ON THE ORBIT SPACE OF AN IRREDUCIBLE REPRESENTATION
OF THE SPECIAL UNITARY GROUP

O. G. STYRT

Abstract. We prove that the quotient of an irreducible representation of a special
unitary group of rank greater than 1 cannot be a smooth manifold.

§ 1. Introduction

This paper is a direct continuation of [1] and [2]. First, we recall three basic definitions
playing a key role in those papers.

Definition. A continuous map between smooth manifolds is said to be piecewise smooth
if it sends any smooth submanifold to a finite union of smooth submanifolds.

In particular, any proper smooth map between smooth manifolds is piecewise smooth.

Consider a differentiable action of a compact Lie group $G$ on a smooth manifold $M$.

Definition. We shall say that the quotient $M/G$ is piecewise diffeomorphic to a smooth
manifold $M'$ if the topological quotient $M/G$ is homeomorphic to $M'$ and the quotient
map $M \to M'$ is piecewise smooth.

Definition. We shall say that the quotient $M/G$ is a smooth manifold if it is piecewise
diffeomorphic to a smooth manifold.

Let us now describe our problem.

Let $V$ be a real vector space and $G \subset \text{GL}(V)$ a compact linear group. As in [1]
and [2], we want to know if the quotient $V/G$ is a topological manifold and also if it is
a smooth manifold. Following [1] and [2], we shall call a topological manifold simply a
manifold.

Let $V_C$ denote the complex space $V \otimes \mathbb{C}$, $\mathfrak{g}$ the linear Lie algebra $\text{Lie} G \subset \mathfrak{gl}(V)$, and $\mathfrak{g}_C$
the complex linear Lie algebra $\mathfrak{g} \otimes \mathbb{C} \subset \mathfrak{gl}(V_C)$.

At the present moment, the following two cases have been dealt with: $[\mathfrak{g}, \mathfrak{g}] = 0$
(see [1]) and $\mathfrak{g} \cong \mathfrak{su}_2$ (see [2]). In this paper, we consider the case when $\mathfrak{g} \cong \mathfrak{su}_{r+1}$, where
$r = \text{rk} \mathfrak{g} > 1$ and the linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is irreducible.

Let $R'$ denote the representation of a complex reductive Lie group dual to the represen-
tation $R$.

When speaking of indecomposable root systems we shall use the numbering of the
simple roots adopted in [3, Table 1] and [4, Table 1], where $(i_1, \ldots, i_m)$ denotes the
subset of simple roots indexed by $i_1, \ldots, i_m$, $\varphi_i$ denotes the fundamental weight indexed
by $i$, and $\varphi(\mathfrak{h})$ is the linear algebra corresponding to the irreducible representation of the
complex simple algebra $\mathfrak{h}$ with highest weight $\varphi$.

The space $V$ admits a $G$-invariant scalar product and hence may (and will) be considered
as a Euclidean space on which $G$ acts by orthogonal operators. Thus, $G \subset \text{O}(V)$.

Let $\mathcal{R}$ be the tautological representation $\mathfrak{g}_C : V_C$. We may have the following cases:

2010 Mathematics Subject Classification. Primary 22E46; Secondary 17B10, 17B20, 17B45.

Key words and phrases. Lie group, topological action quotient.
1) \( \tilde{R} = R \), where \( R \) is a faithful irreducible representation; and
2) \( \tilde{R} = R + R' \), where \( R \) is a faithful irreducible representation.

In the latter case, \( V \) admits a \( \mathfrak{g} \)-invariant complex structure, which naturally yields a (complex) representation \( \mathfrak{g}_C : V \) isomorphic to \( \tilde{R} \).

Suppose now that \( \mathfrak{g} \cong \mathfrak{su}_{r+1} \), where \( r = \text{rk} \mathfrak{g} > 1 \), and the linear Lie algebra \( \mathfrak{g} \subset \mathfrak{gl}(V) \) is irreducible.

In this paper we shall prove the following Theorems 1.1 and 1.2.

**Theorem 1.1.** The quotient \( V/G \) can be a smooth manifold only in the following cases:
1) the representation \( \tilde{R} \) of \( \mathfrak{g}_C \) coincides with the representation \( R \) and is isomorphic to one of the representations \( R_{\varphi_2} \) (\( r = 3 \)), \( R_{\varphi_2} \) (\( r = 3 \)), and \( R_{\varphi_4} \) (\( r = 7 \));
2) \( \tilde{R} = R + R' \) and the representation \( R \) is isomorphic (up to an outer automorphism of the algebra \( \mathfrak{g}_C \)) to one of the representations \( R_{\varphi_1}, R_{\varphi_2}, R_{\varphi_2} \) (\( r > 3 \)), and \( R_{\varphi_3} \) (\( r = 5 \)).

**Theorem 1.2.** If \( G = G^0 \) and the representation \( R \) is isomorphic (up to an outer automorphism of \( \mathfrak{g}_C \)) to one of the representations \( R_{\varphi_1}, R_{\varphi_1}, R_{\varphi_2} \) (\( r > 2 \)), \( R_{\varphi_2} \) (\( r = 3 \)), \( R_{\varphi_3} \) (\( r = 5 \)), and \( R_{\varphi_4} \) (\( r = 7 \)), then \( V/G \) is not a manifold.

**Corollary 1.3.** If \( G = G^0 \), then the quotient \( V/G \) is not a smooth manifold.

In §2 we introduce some notation used later in the text, and also prove some auxiliary results. In §3 we prove Theorem 1.1 and in §4 we prove Theorem 1.2.

**§ 2. Notation and useful facts**

In this section we mention a number of useful notation and results, including the ones taken from [1], [2], and [5] (all new assertions have proofs).

For a linear representation of a Lie group \( G \) (respectively, a Lie algebra \( \mathfrak{g} \)) in the space \( V \), the stabilizer (respectively, the isotropy subalgebra) of a vector \( v \in V \) will be denoted by \( G_v \) (respectively, by \( \mathfrak{g}_v \)).

In [5], for each indecomposable simple root system \( \Pi \) a certain subset \( \partial \Pi \subset \Pi \) was defined.

All indecomposable simple root systems \( \Pi \) such that \( \partial \Pi \neq \Pi \) and \( \partial \Pi \subset \Pi \) are listed in [5, §4, Table 1]. For example, if \( \Pi \cong A_r \), then

\[
\partial \Pi = \begin{cases}
(1, 2, r - 1, r), & r \geq 6, \\
\Pi, & r < 6.
\end{cases}
\]

**2.1. Representations of compact groups.** Suppose we have a Euclidean space \( V \), a compact Lie group \( G \) with Lie algebra \( \mathfrak{g} \), a linear representation \( G \to \text{O}(V) \), and its differential – a representation \( \mathfrak{g} : V \).

For any vector \( v \in V \), let \( N_v \) denote the subspace \((\mathfrak{g}_v)^\perp \subset V \). Clearly \( \mathfrak{g}_v = \text{Lie} G_v \) and \( G_v N_v = N_v \) (\( v \in V \)).

**Lemma 2.1.** If \( V/G \) is a (smooth) manifold, then each quotient \( N_v/G_v, v \in V \), is also a (smooth) manifold.

*Proof.* See Lemma 2.3 in [2, §2].

**Lemma 2.2.** Suppose \( \text{dim} \mathfrak{g} = 1 \).
1) If \( V/G \) is a manifold, then \( \text{dim}(\xi V) \neq 2 \) for any vector \( \xi \in \mathfrak{g} \).
2) If \( V/G \) is a smooth manifold, then \( \text{dim}(\xi V) \leq 6 \) for any vector \( \xi \in \mathfrak{g} \).

*Proof.* See Corollary 2.3 in [2, §2].
Lemma 2.3. If $\mathfrak{g} \cong \mathfrak{su}_2$ and $V/G$ is a smooth manifold, then the sum of the integer parts of halves of the dimensions of all irreducible components of the representation $\mathfrak{g}: V$ (with their multiplicities) is less than or equal to 4.

Proof. See Theorem 1.1 in \cite{2} §1.

\[ \square \]

Corollary 2.4. If $\mathfrak{g} \cong \mathfrak{su}_2$ and $V/G$ is a smooth manifold, then $\dim(\xi V) \leq 8$ for any $\xi \in \mathfrak{g}$.

Corollary 2.5. If $\mathrm{rk} \mathfrak{g} = 1$ and $V/G$ is a smooth manifold, then $\dim(\xi V) \leq \dim \mathfrak{g} + 5$ for any $\xi \in \mathfrak{g}$.

Proof. Follows from Lemma 2.2 and Corollary 2.4.

\[ \square \]

Corollary 2.6. If $\mathrm{rk} \mathfrak{g} = 1$ and $V/G$ is a smooth manifold, then $\dim(\xi V) \leq \dim[\xi, \mathfrak{g}] + 6$ for any nonzero vector $\xi \in \mathfrak{g}$.

Proof. Since $\mathrm{rk} \mathfrak{g} = 1$, we have $\mathrm{Ker}(\text{ad}\xi) = \mathbb{R}\xi$ for any nonzero vector $\xi \in \mathfrak{g}$, whence $\dim[\xi, \mathfrak{g}] = \dim \mathfrak{g} - 1$. Now use Corollary 2.5.

\[ \square \]

Lemma 2.7. Suppose that $V/G$ is a smooth manifold. Then $\dim(\xi V) \leq \dim[\xi, \mathfrak{g}] + 6$ for any $v \in V$ such that $\mathrm{rk} \mathfrak{g}_v = 1$ and for any nonzero $\xi \in \mathfrak{g}_v$.

Proof. By Lemma 2.1 $N_v/G_v$ is a smooth manifold. Applying Corollary 2.6 to the representation $G_v : N_v$, we have $\dim(\xi N_v) \leq \dim[\xi, \mathfrak{g}_v] + 6$ and, therefore,

$$\dim(\xi V) = \dim(\xi(\mathfrak{g}_v)) + \dim(\xi N_v) \leq \dim(\xi(\mathfrak{g}_v)) + \dim[\xi, \mathfrak{g}_v] + 6 = \dim[\xi, \mathfrak{g}] + 6.$$ 

\[ \square \]

Lemma 2.8. Suppose that $V/G$ is a manifold. Then $\dim(\xi V) \neq \dim[\xi, \mathfrak{g}] + 2$ for any $v \in V$ such that $\dim \mathfrak{g}_v = 1$ and for any nonzero $\xi \in \mathfrak{g}_v$.

Proof. By Lemma 2.1 $N_v/G_v$ is a manifold. Applying Lemma 2.2 to the representation $G_v : N_v$, we have $\dim(\xi N_v) \neq 2$. Furthermore, $[\xi, \mathfrak{g}_v] = 0$ and $\dim(\xi(\mathfrak{g}_v)) = \dim[\xi, \mathfrak{g}]$, whence

$$\dim(\xi V) = \dim(\xi(\mathfrak{g}_v)) + \dim(\xi N_v) = \dim[\xi, \mathfrak{g}] + \dim(\xi N_v) \neq \dim[\xi, \mathfrak{g}] + 2.$$ 

\[ \square \]

Lemma 2.9. Let $v_1, \ldots, v_k \in V$ be arbitrary vectors. If $(\mathfrak{g} v_i, v_j) = 0$ for all $i, j \in \{1, \ldots, k\}, i \neq j$, then, in the subspace $\langle v_1, \ldots, v_k \rangle \subset V$, there is a vector with stabilizer

$$\bigcap_{i=1}^k G_{v_i} \subset G.$$ 

Proof. We induct on $k \in \mathbb{N}$.

When $k = 1$ there is nothing to prove.

Now we prove the lemma for $k \in \mathbb{N} \setminus \{1\}$ assuming that it holds for $k - 1 \in \mathbb{N}$.

By the induction assumption, there is $v \in \langle v_1, \ldots, v_{k-1} \rangle$ with stabilizer

$$G_v = \bigcap_{i=1}^{k-1} G_{v_i} \subset G.$$ 

By assumption, $(\mathfrak{g} v_i, v_k) = 0$ ($i = 1, \ldots, k - 1$), and therefore $(\mathfrak{g} v, v_k) = 0$, $v_k \in N_v$.

