}P^1/\{([1, 0], [t_1, t_2]) \sim ([t_1, t_2], [0, 1])\}. \tag{15}$$

Let G, G_1, G_2 be the functions on $\mathbb{C}^4$ of the form:

$$G(z_1, z_2, w_1, w_2) = \alpha_0 z_1 w_1 + \beta_0 z_1 w_2 + \gamma_0 z_2 w_1 + \delta_0 z_2 w_2,\ \alpha, \beta, \gamma, \delta \in \mathbb{C},$$

$$G_i(z_1, z_2, w_1, w_2) = \alpha_i z_1 w_1 + \beta_i z_1 w_2 + \gamma_i z_2 w_1 + \delta_i z_2 w_2,\ \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}.$$

Proposition 2 ([12])**.** *For generic $(\alpha_i, \beta_i, \gamma_i, \delta_i) \in \mathbb{C}^4$, $i = 0, 1, 2$, there exist $B, c_1, c_2 \in \mathbb{C}^*$ such that for every $t = (t_1, t_2) \in \mathbb{C}^2$ the functions G, G_1, G_2 satisfy the following equations:*

$$G(0, 1, 0, 1) \neq 0, \tag{16}$$

$$G(1, 0, t_1, t_2) - BG(t_1, t_2, 0, 1) = 0, \tag{17}$$

$$G_i(1, 0, t_1, t_2) - c_i G_i(t_1, t_2, 0, 1) = 0 \quad i = 1, 2. \tag{18}$$

For any $\Lambda \in \mathbb{C}^*$ the discrete Baker-Akhiezer module $\hat{M}_{\Omega,\Lambda} = \bigcup_{k=0}^\infty \hat{M}_{\Omega,\Lambda}(k)$ in this case is defined by

$$\hat{M}_{\Omega,\Lambda}(k) = \left\{ \varphi = \frac{\tilde{h}(n_1, n_2, P)}{G(P)^k} \prod_{j=1}^2 \left(\frac{G_j(P)}{G(P)} \right)^{n_j} \right\},$$

where $\tilde{h}$ is an arbitrary function of the form (12) and satisfies

$$\tilde{h}(n_1, n_2, 1, 0, t_1, t_2) - \Lambda B^k \tilde{h}(n_1, n_2, t_1, t_2, 0, 1) \prod_{j=1}^2 \left(\frac{B}{c_j} \right)^{n_j} = 0. \tag{19}$$

This equation is equivalent to a set of linear homogeneous equations for $\{h_{j\alpha}\}$ which has nontrivial solutions under the generic conditions in Proposition 2 [12]. Notice that (19) can be written as

$$\varphi(n_1, n_2, 1, 0, t_1, t_2) - \Lambda\varphi(n_1, n_2, t_1, t_2, 0, 1) = 0.$$

Theorem 3. *For an uncountable number of $h \in (\mathbb{C}^*)^2$ the module $\hat{M}_{\Omega,\Lambda}$ is a free $\mathcal{T}_2$-module of rank 2 generated by two functions from $\hat{M}_{\Omega,\Lambda}(1)$.*

Let A denote the ring of the meromorphic functions on Ω with poles only on the curve defined by the equation $G(P) = 0$.

Corollary 3. *For values of h specified in Theorem 3 there is a ring embedding,*

$$A \to Mat(2, \mathcal{T}_2).$$

3. Proofs

Theorems 1 to 3 follow from their differential analogues. Since the schemes of the proofs are similar, we only prove Theorem 1 and Theorem 2.

3.1. **Proof of Theorem 1.** Let

$$\nabla_i = \partial_i - \zeta_i(z), \quad \partial_i = \partial/\partial x_i.$$

It is easy to see that it acts on M_c. Let $\mathcal{D} = \mathcal{K}[\partial_1, \dots, \partial_g]$. Then M_c is a $\mathcal{D}$-module. It is called the Baker-Akhiezer module of (X, Θ) [8]. Let

$$\operatorname{gr} M_c = \oplus_i \operatorname{gr}_i M_c, \quad \operatorname{gr}_i M_c = M_c(i)/M_c(i-1).$$

Since $\partial_i M_c(m) \subset M_c(m+1)$, $\operatorname{gr} M_c$ is also a $\mathcal{D}$-module. Recall that we assume that Θ is nonsingular in this paper. Then the following theorem is proved in [8].

Theorem 4. *The module $\operatorname{gr} M_c$ is a free $\mathcal{D}$-module of rank $g!$.*

More precisely, there exists a $\mathcal{D}$-free basis φ_{ij} such that $\varphi_{ij} \in \operatorname{gr}_i M_c$, $1 \le i \le g$, $1 \le j \le r_j$ with

$$r_i = i^g - (i-1)^g - \sum_{j=1}^{i-1} r_j \begin{pmatrix} g+i-j-1 \\ g-1 \end{pmatrix}, \quad r \ge 2,$$

and $r_1 = 1$. Moreover, for each i, one can find φ_{ij} in $\{F_{i,a}(z,x)\}$; that is, one can write

$$\varphi_{ij} = F_{i,a_{ij}}(z,x)$$

for some $a_{ij} \in \mathbb{Z}^g / i\mathbb{Z}^g$.

We remark that, in Theorem 1, $c = 0$ is not excluded. This is because we consider $\mathcal{K}$, the space of meromorphic functions of x, as a coefficient field of $\mathcal{D}$ and M_c.

