PROPERTIES OF SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS ARISING IN HEAT AND MASS TRANSFER THEORY

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Abstract. The aim of the present paper is to study the asymptotic behavior of solutions of integro-differential equations on the basis of spectral analysis of their symbols. To this end, we obtain representations of strong solutions of these equations in the form of a sum of terms corresponding to the real and nonreal parts of the spectrum of the operator functions that are the symbols of these equations. These representations are new for the class of integro-differential equations considered in the paper.

1. Introduction

The classical heat and mass transfer theory describes heat and mass transfer by differential equations of the parabolic type. These equations are derived from the mass and energy conservation laws on the basis of the Fourier and Fick hypotheses on the relationship between the heat and mass fluxes and the temperature and concentration gradients. These hypotheses have the form

\[ \vec{q}(\vec{x}, t) = -\lambda \cdot \text{grad} T(\vec{x}, t), \]

where \( \vec{q} \) is the specific heat or mass flux, \( \text{grad} T \) is the temperature or concentration gradient, \( \lambda \) is the thermal conductivity or the diffusion coefficient, \( \vec{x} \) is the vector of space variables, and \( t \) is time. Thus, one can obtain the classical linear heat equation in homogeneous isotropic media,

\[ \frac{\partial T(x, t)}{\partial t} = a^2 \Delta T(x, t), \]

where \( a^2 = \lambda \). However, Eq. (2) has the serious disadvantage that a change in temperature at some point of the body results in an instantaneous change in temperature at the other points; i.e., the propagation velocity of a heat perturbation is infinite.

The papers [19] and [29] laid the foundations of a general heat and mass transfer theory for materials with variable memory assuming a finite perturbation propagation velocity. In these papers, the specific heat flux is defined by the formula

\[ \vec{q}(\vec{x}, t) = -\int_0^\infty k(s) \cdot \text{grad} T(\vec{x}, t - s) \, ds, \]

where \( k(s) \) is known as the heat flux relaxation function. In particular, if we interpret the integral in (3) as a Stieltjes integral with respect to the measure \( d(\lambda \theta(s)) \), where \( \theta(s) \)
is the Heaviside function, then we obtain the classical formula \[\hat{q}(\vec{x}, t) = -\int_{0}^{\infty} \text{grad} T(\vec{x}, t - s) d\lambda(s) = -\lambda \cdot \text{grad} T(\vec{x}, t)\].

From (3), we obtain
\[\hat{q}_t(\vec{x}, t) = -k(0) \cdot \text{grad} T(\vec{x}, t) - \int_{0}^{\infty} k'(s) \cdot \text{grad} T(\vec{x}, t - s) ds\].

The expression for the time derivative of the internal energy has the form
\[C\partial_t T(\vec{x}, t) + \beta(0) \cdot T(\vec{x}, t) + \int_{0}^{\infty} \beta'(s) \cdot T(\vec{x}, t - s) ds = \dot{Q}(t),\]

where \[\dot{Q}(t)\] is an external heat source. Equation (4) is referred to as the Gurtin–Pipkin equation in the foreign literature.

If we set \[\beta'(s) \equiv 0, k'(s) \equiv 0,\] and \[\dot{Q}(t) \equiv 0\] in Eq. (4), then we obtain the hyperbolic Cattaneo–Maxwell equation
\[C\frac{\partial^2 T(x, t)}{\partial t^2} + \beta(0) \frac{\partial T(x, t)}{\partial t} + \int_{0}^{\infty} \beta'(s) \frac{\partial T(x, t - s)}{\partial t} ds - k(0) \cdot \Delta T(\vec{x}, t) - \int_{0}^{\infty} k'(s) \Delta T(x, t - s) ds = \dot{Q}(x, t),\]

If we assume that \[\beta(s) \equiv 0,\] then Eq. (4) becomes
\[C\frac{\partial^2 T(x, t)}{\partial t^2} - k(0) \cdot \Delta T(\vec{x}, t) - \int_{0}^{\infty} k'(s) \cdot \Delta T(x, t - s) ds = \dot{Q}(x, t).\]

The resulting Eq. (6) also describes one-dimensional longitudinal vibrations of a viscoelastic rod with density \(C\) and stress relaxation function \(k(s)\).

By integrating Eq. (6) with respect to \(t\), we obtain the first-order integro-differential equation
\[C\frac{\partial T(x, t)}{\partial t} - \int_{0}^{\infty} k(s) \cdot \Delta T(x, t - s) ds = Q(x, t),\]

which is also called the Gurtin–Pipkin equation in the physical literature.

Our paper studies the asymptotic behavior of solutions of integro-differential equations of the form (7) on the basis of spectral analysis of the operator functions that are the symbols (characteristic quasipolynomials) of the corresponding integro-differential operators. In this connection, it is preferable to consider integro-differential equations with operator coefficients in a Hilbert space (abstract integro-differential equations), which can be realized as integro-differential equations with partial derivatives with respect to the space variables where necessary. The self-adjoint positive operator \(A\) occurring in what follows can be realized, say, as \(A^2 y = -y''(x),\) \(x \in (0, \pi),\) \(y(0) = y(\pi) = 0,\) or as \(A^2 y = -\Delta y\) with the Dirichlet conditions in a bounded domain, or as a more general self-adjoint elliptic operator in a bounded domain. The case of \(Ay = -y''(x),\) \(x \in (0, \pi),\) \(y(0) = y(\pi) = 0,\) or \(Ay = -\Delta y\) with the Dirichlet conditions in a bounded domain is possible as well. Furthermore, we obtain representations of solutions of these equations
as a sum of terms corresponding to the real and nonreal parts of the spectrum of the operator functions that are the symbols of these equations.

At present, there are quite a few papers dealing with various models of diffusion and heat propagation in media with memory. Traditionally, these models are based on the Cattaneo–Maxwell hyperbolic regularization of the classical heat equation (the Cattaneo–Maxwell equation \(5\)). The left-hand side of \(5\) is the linear part of the Maclaurin series expansion of the delay heat equation

\[
\frac{\partial T(x, t + \tau)}{\partial t} = \lambda \cdot \Delta T(x, t), \quad \tau > 0,
\]

which is also a special case of Eq. \(7\). In the closing part of the paper, we show the significant difference in the structure of the spectrum between the delay equation \(8\), its hyperbolic regularization \(5\), and the corresponding equation \(2\) that does not take the delay into account. We give an example of a partial delay differential equation whose spectrum contains a sequence of eigenvalues \(\lambda_n\) such that \(\text{Re} \lambda_n \to +\infty\) as \(n \to +\infty\). We say that such equations are unstable. In turn, note that the spectra of the symbols (characteristic operator functions) of hyperbolic and parabolic equations lie in the left complex half-plane \(\{\lambda: \text{Re} \lambda < \omega, \omega \in \mathbb{R}_+\}\). Consequently, hyperbolic and parabolic equations are stable in the sense indicated above. Thus, we show that the occurrence of a delay in Eq. \(2\) results in substantial changes in the behavior of the spectrum of an equation of the parabolic type. Examples of unstable functional-differential equations of the parabolic and hyperbolic types can be found in \([36]\).

At present, there are quite a few papers dealing with integro-differential equations in Banach (in particular, Hilbert) spaces whose leading part is an abstract parabolic or hyperbolic equation (see \([1\]–\([17]\) and the references therein). Much attention is also given to integro-differential equations (equations with partial derivatives as well as abstract equations with operator coefficients) arising in problems of hereditary mechanics and thermal physics. Here we mention only the monographs \([5\], \([7\], and \([35\) (see also the bibliographies therein).

A distinguishing feature of the present paper is that attention is mainly paid to spectral issues (analysis of the spectra of the symbols), while most of the papers known so far (see the monographs \([5\] and \([7\), the papers \([1\]–\([4\], \([11\]–\([13\], \([17\], \([20\], and \([21\], and also the bibliographies therein) deal with solvability issues.

2. Definitions, notation and statements of the main results

Let \(H\) be a separable Hilbert space, and let \(A\) be a self-adjoint positive operator on \(H\) with compact inverse. We make the domain \(\text{Dom}(A^\beta)\) of the operator \(A^\beta, \beta > 0\), a Hilbert space \(H^\beta\) by introducing the norm \(\|\cdot\|_\beta = \|A^\beta \cdot\|\), equivalent to the graph norm of \(A^\beta\), on \(\text{Dom}(A^\beta)\).