Hence, for some $v' = v + \varepsilon v_k \in \langle v_1, \ldots, v_k \rangle \cap N_v, \varepsilon \in \mathbb{R}_{>0}$, we have $G_{v'} \subset G_v$, and, as a consequence,

$$G_{v'} = G_v \cap G_{\varepsilon v_k} = \bigcap_{i=1}^k G_{v_i}.$$ 

\[ \square \]
Assertion 2.10. If $G = \text{SU}_m \times \text{SU}_m$ and $V = \mathbb{R}^d \oplus \mathfrak{gl}_m(\mathbb{C})$ ($d \geq 0$, $m > 1$), and the action $G : V$ is given by $(g_1, g_2) : (x, y) \rightarrow (x, g_1 y g_2^{-1})$ ($g_1, g_2 \in \text{SU}_m$, $x \in \mathbb{R}^d$, $y \in \mathfrak{gl}_m(\mathbb{C})$), then $V/G$ is not a manifold.

Proof. Set $v := E \in \mathfrak{gl}_m(\mathbb{C}) \subset V$. It is easy to see that

$G_v = \{(g, g) : g \in \text{SU}_m\} \subset G$, \hspace{1em} $\mathfrak{g}_v = \mathfrak{su}_m \subset \mathfrak{gl}_m(\mathbb{C}) \subset V$

and

$V = \mathbb{R}^d \oplus (\mathbb{C}E) \oplus \mathfrak{su}_m \oplus (i \cdot \mathfrak{su}_m)$.

This implies (since the groups $\text{SU}_m$ and $G_v$ are naturally isomorphic) that the representation $G_v : N_v$ is isomorphic to the direct sum of the trivial action $G_v : \mathbb{R}^{d+2}$ and the adjoint representation $\text{SU}_m : \mathfrak{su}_m$. Hence,

$N_v \cong \mathbb{R}^{d+2} \times \mathbb{R}^{m-1} \cong \mathbb{R}^{m+d+1}$.

Thus, the quotient $N_v/G_v$ is not a manifold and, by Lemma 2.1, neither is the quotient $V/G$. \hfill \square

Suppose that the representation $\mathfrak{g} : V$ is irreducible.

Let $\mathfrak{g}_C$ denote the complex reductive Lie algebra $\mathfrak{g} \otimes \mathbb{C}$.

It is easy to see that there are a complex space $\widetilde{V}$ and an irreducible representation $\mathfrak{g}_C : \widetilde{V}$ such that the representation $\mathfrak{g} : V$ is either the realification or a real form of the representation $\mathfrak{g} : \widetilde{V}$.

Set $\delta := 1 \in \mathbb{R}$ when $\widetilde{V} = V \otimes \mathbb{C}$ and $\delta := 2 \in \mathbb{R}$ when $\widetilde{V} = V$.

We now fix a maximal commutative subalgebra $t$ of $\mathfrak{g}$ and a Cartan subalgebra $t_C := t \otimes \mathbb{C}$ of $\mathfrak{g}_C$. We then have a root system $\Delta \subset t_C^*$ and the corresponding Weyl group $W \subset \text{GL}(t_C^*)$. We also fix a simple root system $\Pi \subset \Delta \subset t_C^*$.

Let $\Lambda \subset t_C^*$ be the set of weights of the irreducible representation $\mathfrak{g}_C : \widetilde{V}$, $\lambda \in t_C^* \setminus \{0\}$ the highest weight of this representation relative to the simple root system $\Pi \subset \Delta \subset t_C^*$, and $\Lambda$ the subset $W \lambda \subset \Lambda \subset t_C^*$.

Finally, for a subset $\Omega \subset t_C^*$, we set

$\mathfrak{g}_C(\alpha) := \alpha \oplus \left( \bigoplus_{\alpha \in \Delta, \ h_{\alpha} \in a} (\mathfrak{g}_C)_{\alpha} \right) \subset \mathfrak{g}_C$.

We then have

$\mathfrak{g}_{\Omega} := \mathfrak{g}_{\Omega_1 \uplus \Omega_2} \cap \mathfrak{g} \subset \mathfrak{g}$ \hspace{1em} ($\Omega_1, \Omega_2 \subset t_C^*$).

Consider an arbitrary weight $\lambda' \in \Lambda$. For any vector $v \neq 0$ in the (nontrivial) subspace $\widetilde{V}_{\lambda'} \subset \widetilde{V}$, the subalgebra $(\mathfrak{g}_C)_v$ of $\mathfrak{g}_C$ coincides with the direct sum of all subspaces $\text{Ker} \lambda' \subset t_C$ and $(\mathfrak{g}_C)_{\alpha}$ ($\alpha \in \Delta$, $\langle \lambda' | \alpha \rangle \geq 0$), and therefore $\mathfrak{g}_v = \mathfrak{g}^{(\lambda')} \subset \mathfrak{g}$.

If $\widetilde{V} = V \otimes \mathbb{C}$ and $2\lambda' \notin \Delta \cup (\Delta + \Delta)$, then $-\lambda' \in \Lambda$ and we also have

$(\mathfrak{g}_C\widetilde{V}_{\lambda'}) \cap (\mathfrak{g}_C\widetilde{V}_{-\lambda'}) = 0$, \hspace{1em} $(\mathfrak{g}\widetilde{V}_{\lambda'}) \cap (\mathfrak{g}\widetilde{V}_{-\lambda'}) = 0$.

This and Lemma 2.9 imply the following lemma.
Lemma 2.11. Let $\Omega \subset \Lambda$ be a subset such that $(\Omega - \Omega) \cap \Delta = \emptyset$. Suppose that one of the following conditions is satisfied: 1) $\delta = 2$; 2) $\delta = 1$, $2\lambda \notin \Delta \cup (\Delta + \Delta)$ and $(\Omega + \Omega) \cap \Delta = \emptyset$. Then there is a vector $v \in V$ such that $g_v = g^\Omega \subset g$.

Let $\|\cdot\|$ denote the order of a subset of the set $\hat{\Lambda} \subset t^*_c$ of weights with multiplicities of the representation $g_C : \hat{V}$.

Lemma 2.12. Suppose that $V/G$ is a smooth manifold. If $v \in V$, $\text{rk} g_v = 1$, and $\xi \in (g_v \cap t) \setminus \{0\}$, then

$$\delta \cdot \|\{\lambda' \in \hat{\Lambda} : \lambda' (\xi) \neq 0\} \| \leq \|\{\alpha \in \Delta : \alpha (\xi) \neq 0\} \| + 6.$$

Proof. By Lemma 2.7

$$\delta \cdot \text{dim}_C(\xi \hat{V}) = \text{dim}(\xi V) \leq \text{dim}[\xi, g] + 6 = \text{dim}_C[\xi, g_C] + 6,$$

which implies the desired assertion.

Corollary 2.13. Let $\Omega \subset \Lambda$ be an arbitrary subset. Let $H$ denote the subspace $\langle \Omega \rangle_C \subset t^*_C$. If

$$\text{dim}_C(t^*_C/H) = 1; \quad \delta \cdot \|\hat{\Lambda} \setminus H\| > |\Delta \setminus H| + 6; \quad (\Omega - \Omega) \cap \Delta = \emptyset;$$

$$\delta = 1 \quad \Rightarrow \quad \left(2\lambda \notin \Delta \cup (\Delta + \Delta)\right) \text{ and } \left((\Omega + \Omega) \cap \Delta = \emptyset\right),$$

then the quotient $V/G$ is not a smooth manifold.

Proof. Follows from Lemmas 2.11 and 2.12.

Lemma 2.14. Suppose the quotient $V/G$ is a manifold and $\hat{\Lambda} = \Lambda$. If $v \in V$, $\xi \in t \setminus \{0\}$, and $g_v = \mathbb{R} \xi$, then

$$\delta \cdot \|\{\lambda' \in \Lambda : \lambda' (\xi) \neq 0\} \| \neq \|\{\alpha \in \Delta : \alpha (\xi) \neq 0\} \| + 2.$$

Proof. By Lemma 2.8

$$\delta \cdot \text{dim}_C(\xi \hat{V}) = \text{dim}(\xi V) \neq \text{dim}[\xi, g] + 2 = \text{dim}_C[\xi, g_C] + 2,$$

which implies the desired assertion.

Corollary 2.15. Let $\Omega \subset \Lambda$ be an arbitrary subset. Let $H$ denote the subspace $\langle \Omega \rangle_C \subset t^*_C$. If

$$\text{dim}_C(t^*_C/H) = 1; \quad \hat{\Lambda} = \Lambda;$$

$$\delta \cdot |\Lambda \setminus H| = |\Delta \setminus H| + 2; \quad (\Omega - \Omega) \cap \Delta = H^\perp \cap \Delta = \emptyset;$$

$$\delta = 1 \quad \Rightarrow \quad \left(2\lambda \notin \Delta \cup (\Delta + \Delta)\right) \text{ and } \left((\Omega + \Omega) \cap \Delta = \emptyset\right),$$

then $V/G$ is not a manifold.

Proof. Follows from Lemmas 2.11 and 2.14.

2.2. Polar representations.

Definition. A representation of a compact Lie group $G$ in a real space $V$ is polar if there is a subspace $V' \subset V$ such that $GV' = V$ and $V' \cap (T_v(Gv)) = 0$ ($v \in V'$).

Lemma 2.16. The quotient space of an arbitrary nontrivial connected polar compact linear group is homeomorphic to a closed half-space.

Proof. By the results of [6] (see Theorems 2.8, 2.9, and 2.10 in §2), the above quotient space is homeomorphic to a quotient space of a nontrivial finite linear group generated by (real) reflections.
Let $G$ be a connected simply-connected compact semisimple Lie group, $\mathfrak{g}$ its tangent algebra, and $\theta \in \text{Aut}(G)$ a nontrivial involutive automorphism. Now consider the adjoint action $G^\theta : \mathfrak{g}$ and its restriction to the (clearly invariant) subspace $\mathfrak{g}^{-d\theta} \subset \mathfrak{g}$.

**Lemma 2.17.** The quotient $\mathfrak{g}^{-d\theta}/G^\theta$ is homeomorphic to a closed half-space.

**Proof.** By Theorem B of [7, §1], the Lie subgroup $G^\theta \subset G$ is connected. Since $G$ is semisimple, the representation $G^\theta : \mathfrak{g}^{-d\theta}$ is nontrivial. This representation is also polar (see [8, 8.5–8.6]). It remains to apply Lemma 2.16. □

**Remark.** The assertion of Lemma 2.17 also follows from the results of [9].

### 2.3. Combinatorial inequalities.

For natural numbers $n, k, m_1, \ldots, m_k$ such that $m_1 + \ldots + m_k = n$, set

$$\binom{n}{m_1, \ldots, m_k} := \binom{n}{m_1} \cdot \binom{n-m_1}{m_2} \cdot \ldots \cdot \binom{n-m_1-\ldots-m_{k-1}}{m_k} = \frac{n!}{m_1! \cdot \ldots \cdot m_k!} \in \mathbb{N}.$$ 

**Assertion 2.18.** If $n, k, m_1, \ldots, m_k \in \mathbb{N}$ and $m_1 + \ldots + m_k = n$, then the inequality $\binom{n}{m_1, \ldots, m_k} < n$ is possible only when $k = 1$ and $m_1 = n$.

**Proof.** We have $\binom{n}{m_1} \leq \binom{n}{m_1, \ldots, m_k} < n$, whence $m_1 = n$. □

**Assertion 2.19.** If $n, k, m_1, \ldots, m_k \in \mathbb{N}$, $k > 1$, and $m_1 + \ldots + m_k = n$, then the inequality $\binom{n}{m_1, \ldots, m_k} \leq 2(n-1)$ is possible only in the following cases:

1) $k = 2$, $\{m_1, m_2\} = \{1, n-1\}$;
2) $n = 4$, $k = 2$, $m_1 = m_2 = 2$.

