COMPARISON OF THE SINGULAR NUMBERS OF CORRECT RESTRICTIONS OF ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract. The paper is dedicated to finding the asymptotics of singular numbers of a correct restriction of a uniformly elliptic differential operator of order 2l defined on a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, which is in general a non-selfadjoint operator. Conditions are established on a correct restriction, ensuring that its singular numbers $s_k$ are of order $k^{2l/n}$ as $k \to \infty$. As an application of this result certain estimates are obtained for the deviation upon domain perturbation of singular numbers of such correct restrictions.

§ 1. Introduction

Let $l, n \in \mathbb{N}$ and $\mathcal{L}$ be an elliptic differential expression of the following form: for $u \in C^\infty(\mathbb{R}^n)$

$$(\mathcal{L}u)(x) = \sum_{|\alpha|,|\beta| \leq l} (-1)^{|\alpha|+|\beta|} D^\alpha(A_{\alpha\beta}(x)D^\beta u), \quad x \in \mathbb{R}^n,$$

where $A_{\alpha\beta} \in C^l(\mathbb{R}^n)$ are real-valued functions for all multi-indices $\alpha, \beta$ satisfying $|\alpha|, |\beta| \leq l$. Moreover, for any bounded domain $\Omega \subseteq \mathbb{R}^n$ with sufficiently smooth boundary let $L_\Omega$ be a linear operator closed in $L_2(\Omega)$ generated by $\mathcal{L}$ on $\Omega$.

Our main aim is finding a family of correct restrictions $B$ of the operator $L_\Omega$ having the same asymptotics of singular numbers $s_k(B)$.

This is done by comparing $s_k(B)$ with singular numbers $s_k(A)$ of a correct restriction $A$ of the operator $L_\Omega$ whose asymptotics of singular numbers are known and represent the inverse $B^{-1}$ in the form $B^{-1} = A^{-1} + K$, where $A^{-1}$ and $K$ are a leading operator and a non-leading operator respectively.

As an application we establish certain estimates for the deviation of singular numbers $s_k(B)$ of correct restrictions of this family upon domain perturbation. This is done by considering a correct restriction $A$ for which such estimates are known and the fact that $s_k(B)$ and $s_k(A)$ have the same asymptotics which are uniform with respect to a certain family of correct restrictions and a certain class of domains $\Omega$.

The paper is organized in the following way. In Section 2 we recall the notion of a correct restriction of a closed linear operator in a Hilbert space and related facts. In Section 3 the definition of leading and non-leading compact operators is given and, under certain assumptions, it is proved that the singular numbers of the sum of a leading and a non-leading operator have the same asymptotics as the singular numbers of the leading operator (Theorem 3.1). Section 4 is dedicated to some auxiliary estimates of the singular numbers of correct restrictions of elliptic differential operators. In Section 5

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a precise description of the class of domains under consideration is given. The core result on the representation of the inverse of a correct restriction as the sum of a leading and a non-leading operator is proved in Section 6 (Theorem 6.1). In Section 7 on the basis of Theorems 3.1 and 6.1 it is proved that the singular numbers of all correct restrictions of a certain family are of the same order \( k^{2l/n} \) and have the same asymptotics as \( k \to \infty \) (Theorem 7.1). Section 8 is dedicated to applying Corollary 7.1 (a “uniform” version of Theorem 7.1) to obtain spectral stability estimates for correct restrictions belonging to a certain family (Theorem 8.1).

Most of the results of this paper were formulated, without proofs, in [4].

§ 2. Preliminaries on correct restrictions of linear operators

Let \( L : D(L) \to H \) be a closed linear operator in a Hilbert space \( H \). It will always be assumed that the domain \( D(L) \) is dense in \( H \) and that the equation

\[
Lu = f
\]

has a solution for any \( f \in H \) (in general non-unique).

Recall that an operator \( A : D(A) \to H \) is called a restriction of \( L \), briefly \( A \subseteq L \), if \( D(A) \subseteq D(L) \) and \( Au = Lu \) for all \( u \in D(A) \).

**Definition 2.1.** A restriction \( A \) of \( L \) is called correct if the equation

\[
Au = f
\]

has a unique solution \( u \in D(A) \) for any \( f \in H \) and the corresponding inverse operator \( A^{-1} : H \to D(A) \) is bounded, i.e.

\[
\|A^{-1}\| = \sup_{f \in H, f \neq 0} \frac{\|A^{-1}f\|}{\|f\|} = \sup_{u \in D(A), u \neq 0} \frac{\|u\|}{\|Au\|} < \infty.
\]

**Remark 2.1.** Clearly \( LA^{-1} = I \), where \( I \) is the identity operator on \( H \). Moreover, if \( B : H \to D(L) \) is a bounded operator such that \( LB = I \), then the operator \( A \) which is the restriction of \( L \) to \( D(A) = B(H) \) is a correct restriction of \( L \) and \( B = A^{-1} \).

Let \( L \) have a correct restriction \( A \). Then it is possible to describe all correct restrictions of \( L \). Namely, the following statement holds. See [9, 11].

**Lemma 2.1.** Let \( A \) be a correct restriction of \( L \).

a) If an operator \( K : H \to \text{Ker} L = \{u \in D(L) : Lu = 0\} \) is bounded, then the operator \( A^{-1} + K \) is invertible and the operator

\[
B = (A^{-1} + K)^{-1}
\]

is a correct restriction of \( L \).

b) If \( B \) is a correct restriction of \( L \), then there exists a bounded operator

\( K : H \to \text{Ker} L \),

namely \( K = B^{-1} - A^{-1} \), such that equality (3) holds.

**Remark 2.2.** Under the assumptions of part a)

\[
B^{-1} = A^{-1} + K
\]

and

\[
D(B) = R(A^{-1} + K) \subseteq D(A) + R(K) \subseteq D(A) + \text{Ker} L.
\]

**Remark 2.3.** In Lemma 2.1 the operators \( K \) can be nonlinear, which allows us to consider boundary value problems with nonlinear boundary conditions. However, in this paper we restrict ourselves to considering the case of linear operators \( K \).
Example 2.1. Let $B$ be an open ball in $\mathbb{R}^n$ and $L$ be the closure of the Laplacian $-\Delta$ with the domain $C^\infty(\mathbb{R}^n)$ in $H = L_2(B)$; hence $L: D(L) \to L_2(B)$,
\[
D(L) = \{ u \in L_2(B): -\Delta_w u \in L_2(B) \}, \quad Lu = -\Delta_w u \quad \text{for } u \in D(L),
\]
where $-\Delta_w u$ is the weak Laplacian.

Furthermore, in the capacity of a restriction $A$ of $L$, we consider the Dirichlet Laplacian $-\Delta_D$ with the domain
\[
D(-\Delta_D) = \{ u \in D(L), \; \text{tr}_{\partial B} u = 0 \}.
\]

It is well known that for each $f \in L_2(B)$ the equation $-\Delta_D u = f$ has a unique solution in $D(-\Delta_D)$ which continuously depends on $f$. Hence $-\Delta_D$ is a correct restriction of $L$ in the sense of Definition 1.