Let \(\{e_n\}_{n=1}^\infty\) be the orthonormal basis consisting of the eigenvectors of \(A\) corresponding to eigenvalues \(a_n, A e_n = a_n e_n, n \in \mathbb{N}\). The eigenvalues \(a_n\) are arranged in ascending order, \(0 < a_1 < a_2 < \cdots < a_n \ldots\), and \(a_n \to +\infty\) as \(n \to +\infty\).

In what follows, we use the notation

\[
u^{(n)}(t) := \frac{d^n u(t)}{dt^n}, \quad n \in \mathbb{N}.
\]

By \(W_{2,\gamma}^{n}(\mathbb{R}_+, A^n)\) we denote the Sobolev space of \(H\)-valued vector functions on the half-line \(\mathbb{R}_+ = (0, \infty)\) equipped with the norm

\[
\|u\|_{W_{2,\gamma}^{n}(\mathbb{R}_+, A^n)} \equiv \left(\int_0^\infty e^{-2\gamma t} \left(\|\nu^{(n)}(t)\|_H^2 + \|A^n u(t)\|_H^2\right) dt\right)^{1/2}, \quad \gamma \geq 0.
\]
For \( n = 0 \), we set \( W_{2,0}^0(\mathbb{R}_+, A^0) \equiv L_{2,\gamma}(\mathbb{R}_+, H) \), and for \( \gamma = 0 \) we write \( W_{2,0}^n = W_2^n \). More detail on the spaces \( W_{2,\gamma}(\mathbb{R}_+, A^n) \) can be found in [18, Chapter I].

Consider the following problem for a first-order integro-differential equation on the half-line \( \mathbb{R}_+ = (0, \infty) \):

\[
\begin{align*}
\frac{du(t)}{dt} + \int_0^t K(t - s) A^2 u(s) \, ds &= f(t), \quad t \in \mathbb{R}_+, \\
u(+0) &= \varphi.
\end{align*}
\]

We assume that the scalar function \( K(t) \) admits the representation

\[
K(t) = \int_0^\infty \frac{e^{-\tau}}{\tau} \, d\mu(\tau),
\]

where \( d\mu \) is a positive measure with the corresponding increasing right continuous distribution function \( \mu \). The integral is understood in the sense of Stieltjes. We assume that the following condition is satisfied:

\[
K(0) = \int_0^\infty \frac{d\mu(\tau)}{\tau} < \infty,
\]

where the support of \( \mu \) is contained in the half-line \((d_1, +\infty), d_1 > 0\). In a number of cases, we substantially use the condition

\[
-\mathcal{K}(1)(0) = \int_0^\infty d\mu(\tau) \equiv \text{Var} \, \mu |_0^\infty < +\infty.
\]

The symbol of Eq. (9) is the operator function

\[
\mathcal{L}(\lambda) = \lambda I + \hat{K}(\lambda) A^2,
\]

where

\[
\hat{K}(\lambda) = \int_0^\infty \frac{d\mu(\tau)}{\tau(\lambda + \tau)}
\]

is the Laplace transform of the kernel \( K(t) \).

First, let us present a result on the well-posed solvability of problem (9), (10).

### 2.1. Well-posed solvability.

**Definition 1.** A vector function \( u \) is called a **strong solution** of problem (9), (10) if it belongs to the space \( W_{2,\gamma}(\mathbb{R}_+, A^2) \) for some \( \gamma \geq 0 \), satisfies Eq. (9) almost everywhere on the half-line \( \mathbb{R}_+ \), and satisfies the initial condition (10).

**Theorem 1.** Assume that \( Af(1)(t) \in L_{2,\gamma}(\mathbb{R}_+, H) \) for some \( \gamma_1 \geq 0 \), \( f(0) = 0 \), and condition (12) is satisfied.

1. If condition (13) is satisfied and \( \varphi \in H_2 \), then for each \( \gamma > \gamma_1 \) problem (9), (10) is uniquely solvable in \( W_{2,\gamma}(\mathbb{R}_+, A^2) \), and the solution satisfies the estimate

\[
\|u\|_{W_{2,\gamma}(\mathbb{R}_+, A^2)} \leq d \left( \|Af(1)(t)\|_{L_{2,\gamma}(\mathbb{R}_+, H)} + \|A^2 \varphi\|_H \right)
\]

with a constant \( d \) independent of the vector function \( f \) and the vector \( \varphi \).

2. If condition (13) is not satisfied and \( \varphi \in H_3 \), then for each \( \gamma > \gamma_1 \) problem (9), (10) is uniquely solvable in \( W_{2,\gamma}(\mathbb{R}_+, A^2) \), and the solution satisfies the estimate

\[
\|u\|_{W_{2,\gamma}(\mathbb{R}_+, A^2)} \leq d \left( \|Af(1)(t)\|_{L_{2,\gamma}(\mathbb{R}_+, H)} + \|A^3 \varphi\|_H \right)
\]

with a constant \( d \) independent of the vector function \( f \) and the vector \( \varphi \).
2.2. Spectral analysis. The spectral analysis of the operator functions that are the symbols of the integro-differential equations in question was carried out on [14]–[16] and [23]–[27] for the case in which the kernel $K(t)$ can be represented as a linear combination or a series of decaying exponentials with positive coefficients. A number of results of the present paper were announced in [33]. In addition, the spectral analysis of these equations was carried out in [30] under different assumptions about the kernels $K(t)$. Controllability problems for the Gurtin–Pipkin equation were studied in [20, 21].

Theorem 2. Assume that conditions (12) and (13) are satisfied and the measure $d\mu$ is compactly supported. Then for sufficiently large $a_n$ the nonreal eigenvalues $\lambda_n^\pm (\lambda_n^- = \lambda_n^+)$ of the operator function $\mathcal{L}(\lambda)$ have the following asymptotics:

$$
\lambda_n^\pm = \pm i \sqrt{\mathcal{K}(0)} a_n + \frac{\mathcal{K}'(0)}{2\mathcal{K}(0)} + O\left(\frac{1}{a_n}\right), \quad a_n \to +\infty.
$$

Let us present a result on the localization of the spectrum of the operator function $\mathcal{L}(\lambda)$ for the case in which the measure $d\mu$ is compactly supported.

Theorem 3. Let the assumptions of Theorem 2 be satisfied, and let the measure $d\mu$ be compactly supported in an interval $[d_1, d_2]$, $0 < d_1 < d_2$. Then there exists a $y_0$, $0 < y_0 < d_1$, such that the spectrum of the operator function $\mathcal{L}(\lambda)$ can be represented in the form

$$
\sigma(\mathcal{L}) := \sigma_R \cup \sigma_I,
$$

where $\sigma_R$ and $\sigma_I$ are the real and the nonreal part of the spectrum of $\mathcal{L}(\lambda)$, respectively, $\sigma_R \subset [-d_2, -d_1 + y_0]$, and

$$
\sigma_I = \{ \lambda_n^\pm \in \mathbb{C} \setminus \mathbb{R}, \lambda_n^- = \overline{\lambda_n^+} \mid n \in \mathbb{N} \},
$$

where the $\lambda_n^\pm$ are the nonreal eigenvalues of $\mathcal{L}(\lambda)$, which have the asymptotic representation (16).

2.3. Representation of solutions. Let us obtain a representation of the solution of the problem (9), (10) on the basis of Theorem 3 on the structure of the spectrum of the operator function $\mathcal{L}(\lambda)$. Note that results on the representation of solutions for the case in which the kernel $\mathcal{K}(t)$ can be represented as the sum of a series of decaying exponentials with positive coefficients were obtained in [22], [33], and [34] and summarized in [35, Chapter 3]. On the complex plane, consider the clockwise contour $\Gamma = C_1 \cup \Gamma^- \cup C_2 \cup \Gamma^+$, where

$$
\Gamma^+ = \{ x + iy \in \mathbb{C} : -d_2 \leq x \leq -d_1, \quad y = y_0, \quad y_0 > 0 \},
$$

$$
\Gamma^- = \{ x + iy \in \mathbb{C} : -d_2 \leq x \leq -d_1, \quad y = -y_0, \quad y_0 > 0 \},
$$

$$
C_1 = \{ \lambda \in \mathbb{C} : \lambda = -d_1 + y_0 e^{i\varphi}, \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \},
$$

$$
C_2 = \{ \lambda \in \mathbb{C} : \lambda = -d_2 + y_0 e^{i\varphi}, \quad \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2} \}.
$$

Theorem 4. Let $f(t) \equiv 0$ in Eq. (9), and let the assumptions of Theorem 3 be satisfied. Then the strong solution of problem (9), (10) can be represented in the form

$$
u(t) = u_I(t) + u_R(t),$$
where
\[ u_I(t) = \sum_{n=1}^{\infty} \text{Re} \left[ \frac{\exp(\lambda_n^+ t)}{l_n^{(1)}(\lambda_n^+)} \right] \varphi_n e_n, \]
\[ u_R(t) = \frac{1}{2\pi i} \int_{\Gamma} \mathcal{L}^{-1} (\lambda) \varphi e^{\lambda t} d\lambda, \]
and the \( \lambda_n^+ \) are the nonreal eigenvalues of the operator function \( \mathcal{L}(\lambda) \), which have the asymptotic representation \( [16] \).