Recall that T_i acts also on M_c. It satisfies

$$T_i M_c(m) \subset M_c(m+1).$$

Therefore T_i acts on $\operatorname{gr} M_c$ too. For $F(z,x) \in M_c$ we have the expansion

$$T_i F(z,x) = F(z,x) + \nabla_i F(z,x) h_i + O(h_i^2),$$

and it is possible to define the map $\tilde{T}_i = (T_i - 1)/h_i : M_c \to M_c$:

$$\tilde{T}_i F(z,x) = \frac{1}{h_i} \left(T_i F(z,x) - F(z,x) \right).$$

It satisfies

$$\tilde{T}_i F(z,x) = \nabla_i F(z,x) + O(h_i). \tag{20}$$

Notice that, as an action on $\operatorname{gr} M_c$,

$$\tilde{T}_i = \frac{1}{h_i} T_i.$$

We prove

Theorem 5. *For an uncountable number of $h \in (\mathbb{C}^*)^g$, $\operatorname{gr} M_c$ is a free $\mathcal{T}_g$-module of rank $g!$ with a basis $\{\varphi_{ij}\}$.*

Proof. Since $\mathrm{gr}M_c$ is a free $\mathcal{D}$ module, for each k, the set of elements

$$\partial_1^{k_1}\cdots\partial_g^{k_g}\varphi_{ij},\qquad (21)$$
$$k_1+\cdots+k_g=k'-i,\quad 0\le k'\le k,\quad 1\le i\le g,\quad 1\le j\le r_i,$$

is a $\mathcal{K}$-basis of $M_c(k)$. The number of elements in (21) is $N_k:=k^g$. Let us enumerate them as $\psi_1^k,\ldots,\psi_{N_k}^k$.

Expand

$$\theta(z)^k\psi_i^k=\sum a_{i,\mu}^k(x)z^\mu,\quad \mu=(\mu_1,\ldots,\mu_g).$$

Since $\{\psi_i^k\}$ is linearly independent over $\mathcal{K}$, there exist $\mu^{(k,1)},\ldots,\mu^{(k,N_k)}$ such that

$$\det\left(a_{i,\mu^{(k,j)}}^k(x)\right)_{1\le i,j\le N_k}\neq 0,$$

where "$\neq 0$" signifies that it is not identically zero as a function of x.

Consider correspondingly that

$$\tilde{T}_1^{k_1}\cdots\tilde{T}_g^{k_g}\varphi_{ij}.\qquad (22)$$

Let us denote the function in (22) which has the same $(k_1,\ldots,k_g)$ as ψ_i^k by $\tilde{\psi}_i^k$. Then

$$\tilde{\psi}_i^k(z,x,h)=\psi_i^k(z,x)+\sum_{l=1}^g O(h_l).$$

If we expand

$$\theta(z)^k\tilde{\psi}_i^k=\sum\tilde{a}_{i,\mu}^k(x,h)z^\mu,$$

then

$$\tilde{a}_{i,\mu}^k(x,h)=a_{i,\mu}^k(x)+\sum_{l=1}^g O(h_l)$$

and

$$\det\left(\tilde{a}_{i,\mu^{(k,j)}}^k(x,0)\right)=\det\left(a_{i,\mu^{(j)}}^k(x)\right)\neq 0.$$

Notice that $\det\left(a_{i,\mu^{(k,j)}}^k(x)\right)$ is an analytic function of x and the zero set of it is of measure zero. Thus

$$\mathbb{C}^g\setminus\bigcup_{k=0}^\infty\left(\det\left(a_{i,\mu^{(k,j)}}^k(x)\right)=0\right)\qquad (23)$$

has positive measure and contains an uncountable number of elements. Take any x_0 from (23). Since $\tilde{a}_{i,\mu}^k(x,h)$ is an analytic function of (x,h), the set

$$\mathbb{C}^g\setminus\bigcup_{k=0}^\infty\{h|\det\left(\tilde{a}_{i,\mu^{(k,j)}}^k(x_0,h)\right)=0\}\qquad (24)$$

contains an uncountable number of elements. Moreover, it contains elements of the form $h_0=(h_{01},\ldots,h_{0g})$, $h_{0i}\neq 0$, for any i, since $\bigcup_{i=1}^g\{\sum_{j\neq i}h_je_j\in\mathbb{C}^g|h_j\in\mathbb{C}^g\}$ is also of measure zero. Take such an h_0. Then, for any k,

$$\det\left(\tilde{a}_{i,\mu^{(k,j)}}^k(x,h_0)\right)\neq 0.$$

For such an h_0 $\{\tilde{\psi}_i^k\}$ is linearly independent and generates $M_c(k)$ over $\mathcal{K}$ for all $k\ge 0$. Therefore $\mathrm{gr}M_c$ is a free $\mathcal{T}_g$ module with the basis $\{\varphi_{ij}\}$. $\square$

Let us prove Theorem 1. Notice that T_i satisfies the following commutation relation with a function of x:

$$T_i F(x) = F(x + h_i e_i)T_i.$$

By definition the discretization $\hat{\varphi}_{ij}(n)$ of $\varphi_{ij}(z, x)$ is

$$\hat{\varphi}(n) = T^n \varphi_{ij}(z, x).$$

By Theorem 5 any element of M_c can be uniquely written as a linear combination of

$$T^m \varphi_{ij}, \quad m \in \mathbb{Z}^g_{\geq 0}, \quad 1 \leq i \leq g, \quad 1 \leq j \leq r_i,$$

with the coefficients in $\mathcal{K}$. The discretization of the element of the form $f(x)T^m\varphi_{ij}$ with $f(x) \in \mathcal{K}$ is given by

$$T^n\left(f(x)T^m\varphi_{ij}(z, x)\right) = f(x + nh)T^m T^n \varphi_{ij}(z, x) = \hat{f}(n)T^m\hat{\varphi}_{ij}(n).$$

Thus any element of $\hat{M}_c$ can be written as a linear combination of $\{T^m\hat{\varphi}_{ij}(n)\}$ with the coefficients in $\hat{\mathcal{K}}$.

Moreover, this description of an element of $\hat{M}_c$ as a linear combination of $\{T^m\hat{\varphi}_{ij}(n)\}$ is unique. In fact, suppose that

$$\sum \hat{f}_{ij}(n)T^{m_{ij}}\hat{\varphi}_{ij}(n) = 0. \tag{25}$$

Applying T^{-n} to (25) we get

$$\sum f_{ij}(x)T^{m_{ij}}\varphi_{ij}(z, x) = 0.$$

It follows that $f_{ij}(x) = 0$ for any (i, j), since $\{\varphi_{ij}(z, x)\}$ is linearly independent over $\mathcal{K}$. Consequently $\hat{f}_{ij}(n) = 0$ for every (i, j).

Thus $\hat{M}_c$ is proved to be a free $\mathcal{T}_g$-module with a basis $\hat{\varphi}_{ij}$. □

3.2. **Proof of Theorem 2.** We shall first give a construction of functions f, f_i satisfying conditions (5)-(9) together with further conditions and related functions $\tilde{f}_i$.

