NECESSARY AND SUFFICIENT CONDITION FOR THE STABILIZATION OF THE SOLUTION OF A MIXED PROBLEM FOR NONDIVERGENCE PARABOLIC EQUATIONS TO ZERO

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Abstract. We consider the first boundary value problem in a cylindrical domain for a uniformly parabolic second-order equation in nondivergence form. The solution satisfies the homogeneous Dirichlet condition on the lateral surface of the cylinder, and the initial function is bounded. We show that if the coefficients of the equation satisfy the local and global Dini conditions, then a necessary and sufficient condition for the stabilization of the solution to zero coincides with a similar condition for the heat equation.

1. Introduction

In the cylinder \( Z = D \times (0, \infty) \), where \( D \) is an unbounded domain in \( \mathbb{R}^n \), \( n \geq 3 \), consider the first mixed problem

\[
Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} - u_t = 0, \\
u|_{\partial D \times (0, \infty)} = 0, \quad u|_{t=0} = u_0(x),
\]

with a continuous bounded initial function \( u_0 \) for a uniformly parabolic operator \( L \),

\[
\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \leq \lambda |\xi|^2,
\]

\( a_{ij}(x) = a_{ji}(x) \quad \forall x \in D, \quad \xi \in \mathbb{R}^n, \quad \lambda > 0. \)

A solution \( u \) of problem (1.1), (1.2) is understood in the classical sense as a function bounded in \( Z \), continuous on the set \( \bar{Z} \setminus (\partial D \times \{0\}) \), and continuously differentiable in \( Z \) twice with respect to \( x \) and once with respect to \( t \).

We will study the pointwise (uniform on compact subsets of \( \bar{D} \)) stabilization to zero of the solution of problem (1.1), (1.2), i.e., the existence of a limit

\[
\lim_{t \to \infty} u(x, t) = 0.
\]

Note that if \( D \) is bounded, then the stabilization in this sense also holds for the solutions of more general parabolic equations (see [1, Section 12, Theorem 1]).
There are quite a few papers dealing with stabilization of solutions of parabolic equations; surveys of such papers can be found, e.g., in [1, 2, 3, 4]. For the Cauchy problem, the stabilization question was answered in [5, 6]. Later, the stabilization criterion obtained in these papers was extended in [7] to the case of a parabolic equation with the $p$-Laplacian.

The stabilization of solutions of boundary value problems for parabolic equations in nondivergence form was studied in [1, 8, 9].

The influence of the structure of $D$ on the stabilization of solutions of initial–boundary value problems for parabolic equations in divergence form was analyzed in [10, 11, 12, 13]. In particular, the papers [10, 11] deal with the second and the first mixed problem, respectively.

It was established in [12] that the existence of a limit (1.4) of solutions of problem (1.1), (1.2) with an arbitrary bounded initial function for an equation in divergence form with measurable coefficients is solely determined by the structure of $D$: a criterion for the stabilization of the solution of the first mixed problem for the heat operator with an arbitrary bounded initial function was obtained saying that the Wiener capacity (see [14]) of the complement of $D$ should be infinite. In the same paper, a stabilization criterion coinciding with the similar criterion for the heat operator was established for an equation in divergence form with measurable coefficients depending on $x$ alone. The paper [15] gives a similar (but only sufficient) condition for the stabilization to zero of the solution of the first mixed problem for equations in divergence form with bounded measurable coefficients depending on $x$ and $t$.

The aim of the present paper is to describe a class of nondivergence parabolic operators for which the condition for the stabilization to zero of the solutions of problem (1.1), (1.2) is the same as for the heat equation. We also prove the results on the stabilization of solutions of problem (1.1), (1.2) announced by the authors in [16].

The study of the structure of $D$ is closely related to the existence of two-sided global estimates of Nash–Aronson type for the fundamental solution of the operator $L$. For equations in divergence form, such estimates were obtained in [17]. For the application of these estimates in the case of equations in divergence form to stabilization issues, see [13, 15].

When studying nondivergence equations, one has to impose additional conditions on the coefficients of the operator (1.1) to ensure the existence of a classical solution of problem (1.1), (1.2). It is well known that the Dini condition is the minimum requirement on the coefficients. Under this condition (see [18]), the fundamental solution satisfies a two-sided local estimate of Nash–Aronson type. We do not know whether there exists a global estimate, because an additional condition at infinity should be imposed on the coefficients for this estimate to hold. We suggest an approach that is based on the barrier method and does not use estimates of the fundamental solution.

We assume that the coefficients $a_{ij}(x)$ are continuous on $\partial D$ and satisfy the Dini condition; i.e.,

\begin{equation}
|a_{ij}(x) - a_{ij}(y)| \leq \varphi(|x - y|) \quad \forall x \in D, \quad y \in \partial D,
\end{equation}

where $i, j = 1, 2, \ldots, n$ and $\varphi(t)$ is a continuous nondecreasing function on $[0, 1]$ such that

\begin{equation}
\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < \infty.
\end{equation}

Furthermore, we require that the coefficients $a_{ij}(x)$ satisfy the global Dini condition at infinity. Namely, we assume that there exist limits

$$
\lim_{x \in D, x \to \infty} a_{ij}(x) = \alpha_{ij}, \quad i, j = 1, 2, \ldots, n,
$$
where \( \{ \alpha_{ij} \} \) is a constant uniformly elliptic matrix, and the inequality

\[
(1.7) \quad |a_{ij}(x) - \alpha_{ij}| \leq \omega \left( \frac{1}{|x| + 1} \right) \quad \forall x \in D, \quad i, j = 1, 2, \ldots, n,
\]
is satisfied, where \( \omega(1/t) \) is a nonnegative semi-additive function monotone decreasing to zero as \( t \to \infty \) such that

\[
(1.8) \quad \int_1^\infty \omega \left( \frac{1}{\tau} \right) \frac{d\tau}{\tau} < \infty.
\]

In what follows, we use the notion of capacity \( \text{cap}(E) \) of a compact set \( E \subset \mathbb{R}^n \), which we now recall (e.g., see [19, Section 1.2]). Consider the set of Borel measures \( \mu \) such that

\[
\int_E |x - y|^{2-n} \, d\mu(y) \leq 1 \quad \text{for} \quad x \in \mathbb{R}^n \setminus E.
\]
The supremum \( \sup \mu(E) \) over this set is called the capacity of \( E \).

Let us state the main result of the present paper, in which \( B_r^{x_0} \) stands for the open ball of radius \( r \) centered at a point \( x_0 \).

**Theorem 1.1.** If conditions (1.3) and (1.5)–(1.8) are satisfied, then it is necessary and sufficient for the stabilization of the solution of the mixed problem (1.1), (1.2) to zero (i.e., for the limit relation (1.4) to hold) that the following integral be divergent:

\[
(1.9) \quad \int_1^\infty \frac{\text{cap}(B_r^{x_0} \setminus D)}{\tau^{n-1}} \, d\tau = \infty.
\]

The point \( x_0 \) in this theorem can be arbitrary, because the convergence or divergence of the integral (1.9) is independent of the choice of \( x_0 \). Condition (1.9) is necessary and sufficient (see [14, Section 5.1, Theorem 5.1]) for the capacity of \( \mathbb{R}^n \setminus D \) to be infinite.

**Corollary of Theorem 1.1.** Assume that the coefficients of the operator \( L \) satisfy the local and global Hölder conditions; i.e., for some \( \alpha \in (0, 1) \),

\[
|a_{ij}(x) - a_{ij}(y)| \leq C|x - y|^\alpha \quad \forall x, y \in \mathbb{R}^n, \quad i, j = 1, 2, \ldots, n,
\]

\[
|a_{ij}(x) - \alpha_{ij}| \leq \frac{C}{(1 + |x|)^\alpha} \quad \forall x \in \mathbb{R}^n, \quad i, j = 1, 2, \ldots, n,
\]

for some \( \alpha \in (0, 1) \). Then the solution of problem (1.1), (1.2) has zero limit (1.4) if and only if the Wiener integral (1.9) is divergent.

The proof of sufficiency in Theorem 1.1 is given in Section 3. It is based on the construction of a subsolution of the heat potential type for equation (1.1) (the construction is carried out in Section 2) and an application of a growth lemma of Landis type ([19, Section 3.6, Lemma 6.1]). The necessity of condition (1.9) is established in Section 5 and is based on the fact that the elliptic part of equation (1.1) has a supersolution of the type of the fundamental solution of the Laplace equation. The construction of this supersolution is carried out in Section 4.

2. Construction of a Subsolution of the Parabolic Operator

2.1. Auxiliary estimates. In what follows, we assume without loss in generality that the origin in \( \mathbb{R}^n \) is contained in the boundary \( \partial D \). It is convenient to rewrite condition (1.7) in the form

\[
a_{ij}(x) = \alpha_{ij} + b_{ij}(x)\omega \left( \frac{1}{|x| + 1} \right) \quad \forall x \in D.
\]
By (1.7), we have

\[ |b_{ij}(x)| \leq 1 \quad \forall x \in D. \]

(2.1)

Let us extend the coefficients \( a_{ij}(x) \) of equation (1.1) into \( \mathbb{R}^n \setminus \bar{D} \) by setting

\[ a_{ij}(y) = \alpha_{ij} + b_{ij}(y^*)\omega \left( \frac{1}{|y| + 1} \right), \quad i, j = 1, \ldots, n, \quad y \in \mathbb{R}^n \setminus \bar{D}, \]

where \( y^* \) stands for the projection of a point \( y \) onto \( \partial D \). The projection \( y^* \) can be chosen uniquely. If there are several such projections, we select the one with the least first coordinate. If there are still several projections remaining, we select the one with the least second coordinate, etc.

As a result of this extension, for \( x \in \bar{D} \) and \( y \in \mathbb{R}^n \setminus D \) we have

\[ a_{ij}(x) - a_{ij}(y) = a_{ij}(x) - a_{ij}(y^*) - \frac{(a_{ij}(y^*) - \alpha_{ij})(\omega \left( \frac{1}{|y| + 1} \right) - \omega \left( \frac{1}{|y^*| + 1} \right))}{\omega \left( \frac{1}{|y^*| + 1} \right)}. \]

This, in view of (1.5), (1.7) and the monotonicity and semi-additivity of \( \omega \), implies that

\[ |a_{ij}(x) - a_{ij}(y)| \leq \varphi(|x - y^*|) + \omega \left( \frac{|y - y^*|}{1 + |y| + |y^*| + |y| \cdot |y^*|} \right) \leq \varphi(|x - y^*|) + \omega(|y - y^*|). \]

Since \( |y^* - y| \leq |x - y| \), it follows from the monotonicity of \( \varphi \) and \( \omega \) that

\[ |a_{ij}(x) - a_{ij}(y)| \leq \varphi(2|x - y|) + \omega(|x - y|) \]

(2.2)

\[ \forall x \in \bar{D}, \quad y \in \mathbb{R}^n \setminus D, \quad |x - y| \leq \frac{1}{2}. \]

Condition (1.7) after the above-mentioned extension of the coefficients becomes

\[ a_{ij}(x) = \alpha_{ij} + b_{ij}(x)\omega \left( \frac{1}{|x| + 1} \right) \quad \forall x \in \mathbb{R}^n, \quad i, j = 1, \ldots, n, \]

(2.3)

where (see (2.1))

\[ |b_{ij}(x)| \leq 1 \quad \forall x \in \mathbb{R}^n. \]

(2.4)

Note that by (2.2)–(2.4) one has the inequality

\[ |b_{ij}(x) - b_{ij}(y)| \leq \varphi(2|x - y|) + \omega(|x - y|) \quad \forall x \in \bar{D}, \quad y \in \mathbb{R}^n \setminus D, \quad |x - y| \leq \frac{1}{2}. \]

(2.5)

Prior to constructing a barrier, let us present two auxiliary assertions, in which we set

\[ \varphi_r(t) = \begin{cases} 
\varphi(2t) + \omega(t) & \text{for } 0 < t \leq \frac{2}{r}, \\
\varphi \left( \frac{t}{2} \right) + \omega \left( \frac{t}{2} \right) & \text{for } t > \frac{2}{r},
\end{cases} \]

(2.6)

\[ \omega_r(t) = \begin{cases} 
\omega \left( \frac{4t}{r} \right) & \text{for } 0 < t \leq \frac{r}{2}, \\
\omega(1) & \text{for } \frac{r}{2} < t \leq pr, \\
\omega \left( \frac{2t}{r} \right) & \text{for } pr < t \leq 2qr,
\end{cases} \]

(2.7)

where \( q > p \), the constant \( p \geq 2 \) will be defined below, and \( r > \max\{2q, 4\} \).