**Proof.** Suppose $k > 2$. Then

$$0 < m_2 < n - m_1, \quad 0 < m_1 < n - m_2 \leq n - 1.$$ 

Therefore $\binom{n-m_1}{m_2} \geq n - m_1$, and also $\binom{n-1}{m_1} \geq n - 1$. Therefore,

$$2(n-1) \geq (nm_1, \ldots, m_k) \geq \binom{n}{m_1} \cdot (n-m_1) = \binom{n}{m_1} \cdot (n-m_1) = n \cdot \binom{n-1}{m_1} \geq n(n-1),$$

$$(n-2)(n-1) \leq 0, \quad n \leq 2 < k \leq m_1 + \ldots + m_k = n.$$ 

This is a contradiction.

Assume now that $k = 2$ and $\{m_1, m_2\} \neq \{1, n-1\}$.

We have $m_1, m_2 \geq 2$ and $\binom{n}{m_1} \geq \binom{n}{2}$. Moreover, $n = m_1 + m_2 \geq 4$. Therefore,

$$2(n-1) \geq \binom{n}{m_1} \cdot \frac{n}{2} = \frac{n(n-1)}{2} \geq \frac{4(n-1)}{2} = 2(n-1),$$

whence $n = 4$, and also

$$\binom{n}{m_1} = \binom{n}{2}, \quad \binom{4}{m_1} = \binom{4}{2}, \quad m_1 = m_2 = 2.$$
2.4. The orbits of the Weyl group. Let $r > 1$ be a natural number. Setting $n := r + 1 \in \mathbb{N}$, we have $n \geq 3$.

Consider the Euclidean space $\mathbb{R}^n$ in which the standard basis $\{e_i\}_{i=1}^n$ is orthonormal, an indecomposable root system $\Delta := \{e_i - e_j : i \neq j\} \subset \mathbb{R}^n$ of type $A_r$, its Weyl group $W \subset O(\mathbb{R}^n)$, the Weyl chamber $C := \{x \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n\} \subset \mathbb{R}^n$, and the corresponding simple root system $\Pi := \{\alpha_1, \ldots, \alpha_r\} \subset \Delta \ (\alpha_i := e_i - e_{i+1}, i = 1, \ldots, r)$.

The lattices $P := \{\lambda \in (\Delta) : \langle \lambda | \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in \Delta\}$ and $Q := (\Delta)_{\mathbb{Z}}$ in the subspace $\langle \Delta \rangle \subset \mathbb{R}^n$ satisfy the relations $WP = P$, $WQ = Q$, $Q \subset P$, and $|P/Q| = n < \infty$.

Let $\varphi_1, \ldots, \varphi_r \in P$ be the fundamental weights relative to the simple root system $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta$ (in the same order). Clearly $\{\varphi_i\}_{i=1}^r$ is a basis of the lattice $P \subset \langle \Delta \rangle$.

For brevity, the orbits of the action $W : P$ are simply called orbits.

Let $\Lambda \subset P$ be an arbitrary orbit. Set $\hat{\Lambda} := \text{conv}(\Lambda) \cap (\Lambda + Q) \subset P$. It is clear that:

1) $|\Lambda \cap C| = 1$;
2) $\Lambda = \Lambda \subset Q$;
3) $\hat{\Lambda} \supset \Lambda$;
4) $W\hat{\Lambda} = \hat{\Lambda}$;
5) for any orbit $\Lambda' \subset \hat{\Lambda}$, we have $\hat{\Lambda}' \subset \hat{\Lambda}$;
6) for any orbit $\Lambda' \subset \hat{\Lambda}$, $\Lambda' \neq \Lambda$, we have $\hat{\Lambda}' \cap \Lambda = \emptyset$.

Fix an orbit $\Lambda \subset P \setminus \{0\}$.
We have $\Lambda \cap C = \{\lambda\}$, where $\lambda \in (P \cap C) \setminus \{0\}$.

2.4.1. Main assertions. Let $\Pi' \subset \Pi \subset \mathbb{R}^n$ be an indecomposable simple root system of order $r - 2$.

It is easy to see that $H := (\{\lambda\} \cup \Pi') \subset \langle \Delta \rangle \subset \mathbb{R}^n$ is an $(r - 1)$-dimensional subspace.

In this subsection, we prove the following Lemmas 2.20 and 2.21.

Lemma 2.20. If $\lambda \not\in \Delta$, $\lambda \neq \varphi_1, \ldots, \varphi_r$, $|\hat{\Lambda} \setminus H| \leq 4n$, and $(-\lambda + \Lambda) \cap (|\hat{\Lambda} \setminus H| \leq 2n)$, then either $\lambda \in \{2\varphi_1, 2\varphi_r, \varphi_1 + \varphi_r, \varphi_1 + \varphi_r, \varphi_2 + \varphi_r\}$ or ($r = 3$) and ($\lambda = 2\varphi_2$).

Lemma 2.21. Suppose that $r \geq 8$, $\lambda = \varphi_j$, where $j \in \mathbb{N}$ and $3 \leq j \leq r - 2$, and also $\Pi' = \{\alpha_1, \ldots, \alpha_{r-2}\} \subset \Pi$. Then $|\hat{\Lambda} \setminus H| > 4n$.

We have $\lambda \in (\langle \Delta \rangle \cap C) \setminus \{0\}$, $\lambda_1 > 0 > \lambda_n$, and therefore

$$\langle \Pi' \rangle = \{x \in (\Delta) : x_p = x_q = 0\}, \quad H = \{x \in (\Delta) : x_q = cx_p\},$$

where $(p, q) \in \{1, \ldots, n\}^2$ is one of the pairs $(1, 2), (1, n)$ and $(n, n - 1)$ (and, therefore, $\lambda_p \neq \lambda_n$), and $c$ is the number $\frac{\lambda_{n}}{\lambda_{p}} \in \mathbb{R}$.

We begin with a proof of Lemma 2.20. First we need some auxiliary results.

Let $x \in P \cap C$ be a vector.

Set $K := \{x_1, \ldots, x_n\} \subset \mathbb{R}$, $k := |K| \in \mathbb{N}$, $m(t) := |\{i \in \{1, \ldots, n\} : x_i = t\}| \in \mathbb{N}$ ($t \in K$) and $m_0 := \max\{m(t)\}_{t \in K} \in \mathbb{N}$.

Proposition 2.22. Suppose that $k \geq 3$. Then

1) $|(Wx) \setminus H| \geq 2n - 2$;
2) if $m_0 \neq n - 2$, then $|(Wx) \setminus H| \geq 5(n - 2)$;
3) if $k \geq 4$, then $|(Wx) \setminus H| \geq 4n$.

Proof. Set

$$L(t') := \{t'' \in K \ \{ct'\} : (m(t') = 1) \Rightarrow (t'' \neq t') \subset K \ (t' \in K),$$

$$L := \{(t', t'') \in K^2 : t'' \in L(t') \} \subset K^2,$$
and
\[ K_0 := \{ t \in K : ct \in K \setminus \{ t \}, m(t) = 1 \} \subset K. \]

It is clear that
\[ \forall \tau \in L \quad d(\tau) := \left| \{ y \in Wx: (y_p, y_q) = \tau \} \right| \geq 1; \quad |(Wx) \setminus H| = \sum_{\tau \in L} d(\tau); \]

\begin{equation}
\forall t \in K \quad |L(t)| \geq k - 2; \quad \forall t \in K \setminus K_0 \quad |L(t)| > k - 2.
\end{equation}

Hence
\[ |L| \geq k(k - 2) + (k - |K_0|) \geq k(k - 2), \]
\[ |K_0| \geq k(k - 2) + k - |L| = k(k - 1) - |L|. \]

Let us show that
\begin{equation}
|L| \geq 4; \quad (|L| = 4) \Rightarrow (m_0 = n - 2).
\end{equation}

Suppose \(|L| \leq 4\). Then
\[ k(k - 2) \leq |L| \leq 4 < 4 \cdot (4 - 2), \quad k = 3, \]
\[ |K_0| \geq 6 - |L| \geq 2, \quad \left| \{ t \in K : m(t) = 1 \} \right| \geq |K_0| \geq 2. \]

Moreover, \( n \geq |K| = k = 3 \). Hence, \( m_0 = n - 2 \). Now if \(|L| \leq 3\), then \(|K_0| \geq 6 - |L| \geq 3 = |K|, K_0 = K, \)
\[ \forall t \in K \quad ct \in K \setminus \{ t \}, \quad ct \neq t, \quad t \neq 0, \quad c \neq 1, \quad c^3 \neq 1, \quad c^3 t \neq t; \quad K \subset \mathbb{R} \setminus \{ 0 \}; \]
\[ \forall t \in K \quad ct \in K \setminus \{ t \} \subset \mathbb{R} \setminus \{ 0 \}, \quad c \neq 0, \]
and we thus have a bijective map \( K \rightarrow K, t \rightarrow ct \), whose third power has no fixed points, whereas \(|K| = 3\). This is a contradiction.

We have thus established \(2.2\).

Let us show that
\begin{equation}
l := \left| \{ \tau \in L : d(\tau) < n - 2 \} \right| \leq 2; \quad (l > 0) \Rightarrow (m_0 = n - 2).
\end{equation}

Fix a number \( t_0 \in K \) such that \( m(t_0) = m_0 \) (it does exist). We also set \( L_0 := \{(t', t'') \in K^2 : t', t'' \neq t_0, t' \neq t''\} \subset K^2 \).

Let \((t', t'') \in L\) be an arbitrary pair such that \( d(t', t'') < n - 2 \). By Assertion \(2.18\) and inequalities \( k \geq 3 \), we have \( k = 3, K = \{ t', t'' \} \subset \mathbb{R}, t' \neq t, t'' \neq t, t' \neq t'', m(t) = n - 2 > d(t', t'') \geq 1 = m(t') = m(t''), m_0 = n - 2, t_0 = t \in K \setminus \{ t', t'' \}, (t', t'') \in L_0, \) where \(|L_0| = (k - 1)(k - 2) = 2\).

We have thus established \(2.3\).

It is clear that \( n \geq |K| = k \geq 3 \). Now, using \(2.1\) - \(2.3\), we have that
\[ |(Wx) \setminus H| \geq (n - 2) \cdot (|L| - l) + l = (n - 2) \cdot |L| - (n - 3)l \]
\[ = 4(n - 2) - 2(n - 3) + (n - 2) \cdot (|L| - 4) + (n - 3) \cdot (2 - l) \]
\[ \geq 4(n - 2) - 2(n - 3) = 2n - 2, \]
and, in the case \( m_0 \neq n - 2 \), we also have \(|L| \geq 5\) and \( l = 0 \), whence
\[ |(Wx) \setminus H| \geq (n - 2) \cdot |L| - (n - 3)l = (n - 2) \cdot |L| \geq 5(n - 2). \]

Suppose \( k \geq 4 \). Then \( n \geq k \geq 4, |L| \geq k(k - 2) \geq 8, \) and also \( m_0 \leq n - (k - 1) < n - 2 \). Therefore, \(|(Wx) \setminus H| \geq (n - 2) \cdot |L| \geq 8(n - 2) = 4n + 4(n - 4) \geq 4n. \)

This finishes the prof. \(\square\)

**Corollary 2.23.** \( \Delta \setminus H \geq 2n - 2. \)
Proof. We have $x := \varepsilon_1 - \varepsilon_n \in \Delta \cap C$ and also $|\{x_1, \ldots, x_n\}| = 3$. Now use Proposition 2.22. \hfill \Box

Proposition 2.24. For any vector $x \in (P \cap C) \setminus \{0\}$, we have the inequality $|(Wx) \setminus H| \geq 2$, which may become an equality only in the following cases:

1) $x_2 = \ldots = x_n$;
2) $x_1 = \ldots = x_{n-1}$;
3) $n = 4$, $x_1 = x_2$, $x_3 = x_4$.