Consider the function $F(z_1, z_2, t, s)$, $t \in \mathbb{C}^g$, $s \in \mathbb{C}$, of the form

$$F(z_1, z_2, t, s) = \sum_{k=1}^{g}(\gamma_k(s)z_1 + \delta_k(s)z_2)t_k$$

and the following equation for F:

$$F(a_1, b_1, t, s) = Ae^s F(a_2, b_2, \mathcal{P}(t), s). \tag{26}$$

This equation gives g linear homogeneous equations for $2g$ unknown variables γ_k, δ_k. By examining the case $(a_1, b_1) = (1, 0)$, $(a_2, b_2) = (0, 1)$, $\mathcal{P}(t) = (\lambda_1 t_1, \ldots, \lambda_g t_g)$, we see easily that, for generic choice of a_i, b_i, $\mathcal{P}$, there exist g linearly independent solutions $\{F_i\}$ of (26) such that the following conditions are satisfied:

(i) $F_i(z_1, z_2, t, 0)$ is independent of i. Set $f(z_1, z_2, t) = F_i(z_1, z_2, t, 0)$.

(ii) The set of functions $\{f, \partial_s F_i(z_1, z_2, t, 0)\}$ is linearly independent.

(iii) $f(a_1, b_1, v_j) \neq 0$ for any j.

We take $\tilde{c}_i, h_i \in \mathbb{C}^*$ and set

$$c_i = Ae^{\tilde{c}_i h_i}.$$

We define f_i and $\tilde{f}_i$ by

$$f_i(z_1, z_2, t, h_i) = F_i(z_1, z_2, t, \tilde{c}_i h_i), \tag{27}$$

$$\tilde{f}_i(z_1, z_2, t) = \tilde{c}_i \partial_s F_i(z_1, z_2, t, 0). \tag{28}$$

Then f, f_i satisfy (5)-(9) and $\tilde{f}_i$ satisfies

$$\tilde{f}_i(z_1, z_2, t) = \partial_{h_i} f_i(z_1, z_2, t, 0), \tag{29}$$

$$\tilde{f}_i(a_1, b_1, t) - A\tilde{f}_i(a_2, b_2, \mathcal{P}(t)) - \tilde{c}_i f(a_1, b_1, t) = 0, \tag{30}$$

due to (26). Moreover, by property (ii), $\{f, \tilde{f}_1, \dots, \tilde{f}_g\}$ is linearly independent.

Next we consider, for k fixed, a function $h(x, P)$, $P \in \mathbb{C}^{g+2}$, such that

$$h(x, P) = \sum_{0 \le j \le k, |\alpha| = k} h_{j\alpha}(x) z_1^j z_2^{k-j} t^\alpha, \tag{31}$$

$$\frac{h(x, a_1, b_1, t)}{f(a_1, b_1, t)^k} - \Lambda e^{-\sum_{i=1}^g \tilde{c}_i x_i} \frac{h(x, a_2, b_2, \mathcal{P}(t))}{f(a_2, b_2, \mathcal{P}(t))^k} = 0. \tag{32}$$

Equation (32) is equivalent to the system of linear homogeneous equations for $\{h_{j\alpha}\}$. As shown in [12],

$$\sharp\{h_{j\alpha}\} - \sharp\{\text{equations}\} = g \begin{pmatrix} g+k-1 \\ g \end{pmatrix},$$

where $\sharp S$ denotes the number of elements of S. Therefore (32) has non-trivial solutions. Moreover, it is possible take a basis of solutions such that each element of a basis is a rational function of $e^{\sum_{i=1}^g \tilde{c}_i x_i}$ and is analytic at $x = 0$.

Let $\mathcal{K}$ be the ring of rational functions of $e^{\sum_{i=1}^g \tilde{c}_i x_i}$ and

$$M_\lambda = \bigcup_{k=0}^\infty M_\lambda(k), \qquad M_\Lambda(k) = \left\{ \frac{h(x, P)}{f(P)^k} \right\},$$

where $h(x, P)$ is an arbitrary rational function of $e^{\sum_{i=1}^g \tilde{c}_i x_i}$ that satisfies (31), (32). Obviously M_Λ is a vector space over $\mathcal{K}$. As remarked above we can take a basis of each $M_\Lambda(k)$ over $\mathcal{K}$ such that each element of the basis is analytic at $x = 0$.

Equation (32) signifies that an element $\varphi(x, P)$ of M_Λ satisfies

$$\varphi(x, a_1, b_1, t) - \Lambda e^{-\sum_{i=1}^g \tilde{c}_i x_i} \varphi(x, a_2, b_2, \mathcal{P}(t)) = 0.$$

Let

$$\xi_i(P) = \frac{\tilde{f}_i(P)}{f_i(P)}.$$

Then it satisfies that

$$\xi_i(a_1, b_1, t) - \xi_i(a_2, b_2, \mathcal{P}(t)) - \tilde{c}_i = 0, \tag{33}$$

due to (30). We set

$$\nabla_i = \partial_i + \xi_i(P).$$

Using (33) one can easily check that ∇_i acts on M_Λ and satisfies

$$\nabla_i M_\Lambda(k) \subset M_\Lambda(k+1).$$

Thus M_Λ and $\mathrm{gr} M_\Lambda$ become modules over the ring of differential operators $\mathcal{D} := \mathcal{K}[\partial_1, \dots, \partial_g]$, where ∂_i acts by ∇_i. The $\mathcal{D}$-module M_Λ is the Baker-Akhiezer module of $(\Gamma, (f = 0))$ constructed in [12].

The following theorem had been proved in [12].

Theorem 6. *The module* $\mathrm{gr} M_\Lambda$ *is a free* $\mathcal{D}$*-module of rank* g *generated by* g *functions from* $M_\Lambda(1)$.

Similarly to the case of abelian varieties we define the operator T_i by

$$T_i = \frac{f_i(P, h_i)}{f(P)} \mathrm{e}^{h_i \partial_i},$$

where $\mathrm{e}^{h_i \partial_i}$ is the shift operator:

$$\mathrm{e}^{h_i \partial_i} G(\ldots, x_i, \ldots) = G(\ldots, x_i + h_i, \ldots).$$

By (8) and (9) T_i acts on M_Λ and satisfies $T_i M_\Lambda(k) \subset M_\Lambda(k+1)$. Therefore T_i acts on $\mathrm{gr} M_\Lambda$.