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Lemma 2.1. If conditions (1.5) and (1.7) are satisfied, then the inequality
\[ |a_{ij}(x) - a_{ij}(y)| \leq C \varphi_r(|x-y|) + C \omega_r(|x-y|), \]
where the constant $C$ depends only on $\omega(1)$, holds for $x \in D \cap B^0_{qr}$ and $y \in (\mathbb{R}^n \setminus D) \cap \left( B^0_r \setminus B^0_{r/2} \right)$, $r \geq 4q$.

Proof. From (2.3), we have
\[ (2.9) \]
\[ |a_{ij}(x) - a_{ij}(y)| \leq |b_{ij}(x) - b_{ij}(y)| \omega \left( \frac{1}{|y| + 1} \right) + |b_{ij}(x)| \cdot \left[ \omega \left( \frac{1}{|x| + 1} \right) - \omega \left( \frac{1}{|y| + 1} \right) \right]. \]

Let us estimate the first term on the right-hand side in (2.9). It follows from (2.5) and the monotonicity of $\omega$ that
\[ (2.10) \]
\[ |b_{ij}(x) - b_{ij}(y)| \omega \left( \frac{1}{|y| + 1} \right) \leq \varphi(2|x-y|) + (1 + \omega(1))\omega(|x-y|) \quad \text{for } |x-y| \leq \frac{2}{r}, \]
where $r \geq 4$. Furthermore, according to (2.4), the monotonicity of $\omega$ and the conditions imposed on $y$ imply that for $r > 4$ one has
\[ (2.11) \]
\[ |b_{ij}(x) - b_{ij}(y)| \omega \left( \frac{1}{|y| + 1} \right) \leq 2\omega \left( \frac{2}{r} \right) \quad \text{for } |x-y| > \frac{2}{r}. \]

Thus, from (2.10) and (2.11) we obtain
\[ (2.12) \]
\[ |b_{ij}(x) - b_{ij}(y)| \omega \left( \frac{1}{|y| + 1} \right) \leq (2 + \omega(1))\varphi_r(|x-y|), \]
where $\varphi_r$ is the function defined in (2.6).

Let us estimate the second term in (2.9). It follows from the monotonicity and semi-additivity of $\omega$ that
\[ \left| \omega \left( \frac{1}{|x| + 1} \right) - \omega \left( \frac{1}{|y| + 1} \right) \right| \leq \omega \left( \frac{1}{|x| + 1} - \frac{1}{|y| + 1} \right) \leq \omega \left( \frac{|x-y|}{1 + |x| + |y| + |x| \cdot |y|} \right) \leq \omega \left( \frac{2|x-y|}{1 + |y|} \right) \leq \omega \left( \frac{2|x-y|}{r} \right) \]
for $|x-y| \leq r/2$, so that (see (2.4))
\[ (2.13) \]
\[ |b_{ij}(x)| \cdot \omega \left( \frac{1}{|x| + 1} \right) - \omega \left( \frac{1}{|y| + 1} \right) \leq \omega \left( \frac{2|x-y|}{r} \right) \quad \text{for } |x-y| \leq \frac{r}{2}. \]

If $r/2 < |x-y| \leq pr$, then
\[ (2.14) \]
\[ |b_{ij}(x)| \cdot \omega \left( \frac{1}{|x| + 1} \right) - \omega \left( \frac{1}{|y| + 1} \right) \leq 2\omega(1) \quad \text{for } \frac{r}{2} < |x-y| < pr. \]

If $|x-y| \geq pr$, then $|x| \geq (p-1)r$, and since $p \geq 2$, we can use the same argument as in the derivation of (2.13) to conclude that
\[ (2.15) \]
\[ |b_{ij}(x)| \cdot \omega \left( \frac{1}{|x| + 1} \right) - \omega \left( \frac{1}{|y| + 1} \right) \leq \omega \left( \frac{2|x-y|}{(p-1)r^2} \right) \leq \omega \left( \frac{4q}{r} \right) \]
for $pr \leq |x-y| \leq 2qr$
provided that $r \geq 4q$. It follows from (2.13)–(2.15) that
\[ (2.16) \]
\[ |b_{ij}(x)| \cdot \omega \left( \frac{1}{|x| + 1} \right) - \omega \left( \frac{1}{|y| + 1} \right) \leq 2\omega_r(|x-y|), \]
where $\omega_r(t)$ is the function in (2.7). By taking into account (2.12) and (2.16), we arrive from (2.9) at the desired estimate (2.8). The proof of the lemma is complete.  

We set
\begin{equation}
\phi_r(t) = \varphi_r(t) + \omega_r(t)
\end{equation}
in \((2.8)\); then, under the assumptions of Lemma \(2.1\)
\begin{equation}
|a_{ij}(x) - a_{ij}(y)| \leq C\phi_r(|x - y|).
\end{equation}

**Lemma 2.2.** If conditions \((1.6)\) and \((1.8)\) are satisfied, then there exists a constant \(C\) independent of \(q\) and \(r\) such that
\begin{equation}
\int_0^{2qr} \phi_r(t) \frac{dt}{t} \leq C
\end{equation}
for \(r \geq r_0(q)\).

**Proof.** First, we use condition \((1.6)\) and set
\[K_1(r) = \int_r^\infty \varphi(t^{-1}) \frac{dt}{t},\]
where \(r \geq 1\). Let us show that
\begin{equation}
\varphi(r^{-1}) \leq CK_1^{1/2}(\sqrt{r}) \ln^{-1} r \quad \text{for } r \geq 1
\end{equation}
with a constant \(C\) independent of \(r\). It is easily seen that
\[
\int_1^\infty \varphi(t^{-1})K_1^{-1/2}(t) \frac{dt}{t} = \int_0^{K_1(1)} K_1^{-1/2}(t) dK_1 < \infty,
\]
and the monotonicity of the functions \(\varphi(t^{-1})\) and \(K_1(t)\) for \(r \geq 1\) implies that
\[
\int_1^\infty \varphi(t^{-1})K_1^{-1/2}(t) \frac{dt}{t} > \int_{1/r}^r \varphi(t^{-1})t^{-1}K_1^{-1/2}(t) dt
\]
\[
\geq \varphi(r^{-1})K_1^{-1/2}(\sqrt{r}) \int_{1/r}^r dt = \frac{1}{2} \varphi(r^{-1})K_1^{-1/2}(\sqrt{r}) \ln r,
\]
whence \((2.20)\) follows. In the same way, by using condition \((1.8)\) and by setting
\[K_2(r) = \int_r^\infty \omega(t^{-1}) \frac{dt}{t},\]
one can show that
\begin{equation}
\omega(r^{-1}) \leq CK_2^{1/2}(\sqrt{r}) \ln^{-1} r \quad \text{for } r > 1
\end{equation}
with a constant \(C\) independent of \(r\).

Let us proceed to the proof of \((2.19)\). First, we estimate the integral of the function \(\varphi_r\) in \((2.6)\) occurring in \((2.17)\). By conditions \((1.6)\) and \((1.8)\), we have
\begin{equation}
\int_0^{2qr} \varphi_r(t) \frac{dt}{t} = \int_0^{2/r} (\varphi(2t) + \omega(t)) \frac{dt}{t} + \left( \varphi \left( \frac{4}{r} \right) + \omega \left( \frac{2}{r} \right) \right) \int_{2/r}^{2qr} \frac{dt}{t} \leq C + \left( \varphi \left( \frac{4}{r} \right) + \omega \left( \frac{2}{r} \right) \right) \ln (qr^2).
\end{equation}
By using \((2.20)\) and \((2.21)\), we obtain
\[
\int_0^{2qr} \varphi_r(t) \frac{dt}{t} \leq C + C \left( K_1^{1/2} \left( \sqrt{r} \frac{r}{4} \right) \ln^{-1} r \frac{r}{4} + K_2^{1/2} \left( \sqrt{r} \frac{r}{2} \right) \ln^{-1} r \frac{r}{2} \right) \ln (qr^2).
\]
Thus, since \(K_i(t) \to 0\) as \(t \to \infty\), \(i = 1, 2\), we obtain
\begin{equation}
\int_0^{2qr} \varphi_r(t) \frac{dt}{t} \leq C \quad \text{for } r \geq c(q)
\end{equation}
with a constant $C$ independent of $r$ and $q$. It remains to estimate the integral of the function $\omega_r$ in (2.7) occurring in (2.17). We have
\[
\int_0^{2qr} \omega_r(t) \frac{dt}{t} = \int_0^{r/2} \omega \left( \frac{2t}{r} \right) \frac{dt}{t} + \omega(1) \ln(2p) + \omega \left( \frac{4q}{r} \right) \ln(2qp^{-1}),
\]
and
\[
\int_0^{2qr} \omega_r(t) \frac{dt}{t} \leq C + CK_2^{1/2} \left( \sqrt{\frac{r}{4q}} \right) \ln^{-1} \frac{r}{4q} \ln(2qp^{-1}) \leq C \quad \text{for } r \geq c(q)
\]
by (2.19). By comparing (2.23) with (2.24), we arrive at (2.19). The proof of the lemma is complete.

We introduce the notation
\[
h_r(t) = \begin{cases} 
\omega \left( \frac{2t}{r} \right) & \text{for } 0 < t \leq \frac{r}{2}, \\
\omega(1) & \text{for } \frac{r}{2} < t \leq 2pr, \\
\omega \left( \frac{4q}{r} \right) & \text{for } 2pr < t \leq 2qr
\end{cases}
\]
and set
\[
w_r(t) = \varphi_r(t) + h_r(t).
\]
It is easily seen that
\[
\phi_r(t_1) \leq w_r(t_2) \quad \text{if } t_1 \leq t_2 \leq 2pr.
\]
Just as in the proof of Lemma 2.2, we establish that the inequality
\[
\int_0^{2qr} w_r(t) \frac{dt}{t} \leq C
\]
holds for $r \geq r_1(q)$ with a constant $C$ independent of $r$ and $q$.