**Theorem 5.** Let the assumptions of Theorem 3 be satisfied. Then there exists a \( y_0 > 0 \), \( 0 < y_0 < d_1 \), such that the vector functions \( u_I(t) \) and \( u_R(t) \) satisfy the estimates
\[ \|u_I(t)\|_H \leq C_1 e^{\kappa t} \|\varphi\|_H, \quad \|u_R(t)\|_H \leq C_2 \eta(t) A^{-2} \|\varphi\|_H, \]
where \( \kappa = \sup_{n \in \mathbb{N}} \text{Re} \lambda_n^+ \), the \( C_j, j = 1, 2 \), are some positive constants, and
\[ \eta(t) = \frac{1}{2\pi} \left[ \frac{2r_0^2(y_0)}{y_0^2} \xi^2(t) + 2\pi^2 \mu^2(y_0)(e^{-2(d_1-y_0)t} + e^{-2(d_2-y_0)t}) \right]^{1/2}, \]
\[ \xi(t) = \frac{e^{d_2 t} - e^{d_1 t}}{t}, \]
\[ r_0(y_0) = \left( \int_{d_1}^{d_2} \frac{d\mu(\tau)}{\tau^2 + y_0^2} \right)^{-1}, \quad \mu(y_0) = \left( \int_{d_1}^{d_2} \frac{d\mu(\tau)}{\tau((d_2-d_1+y_0)^2+y_0^2)} \right)^{-1}. \]

**Theorem 6.** Let \( \varphi = 0 \) in problem [9], [10] and let the assumptions of Theorems 1 and 3 be satisfied. Then the strong solution of problem [9], [10] can be represented in the form
\[ (19) \quad u(t) = w_I(t) + w_R(t), \]
where
\[ w_I(t) = \sum_{n=1}^{\infty} \left( \int_0^t \left[ \frac{\exp(\lambda_n^+(t-\tau))}{l_n^{(1)}(\lambda_n^+)} + \frac{\exp(\lambda_n^-(t-\tau))}{l_n^{(1)}(\lambda_n^-)} \right] f_n(\tau) d\tau \right) e_n, \]
\[ w_R(t) = \frac{1}{2\pi i} \int_0^t \left( \int_{\Gamma} \mathcal{L}^{-1} (\lambda) e^{\lambda(t-\tau)} d\lambda \right) f(\tau) d\tau, \]
and the \( \lambda_n^\pm \) are the nonreal eigenvalues of the operator function \( \mathcal{L}(\lambda) \), which have the asymptotic representation \( [16] \).

Note that the series \( u_I(t) \) and \( w_I(t) \) corresponding to the nonreal part of the spectrum of the operator function \( \mathcal{L}(\lambda) \) in the representations (18) and (19) of the solutions are close in their structure and behavior to the Fourier series representation of the solution of the wave equation and have the wave type of behavior in this sense. In turn, the terms \( u_R(t) \) and \( w_R(t) \) corresponding to the real part of the spectrum of the operator function \( \mathcal{L}(\lambda) \) are in this sense close to the solution of the heat equation. It is noteworthy that the function \( u_R(t) \) is infinitely differentiable. Thus, according to their properties, the solutions of problem [9], [10] occupy an intermediate position between the solutions of the wave equation and the heat equation.

2.4. **Example.** On the positive half-line \( \mathbb{R}_+ = (0, \infty) \), consider the following initial value problem for a first-order integro-differential equation:
\[ (20) \quad \frac{du}{dt} + \int_{-\infty}^{t} A^2 u(t-s) d\sigma(s) = f(t), \quad t \in \mathbb{R}_+, \quad u(t) = 0, \quad t \in (-\infty, 0), \]
where \( d\sigma \) is a positive measure with nondecreasing right continuous distribution function \( \sigma(s) \). The integral is understood as a Stieltjes integral.

Consider the important special case of Eq. (20) in which the distribution function \( \sigma(t) \) can be represented in the form \( \sigma(t) = \theta(t-h) \), where \( h > 0 \) and \( \theta(t) \) is the Heaviside
function. In this model case, the integral term has the form $A^2 u(t - h)$. Thus, Eq. (20) is an equation with retarded argument. The operator function

$$\mathbb{L}(\lambda) = \lambda I + A^2 e^{-\lambda h}$$

is the symbol of this equation. (Under the assumption that the kernel $K(t)$ in Eq. (9) can be represented in the form $K(t) = \delta(t - h)$, where $\delta(t)$ is the Dirac function, we arrive at the same differential-difference equation.)

For the case of a finite-dimensional space $H$, equations of the form (20) were studied by numerous authors. However, the spectrum of the operator function $\mathbb{L}(\lambda)$ has apparently not been studied earlier for the case in which $H$ is infinite-dimensional and $A$ is an unbounded self-adjoint operator on $H$.

Set $h = 1$. If $a_n = n$, then the spectrum of $\mathbb{L}(\lambda)$ is the closure of the set of zeros of the functions $l_n(\lambda) = \lambda + n^2 e^{-\lambda}$, $n \in \mathbb{N}$; i.e.,

$$\sigma(\mathbb{L}) = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} \lambda_{nk}, \quad l_n(\lambda_{nk}) = 0,$$

and for given $n$ the sequence $\{\lambda_{nk}\}_{k \in \mathbb{Z}}$ has the asymptotics

$$\lambda_{nk} = \ln n^2 - \ln \left| \frac{\pi}{2} + 2\pi k \right| + \frac{1}{2k} + o \left( \frac{1}{k} \right) \pm i \left( \frac{\pi}{2} + 2\pi k + O \left( \frac{\ln k}{k} \right) \right), \quad k \to +\infty$$

(see [36]).

Claim 1. There exists a family of solutions $\lambda_n = x_n + iy_n$ of the equations $l_n(\lambda) = 0$ such that $x_n \sim \ln n^2 - \ln \ln n^2$ as $n \to +\infty$.

Remark. The spectrum of Eq. (5) in the one-dimensional case under the conditions

$$\lambda = 1, \quad \tau = 1, \quad \Delta T = T_{xx}(t, x), \quad T(t, 0) = T(t, \pi) = 0$$

can be represented in the form (see [36] for details)

$$\sigma = \bigcup_{n=1}^{\infty} \lambda_n^\pm, \quad \lambda_n^\pm = -\frac{1}{2} \pm in \left( 1 - \frac{1}{8n^2} + o \left( \frac{1}{n^3} \right) \right), \quad n \to +\infty.$$

Thus, there is a dramatic difference in the structure of the spectrum between Eq. (5) and Eq. (8).

3. Proof of the main assertions

3.1. Proof of Theorem I. We start by proving Theorem I for the case of the homogeneous initial condition $u(+0) = 0$. First, note that the Laplace transform $\hat{u}(\lambda)$ of every strong solution of Eq. (9) such that $u(+0) = 0$ has the form

$$\hat{u}(\lambda) = \mathcal{L}^{-1}(\lambda) \hat{f}(\lambda),$$

where the operator function $\mathcal{L}(\lambda)$ is the symbol of the equation (9) and can be represented in the form

$$\mathcal{L}(\lambda) = \lambda I + \hat{K}(\lambda) A^2.$$

If we establish that the vector function on the right-hand side in (21) is such that the vector functions $A^2 \hat{u}(\lambda)$ and $\lambda \hat{u}(\lambda)$ belong to the Hardy space $H_2(\Re \lambda > \gamma, H)$ for some $\gamma > \gamma_0 \geq 0$, then we can conclude by the Paley–Wiener theorem that the vector functions $A^2 u(t)$ and $du/dt$ belong to the space $L_{2,\gamma}(\mathbb{R}^+, H)$ and hence $u(t) \in W^1_{2,\gamma}(\mathbb{R}^+, A^2)$. Thus, we will establish the solvability of problem (9), (10) in the space $W^1_{2,\gamma}(\mathbb{R}^+, A^2)$.