By (29) we have

$$f_i(P, h_i) = f(P) + \tilde{f}_i(P) h_i + O(h_i^2).$$

Consequently,

$$T_i = 1 + h_i \nabla_i + O(h_i^2).$$

We set

$$\tilde{T}_i = \frac{1}{h_i}(T_i - 1) = \nabla_i + O(h_i).$$

On $\mathrm{gr} M_\Lambda$ we have

$$\tilde{T}_i = \frac{1}{h_i} T_i.$$

The discretization $\hat{M}_\Lambda$ of M_Λ is defined similarly to the case of Abelian varieties using T_i. Explicitly, $\hat{M}_\Lambda$ is given by (10) and (11).

The proof of Theorem 2 is completely similar to that of Theorem 1 and reduces to Theorem 6 using $\tilde{T}_i$. □

4. Commuting difference operators

In this section we give examples of explicit forms of commuting difference operators.

4.1. **Two-point operators:** $g = 1$. Let $g = 1$ in Theorem 1, $X = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. In this case the DBA module $\hat{M}_0$ is generated over $\mathcal{T}_1$ by the function

$$\psi(n, z) = \frac{\theta(z + x + nh)}{\theta(z)} \left(\frac{\theta(z - h)}{\theta(z)} \right)^n \in \hat{M}_c(1).$$

Let

$$\lambda = \frac{\theta(z - h)\theta(z + h)}{\theta^2(z)}.$$

There is a unique operator of the form

$$L_1 = v_2(n) T^2 + v_1(n) T + v_0(n)$$

such that

$$L_1 \psi(n, z) = \lambda(z) \psi(n, z). \tag{34}$$

Let us find the coefficients $v_i(n)$. We divide (34) by $(\theta(z - h)/\theta(z))^n$ and multiply by $\theta(z)^3$:

$$\begin{aligned} &v_2(n)\theta(z + x + (n+2)h)\theta^2(z - h) + v_1(n)\theta(z + x + (n+1)h)\theta(z - h)\theta(z) \\ &+ v_0(n)\theta(z + x + nh)\theta(z)^2 = \theta(z - h)\theta(z + h)\theta(z + x + nh). \end{aligned} \tag{35}$$

We recall that $\theta(\frac{1}{2}+\frac{1}{2}\tau)=0$. Let us substitute $z=p=\frac{1}{2}+\frac{1}{2}\tau+h$ in (35). We obtain $v_0=0$. Let us divide (35) by $\theta(z-h)$ and again substitute $z=p=\frac{1}{2}+\frac{1}{2}\tau+h$. We obtain

$$v_1(n)=\frac{\theta(p+x+nh)\theta(p+h)}{\theta(p+x+(n+1)h)\theta(p)}.$$

We put $z=q=\frac{1}{2}+\frac{1}{2}\tau$ in (35) and obtain

$$v_2(n)=\frac{\theta(q+x+nh)\theta(q+h)}{\theta(q+x+(n+2)h)\theta(q-h)}.$$

Similarly, for

$$\mu=\frac{\theta(z-h)\theta^2(z+\frac{h}{2})}{\theta^3(z)}$$

we have

$$L_2\psi=\left(u_3(n)T^3+u_2(n)T^2+u_1(n)T+u_0(n)\right)\psi=\mu\psi. \tag{36}$$

From (36) we obtain $u_0=0$,

$$u_1(n)=\frac{\theta(p+x+nh)\theta^2(p+\frac{h}{2})}{\theta(p+x+(n+1)h)\theta^2(p)},\quad u_3(n)=\frac{\theta(q+x+nh)\theta^2(q+\frac{h}{2})}{\theta(q+x+(n+3)h)\theta^2(q-h)}.$$

To find $u_2(n)$ let us substitute $z=r=\frac{1}{2}+\frac{1}{2}\tau-\frac{h}{2}$ in (36):

$$u_2(n)=-u_1(n)\frac{\theta(r+x+(n+1)h)\theta(r)}{\theta(r+x+(n+2)h)\theta(r-h)}-u_3(n)\frac{\theta(r+x+(n+3)h)\theta(r-h)}{\theta(r+x+(n+2)h)\theta(r)}.$$

Operators L_1 and L_2 commute.

It is easy to see that for the meromorphic function η with poles at p and q there is an operator of the form

$$L=\sum_{i=N_-}^{N_+}v_i(n)T^i$$

such that

$$L\psi=\eta\psi.$$

We see that in the case $g=1$ our construction is involved in the two-points construction [2].

4.2. 2×2**-matrix operators: Abelian varieties.** Let $g=2$ in Theorem 1, $X=\mathbb{C}^2/(\mathbb{Z}^2+\tau\mathbb{Z}^2)$. The functions

$$\begin{aligned}\psi_1 &= \frac{\theta(z+x+nh)}{\theta(z)}\prod_{j=1}^{2}\left(\frac{\theta(z-h_je_j)}{\theta(z)}\right)^{n_j}\in\hat{M}_0(1),\\ \psi_2 &= \frac{\theta(z+x+nh+\beta)\theta(z-\beta)}{\theta^2(z)}\prod_{j=1}^{2}\left(\frac{\theta(z-h_je_j)}{\theta(z)}\right)^{n_j}\in\hat{M}_0(2)\end{aligned} \tag{37}$$

form a basis in $\hat{M}_0$, where β belongs to some open everywhere dense subset in $\mathbb{C}^2$. Let us find the operator corresponding to the function

$$\lambda=\frac{\theta(z-h_1e_1)\theta(z+h_1e_1)}{\theta^2(z)}.$$

We have

$$\begin{aligned}L_{11}\psi_1+L_{12}\psi_2&=\lambda\psi_1,\\ L_{21}\psi_1+L_{22}\psi_2&=\lambda\psi_2.\end{aligned} \tag{38}$$