2.2. Construction of a subsolution of the heat potential type. In what follows, the constants $q$ and $p$ have the same meaning as in Lemma 2.1. Let
\[
Z_r = B^0_{qr} \times (t_0, t_0 + 2r^2), \quad t_0 > 0,
\]
let $A_{ij}(y)$ be the entries of the inverse of the matrix $\{a_{ij}(y)\}$, and let
\[
\rho(x, y) = \left( \sum_{i,j=1}^n A_{ij}(y)(x_i - y_i)(x_j - y_j) \right)^{1/2}.
\]
Set
\[
{\mathcal R} = {\mathcal R}(x, y, t, \tau) = \frac{\rho^2(x, y)}{4(t - \tau)} - \frac{n}{2} \ln \frac{4r^2}{t - \tau},
\]
assuming that $(y, \tau) \in Z_r \cap (\mathbb{R}^{n+1} \setminus Z)$, $y \in (\mathbb{R}^n \setminus D) \cap (\overline{B^0_{qr}} \setminus B_{r/2}^0)$, and $t > \tau$. In what follows, $\chi(t)$ stands for the Heaviside function; i.e., $\chi(t) = 0$ for $t \leq 0$ and $\chi(t) = 1$ for $t > 0$.

**Lemma 2.3.** If conditions (1.3), (1.5), and (1.8) are satisfied, then for $r \geq r_0(q)$ there exists a function $F(x, y, t, \tau)$ such that $LF \geq 0$ in $Z_r \cap Z$ and the inequalities
\[
F(x, y, t, \tau) \leq C_1 \chi(t - \tau)(t - \tau)^{-n/2} \exp \left( -\frac{\rho^2(x, y)}{4(t - \tau)} \right),
\]
\[
F(x, y, t, \tau) \geq \begin{cases} 
C_2 \chi(t - \tau)(t - \tau)^{-n/2} \exp \left( -\frac{\rho^2(x, y)}{4(t - \tau)} \right) & \text{if } {\mathcal R} \leq 0, \\
0 & \text{if } {\mathcal R} > 0
\end{cases}
\]
hold for \((x,t) \in Z \cap Z\), where the positive constants \(C_1\) and \(C_2\) depend only on the dimension \(n\) of the space and the coefficients of the equation.

**Proof.** We make the scaling transformation \(x = \tilde{x}r, t - t_0 = \tilde{t}^2\) and set \(\tilde{a}_{ij}(\tilde{x}) = a_{ij}(r\tilde{x})\). Then (see (2.18))

\[
|\tilde{a}_{ij}(\tilde{x}) - \tilde{a}_{ij}(\tilde{y})| \leq C\phi_r(r|\tilde{x} - \tilde{y}|)
\]

in the new variables, and the cylinder \(Z_r\) is transformed into the cylinder \(Z = B_0 \times (0, 2)\). From now on, we omit the tildes and retain the old notation for the new variables and the transformed coefficients as well as for the cylinder \(Z\) in which the equation is given. In what follows, we assume for simplicity that \(a_{ij}(y) = \delta_{ij}\), where \(\delta_{ij}\) is the Kronecker delta, and set

\[
R = R(x, y, t, \tau) = \frac{|x - y|^2}{4(t - \tau)} - \frac{n}{2} \ln \frac{4}{t - \tau}
\]

for \((x, t) \in Z \cap Z\) and \((y, \tau) \in Z \cap (\mathbb{R}^{n+1} \setminus Z)\).

We need the inequality

\[
|\xi| \leq c_0(n) \exp \left(\frac{|\xi|^2}{4n}\right),
\]

which implies that \(|x - y| \leq 2c_0e^{R/n}\). In particular,

\[
|x - y| \leq p(n)e^{R/n}, \quad \text{where } p = \max\{2c_0, 2\}.
\]

In (2.34), we take the constant \(p\) in (2.33) and estimate the function \(\phi_r\) of the variable \(|x - y|\) (see (2.32)) from above via a function of the argument \(R\). It follows from (2.33) and the monotonicity of \(\phi_r\) that

\[
\varphi_r(r|x - y|) \leq \varphi_r(R) = \begin{cases} 
\varphi_r(pre^{R/n}) & \text{if } R \leq 0, \\
\varphi_r(pr) & \text{if } R > 0.
\end{cases}
\]

Likewise, since the function \(\omega_r\) is nondecreasing on \((0, p]\) and \(e^{R/n} \leq 1\) for \(R \leq 0\), we obtain

\[
\omega_r(r|x - y|) \leq \omega_r(R) = \begin{cases} 
\omega_r(pre^{R/n}) & \text{if } R \leq 0, \\
\omega(1) & \text{if } R > 0.
\end{cases}
\]

We introduce the notation \(\psi_r(R) = \varphi_r(R) + \omega_r(R)\) and arrive at the inequality

\[
\phi_r(r|x - y|) \leq \psi_r(R).
\]

Let us proceed to the construction of a subsolution of equation (1.1). Let

\[
Q = Q(t - \tau) = Cn \int_{\sqrt{t-\tau}}^2 w_r(prz^{-1}) \, dz,
\]

where the function \(w_r\) is defined in (2.23) and the positive constant \(C\) will be defined later. We seek a subsolution of equation (1.1) in the form

\[
F(x, y, t, \tau) = f(R(x, y, t, \tau)) \exp(Q(t - \tau)),
\]

a priori assuming that \(f(R) > 0\), \(f'(R) \leq 0\), and \(f''(R) \geq 0\). The substitution of \(F\) into the equation gives

\[
LF = Q \left(\frac{|x - y|^2}{4(t - \tau)}\right)^2 \left(\sum_{i,j=1}^n a_{ij}(x)\gamma_i\gamma_j + f'(R)\right)
\]

\[
+ \frac{f'(R)}{2(t - \tau)} \left(\sum_{i=1}^n a_{ii}(x) - n\right) + f(R) \frac{Cnw_r(pr\sqrt{t-\tau})}{2(t - \tau)},
\]

\[
(2.38)
\]
where $\gamma_i = \frac{x_i - y_i}{|x - y|}$. Note that
\[
\sum_{i=1}^{n} a_{ii}(y) = n, \quad \sum_{i,j=1}^{n} a_{ij}(y)\gamma_i\gamma_j = 1.
\]

It follows from the first equation by (2.32) and (2.36) that
\[
\left| \sum_{i=1}^{n} a_{ii}(x) - n \right| \leq Cn\phi_r(r|x - y|) \leq Cn\psi_r(R)
\]
and from the second equation by (1.3) and (2.36) that
\[
\left| \frac{1}{\sum_{i,j=1}^{n} a_{ij}(x)\gamma_i\gamma_j} - 1 \right| \leq C(n, \lambda)\phi_r(r|x - y|) \leq C(n, \lambda)\psi_r(R).
\]

Since $f'(R) \leq 0$, from (2.38) and (2.39) we obtain
\[
LF \geq Q \left( \frac{|x - y|^2}{4(t - \tau)^2} \left( f''(R) \sum_{i,j=1}^{n} a_{ij}(x)\gamma_i\gamma_j + f'(R) \right) + \frac{n f'(R) \psi_r(R)}{2(t - \tau)} + f(R) \frac{Cn \psi_r(pr\sqrt{t - \tau})}{2(t - \tau)} \right).
\]

Let us verify whether the inequality
\[
\psi_r(R) \frac{|x - y|^2}{2(t - \tau)^2} + \frac{w_r(pr\sqrt{t - \tau})}{2(t - \tau)} \geq \psi_r(R) \frac{2n(t - \tau)\ln \frac{4}{t - \tau}}{2(t - \tau)^2} \geq \psi_r(R) \frac{2n(t - \tau)}{2(t - \tau)^2} \geq \psi_r(R)
\]
holds. If $R > 0$, then
\[
\psi_r(R) \frac{|x - y|^2}{2(t - \tau)^2} \geq \psi_r(R) \frac{2n(t - \tau)\ln \frac{4}{t - \tau}}{2(t - \tau)^2} \geq \psi_r(R) \frac{2n(t - \tau)}{2(t - \tau)^2} \geq \psi_r(R),
\]
because $2n \ln \frac{4}{t - \tau} > 1$ for $0 < t - \tau < 2$, and inequality (2.42) is satisfied. If $R \leq 0$ and $|x - y|^2 \geq (t - \tau)$, then (2.42) also holds. Finally, if $R \leq 0$ and $|x - y|^2 < (t - \tau)$, then
\[
\exp \frac{|x - y|^2}{4n(t - \tau)} \leq \exp \frac{1}{4n} < 2,
\]
and it follows from the relation
\[
e^{R/n} = 2^{-1} \exp \left( \frac{|x - y|^2}{4n(t - \tau)} \right) (t - \tau)^{1/2}
\]
that $pe^{R/n} \leq p(t - \tau)^{1/2} \leq 2pe^{R/n} \leq 2p$. Now we use the structure of the functions $\psi_r$ and $w_r$ to conclude from (2.26) that $w_r(pr\sqrt{t - \tau}) \geq \psi_r(R)$, which implies (2.42) in the remaining case.
In view of (2.42), inequality (2.41) can be rewritten in the form

\[ LF \geq Q \left( \frac{|x - y|^2}{4(t - \tau)^2} \left( f''(R) \sum_{i,j=1}^{n} a_{ij}(x)\gamma_i\gamma_j + f'(R) \right) \right. \]
\[ + f'(R) \left( \frac{|x - y|^2}{2(t - \tau)^2} n\psi_r(R) + \frac{nw_r(pr\sqrt{t - \tau})}{2(t - \tau)} \right) + f(R) \left( \frac{Cnw_r(pr\sqrt{t - \tau})}{2(t - \tau)} \right) \]
\[ = Q \left( \frac{|x - y|^2}{4(t - \tau)^2} \left( f''(R) \sum_{i,j=1}^{n} a_{ij}(x)\gamma_i\gamma_j + f'(R) + 2f'(R)n\psi_r(R) \right) \right. \]
\[ + \frac{nw_r(pr\sqrt{t - \tau})}{2(t - \tau)}(f'(R) + Cf(R)) \]
\[ = Q \left( \frac{|x - y|^2}{4(t - \tau)^2} \sum_{i,j=1}^{n} a_{ij}(x)\gamma_i\gamma_j \right. \]
\[ \times \left( f''(R) + f'(R)(1 + 2n\psi_r(R))(1 + C(n, \lambda)\psi_r(R(1))) \right. \]
\[ + \frac{nw_r(pr\sqrt{t - \tau})}{2(t - \tau)}(f'(R) + Cf(R)) \right). \]

Now we use inequalities (2.36) and (2.40), by which

\[ (2.43) \quad LF \geq Q \left( \frac{|x - y|^2}{4(t - \tau)^2} \sum_{i,j=1}^{n} a_{ij}(x)\gamma_i\gamma_j \right. \]
\[ \times \left( f''(R) + f'(R)(1 + 2n\psi_r(R))(1 + C(n, \lambda)\psi_r(R(1))) \right. \]
\[ \left. + \frac{nw_r(pr\sqrt{t - \tau})}{2(t - \tau)}(f'(R) + Cf(R)) \right). \]

Next, we solve the ordinary differential equation

\[ f''(R) + f'(R)(1 + 2n\psi_r(R))(1 + C(n, \lambda)\psi_r(R)) = 0, \]

and take the solution of the form

\[ f(R) = \int_{R}^{\infty} \exp\left( - \int_{0}^{z} (1 + 2n\psi_r(s))(1 + C(n, \lambda)\psi_r(s)) \, ds \right) \, dz. \]

Thus, the right-hand side of (2.43) is nonnegative provided that

\[ (2.44) \quad f'(R) + Cf(R) \geq 0. \]

