DISTRIBUTION OF THE EIGENVALUES OF SINGULAR DIFFERENTIAL OPERATORS IN A SPACE OF VECTOR-FUNCTIONS

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Abstract. A significant part of B. M. Levitan’s scientific activity dealt with questions on the distribution of the eigenvalues of differential operators [1]. To study the spectral density, he mainly used Carleman’s method, which he perfected. As a rule, he considered scalar differential operators. The purpose of this paper is to study the spectral density of differential operators in a space of vector-functions. The paper consists of two sections. In the first we study the asymptotics of a fourth-order differential operator
\[ y^{(4)} + Q(x)y = \lambda y, \]
both taking account of the rotational velocity of the eigenvectors of the matrix \( Q(x) \) and without taking the rotational velocity of these vectors into account. In Section 2 we study the asymptotics of the spectrum of a non-semi-bounded Sturm–Liouville operator in a space of vector-functions of any finite dimension.

§ 1. THE ASYMPTOTICS OF THE SPECTRUM OF A NON-SEMI-BOUNDED SINGULAR DIFFERENTIAL OPERATOR OF FOURTH ORDER IN A SPACE OF VECTOR-FUNCTIONS

In the space \( H = L^2(0, \infty) \oplus L^2(0, \infty) \) we consider the minimal differential operator \( L_0 \) generated by a differential expression
\[ L_0 = y^{(4)} + Q(x)y = \lambda y, \]
where \( y = (y_1(x), y_2(x)) \), \( 0 < x < \infty \). Here, \( Q(x) = \|q_{ij}(x)\|_{i,j=1}^2 \) is a real symmetric matrix whose eigenvalues satisfy \( \mu_i(x) \rightarrow -\infty \) as \( x \rightarrow \infty \). We set
\[ \phi(x) = \frac{1}{2} \arctan \frac{q_{22} - q_{11}}{2q_{12}}. \]
We call the function \( \phi'(x) \) the rotational velocity of the eigenvectors of the matrix \( Q(x) \).

Theorem 1.1. Suppose that the following conditions hold for sufficiently large \( x_0 \) and for \( x > x_0 \):
1) \( |\phi'(x_0)| \leq \text{const} \);
2) \( 0 < A \leq \frac{\mu_i(x)}{\mu_j(x)} \leq B, \ i, j = 1, 2; \)
3) \( \int_{x_0}^{\infty} |\mu_i^{-1/4}(x)| \, dx < \infty, \int_{x_0}^{\infty} \left| \frac{\mu_i^2(x)}{\mu_i^{9/4}(x)} \right| \, dx < \infty, \int_{x_0}^{\infty} \left| \frac{\phi''(x)}{\mu_i^{1/4}(x)} \right| \, dx < \infty, \ i = 1, 2; \)
4) \( |\mu_i'(x)| \leq C|\mu_i(x)|^\alpha, \ C = \text{const}, \ i = 1, 2, \ 0 < \alpha < 5/4. \)

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Then the system \((1.1)\) has eight linearly independent solutions \(y_j(x, \lambda)\) such that
\[
y_j = \psi_1(x, \lambda) \exp \left\{ \varepsilon_j \int_0^x (\lambda - \mu_1(t))^{1/4} \, dt \right\} (1 + o(1)),
\]
\[
y_{j+4} = \psi_2(x, \lambda) \exp \left\{ \varepsilon_j \int_0^x (\lambda - \mu_2(t))^{1/4} \, dt \right\} (1 + o(1))
\]
as \(x \to \infty\), where \(j = 1, 2, 3, 4\), the \(\varepsilon_j\) are the fourth roots of unity,
\[
\psi_1(x, \lambda) = \frac{1}{8(\lambda - \mu_1(x))^{3/4}} \begin{pmatrix} \cos \phi(x) \\ - \sin \phi(x) \end{pmatrix},
\]
\[
\psi_2(x, \lambda) = \frac{1}{8(\lambda - \mu_2(x))^{3/4}} \begin{pmatrix} \sin \phi(x) \\ \cos \phi(x) \end{pmatrix}.
\]

We shall explain the meaning of the conditions in Theorem 1.1. Conditions 3 and 4 mean that the functions \(\mu_i(x)\) satisfy the Titchmarsh–Levitan condition for regularity of growth, the functions \(|\mu_i(x)|\) have a definite growth at infinity, and the third estimate in condition 3 means that we are looking at the case of ‘slow’ rotation of the eigenvectors of the matrix \(Q(x)\). Condition 2 means that the eigenvalues of the matrix \(Q(x)\) are ‘of the same strength’. We also observe that in the case when the functions have power growth: \(|\mu_j(x)| \sim x^\gamma\) and \(\phi(x) \sim x^\beta\) as \(x \to \infty\), all the conditions of the theorem hold for \(\gamma > 4\), \(\beta \leq 1\).