Operators L_{11} and L_{12} have the form

$$L_{11} = v_{20}T_1^2 + v_{11}T_1T_2 + v_{02}T_2^2 + v_1T_1 + v_2T_2 + v_0, \quad L_{12} = u_1T_1 + u_2T_2 + u_0.$$

Let us divide (38) by $\prod_{j=1}^{2}(\theta(z - h_je_j)/\theta(z))^{n_j}$ and multiply by $\theta(z)^3$. Then we get

$$\begin{aligned}
&v_{20}\theta(z + x + nh + 2h_1e_1)\theta(z - h_1e_1)^2\\
&+v_{11}\theta(z + x + (n + 1)h)\theta(z - h_1e_1)\theta(z - h_2e_2)\\
&+v_{02}\theta(z + x + nh + 2h_2e_2)\theta(z - h_2e_2)^2\\
&+v_1\theta(z + x + nh + h_1e_1)\theta(z - h_1e_1)\theta(z)\\
&+v_2\theta(z + x + nh + h_2e_2)\theta(z - h_2e_2)\theta(z) + v_0\theta(z + x + nh)\theta(z)^2\\
&+u_1\theta(z + x + nh + h_1e_1 + \beta)\theta(z - \beta)\theta(z - h_1e_1)\\
&+u_2\theta(z + x + nh + h_2e_2 + \beta)\theta(z - \beta)\theta(z - h_2e_2)\\
&+u_0\theta(z + x + nh + \beta)\theta(z - \beta)\theta(z) = \theta(z - h_1e_1)\theta(z + h_1e_1)\theta(z + x + nh).
\end{aligned} \tag{39}$$

Lemma 1. *The equalities*

$$v_0 = u_0 = 0$$

are valid.

Proof. Let p_1 and p_2 be the points of intersection of the curves $\theta(z - h_1e_1) = 0$ and $\theta(z - h_2e_2) = 0$. Let us substitute $z = p_1$ and $z = p_2$ in (39):

$$v_0\theta(p_i + x + nh)\theta(p_i)^2 + u_0\theta(p_i + x + nh + \beta)\theta(p_i - \beta)\theta(p_i) = 0.$$

These equations can be considered as a system of linear equations for v_0, u_0. If $v_0 \neq 0$ or $u_0 \neq 0$, then

$$\begin{aligned}
&\theta(p_1 + x + nh)\theta(p_2 + x + nh + \beta)\theta(p_2 - \beta)\theta(p_1)\\
&-\theta(p_2 + x + nh)\theta(p_1 + x + nh + \beta)\theta(p_1 - \beta)\theta(p_2) = 0.
\end{aligned}$$

If β is a solution of $\theta(p_1 - \beta) = 0$, then this equality is not valid. Consequently, for β in general position this equality is not valid. Thus Lemma 1 is proved. □

Let us restrict (39) on the curve $\theta(z - h_1e_1) = 0$ and divide by $\theta(z - h_2e_2)$:

$$\begin{aligned}
&v_{02}\theta(z + x + nh + 2h_2e_2)\theta(z - h_2e_2) + v_2\theta(z + x + nh + h_2e_2)\theta(z)\\
&+u_2\theta(z + x + nh + h_2e_2 + \beta)\theta(z - \beta) = 0.
\end{aligned} \tag{40}$$

Let q_1 and q_2 be the points of intersection of $\theta(z - h_1e_1) = 0$ and $\theta(z) = 0$. Then

$$v_{02}\theta(q_i + x + nh + 2h_2e_2)\theta(q_i - h_2e_2) + u_2\theta(q_i + x + nh + h_2e_2 + \beta)\theta(q_i - \beta) = 0.$$

By a similar argument as in the proof of Lemma 1 we obtain

$$v_{02} = v_2 = u_2 = 0.$$

We divide (39) by $\theta(z - h_1e_1)$ and get

$$\begin{aligned}
&v_{20}\theta(z + x + nh + 2h_1e_1)\theta(z - h_1e_1) + v_{11}\theta(z + x + (n + 1)h)\theta(z - h_2e_2)\\
&+v_1\theta(z + x + nh + h_1e_1)\theta(z) + u_1\theta(z + x + nh + h_1e_1 + \beta)\theta(z - \beta)\\
&= \theta(z + x + nh)\theta(z + h_1e_1).
\end{aligned} \tag{41}$$

Let us substitute $z = p_1$ and $z = p_2$ in (41). Then we obtain

$$\begin{pmatrix} v_1 \\ u_1 \end{pmatrix} = A_1^{-1}\begin{pmatrix} \theta(p_1 + x + nh)\theta(p_1 + h_1e_1) \\ \theta(p_2 + x + nh)\theta(p_2 + h_1e_1) \end{pmatrix},$$

$$A_1 = \begin{pmatrix} \theta(p_1+x+nh+h_1e_1)\theta(p_1) & \theta(p_1+x+nh+h_1e_1+\beta)\theta(p_1-\beta) \\ \theta(p_2+x+nh+h_1e_1)\theta(p_2) & \theta(p_2+x+nh+h_1e_1+\beta)\theta(p_2-\beta) \end{pmatrix}.$$

Let r_1 and r_2 be the points of intersection of $\theta(z)=0$ and $\theta(z-\beta)=0$. From (41) we obtain

$$\begin{pmatrix} v_{20} \\ v_{11} \end{pmatrix} = A_2^{-1} \begin{pmatrix} \theta(r_1+x+nh)\theta(r_1+h_1e_1) \\ \theta(r_2+x+nh)\theta(r_2+h_1e_1) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \theta(r_1+x+nh+2h_1e_1)\theta(r_1-h_1e_1) & \theta(r_1+x+(n+1)h)\theta(r_1-h_2e_2) \\ \theta(r_2+x+nh+2h_1e_1)\theta(r_2-h_1e_1) & \theta(r_2+x+(n+1)h)\theta(r_2-h_2e_2) \end{pmatrix}.$$