Let us show that (2.45) holds for an appropriate choice of the constant $C$. To this end, note that

\[ -f'(R) = \exp\left( - \int_{0}^{R} (1 + 2n\psi_r(s))(1 + C(n, \lambda)\psi_r(s)) \, ds \right) \]
\[ = \int_{0}^{\infty} \exp\left( - \int_{0}^{z} (1 + 2n\psi_r(s))(1 + C(n, \lambda)\psi_r(s)) \, ds \right) \]
\[ \times (1 + 2n\psi_r(z))(1 + C(n, \lambda)\psi_r(z)) \, dz. \]

Since the expression

\[ (1 + 2n\psi_r(z))(1 + C(n, \lambda)\psi_r(z)) \]

is bounded above by a constant $C$ independent of $r$ (the coefficients of the equation are bounded), we have

\[ -f'(R) \leq Cf(R). \]
We take $C = C$ and obtain $LF \geq 0$. By (2.44), the subsolution that we have found has the form
\[
F = Q \int_{R}^{\infty} \exp\left( - \int_{0}^{z} (1 + 2n \psi_{r}(s))(1 + C(n, \lambda \psi_{r}(s)) ds \right) dz,
\]
where $Q$ is the function in (2.37). Let us estimate the function $F$. We have
\[
F = Q \int_{R}^{\infty} \exp\left( - \int_{0}^{z} (1 + 2n \psi_{r}(s))(1 + C(n, \lambda \psi_{r}(s)) ds \right) dz \\
\leq C_{1} Q \int_{R}^{\infty} \exp\left( - \int_{0}^{z} (1 + 2n \psi_{r}(s))(1 + C(n, \lambda \psi_{r}(s)) ds \right) \\
\times (1 + 2n \psi_{r}(z))(1 + C(n, \lambda \psi_{r}(z)) dz \\
= C_{1} Q \exp\left( - \int_{0}^{R} (1 + 2n \psi_{r}(s))(1 + C(n, \lambda \psi_{r}(s)) ds \right).
\]
Thus,
\[
F \leq C_{1} Q \exp\left( - \int_{0}^{R} (1 + 2n \psi_{r}(s))(1 + C(n, \lambda \psi_{r}(s)) ds \right).
\]
By a similar argument, we obtain
\[
F \geq C_{2} Q \exp\left( - \int_{0}^{R} (1 + 2n \psi_{r}(s))(1 + C(n, \lambda \psi_{r}(s)) ds \right).
\]
We set
\[
G(x, y, t, \tau) = \chi(t - \tau)(t - \tau)^{-n/2} \exp\left( - \frac{|x - y|^{2}}{4(t - \tau)} \right),
\]
and use the relation
\[
e^{-R} = 4^{n/2} G(x, y, t, \tau),
\]
by which
\[
\exp\left( - \int_{0}^{R} (1 + 2n \psi_{r}(s))(1 + C(n, \lambda \psi_{r}(s)) ds \right) \\
= 4^{n/2} G \exp\left( - \int_{0}^{R} (C(n, \lambda) + 2n + 2C(n, \lambda) \psi_{r}(s)) \psi_{r}(s) ds \right).
\]
Now if $R > 0$, then (see (2.46))
\[
0 \leq F \leq C_{1} Q G.
\]
If $R \leq 0$, then (see (2.34)–(2.36))
\[
- \int_{0}^{R} (C(n, \lambda) + 2n + 2C(n, \lambda) \psi_{r}(s)) \psi_{r}(s) ds \\
\leq \int_{-\infty}^{0} (C(n, \lambda) + 2n + 2C(n, \lambda) \phi_{r}(pr^{s/n})) \phi_{r}(pr^{s/n}) ds,
\]
and after the change of variable $\theta = pr^{s/n}$ we obtain, in view of (2.49),
\[
- \int_{0}^{R} (C(n, \lambda) + 2n + 2C(n, \lambda) \psi_{r}(s)) \psi_{r}(s) ds \\
\leq n \int_{0}^{pr} (C(n, \lambda) + 2n + 2C(n, \lambda) \phi_{r}(\theta)) \phi_{r}(\theta) d\theta \leq C.
\]
Thus, it follows from (2.46)–(2.48) that
\[
C_{2} G \leq C_{2} Q G \leq F \leq C_{3} Q G \quad \text{for } R \leq 0.
\]
It remains to estimate the integral $Q$ in (2.37) from above. By (2.27), one has
\[ \int_{\sqrt{t_0}}^{2} \sqrt{t - \tau} w_r(\text{prz}) \frac{dz}{z} \leq \int_{0}^{2pr} w_r(z) \frac{dz}{z} \leq C; \]
as a result, it follows from (2.49) and (2.50) that
\[ F \leq C_4 G, \tag{2.51} \]
and
\[ F \geq \begin{cases} C_5 G & \text{if } R \leq 0, \\ 0 & \text{if } R > 0. \end{cases} \tag{2.52} \]

These estimates have been obtained in the variables $\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}$. (We return to the notation with tildes.) Let us pass to the original coordinates (i.e., replace $x$ by $x_{r-1}$, $t$ by $(t - t_0)r^{-2}$, and likewise for $y$ and $\tau$.) Then the function $G$ acquires the form $r^n G(x, y, t, \tau)$. We multiply the barrier function $F$ by $r^{-n}$ and arrive at (2.30) and (2.31) by virtue of (2.51) and (2.52).

So far, we have assumed that $a_{ij}(y) = \delta_{ij}$. If this is not the case, then one should make a nonsingular linear transformation of the space variables after which the elliptic part of the parabolic operator at a point $y$ coincides with the Laplace operator. Under this transformation, the conditions on the coefficients acquire a form similar to (1.3) and (1.5)–(1.8). As a result, we arrive at the estimates (2.30) and (2.31) with the squared Euclidean distance $|x - y|^2$ occurring in the exponent. By making the inverse transformation in these estimates, we obtain the desired estimates (2.30) and (2.31). The proof of the lemma is complete. \(\square\)

Note that the elliptic distance $\rho(x, y)$ (see (2.29)) satisfies the estimates
\[ \alpha|x - y| \leq \rho(x, y) \leq \beta|x - y| \tag{2.53} \]
with constants $\alpha$ and $\beta$ depending on the ellipticity constants of the matrix $\{a_{ij}(x)\}$ alone. To simplify the computations in what follows, it is convenient to make a homothety transformation with respect to the spatial variable, after which equation (1.1) acquires the form
\[ \sum_{i,j=1}^{n} 4\beta a_{ij}(x) u_{x_i x_j} - u_t = 0. \]
As a result of this transformation, the conditions of the form (1.3) and (1.5)–(1.8) on the coefficients are satisfied, and relation (2.53) becomes
\[ a|x - y| \leq \rho(x, y) \leq \frac{|x - y|}{4}, \]
where $a = 4^{-1}\alpha\beta^{-1}$. Accordingly, inequalities (2.30) and (2.31) can be rewritten as
\[ 0 \leq F(x, y, t, \tau) \leq C_1 \chi(t - \tau)(t - \tau)^{-n/2} \exp \left( -\frac{a^2|x - y|^2}{4(t - \tau)} \right) \tag{2.54} \]
and
\[ F(x, y, t, \tau) \geq C_2 \chi(t - \tau)(t - \tau)^{-n/2} \exp \left( -\frac{|x - y|^2}{64(t - \tau)} \right) \tag{2.55} \]
provided that
\[ |x - y|^2 \leq 32n(t - \tau) \ln \frac{4r^2}{l - \tau}. \tag{2.56} \]
3. Sufficient condition for stabilization to zero

3.1. Growth lemma. The proof of sufficiency of condition (1.9) in Theorem 1.1 is based on an analog of the growth lemma due to Landis [19 Section 3.6, Lemma 6.1]. Before stating this lemma, set

\[ Z_{r_1}^{t_1, t_2} = \{(x, t) : t_1 < t < t_2, |x| < r\} \]

and consider the cylinders

\[ Z_1 = Z_{r_0 - 2r}^{t_0, t_0}, \quad Z_2 = Z_{r_0 - 3r^2/4}^{t_0, t_0}, \quad t_0 > 2r^2 > 0, \]

where the constant \( \theta > 1 \) will be chosen later.

Further, we need the notion of heat capacity \( \gamma(E) \) of a compact set \( E \subset \mathbb{R}^{n+1} \), which is defined as the supremum of \( \mu(E) \) over all Borel measures \( \mu \) satisfying the condition

\[ \int_E \chi(t - \tau)(t - \tau)^{-n/2} \exp\left(-\frac{|x - y|^2}{4(t - \tau)}\right) \, d\mu(y, \tau) \leq 1 \quad \text{for } (x, t) \notin E. \]

It suffices to establish the stabilization to zero of the solution of problem (1.1), (1.2) for a nonnegative initial function \( u_0(x) \), which is assumed in what follows. We also assume that conditions (1.3) and (1.5)–(1.8) are satisfied.

Lemma 3.1 (Landis lemma). If \( u(x, t) \) is a nonnegative solution of problem (1.1), (1.2), then there exist constants \( \theta_1 > 1 \) and \( \eta > 0 \) depending only on the coefficients of equation (1.1) such that

\[ \sup_{Z \cap Z_1} u \geq \left(1 + \frac{\eta}{r^n} \gamma(E_r)\right) \sup_{Z \cap Z_2} u \]

for \( \theta \geq \theta_1 \) and \( r \geq r_0(\theta) \), where

\[ E_r = G_r \times \left[t_0 - 2r^2, t_0 - \frac{7r^2}{4}\right], \quad G_r = (\mathbb{R}^n \setminus D) \cap (\bar{B}_{r/2}^0 \setminus B_r^0). \]

Proof. Without loss of generality, we assume that the boundary \( \partial D \) of the base \( D \) of the cylinder \( Z \) in which the boundary value problem (1.1), (1.2) is considered contains the origin. Consider the function \( F \) defined in Lemma 2.3 for the cylinder \( Z_1 = B_{q}r \times (t_0 - 2r^2, t_0) \) with a constant \( q \) to be defined below.