Proof. Since the tautological representation $W : \langle \Delta \rangle$ is irreducible, we have

\begin{align*}
(2.4) & \sum_{y \in Wx} y = 0; \\
(2.5) & \langle Wx \rangle = \langle \Delta \rangle; \\
(2.6) & \bigcap_{w \in W} (wH) = 0.
\end{align*}

By (2.4), if $|(Wx) \setminus H| \leq 1$, then $Wx \subset H$ and $\langle Wx \rangle \subset H$, contrary to (2.5). This proves that $|(Wx) \setminus H| \geq 2$.

Suppose that $|(Wx) \setminus H| = 2$.

By (2.6), there are elements $w_1, \ldots, w_r \in W$ such that $\bigcap_{i=1}^r (w_iH) = 0$. We have

$$0 \notin Wx, \quad Wx = \bigcup_{i=1}^r ((Wx) \setminus (w_iH)),$$

and also

$$|Wx| \leq \sum_{i=1}^r |(Wx) \setminus (w_iH)| = \sum_{i=1}^r |(Wx) \setminus H| = 2(n - 1).$$

Now use Assertion 2.23. \hfill \Box

Proposition 2.25. Let $x \in (P \cap C) \setminus \{0\}$ be a vector and $\Lambda'$ be the orbit $Wx \subset P$. Then $|\hat{\Lambda}' \setminus H| \geq 2$, and the equality $|\hat{\Lambda}' \setminus H| = 2$ is possible only in the following cases:

1) $x_1 - x_2 = 1$, $x_2 = \ldots = x_n$;
2) $x_1 = \ldots = x_{n-1}$, $x_{n-1} - x_n = 1$;
3) $n = 4$, $x_1 = x_2$, $x_2 - x_3 = 1$, $x_3 = x_4$.

Proof. Proposition 2.24 and the relation $\hat{\Lambda}' \supset \Lambda'$ imply the inequality $|\hat{\Lambda}' \setminus H| \geq 2$, which may become an equality only in cases 1) and 3) mentioned in the statement of Proposition 2.24, provided the (W-invariant) subset $\hat{\Lambda}' \setminus (\Lambda' \cup \{0\}) \subset P$ contains no orbits, i.e., when $\hat{\Lambda}' \setminus \{0\} = \Lambda'$.

Suppose $|\hat{\Lambda}' \setminus H| = 2$. Then $\hat{\Lambda}' \setminus \{0\} = \Lambda'$, and one of conditions 1) and 3) from Proposition 2.24 is satisfied, whence $x \notin \Delta$. Furthermore, for any $\alpha \in \Delta$ such that $\langle x|\alpha \rangle > 0$ we have $x - \alpha \in \hat{\Lambda}' \setminus \{0\} = \Lambda'$, which implies that $\langle x|\alpha \rangle = 1$. In particular, $\langle x|\varepsilon_1 - \varepsilon_n \rangle = 1$ and $x_1 - x_n = 1$.

This finishes the proof of the proposition. \hfill \Box

Proposition 2.26. Let $x \in P \cap C$ be a vector and $\Lambda'$ be the orbit $Wx \subset P$. If $x \notin \Delta$ and $|\{x_1, \ldots, x_n\}| \geq 3$, then we have the inequality $|\hat{\Lambda}' \setminus H| \geq 2n$, which may become an equality only in the following cases:

1) $x_1 = x_2$, $x_3 = \ldots = x_{n-1}$, $x_2 - x_3 = x_{n-1} - x_n = 1$;
2) $x_2 = \ldots = x_{n-2}$, $x_{n-1} = x_n$, $x_1 - x_2 = x_{n-2} - x_{n-1} = 1$. 
Proof. Since \( |\{x_1, \ldots, x_n\}| \geq 3 \), we have \( x_1 - x_n > 1 \). Thus, among all the pairs \((i_1, i_2) \in \{1, \ldots, n\}^2\) such that \( x_{i_1} - x_{i_2} > 1 \), we can choose a pair \((i_1, i_2)\) with smallest \( i_2 - i_1 \).

Clearly, \( i_2 > i_1 \). Moreover, if \( i_1 = i_2 + 1, \ldots, i_2 - 1 \) is an arbitrary number, then \( x_{i_1} - x_i \leq 1 \) and \( x_{i_2} - x_i \leq 2 \), whereas \( x_{i_1} - x_i + x_{i_2} - x_i = x_{i_1} - x_{i_2} > 1 \), whence \( x_{i_1} - x_i = x_{i_2} - x_i = 1 \). Thus,

\[
\begin{align*}
(i_2 - i_1 > 1) & \Rightarrow (x_{i_1} - 1 = x_{i_1+1} = \ldots = x_{i_2-1} = x_{i_2} + 1). \\
\end{align*}
\]

(2.7)

Set \( \alpha := e_{i_1} - e_{i_2} \in \Delta \). Since \( \langle x | \alpha \rangle = x_{i_1} - x_{i_2} > 1 \) and \( x \notin \Delta \) it follows that

\[
y := x - \alpha \in \hat{\Lambda}' - \{0\}. \]

Let \( \Lambda'' \) denote the orbit \( \text{W}_y \subset P \).

We have \( x \in C, \ x_1 \geq \ldots \geq x_n, \) and, according to (2.7),

\[
\begin{align*}
x_1 \geq \ldots \geq x_{i_1-1} > x_{i_1} - 1 \geq x_{i_1+1} \geq \ldots \geq x_{i_2-1} \geq x_{i_2} + 1 \geq x_{i_2+1} \geq \ldots \geq x_n; \\
(i_2 - i_1 > 1) & \Rightarrow (y_{i_1} = \ldots = y_{i_2}), \\
y_1 \geq \ldots \geq y_{i_1-1} > y_{i_1} \geq \ldots \geq y_{i_2} > y_{i_2+1} \geq \ldots \geq y_n. \quad \text{Therefore, } y \subset C. \quad \text{Moreover, } \hat{\Lambda}' \subset \Lambda' \cup \Lambda'', \text{ and therefore,}
\end{align*}
\]

\[
|\hat{\Lambda}' \setminus H| \geq |\Lambda' \setminus H| + |\Lambda'' \setminus H|. 
\]

Suppose \( |\hat{\Lambda}' \setminus H| \leq 2n \).

It follows from Propositions 2.22 and 2.25 that \( |\hat{\Lambda}' \setminus H| = 2n, \ y_1 = \ldots = y_j, \ y_{j+1} = \ldots = y_n, \ y_j - y_{j+1} = 1, \ j \in \{1, \ldots, n - 1\}, \) and either \( j \in \{1, n - 1\} \) or \( n = 4 \) and \( j = 2 \).

Notice the following:

1) as mentioned before, \( y \neq 0 \), and, by (2.3), \((i_1, i_2) \neq (1, n)\);
2) if \( i_1 > 1 \), then \( y_{i_1-1} > y_{i_1}, j = i_1 - 1 < i - 1 \leq n - 1, (j = 1) \lor ((n = 4) \text{ and } (j = 2))\);
3) if \( i_2 < n \), then \( y_{i_2} > y_{i_2+1}, j = i_2 > i_1 \geq 1, (j = n - 1) \lor ((n = 4) \text{ and } (j = 2))\).

Since \( i_1 - 1 < i_1 < i_2 \), exactly one of the following equalities \( i_1 = 1 \) and \( i_2 = n \) holds.

We can have the following cases.

Case 1. \( i_1 > 1, i_2 = n, j = i_1 - 1, \) and \( j = 1 \).

We have \( i_1 = 2, y_1 - 1 = y_2 = \ldots = y_n, x_1 - 1 = x_2 - 1 = x_3 = \ldots = x_{n-1} = x_n + 1 \).

Case 2. \( i_1 > 1, i_2 = n, j = i_1 - 1, n = 4, \) and \( j = 2 \).

We have \( i_1 = 3, y_1 = y_2 = y_3 + 1 = y_4 + 1, x_1 = x_2 = x_3 = x_4 + 2 \).

Case 3. \( i_1 = 1, \ i_2 < n, \ j = i_2, \) and \( j = n - 1 \).

We have \( i_2 = n - 1, y_1 = \ldots = y_{n-1} = y_n + 1, x_1 - 1 = x_2 = \ldots = x_{n-2} = x_{n-1} + 1 = x_n + 1 \).

Case 4. \( i_1 = 1, \ i_2 < n, \ j = i_2, n = 4, \) and \( j = 2 \).

We have \( i_2 = 2, y_1 - 1 = y_2 - 1 = y_3 = y_4, x_1 - 2 = x_2 = x_3 = x_4 \).

Now use the fact that \( |\{x_1, \ldots, x_n\}| \geq 3 \). \( \square 

Proposition 2.27. Let \( x \in P \cap C \) be a vector and let \( \Lambda' \) denote the orbit \( \text{W}_x \subset P \). If

(2.9)

\[
|\{x_1, \ldots, x_n\}| = 2,
\]

\( x_1 - x_n > 1 \), and \( |\hat{\Lambda}' \setminus H| \leq 2n \), then one of the following holds:

1) \( x_1 - x_2 = 2, \ x_2 = \ldots = x_n \);
2) \( x_1 = \ldots = x_{n-1}, \ x_{n-1} - x_n = 2 \);
3) \( n = 4, \ x_1 = x_2 = 1, \ x_3 = x_4 = -1 \).
Proof. By assumption, \( x_1 = \ldots = x_j, x_{j+1} = \ldots = x_n \), and \( x_j - x_{j+1} \geq 2 \), where \( j \in \{1, \ldots, n-1\} \).

Set \( \alpha := \alpha_j \in \Delta \). We have \( \langle x | \alpha \rangle = x_j - x_{j+1} > 1 \), and therefore \( y := x - \alpha \in \hat{\Lambda} \setminus \Lambda \). Moreover, \( x \in C, x_1 \geq \ldots \geq x_n, x_1 \geq \ldots \geq x_{j-1} > x_j - 1 \geq x_{j+1} + 1 > x_{j+2} \geq x_n, \)
\( y_1 \geq \ldots \geq y_{j-1} > y_j \geq y_{j+1} > y_{j+2} \geq \ldots \geq y_n, y \in C \).

Let \( \Lambda'' \) be the orbit \( Wy \subset P \). It is easy to see that \( \hat{\Lambda} \supset \Lambda' \sqcup \Lambda'' \). Therefore, \( 2n \geq |\hat{\Lambda} \setminus H| \geq |\Lambda' \setminus H| + |\Lambda'' \setminus H| \). By Proposition 2.21, \( |\Lambda' \setminus H| \geq 2 \), which implies the inequality \( |\Lambda'' \setminus H| \leq 2n - 2 < 2n \). By Proposition 2.26 we have that either \( \{(y_1, \ldots, y_n)\} \leq 2 \) or \( y \in \Delta \).

Suppose that \( \{(y_1, \ldots, y_n)\} \leq 2 \).

For the integers \( k_1 := |\{y_1, \ldots, y_{j-1}\}|, k_2 := |\{y_j, y_{j+1}\}|, k_3 := |\{y_{j+2}, \ldots, y_n\}| \) we have \( k_1, k_3 \geq 0, k_2 \geq 1, (k_1 = 0) \Rightarrow (j = 1), \) and \( (k_3 = 0) \Rightarrow (j = n - 1) \). Furthermore, \( n - 1 > 1 \), whence
\[ k_1 + k_3 \geq 1; \quad (k_1 + k_3 = 1) \Rightarrow (j \in \{1, n - 1\}). \]

Moreover, \( (k_1 + k_3) + k_2 = |\{y_1, \ldots, y_n\}| \leq 2 \) and \( k_2 \geq 1 \). Therefore, \( k_1 + k_3 = 1, j \in \{1, n - 1\}, \) and \( k_2 = 1, y_j = y_{j+1}, x_j - x_{j+1} = 2 \).

Suppose now that \( y \in \Delta \).