Similarly it is possible to find operators L_{21}, L_{22} and an operator corresponding to

$$\frac{\theta(z-h_2e_2)\theta(z+h_2e_2)}{\theta^2(z)}.$$

4.3. **2×2-matrix operators with rational coefficients.** It is a well known fact that the Lame identity

$$(\partial_x^2 - 2\wp(x))\psi(x,z) = \wp(z)\psi(x,z),$$
$$\psi(x,z) = \frac{\sigma(z+x)}{\sigma(x)\sigma(z)}e^{-x\zeta(z)},$$

where $\sigma, \zeta, \wp$ are Weierstrass functions of the elliptic curve $w^2 = 4y^3 + \alpha_1 y + \alpha_0$, takes the form

$$\left(\partial_x^2 - \frac{2}{x^2}\right)\psi^\vee(x,z) = \frac{1}{z^2}\psi^\vee(x,z),$$
$$\psi^\vee(x,z) = \frac{z+x}{xz}e^{-\frac{x}{z}}$$

under the degeneration $\alpha_i \to 0$. The Lame potential becomes the rational function $-\frac{2}{x^2}$. In this section we shall consider the spectral variety $X^\vee$ obtained from the Abelian variety $X = \mathbb{C}^2/(\mathbb{Z}^2 + \tau\mathbb{Z}^2)$ by a similar degeneration. Elements of the corresponding DBA module are expressed in terms of elementary functions; coefficients of commuting difference operators are rational functions. To describe $X^\vee$ we recall Mumford's construction of the affine part of the Jacobian variety of a hyperelliptic curve Σ of genus g (see [13]):

$$w^2 = f(y) = 4y^{2g+1} + \alpha_{2g}y^{2g} + \cdots + \alpha_0.$$

Let us introduce the polynomials

$$a(y) = \sum_{i=1}^{g} a_{2i+1}y^{g-i},\ b(y) = \sum_{i=0}^{g} b_{2i}y^{g-i},\ c(y) = \sum_{i=0}^{g+1} c_{2i}y^{g+1-i},$$

$b_0 = 1, c_0 = 4, a_1 = 0$. We shall consider the affine space $\mathbb{C}^{3g+1}$ with the coordinates $(a_{2i+1}, b_{2i}, c_{2i})$. The affine part $J(\Sigma)\backslash\Theta$ is given in $\mathbb{C}^{3g+1}$ by the following system of equations for a_{2i+1}, b_{2i}, c_{2i}:

$$a^2(y) + b(y)c(y) = f(y).$$

In the case $g=2$ we have the following equations:

$$\alpha_0 - a_5^2 - b_4c_6 = 0,\ \alpha_1 - 2a_3a_5 - b_4c_4 - b_2c_6 = 0,$$

$$\alpha_2 - a_3^2 - b_4c_2 - b_2c_4 - c_6 = 0,\ \alpha_3 - 4b_4 - b_2c_2 - c_4 = 0,\ \alpha_4 - 4b_2 - c_2 = 0.$$

We define the spectral variety $X^\vee$ by the conditions $\alpha_i = 0$. From the last three equations one can find c_2, c_4, c_6, and substitute it in the first two equations. One gets that $X^\vee$ is isomorphic to the variety defined in $\mathbb{C}^4$ by the two equations

$$b_4(a_3^2 + 4b_2^3 - 8b_2b_4) - a_5^2 = 0, \quad a_3^2b_2 - 2a_3a_5 + 4(b_2^4 - 3b_2^2b_4 + b_4^2) = 0. \tag{42}$$

Analytically this degeneration of the Jacobian variety is well described by using the sigma function of X. The sigma function is a certain modification of Riemann's theta function which was originally introduced by Klein [14, 15]. The important property for us now is that the sigma function $\sigma(z_1, z_2)$ becomes the Schur function

$$\sigma^\vee = \frac{z_1^3}{3} - z_2$$

under the limit $\alpha_i \to 0$ [16, 17]. The a_i, b_i, c_i coordinates of the Jacobian can be described explicitly using the sigma function (see [13]), and, consequently, those of the variety given by (42) are described by the Schur function.

One can replace Riemann's theta function by the sigma function in the description of the DBA module on X as in the previous section. A free basis of the DBA module is given by

$$\begin{aligned} \psi_1 &= \frac{\sigma(z+x+nh)}{\sigma(z)} \prod_{j=1}^{2} \left(\frac{\sigma(z-h_je_j)}{\sigma(z)}\right)^{n_j} \in \hat{M}_0(1), \\ \psi_2 &= \frac{\sigma(z+x+nh+\beta)\sigma(z-\beta)}{\sigma^2(z)} \prod_{j=1}^{2} \left(\frac{\sigma(z-h_je_j)}{\sigma(z)}\right)^{n_j} \in \hat{M}_0(2). \end{aligned}$$

Taking the limit $\alpha_i \to 0$ we get a new free DBA module on $X^\vee$ generated by the functions

$$\begin{aligned} \psi_1^\vee &= \frac{\sigma^\vee(z+x+nh)}{\sigma^\vee(z)} \prod_{j=1}^{2} \left(\frac{\sigma^\vee(z-h_je_j)}{\sigma^\vee(z)}\right)^{n_j}, \\ \psi_2^\vee &= \frac{\sigma^\vee(z+x+nh+\beta)\sigma^\vee(z-\beta)}{(\sigma^\vee(z))^2} \prod_{j=1}^{2} \left(\frac{\sigma^\vee(z-h_je_j)}{\sigma^\vee(z)}\right)^{n_j}. \end{aligned}$$

Let $\Psi = {}^t(\psi_1, \psi_2)$ and $\Psi^\vee = (\psi_1^\vee, \psi_2^\vee)$. As a limit of the identity $L(\lambda)\Psi = \lambda\Psi$ we get $L^\vee(\lambda^\vee)\Psi^\vee = \lambda^\vee\Psi^\vee$, where $\lambda^\vee$ is the corresponding limit of λ. For different $\lambda^\vee$ and $\mu^\vee$ operators $L^\vee(\lambda^\vee)$ and $L^\vee(\mu^\vee)$ commute. By the method explained in the previous section we can directly compute the operator corresponding to the function

$$\lambda^\vee = \frac{\sigma^\vee(z-h_1e_1)\sigma^\vee(z+h_1e_1)}{(\sigma^\vee(z))^2} = \frac{((z_1-h_1)^3/3 - z_2)((z_1+h_1)^3/3 - z_2)}{(z_1^3/3 - z_2)^2}.$$