Let a measure \( \mu \) realize the heat capacity \( \gamma_a(E_r) \) of the compact set \( E_r \) for the kernel

\[ F_1(x, y, t, \tau) = \chi(t - \tau)(t - \tau)^{-n/2} \exp\left(-\frac{q^2|x - y|^2}{4(t - \tau)}\right). \]

It is well known (see [20]) that there exists a positive constant \( C_3(n, a) \) such that

\[ \gamma_a(E) \geq C_3 \gamma(E) \]

for each compact set \( E \subset \mathbb{R}^n \), where \( \gamma(E) \) is the classical heat capacity defined above. It follows from the upper bound (2.5.1) that the function

\[ V(x, t) = \frac{1}{C_1} \int_E F(x, y, t, \tau) \, d\mu(y, \tau) \]

satisfies the inequality

\[ V(x, t) \leq 1, \quad (x, t) \in \mathbb{R}^{n+1} \setminus E_r. \]

It is subparabolic on \( \mathbb{R}^{n+1} \setminus E_r \) and satisfies inequality \( V \leq 1 \) in the cylinder \( Z_1 \).
Let $S_1$ be the lateral surface of the cylinder $Z_1$. Let us estimate the function $F$ for $(x, t) \in S_1$ for an arbitrary point $(y, \tau) \in E_r$ with the use of the upper bound (2.54). By virtue of this bound,

\begin{equation}
(3.6) \quad \sup_{(x, t) \in S_1} F \leq C_1 \sup_{(x, t) \in S_1} (t - \tau)^{-n/2} \exp \left( -\frac{a^2|x - y|^2}{4(t - \tau)} \right).
\end{equation}

Let us fix a point $x \in \partial B^0_{qr}$ and find the value of $t > \tau$ at which the function on the right-hand side in (3.6) attains its maximum,

\[ \frac{\partial}{\partial t} \left[ (t - \tau)^{-n/2} \exp \left( -\frac{a^2|x - y|^2}{4(t - \tau)} \right) \right] = 0. \]

We obtain

\[ t - \tau = \frac{a^2|x - y|^2}{2n}. \]

For $|x| = qr$ and $|y| \leq r$, where $q > 1$, we have $|x - y| \geq (q - 1)r$. Hence

\[ t - \tau \geq \frac{a^2(q - 1)^2r^2}{2n}. \]

It follows from (3.6) and the monotonicity of $F_1$ until the first maximum that if $q > 2$, then

\begin{equation}
(3.7) \quad \sup_{(x, t) \in S_1} F \leq C_1 \left( \frac{8n}{a^2q^2} \right)^{n/2} \exp \left( -\frac{n}{2} \right) r^{-n}.
\end{equation}

Let us estimate $\inf_{(x, t) \in Z_2} F(x, y, t, \tau)$ under the assumption that $(y, \tau) \in E_r$. Since $|x - y|^2 \leq 4r^2$ and $r^2/4 \leq t - \tau \leq 2r^2$ for $(x, t) \in Z_2$ and $(y, \tau) \in E_r$, we have inequality (2.56). By applying the lower bound (2.55) and the estimates given above, we obtain

\begin{equation}
(3.8) \quad \inf_{(x, t) \in Z_2} F \geq C_2 2^{-n/2} \exp \left( -\frac{1}{16} \right) r^{-n}.
\end{equation}

In view of (3.7) and (3.8), we see that the potential $V$ in (3.5) satisfies the inequalities

\begin{equation}
(3.9) \quad \sup_{(x, t) \in S_1} V \leq \left( \frac{8n}{a^2q^2} \right)^{n/2} \exp \left( -\frac{n}{2} \right) r^{-n} \gamma_0(E_r),
\end{equation}

\begin{equation}
(3.10) \quad \inf_{(x, t) \in Z_2} V \geq C_1^{-1} C_2 2^{-n/2} \exp \left( -\frac{1}{16} \right) r^{-n} \gamma_0(E_r).
\end{equation}

In the cylinder $Z_1 \cap Z$, consider the function

\begin{equation}
(3.11) \quad U(x, t) = M \left( 1 - V(x, t) + \sup_{(x, t) \in S_1} V \right),
\end{equation}

where $M = \sup_{Z_1 \cap Z} u(x, t)$. Let us compare $U$ with the solution $u$ of problem (1.1), (1.2) on the parabolic boundary of the domain $Z_1 \cap Z$. Clearly, the function (3.11) is superparabolic in $Z_1 \cap Z$. We have $U(x, t) \geq M$ on the lateral surface of $Z_1 \cap Z$, because $V \leq 1$, and $U = M$ on the lower base, because $V = 0$ on the lower face of the cylinder $Z_1 \cap Z$. By the maximum principle, $u \leq U$ in $Z_1 \cap Z$ and

\[ \sup_{Z_2 \cap Z} u \leq M \left( 1 - \inf_{Z_2 \cap Z} V + \sup_{(x, t) \in S_1} V \right). \]

Here we use the estimates (3.9) and (3.10) and take $q = \theta_1$ to satisfy the inequality

\[ C_1^{-1} C_2 2^{-n/2} \exp \left( -\frac{1}{16} \right) \geq 2 \left( \frac{8n}{a^2q^2} \right)^{n/2} \exp \left( -\frac{n}{2} \right). \]
Thus, for \( q = \theta \geq \theta_1 \) in view of (3.1) we obtain
\[
\sup_{Z_2 \cap Z} u \leq M \left( 1 - \frac{\eta}{r^n} \gamma(E_r) \right),
\]
where
\[
\eta = 2^{-1} C_1^{-1} C_2^2 2^{-n/2} \exp \left( -\frac{1}{16} \right).
\]
It remains to note that (3.12) implies the desired estimate (3.2). The proof of the lemma is complete. \( \square \)

We need yet another assertion based on Lemma 3.1 and which plays a key role in the proof of sufficiency in Theorem 1.1. Namely, we need a lower bound for the heat capacity of the cylinder \( Q = G \times [t_1, t_2] \), where \( G \subset \mathbb{R}^n \) is a compact set, via the capacity \( \text{cap}(G) \) of the base of \( Q \). It is well known that (e.g., see [13, Section 4, Lemma 4])
\[
\gamma(Q) \geq C_4 (t_2 - t_1) \text{cap}(G)
\]
with a constant \( C_4 \) independent of \( G, t_1 \), and \( t_2 \). Using (3.13) and (3.3), we rewrite (3.2) in the form
\[
\sup_{Z \cap Z_1} u \geq \left( 1 + \frac{C_4 \eta}{4r^{n-2}} \text{cap}(G_r) \right) \sup_{Z \cap Z_2} u.
\]
An estimate similar to (3.14) holds in the cylinders
\[
Z_1(k) = Z_{\theta r_k}^{t_0 - 2r^2 k^2 t_0}, \quad Z_2(k) = Z_{r_k}^{t_0 - 3r^2 k^2/4, t_0}, \quad r_k = 2^{-k} r, \quad k = 1, 2, \ldots, k_0.
\]
Namely,
\[
\sup_{Z \cap Z_1(k)} u \geq \left( 1 + \frac{C_4 \eta}{4r_k^{n-2}} \text{cap}(G_{r_k}) \right) \sup_{Z \cap Z_2(k)} u, \quad k = 1, 2, \ldots, k_0,
\]
for \( r \geq r_0(\theta, k_0) \). In particular for \( k = 0 \), we obtain (3.14). Using the maximum principle and the numbers \( r_k \), we rewrite (3.14) and (3.15) in the form
\[
\sup_{Z \cap Z_1} u \geq \left( 1 + \frac{C_4 \eta}{4k^{n-2}} \text{cap}(G_{r_k}) \right) \sup_{Z \cap Z_2(k_0)} u, \quad k = 0, 1, \ldots, k_0,
\]
and sum all these relations. Then we divide both sides of the resulting inequality by \( k_0 + 1 \) and obtain
\[
\sup_{Z \cap Z_1} u \geq \left( 1 + \frac{C_4 \eta}{4(k_0 + 1)^{n-2}} \sum_{k=0}^{k_0} 2^{k(n-2)} \text{cap}(G_{r_k}) \right) \sup_{Z \cap Z_2(k_0)} u
\]
\[
\geq \left( 1 + \frac{C_4 \eta}{4(k_0 + 1)^{n-2}} \sum_{k=0}^{k_0} \text{cap}(G_{r_k}) \right) \sup_{Z \cap Z_2(k_0)} u.
\]
Since the Wiener capacity is semi-additive, we have
\[
\sum_{k=0}^{k_0} \text{cap}(G_{r_k}) \geq \text{cap}((\mathbb{R}^n \setminus D) \cap (\bar{B}_r^0 \setminus B_{r_{k_0+1}}^0)).
\]
Take \( k_0 \) to be the first positive integer such that \( 2^{k_0+1} \geq \theta \), and set \( H = (\mathbb{R}^n \setminus D) \cap (\bar{B}_r^0 \setminus B_{r_{k_0+1}}^0) \). Then the estimate (3.16), which already holds for \( r > r_1(\theta) \), acquires the form
\[
\sup_{Z \cap Z_1} u \geq \left( 1 + \eta_1 \frac{\text{cap}(H)}{r^{n-2}} \right) \sup_{Z \cap Z_2(k_0)} u.
\]
Set \( Z_1 = Z_1 \) and \( Z_2 = Z_{g^{-2}}^{t_0 \to 3g^2 t_0 / 4} \). Then it follows from (3.17) by the maximum principle that

\[
(3.18) \quad \sup_{Z \cap Z_1} u \geq \left( 1 + \eta_1 \frac{\text{cap}(H)}{\nu n^{-2}} \right) \sup_{Z \cap Z_2} u.
\]

Before continuing the exposition, let us present an auxiliary assertion. This assertion is well known (e.g., see [14, Section 5.1, p. 356 of the Russian edition]), and we give it for the reader’s convenience. Let

\[
(3.19) \quad H_m = (\mathbb{R}^n \setminus D) \cap \{ x : \theta^{m-1} \leq |x| \leq \theta^m \}, \quad \theta > 1.
\]

**Lemma 3.2.** If condition (1.9) is satisfied, then

\[
(3.20) \quad \sum_{m=m_0}^\infty \frac{\text{cap}(H_m)}{\theta^{m(n-2)}} = \infty
\]

for each \( \theta > 1 \).

**Proof.** We set \( F_m = (\mathbb{R}^n \setminus D) \cap \{ x : |x| \leq \theta^m \} \) and rewrite (1.9) in the equivalent form

\[
(3.21) \quad \sum_{m=m_0}^\infty \frac{\text{cap}(F_m)}{\theta^{m(n-2)}} = \infty.
\]

Since the capacity is semi-additive, we have

\[
\text{cap}(F_m) \leq \text{cap}(H_m) + \text{cap}(F_{m-1}),
\]

and hence

\[
\sum_{m=m_0}^N \frac{\text{cap}(H_m)}{\theta^{m(n-2)}} \geq \sum_{m=m_0}^N \frac{\text{cap}(F_m)}{\theta^{m(n-2)}} - \theta^{2-n} \sum_{m=m_0}^N \frac{\text{cap}(F_{m-1})}{\theta^{(m-1)(n-2)}} = \sum_{m=m_0}^N \frac{\text{cap}(F_m)}{\theta^{m(n-2)}} - \theta^{2-n} \sum_{m=m_0}^{N-1} \frac{\text{cap}(F_m)}{\theta^{m(n-2)}} = (1 - \theta^{2-n}) \sum_{m=m_0}^N \frac{\text{cap}(F_m)}{\theta^{m(n-2)}} + \theta^{2-n} \frac{\text{cap}(F_{N})}{\theta^{N(n-2)}} - \theta^{2-n} \frac{\text{cap}(F_{m_0-1})}{\theta^{(m_0-1)(n-2)}}.
\]

Thus,

\[
\sum_{m=m_0}^N \frac{\text{cap}(H_m)}{\theta^{m(n-2)}} \geq (1 - \theta^{2-n}) \sum_{m=m_0}^N \frac{\text{cap}(F_m)}{\theta^{m(n-2)}} - \theta^{2-n} \frac{\text{cap}(F_{m_0-1})}{\theta^{(m_0-1)(n-2)}},
\]

and in the limit as \( N \to \infty \) we find that (3.21) implies (3.20). The proof of the lemma is complete. \( \square \)

3.2. **Proof of the sufficient stabilization condition.** Assume that condition (1.9) is satisfied. Then the series (3.20) diverges for each \( \theta > 1 \). We set \( r = \theta^{m+2} \) in (3.1) and introduce the system of coaxial cylinders

\[
Z_1^{m} = Z_{\theta_{m+3}}^{t_0 \to 2g^2 t_0} / 4, \quad Z_2^{m} = Z_{\theta_{m+2}}^{t_0 \to 3g^2 t_0 / 4}.
\]

Take a positive constant \( \theta \) such that \( \theta^{-2} \leq \min\{1/8, \theta_1^{-2}\} \), where \( \theta_1 > 1 \) is the constant in Lemma 3.1. This choice of \( \theta \) ensures the inclusions \( Z_2^{m} \supset Z_1^{m-1} \) for all \( m = 1, 2, \ldots \).