By Lemma 2.1 for each bounded operator $K: L_2(B) \to \text{Ker} L$, the space of all solutions $u \in D(L)$ of the equation $\Delta_w u = 0$, and the operator $((-\Delta_D)^{-1} + K)^{-1}$ is a correct restriction of $L$. This implies that for each $f \in L_2(B)$ the equation
\[
((-\Delta_D)^{-1} + K)^{-1} u = f
\]
has a unique solution
\[
(\Delta_D)^{-1} f \in D((\Delta_D)^{-1} + K)^{-1} = R((\Delta_D)^{-1}) + R(K) \subseteq D(-\Delta_D) + R(K) \subseteq D(-\Delta_D) + \text{Ker} L.
\]

Moreover,
\[
\begin{align*}
\begin{cases}
  u \in D((\Delta_D)^{-1} + K)^{-1}, \\
  ((\Delta_D)^{-1} + K)^{-1} u = f
\end{cases}
\iff 
\begin{cases}
  u \in L_2(B), -\Delta_w u \in L_2(B), \\
  -\Delta_w u = f \text{ on } B, \\
  \text{tr}_{\partial B} (u + K(\Delta_w u)) = 0.
\end{cases}
\end{align*}
\]

The implication $\Rightarrow$ follows since by (6) and (7) $-\Delta_w u = f$ on $\Omega$ and $\text{tr}_{\partial B} u = \text{tr}_{\partial B} (-\Delta_D)^{-1} f + \text{tr}_{\partial B} K f = \text{tr}_{\partial B} K f$.

Conversely, assume that $u \in L_2(B), -\Delta_w u \in L_2(B), -\Delta_w u = f$ on $B$, $\text{tr}_{\partial B} u = \text{tr}_{\partial B} K f$, and $v = u - K f$. Then $v \in D(L)$ (since $K f \in D(L)$), $-\Delta_w v = f$ on $B$, and $\text{tr}_{\partial B} v = 0$.

Therefore, $v = ((\Delta_D)^{-1} f$. Consequently,
\[
(\Delta_D)^{-1} f + K f \iff u \in R((\Delta_D)^{-1} + K) = D((\Delta_D)^{-1} + K)^{-1})
\]
and $((\Delta_D)^{-1} + K)^{-1} u = f$.

Example 2.2. Let $H$, $L$, $A$ be the same as in Example 2.1 $n \geq 2$ and $\varphi_1, \varphi_2 \in L_2(B)$. Assume that $\varphi_1$ is a harmonic function on $B$ and that for all $f \in L_2(B)$
\[
(\varphi_1(x)) = \varphi_1(x) \int_B \varphi_2(y) f(y) \, dy, \quad x \in B.
\]

Then $K$ is bounded from $L_2(B)$ to $\text{Ker} L$, and all the statements of Example 2.1 are applicable to $K$. In particular this implies that the following problem
\[
\begin{align*}
\begin{cases}
  u \in L_2(B), -\Delta_w u \in L_2(B), \\
  -\Delta_w u = f \text{ on } B, \\
  \text{tr}_{\partial B} \left( u(x) + \varphi_1(x) \int_B \varphi_2(y) \Delta_w u(y) \, dy \right) = 0
\end{cases}
\end{align*}
\]
with a non-local boundary condition is well posed.
§ 3. SOME ESTIMATES FOR SINGULAR NUMBERS OF GENERAL COMPACT LINEAR OPERATORS

Let $A$ be a compact linear operator in a Hilbert space $H$. Recall that the eigenvalues of the self-adjoint operator $(A^* A)^{1/2}$ are called the singular numbers of the operator $A$. As usual all singular numbers of $A$ will be written in the form of the sequence $\{s_n(A)\}_{n=1}^{\infty}$, where

$$s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq \cdots$$

and each singular number is repeated as many times as its multiplicity. Note that $\lim_{n \to \infty} s_n(A) = 0$.

**Lemma 3.1.** Let $A$ and $B$ be compact linear operators in a Hilbert space $H$, $0 < \alpha < \beta$ and $\alpha/\beta < \theta < 1$.

Moreover, let for some $c_1, c_2 > 0$ for all $k \in \mathbb{N}$

$$c_1 k^{-\alpha} \leq s_k(A), \quad s_k(B) \leq c_2 k^{-\beta}. \quad (9)$$

Then for all $k \in \mathbb{N}$

$$s_k(A + B) \leq s_{k - [k^\theta]}(A)(1 + c_3 k^{\alpha-\theta\beta}), \quad (10)$$

where $c_3 = c_1^{-1} c_2 2^\beta$ and $[a]$ denotes the integer part of a real number $a$.

**Proof.** We apply the inequality

$$s_k(A + B) \leq \min_{m+n=k+1} \left( s_m(A) + s_n(B) \right). \quad (11)$$

(5 6 7; see also 8 Chapter 2, Corollary 2.2). Taking into account that $\frac{\mu}{\mu + 1} \leq 2$ for each real number $\mu \geq 1$, we have

$$s_k(A + B) \leq s_{k - [k^\theta]}(A) + s_{[k^\theta]}(B) \leq s_{k - [k^\theta]}(A) + c_2 [k^\theta]^{-\beta} \frac{s_{k - [k^\theta]}(A)}{c_1 (k - [k^\theta] + 1)^{-\alpha}} \leq s_{k - [k^\theta]}(A)(1 + c_1^{-1} c_2 [k^\theta]^{-\beta} k^\alpha) \leq s_{k - [k^\theta]}(A)(1 + c_1^{-1} c_2 2^\beta k^{\alpha-\theta\beta}).$$

Hence, inequality (10) follows. \hfill \Box

**Remark 3.1.** Assume that for some $k_1 \in \mathbb{N}$ inequalities (9) are satisfied with the same $c_1, c_2$ only for natural $k \geq k_1$. Since $[k^\theta], k - [k^\theta] \to \infty$ as $k \to \infty$, there exists $k_2 \in \mathbb{N}$, depending only on $k_1$ and $\theta$, such that $[k^\theta], k - [k^\theta] + 1 \geq k_1$ for all $k \geq k_2$. Hence, by the above proof, it follows that inequality (10) is satisfied with the same $c_3$ for natural $k \geq k_2$.

**Lemma 3.2.** Under the assumptions of Lemma 3.1 there exists $c_4 > 0$, depending only on $\alpha, \beta, \theta, c_1, c_2$ and for each $\gamma > 1$ there exists $\varkappa_\gamma \in \mathbb{N}$, depending only on $\gamma, \alpha, \beta, \theta, c_1, c_2$ such that

$$s_{k + [\gamma k^\theta]}(A)(1 - c_4 k^{\alpha-\theta\beta}) \leq s_k(A + B) \leq s_{k - [k^\theta]}(A)(1 + c_4 k^{\alpha-\theta\beta}) \quad (12)$$

for all $k \geq \varkappa_\gamma$.

**Proof.** 1. By inequality (11), with $B$ replaced by $-B$, $A$ replaced by $A + B$, and $k$ replaced by $2k$, we have

$$s_{2k}(A) \leq s_k(A + B) + s_{k+1}(B).$$

Hence, by assumption (9)

$$s_k(A + B) \geq s_{2k}(A) - s_{k+1}(B) \geq c_1 (2k)^{-\alpha} - c_2 (k+1)^{-\beta} \geq k^{-\alpha} (c_1 2^{-\alpha} - c_2 k^{\alpha-\beta}) \geq c_5 k^{-\alpha},$$
where $c_5 = c_1 2^{-\alpha - 1}$, if $k \geq k_1 = [(c_1^{-1}c_2 2^{\alpha + 1})^{1/(\beta - \alpha)}] + 1$.

Therefore by Lemma 3.4 and Remark 3.1 it follows that there exists $k_2 \in \mathbb{N}$, depending only on $\alpha, \beta, \theta, c_1, c_2$, such that inequality (13) holds, with $B$ replaced by $-B$, $A$ replaced by $A + B$, $k$ replaced by $m$, and $c_1$ replaced by $c_5$, for all $m \geq k_2$. So

$$ (13) \quad s_m(A) \leq s_{m-[m^\theta]}(A+B)(1 + c_4 m^{\alpha - \theta \beta}), $$

where $c_4 = c_5^{-1} c_2 2^\beta > c_3$, for all $m \geq k_2$.