Consider the projection $\hat{u}_n(\lambda)$ of the vector function $\hat{u}(\lambda)$ onto the one-dimensional subspace spanned by the vector $e_n$,

$$\hat{u}_n(\lambda) = l_n^{-1}(\lambda) \hat{f}_n(\lambda),$$
where 
\[ \hat{f}_n(\lambda) = (\hat{f}(\lambda), e_n), \quad l_n(\lambda) = (L(\lambda)e_n, e_n) = \lambda + a_n^2 \hat{K}(\lambda). \]

The restriction of the vector function \( A^2 \hat{u}(\lambda) \) to the one-dimensional subspace spanned by \( e_n \) has the form
\[ (A^2 \hat{u}(\lambda), e_n) = \frac{a_n \hat{g}_n(\lambda)}{\lambda \mu_n(\lambda)}, \]

where \( \hat{g}_n(\lambda) \) is the \( n \)th coordinate function of the vector function \( \hat{g}(\lambda) = A\lambda \hat{f}(\lambda) \). By the assumptions of the theorem, the vector function \( g(t) = Af^{(1)}(t) \) lies in the space \( L_{2,\gamma_1}(\mathbb{R}_+, H) \), and hence its Laplace transform \( A\lambda f(\lambda) \) lies in \( H_2(\text{Re} \lambda > \gamma_1, H) \).

It follows that to prove that \( A^2 \hat{u}(\lambda) \in H_2(\text{Re} \lambda > \gamma, H) \), it suffices to establish the following estimate uniform in \( \lambda \) (\( \text{Re} \lambda > \gamma \)) and \( n \in \mathbb{N} \):
\[ \sup_{\text{Re} \lambda > \gamma} \frac{a_n}{\mu_n(\lambda)} \leq \text{const}. \]

To this end, we consider the function
\[ \mathfrak{M}_n(\lambda) = \frac{\lambda \mu_n(\lambda)}{a_n^2} \]
and estimate \( |\mathfrak{M}_n(\lambda)| \) from below. By computing the real and imaginary parts of \( \mathfrak{M}_n(\lambda) \), we obtain
\[ \text{Re} \mathfrak{M}_n(\lambda) = \frac{x^2 - y^2}{a_n^2} + \int_{d_1}^{\infty} \frac{d\mu(\tau)}{\tau} - \int_{d_1}^{\infty} \frac{(x + \tau) d\mu(\tau)}{(x + \tau)^2 + y^2}; \quad \lambda = x + iy, \]
\[ \text{Im} \mathfrak{M}_n(\lambda) = \frac{2xy}{a_n^2} + y \int_{d_1}^{\infty} \frac{d\mu(\tau)}{(x + \tau)^2 + y^2}; \quad n \in \mathbb{N}. \]

First, let us estimate \( |\text{Im} \mathfrak{M}_n(\lambda)| \) from below for \( |y| > x \), where \( x > \gamma > 0 \). We have
\[ |\text{Im} \mathfrak{M}_n(\lambda)| > |y| \left( \frac{2\gamma}{a_n^2} + \frac{1}{y^2} \int_{d_1}^{\infty} \frac{d\mu(\tau)}{1 + (x + \tau)^2/y^2} \right) \]
\[ > \frac{2\gamma |y|}{a_n^2} + \frac{1}{y} \int_{d_1}^{\infty} \frac{d\mu(\tau)}{1 + (1 + \tau/|y|)^2} > \frac{2\gamma |y|}{a_n^2} + \frac{\eta_0(\gamma)}{|y|}, \]
where
\[ \eta_0(\gamma) = \int_{d_1}^{\infty} \frac{d\mu(\tau)}{1 + (1 + \tau/\gamma)^2} \]
and \( 0 < d_1 < +\infty \). Hence we obtain
\[ |\text{Im} \mathfrak{M}_n(\lambda)| > \frac{2\gamma y^2 + \eta_0(\gamma) a_n^2}{a_n^2 |y|} > \frac{2\sqrt{2\gamma \eta_0}}{a_n}. \]

The last inequality for \( |y| > x > \gamma > 0 \) implies the estimate
\[ \left| \frac{a_n}{\lambda \mu_n(\lambda)} \right| \leq \frac{1}{\sqrt{2\gamma \cdot \eta_0(\gamma)}}. \]

Then we estimate \( |\text{Re} \mathfrak{M}_n(\lambda)| \) from below for \( |y| < x \), where \( x > \gamma > 0 \). Note that
\[ \int_{d_1}^{\infty} \frac{(x + \tau) d\mu(\tau)}{(x + \tau)^2 + y^2} \leq \int_{d_1}^{\infty} \frac{d\mu(\tau)}{x + \tau} \leq \int_{d_1}^{\infty} \frac{d\mu(\tau)}{\gamma + \tau}. \]

Thus, for \( x > |y| \) we have the estimate
\[ \text{Re} \mathfrak{M}_n(\lambda) \geq \frac{x^2 - y^2}{a_n^2} + \int_{d_1}^{\infty} \frac{d\mu(\tau)}{\tau} - \int_{d_1}^{\infty} \frac{d\mu(\tau)}{\tau + \gamma} \geq \gamma \int_{d_1}^{\infty} \frac{d\mu(\tau)}{\tau + \gamma} =: \theta_0 > 0. \]
Consequently, for $|y| > x > \gamma > 0$ we obtain the estimate
\begin{equation}
\frac{a_n}{|\lambda l_n(\lambda)|} < \frac{1}{a_n |\text{Re} M_n(\lambda)|} < \frac{1}{a_n \theta_0}.
\end{equation}
By combining inequalities \[24\] and \[25\], we see that the following estimate holds in the half-plane $\{\lambda: \text{Re} \lambda = x > \gamma\}$:
\begin{equation}
\sup_{\lambda > \gamma, n \in \mathbb{N}} \frac{a_n}{|\lambda l_n(\lambda)|} \leq \max \left( \frac{1}{\sqrt{2\gamma \theta_0}}, \frac{1}{a_1 \theta_0} \right).
\end{equation}
Thus, we have proved inequality \[28\].

Now let us estimate the norm of the vector function $A^2 u(t)$ in $L_{2,\gamma}(\mathbb{R}_+, H)$. Indeed, it follows from \[21\] that
\begin{equation}
A^2 \hat{u}(\lambda) = A^2 \mathcal{L}^{-1}(\lambda) \hat{f}(\lambda) = A \lambda^{-1} \mathcal{L}^{-1}(\lambda) A \lambda \hat{f}(\lambda)
\end{equation}
\begin{equation}
= A^2 \hat{u}(\lambda) = \sum_{j=1}^{\infty} a_j \lambda^{-1} t_j^{-1}(\lambda) a_j \lambda \hat{f}_j(\lambda) e_j.
\end{equation}
By the assumption of Theorem \[1\] the vector function $Af^{(1)}(t)$ lies in $L_{2,\gamma}(\mathbb{R}_+, H)$, and $q(0) = 0$. Consequently, the following relations hold by virtue of inequality \[26\], relation \[27\], and the Paley–Wiener theorem:
\begin{equation}
\|A^2 u(t)\|_{L_{2,\gamma}(\mathbb{R}_+, H)}^2 = \|A^2 \hat{u}(\lambda)\|_{H_2(\text{Re} \lambda > \gamma, H)}^2 = \|A \lambda^{-1} \mathcal{L}^{-1}(\lambda) A \lambda \hat{f}(\lambda)\|_{H_2(\text{Re} \lambda > \gamma, H)}^2
\end{equation}
\begin{equation}
= \sup_{\mu > \gamma} \int_{-\infty}^{+\infty} \left[ \sum_{j=1}^{\infty} \left| \frac{a_j}{\mu + iv} l_j(\mu + iv) \cdot a_j(\mu + iv) \hat{f}_j(\mu + iv) \right|^2 \right] dv
\end{equation}
\begin{equation}
\leq \sup_{\lambda > \gamma, j \in \mathbb{N}} \frac{a_j}{\lambda \lambda_j(\lambda)} \|A \lambda \hat{f}(\lambda)\|_{H_2(\text{Re} \lambda > \gamma, H)}^2 \leq C_1 \|Af^{(1)}(t)\|_{L_{2,\gamma}(\mathbb{R}_+, H)}^2,
\end{equation}
where
\begin{equation}
C_1 = \sup_{\lambda > \gamma, j \in \mathbb{N}} \frac{a_j}{\lambda \lambda_j(\lambda)}, \quad \lambda = \mu + iv.
\end{equation}
Consequently,
\begin{equation}
\|A^2 \hat{u}(\lambda)\|_{L_{2,\gamma}(\mathbb{R}_+, H)} \leq C_1 \|Af^{(1)}(t)\|_{L_{2,\gamma}(\mathbb{R}_+, H)}.
\end{equation}
Now let us show that the vector function $\lambda \hat{u}(\lambda)$ lies in the space $H_2(\text{Re} \lambda > \gamma, H)$. To this end, consider the restriction of $\lambda \hat{u}(\lambda)$ to the one-dimensional subspace spanned by $e_n$,
\begin{equation}
(\lambda \hat{u}(\lambda)e_n, e_n) = \frac{\lambda \hat{f}_n(\lambda)}{l_n(\lambda)},
\end{equation}
where $f_n(\lambda)$ is the $n$th coordinate function of $\hat{f}(\lambda)$. By the assumption of the theorem, $f^{(1)}(t) \in L_{2,\gamma_1}(\mathbb{R}_+, H)$, and since $f(0) = 0$, it follows that its Laplace transform $\hat{h}(\lambda) = \lambda \hat{f}(\lambda)$ lies in $H_2(\text{Re} \lambda > \gamma_1, H)$.