We now turn to the proof of the theorem.

**Proof.** Using the change of variables \(z = (y, y', y'', y''')^T\) in the system \((1.1)\) we obtain the system of first-order differential equations \(z' = Az\), where
\[
z = (z_1(x, \lambda), z_2(x, \lambda), z_3(x, \lambda), z_4(x, \lambda))
\]
is a new unknown vector-function,
\[
A = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -Q + \lambda I & 0 & 0 & 0 \end{pmatrix},
\]
and \(I\) is the two-dimensional identity matrix.

We introduce into consideration an orthogonal matrix \(U_1\) such that
\[
U_1^{-1}QU_1 = \Lambda = \text{diag}\{\mu_1, \mu_2\}, \quad \mu_{1,2} = \frac{q_{11} + q_{22} \pm \sqrt{(q_{11} - q_{22})^2 + 4q_{12}^2}}{2}.
\]
We further perform the change of variables
\[
z = \text{diag}\{U_1, U_1, U_1, U_1\} w = Uw,
\]
\[
z' = U'w + Uw', \quad U'w + Uw' = Auw,
\]
\[
w' = (U^{-1}Au)w - U^{-1}U'w,
\]
\[
U^{-1}Au = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -\Lambda + \lambda I & 0 & 0 & 0 \end{pmatrix},
\]
\[
U^{-1}U' = \text{diag}\{\tilde{p}, \tilde{p}, \tilde{p}, \tilde{p}\} = P, \quad \tilde{p} = \phi'(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Next, since condition 1 of Theorem 1 holds, the leading elements in the system (1.2) will be the elements of the matrix $U^{-1} AU$.

It is known that there exists a matrix that reduces $U^{-1} AU$ to diagonal form; we denote it by $C(x, \lambda)$:

$$C^{-1}(U^{-1} AU)C = M = \text{diag}\{ \bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4, \bar{\mu}_5, \bar{\mu}_6, \bar{\mu}_7, \bar{\mu}_8 \},$$

$$\bar{\mu}_1 = (\lambda - \mu_1)^{1/4}, \quad \bar{\mu}_2 = (\lambda - \mu_2)^{1/4},$$

$$\bar{\mu}_3 = -(\lambda - \mu_1)^{1/4}, \quad \bar{\mu}_4 = -(\lambda - \mu_2)^{1/4},$$

$$\bar{\mu}_5 = i(\lambda - \mu_1)^{1/4}, \quad \bar{\mu}_6 = i(\lambda - \mu_2)^{1/4},$$

$$\bar{\mu}_7 = -i(\lambda - \mu_1)^{1/4}, \quad \bar{\mu}_8 = -i(\lambda - \mu_2)^{1/4}.$$

The elements of the matrix $C$ are determined from the system of equations

$$c_{1i} \bar{\mu}_i = c_{2i}, \quad c_{2i} \bar{\mu}_i = c_{3i}, \quad c_{3i} \bar{\mu}_i = c_{4i}, \quad c_{4i} \bar{\mu}_i = (-\Delta + \lambda I) c_{1i},$$

where the elements of $C$, $c_{ij}$, $i = 1, \ldots, 4$, $j = 1, \ldots, 4$, are two-dimensional matrices,

$$\bar{\mu}_1 = \begin{pmatrix} \bar{\mu}_1(x) & 0 \\ 0 & \bar{\mu}_2(x) \end{pmatrix}, \quad \bar{\mu}_2 = \begin{pmatrix} \bar{\mu}_3(x) & 0 \\ 0 & \bar{\mu}_4(x) \end{pmatrix},$$

$$\bar{\mu}_3 = \begin{pmatrix} \bar{\mu}_5(x) & 0 \\ 0 & \bar{\mu}_6(x) \end{pmatrix}, \quad \bar{\mu}_4 = \begin{pmatrix} \bar{\mu}_7(x) & 0 \\ 0 & \bar{\mu}_8(x) \end{pmatrix}.$$