By (2.9), \( 0 \notin \{x_1, \ldots, x_n\} \). It now follows from the relations \( x = y + \alpha \in \Delta + \Delta \) and \( x \in C \) that
\[ n \leq 4; \quad (n = 4) \Rightarrow (x_1 = x_2 = 1, x_3 = x_4 = -1). \]

Suppose \( n = 3 \). Then for \( r = 2 \),
\[ (2.10) \quad \dim H = r - 1 = 1. \]

Furthermore,
\[ \Lambda'' = \Delta, \quad \hat{\Lambda} \supset \Lambda' \sqcup \Delta, \]
\[ |\Lambda'| + |\Delta| \leq |\Lambda' \cap H| + |\Delta \cap H| + |\hat{\Lambda}' \setminus H| \leq |\Lambda' \cap H| + |\Delta \cap H| + 2n, \]
\[ 3 + n(n - 1) \leq |\Lambda' \cap H| + |\Delta \cap H| + 2n, \]
\[ |\Lambda' \cap H| + |\Delta \cap H| \geq n(n - 3) + 3 = 3, \]
and, according to (2.10),
\[ |\Lambda' \cap H|, |\Delta \cap H| \leq 2 < 3 \leq |\Lambda' \cap H| + |\Delta \cap H|, \]
\[ |\Lambda' \cap H|, |\Delta \cap H| > 0, \quad \Lambda' \cap H, \Delta \cap H \neq \emptyset. \]

But, by (2.9), no root of \( \Delta \subset \mathbb{R}^n \) is proportional to a vector from the orbit \( \Lambda' = Wx \subset P \), contrary to (2.10).

Thus, the proposition is proved. \( \square \)

Corollary 2.28. Let \( x \in P \cap C \) be a vector and let \( \Lambda' \) denote the orbit \( Wx \subset P \). If \( x_1 - x_n > 2 \), then \( |\hat{\Lambda}' \setminus H| \geq 2n \) and \( |\Lambda' \setminus H| \geq 2 \).

Proof. Follows from Propositions 2.21, 2.26, and 2.27. \( \square \)

Proposition 2.29. Let \( x \in P \cap C \) be a vector, and let \( \Lambda' \) denote the orbit \( Wx \subset P \). If \( x \notin \Delta, x_1 - x_n > 1, -\Lambda' = \Lambda', \) and \( |\hat{\Lambda}' \setminus H| \leq 4n, \) then \( n = 4, x_1 = x_2 = 1, \) and \( x_3 = x_4 = -1. \)

Proof. We have \( x_{n+1-i} = -x_i \) for any \( i = 1, \ldots, n \). Furthermore, since \( n - 1 \geq 2 \), we have \( x_2 \geq x_{n-1} = -x_2, x_2 \geq 0. \)

We can have the following cases.
Case 1. \( x_1 - x_n > 2 \) and \( x_2 > 0 \).

Case 2. \( x_1 - x_n > 2 \) and \( x_2 = 0 \).

Case 3. \( x_1 - x_n = 2 \).

First consider Case 1.

We have \( x_1 = \frac{(x_1 - x_n)}{2} > 1 \).

Set \( \alpha := \varepsilon_1 - \varepsilon_n \in \Delta \). Because \( \langle x | \alpha \rangle = x_1 - x_n > 2 \) and \( x \notin \Delta \), we have \( y := x - \alpha \in \hat{\Lambda}' \setminus (\hat{\Lambda}' \cup \{0\}) \). Let \( \Lambda'' \) denote the orbit \( Wy \subset P \). We have \( \Lambda'' \cap C = \{ \bar{y} \} \) and \( \bar{y} \in (P \cap C) \setminus \{0\} \). Moreover,

\[
\hat{\Lambda}' \cup \Lambda' \cap \Lambda'' \text{, } 4n \geq |\hat{\Lambda}' \setminus H| \geq |\Lambda' \setminus H| + |\Lambda'' \setminus H|,
\]

and, by Proposition 2.24 \( |\Lambda' \setminus H| \) and \( |\Lambda'' \setminus H| < 4n \).

By Proposition 2.22 \(|\{x_1, \ldots, x_n\}, \{\bar{y}_1, \ldots, \bar{y}_n\}| < 4 \), \(|\{y_1, \ldots, y_n\}| < 4 \). Moreover,

\[
\{x_1, \ldots, x_n\} \supset \{x_1, x_2, x_{n-1}, x_n\} = \{x_1, x_2, \} \text{, } \{y_1, \ldots, y_n\} = \{y_1, x_2, \ldots, x_{n-1}, y_n\},
\]

\( y_1 = x_1 - 1 > 0 \), \( y_n = x_n + 1 = -x_1 + 1 = -y_1 \), \( \{y_1, \ldots, y_n\} \supset \{\pm y_1, \pm x_2\} \). Thus, \(|\{\pm x_1, \pm x_2\}| < 4 \) and \(|\{\pm y_1, \pm x_2\}| < 4 \). On the other hand, \( x_1, y_1, x_2 > 0 \). This implies that \( x_2 = x_1 \) and \( x_2 = y_1 = x_1 - 1 \).

Thus Case 1 leads to a contradiction.

Now consider Case 2.

We have

\[
x_{n-1} = -x_2 = 0, \quad x_2 = \ldots = x_{n-1} = 0, \quad x = x_1(\varepsilon_1 - \varepsilon_n).
\]

Furthermore, \( x_1 = x_1 - x_2 \in \mathbb{Z} \), and also \( x_1 = \frac{(x_1 - x_n)}{2} > 1 \), whence \( x_1 \geq 2 \). Notice that \( x_1 - 1 \geq 1 > 0 \) and, as a consequence,

\[
y := (x_1 - 1)\varepsilon_1 + \varepsilon_2 - x_2\varepsilon_n \in (\hat{\Lambda}' \setminus \Lambda') \cap C \subset P.
\]

The orbits \( \Lambda' := Wy \subset P \) and \( \Lambda'' \subset P \) are distinct, since \( y_1 + y_n = (x_1 - 1) - x_1 \neq 0 \).

Therefore \( \hat{\Lambda}' \supset \Lambda' \cap (\Lambda'' \setminus \Lambda') \) and \( 4n \geq |\hat{\Lambda}' \setminus H| \geq |\Lambda' \setminus H| + |\Lambda'' \setminus H| \). Moreover,

1) \( |\{x_1, \ldots, x_n\}| = |\{ \pm x_1, 0 \}| = 3 \), and, according to Proposition 2.22 \( |\Lambda' \setminus H| \geq 2n - 2 \);

2) \( y_1 - y_n = (x_1 - 1) + x_1 = 2x_1 - 1 \geq 3 \), and, by Corollary 2.28 \( |\Lambda'' \setminus H| + |\Lambda'' \setminus H| > 2n + 2 \).

Hence,

\[
|\Lambda' \setminus H| + |\Lambda'' \setminus H| + \hat{\Lambda}' \setminus H > (2n - 2) + (2n + 2) = 4n.
\]

This is a contradiction.

Finally, consider Case 3.

We have \( x_1 = \frac{(x_1 - x_n)}{2} = 1, \quad x = (\varepsilon_1 + \ldots + \varepsilon_j) - (\varepsilon_{n+1-j} + \ldots + \varepsilon_n), \quad j \in \mathbb{N}, \quad j \leq \frac{n}{2}. \)

By assumption, \( x \notin \Delta \), whence \( j \geq 2, \ n \geq 4 \). If \( n = 4 \), then \( j = 2, \ x_1 = x_2 = 1, \) and \( x_3 = x_4 = -1 \).

Suppose that \( n > 4 \). Set \( y := (\varepsilon_1 + \varepsilon_2) - (\varepsilon_{n-1} + \varepsilon_n) \in P \cap C \). It is easy to see that \( \hat{\Lambda}' \supset (Wy) \cup \Delta \). Hence, \( 4n \geq |\hat{\Lambda}' \setminus H| \geq |(Wy) \setminus H| + |\Delta \setminus H| \) and, according to Corollary 2.28 \( |(Wy) \setminus H| \leq 4n - |\Delta \setminus H| \leq 4n - (2n - 2) = 2n + 2 \).

Furthermore, \( n - 4 > 0 \), \( |\{y_1, \ldots, y_n\}| = 3 \) and, also, \( n - 2 > 2, \max\{2, n - 4\} < n - 2 \). This together with Proposition 2.22 implies that

\[
|(Wy) \setminus H| \geq 5(n - 2) = (2n + 2) + 3(n - 4) > 2n + 2.
\]

This is a contradiction.

The proposition is proved. \( \square \)
Applying Propositions 2.2.26, 2.2.27 and 2.2.28 to the vector $\lambda \in (P \cap C) \setminus \{0\}$ and its orbit $\Lambda \subset P \setminus \{0\}$, we have Lemma 2.2.20.

Now let us prove Lemma 2.2.21.

By assumption, $j \in \{3, \ldots, n-3\}$, $\lambda_1 = \ldots = \lambda_j$, $\lambda_{j+1} = \ldots = \lambda_n$, and $\lambda_j - \lambda_{j+1} = 1$. In particular, $\lambda_{n-1} = \lambda_n$, and therefore

$$H = \{ x \in (\Delta) : x_{n-1} = x_n \}, \quad |\Lambda \setminus H| = \binom{2}{1} \cdot \binom{n-2}{j-1}.$$\[\text{Moreover } 2 \leq j-1 \leq (n-2) - 2, \text{ whence } \binom{n-2}{j-1} = \binom{n-2}{2} = \frac{(n-2)(n-3)}{2}.\]

Therefore,

$$|\Lambda \setminus H| \geq (n-2)(n-3) > n(n-5) = n(r-4) = 4n + n(r-8) \geq 4n.$$ \[\text{Thus Lemma 2.2.21 is proved.}\]

2.4.2. Additional results. Here we prove Propositions 2.30, 2.33

For arbitrary, pairwise distinct numbers $i_1, \ldots, i_k \in \{1, \ldots, n\}$, where $k = 1, \ldots, n$, we set

$$\lambda(i_1, \ldots, i_k) := (\varepsilon_{i_1} + \ldots + \varepsilon_{i_k}) - \frac{k}{n} \cdot (\varepsilon_1 + \ldots + \varepsilon_n) \in P.$$\[\text{Furthermore, for arbitrary pairwise distinct numbers } i, i_1, i_2 \in \{1, \ldots, n\} \text{ we set } \lambda^{(i)}(i_1, i_2) := \lambda(i_1) + (\varepsilon_{i_2} - \varepsilon_i) = \lambda(i_2) + (\varepsilon_i - \varepsilon_i) = \left(\varepsilon_{i_1} + \varepsilon_{i_2} - \varepsilon_i\right) - \frac{1}{n} \cdot (\varepsilon_1 + \ldots + \varepsilon_n) \in P.\]

**Proposition 2.30.** If $r > 2$ and $\lambda = \varphi_2 + \varphi_r$, then there are a subset $\Omega \subset \Lambda$ and a hyperplane $H \subset \langle \Delta \rangle$ such that $\langle \Omega \rangle = H$, $(\Omega - \Omega) \cap \Delta = \emptyset$ and $2 \cdot |\Lambda \setminus H| > |\Delta \setminus H| + 6$.

**Proof.** We have $n = r + 1 \geq 4$, $\lambda = \lambda^{(n)}_{(1,2)}$, and $\hat{\Lambda} = \Lambda \cup \{\lambda_1, \ldots, \lambda_n\}$.

Set $\Omega := \{ \lambda^{(i+1)}_{(i+1)} : i = 2, \ldots, r \} \subset \Lambda$ and $H := \langle \Omega \rangle \subset \langle \Delta \rangle$. Let us prove that the subset $\Omega \subset \Lambda$ and the subspace $H \subset \langle \Delta \rangle$ satisfy the desired requirements.

An arbitrary vector of the form $(\varepsilon_{i-1} + \varepsilon_{j-1} + \varepsilon_{i+1}) - (\varepsilon_{i-1} + \varepsilon_{i+1}) + P (2 \leq i < j \leq r)$ has its $(i+1)$-st coordinate equal to 2 when $j \leq i + 2$, and six nonzero coordinates when $j > i + 2$. Hence, $(\Omega - \Omega) \cap \Delta = \emptyset$.