Thus, to prove that $\lambda \hat{u}(\lambda) \in H_2(\text{Re} \lambda > \gamma, H)$, where $\gamma > \gamma_1$, it suffices to establish the following estimate uniform in $\lambda (\text{Re} \lambda > \gamma)$ and $n \in \mathbb{N}$:
\begin{equation}
\sup_{\lambda > \gamma, n \in \mathbb{N}} \left| \frac{1}{l_n(\lambda)} \right| \leq \text{const}.
\end{equation}
Consider the function
\[ l_n(\lambda) = \lambda + a_n^2 \hat{\mathcal{E}}(\lambda) = \lambda + a_n^2 \int_0^\infty \frac{d\mu(\tau)}{\tau(\lambda + \tau)}. \]
Let us estimate \(|l_n(\lambda)|\) from below. By computing the real part of \(l_n(\lambda)\), we obtain
\[ \text{Re} \ l_n(\lambda) = x + a_n^2 \int_0^\infty \frac{(x + \tau) \, d\mu(\tau)}{\tau((x + \tau)^2 + y^2)}, \quad \lambda = x + iy. \]
Let us estimate \(|\text{Re} \ l_n(\lambda)|\) from below for \(x > \gamma > \gamma_1 \geq 0\). We have
\[ \text{Re} \ l_n(\lambda) = \left| x + a_n^2 \int_0^\infty \frac{(x + \tau) \, d\mu(\tau)}{\tau((x + \tau)^2 + y^2)} \right| > \gamma. \]
By using the estimate (32), for \(x > \gamma > \gamma_1 \geq 0\) we obtain
\[ \left| \frac{1}{l_n(\lambda)} \right| < \frac{1}{|\text{Re} \ l_n(\lambda)|} < \frac{1}{\gamma}. \]
Consequently,
\[ \sup_{\text{Re} \lambda > \gamma} \left| \frac{1}{l_n(\lambda)} \right| < \frac{1}{\gamma}. \]
Thus, we obtain the desired estimate (31).

**Remark 1.** The estimate (33) implies the inequality
\[ \sup_{\text{Re} \lambda > \gamma} \| \mathcal{L}^{-1}(\lambda) \| \leq \text{const}. \]

Inequality (33), the fact that \(f(0) = 0\), and the Paley–Wiener theorem imply the following relations:
\[ \|u^{(1)}(t)\|_{L^2_{\gamma, \gamma}([\mathbb{R}+, H])} = \|\lambda \hat{u}(\lambda)\|_{H^2_2(\text{Re} \lambda > \gamma, H)} = \|\mathcal{L}^{-1}(\lambda) \lambda \hat{f}(\lambda)\|_{H^2_2(\text{Re} \lambda > \gamma, H)} \]
\[ = \sup_{\mu > \gamma} \left[ \int_{-\infty}^{+\infty} \sum_{j=1}^{+\infty} \frac{1}{l_j(\mu + i\nu)} \cdot (\mu + i\nu) \hat{f}_j(\mu + i\nu) \right]^2 d\nu \]
\[ \leq \sup_{\text{Re} \lambda > \gamma} \left| \frac{1}{l_j(\lambda)} \right|^2 \|\lambda \hat{f}(\lambda)\|_{H^2_2(\text{Re} \lambda > \gamma, H)} \leq C_2 \|f^{(1)}(t)\|_{L^2_{\gamma, \gamma}([\mathbb{R}+, H])}, \]
where
\[ C_2 = \sup_{\text{Re} \lambda > \gamma} \left| \frac{1}{l_j(\lambda)} \right|, \quad \lambda = \mu + i\nu. \]
Thus, we find that \(\lambda \hat{u}(\lambda) \in H^2_2(\text{Re} \lambda > \gamma, H)\), and one has the estimate
\[ \left\| \frac{du}{dt} \right\|_{L^2_{\gamma, \gamma}([\mathbb{R}+, H])} \leq C_2 \|f^{(1)}(t)\|_{L^2_{\gamma, \gamma}([\mathbb{R}+, H])}. \]
Consequently, by combining the estimates (36) and (29), we obtain the desired inequality
\[ \|u\|_{W^1_{\gamma, \gamma}([\mathbb{R}+, A^2])} \leq C \|f^{(1)}(t)\|_{L^2_{\gamma, \gamma}([\mathbb{R}+, H])} \]
with a constant \(C\) independent of \(f\).

Thus, we have proved that Eq. (4) has a solution \(u(t) \in W^1_{\gamma, \gamma}([\mathbb{R}+, A^2])\) and the estimate (37) holds.
A straightforward verification shows that the vector function
\[ u(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} L^{-1}(\lambda) \hat{f}(\lambda)e^{\lambda t} d\lambda \]
satisfies Eq. (9) and the initial condition \( u(+0) = 0 \). Here the argument is completely similar to the corresponding argument in [31].

Let us proceed to the proof of Theorem 1 with inhomogeneous initial conditions. To this end, in problem (9), (10) we set
\[ u(t) = \cos(\sqrt{S}At)\varphi + \omega(t), \]
where \( S := K(0) \). Then for the function \( \omega(t) \) we obtain the problem
\[
\begin{align*}
\frac{d\omega(t)}{dt} + \int_0^t K(t-s)A^2\omega(s) \, ds &= f_1(t), \quad t \in \mathbb{R}_+, \\
\omega(+0) &= 0,
\end{align*}
\]
where
\[ f_1(t) = f(t) - \int_0^t K(t-s)A^2 \cos(\sqrt{S}At)\varphi \, ds + \sqrt{S}A \sin(\sqrt{S}At)\varphi, \]
and consequently,
\[
A f_1^{(1)}(t) = A f^{(1)}(t) - \int_0^t K^{(1)}(t-s)A^3 \cos(\sqrt{S}At)\varphi \, ds.
\]
Let us show that the vector function \( f_1(t) \) satisfies the assumptions of Theorem 1 with zero initial conditions. Set
\[
h(t) := \int_0^t K^{(1)}(t-s)A^3 \cos(\sqrt{S}At)\varphi \, ds.
\]

1. Assume that (13) is satisfied. Let us estimate the norm \( \|Ah(t)\|_{L_2,\gamma(\mathbb{R}_+,H)} \). A straightforward integration shows that
\[
\begin{align*}
\int_0^t e^{-\tau(t-s)} \cos(\sqrt{S}As) \, ds &= (SA^2 + \tau^2 I)^{-1}(\tau(\cos(\sqrt{S}At) - e^{-\tau t}I) + \sqrt{S}A \sin(\sqrt{S}At)).
\end{align*}
\]
We need the following easy-to-verify remark (see [31]).