The matrix $C$ is determined from this system non-uniquely, up to multiplication on the right by a block-diagonal matrix $\delta(x) = \text{diag}\{ \delta_1(x), \delta_2(x), \delta_3(x), \delta_4(x) \}$. Then elements of $C$ have the form

$$c_{1i} = \delta_1(x), \quad c_{2i} = \delta_1(x) \bar{\mu}_i(x), \quad c_{3i} = \delta_1(x) \bar{\mu}_i^2(x), \quad c_{4i} = \delta_1(x) \bar{\mu}_i^3(x).$$

We find the matrix $C^{-1}$ from the condition $C^{-1} C = E$. We let $T$ denote the matrix $C^{-1} C'$ and find its elements. We choose the matrix $\delta(x)$ in such a way that the condition $(C^{-1} C')_{ii} = 0$, $i = 1, \ldots, 8$, hold. Then the blocks of the matrix $\delta(x)$ have the form

$$\delta_1(x) = (\bar{\mu}_1(x))^{-3/2}, \quad \delta_2(x) = (\bar{\mu}_2(x))^{-3/2},$$

$$\delta_3(x) = -i \bar{\mu}_3(x)^{-3/2}, \quad \delta_4(x) = i \bar{\mu}_4(x)^{-3/2},$$

$$C = \begin{pmatrix}
\bar{\mu}_1(x)^{-1/2} & (-\bar{\mu}_2(x))^{-1/2} & i(-i \bar{\mu}_2(x))^{-1/2} & -i(i \bar{\mu}_2(x))^{-1/2} \\
(-\bar{\mu}_2(x)^{-1/2}) & \bar{\mu}_2(x)^{-1/2} & i(i \bar{\mu}_4(x))^{-1/2} & -i(i \bar{\mu}_4(x))^{-1/2} \\
(-i \bar{\mu}_3(x))^{-1/2} & (-i \bar{\mu}_3(x))^{-1/2} & \bar{\mu}_3(x)^{-1/2} & (-i \bar{\mu}_3(x))^{-1/2} \\
(-i \bar{\mu}_3(x))^{-1/2} & (-i \bar{\mu}_3(x))^{-1/2} & i(i \bar{\mu}_4(x))^{-1/2} & \bar{\mu}_4(x)^{-1/2} 
\end{pmatrix}.$$
Next, as in Chapter 5 in [3], setting
\[ w = C(I + G)u, \]
where the matrix \( G \) with elements \( g_{ij} \) satisfies the equation
\[ GM - MG = -T - C^{-1}PC, \]
we obtain the system of equations
\[ (1.3) \quad u' = (M + \Theta(x, \lambda))u, \]
where
\[ \Theta(x, \lambda) = (I + G)^{-1}(-TG - G' - C^{-1}PCG). \]

Setting
\[ u = s \cdot \exp \left\{ \int_0^x \bar{\mu}_i(t, \lambda) \, dt \right\}, \]
in (1.3) for a fixed \( i (i = 1, \ldots, 8) \), where \( s = (s_1, s_2, s_3, \ldots, s_8) \) is an unknown vector-function, we arrive at the system of first-order equations
\[ (1.3') \quad \frac{d}{dx} s_i(x, \lambda) = \eta_i(x, \lambda) s_i(x, \lambda) + \sum_{m=1}^8 \Theta_{im}(x, \lambda) s_i(x, \lambda), \quad i = 1, \ldots, 8, \]
where \( \eta_i(x, \lambda) = \bar{\mu}_i(x, \lambda) - \bar{\mu}_j(x, \lambda) \) and the \( \Theta_{ij}(x, \lambda) \) are elements of the matrix \( \Theta(x, \lambda) \).

We claim that
\[ (1.4) \quad \int_0^\infty \| \Theta(x, \lambda) \| \, dx < \infty. \]

Here and in what follows, by the norm of a matrix we mean the sum of the absolute values of its elements.