For simplicity we put $h_1 = h_2 = 1, x = 0, \beta = (1, 1/3)$. We have

$$L_{11}^\vee(\lambda^\vee) = v_{20}T_1^2 + v_{11}T_1T_2 + v_1T_1, \quad L_{12}^\vee(\lambda^\vee) = u_1T_1,$$

$$u_1 = \frac{-2n_1^2(n_1+1)(n_1+2)(n_1(n_1+3)+5) + 6n_2(2n_1(n_1+1)(n_1+2)-3) - 18n_2^2}{(n_1+2)(6n_2+n_1(n_1(n_1+6)+13)+14)},$$

$$v_{20} = -u_1 - \frac{n_1}{n_1+2}, \quad v_{11} = \frac{(n_1+2)(2n_1(n_1+1) - u_1(n_1+3)) - 6n_2}{3(n_1+2)(n_1+1)},$$

$$v_1 = 2 - v_{11} - \frac{2}{n_1+2},$$

$$\begin{aligned} L_{21}^\vee(\lambda^\vee) &= q_{30}T_1^3 + q_{21}T_1^2T_2 + q_{12}T_1T_2^2 + q_{20}T_1^2 + q_{11}T_1T_2 + q_1T_1, \\ L_{22}^\vee(\lambda^\vee) &= p_{20}T_1^2 + p_{11}T_1T_2 + p_1T_1, \end{aligned}$$

$$\begin{aligned} p_1 = {} & \frac{2(n_1^6 + 9n_1^5 + 37n_1^4 + 48 + n_1^2(106 - 27n_2))}{3(n_1+2)(n_1^3 + 6n_1^2 + 13n_1 + 6n_2 + 14)} \\ & + \frac{2(n_1(88 - 21n_2) + n_1^3(83 - 6n_2) + 21n_2 + 9n_2^2)}{3(n_1+2)(n_1^3 + 6n_1^2 + 13n_1 + 6n_2 + 14)}, \end{aligned}$$

$$p_{11} = \frac{2(5 + n_1(11 + n_1(n_1 + 6)) - 3n_2) - 9(n_1 + 1)(n_1 + 2)q_{12}}{3(n_1 + 2)(n_1 + 3)},$$

$$q_1 = \frac{3p_{12} + 3p_1 - 4 + n_1(p_{11} + p_1 - 2)}{3(n_1 + 1)} + q_{12},$$

$$p_{20} = \frac{-2n_1^6 - 24n_1^5 - 123n_1^4 + 12n_1^3(n_2 - 28)}{(n_1 + 3)(n_1^3 + 9n_1^2 + 28n_1 + 6n_2 + 34)} + \frac{n_1^2(72n_2 - 501) + 6n_1(21n_2 - 64) - 18(n_2^2 - 2n_2 + 7)}{(n_1 + 3)(n_1^3 + 9n_1^2 + 28n_1 + 6n_2 + 34)},$$

$$q_{11} = -q_1 - q_{12}, \quad q_{20} = \frac{3(n_1 + 1)(q_{12} - q_1)}{n_1 + 3} - q_{21}, \quad q_{30} = \frac{2}{n_1 + 3} - p_{20} - 1,$$

$$q_{21} = \frac{9(n_1(n_1(n_1 + 3) + 3) - 3n_2 - 5)q_{12} + ((n_1 + 3)^3 - 3n_2)q_{30}}{3(5 + n_1(n_1(n_1 + 6) + 12) - 3n_2)},$$

$$q_{12} = \frac{2(46 + 61n_1^4 + 12n_1^5 + n_1^6 + n_1(161 - 66n_2))}{9(n_1 + 2)(n_1^3 + 6n_1^2 + 13n_1 + 20)} + \frac{2(n_1^3(163 - 6n_2) - 21n_2 + 9n_2^2 - 4n_1^2(9n_2 - 58))}{9(n_1 + 2)(n_1^3 + 6n_1^2 + 13n_1 + 20)}.$$

Similarly, for the function

$$\mu^\vee = \frac{\sigma^\vee(z - h_2e_2)\sigma^\vee(z + h_2e_2)}{(\sigma^\vee(z))^2} = \frac{(z_1^3/3 - (z_2 - h_2))(z_1^3/3 - (z_2 + h_2))}{(z_1^3/3 - z_2)^2}$$

we have

$$L_{11}^\vee(\mu^\vee) = f_{11}T_1T_2 + f_{02}T_2^2 + f_2T_2, \quad L_{12}^\vee(\mu^\vee) = g_2T_2,$$

$$f_{11} = \frac{18n_1}{n_1(n_1(n_1 + 3) + 4) + 6(n_2 + 2)}, \qquad f_{02} = \frac{f_{11}(n_1 + 2)}{3n_1} - 1,$$

$$f_2 = 1 - f_{02}, \qquad g_2 = -f_{11},$$

$$\begin{aligned} L_{21}^\vee(\mu^\vee) &= r_{21}T_1^2T_2 + r_{12}T_1T_2^2 + r_{03}T_2^3 + r_{11}T_1T_2 + r_{02}T_2^2 + r_2T_2, \\ L_{22}^\vee(\mu^\vee) &= j_{11}T_1T_2 + j_{02}T_2^2 + j_2T_2, \end{aligned}$$

$$r_{21} = \frac{18(n_1 + 1)}{n_1^3 + 6n_1^2 + 13n_1 + 20 + 6n_2}, \quad r_{03} = \frac{2(n_1 + 2)}{n_1^3 + 3n_1^2 + 4n_1 + 6(n_2 + 2)},$$

$$r_{12} = \frac{9n_1r_{03} + 9n_1^2r_{03} + 6r_{21} + 5n_1r_{21} + n_1^2r_{21}}{3n_1^2 + 9n_1 + 6},$$

$$r_{11} = \frac{-n_1^3r_{12} - 6(n_2 + 2)r_{12} + n_1^2(9r_{03} - 6r_{12} + r_{21}) + n_1(3r_{21} - 9r_{03} - 7r_{12})}{n_1^3 + 3n_1^2 + 4n_1 + 6(n_2 + 2)},$$

$$r_{02} = -\frac{2(n_1^3 + 3n_1^2 + 4n_1 + 15 + 6n_2)r_{03}}{n_1^3 + 3n_1^2 + 4n_1 + 6(n_2 + 2)}, \quad r_2 = -r_{02} - r_{03}, \quad j_{11} = -r_{21},$$

$$j_{02} = -\frac{2 + n_1 + 3n_1r_{03}}{n_1 + 2}, \quad j_2 = \frac{n_1 + 2 - j_{02}(n_1 + 2) - 3n_1r_{02} - 6n_1r_{03}}{n_1 + 2}.$$