**Remark 2.** One has the inequality
\[
\|(SA^2 + \tau^2 I)^{-1}\|_H^2 \lesssim \tau^{-2} \|A^{-1}\|_H^2.
\]
Now we use (11), (13), and the inequalities \( \| \cos(At) \| \leq 1 \) and \( \| \sin(At) \| \leq 1 \), take into account the preceding Remark 2 and arrive at the following chain of inequalities:

\[
\| Ah(t) \|_{L_2,\gamma}(\mathbb{R}^+, H) = \left\| \int_0^t \int_0^\infty e^{-(t-s)\tau} d\mu(\tau) \left( A^3 \cos(\sqrt{S} As) \phi \right) ds \right\|_{L_2,\gamma}(\mathbb{R}^+, H) \\
= \left\| \int_0^\infty \left( A^3 \int_0^t e^{-\tau(t-s)} \cos(\sqrt{S} As) ds \right) d\mu(\tau) \phi \right\|_{L_2,\gamma}(\mathbb{R}^+, H) \\
\leq \left\| \int_0^\infty [(SA^2 + \tau^2 I)^{-1} A^3 (\cos(\sqrt{S} At) - e^{-\tau t} I) + \sqrt{S} A \sin(\sqrt{S} At)] d\mu(\tau) \phi \right\|_{L_2,\gamma}(\mathbb{R}^+, H) \\
= \left\| \cos(\sqrt{S} At) \int_0^\infty (SA^2 + \tau^2 I)^{-1} A^3 \tau d\mu(\tau) \phi \\
- \int_0^\infty (SA^2 + \tau^2 I)^{-1} e^{-\tau t} A^3 d\mu(\tau) \phi \\
+ \sin(\sqrt{S} At) \int_0^\infty (SA^2 + \tau^2 I)^{-1} \sqrt{S} A^4 d\mu(\tau) \phi \right\|_{L_2,\gamma}(\mathbb{R}^+, H) \\
\lesssim \left\| \int_0^\infty (SA^2 + \tau^2 I)^{-1} A^3 \tau d\mu(\tau) \phi \right\|_H \\
+ \left\| \int_0^\infty (SA^2 + \tau^2 I)^{-1} A^4 d\mu(\tau) \phi \right\|_H \\
\lesssim \left\| \int_0^\infty \tau (\tau^{-1} A^{-1}) A^3 d\mu(\tau) \phi \right\|_H + \left\| \int_0^\infty d\mu(\tau) A^2 \phi \right\|_H \\
= 2 \int_0^\infty d\mu(\tau) \left\| A^2 \phi \right\|_H.
\]

Thus, it follows from (40), (13), and the last estimate that

\[
\| \omega \|_{W_{2,\gamma}(\mathbb{R}^+, A^2)} \leq d \| Af_1^{(1)}(t) \|_{L_2,\gamma}(\mathbb{R}^+, H) \leq d \left( \| Af^{(1)}(t) \|_{L_2,\gamma}(\mathbb{R}^+, H) + \| A^2 \phi \|_H \right).
\]

From this, in turn, we obtain the estimate (11).

2. Assume that condition (13) is not satisfied. Then it follows from (40) and condition (13) that

\[
\| Af_1^{(1)}(t) \|_{L_2,\gamma}(\mathbb{R}^+, H) \leq \| Af^{(1)}(t) \|_{L_2,\gamma}(\mathbb{R}^+, H) \\
+ \left\| \int_0^t \int_0^\infty \frac{e^{-(t-s)\tau}}{\tau} d\mu(\tau) \left( A^3 \cos(\sqrt{S} As) \phi \right) ds \right\|_{L_2,\gamma}(\mathbb{R}^+, H) \\
\leq \| Af^{(1)}(t) \|_{L_2,\gamma}(\mathbb{R}^+, H) + \frac{1}{\sqrt{2\gamma}} \int_0^\infty \frac{e^{-\tau t}}{\tau} d\mu(\tau) \left\| A^3 \psi_0 \right\|_H.
\]

Thus,

\[
\| \omega \|_{W_{2,\gamma}(\mathbb{R}^+, A^2)} \leq d \| Af_1^{(1)}(t) \|_{L_2,\gamma}(\mathbb{R}^+, H) \leq d \left( \| Af^{(1)}(t) \|_{L_2,\gamma}(\mathbb{R}^+, H) + \| A^3 \phi \|_H \right).
\]

Hence we obtain the estimate (15). The proof of Theorem 1 is complete.

3.2. Proof of Theorems 2–6 and Claim 1. We will use the following assertions.

Lemma 1 (Schwarz). Let \( f \) be an analytic function that maps the upper half-plane \( \mathbb{C}^+ \) into itself. Then the equation \( z = f(z) \) has at most one solution, and if there exists a solution \( w \), then \( |f'(w)| < 1 \). Otherwise, \( f \) is an elliptic linear-fractional transformation.
Theorem 7 (Denjoy–Wolff [32]). Suppose that an analytic function \( f \) maps the upper half-plane \( \mathbb{C}^+ \) into itself and is not an elliptic linear-fractional transformation. Then there exists a unique point \( w \in \mathbb{C}^+ \cup \{\infty\} \) such that the iterations \( f^{*n} \) converge to \( w \) uniformly on compact subsets of \( \mathbb{C}^+ \). There exists a sectorial limit \( \lim_{z \to w} f(z) \), and it satisfies the equation \( w = f(w) \). Moreover, there exists a sectorial derivative \( f'(w) \) which satisfies the inequality \( |f'(w)| \leq 1 \).

Remark. The sectorial limit means that the point \( z \) belongs to an arbitrary sector \( \varepsilon < \arg(z - w) < \pi - \varepsilon \), where \( \varepsilon > 0 \) if \( w \in \mathbb{R} \). The sectorial derivative is defined as \( f'(w) = \lim_{z \to w} (f(z) - f(w))/(z - w) \) if \( w \in \mathbb{R} \).

Lemma 2. Assume that the measure \( d\mu \) is compactly supported in an interval \([d_1,d_2] \), \( 0 < d_1 < d_2 \). Then for sufficiently large \( |\lambda| \) one has the representation

\[
\hat{K}(\lambda) = \frac{B}{\lambda} - \frac{C}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right), \quad |\lambda| \to +\infty,
\]

where

\[
B = \int_{d_1}^{d_2} \frac{d\mu(\tau)}{\tau}, \quad C = \int_{d_1}^{d_2} d\mu(\tau).
\]

Lemma 2 readily follows from the Laurent series expansion of the function \( \hat{K}(\lambda) \).

Proof of Theorem 2. Consider the restriction

\[
l_n(\lambda) := (\mathcal{L}(\lambda) e_n, e_n) = \lambda + a_n^2 \hat{K}(\lambda)
\]

of the operator function \( \mathcal{L}(\lambda) \) to the one-dimensional subspace spanned by the vector \( e_n \). Thus, we obtain a countable family of functions \( l_n(\lambda), n \in \mathbb{N} \). The nonreal eigenvalues of \( \mathcal{L}(\lambda) \) satisfy the equations

\[
\lambda = F_n(\lambda),
\]

where \( F_n(\lambda) := -a_n^2 \hat{K}(\lambda) \). The function \( F_n(\lambda) \) maps the upper half-plane \( \mathbb{C}^+ \) into itself, and the equation has at most one solution by the Schwarz lemma. Note that

\[
\Re(\lambda + a_n^2 \hat{K}(\lambda)) = \Re \lambda + a_n^2 \int_{d_1}^{d_2} \frac{\tau + \Re \lambda}{|\lambda + \tau|^2} d\mu(\tau) > 0
\]

for \( \Re \lambda \geq 0 \). Consequently, (46) has roots in the left half-plane by Theorem 7.