2. For a given $k \in \mathbb{N}$, let a natural number $m$ be such that

$$ (14) \quad m - [m^\theta] + 1 = k. $$

2a. Note that the function $f$ defined by $f(m) = m - [m^\theta] + 1$ for $m \in \mathbb{N}$ is non-decreasing on $\mathbb{N}$ and $f(\mathbb{N}) = \mathbb{N}$. Therefore this equation has a solution for each $k \in \mathbb{N}$, although the solution, in general, is non-unique.

Indeed, first note that for all $m, n \in \mathbb{N}$ the inequalities

$$ n - 1 < m^\theta \leq n, \quad (n - 1)^{1/\theta} < m^\theta \leq n^{1/\theta} \quad \text{and} \quad [(n - 1)^{1/\theta}] + 1 \leq m \leq [n^{1/\theta}] $$

are equivalent. Hence

$$ f(\mathbb{N}) = \bigcup_{n=1}^\infty f(\{m \in \mathbb{N}: n - 1 < m^\theta \leq n\}) $$

$$ = \bigcup_{n=1}^\infty f(\{m \in \mathbb{N}: [(n-1)^{1/\theta}] + 1 \leq m \leq [n^{1/\theta}]\}). $$

Assume that $n$ is fixed and $[(n-1)^{1/\theta}] + 1 \leq [n^{1/\theta}]$. If $[(n-1)^{1/\theta}] + 1 \leq m^\theta \leq [n^{1/\theta}] - 1$, then $n - 1 < m^\theta < n$; hence $[m^\theta] = n - 1$ and

$$ f(m) = m - [m^\theta] + 1 = m - n + 2. $$

If $n^{1/\theta} \notin \mathbb{N}$, then $n - 1 \leq (n^{1/\theta} - 1)^{1/\theta} < [n^{1/\theta}]^\theta < n$ hence $[n^{1/\theta}]^\theta = n - 1$, and this equality also holds for $m = [n^{1/\theta}]$. If $n^{1/\theta} \notin \mathbb{N}$, then $[n^{1/\theta}]^\theta = n$ and

$$ f([n^{1/\theta}]) = [n^{1/\theta}] - n + 1 = f([n^{1/\theta}] - 1). $$

In both cases

$$ f(\{m \in \mathbb{N}: [(n-1)^{1/\theta}] + 1 \leq m \leq [n^{1/\theta}]\}) = [f([(n-1)^{1/\theta}] + 1), f([n^{1/\theta}])] \cap \mathbb{N}. $$

Clearly this equality is also satisfied if $[(n-1)^{1/\theta}] + 1 = [n^{1/\theta}]$.

Therefore

$$ f(\mathbb{N}) = \bigcup_{n=1}^\infty \left( [f([(n-1)^{1/\theta}] + 1), f([n^{1/\theta}])] \cap \mathbb{N} \right) $$

$$ = \left( \bigcup_{n=1}^\infty [f([(n-1)^{1/\theta}] + 1), f([n^{1/\theta}])] \right) \cap \mathbb{N} = \mathbb{N}, $$

because $f(1) = 1$ and for each $n \in \mathbb{N}$

$$ f([n^{1/\theta}] + 1) = [n^{1/\theta}] - [(n^{1/\theta} + 1)^\theta] + 2 \leq [n^{1/\theta}] - [(n^{1/\theta})^\theta] + 2 = f([n^{1/\theta}]) + 1. $$

Also note that

$$ f([n^{1/\theta}] + 1) \geq [n^{1/\theta}] - [((n^{1/\theta})^\theta + 1) + 2 = f([n^{1/\theta}]). $$

The above argument also implies that the function $f$ is non-decreasing on $\mathbb{N}$, but is not strictly increasing.

2b. Assume that $m \in \mathbb{N}$ satisfies equation (14). Since $m = k + [m^\theta] - 1 \geq k$ it also follows that

$$ m \geq k + [k^\theta] - 1. $$
Next we claim that for each $\gamma > 1$ there exists $\nu_\gamma > 0$, depending only on $\gamma$ and $\theta$ such that
\begin{equation}
(15) \quad m \leq k + \lfloor \gamma k^\theta \rfloor
\end{equation}
for all natural $k \geq \nu_\gamma$.

Indeed, equality (11) implies that for each $0 < \delta < 1$
\[ k \geq m - m^\theta = m(1 - m^{\theta - 1}) \geq \delta m \quad \text{if} \quad m \geq (1 - \delta)^{(1/(\theta - 1))} \]
Hence for all natural $k \geq (1 - \delta)^{(1/(\theta - 1))}$ ($\Rightarrow m \geq (1 - \delta)^{(1/(\theta - 1))}$)
\[ k \geq m - \lfloor m^\theta \rfloor \geq m - \lfloor \delta^{-\theta} k^\theta \rfloor \Rightarrow m \leq k + \lfloor \delta^{-\theta} k^\theta \rfloor = k + \lfloor \gamma k^\theta \rfloor \]
if $\delta^{-\theta} = \gamma$. So inequality (15) holds for $k \geq \nu_\gamma = (1 - \gamma^{-1/\theta})^{1/(\theta - 1)}$.

3. Inequalities (13) and (15) imply that for natural $k \geq \max\{k_2, \nu_\gamma\}$
\[ s_k(\gamma k^\theta)(A) \leq s_k(A + B)(1 + c_4(k + \lfloor \gamma k^\theta \rfloor)^{\alpha/\theta} \beta) \leq s_k(A + B)(1 + c_4 k^{\alpha/\theta - \beta}) \]
Therefore
\[ s_k(A + B) \geq s_k(\gamma k^\theta)(A)(1 - c_4 k^{\alpha/\theta - \beta}) \]
If $k \geq (2c_4)^{1/(\theta\beta - \alpha)}$, then $1 - c_4 k^{\alpha/\theta - \beta} \geq 1/2$.
Also by (10) for $k \geq 2$
\[ s_k(A + B) \leq s_{k - [\gamma k^\theta]}(A)(1 + c_4 k^{\alpha/\theta - \beta}) \]
since $c_4 > c_3$.

Hence, the statement follows for
\[ k \geq x_\gamma = \lfloor \max\{k_2, \nu_\gamma, (2c_4)^{1/(\theta\beta - \alpha)}, 2\} \rfloor + 1. \]

Remark 3.2. Applying more delicate estimates one can prove that inequality (15) follows from the inequality $x \leq \gamma((x + 1)^{1/\theta} - x)^\theta$, $x \geq x_\gamma$, which implies that for $0 < \theta \leq 1/2$ inequality (15) and hence inequality (12) also hold with $\gamma = 1$.

Definition 3.1. Let $A$ and $B$ be compact linear operators in a Hilbert space $H$.

If $c_1, c_2 > 0$, $0 < \alpha < \beta$, and condition (2) holds, we say that in the representation $C = A + B$ the operator $A$ is a leading operator and the operator $B$ is a non-leading operator with the parameters $c_1, c_2 > 0$, $\alpha$, and $\beta$.

If $0 < \alpha < \beta$ and there exist $c_1, c_2 > 0$ such that condition (3) holds, we say that in the representation $C = A + B$ the operator $A$ is a leading operator and the operator $B$ is a non-leading operator with the parameters $\alpha$ and $\beta$.

If there exist $c_1, c_2 > 0$ and $0 < \alpha < \beta$ such that condition (4) holds, we say that in the representation $C = A + B$ the operator $A$ is a leading operator and the operator $B$ is a non-leading operator.

Theorem 3.1. Let $A$ and $B$ be compact linear operators in a Hilbert space $H$. If in the representation $C = A + B$ the operator $A$ is a leading operator and the operator $B$ is a non-leading operator and for all $0 < \sigma < 1$
\begin{equation}
(16) \quad \lim_{k \to \infty} \frac{s_{k + [\sigma k]}(A)}{s_k(A)} = 1,
\end{equation}
then
\begin{equation}
(17) \quad \lim_{k \to \infty} \frac{s_k(A + B)}{s_k(A)} = 1.
\end{equation}
Proof. Let \( A \) be a leading operator and \( B \) a non-leading operator with the parameters \( \alpha, \beta \) and in Lemma 3.2 let \( \gamma = 2 \) and \( \theta = \frac{1}{2} \left( \frac{2}{3} + 1 \right) \). Moreover, let \( \sigma = \frac{1}{2}(\theta + 1) \).