The subspace $H' := H \setminus \langle \Delta \rangle \subset \mathbb{R}^n$ is given by the equations $x_{i-1} + x_i = x_{i+1}$, where $i = 2, \ldots, r$, and $x_1 + \ldots + x_n = 0$. Moreover,

$$\dim H \leq |\Omega| = r - 1, \quad \dim H' = r - \dim H \geq 1.$$\[\text{Let } x \in H' \text{ be an arbitrary nonzero vector.}\]

Let us show that $x_1, x_2, x_3, x_4 \neq 0$. We have $x_{i-1} + x_i = x_{i+1}$ ($i = 2, \ldots, r$) and $x_1 + \ldots + x_n = 0$, whence

$$\forall i = 2, \ldots, r \quad x_{i-1}x_i \leq x_{i-1}x_i + x_i^2 = x_{i+1}x_i;$$\[\text{and } 2x_3 + \sum_{i=4}^{n} x_i = (x_1 + x_2 + x_3) + \sum_{i=4}^{n} x_i = \sum_{i=1}^{n} x_i = 0;\]

$$4x_3^2 + 4 \cdot \sum_{i=4}^{n} x_3x_i + \sum_{i_1, i_2=4}^{n} x_{i_1}x_{i_2} = 0.$$
Suppose that there is a pair \((i_1, i_2) \in \{3, \ldots, n\}\) such that \(i_1 \leq i_2\) and \(x_{i_1} x_{i_2} < 0\). Among all such pairs choose a pair \((i_1, i_2)\) with smallest \(i_1 + i_2\). We have \(i_1 < i_2\). Furthermore, if \(i_2 \geq 5\), then

\[
3 \leq i_2 - 2 < i_2 - 1 < n \quad \text{and} \quad i_1 + (i_2 - 2) < i_1 + (i_2 - 1) < i_1 + i_2,
\]

whence \(x_{i_1} x_{i_2-2}, x_{i_1} x_{i_2-1} \geq 0\), \(x_{i_1} (x_{i_2-2} + x_{i_2-1}) \geq 0\), and \(x_{i_1} x_{i_2} \geq 0\), contrary to the assumption. Hence, \(i_2 < 5\), \(3 \leq i_2 < 5\), and therefore \((i_1, i_2) = (3, 4)\) and \(x_3 x_4 < 0\).

By \((2.11)\), \(x_1 x_2 \leq x_2 x_3 \leq x_3 x_4 < 0\), \(x_1, x_2, x_3, x_4 \neq 0\).

Suppose now that \(x_1 x_{i_2} \geq 0\) for all \(i_1, i_2 \in \{3, \ldots, n\}\) \((i_1 \neq i_2)\). Then, according to \((2.12)\), \(x_3^2 = \ldots = x_{n-1}^2 = 0\), \(x_3 = \ldots = x_n = 0\), \(x_2 = x_4 - x_3 = 0\), \(x_1 = x_3 - x_2 = 0\), \(x_1 = \ldots = x_n = 0\), whereas \(x \neq 0\). This is a contradiction.

We have thus established that \(\dim H' \geq 1\) and, moreover, \(x_1, x_2, x_3, x_4 \neq 0\) for any nonzero vector \(x \in H'\). As a consequence, \(H' = \mathbb{R} x \subset \langle \Delta \rangle\) \((x \in \Delta, x_1, x_2, x_3, x_4 \neq 0)\), \(\dim H' = 1\), \(H = (H')^\perp \cap \langle \Delta \rangle = \{y \in \langle \Delta \rangle : (y, x) = 0\} \subset \mathbb{R}^n\), \(\dim H = r - 1\).

It remains to show that \(2 \cdot |\Lambda \setminus H| > |\Delta \setminus H| + 6\).

For any \(i = 1, 2, 3, 4\), we have \((\lambda(i), x) = (\epsilon(i), x) = x_i \neq 0\), whence \(\lambda(i) \in \Lambda \setminus H\).

Therefore, \(|\Lambda \setminus H| \geq |\Lambda \setminus H| + 4 > |\Delta \setminus H| + 3\).

Let \(I = \{i_1, i_2\} \subset \{1, \ldots, n\}\) be an arbitrary two-element subset. Let us prove that there is a number \(i \in \{1, \ldots, n\} \setminus I\) such that \(x_i \neq x_{i_1} + x_{i_2}\).

Suppose that \(x_i = x_{i_1} + x_{i_2}\) for any \(i \in \{1, \ldots, n\} \setminus I\). Then

\[
0 = \sum_{i=1}^n x_i = (x_{i_1} + x_{i_2}) + (n - 2)(x_{i_1} + x_{i_2}) = (n - 1)(x_{i_1} + x_{i_2}), \quad x_{i_1} + x_{i_2} = 0.
\]

Thus, all numbers \(x_i \in \mathbb{R}\) (where \(i \in \{1, \ldots, n\}\) and \(i \neq i_1, i_2\)) are zero, which is impossible since \(x_1, x_2, x_3, x_4 \neq 0\).

According to the above, for all distinct numbers \(i_1, i_2 \in \{1, \ldots, n\}\) there is a number \(i \in \{1, \ldots, n\} \setminus \{i_1, i_2\}\) such that \(x_i \neq x_{i_1} + x_{i_2}\) (and therefore \((\epsilon_{i_1} + \epsilon_{i_2} - \epsilon_i, x) \neq 0\), \((\lambda_{i_1, i_2}^{(i)}, x) \neq 0\), \(\lambda_{i_1, i_2}^{(i)} \in \Lambda \setminus H\)). Hence,

\[
|\Lambda \setminus H| \geq C_n^2, \quad |\Lambda \setminus H| > |\Lambda \setminus H| + 3 \geq C_n^2 + 3
\]

and, therefore, \(2 \cdot |\Lambda \setminus H| > 2 \cdot C_n^2 + 6 = |\Delta| + 2 \geq |\Delta \setminus H| + 6\). \(\square\)

**Proposition 2.31.** If \(r = 3\) and \(\lambda = \varphi_1 + \varphi_2 + \varphi_3\), then

1) \(2\lambda \notin \Delta \cup (\Delta + \Delta)\);
2) there is a subset \(\Omega \subset \Lambda\) and a hyperplane \(H \subset \langle \Delta \rangle\) such that \(\langle \Omega \rangle = H\), \((\Omega + \Omega) \cap \Delta = (\Omega - \Omega) \cap \Delta = \emptyset\), and \(|\Lambda \setminus H| > |\Delta \setminus H| + 6\).

**Proof.** We have

\[
\lambda = \frac{1}{2} \cdot (3(\epsilon_1 - \epsilon_4) + (\epsilon_2 - \epsilon_3)), \quad 2\lambda = 3(\epsilon_1 - \epsilon_4) + (\epsilon_2 - \epsilon_3) \notin \Delta \cup (\Delta + \Delta),
\]

\(|\Lambda| = 24\) and \(|\Delta| = 12\).

Set

\[
\Omega := \left\{ \frac{1}{2} \cdot (3(\epsilon_1 - \epsilon_4) + (\epsilon_2 - \epsilon_3)), \frac{1}{2} \cdot (3(\epsilon_1 - \epsilon_2) + (\epsilon_4 - \epsilon_3)) \right\} \subset \Lambda.
\]

As is easily seen,

\((\Omega + \Omega) \cap \Delta = (\Omega - \Omega) \cap \Delta = \emptyset\) and \(H := \langle \Omega \rangle = \{x \in \langle \Delta \rangle : x_1 + 3x_3 = 0\} \subset \langle \Delta \rangle\).

Furthermore,

\[\Lambda \cap H = \Omega \cup (-\Omega) \subset \Lambda, \quad |\Lambda \cap H| = 4, \quad |\Lambda \setminus H| = 20 > |\Delta| + 6 \geq |\Delta \setminus H| + 6.\] \(\square\)
Proposition 2.32. If $r = 6$ and $\lambda = \varphi_3$, then there are a subset $\Omega \subset \Lambda$ and a hyperplane $H \subset \langle \Delta \rangle$ such that $\langle \Omega \rangle = H$, $(\Omega - \Omega) \cap \Delta = \emptyset$, and $2 \cdot |\hat{A} \setminus H| > |\Delta \setminus H| + 6$.

Proof. By assumption, $\lambda = \lambda_{(1,2,3)}$ and $|\Lambda| = 35$.

Set

$$\Omega := \{\lambda(1,2,5), \lambda(3,4,5), \lambda(1,4,6), \lambda(2,3,6), \lambda(5,6,7)\} \subset \Lambda.$$ 

We have

$$(\Omega - \Omega) \cap \Delta = \emptyset \quad \text{and} \quad H := \langle \Omega \rangle = \{x \in \langle \Delta \rangle: x_1 + x_3 = x_2 + x_4\} \subset \langle \Delta \rangle.$$ 

Hence, $|\Delta \setminus H| = n^2 - (3^2 + 2^2 + 2^2) = 32$. Moreover,

$\Lambda \cap H = \{\lambda(i_1,i_2,i_3) \in P: i_1 \in \{1,3\}, i_2 \in \{2,4\}, i_3 \in \{5,6,7\}\} \cup \{\lambda(5,6,7)\} \subset \Lambda,$

$|\Lambda \cap H| = 13, \quad |\Lambda \setminus H| = 22, \quad 2 \cdot |\Lambda \setminus H| = 44 > |\Delta \setminus H| + 6. \qed$

Proposition 2.33. If $r = 7$ and $\lambda = \varphi_3$, then there are a subset $\Omega \subset \Lambda$ and a hyperplane $H \subset \langle \Delta \rangle$ such that $\langle \Omega \rangle = H$, $(\Omega - \Omega) \cap \Delta = \emptyset$ and $2 \cdot |\hat{A} \setminus H| > |\Delta \setminus H| + 6$.

Proof. We have $\lambda = \lambda_{(1,2,3)}$ and $|\Lambda| = |\Delta| = 56$.

Set

$$\Omega := \{\lambda(1,2,7), \lambda(3,4,7), \lambda(5,6,7), \lambda(4,5,8), \lambda(1,6,8), \lambda(2,3,8)\} \subset \Lambda.$$ 

As is easily seen,

$$(\Omega - \Omega) \cap \Delta = \emptyset \quad \text{and} \quad H := \langle \Omega \rangle = \{x \in \langle \Delta \rangle: x_1 + x_3 + x_5 = x_2 + x_4 + x_6\} \subset \langle \Delta \rangle.$$ 

Furthermore,

$$\Lambda \cap H = \{\lambda(i_1,i_2,i_3) \in P: i_1 \in \{1,3,5\}, i_2 \in \{2,4,6\}, i_3 \in \{7,8\}\} \subset \Lambda,$$

whence $|\Lambda \cap H| = 18, \quad |\Lambda \setminus H| = 38$, and $2 \cdot |\Lambda \setminus H| = 76 > |\Delta| + 6 \geq |\Delta \setminus H| + 6. \qed$

§ 3. Proofs of the main results

This section is devoted to proving Theorem 1.1.

We return to the notation and the assumptions from §1.

Set $\delta := 1 \in \mathbb{R}$ if the representation $R$ of the algebra $\mathfrak{g}_C$ is orthogonal, and $\delta := 2 \in \mathbb{R}$ otherwise.

Fix a maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t}_C := \mathfrak{t} \otimes \mathbb{C}$ of $\mathfrak{g}_C$. As a result, we have a root system $\Delta \subset \mathfrak{t}_C^\ast$ and its Weyl group $W \subset GL(\mathfrak{t}_C^\ast)$. We have $\mathfrak{g} \cong su_{r+1}$, $r > 1$, and therefore $\Delta \subset \mathfrak{t}_C^\ast$ is an indecomposable root system of type $A_r$. Moreover,

$$\langle \Delta \rangle = \{\lambda \in \mathfrak{t}_C^\ast: \lambda(t) \subset i\mathbb{R}\} \subset \mathfrak{t}_C^\ast = \langle \Delta \rangle \oplus i\langle \Delta \rangle.$$ 

Fix a simple root system $\Pi \subset \Delta$ and the corresponding Weyl chamber $C \subset \langle \Delta \rangle \subset \mathfrak{t}_C^\ast$.