Set

\[
M_{m+3} = \sup_{Z_2^{m} \cap Z} u(x, t),
\]

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where \( u(x, t) \) is the positive solution of problem (1.1), (1.2). Inequality (3.18) and notation (3.19) imply the estimate

\[
M_{m+3} \geq \left( 1 + \eta_1 \frac{\text{cap}(H_{m+2})}{\theta(m+2)(n-2)} \right) M_m
\]

for \( m \geq m_0(\theta) \). By (3.20), at least one of the three series

\[
\sum_{k=k_0}^{\infty} \frac{\text{cap}(H_{3k+i-2})}{\theta(3k+i-2)(n-2)}, \quad i = 0, 1, 2,
\]
diverges; let this be the case for \( i = i_0 \). We take \( m = 3k + i_0 - 4 \) in (3.22) and obtain

\[
M_{3k+i_0-1} \geq \left( 1 + \eta_1 \frac{\text{cap}(H_{3k+i_0-2})}{\theta(3k+i_0-2)(n-2)} \right) M_{3(k-1)+i_0-1}
\]

for \( 3k + i_0 - 4 \geq m_0(\theta) \).

By iterating relation (3.24), we find that

\[
M_{3k+i_0-1} \geq M_{3k_0+i_0-1} \prod_{m=k_0}^{k} \left( 1 + \eta_1 \frac{\text{cap}(H_{3m+i_0-2})}{\theta(3m+i_0-2)(n-2)} \right)
\]

\[
= M_{3k_0+i_0-1} \cdot \exp \left( \sum_{m=k_0}^{k} \ln \left( 1 + \eta_1 \frac{\text{cap}(H_{3m+i_0-2})}{\theta(3m+i_0-2)(n-2)} \right) \right).
\]

Hence, since

\[
\text{cap}(H_{3m+i_0-2}) \leq \theta(3m+i_0-2)(n-2),
\]
we can use the inequality \( \ln(1 + \eta_1 x) > ax, x \in (0, 1] \), where the constant \( a > 0 \) depends on \( \eta_1 \), to obtain

\[
M_{3k+i_0-1} \geq M_{3k_0+i_0-1} \exp \left( a \sum_{m=k_0}^{k} \frac{\text{cap}(H_{3m+i_0-2})}{\theta(3m+i_0-2)(n-2)} \right).
\]

Note that

\[
M_{3k+i_0-1} \leq M,
\]
because the solution of problem (1.1), (1.2) is bounded; thus, we can rewrite (3.25) in the form

\[
M_{3k_0+i_0-1} \leq M \exp \left( -a \sum_{m=k_0}^{k} \frac{\text{cap}(H_{3m+i_0-2})}{\theta(3m+i_0-2)(n-2)} \right).
\]

Fix an \( \varepsilon > 0 \) and take a positive integer \( k(\varepsilon) \) such that the right-hand side of (3.26) does not exceed \( \varepsilon/2 \). This is possible, because the solution of problem (1.1), (1.2) is bounded and the series (3.23) diverges for \( i = i_0 \). As a result, we obtain

\[
M_{3k_0+i_0-1} = \sup_{Z \cap Z_1^{3k+i_0-1}} u \leq \varepsilon.
\]

Given \( k(\varepsilon) \), we find a \( t(\varepsilon) \) from the inequality

\[
t(\varepsilon) \geq 2\theta^{2(3k(\varepsilon)+i_0-1)}.
\]

Now if \( t > t(\varepsilon) \) and

\[
t \geq 2\theta^{2(3k+i_0-1)},
\]
then we arrive at the inequality

\[
\sup_{Z \cap Z_1^{3k+i_0-1}} u \leq \varepsilon,
\]
since the number of terms in the exponent in (3.26) does not decrease under assumption (3.27). Thus, we have proved the limit relation (1.4) and hence the sufficiency in Theorem 1.1.

4. CONSTRUCTION OF A SUPERSOLUTION OF THE ELLIPTIC OPERATOR

In an unbounded domain $D \subset \mathbb{R}^n$, $n > 2$, consider a uniformly elliptic operator

$$
A = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}
$$

whose coefficients satisfy conditions (1.3) and (1.5)–(1.8). Just as before, we assume that the origin lies on the boundary $\partial D$. We extend the coefficients of the operator (4.1) to the entire $\mathbb{R}^n$ in the same way as in Section 2.1 and rewrite conditions (1.5) and (1.7) in the form (2.2) and (2.3), respectively.

Before continuing our exposition, let us present simple corollaries of conditions (2.2) and (2.3). In what follows, we set

$$
\Lambda = \max_{i,j=1,...,n} \sup_{x \in \mathbb{R}^n} |a_{ij}(x)|,
$$

and $\rho(x,y)$ has the same meaning as in (2.29). By (2.2), the monotonicity of $\varphi$ and $\omega$, and the estimate (2.53), we have

$$
|a_{ij}(x) - a_{ij}(y)| \leq \bar{\varphi}(\rho(x,y)) \quad \forall x, y \in \mathbb{R}^n \setminus D,
$$

where

$$
\bar{\varphi}(t) = \begin{cases} 
\varphi(2\alpha^{-1}t) + \omega(\alpha^{-1}t) & \text{for } 0 < t \leq \frac{\alpha}{2}, \\
2\Lambda & \text{for } t > \frac{\alpha}{2}.
\end{cases}
$$

Next, if $y \in B_{64}^0 \setminus D$, then

$$
|a_{ij}(x) - \alpha_{ij}| \leq \bar{\omega}\left(\frac{1}{|x-y|}\right) = 64\omega\left(\frac{1}{|x-y|}\right) \quad \forall x \in D \setminus B_2^y.
$$

For the proof, it suffices to note that if $y \in B_{64}^0$, then $\frac{1}{|x|+1} \leq \frac{64}{|x-y|}$. Now we use (2.3), (2.4), and the monotonicity and semi-additivity of $\omega$ and arrive at (4.3).

In the same way, we establish the inequality

$$
|a_{ij}(x) - \alpha_{ij}| \leq \bar{\omega}\left(\frac{1}{|x-y|}\right) \quad \forall x \in D \cap (B_{|y|/4}^y \setminus B_2^y) \text{ and } \forall x \in D \setminus B_{2|y|}^y
$$

under the assumption that $|y| \geq 16$ and $y \in \mathbb{R}^n \setminus D$. If $x \in D \cap B_{|y|/4}^y$, then $|y| \leq \frac{4}{3}|x|$. Indeed, since $|y| \leq |x-y| + |x| \leq \frac{4|y|}{3} + |x|$, we obtain $|y| \leq \frac{4}{3}|x|$ and $|x-y| \leq |x| + |y| \leq \frac{7}{3}(|x|+1) < 3(|x|+1)$, and so $\frac{1}{|x|+1} \leq \frac{3}{|x-y|}$. We again use (2.3), (2.4), and the properties of $\omega$ and obtain (4.4). If $x \in D \setminus B_2^y$, then $|y| \leq |x|$ and $\frac{1}{|x|+1} \leq \frac{2}{|x-y|}$, which again implies (4.4).

**Lemma 4.1.** If conditions (1.3) and (1.5)–(1.8) are satisfied, then there exists a function $G(x,y)$ such that

$$
AG = \sum_{i,j=1}^{n} a_{ij}(x)G_{x,x_j} \leq 0 \quad \text{in } D \text{ for } y \in \mathbb{R}^n \setminus D
$$

and the double inequality

$$
C_1(n,A)|x-y|^{2-n} \leq G(x,y) \leq C_2(n,A)|x-y|^{2-n}
$$

holds for $x \in D$ and $y \in \mathbb{R}^n \setminus D$, where the positive constants $C_1$ and $C_2$ depend only on the space dimension $n$ and the coefficients of the operator $A$. 
Proof. First, assume that \( y \in B_{54}^0 \setminus D \). In this case, the argument is based on the construction of three auxiliary supersolutions \( G_i(x, y) \), \( i = 1, 2, 3 \), in the respective domains \( D \cap B_3^y \), \( D \setminus B_2^y \), and \( D \). In what follows, we assume for simplicity that the matrix \( \{ \alpha_{ij} \} \) in condition (1.4) is the identity matrix.

Supersolution \( G_1(x, y) \) in the domain \( D \cap B_3^y \). We set \( E_r^y = \{ x : \rho(x, y) < r \} \) and note that \( B_2^y \subset E_{\beta r}^y \subset E_{\beta r/\alpha}^y \). We seek a supersolution \( G_1(x, y) \) of the operator (1.1) in \( D \cap E_{4 \beta/\alpha}^y \), in the form \( G_1(x, y) = f(\rho(x, y)) \). Set

\[
e_y(x) = \frac{\sum_{i,j=1}^n a_{ij}(x) A_{ij}(y)}{\sum_{i,j=1}^n a_{ij}(x) \rho(x, y) \rho_x(x, y)},
\]

and note that

\[
Af(\rho(x, y)) = f''(\rho) + \frac{e_y(x) - 1}{\rho} f'(\rho).
\]

It is easily seen that

\[
|e_y(x) - n| \leq c(n) \bar{\varphi}(\rho)
\]

by (4.2), and we find the desired function \( G_1(x, y) = f(\rho(x, y)) \) under the assumption that \( f'(\rho) \leq 0 \) as a solution of the equation

\[
f''(\rho) + \frac{n - 1 - c \bar{\varphi}(\rho)}{\rho} f'(\rho) = 0
\]

in the form

\[
G_1(x, y) = \int_{\rho(x, y)}^{4 \beta/\alpha} t^{1-n} \exp\left(-c \int_{t}^{4 \beta/\alpha} \bar{\varphi}(\tau) \tau^{-1} d\tau \right) dt.
\]

It follows from conditions (1.6) and (1.8) and relation (4.2) that

\[
c_1(n, A)|x - y|^{2-n} \leq G_1(x, y) \leq c_2(n, A)|x - y|^{2-n},
\]

\( x \in D \cap B_3^y \subset D \cap E_{4 \beta/\alpha}^y \).