Given that \( k \in \mathbb{N} \), let \( m = m - [k^\theta] \). By the proof of Lemma 3.2 (see (14) and (15)) \( k \leq m + 1 + [2(m + 1)^\theta] \) for all \( k \geq (1 - 2^{-1/\theta})^{1/(\theta - 1)} \).

Hence there exists \( \varkappa \in \mathbb{N} \), depending only on \( \alpha \) and \( \beta \) such that for all \( k \geq \varkappa \)
\[ k + [2k^\theta] \leq k + [k^\sigma], \quad k \leq m + [m^\sigma]. \]

Therefore
\[
\frac{s_{k+[k^\sigma]}(A)}{s_k(A)} \leq \frac{s_{k+[2k^\sigma]}(A)}{s_k(A)} \leq 1 \leq \frac{s_{k-[k^\sigma]}(A)}{s_k(A)} = \frac{s_m(A)}{s_k(A)} \leq \frac{s_{m+[m^\sigma]}(A)}{s_m(A)}.
\]

Consequently
\[
\lim_{k \to \infty} \frac{s_{k+[2k^\sigma]}(A)}{s_k(A)} = \lim_{k \to \infty} \frac{s_{k-[k^\sigma]}(A)}{s_k(A)} = 1,
\]
and the statement follows by inequality (12). \( \square \)

Remark 3.3. Assume that, in addition to assumption (9)
\begin{equation}
\tag{18}
\lim_{k \to \infty} k^\alpha s_k(A) = a > 0
\end{equation}
or, more generally,
\begin{equation}
\tag{19}
\lim_{k \to \infty} k^\alpha L(k)s_k(A) = a > 0,
\end{equation}
where \( L \) is an arbitrary positive function on \([1, \infty)\) for which for any \( 0 < b < \infty \)
\[
\lim_{k \to \infty} \frac{L(bk)}{L(k)} = 1.
\]

Then the statement of Theorem 3.1 follows from Theorem 2.3 in [8], Chapter 2, according to which if (18) or (19) is satisfied and
\[
\lim_{k \to \infty} \frac{s_k(B)}{s_k(A)} = 0,
\]
then equality (17) holds.

Corollary 3.1. Let \( \mathcal{A}, \mathcal{B} \) be families of compact linear operators \( A, B \) respectively, in a Hilbert space \( H \). If there exist \( c_1, c_2 > 0 \) and \( 0 < \alpha < \beta \) such that in the representations \( C = A + B \) for all \( A \in \mathcal{A}, B \in \mathcal{B} \) the operators \( A \) are leading operators and the operators \( B \) are non-leading operators with the parameters \( c_1, c_2 > 0 \) and \( 0 < \alpha < \beta \) and, for all \( 0 < \sigma < 1 \), in (16) convergence is uniform with respect to \( A \in \mathcal{A}, B \in \mathcal{B} \).

Proof. This proof follows from the proof of Theorem 3.1 and from Lemma 3.2 because the parameter \( \varkappa \) in the above proof depends only on \( \alpha \) and \( \beta \), and the parameter \( c_4 \) in inequality (10) depends only on \( c_1, c_2 > 0 \), \( \alpha \), and \( \beta \). \( \square \)

§ 4. Some estimates for singular numbers of elliptic operators

Let \( l \in \mathbb{N} \) and for \( u \in C^\infty(\mathbb{R}^n) \)
\begin{equation}
\tag{20}
(\mathcal{L}u)(x) = \sum_{|\alpha|,|\beta| \leq l} (-1)^{|\alpha|+|\beta|} D^\alpha(A_{\alpha\beta}(x)D^\beta u), \quad x \in \mathbb{R}^n,
\end{equation}
where \( A_{\alpha\beta} \in C^l(\mathbb{R}^n) \) are real-valued functions for all multi-indices \( \alpha, \beta \) satisfying \(|\alpha|,|\beta| \leq l \). We assume the uniform ellipticity condition is satisfied, i.e. for some \( \nu > 0 \)
\begin{equation}
\tag{21}
\sum_{|\alpha| = |\beta| = l} A_{\alpha\beta}(x)\xi^\alpha \xi^\beta \geq \nu |\xi|^{2l}
\end{equation}
for all \( x, \xi \in \mathbb{R}^n \).
We shall consider the closures of \( \mathcal{L} \) in \( L_2(\Omega) \) for bounded domains \( \Omega \subseteq \mathbb{R}^n \) which will be denoted by \( L_\Omega \).

**Lemma 4.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded domain.

1. The singular numbers \( s_k(A) \) of any correct restriction \( A \) of the operator \( L_\Omega \) defined above whose inverse \( A^{-1} \) is compact, satisfy the following inequality: there exists \( c_1 > 0 \) such that for all \( k \in \mathbb{N} \)

\[
s_k(A) \leq c_1 k^{2l/n}.
\]

2. There exists a correct restriction \( A \) of the operator \( L_\Omega \), \( k_0 \in \mathbb{N} \), and \( c_2, c_3 > 0 \) such that for all \( k \in \mathbb{N} \), \( k \geq k_0 \)

\[
c_2 k^{2l/n} \leq s_k(A) \leq c_3 k^{2l/n}.
\]

**Proof.**
1. Let \( B_1 \subseteq \Omega \) be an open ball. Moreover, let \( P_1 : D(P_1) \to L_2(\Omega) \) be the operator generated by \( L_\Omega \) with the homogeneous Dirichlet boundary conditions, i.e.

\[
D(P_1) = \left\{ u \in W^2_2(B_1) : \text{tr}_{\partial B_1} \left( \frac{\partial^m f}{\partial \nu^m} \right) = 0, \ m = 0, 1, \ldots, l-1 \right\}
\]

and \( P_1 u = L_\Omega u \) for each \( u \in D(P_1) \). It is well known that the operator \( P_1 \) has discrete spectrum and its singular numbers satisfy the following inequality: there exist \( k_0 \in \mathbb{N} \) and \( c_4, c_5 > 0 \) such that for all \( k \in \mathbb{N} \), \( k \geq k_0 \)

\[
c_4 k^{2l/n} \leq s_k(P_1) \leq c_5 k^{2l/n}.
\]

2. Let \( T : W^2_2(B_1) \to W^2_2(\mathbb{R}^n) \) be a bounded linear extension operator. Then for any correct restriction \( A \) of \( L_\Omega \) and for any \( u \in D(P_1) \)

\[
\| ATu \|_{L_2(\Omega)} \leq c_6 \| T u \|_{W^2_2(B_1)} \leq c_6 c_7 \| u \|_{W^2_2(B_1)} \leq c_6 c_7 c_8 \| P_1 u \|_{L_2(B_1)}.
\]

Here

\[
c_6 = \bar{c}_6 \max_{|\alpha|, |\beta| \leq l} \max_{|\gamma| \leq 1} \sup_{x \in \Omega} |(D^\gamma A_{\alpha\beta})(x)|,
\]

where \( \bar{c}_6 \) depends only on \( n \) and \( l \), \( c_7 \) is the norm of the extension operator \( T \), and \( c_8 \) is the norm of the inverse operator \( P_1^{−1} : L_2(B_1) \to W^2_2(B_1) \).