We will estimate the elements \( g_{ij} \) of the matrix \( G \). All the \( g_{ij}(x, \lambda) \) are bounded above by linear combinations of functions of the form
\[ \frac{\mu_i'(x)}{(\lambda - \mu_i(x))^{5/4}}, \]
\[ \phi'(x) \left[ \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \right] \left( \sum_{i=1}^2 \bar{c}_i (\lambda - \mu_i(x))^{1/4} \right)^{-1}, \quad \bar{c}_i = \text{const}. \]

We have
\[ \left| \frac{\mu_i'(x)}{(\lambda - \mu_i(x))^{5/4}} \right| \leq C \left| \frac{\mu_i^0(x)}{\mu_i^{5/4}(x)} \right| = o(1) \]
as \( x \to \infty \) by condition 4 of Theorem 11.\( ^{11} \) Note that
\[ \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} = \frac{\mu_i(x)(1 + o(1))}{\mu_j(x)(1 + o(1))}, \]

\[ \left| \phi'(x) \left[ \left( \frac{\lambda - \mu_i}{\lambda - \mu_j} \right)^{3/8} + \left( \frac{\lambda - \mu_i}{\lambda - \mu_j} \right)^{1/8} \right] \left( \sum_{i=1}^2 \bar{c}_i (\lambda - \mu_i^{1/4}) \right)^{-1} \right| \leq \left| \frac{\phi'(x)}{C_i(\lambda - \mu_i)^{1/4}} \left[ \left( \frac{\lambda - \mu_i}{\lambda - \mu_j} \right)^{3/8} + \left( \frac{\lambda - \mu_i}{\lambda - \mu_j} \right)^{1/8} \right] \right| \]
\[ \leq \left| \frac{\phi'(x)}{C_i(\lambda - \mu_i^{1/4})} \left( \frac{\mu_i}{\mu_j} \right)^{3/8} + \left( \frac{\mu_i}{\mu_j} \right)^{1/8} \right| \leq C_1 \frac{B^{3/8} + B^{1/8}}{|\mu_i|^{1/4}} = o(1), \quad x \to \infty, \]
since conditions 1, 2 of Theorem 1.1 hold. This means that we can make \( \|G(x, \lambda)\| \) smaller than 1/2, say, as \( x \to \infty \); consequently, for large \( x \) the matrix \( I + G \) has bounded inverse matrix \( (I + G)^{-1} \).

Thus, \( \|I + G\| \leq c, \|(I + G)^{-1}\| \leq c, x \to \infty \). Therefore, in order to verify the validity of 1.4, it is sufficient to prove the integrability of the elements of the matrices \( TG, G', C^{-1}PCG \).

We next estimate the elements of the matrix \( TG \). They are bounded above by linear combinations of functions of the form

\[
|\mu_i^2(x)(\lambda - \mu_i(x))^{-9/4}|,
\]
\[
\left| \frac{\mu_i'(x)\phi'(x)}{(\lambda - \mu_i(x))^{5/4}} \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \right|, \quad i, j = 1, 2.
\]

We claim that

\[
\int_0^\infty (TG)_{ij} \, dx < \infty,
\]
\[
\int_0^\infty \left| \frac{(\mu_i'(x))^2}{(\lambda - \mu_i(x))^{9/4}} \right| \, dx = \int_0^x \left| \frac{(\mu_i'(x))^2}{(\lambda - \mu_i(x))^{9/4}} \right| \, dx + \int_x^\infty \left| \frac{(\mu_i'(x))^2}{(\lambda - \mu_i(x))^{9/4}} \right| \, dx.
\]

The first integral converges, since the integrand is continuous. We can estimate the second integral using condition 3:

\[
\int_x^\infty \left| \frac{(\mu_i'(x))^2}{(\lambda - \mu_i(x))^{9/4}} \right| \, dx \leq C \int_x^\infty \frac{|\mu_i'(x)| \cdot |\mu_i''(x)|}{|\mu_i^{9/4}(x)|} \, dx < \infty.
\]

As \( \phi'(x) \) is bounded, applying condition 2 from Theorem 1.1 similar arguments show that the integral

\[
\int_0^\infty \left| \frac{\mu_i'(x)\phi'(x)}{(\lambda - \mu_i(x))^{5/4}} \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \right| \, dx
\]

is absolutely convergent.

We will now prove that

\[
\int_0^\infty \|G'(x, \lambda)\| \, dx < \infty.
\]

To do this we use the estimates obtained earlier for the elements of the matrix \( G \).