Supersolution \( G_2(x, y) \) in the domain \( D \setminus B_2^y \). For \( x \in D \setminus B_2^y \), we seek a supersolution in the form

\[
G_2(x, y) = f(|x - y|) = f(r).
\]

In this case,

\[
Af(r) = f''(r) + \frac{\tilde{e}_y(x) - 1}{r} f'(r),
\]

where

\[
\tilde{e}_y(x) = \frac{\sum_{i,j=1}^n a_{ij}(x)}{\sum_{i,j=1}^n a_{ij}(x) r_x(x, y) r_{x_j}(x, y)}.
\]

We again assume that \( f'(r) \leq 0 \). By (4.3),

\[
|\tilde{e}_y(x) - n| \leq c(n) \tilde{\omega} \left( \frac{1}{r} \right);
\]

in view of (1.8), we find \( f \) as a solution of the differential equation

\[
f''(r) + \frac{n - 1 - c \tilde{\omega}(1/r)}{r} f'(r) = 0
\]

in the form

\[
f(r) = G_2(x, y) = \int_{|x - y|}^{\infty} t^{1-n} \exp\left(-c \int_{t}^{\infty} \tilde{\omega} \left( \frac{T}{\tau} \right) \tau^{-1} d\tau \right) dt,
\]

for which we have the estimates

\[
c_3(n, A)|x - y|^{2-n} \leq G_2(x, y) \leq c_4(n, A)|x - y|^{2-n}, \quad x \in D \setminus B_2^y.
\]
Supersolution $G_3(x, y)$ in the domain $D$ and the construction of the function $G(x, y)$. Let $\eta \in C_0^\infty(B_2^1)$, $0 \leq \eta \leq 1$, and $\eta(x) = 1$ for $x \in B_3^1$. Consider the function
\begin{equation}
F(x, y) = \eta(x)G_1(x, y) + (1 - \eta(x))G_2(x, y).
\end{equation}

It is easily seen that
\begin{equation}
AF(x, y) \leq 0 \quad \text{for } x \in D \cap B_3^1\text{ and for } x \in D \setminus B_4^1,
\end{equation}
\[|AF(x, y)| \leq C_0(n, A) \quad \text{for } x \in D \cap (B_1^3 \setminus B_4^3),\]
where $C_0$ is the constant in (4.14), then, by (4.7) and (4.12), the desired supersolution $G(x, y)$ can be defined as
\[G(x, y) = F(x, y) + G_3(x, y).\]

We seek $G_3(x, y)$ in the form $f(|x - y|)$ assuming that $f'(r) \leq 0$. Then equation (4.8) holds for $Af(r)$. Let
\begin{equation}
l = \inf_{|z| = 1, \, x \in \mathbb{R}^n} \frac{\sum_{i=1}^n a_{ii}(x)}{\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j}.
\end{equation}

We replace $\bar{\omega}(x)$ in (4.8) by the infimum $l$ for $r \leq 4$ and use inequality (4.9) for $r > 4$. If we set
\[\psi(r) = \begin{cases} n - l, & \text{for } 0 < r \leq 4, \\ \bar{\omega}(\frac{1}{r}), & \text{for } r > 4, \end{cases}\]
\[g(r) = \begin{cases} 0, & \text{for } r \geq 4, \\ C_0, & \text{for } 0 < r < 4, \end{cases}\]
we have
\[f''(r) + \frac{n - 1 - \psi(r)}{r} f'(r) = -g(r),\]
for which the desired solution has the form
\begin{equation}
G_3(x, y) = \int_r^\infty \tau^{1-n} \exp\left(-\int_\tau^\infty \psi(z)z^{-1}dz\right)\times \left(\int_0^\tau \tau^{n-1} \exp\left(\int_s^\infty \psi(z)z^{-1}dz\right)g(s)ds\right)d\tau.
\end{equation}

First, let us show that (4.15) holds for $x \in D \cap B_1^3$. For $\tau < 4$, we use the relation
\[\exp\left(\mp \int_\tau^\infty \psi(z)z^{-1}dz\right) = \exp\left(\mp \int_\tau^4 \psi(z)z^{-1}dz\right) \exp\left(\mp \int_4^\infty \psi(z)z^{-1}dz\right) = \left(\frac{\tau}{4}\right)^{(n-1)} \exp\left(\mp c \int_4^\infty \bar{\omega}\left(\frac{1}{z}\right)z^{-1}dz\right),\]
by which
\[G_3(x, y) = \frac{C_0}{2l}(16 - r^2) + \frac{4^n C_0}{l} \exp\left(c \int_4^\infty \bar{\omega}\left(\frac{1}{z}\right)z^{-1}dz\right)\int_4^\infty \tau^{1-n} \exp\left(-c \int_\tau^\infty \bar{\omega}\left(\frac{1}{z}\right)z^{-1}dz\right)d\tau,\]
which implies (4.15) for \( x \in D \cap B^0_4 \). Now let \( x \in D \setminus B^0_4 \). An easy computation shows that

\[
G_3(x, y) = \frac{4^n C_0}{l} \exp \left( c \int_4^{\infty} \bar{\omega} \left( \frac{1}{z} \right) z^{-1} dz \right) \int_{r}^{\infty} \tau^{1-n} \exp \left( -c \int_{\tau}^{\infty} \bar{\omega} \left( \frac{1}{\tau} \right) \tau^{-1} d\tau \right) d\tau
\]

in this case, which implies (4.15) holds for \( x \in D \setminus B^0_4 \).

Now let us prove the existence of the desired supersolution \( G(x, y) \) for \( y \in (\mathbb{R}^n \setminus D) \cap \{y \in B^0_{64}\} \). In this case, the proof is also based on the construction of three auxiliary supersolutions \( \Gamma_i(x, y), i = 1, 2, 3 \), of the operator (4.1) in the respective domains \( D \cap B^y_{|y|/8}, D \setminus B^y_{|y|/4}, \) and \( D \).  

Supersolution \( \Gamma_1(x, y) \) in the domain \( D \cap B^y_{|y|/8} \). The function \( \Gamma_1(x, y) \) is constructed in exactly the same way as the supersolution \( G(x, y) \) in the case considered above for \( y \in B^0_{64} \setminus D \). First, we use (4.6) to define a supersolution \( G_1(x, y) \) in \( D \cap B^y_{3} \). It is hard to construct a supersolution in \( D \setminus B^y_{3} \), but in the domain \( D \cap (B^y_{|y|/4} \setminus B^y_{3}) \) one can present an analog of the function \( G_2(x, y) \) in (4.11), for which we retain the same notation. If we seek \( G_2(x, y) \) in the form \( G_2(x, y) = f(|x - y|) = f(r) \), then we arrive at (4.7). By (4.4), inequality (4.9) holds, and we find \( f \) under the assumption that \( f' \leq 0 \) as a solution of the differential equation (4.10) in the form (4.11). The subsequent argument does not differ in any respect from that used earlier when finding the supersolution \( G(x, y) \) for the case in which \( y \in B^0_{64} \setminus D \). The desired supersolution has the form \( \Gamma_1(x, y) = F(x, y) + G_3(x, y) \), where the functions \( F \) and \( G_3 \) are defined in (4.13) and (4.17), respectively. One has

\[
c_6(n, A)|x - y|^{2-n} \leq \Gamma_1(x, y) \leq c_7(n, A)|x - y|^{2-n} \quad \text{for} \quad x \in D \cap B^y_{|y|/8}.
\]

Supersolution \( \Gamma_2(x, y) \) in the domain \( D \setminus B^y_{2|y|} \). In this case, for \( x \in D \setminus B^y_{2|y|} \) one has inequality (4.12), which implies (4.9), and \( \Gamma_2(x, y) \) is defined in exactly the same way as the supersolution \( G_2(x, y) \) constructed earlier for \( y \in B^0_{64} \setminus D \). As a result, we have (see (4.11))

\[
\Gamma_2(x, y) = \int_{|x - y|}^{\infty} t^{1-n} \exp \left( -c \int_{t}^{\infty} \bar{\omega} \left( \frac{1}{\tau} \right) \tau^{-1} d\tau \right) dt,
\]

whence it follows that

\[
c_8(n, A)|x - y|^{2-n} \leq \Gamma_1(x, y) \leq c_9(n, A)|x - y|^{2-n} \quad \text{for} \quad x \in D \setminus B^y_{2|y|}.
\]

Supersolution \( \Gamma_3(x, y) \) in the domain \( D \) and the construction of the function \( G(x, y) \). Let

\[
\eta \in C^\infty(\mathbb{R}^n), \quad 0 \leq \eta \leq 1, \quad \eta(x) = 1 \quad \text{for} \quad x \in B^y_{|y|/9} \cup (\mathbb{R}^n \setminus B^y_{3|y|}),
\]

\[
\eta(x) = 0 \quad \text{for} \quad x \in B^y_{2|y|} \setminus B^y_{|y|/8} \quad \text{and} \quad |\nabla \eta| \leq C|y|^{-1}.
\]

We set

\[
F(x, y) = (\Gamma_1(x, y) + \Gamma_2(x, y))\eta(x)
\]

and note that

\[
AH(x, y) \leq 0 \quad \text{for} \quad x \in (D \cap B^y_{|y|/9}) \cup (D \setminus B^y_{3|y|}),
\]

\[
|AH(x, y)| \leq C_0(n, A)|y|^{-n} \quad \text{for} \quad x \in D \cap (B^y_{3|y|} \setminus B^y_{|y|/9}).
\]
Our aim is to construct a nonnegative function $\Gamma_3(x, y)$ satisfying the following conditions:

\begin{align}
(4.21) & \quad A\Gamma_3(x, y) \leq 0 \quad \text{for} \quad x \in (D \cap B^y_{|y|/9}) \cup (D \setminus B^y_{3|y|}), \\
(4.22) & \quad A\Gamma_3(x, y) \leq -C_0|y|^{-n} \quad \text{for} \quad x \in D \cap (B^y_{3|y|} \setminus B^y_{|y|/9}), \\
(4.23) & \quad 0 \leq \Gamma_3(x, y) \leq c_{10}(n, A)|x - y|^{2-n} \quad \text{for} \quad x \in D \cap B^y_{|y|/8}, \\
(4.24) & \quad c_{11}(n, A)|x - y|^{2-n} \leq \Gamma_3(x, y) \leq c_{12}(n, A)|x - y|^{2-n} \quad \text{for} \quad x \in D \setminus B^y_{|y|/9},
\end{align}

where $C_0$ is the constant in (4.20). Once we write out such a function, we can define the desired supersolution $G(x, y)$ by the formula

$$G(x, y) = H(x, y) + \Gamma_3(x, y).$$

The function $\Gamma_3(x, y)$ can be determined by the same line of argument as $G_3(x, y)$ (see (4.17)). Let $l$ be the constant in (4.16). Set

$$\theta(r) = \begin{cases} 
  n - l & \text{for} \quad 0 < r \leq 3|y|, \\
  c\bar{\omega}(\frac{1}{r}) & \text{for} \quad r \geq 3|y|,
\end{cases}$$

$$h(r) = \begin{cases} 
  0 & \text{for} \quad r \geq 3|y|, \\
  C_0|y|^{-n} & \text{for} \quad 0 < r < 3|y|,
\end{cases}$$

where $C_0$ is the constant in (4.20). We seek $\Gamma_3(x, y)$ in the form $f(|x - y|)$, a priori assuming that $f' \leq 0$. For $Af(r)$, we have relation (4.8), where we replace $\bar{\omega}(x)$ by the infimum $l$ for $r < 3|y|$ and use inequality (4.9) for $r \geq 3|y|$. Now finding $\Gamma_3(x, y)$ amounts to integrating the equation

$$f''(r) + \frac{n - 1 - \theta(r)}{r} f'(r) = -h(r),$$

whose desired solution has the form

$$\Gamma_3(x, y) = \int_r^\infty \tau^{1-n} \exp \left(-\int_\tau^\infty \theta(z) z^{-1} \, dz\right) \times \left(\int_0^\tau s^{n-1} \exp \left(\int_s^\infty \theta(z) z^{-1} \, dz\right) h(s) \, ds\right) d\tau.$$

Let us prove the estimates (4.23) and (4.24) for $\Gamma_3(x, y)$. Let $x \in D \cap B^y_{3|y|}$. We split the integral on the right-hand side in (4.25) into the sum of integrals over $(r, 3|y|]$ and $(3|y|, \infty)$. For $\tau < 3|y|$, we use the relations

$$\exp \left(\mp \int_\tau^\infty \theta(z) z^{-1} \, dz\right) = \exp \left(\mp \int_\tau^{3|y|} \psi(z) z^{-1} \, dz\right) \exp \left(\mp \int_{3|y|}^\infty \theta(z) z^{-1} \, dz\right) = \left(\frac{\tau}{3|y|}\right)^{\pm(n-l)} \exp \left(\mp c \int_{3|y|}^\infty \bar{\omega} \left(\frac{1}{z}\right) z^{-1} \, dz\right).$$