By the Min-Max principle it follows that for all \( k \in \mathbb{N} \)

\[
s_k(A) = \inf_{\substack{M \subseteq D(A) \ \text{dim} M = k}} \sup_{v \in M, u \approx 0} \frac{\| Av \|_{L_2(\Omega)}}{\| u \|_{L_2(\Omega)}} \leq \inf_{\substack{M \subseteq D(P_1) \ \text{dim} M = k}} \sup_{u \in M, u \approx 0} \frac{\| ATu \|_{L_2(\Omega)}}{\| T u \|_{L_2(\Omega)}}
\]

\[
\leq c_6 c_7 c_8 \inf_{\substack{M \subseteq D(P_1) \ \text{dim} M = k}} \sup_{u \in M, u \approx 0} \frac{\| P_1 u \|_{L_2(B_1)}}{\| u \|_{L_2(B_1)}} = c_6 c_7 c_8 s_k(P_1) \leq c_5 c_6 c_7 c_8 k^{2l/n}.
\]

3. Let \( B_2 \) be an open ball such that \( \Omega \subseteq B_2 \) and let \( P_2 \) be the operator similar to \( P_1 \) but defined on \( B_2 \). Moreover, let \( T_0 : L_2(\Omega) \to L_2(B_2) \) be the extension-by-zero operator and \( R_\Omega : L_2(B_2) \to L_2(\Omega) \) be the restriction operator. We define

\[
(25) \quad B = R_\Omega P_2^{-1} T_0.
\]

Then \( B \) is a bounded linear operator for which

\[
L_\Omega B = L_\Omega R_\Omega P_2^{-1} T_0 = R_\Omega P_2^{-1} T_0 = R_\Omega T_0 = I
\]

\[\quad \]

---

1. As usual it is assumed that all singular numbers of \( A \) are written in the form of a sequence \( \{s_k(A)\}_{k=1}^\infty \), where \( s_1(A) \leq s_2(A) \leq \cdots \leq s_k(A) \leq \cdots \) and each singular number is repeated as many times as its multiplicity.

2. Here and in similar situations in the sequel we assume, without loss of generality, that \( 0 \) does not belong to the spectrum of \( P_1 \). Otherwise, we may consider the operator \( P_1 + \mu I \) with the appropriately chosen \( \mu \in \mathbb{R} \).
and
\[ R(B) \subseteq R_\Omega(D(P_2)) \subseteq W^{2l}_2(\Omega) \subseteq D(L_\Omega). \]

By Remark 2.1, \( B \) is the inverse of a certain correct restriction \( A \) of the operator \( L_\Omega \). So
\[(26) \quad A^{-1} = R_\Omega P_2^{-1} T_0. \]

This implies that there exist \( c_9, c_{10} > 0 \) such that \( c_9 s_k(P_2) \leq s_k(A) \leq c_{10} s_k(P_2) \) for all \( k \in \mathbb{N} \). Hence, by (24), where the operator \( P_1 \) and the ball \( B_1 \) are replaced by \( P_2 \) and \( B_2 \), the statement follows.

\[ \square \]

Remark 4.1. Let \( \Omega \) contain an open ball \( B(x, r) \) of radius \( r > 0 \) for some \( x \in \Omega \) and be contained in the ball \( B(0, R) \). Moreover, let
\[ \tau = \max_{|\alpha|,|\beta| \leq l} \max_{|\gamma| \leq l} \sup_{x \in B(0, R)} |(D^\gamma A_{\alpha,\beta})(x)|. \]

Then the above proof implies that it may be assumed that in inequality (22) \( c_1 \) depends only on \( n, l, r, R, \tau \), and the ellipticity constant \( \nu \).

\section{§ 5. Classes of domains}

For any set \( V \) in \( \mathbb{R}^n \) and \( \delta > 0 \) we denote by \( V_\delta \) the set \( \{ x \in V : d(x, \partial V) > \delta \} \) and by \( V^\delta \) the \( \delta \)-neighbourhood of \( V \).

Let \( n \in \mathbb{N}, n \geq 2, \rho > 0, s, s' \in \mathbb{N}, s' \leq s \) and \( \{V_j\}_{j=1}^s \) be a family of bounded open cuboids and \( \{r_j\}_{j=1}^s \) be a family of rotations in \( \mathbb{R}^n \).

We say that that \( A = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s) \) is an atlas in \( \mathbb{R}^n \) with the parameters \( \rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s \), briefly an atlas in \( \mathbb{R}^n \).

Let \( l \in \mathbb{N} \) and \( M > 0 \). We denote by \( C_M^l(\mathcal{A}) \) the family of all bounded domains \( \Omega \) in \( \mathbb{R}^n \) satisfying the following properties:

(i) \( \Omega \subseteq \bigcup_{j=1}^s (V_j)_\rho \) and \( (V_j)_\rho \cap \Omega \neq \emptyset \);

(ii) \( V_j \cap \partial\Omega \neq \emptyset \) for \( j = 1, \ldots, s' \), \( V_j \cap \partial\Omega = \emptyset \) for \( s' < j \leq s \);

(iii) for \( j = 1, \ldots, s' \)
\[ r_j(V_j) = \{ x \in \mathbb{R}^n : a_{ij} < x_i < b_{ij}, i = 1, \ldots, n \} \]

and
\[ r_j(\Omega \cap V_j) = \{ x \in \mathbb{R}^n : a_{nj} < x_n < g_j(\bar{x}), \bar{x} \in W_j \}, \]
where \( \bar{x} = (x_1, \ldots, x_{n-1}) \), \( W_j = \{ \bar{x} \in \mathbb{R}^{n-1} : a_{ij} < x_i < b_{ij}, i = 1, \ldots, n-1 \} \) and \( g_j \in C^l(W_j) \) (it is meant that if \( s' < j \leq s \), then \( g_j(\bar{x}) = b_{nj} \) for all \( \bar{x} \in W_j \)); moreover, for \( j = 1, \ldots, s' \)
\[ a_{nj} + \rho \leq g_j(\bar{x}) \leq b_{nj} - \rho \]

and
\[ |D^\alpha g_j(\bar{x})| \leq M, \quad 1 \leq |\alpha| \leq l, \]

for all \( \bar{x} \in \bar{W}_j \).

We say that a domain \( \Omega \) in \( \mathbb{R}^n \) is of class \( C^l \) (or with boundary of class \( C^l \)) if \( \Omega \in C_M^l(\mathcal{A}) \) for some atlas \( \mathcal{A} \) and \( M > 0 \).
§ 6. Representation of the inverse of a correct restriction as the sum of a leading and a non-leading operator

In this section we prove the core result of the paper.

**Theorem 6.1.** Let \( l, n \in \mathbb{N} \), \( n \geq 2 \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with the boundary \( \partial \Omega \) of class \( C^{2l} \).

Let \( A \) and \( B \) be correct restrictions of the operator \( L_\Omega \), defined in Section 4, such that

\[
\begin{align*}
D(A) \subseteq W^{2l}_2(\Omega), \\
D(B) \subseteq W^s_2(\Omega),
\end{align*}
\]

where

\[
2l\left(1 - \frac{1}{n}\right) < s \leq 2l,
\]

and the operators

\[
\begin{align*}
A^{-1} : L_2(\Omega) &\rightarrow W^{2l}_2(\Omega), \\
B^{-1} : L_2(\Omega) &\rightarrow W^s_2(\Omega)
\end{align*}
\]

are bounded.

Then in the representation

\[
B^{-1} = A^{-1} + K,
\]

the operator \( A^{-1} \) is a leading operator and the operator \( K = B^{-1} - A^{-1} \) is a non-leading operator with the parameters \( \alpha = 2l/n \) and any \( 2l/n < \beta < s/(n-1) \).

**Proof.** 1. We start by stating several corollaries of the general results on the solvability of the boundary value problems proved in [10]. Here we use the definitions of Sobolev spaces \( W^{r,2}_2(\partial \Omega) \) for arbitrary real \( r \) given in that book.