Let $P$ and $Q$ denote the lattices $\{\lambda \in \langle \Delta \rangle: \langle \lambda|\alpha \rangle \in \mathbb{Z} \forall \alpha \in \Delta\} \subset \langle \Delta \rangle$ and $\langle \Delta \rangle_\mathbb{Z} \subset \langle \Delta \rangle$, respectively. We have $Q \subset P$. Let $\lambda \in (P \cap C) \setminus \{0\}$ be the highest weight representation $R$ of $\mathfrak{g}_C$ relative to the simple root system $\Pi \subset \Delta \subset \mathfrak{t}_C^\ast$.

Set $\Lambda := W\lambda \subset P, \hat{\Lambda} := conv(\Lambda) \cap (\lambda + Q) \subset P$, and $n := r + 1 \in \mathbb{N}$. It is easy to see that the set of weights of the representation $R$ of $\mathfrak{g}_C$ coincides with the subset $\hat{\Lambda} \subset P$.

To prove Theorem 1.1 assume the opposite: suppose that $V/G$ is a smooth manifold and the linear algebra $R(\mathfrak{g}_C)$ is not isomorphic to any of the linear algebras $ad(\mathfrak{g}_C)$,
\( \varphi_1(A_r), \varphi_2(A_r), (2\varphi_1)(A_r) \) \((r > 1)\), \((2\varphi_2)(A_3), \varphi_3(A_5)\) and \(\varphi_4(A_7)\). The latter condition means that

\[
\lambda \notin \Delta \cup \{\varphi_1, \varphi_2, \varphi_{r-1}, \varphi_r, 2\varphi_1, 2\varphi_r\};
\]

\[
(r = 3) \implies (\lambda \neq 2\varphi_2);
\]

\[
(r = 5) \implies (\lambda \neq \varphi_3);
\]

\[
(r = 7) \implies (\lambda \neq \varphi_4).
\]

Moreover, since the linear algebras \(\varphi_j(A_r)\) and \(\varphi_{n-j}(A_r)\) \((j = 1, \ldots, r)\), and also the linear algebras \((\varphi_1 + \varphi_{r-1})(A_r)\) and \((\varphi_2 + \varphi_r)(A_r)\), are isomorphic, we may (and will) assume without loss of generality that

\[
(\lambda = \varphi_j, j = 1, \ldots, r) \implies \left(j \leq \frac{n}{2}\right); \quad \lambda \neq \varphi_1 + \varphi_{r-1}.
\]

Set \(\Pi_\lambda := \{\alpha \in \Pi : \langle \lambda|\alpha \rangle \neq 0\} \subset \Pi\). Also, let \(\mathcal{P}\) denote the family of all indecomposable simple root systems \(\Pi' \subset \Pi \subset \mathfrak{t}_\mathbb{C}\) of order \(r - 2\).

**Assertion 3.1.** Suppose that \(r > 2\). Then the simple root system \(\Pi'\) coincides with the union of all of its subsets \(\Pi'' \in \mathcal{P}\).

**Proof.** See 3.1 of [5] §3. \(\square\)

**Lemma 3.2.** Let \(\Pi' \in \mathcal{P}\) be a simple root system such that \((r > 2) \implies (\Pi_\lambda \cap \Pi' \neq \emptyset)\). If \(\lambda \neq \varphi_2 + \varphi_r\), then at least one of the following conditions holds:

\[
(\lambda \in \{\varphi_3, \ldots, \varphi_{r-2}\}); \quad (\Pi' = (1, \ldots, r - 2) \subset \Pi) \implies (r < 8);
\]

\[
(\Pi_\lambda \cap \Pi') = \{\alpha \} \subset \Pi, \quad \alpha \in \partial \Pi' \subset \Pi, \quad \langle \lambda|\alpha \rangle = 1.
\]

**Proof.** It is clear that \(H := \{\lambda\} \cup \Pi' \subset \langle \Delta \rangle \subset \mathfrak{t}_\mathbb{C}\) is an \((r - 1)\)-dimensional (real) subspace and, therefore, the intersection of the kernels of all linear functions on that subspace is of the form \(\mathbb{C}\xi \subset \mathfrak{t}_\mathbb{C}, \xi \in \mathfrak{t} \setminus \{0\}\).

We have \(\Pi \cong A_r\) and \(\Pi' \cong A_{r-2}\). As a consequence,

\[
|\Delta| = r(r + 1) \quad \text{and} \quad |\Delta \cap \langle \Pi' \rangle| = (r - 2)(r - 1).
\]

Hence,

\[
|\Delta \cap H| \geq |\Delta \cap \langle \Pi' \rangle| = (r - 2)(r - 1),
\]

\[
|\Delta \setminus H| \leq r(r + 1) - (r - 2)(r - 1) = 4r - 2.
\]

**Suppose**

\[
(\exists v \in V \quad \xi \in \mathfrak{g}_v, \quad \text{rk } \mathfrak{g}_v = 1).
\]

By Lemma 2.12

\[
\delta \cdot |\hat{\Lambda} \setminus H| \leq |\Delta \setminus H| + 6 \leq 4r + 4 = 4n.
\]

Moreover, \((\delta = 1) \implies (\Lambda = \Lambda)\). Thus

\[
|\hat{\Lambda} \setminus H| \leq 4n; \quad (\Lambda = \Lambda) \lor (|\hat{\Lambda} \setminus H| \leq 2n).
\]

Furthermore, according to \((3.2)\), \(\lambda \notin \{\varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r\}\). Now, using Lemmas 2.20 and 2.21 and the relations \((3.1)\), we have \((3.3)\).

If \((3.5)\) does not hold, then, by Lemma 3.4 of [5] §3, we have the relations \((3.3)\). \(\square\)

**Corollary 3.3.** Let \(\Pi' \in \mathcal{P}\) be a simple root system which does not satisfy \((3.3)\). If \(\lambda \neq \varphi_2 + \varphi_r\), then

\[
\Pi_\lambda \cap \Pi' \leq 1, \quad \Pi_\lambda \cap (\Pi' \setminus (\partial \Pi')) = \emptyset, \quad \forall \alpha \in \Pi_\lambda \cap \Pi' \quad \langle \lambda|\alpha \rangle = 1;
\]

\[
((r > 2) \implies (\Pi_\lambda \cap \Pi' \neq \emptyset)) \implies r > 2.
\]
Corollary 3.4. Let \( \Pi' \subset P \) be a simple root system which does not satisfy (3.3). If \( \lambda \neq \varphi_2 + \varphi_r \), then \( r > 2 \) and (3.6) holds.

Suppose that \( \lambda = \varphi_2 + \varphi_r \).

By (3.1), \( \lambda \neq 2\varphi_2 \), whence \( r > 2 \). Furthermore, the representation \( R \) of \( \mathfrak{g}_C \) is not self-adjoint, and therefore \( \delta = 2 \). Now, using Proposition 2.31 and Corollary 2.13, we obtain a contradiction with the fact that \( V/G \) is a smooth manifold.

Hence, \( \lambda \neq \varphi_2 + \varphi_r \). By (3.2), \( \lambda \notin \{ \varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r \} \).

Suppose \( \lambda \notin \{ \varphi_3, \ldots, \varphi_{r-2} \} \).

None of the simple root systems \( \Pi' \subset P \) satisfies (3.3). Moreover, as is easily seen, \( P \neq \emptyset \). Corollary 3.4 implies that \( r > 2 \) and that (3.6) holds for any simple root system \( \Pi' \subset P \).

Therefore,

1) for any \( \alpha \in \Pi_\lambda \), we have \( \langle \lambda | \alpha \rangle = 1 \) (see Assertion 3.3);

2) in the Dynkin diagram of the simple root system \( \Pi \), the path between the vertices corresponding to two distinct roots from the subset \( \Pi_\lambda \subset \Pi \) has at least \( r - 2 \) edges.

Hence, either \( \lambda \in \{ \varphi_1, \ldots, \varphi_r \} \cup \{ \varphi_1 + \varphi_r, \varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r \} \) or \( r = 3 \) and \( \lambda = \varphi_1 + \varphi_2 + \varphi_3 \). At the same time, \( \lambda \notin \{ \varphi_3, \ldots, \varphi_{r-2} \} \cup \{ \varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r \} \). By (3.1), \( r = 3 \) and \( \lambda = \varphi_1 + \varphi_2 + \varphi_3 \). The representation \( R \) of \( \mathfrak{g}_C \) is orthogonal, and therefore \( \delta = 1 \). By Proposition 2.31 and Corollary 2.13, the quotient \( V/G \) is not a smooth manifold. On the other hand, \( V/G \) is a smooth manifold. This is a contradiction.

Thus, we have established that \( \lambda \in \{ \varphi_3, \ldots, \varphi_{r-2} \} \).

We have \( \lambda = \varphi_j, \; j \in \mathbb{N}, \; 3 \leq j \leq r - 2 \), whence \( r \geq 5 \). By (3.2), \( j \leq \frac{n}{2} \).

If \( r = 5 \), then \( j = 3 \), \( \lambda = \varphi_3 \), contrary to (3.1).

Thus, \( r \geq 6 \), \( 3 \leq j \leq \frac{n}{2} \), and \( \lambda = \varphi_j \). Moreover, by (3.1), \( (r = 7) \Rightarrow (\lambda \neq \varphi_4) \).

Case 1. \( r = 6 \) and \( \lambda = \varphi_3 \).

The representation \( R \) of \( \mathfrak{g}_C \) is not self-adjoint. Therefore \( \delta = 2 \). Applying Proposition 2.32 and Corollary 2.13, we obtain a contradiction with the fact that \( V/G \) is a smooth manifold.

Case 2. \( r = 7 \) and \( \lambda = \varphi_3 \).

The representation \( R \) of \( \mathfrak{g}_C \) is not self-adjoint. Hence, \( \delta = 2 \). Using Proposition 2.33 and Corollary 2.13, we obtain a contradiction with the fact that \( V/G \) is a smooth manifold.

Case 3. \( r \geq 8 \).

The simple root system \( \Pi' := (1, \ldots, r - 2) \subset \Pi \), belonging to the family \( P \), does not satisfy (3.3) and, by Corollary 3.4, satisfies (3.6). In particular, \( \Pi_\lambda \cap (\Pi' \setminus (\partial \Pi')) = \emptyset \). Moreover, \( \Pi' \cong A_{r-2} \) and \( r - 2 \geq 6 \), whence \( \partial \Pi' = (1, 2, r - 3, r - 2) \subset \Pi \) and \( \Pi' \setminus (\partial \Pi') = (3, \ldots, r - 4) \subset \Pi \). Furthermore,

\[
3 \leq j \leq r - 4,
\]

and also \( \Pi_\lambda = (j) \subset \Pi \). Hence, \( \Pi_\lambda \cap (\Pi' \setminus (\partial \Pi')) = (j) \neq \emptyset \). We have obtained a contradiction. Thus we have a complete ("contrapositive") proof of Theorem 1.11.

§ 4. PARTICULAR CASES

In this section, we prove Theorem 1.2.

As before, we shall use the notation and the assumptions from § 1.

Set \( n := r + 1 \in \mathbb{N} \).

Suppose the linear Lie group \( G \subset \text{GL}(V) \) is connected and the linear algebra \( R(\mathfrak{g}_C) \) is isomorphic to one of the following linear algebras:
such that (4.1)

1) $\text{ad}(g_C)$;  
2) $\varphi_1(A_r) \ (r > 1)$;  
3) $(2\varphi_1)(A_r) \ (r > 1)$;  
4) $\varphi_2(A_r) \ (r > 3)$;  
5) $\varphi_2(A_3) \cong \varphi_1(D_3)$;  
6) $(2\varphi_2)(A_3) \cong (2\varphi_1)(D_3)$;  
7) $\varphi_3(A_5)$;  
8) $\varphi_4(A_7)$.

First, consider cases [1], [2], [5], [9] and [3].

The following representations of the complex simple Lie groups are polar (see [6, §3]):

- the adjoint representation of an arbitrary complex simple Lie group;
- the representation $R_{\varphi_1} + R'_{\varphi_1}$ of the complex simple Lie group $\text{SL}_n(\mathbb{C})$;
- the representation $R_{\varphi_1}$ and $R_{2\varphi_1}$ of the complex simple Lie group $\text{SO}_6(\mathbb{C})$;
- the representation $R_{\varphi_4}$ of the complex simple Lie group $\text{SL}_8(\mathbb{C})$.