To find the nonreal roots of Eq. (46), we use the representation (45) to rewrite this equation in the form

\[
\lambda_n = F_n(\lambda_n) = a_n^2 \left( \frac{B}{\lambda_n} - \frac{C}{\lambda_n^2} + o\left(\frac{1}{\lambda_n^2}\right) \right), \quad |\lambda| \to +\infty.
\]

Hence we obtain the equation

\[
\lambda_n^3 + a_n^2 B \lambda_n - a_n^2 C + o(1) = 0.
\]

We seek a solution of Eq. (47) in the form

\[
\lambda_n = i\sqrt{B} a_n + d + o\left(\frac{1}{a_n}\right), \quad a_n \to +\infty,
\]

with an unknown coefficient \( d \). By substituting the representation (48) into Eq. (47) and by matching the coefficients of \( a_n^2 \), we obtain \( d = K'(0)/(2K(0)) \). The proof of Theorem 2 is complete. □
Proof of Theorem 3 The function \( F_n(\lambda) := -a_n^2 \hat{K}(\lambda) \) maps the upper half-plane \( \mathbb{C}^+ \) into itself, and hence Eq. (46) has at most one solution by the Schwarz lemma. By Lemma 2, Eq. (46) has a root \( \lambda_n^+ \) in the left upper quadrant for sufficiently large \( a_n \). Thus, to complete the proof of the theorem, it remains to show that there exists a \( y_0 > 0 \) such that \( \sigma_R \in [-d_2, -d_1 + y_0] \), where \( 0 < y_0 < d_1 \). Indeed, for each \( n \in \mathbb{N} \) one has the representation

\[
\Re l_n(\lambda) = x + a_n^2 \int_{d_1}^{d_2} \frac{x + \tau}{\tau((x + \tau)^2 + \sigma^2)} d\mu(\tau),
\]

which implies that \( \Re l_n(\lambda) > 0 \) for \( \Re \lambda = x > 0 \). Now consider various cases in which \( \Re(\lambda) = x < 0 \). If \( x < -d_2 \) and \( \tau \in [d_1, d_2] \), then \( x + \tau < 0 \) and hence \( \Re l_n(\lambda) < 0 \). If \( -d_1 < x < 0 \) and \( \tau \in [d_1, d_2] \), then \( x + \tau > 0 \) and hence \( \Re l_n(\lambda) > 0 \) starting from some term of the sequence \( a_n \). Thus, the function \( l_n(\lambda) \) can have zeros in the interval \( -d_1 < x < 0 \) only for finitely many terms of the sequence \( a_n \). Consequently, there exists a \( y_0 > 0 \) such that the estimate \( 0 < |l_n^{-1}(\lambda)| < \infty \) holds for all \( n \in \mathbb{N} \) provided that \( \Re \lambda \in (-d_1 + y_0, 0) \).

It follows from the preceding argument that \( \sigma_R \in [-d_2, -d_1 + y_0] \), where \( 0 < y_0 < d_1 \). The proof of Theorem 3 is complete. \( \square \)

Remark. The existence of a solution of the equation \( \lambda = F_n(\lambda) \) in the upper half-plane \( \mathbb{C}^+ \) can be established with the use of iterations of the mapping \( F_n(\lambda) := -a_n^2 \hat{K}(\lambda) \) starting from an arbitrary point in \( \mathbb{C}^+ \). The sequence \( \lambda_k = F_n(\lambda_{k-1}) \), \( k = 0, 1, \ldots \), converges by the Denjoy–Wolff Theorem 7. If this sequence \( \lambda_k \) converges to a point in the upper half-plane, then this point is unique. If the sequence \( \lambda_k \) converges to a point on the negative real half-line \( \mathbb{R}_- \), then the equation \( \lambda = F_n(\lambda) \) has no solutions in \( \mathbb{C}^+ \) by Theorem 7.

Proof of Theorem 4 The proof of Theorem 4 heavily relies on Theorem 3. The solution of problem (9), (10) can be represented in the form

\[
\int_{\gamma - i\infty}^{\gamma + i\infty} \hat{\mu}(\lambda) e^{\lambda t} d\lambda = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} L^{-1}(\gamma) \varphi e^{\lambda t} d\lambda, \quad t > 0.
\]

Fix a vector \( e_n \) and consider the projection of \( u(t) \) onto the one-dimensional subspace spanned by \( e_n \).

\[
u(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{\mu}(\lambda) e^{\lambda t} d\lambda = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\varphi_n e^{\lambda t}}{l_n(\lambda)} d\lambda,
\]

where \( \hat{\mu}(\lambda) = (\hat{\mu}(\lambda), e_n) \) and \( \varphi_n = (\varphi, e_n) \). By Theorem 3 the function

\[
\hat{\mu}(\lambda) = \frac{\varphi_n}{l_n(\lambda)}
\]

has simple poles in the left half-plane at the points \( \lambda_n^\pm \) and also singular points on the segment \( [-d_2, -d_1 + y_0], \) where \( 0 < y_0 < d_1 \).

Let us show that the function \( \varphi_n(t) \) can be represented in the form

\[
u_n(t) = 2 \Re \left[ \frac{\exp(\lambda_n^+ t)}{l_n^{(1)}(\lambda_n^+)} \right] \varphi_n + \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi_n e^{\lambda t}}{l_n(\lambda)} d\lambda.
\]
On the complex plane \( \mathbb{C} \), consider the counterclockwise contour

\[
\Gamma_{R, \gamma} = \Gamma_1 \cup \Gamma_2 \cup C_R \cup \Gamma_3,
\]

where

\[
\begin{align*}
\Gamma_1 &= \{ \lambda \in \mathbb{C} : \Re \lambda = \gamma, \, |\Im \lambda| \leq R \}, \\
\Gamma_2 &= \{ \lambda \in \mathbb{C} : \Im \lambda = R, \, 0 \leq \Re \lambda \leq \gamma \}, \\
C_R &= \{ \lambda \in \mathbb{C} : |\lambda| = R, \, \Re \lambda < 0 \}, \\
\Gamma_3 &= \{ \lambda \in \mathbb{C} : \Im \lambda = -R, \, 0 \leq \Re \lambda \leq \gamma \}.
\end{align*}
\]

Let us show that the integrals over the segments \( \Gamma_2 \) and \( \Gamma_3 \) tend to zero,

\[
\int_{\Gamma_2} l^{-1}_n (\lambda) e^{\lambda t} d\lambda \to 0, \quad t > 0, \quad R \to +\infty.
\]

Now let us show that the integrals over the segments \( \Gamma_2 \) and \( \Gamma_3 \) tend to zero,

\[
\int_{\Gamma_3} l^{-1}_n (\lambda) e^{\lambda t} d\lambda \to 0, \quad t > 0, \quad R \to +\infty.
\]

By using the estimate (52), we obtain the chain of inequalities

\[
\left| \int_{\pm iR + \gamma} l^{-1}_n (\lambda) e^{\lambda t} d\lambda \right| \leq \int_0^\infty |l^{-1}_n (x \pm iR)| e^{xt} dx \leq c \int_0^\infty e^{xt} dx = c \frac{e^{\gamma t} - 1}{tR} \to 0
\]

as \( R \to +\infty \), where \( x = \Re \lambda \).

Relations (53) and (54) imply (51). We obtain the desired representation (18) from (51), Theorem 3 on the localization of the spectrum, and the Cauchy integral theorem. The proof of Theorem 4 is complete.

**Proof** of Theorem 5. We start by estimating the norm of \( u_I(t) \) in \( H \). Let us estimate \( |l^{-1}_n (\lambda^+) \lambda| \) from below by using the asymptotics (16) of \( \lambda^+ \). A straightforward verification shows that

\[
|l^{-1}_n (\lambda^+) \lambda| \geq 1 - a^2_2 \int_{d_1}^{d_2} \frac{d\mu(\tau)}{\tau (\tau + \lambda_n^+)^2} \geq 1 + \frac{1}{2} \int_{d_1}^{d_2} \frac{(1 - 2\tau^2 / a^2_2) d\mu(\tau)}{\tau (1 + \tau^2 / a^2_2)^2} \geq 1.
\]
Hence we obtain
\[
\|u_t(t)\|_H^2 = \left\| \sum_{n=1}^{\infty} \Re \left[ \frac{\exp(\lambda_n^+ t)}{l_n^{(1)}(\lambda_n^+)} \right] \varphi_n e_n \right\|_H^2 \leq \sum_{n=1}^{\infty} \left| \frac{\exp(\lambda_n^+ t)}{l_n^{(1)}(\lambda_n^+)} \right|^2 |\varphi_n|^2
\]
(55)
\[
= \sum_{n=1}^{\infty} \frac{\exp(2 \Re \lambda_n^+ t)}{|l_n^{(1)}(\lambda_n^+)|^2} |\varphi_n|^2 \leq \sum_{n=1}^{\infty} \exp(2 \Re \lambda_n^+ t)|\varphi_n|^2 \leq C \exp(2\kappa t) \|\varphi\|_H^2,
\]
where \(\kappa = \sup_{n \in \mathbb{N}} \Re \lambda_n^+\).

Let us proceed to estimating \(u_R(t)\). Let us estimate the function \(|l_n^{-1}(\lambda)|\) on the contour \(\Gamma\).

We start from the estimates on the horizontal sides \(\Gamma^{\pm}\). Set \(\lambda = s \pm iy_0\), \(s \in [-d_2, -d_1]\).