Let

\[
B_m u = \text{tr}_{\partial \Omega} \left( \frac{\partial^m u}{\partial \nu^m} \right), \quad m = 0, 1, \ldots, l - 1
\]

and \( 0 < \sigma \leq 2l \). Then the following Dirichlet problem

\[
\begin{align*}
\begin{cases}
u \in W^{\sigma}_2(\Omega), \\
L_\Omega u = 0 \text{ on } \Omega, \\
B_m u = h_m \text{ on } \partial \Omega, \quad m = 0, 1, \ldots, l - 1, \\
h_m \in W^{\sigma-m-1/2}_2(\partial \Omega)
\end{cases}
\end{align*}
\]

has a unique solution\(^3\). Moreover, there exists \( c_1 > 0 \) such that

\[
\|u\|_{W^{\sigma}_2(\Omega)} \leq c_1 \sum_{m=0}^{l-1} \|h_m\|_{W^{\sigma-m-1/2}_2(\partial \Omega)}
\]

for all collections of functions \( h_m \in W^{\sigma-m-1/2}_2(\partial \Omega), m = 0, 1, \ldots, l - 1 \). Note that, for sufficiently small positive \( \sigma, \sigma - m - 1/2 < 0 \).

---

\(^3\) For \( s > 0, s \notin \mathbb{N} \), \( W^{\sigma}_2(\Omega) \) is the Sobolev space of fractional order with the norm

\[
\|u\|_{W^{\sigma}_2(\Omega)} = \left( \sum_{|\alpha| \leq |\sigma|} \int_\Omega |D^\alpha u(x)|^2 \, dx + \sum_{|\alpha| = |\sigma|} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2(s-|\sigma|)}} \, dx \, dy \right)^{1/2},
\]

where \( D^\alpha u \) denotes the weak derivative of \( u \) of order \( \alpha \) on \( \Omega \).

\(^4\) We assume that 0 does not belong to the spectrum of the corresponding Dirichlet operator. See footnote 2.
Assume that $0 < \mu < s$. We shall use the above statement for $\sigma = s$ and $\sigma = \mu$. If $\sigma = \mu$ this implies, in particular, that for each $j = 0, 1, \ldots, l - 1$ the Dirichlet problem
\[
\begin{cases}
u \in W_2^s(\Omega), \\
L_\Omega u = 0 \text{ on } \Omega, \\
B_m u = 0 \text{ on } \partial \Omega, \quad m = 0, 1, \ldots, j - 1, j + 1, \ldots, l - 1, \\
B_j u = h \text{ on } \partial \Omega, \\
h \in W_2^{s-j-1/2}(\partial \Omega)
\end{cases}
\] (35)

has a unique solution and the corresponding inverse operator
\[
R_j : W_2^{s-j-1/2}(\partial \Omega) \to W_2^\mu(\Omega)
\]
is bounded.

Note that the solution of the problem (33) with $\sigma = \mu$ can be represented in the form
\[
u = \sum_{j=0}^{l-1} R_j h_j.
\]

Since for each function $f \in L_2(\Omega)$ by (30) and (31) we have $K f = B^{-1} f - A^{-1} f \in W_2^s(\Omega)$, $L_\Omega K f = 0$ on $\Omega$ and by the trace theorem for Sobolev spaces (note that by (29) $s > l - 1/2$)
\[
B_m K f \in W_2^{s-m-1/2}(\partial \Omega) \subseteq W_2^{\mu-m-1/2}(\partial \Omega), \quad m = 0, 1, \ldots, l - 1,
\]
is implied that
\[
K f = \sum_{j=0}^{l-1} R_j B_j K f, \quad f \in L_2(\Omega).
\] (36)

2. We shall make use of the following fact: for a compact linear operator $G : H_1 \to H_2$ acting from a Hilbert space $H_1$ to a Hilbert space $H_2$,
\[
s_1(G) = \|G\|; \quad s_k(G) = \min_{\dim F=k-1} \|A - F\|, \quad k \geq 2,
\]
where the minimum is taken with respect to all linear operators $F : H_1 \to H_2$ of dimension $k - 1$. (See [1] and [8, Theorem 2.1].)

3. Consider the embedding operator
\[
E_1 : W_2^\mu(\Omega) \to L_2(\Omega).
\]
This operator is compact and its singular numbers $s_k(E_1)$ have the order $k^{-\mu/n}$. Hence, there exist linear operators $M_k : W_2^\mu(\Omega) \to L_2(\Omega)$ of dimension $k$, $k \in \mathbb{N}$, such that
\[
\|(E_1 - M_k)g\|_{L_2(\Omega)} = s_{k+1}(E_1) \leq c_2 (k + 1)^{-\mu/n} \|g\|_{W_2^\mu(\Omega)},
\] (37)

where $c_2 > 0$ is independent of $k$ and $g \in W_2^\mu(\Omega)$.

Consider also the embedding operators
\[
E_{2j} : W_2^{s-j-1/2}(\partial \Omega) \to W_2^{\mu-j-1/2}(\partial \Omega), \quad j = 0, 1, \ldots, l - 1.
\]
These operators are compact and their singular numbers $s_k(E_{2j})$ have the same order $k^{-(s-\mu)/(n-1)}$. (See [12].) Hence, there exist linear operators
\[
F_{kj} : W_2^{s-j-1/2}(\partial \Omega) \to W_2^{\mu-j-1/2}(\partial \Omega)
\]
of dimension $k$, $k \in \mathbb{N}$, such that
\[
\|(E_{2j} - F_{kj})g\|_{W_2^{s-j-1/2}(\partial \Omega)} = s_{k+1}(E_{2j}) \leq c_3 (k + 1)^{-(s-\mu)/(n-1)} \|g\|_{W_2^{\mu-j-1/2}(\partial \Omega)},
\] (38)
where $c_3 > 0$ is independent of $j, k$ and $g \in W^{\mu-j-1/2}_2(\partial \Omega)$.

4. Denote by $T_k$, $k \in \mathbb{N}$, the operators defined by

$$T_k f = K f - \sum_{j=0}^{l-1} (E_1 - M_k) R_j (E_{2j} - F_{kj}) B_j K f, \quad f \in L^2(\Omega).$$

By (36)

$$\sum_{j=0}^{l-1} E_1 R_j E_{2j} B_j K f = \sum_{j=0}^{l-1} R_j B_j K f = K f.$$ 

Therefore

$$T_k f = \sum_{j=0}^{l-1} E_1 R_j F_{kj} B_j K f + M_k \left( \sum_{j=0}^{l-1} R_j (E_{2j} - F_{kj}) B_j K f \right).$$

Hence, $T_k : L^2(\Omega) \to L^2(\Omega)$ are linear operators of dimension not exceeding $(l + 1)k$.