Hence, in each of the cases [1], [2], [5], [9], and [3] the linear Lie group $G \subset \text{GL}(V)$ is polar and, by Lemma 2.16, the quotient $V/G$ is homeomorphic to a closed half-space (in particular, it is not a manifold).

Now we consider case [7].

Suppose that the linear group $G \subset \text{GL}(V)$ is connected and the linear algebra $R(g_C)$ is isomorphic to the linear algebra $\varphi_3(A_3)$.

Consider an $n$-dimensional Hermitian space $E$. It admits a direct sum decomposition into two orthogonal three-dimensional subspaces $E_+$ and $E_-$. It is clear that the tautological representation $G : V$ is isomorphic to the natural action $\text{SU}(E) : E^3$. Without loss of generality we may assume that $G = \text{SU}(E)$, $V = E^3$, and the tautological representation $G : V$ coincides with the natural action $\text{SU}(E) : E^3$.

As is easily seen, the space $V$ decomposes into a direct sum of the subspaces $V_j := E^3_j \cap E^3_{-j} \ (j = 0, 1, 2, 3)$, and $V_0 = C_v, \ v \in V_0 \setminus \{0\}$. Furthermore, $G_v = \text{SU}(E_+) \times \text{SU}(E_-) \subset G$, where $\text{SU}(E_{\pm}) := \{g \in G : E^g \supset E_+ \subset G\}$. As a result, $G_v V_j = V_j \ (j = 0, 1, 2, 3)$, $g_v = (iRv) \oplus V_1$, and $V = (g_v) \oplus (Rv) \oplus V_2 \oplus V_3, \ V_3 \subset V^{G_v}, \ \dim_{\mathbb{C}} V_3 = 1$. Therefore, the representation $G_v : N_v$ is isomorphic to the direct sum of the trivial action $G_v : \mathbb{R}^3$ and the representation $G_v : V_2$. The latter is, in turn, isomorphic to the natural action $(\text{SU}(E_) \times \text{SU}(E_-)) : (E_+ \otimes E_-^2)$ and, equivalently, to the natural action $(\text{SU}(E_+) \times \text{SU}(E_-)) : (E_+ \otimes E_-^2)$. Now, setting $d := m := 3 \in \mathbb{N}$ and applying Assertion 2.10 to the representation $G_v : N_v$, we have that the quotient of that representation is not a manifold. By Lemma 2.11, neither is the quotient $V/G$.

Now consider the remaining cases [3] and [4].

Henceforth, we shall assume that the linear group $G \subset \text{GL}(V)$ is connected, and the linear algebra $R(g_C)$ is isomorphic to one of the linear algebras $(2\varphi_1)(A_r) \ (r > 1)$ and $\varphi_2(A_r) \ (r > 3)$.

Set $\delta := 1 \in \mathbb{R}$ (respectively, $\delta := -1 \in \mathbb{R}$) if the linear algebra $R(g_C)$ is isomorphic to the linear algebra $(2\varphi_1)(A_r)$ (respectively, the linear algebra $\varphi_2(A_r)$).

Suppose $\delta = -1$ and $n \in 2\mathbb{Z} + 1$.

The representation $R_{\varphi_2} + R'_{\varphi_2}$ of the complex simple Lie group $\text{SL}_n(\mathbb{C})$ is polar (see [6, §3]). By Lemma 2.16, the quotient $V/G$ is homeomorphic to a closed half-space and, therefore, is not a manifold.

Henceforth we shall assume that $(\delta = 1) \vee (n \in 2\mathbb{Z})$.

Consider an $n$-dimensional Hermitian space $E$ with scalar multiplication $f_0(\cdot, \cdot)$, the complex space $B$ of all bilinear forms on $E$ and the subspace $B_5 \subset B$ of all forms $f \in B$ such that $f(x, y) = \delta \cdot f(y, x) \ (x, y \in E)$.

The tautological representation $G : V$ is isomorphic to the representation

$$
\text{SU}(E) : B_5, \quad (gf)(x, y) := f(g^{-1} x, g^{-1} y) \in \mathbb{C}
$$

$$(g \in \text{SU}(E), \ f \in B_5, \ x, y \in E).$$

(4.1)
There is a form $f_1 \in B_\delta$ such that the (fixed) orthonormal basis of the Hermitian space $E$ is orthonormal when $\delta = 1$ and symplectic when $\delta = -1$. Moreover, there is clearly an antilinear operator $F: E \to E$ such that $F^2 = \delta E \in U(E)$ and $f_1(x, y) = f_0(Fx, y)$ $(x, y \in E)$.

Let $\theta$, $\theta_0$, and $\theta_1$ be involutive automorphisms of the real Lie group $GL_C(E)$, uniquely determined by the relations

$$\theta(g)F = Fg \quad \text{and} \quad f_1(\theta_1(g)x, gy) = f_1(x, y) \quad (i = 0, 1, \ g \in GL_C(E), \ x, y \in E).$$

For arbitrary $g \in GL_C(E)$ and $x, y \in E$, we have

$$f_0(F\theta_1(g)x, gy) = f_1(\theta_1(g)x, gy) = f_1(x, y) = f_0(Fx, y) = f_0(\theta_0(g)Fx, gy),$$

whence $F\theta_1(g) = \theta_0(g)F$. Therefore, $\theta_0 \equiv \theta \circ \theta_1$ and $\theta_0 \circ \theta_1 \equiv \theta_1 \circ \theta_0 \equiv \theta$. The real Lie algebra $Lie(GL_C(E)) = gl_C(E)$ has commuting involutive automorphisms $\sigma_i := d\theta_i$ ($i = 0, 1$). It is easy to see that $f_1(\sigma_i(x, y)) + f_1(x, \xi y) = 0$ ($i = 0, 1, \ \xi \in gl_C(E), \ x, y \in E$). Hence, the automorphism $\sigma_0$ (respectively, $\sigma_1$) of the real Lie algebra $gl_C(E)$ is antilinear (respectively, linear) over $C$, and the subspace $(gl_C(E))^{-\sigma_1} \subset gl_C(E)$ contains the subspace $CE \subset gl_C(E)$.

It is clear that $(GL_C(E))^{\theta_0} = U(E) \subset GL_C(E)$. Furthermore, the map

$$S: gl_C(E) \to B, \quad (S(\xi))(x, y) := f_1(x, \xi y) \in C \quad (\xi \in gl_C(E), \ x, y \in E)$$

is an isomorphism of complex linear spaces. Moreover, for all $g \in SU(E)$, $\xi \in gl_C(E)$, and $x, y \in E$ we have

$$(S(\xi))(g^{-1}x, g^{-1}y) = f_1(g^{-1}x, \xi g^{-1}y) = f_1(x, \theta_1(g)\xi g^{-1}y)$$

$$= (S(\theta_1(g)\xi g^{-1}))(x, y);$$

$$\delta \cdot (S(\xi))(y, x) = \delta \cdot f_1(y, \xi x) = f_1(\xi x, y)$$

$$= -f_1(x, \sigma_1(\xi)y) = -(S(\sigma_1(\xi)))(x, y).$$

This implies that

$$S^{-1}(B_\delta) = (gl_C(E))^{-\sigma_1} \subset gl_C(E),$$

and the representation (4.1) is isomorphic to the representation

(4.2)

$$SU(E) : (gl_C(E))^{-\sigma_1}, \quad g : \xi \to \theta_1(g)\xi g^{-1}.$$ 

As we already mentioned, the tautological representation $G : V$ is isomorphic to the representation (4.1), and therefore to the representation (4.2). Without loss of generality we may assume that $G = SU(E) \subset GL_C(E)$, $V = (gl_C(E))^{-\sigma_1} \subset gl_C(E)$, and the tautological representation $G : V$ coincides with the representation (4.2).

The commuting automorphisms $\theta_0$ and $\theta_1$ of the Lie group $GL_C(E)$ preserve the commutant $SL_C(E)$ and, therefore, the subgroup $(SL_C(E))^{\theta_0} = SL_C(E) \cap U(E) = G$.

The commuting involutive automorphisms $\sigma_0$ and $\sigma_1$ of $gl_C(E)$ preserve the center $CE$ and the commutant $sl_C(E)$, which allows us to decompose this algebra into a direct sum of $\sigma_1$-invariant subspaces $CE$, $(sl_C(E))^{\sigma_0}$, and $(sl_C(E))^{-\sigma_0}$. Furthermore, since $(SL_C(E))^{\theta_0} = G \subset GL_C(E)$ we have that $(sl_C(E))^{\sigma_0} = g \subset gl_C(E)$. Since $C(sl_C(E)) = sl_C(E) \subset gl_C(E)$, and $\sigma_0$ is an antilinear over $C$ automorphism of $gl_C(E)$, we have

$$(sl_C(E))^{-\sigma_0} = i \cdot (sl_C(E))^{\sigma_0} = ig \subset gl_C(E).$$

By the above,

$$V = (gl_C(E))^{-\sigma_1} = (CE)^{-\sigma_1} \oplus g^{-\sigma_1} \oplus (ig)^{-\sigma_1} \subset gl_C(E).$$
and \((\sigma_1 - \id_g)\mathfrak{g} = \mathfrak{g}^{-\sigma_1} \subset \mathfrak{g}\). Since
\[
(C \mathfrak{E})^{-\sigma_1} = C \mathfrak{E} \subset \mathfrak{g} \mathfrak{I}_C(\mathfrak{E}) \quad \text{and} \quad \sigma_1 \in \text{Aut}_C(\mathfrak{g} \mathfrak{I}_C(\mathfrak{E})),
\]
we have that
\[
(\sigma_1 - \id_g)\mathfrak{g} = \mathfrak{g}^{-\sigma_1} \subset \mathfrak{g}.
\]
Since \((C \mathfrak{E})^{-\sigma_1} = C \mathfrak{E} \subset \mathfrak{g} \mathfrak{I}_C(\mathfrak{E})\) and \(\sigma_1 \in \text{Aut}_C(\mathfrak{g} \mathfrak{I}_C(\mathfrak{E}))\), we have that
\[
(4.3) \quad V = (C \mathfrak{E}) \oplus \mathfrak{g}^{-\sigma_1} \oplus i(\mathfrak{g}^{-\sigma_1}) \subset \mathfrak{g} \mathfrak{I}_C(\mathfrak{E}).
\]
It is clear that \(v := E \in V\). Moreover (see (4.2)),
\[
G_v = G^{\sigma_1} \subset G, \quad g_v = (\sigma_1 - \id_g)g = g^{-\sigma_1} \subset g,
\]
and the action \(G_v : V\) is given by \(g : \xi \mapsto g\xi g^{-1}\). By (4.3), the representation \(G_v : N_v\) is isomorphic to the direct sum of the trivial action \(G^{\sigma_1} : \mathbb{R}^2\) and the adjoint action \(G^{\sigma_1} : g^{-\sigma_1}\).

By Lemma 2.17, the quotient \(\mathfrak{g}^{-\sigma_1}/G^{\sigma_1}\) is homeomorphic to a closed half-space. Moreover, \(N_v/G_v \cong \mathbb{R}^2 \times (\mathfrak{g}^{-\sigma_1}/G^{\sigma_1})\) and, therefore, the quotient \(N_v/G_v\) is also homeomorphic to a closed half-space.

Thus we have established that the quotient \(N_v/G_v\) is not a manifold. By Lemma 2.1 neither is the quotient \(V/G\).

Therefore, in each of cases 3) and 4) the quotient \(V/G\) is not a manifold. Now Theorem 1.2 is fully proved. □

Theorems 1.1 and 1.2 immediately imply Corollary 1.3.

ACKNOWLEDGMENTS

The author is grateful to Professor E. B. Vinberg for constant scientific support and numerous valuable advice.

REFERENCES


Department of Mechanics and Mathematics, Moscow State University, Moscow
E-mail address: oleg.styrt@mail.ru