4.4. 2×2-**matrix operators: Rational spectral variety.** Let us consider the DBA module $\hat{M}_{\Omega,1}$ of the case of $\Lambda = 1$. We set

$$G = z_1w_1 + z_1w_2 + z_2w_2, \; G_1 = 4z_1w_1 + 2z_1w_2 + z_2w_2, \; G_2 = z_1w_1 - z_1w_2 + z_2w_2.$$

Here $B = 1$ in (17) and $c_1 = 2, c_2 = -1$ in (18). We choose the following basis of $\hat{M}_{\Omega,1}$:

$$\psi_1 = \frac{z_2 w_1}{G}\left(\frac{G_1}{G}\right)^{n_1}\left(\frac{G_2}{G}\right)^{n_2},$$

$$\psi_2 = \frac{z_1 w_1 + (-1)^{n_2} 2^{n_1} z_1 w_2 + 2^{2n_1} z_2 w_2}{G}\left(\frac{G_1}{G}\right)^{n_1}\left(\frac{G_2}{G}\right)^{n_2}.$$

We have

$$\psi_i(n_1, n_2, [1,0], [t_1, t_2]) - \psi_i(n_1, n_2, [t_1, t_2], [0,1]) = 0.$$

Let

$$\lambda_1 = \frac{z_2 w_1}{G},\ \lambda_2 = \frac{z_1 z_2 w_1 w_2}{G^2}.$$

These functions satisfy the identity

$$\lambda_i([1,0], [t_1, t_2]) - \lambda_i([t_1, t_2], [0,1]) = 0.$$

It is easy to check that

$$\begin{pmatrix} T_1 + a_2 T_2 + a & b_1 T_1 + b_2 T_2 + b \\ c_2 T_2 + c & d_1 T_1 + d_2 T_2 + d \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where

$$a_2 = -1 + (-2 + (-1)^{n_2} 2^{n_1} + 3 \cdot 2^{1+2n_1}) b_1, \quad a = -4 - a_2 - b_1,$$

$$b_1 = \frac{3}{-1 + 4^{1+n_1}}, \quad b_2 = \frac{3(-1 + (-1)^{n_2} 2^{1+n_1})}{(1 + (-1)^{n_2} 2^{n_1})(-1 + 4^{1+n_1})}, \quad b = -4b_1 - b_2,$$

$$d_1 = \frac{-1 + 4^{n_1}}{-1 + 4^{1+n_1}}, \quad c_2 = \frac{1}{2}(1 - (-1)^{n_2} 2^{n_1}) + (-2 + (-1)^{n_2} 2^{n_1} + 3 \cdot 2^{1+2n_1}) d_1,$$

$$c = 1 - c_2 - d_1, \quad d_2 = \frac{(-1 + (-1)^{n_2} 2^{1+n_1}) d_1}{1 + (-1)^{n_2} 2^{n_1}}, \quad d = -4d_1 - d_2.$$

In a similar way we get

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$L_{11} = -\frac{1}{2} T_1 T_2 + a'_{22} T_2^2 + \frac{1}{2} T_1 + a'_2 T_2 + a',$$

$$L_{12} = b'_{12} T_1 T_2 + b'_{22} T_2^2 + b'_1 T_1 + b'_2 T_2 + b',$$

$$L_{21} = c'_{22} T_2^2 + c'_2 T_2 + c', \quad L_{22} = d'_{12} T_1 T_2 + d'_{22} T_2^2 + d'_1 T_1 + T_2 + d',$$

where

$$a'_{22} = \frac{1}{2}(1 - 4(1 + (-1)^{n_2} 2^{1+3n_1} - 3 \cdot 4^{n_1}) b'_{12} + (-1)^{n_2} 2^{n_1} b'_2 - (-1)^{n_2} 8^{n_1} b'_2),$$

$$a'_2 = (2 - 2a'_{22} - b'_{12} - 2(-1)^{n_2} 2^{n_1} b'_{12}), \quad a' = -a'_2 - a'_{22}, \quad b'_{12} = \frac{3}{2(-1 + 4^{1+n_1})},$$

$$b'_{22} = \frac{1}{2}(-4(1 + (-1)^{n_2} 2^{1+n_1}) b'_{12} - (1 + (-1)^{n_2} 2^{n_1}) b'_2), \quad b'_1 = -b'_{12},$$

$$b'_2 = \frac{3}{-1 + 4^{n_1}}, \quad b' = -b'_2 - b'_{22},$$

$$c'_{22} = -\frac{1}{4}(-1 + (-1)^{n_2} 2^{n_1})(-1 + (-8 - (-1)^{n_2} 2^{3+n_1} + 4^{2+n_1}) d'_{12} + (-1)^{n_2} 2^{1+n_1}(1 + (-1)^{n_2} 2^{n_1})),$$

$$c_2' = -\frac{1}{2} - 2c_{22}' - (1 + (-1)^{n_2} 2^{1+n_1}) d_{12}', \quad c' = -c_2' - c_{22}', \quad d_1' = -d_{12}',$$
$$d_{12}' = \frac{-1 + 4^{n_1}}{2(-1 + 4^{1+n_1})}, \quad d' = -1 - d_{22}',$$
$$d_{22}' = \frac{1}{2}(-4(1 + (-1)^{n_2} 2^{1+n_1}) d_{12}' - (1 + (-1)^{n_2} 2^{n_1})).$$

Acknowledgments

This research was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 23540245. The first author was also partially supported by RFBR grant 11-01-12106-ofi-m-2011. The second author would like to thank Koji Cho for useful suggestions on the proofs of theorems. A part of this work was done while the the first author stayed at Tsuda College. He is grateful to this institution for kind hospitality. The authors are also grateful to the referee for a number of useful comments that improved the paper.

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Originally published in English