Moreover, by (37)

$$\| (K - T_k) f \|_{L^2(\Omega)} = \left\| \sum_{j=0}^{l-1} (E_1 - M_k) R_j (E_{2j} - F_{kj}) B_j K f \right\|_{L^2(\Omega)}$$

$$\leq \sum_{j=0}^{l-1} \left\| (E_1 - M_k) R_j (E_{2j} - F_{kj}) B_j K f \right\|_{L^2(\Omega)}$$

$$\leq c_2 (k + 1)^{-\mu/n} \sum_{j=0}^{l-1} \left\| R_j (E_{2j} - F_{kj}) B_j K f \right\|_{W^{\mu-j-1}_2(\Omega)}$$

$$\leq c_2 c_4 (k + 1)^{-\mu/n} \sum_{j=0}^{l-1} \left\| (E_{2j} - F_{kj}) B_j K f \right\|_{W^{\mu-j-1/2}_2(\partial \Omega)},$$

where

$$c_4 = \max_{j=0, \ldots, l-1} \left\| R_j \right\|_{W^{\mu-j-1/2}_2(\partial \Omega) \to W^\mu_2(\Omega)}.$$

Next, by (38) and the trace theorem for Sobolev spaces

$$\| (K - T_k) f \|_{L^2(\Omega)} \leq c_2 c_3 c_4 (k + 1)^{-\mu/n - (s-\mu)/(n-1)} \sum_{j=0}^{l-1} \| B_j K f \|_{W^{\mu-j-1/2}_2(\partial \Omega)}$$

$$\leq c_2 c_3 c_4 c_5 (k + 1)^{-\frac{1}{n-1}(s-\mu/n)} \| K f \|_{W^\mu_2(\Omega)} \leq c_2 c_3 c_4 c_5 c_6 (k + 1)^{-\frac{1}{n-1}(s-\mu/n)} \| f \|_{L^2(\Omega)},$$

where

$$c_5 = \sum_{j=0}^{l-1} \| B_j \|_{W^\mu_2(\Omega) \to W^{\mu-j-1/2}_2(\partial \Omega)},$$

$$c_6 = \| K \|_{L^2(\Omega) \to W^\mu_2(\Omega)} \leq \| B^{-1} \|_{L^2(\partial \Omega) \to W^\mu_2(\Omega)} + \| A^{-1} \|_{L^2(\Omega) \to W^\mu_2(\Omega)}$$

$$\leq \| B^{-1} \|_{L^2(\partial \Omega) \to W^\mu_2(\Omega)} + c_7 \| A^{-1} \|_{L^2(\Omega) \to W^\mu_2(\Omega)}$$

and $c_7$ is the norm of the embedding operator $E : W^{2l}_2(\Omega) \to W^\mu_2(\Omega)$.

5. Given $\frac{2l}{n} < \beta < \frac{s}{n-1}$, we choose $0 < \mu < s$ such that $\frac{1}{n-1} \left( s - \frac{\mu}{n} \right) = \beta$. This is possible if and only if $s > 2l \left( 1 - \frac{1}{n} \right)$. So for all $\frac{2l}{n} < \beta < \frac{s}{n-1}$

$$\| (K - T_k) f \|_{L^2(\Omega)} \leq c_8 (k + 1)^{-\beta} \| f \|_{L^2(\Omega)},$$
where
\[ c_8 = c_2c_3c_4c_5\left(\|B^{-1}\|_{L_2(\Omega)\rightarrow W_2^1(\Omega)} + c_7\|A^{-1}\|_{L_2(\Omega)\rightarrow W_2^1(\Omega)}\right) \]
is independent of \( k \in \mathbb{N} \) and \( f \in L_2(\Omega) \).

Since \( \dim T_{\frac{k-1}{k+1}} \leq k-1 \) we have
\[ s_k(K) \leq \|K - T_{\frac{k-1}{k+1}}\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq c_8\left(\frac{k-1}{l+1} + 1\right)^{-\beta} \leq c_8(k+1)^{-\beta}(k-1)^{-\beta} \leq c_8(2l+1)^{-\beta}k^{-\beta} \]
for all \( k \geq 2 \). This inequality also holds for \( k = 1 \) because without loss of generality one may assume that \( c_2, c_3, c_4, c_5 \geq 1 \), in which case \( s_1(K) = \|K\| \leq c_8 \).

Finally, by Lemma 4.1 it follows that for all \( k \in \mathbb{N} \)
\[ s_k(A^{-1}) \geq c_7^{-1}k^{-2l/n}, \]
where \( c_1 \) is from inequality (22), which implies the statement. \( \square \)

**Analysis of the proof.** Estimate (37) with \( \mu = s \) implies immediately that for \( k \geq 2 \)
\[ s_k(K) \leq \|K - M_{k-1}K\|_{L_2(\Omega)\rightarrow L_2(\Omega)} = (E_1 - M_{k-1})K\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq c_2\|K\|_{W_2^1(\Omega)\rightarrow L_2(\Omega)}k^{-s/n}. \]
This is a rather rough estimate and combined with estimate (12) it does not allow us to claim that \( K \) is a non-leading operator. However, this estimate holds for any bounded linear operator \( K : W^s_2(\Omega) \rightarrow L_2(\Omega) \) whilst the operator \( K = B^{-1} - A^{-1} \) under consideration apart from that is such that \( L_1K = 0 \). This essential fact enables us to improve significantly estimate (41). In the above proof this is done by considering \( Kf \) as a solution of the Dirichlet boundary value problem for the operator \( L_1 \) with the boundary data \( B_mKf \in W^{2m-1/2}_2(\partial\Omega) \), \( m = 0, \ldots, l-1 \), where \( s - m - 1/2 > 0 \), using estimate (38) with any \( 0 < \mu < s \) and deep results from [10] on solvability of this Dirichlet problem for the case in which the boundary data belongs to Sobolev spaces of negative order.

**Example 6.1.** In Example 2.2 let \( \varphi_1 \in W^s_2(B) \) where \( 2 - 2/n < s \leq 2 \). Consider the operator \( L_s : D(L_s) \rightarrow L_2(B) \) where
\[ D(L_s) = \left\{ u \in L_2(B) : -\Delta_w u \in L_2(B), \operatorname{tr}_{\partial B} \left( u(x) + \varphi_1(x) \int_B \varphi_2(y) \Delta_w u(y) dy \right) = 0 \right\} \]
and \( L_su = -\Delta_w u \) for \( u \in D(L_s) \). By Examples 2.1 and 2.2 \( L_s \) is a correct restriction of the operator \( L \). Moreover, the operator \( L^{-1}_s : L_2(B) \rightarrow W^s_2(B) \) is bounded. Therefore by Theorem 6.1 the singular numbers \( s_k(L_s) \) have the same asymptotics as those of the Dirichlet Laplacian:
\[ \lim_{k \to \infty} s_k(L_s)k^{-2/n} = \lim_{k \to \infty} s_k(-\Delta_D)k^{-2/n}. \]

**Corollary 6.1.** Let \( l, n \in \mathbb{N}, n \geq 2, 2l\left(1 - \frac{1}{n}\right) < s \leq 2l, \alpha = \frac{2l}{n}, \) and \( \frac{2l}{n} < \beta < \frac{s}{n-1} \).

Moreover, let \( \mathcal{A} \) be a fixed atlas in \( \mathbb{R}^n \) and \( M > 0 \).

Assume that \( G, H \subseteq C^2_M(A) \) and \( \mathfrak{A}(A) = \{A_\Omega\}_{\Omega \in G}, \mathfrak{B}(A) = \{B_\Omega\}_{\Omega \in H} \) are families of correct restrictions \( A_\Omega \) and \( B_\Omega \) of the operator \( L_\Omega \) defined in Section 4, satisfying the conditions
\[ D(A_\Omega) \subseteq W^s_2(\Omega), \quad A_\Omega \in \mathfrak{A}(A), \]
\[ D(B_\Omega) \subseteq W^s_2(\Omega), \quad B_\Omega \in \mathfrak{B}(A), \]
Proof. Let the correct restriction $A_{\Omega}$ be any correct restriction of the operator $L_{\Omega}$ defined in Section 3, satisfying conditions (28) and (31).

Then there exists $b > 0$ such that for all correct restrictions $B$ of the operator $L_{\Omega}$, defined in Section 4, satisfying conditions (28) and (31),

(48) \[ \lim_{k \to \infty} s_k(B)k^{-2l/n} = b. \]

Proof. Let the correct restriction $A$ of the operator $L_{\Omega}$ be defined by

\[ D(A) = \left\{ u \in W^{2l}_2(\Omega) : \text{tr}_{\partial\Omega} \left( \frac{\partial^m u}{\partial n^m} \right) = 0, m = 0, 1, \ldots, l - 1 \right\} \]

and $Au = L_{\Omega}u$ for each $u \in D(A)$. Without loss of generality we assume that the operator $A$ has compact inverse $A^{-1}$ (see footnote 2). It is well known that its singular numbers $s_k(A^{-1})$ are such that

(49) \[ \lim_{k \to \infty} s_k(A^{-1})k^{2l/n} = a \]

for some $a > 0$.