For sufficiently large \(n \in \mathbb{N}\), we have the inequality
\[
|l_n^{-1}(s \pm iy_0)| \leq \frac{r(s, y_0)}{y_0a_n^2},
\]
where
\[
r(s, y_0) = \left( \int_{d_1}^{d_2} \frac{d\mu(\tau)}{\tau((s + \tau)^2 + y_0^2)} \right)^{-1}.
\]
(57)

On the horizontal sides \(\Gamma^{\pm}: \lambda = s \pm iy_0, -d_2 \leq s \leq -d_1, d\lambda = ds\), we have
\[
\int_{\Gamma^{\pm}} e^{\lambda} l_n^{-1}(\lambda) d\lambda \leq \int_{-d_2}^{-d_1} e^{st} |l_n^{-1}(s \pm iy_0)| ds \leq \frac{r(0, y_0)}{a_n^2 y_0} \cdot \xi(t) = \frac{r_0(y_0)}{a_n^2 y_0} \cdot \xi(t),
\]
where
\[
\xi(t) = \frac{e^{-d_2 t} - e^{-d_1 t}}{t}.
\]
(58)

In turn, the following estimates hold on the semicircles \(C_j, j = 1, 2\), for sufficiently large \(n\):
\[
|l_n(-d_j + y_0 e^{i\varphi})| \geq a_n^2 \frac{y_0}{\sqrt{2}} \mu^{-1}(y_0),
\]
where
\[
\mu(y_0) = \left( \int_{d_1}^{d_2} \frac{d\mu(\tau)}{(d_2 - d_1 + y_0^2 + y_0^2)} \right)^{-1}.
\]
Consequently, for \(\lambda = -d_j + y_0 e^{i\varphi}, j = 1, 2\), we have
\[
\int_{C_j} e^{\lambda} l_n^{-1}(\lambda) d\lambda \leq \left| \int_{-\frac{\pi}{2} + \pi(j-1)}^{\frac{\pi}{2} + \pi(j-1)} e^{(-d_j + y_0 e^{i\varphi})t} l_n^{-1}(-d_j + y_0 e^{i\varphi}) y_0 e^{i\varphi} d\varphi \right|
\]
\[
\leq \frac{\pi \sqrt{2}}{a_n^2} e^{(-d_j + y_0) t} \cdot \mu(y_0).
\]
(59)
Hence, in view of the estimates (58) and (59), we have the chain of inequalities
\[
\|u_R(t)\|_H^2 = \left\| \frac{1}{2\pi i} \int_{\Gamma} L^{-1}(\lambda)\varphi e^{\lambda t} d\lambda \right\|^2 = \left\| \frac{1}{2\pi i} \sum_{n=1}^{\infty} \left( \int_{\Gamma} \frac{\varphi_n}{l_n(\lambda)} e^{\lambda t} d\lambda \right) e_n \right\|^2
\]
\[
= \frac{1}{(2\pi)^2} \sum_{n=1}^{\infty} \left\| \int_{\Gamma} \frac{\varphi_n}{l_n(\lambda)} e^{\lambda t} d\lambda \right\|^2 \leq \frac{1}{(2\pi)^2} \sum_{n=1}^{\infty} \left\| \frac{\varphi_n}{l_n(\lambda)} e^{\lambda t} \right\|^2 |d\lambda|
\]
\[
\leq \frac{1}{(2\pi)^2} \left( 2 \frac{\xi^2(t)}{\xi_0^2} + 2 \pi^2 \mu^2(y_0) (e^{-2(2-y_0)t} + e^{-2(2-y_0)t}) \right) \sum_{n=1}^{\infty} a_n^{-4} |\varphi_n|^2.
\]

The proof of Theorem 5 is complete.

\textbf{Proof of Theorem 6} Let us present a concise proof of Theorem 6 based on the Duhamel principle. Consider problem (9), (10) with the initial condition \( \varphi = 0 \). By Theorem 1 problem (9), (10) has a unique solution \( u(t) \in W_{2, \gamma}^{1}(\mathbb{R}^+, A^2) \). Let us show that this solution can be represented in the form
\[
(60) \quad u(t) = \int_0^t v(t, \tau) d\tau,
\]
where the vector function \( v(t, \tau) \) is the solution of the problem
\[
(61) \quad \frac{dv(t, \tau)}{dt} + \int_\tau^t K(t-s)A^2 v(s, \tau) ds = 0, \quad t > \tau,
\]
\[
(62) \quad v(\tau, \tau) = f(\tau).
\]
We differentiate the solution \( u(t) \) of the form (60) with respect to \( t \) and take into account condition (62), thus obtaining
\[
u_t = v(t, t) + \int_0^t v_t(t, \tau) d\tau = f(t) + \int_0^t v_t(t, \tau) d\tau,
\]
\[
\int_0^t K(t-s)A^2 \left( \int_\tau^s v(s, \tau) d\tau \right) ds = \int_0^t \left( \int_\tau^t K(t-s)A^2 v(s, \tau) ds \right) d\tau.
\]
Hence we have
\[
u_t + \int_0^t K(t-s)A^2 u(s) ds = f(t) + \int_0^t \left( v_t(t, \tau) + \int_\tau^t K(t-s)A^2 v(s, \tau) ds \right) d\tau = f(t),
\]
\[
u(+0) = 0.
\]
A straightforward verification shows that the Laplace transform of the solution of problem (61), (62) has the form
\[
(63) \quad \widehat{v}(\lambda, \tau) = \mathcal{L}^{-1}(\lambda)e^{-\lambda \tau} f(\tau).
\]
By Theorem 6 \( v(t, \tau) \) admits the representation
\[
(64) \quad v(t, \tau) = \sum_{n=1}^{\infty} \frac{\exp(\lambda_n^+ t)}{l_n^{(1)}(\lambda_n^+)} f_n(\tau) e_n
\]
\[
+ \sum_{n=1}^{\infty} \frac{\exp(\lambda_n^- t)}{l_n^{(1)}(\lambda_n^-)} f_n(\tau) e_n + \frac{1}{2\pi i} \int_{\Gamma} \mathcal{L}^{-1}(\lambda)e^{\lambda(t-\tau)} f(\tau) d\tau.
\]
In turn, relations (64) and (60) imply the desired representation (19).

The proof of Theorem 6 is complete. \qed
Proof of Claim. We can rewrite the equation
\[ l_n(\lambda) = \lambda + n^2 e^{-\lambda} = 0, \quad \lambda = x + iy, \]
as the following system by separating the real and imaginary parts:
\[
\begin{cases}
  e^x (x \cos y - y \sin y) = -n^2, \\
  y \cos y + x \sin y = 0.
\end{cases}
\]
If \( y = 0 \), then Eq. \((65)\) has the unique solution \((0, 0)\) for \( n = 0 \). Take \( y \neq 0 \); then we have
\[
(66) \quad x = -\frac{y \cos y}{\sin y}, \quad e^{-\frac{y \cos y}{\sin y}} \frac{y}{\sin y} = n^2.
\]
Note that \( x = g(y) = -\frac{y \cos y}{\sin y} \to +\infty \) as \( y \to \pi - 0 \).
If we set \( y = \pi - \delta, \delta > 0 \), then Eq. \((66)\) acquires the form
\[
(67) \quad e^{(\frac{\pi}{\delta} - 1)} (\frac{\pi}{\delta} - 1) = n^2,
\]
as \( \delta \to 0 \). Set \( \theta = \frac{\pi}{\delta} - 1 \). Then Eq. \((67)\) has the form \( \theta e^\theta = n^2 \). By using the results in Fedoryuk’s monograph [37, pp. 51–52], we obtain the following asymptotic representation for \( \theta \):
\[
\theta = \ln n^2 - \ln \ln n^2 + O \left( \frac{\ln \ln n^2}{\ln n^2} \right), \quad n \to +\infty.
\]
Thus,
\[
\frac{\pi}{\delta} = \ln n^2 - \ln \ln n^2 + O \left( \frac{\ln \ln n^2}{\ln n^2} \right), \quad \delta \sim \sin \delta \sim \frac{\pi}{\ln n^2 - \ln \ln n^2} \sim \frac{\pi}{2 \ln n}
\]
as \( n \to +\infty \). Then
\[
\begin{align*}
y_n &= \pi \left( 1 - \frac{1}{\ln n^2 - \ln \ln n^2} + O \left( \frac{1}{\ln n} \right) \right), \\
x_n &= -y_n \cos y_n \sim \ln n^2 - \ln \ln n^2 \to +\infty, \quad n \to +\infty. \quad \square
\end{align*}
\]

References


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