An easy computation shows that

$$\Gamma_3(x, y) = \frac{C_0|y|^2 - r^2}{2l|y|^n} + \frac{3^n C_0}{l} \exp \left(c \int_{3|y|}^\infty \bar{\omega} \left(\frac{1}{z}\right) z^{-1} \, dz\right) \int_{3|y|}^\infty \tau^{1-n} \exp \left(-c \int_\tau^\infty \bar{\omega} \left(\frac{1}{z}\right) z^{-1} \, dz\right) d\tau,$$

and we obtain (recall that $|y| \geq 64$)

$$c_{13}(n, A)|y|^{2-n} \leq \Gamma_3(x, y) \leq c_{14}(n, A)|y|^{2-n}.$$
Hence from \((4.18)\) and \((4.19)\) we arrive at \((4.23)\) and inequalities \((4.24)\) for \(x \in D \cap (B^y_{3|y|} \setminus B^y_{3|y|/9})\). Now let \(x \in D \setminus B^y_{3|y|/9}\). Then

\[
\Gamma_\delta(x, y) = \frac{C_0}{9^n} \exp \left( c\int_{|y|}^\infty \tilde{\omega} \left( \frac{1}{z} \right) z^{-1} \,dz \right) \int_0^\infty r^{1-n} \exp \left( -c\int_r^\infty \tilde{\omega} \left( \frac{1}{z} \right) z^{-1} \,dz \right) \,dr,
\]

which implies inequalities \((4.24)\) for \(x \in D \setminus B^y_{3|y|/9}\) by virtue of \((4.18)\) and \((4.19)\). The proof of the lemma is complete. \(\square\)

5. NECESSARY CONDITION FOR STABILIZATION TO ZERO

Let us determine a solution \(v\) of the Dirichlet problem

\[(5.1)\]

\[Av = 0 \quad \text{in } D, \quad v|_{\partial D} = 1\]

for the operator \((4.1)\) in an unbounded domain \(D \subset \mathbb{R}^n, n > 2\). The function \(v \equiv 1\) is a natural solution of this problem. Our aim is to show that if condition \((1.9)\) is violated, then there exists a different solution of problem \((5.1)\) such that

\[(5.2)\]

\[0 < v(x) < 1 \quad \text{for } x \in D.\]

This will imply the necessity of condition \((1.9)\) in Theorem \((1.1)\). Indeed, the function \(w = 1 - v\) is a solution of problem \((1.1), (1.2)\) and does not stabilize to zero.

Let us determine the solution \(v\) of problem \((5.1)\). In what follows, we assume without loss in generality that the origin lies on \(\partial D\). Consider the sequence of domains \(D_m = D \cap B^0_{4m+1}, m = 1, 2, \ldots\), and the corresponding sequence \(\{v_m\}\) of Wiener generalized solutions (e.g., see \([19\text{ Section 1.5}])\) of the Dirichlet problems

\[(5.3)\]

\[Av_m = 0 \quad \text{in } D_m, \quad v_m|_{\partial D_m} = f_m,\]

where \(f_m\) is a continuous function on \(\partial D_m\) such that \(0 \leq f_m \leq 1, f_m(x) = 1\) for \(x \in B^0_{4m} \cap \partial D\), and \(f_m(x) = 0\) for \(x \in \partial D \setminus B^0_{4m+1/2}\). It is easily seen that the sequence \(\{v_m\}, m = k, k+1, \ldots\), in the domain \(D_k\) is monotone increasing and bounded above and hence converges. We understand a solution \(v\) of problem \((5.1)\) as the function defined at each point \(x \in D\) by the formula \(v(x) = \lim_{m \to \infty} v_m(x)\).

Remark. Note that if the Dirichlet problem for the elliptic operator \((4.1)\) is solvable in the classical sense for every interior subdomain of \(D_m\) with sufficiently smooth boundary, then the Wiener generalized solution \(v_m\) of problem \((5.3)\) also satisfies the equation in \((5.3)\) in the classical sense. As was shown in \([21\text{]}\), to this end it suffices to assume that the coefficients of the operator \((4.1)\) satisfy the local Dini condition in \(D\). Under this condition, the function \(v\) defined above satisfies the equation in \((5.1)\) in the classical sense as well.

In what follows, the potentials of a measure \(\mu\) concentrated on a set \(E\) will be denoted by

\[U^\mu(x) = \int_E |x - y|^{2-n} \,d\mu(y), \quad W^\mu(x) = \int_E G(x, y) \,d\mu(y),\]

where \(G(x, y)\) is a supersolution of \(\mathcal{L}\) satisfying the estimates \((4.5)\). By these estimates,

\[(5.4)\]

\[C_1 U^\mu(x) \leq W^\mu(x) \leq C_2 U^\mu(x) \quad \text{for } x \in D.\]

Let \(F_m = (\mathbb{R}^n \setminus D) \cap \bar{B}^0_{4m}\), and let \(H_m = (\mathbb{R}^n \setminus D) \cap (\bar{B}^0_{4m+1} \setminus B^0_{4m})\). In what follows, by \(\mu_m\) and \(\nu_m\) we denote the equilibrium measures realizing the capacities of the compact sets \(F_m\) and \(H_m\), respectively. Recall that

\[(5.5)\]

\[U^{\mu_m}(x) = 1 \quad \text{for } \mu_m\text{-almost all } x \in F_m,\]

\[(5.6)\]

\[U^{\nu_m}(x) = 1 \quad \text{for } \nu_m\text{-almost all } x \in H_m.\]
Before proceeding further, let us present an auxiliary assertion similar to Lemma 5.2 in the book [19, Section 1.5].

**Proposition 5.1.** If

\[(5.7) \quad \text{cap}(F_m) < \frac{4^{(m-2)(n-2)} C_1}{8 C_2^2},\]

where \( m > 4 \), then there exists a point \( x' \in D \cap B_{4m-2}^0 \) such that \( W^{\mu m}(x') < C_1/4 \), where \( C_1 \) and \( C_2 \) are the constants in (4.5).

**Proof.** Assume the contrary: The inequality \( W^{\mu m}(x) \geq C_1/4 \) holds for all \( x \in D \cap B_{4m-2}^0 \). Then \( U^{\mu m} \geq C_1/(4C_2) \) for \( x \in D \cap B_{4m-2}^0 \) by (5.4). Set

\[ w(x) = \frac{4^{(m-2)(n-2)} C_1}{4C_2 |x|^{n-2}}. \]

In view of (5.5), it follows from the maximum principle that

\[ U^{\mu m}(x) \geq w(x) \quad \text{for} \quad x \in \mathbb{R}^n \setminus (F_m \cup (D \cap B_{4m-2}^0)), \]

and we find from (5.4) that

\[ W^{\mu m}(x) \geq C_1 w(x) \quad \text{for} \quad x \in D \setminus B_{4m-2}^0. \]

In particular, if \( R_0 > 4^{m_0} \), then

\[ (5.8) \quad W^{\mu m}(x)|_{D \cap \partial B_R^0} \geq \frac{4^{(m-2)(n-2)} C_2^2}{4C_2 R^{n-2}} \quad \text{for} \quad R > R_0. \]

On the other hand,

\[ W^{\mu m}(x)|_{D \cap \partial B_R^0} \leq \frac{4^{(m-2)(n-2)} C_1}{8 C_2 (R - R_0)^{n-2}} \]

by (5.4) and (5.7), and we have

\[ W^{\mu m}(x)|_{D \cap \partial B_R^0} < \frac{4^{(m-2)(n-2)} C_1}{4C_2 R^{n-2}} \]

for sufficiently large \( R > R_0 \), which contradicts (5.8). The proof of the proposition is complete. \( \square \)

**Theorem 5.1.** If the integral on the left-hand side in (1.9) converges, then the solution of problem (5.1) satisfies inequalities (5.2).

**Proof.** Since the integral in (1.9) converges, we have

\[ (5.9) \quad \sum_{m=1}^{\infty} \frac{\text{cap}(F_m)}{4m^{n-2}} < \infty. \]

First, note that the solution \( v \) of problem (5.1) is positive in \( D \). This readily follows from the definition of \( v \) and the maximum principle. Let us show that the upper bound in (5.2) holds.

Consider the auxiliary function

\[ \Gamma_k(x) = C_1^{-1} \left( W^{\mu m_0}(x) + \sum_{m=0}^{m_0+k} W^{\nu m}(x) \right), \quad m_0 > 4, \]

where \( C_1 \) is the constant in (4.5). By (5.5) and the maximum principle (see (5.5) and (5.6)), the solutions \( v_m \) of the Dirichlet problems (5.3) satisfy the inequalities

\[ (5.10) \quad v_m(x) \leq \Gamma_k(x) \quad \text{for} \quad x \in D_m, \quad m = m_0, m_0 + 1, \ldots, m_0 + k - 1. \]
Let us estimate the right-hand side of (5.10) for $x \in D \cap B_{4m_0-2}^0$. Using (4.5), we obtain
\[
C_1^{-1} \sum_{m=m_0}^{m_0+k} W^{\nu_m}(x) \leq C_1^{-1} C_2 16^{n-2} \sum_{m=m_0}^{m_0+k} \frac{\text{cap}(H_m)}{4m(n-2)}
\leq C_1^{-1} C_2 64^{n-2} \sum_{m=m_0+1}^{\infty} \frac{\text{cap}(F_m)}{4m(n-2)}, \quad \forall x \in D \cap B_{4m_0-2}^0.
\]

It follows from the convergence of the series (5.9) that there exists a number $m_0$ starting from which
\[
(5.11) \quad C_1^{-1} \sum_{m=m_0}^{m_0+k} W^{\nu_m}(x) \leq \frac{1}{4} \quad \forall x \in D \cap B_{4m_0-2}^0, \quad k = 1, 2, \ldots.
\]

On the other hand, we can again use condition (5.9) to choose a number $m_0$ large enough that inequality (5.7) holds for $x \in D \cap B_{4m_0-2}^0$. By Proposition 5.1, there exists a point $x' \in D \cap B_{4m_0-2}^0$ such that $W^{\nu_{m_0}}(x') < C_1/4$. Hence it follows from (5.11) and (5.10) that
\[
v_{m}(x') < \frac{1}{2}, \quad x' \in D \cap B_{4m_0-2}^0, \quad m = m_0, m_0 + 1, \ldots, m_0 + k - 1.
\]

By passing to the limit first as $k \to \infty$ and then as $m \to \infty$ in this inequality, we conclude that the solution $v$ of problem (5.4) satisfies the estimate $v(x') < 1/2$. Since $v$ is positive in $D$, we obtain the upper bound in (5.2). Indeed, assume the contrary: There exists an $x_0 \in D$ such that $v(x_0) = 1$. Consider an arbitrary bounded subdomain $D' \subseteq D$ containing $x_0$ and $x'$ and the function $w = 1 + \varepsilon - v$, which is positive in $D$. By Harnack’s inequality,
\[
\inf_{D'} w \geq C(n, A, D') \sup_{D'} w,
\]
and so $\varepsilon \geq C(n, A, D')(1 + \varepsilon - v(x'))$. By passing to the limit as $\varepsilon \to 0$, we arrive at a contradiction. This proves the necessity of condition (1.9). The proof of Theorem 1.1 is complete. 

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