Let $B$ be any correct restriction of the operator $L_{\Omega}$ satisfying conditions (28) and (31). By Theorem 6.1 $B^{-1} = A^{-1} + K$, where $A^{-1}$ is a leading operator and $K$ is a non-leading operator. Since condition (16) is satisfied by Theorem 3.1

(50) \[ \lim_{k \to \infty} \frac{s_k(B^{-1})}{s_k(A^{-1})} = 1. \]

Consequently

(51) \[ \lim_{k \to \infty} s_k(B^{-1})k^{2l/n} = \lim_{k \to \infty} s_k(A^{-1})k^{2l/n} = a \]
and

$$\lim_{k \to \infty} s_k(B)k^{-2l/n} = b \equiv \frac{1}{a}. \quad \square$$

**Corollary 7.1.** Let \( l, n \in \mathbb{N}, n \geq 2, 2l(1 - 1/n) < s \leq 2l, A \) be a fixed atlas in \( \mathbb{R}^n \), and \( M > 0 \). Assume that \( H \subseteq C^2_M(A) \) and \( \mathcal{B}(A) = \{B_\Omega\}_{\Omega \in H} \) is a family of correct restrictions \( B_\Omega \) of the operator \( L_\Omega \) defined in Section 4, satisfying conditions (28) and (47).

Then the expressions \( s_k(B_\Omega)k^{-2l/n} \) converge to \( b \) as \( k \to \infty \) uniformly with respect to \( B_\Omega \in \mathcal{B}(A) \).

**Proof.** Let the family \( \mathfrak{A}(A) \) consist of only one operator \( A \) defined above. Since condition (16) is satisfied the statement follows by Corollaries 3.1 and 6.1. \( \square \)

§ 8. Application to spectral stability estimates

In this section we restrict ourselves to considering the operator \( \mathcal{L} \) for the case in which \( l = 1 \), \( A_{\alpha\beta} = A_{\beta\alpha} \) and there are no lower terms. So for \( u \in C^\infty(\mathbb{R}^n) \)

$$\mathcal{L}u = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}), \quad x \in \mathbb{R}^n,$$

where \( a_{ij} \in C^1(\mathbb{R}^n) \) are real-valued functions satisfying \( a_{ij} = a_{ji} \) \( \forall i, j = 1, \ldots, n \), and

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$$

for all \( \xi, x \in \mathbb{R}^n \). As in previous sections, for bounded domains \( \Omega \subseteq \mathbb{R}^n \), \( L_\Omega \) denotes the closure of \( \mathcal{L} \) in \( L_2(\Omega) \).

**Theorem 8.1.** Let \( n \in \mathbb{N}, n \geq 2, 2 - 2/n < s \leq 2, A \) be a fixed atlas in \( \mathbb{R}^n \), and \( M > 0 \).

Moreover, let \( \mathcal{B}(A) = \{B_\Omega\}_{\Omega \in C^2_M(A)} \) be a family of correct restrictions \( B_\Omega \) of the operator \( L_\Omega \) defined above, satisfying conditions (28) and (47).

Then there exist \( \varepsilon_1, c_1 > 0 \) and for each \( \varepsilon \in (0, \varepsilon_1] \) there exists \( k(\varepsilon) \in \mathbb{N} \) such that for all \( k \geq k(\varepsilon) \)

$$|s_k(B_{\Omega_1}) - s_k(B_{\Omega_2})| \leq c_1 k^{2/n} \varepsilon \quad (49)$$

for all \( \Omega_1, \Omega_2 \in C^2_M(A) \) satisfying

$$\Omega_1 \varepsilon \subseteq \Omega_2 \subseteq (\Omega_1)^\varepsilon \quad \text{or} \quad (\Omega_2)\varepsilon \subseteq \Omega_1 \subseteq (\Omega_2)^\varepsilon. \quad (50)$$

**Proof.** 1. Let \( \mathfrak{A}(A) = \{A_\Omega\}_{\Omega \in C^2_M(A)} \) be the family of the correct restrictions \( A_\Omega \) of the operator \( L_\Omega \) defined by

$$D(A_\Omega) = \{u \in W_2(\Omega): \text{tr}_{\partial\Omega} u = 0\}$$

and \( A_\Omega u = L_\Omega u \) for each \( u \in D(A_\Omega) \). By Theorem 7.15 in [3] there exist \( \varepsilon_2, c_2 > 0 \), depending only on \( A \) and \( L \), such that the eigenvalues \( \lambda_k(A_\Omega) \) satisfy the following estimate: for all \( k \in \mathbb{N} \) for all \( \varepsilon \in (0, \varepsilon_2] \)

$$|\lambda_k(A_{\Omega_1}) - \lambda_k(A_{\Omega_2})| \leq c_2 \lambda_k(A_{\Omega_1}) \varepsilon \quad \text{for all} \quad \Omega_1, \Omega_2 \in C^2_M(A) \text{ satisfying condition (50).}$$

Note also that there exists \( c_3 > 0 \), depending only on \( n, l, A \) and \( L \), such that for all \( k \in \mathbb{N} \)

$$\lambda_k(A_\Omega) \leq c_3 k^{2/n} \quad \text{for all} \quad A_\Omega \in \mathfrak{A}(A). \quad (51)$$
This follows from Remark 4.1 since each $\Omega \in C^2_M(\mathcal{A})$ contains an open ball $B(x, \varrho/2)$ for some $x \in \Omega$ and is contained in the ball $B(0, R)$, where $R > 0$ depends only on $\{V_j\}_{j=1}^3$.

2. Next for all $k \in \mathbb{N}$ and for all $\Omega_1, \Omega_2 \in C^2_M(\mathcal{A})$

$$|s_k(B_{\Omega_1}) - s_k(B_{\Omega_2})| \leq \left|\frac{s_k(B_{\Omega_1})}{\lambda_k(A_{\Omega_1})} - 1\right| \lambda_k(A_{\Omega_1}) + c_2 \frac{\lambda_k(A_{\Omega_1})}{\lambda_k(A_{\Omega_1})} + \left|\frac{s_k(B_{\Omega_2})}{\lambda_k(A_{\Omega_2})} - 1\right| \lambda_k(A_{\Omega_2})$$

$$\leq c_6 \left(\left|\frac{s_k(B_{\Omega_1})}{s_k(A_{\Omega_1})} - 1\right| + \varepsilon + \left|\frac{s_k(B_{\Omega_2})}{s_k(A_{\Omega_2})} - 1\right|\right) k^{2/n},$$

where $c_6 = \max\{c_2, c_3\}$.

The statement follows since by Corollary 7.1

$$\frac{s_k(B_{\Omega_1})}{s_k(A_{\Omega_1})} \to k^{-2/n} \quad \text{and} \quad \frac{s_k(B_{\Omega_2})}{s_k(A_{\Omega_2})} \to k^{-2/n}$$

converge as $k \to \infty$ to 1 uniformly with respect to $A_{\Omega_1}, A_{\Omega_2} \in \mathfrak{A}(\mathcal{A})$ and $B_{\Omega_1}, B_{\Omega_2} \in \mathfrak{B}(\mathcal{A})$. □

**Concluding remarks.** A natural question arises related to the statements of Theorems 6.1 and 7.1. Is the assumption $s > 2(1 - 1/n)$ sharp? We conjecture that it is.

In this paper we are dealing with linear operators in Hilbert spaces. It is of interest to extend the results of the paper to the non-Hilbert case and to some classes of nonlinear operators.

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