In the case where \( g_{ij}(x, \lambda) = \frac{C_{ij}\mu_i'(x)}{(\lambda - \mu_i(x))^{5/4}} \), we obtain

\[
(g_{ij})' = C_{ij} \frac{\mu_i''(x)}{(\lambda - \mu_i(x))^{5/4}} + C_{ij} \frac{5\mu_i'(x)\mu_i'(x)}{4(\lambda - \mu_i(x))^{9/4}},
\]

\[
\int_0^\infty |g_{ij}'| \, dx \leq C_{ij} \int_0^\infty \left| \frac{\mu_i''(x)}{\mu_i^{5/4}(x)} + \frac{\mu_i^2}{\mu_i^{9/4}(x)} \right| \, dx < \infty.
\]

by the conditions in part 3 of Theorem 1.1.

If, instead,

\[
g_{ij}' = \left( \phi'(x) \left[ \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right]^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \left( \sum_{i=1}^2 \tilde{c}_i(\lambda - \mu_i(x))^{1/4} \right)^{-1} \right)',
\]

the estimates are again similar, taking account of conditions 3 in Theorem 1.1.

We will prove that

\[
\int_0^\infty \|C^{-1}PCG\| \, dx < \infty.
\]
Since the elements of the matrix $C^{-1}PC$ are bounded above by a linear combination of functions of the form

$$\left| \phi'(x) \left[ \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \right] \right|,$$

using the expressions we have obtained for the elements of the matrix $G$ it is easy to write out the elements of the matrix $C^{-1}PC$:

$$\left[ \begin{array}{c} \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \\ \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \end{array} \right] \tilde{C}_{ij} \phi'(x) \mu_i'(x) (\lambda - \mu_i(x))^{5/4},$$

$$\left[ \begin{array}{c} \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \end{array} \right]^2 \sum_{i=1}^{2} \tilde{C}_{ij} \phi^2(x).$$

We again split the integral of the first expression into two integrals over the intervals $[0,x_0]$ and $[x_0,\infty)$. Now, by the hypotheses of Theorem 1.1 the integral over the interval $[x_0,\infty)$ satisfies the estimate

$$\int_{x_0}^{\infty} \left| \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \right| \tilde{C}_{ij} \phi'(x) \mu_i'(x) (\lambda - \mu_i(x))^{5/4} \, dx$$

$$\leq \tilde{C}_{ij} C (B^{3/8} + B^{1/8}) \int_{x_0}^{\infty} \mu_i'(x) (\lambda - \mu_i(x))^{5/4} \, dx < \infty \quad \text{as} \quad x \to \infty,$$

while the integral over the interval $[0,x_0]$ is clearly convergent. Thus, we finally obtain that the integral of the first expression is absolutely convergent.

Similarly,

$$\int_{x_0}^{\infty} \left| \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{3/8} + \left( \frac{\lambda - \mu_i(x)}{\lambda - \mu_j(x)} \right)^{1/8} \right|^2 \tilde{C}_{ij} \phi^2(x) \sum_{i=1}^{2} \mu_i'(x) (\lambda - \mu_i(x))^{1/4} \, dx$$

$$\leq C^2 (B^{3/8} + B^{1/8}) \tilde{C}_{ij} \int_{x_0}^{\infty} |\mu_i(x)|^{-1/4} \, dx < \infty,$$

since conditions 1 and 3 hold.

We claim that in our case all the hypotheses of Lemma 1 on p. 288 of [2] hold.

It is easy to show that $\text{Re}(\tilde{\mu}_i(x) - \tilde{\mu}_j(x))$ does not change sign for sufficiently large $x_0, i, j = 1, \ldots, 8, i \neq j$. We show this, for example, for the case $i = 1, j = 2$:

$$\text{Re}(\tilde{\mu}_1(x) - \tilde{\mu}_2(x)) = \text{Re}((\lambda - \mu_1)^{1/4} + (\lambda - \mu_1)^{1/4})$$

$$= \text{Re}(2(\sigma + i\tau - \mu_1)^{1/4}) = 2 \text{Re}((\lambda - \mu_1)^{1/4} + (\lambda - \mu_1)^{1/4}).$$

Since $\lambda = \sigma + i\tau, \tau = \sigma\gamma, 0 < \gamma < 1, \sigma > 0, \tau > 0$, it follows that as $x \to \infty$ we have

$$2 \text{Re}\{\sigma + i\tau - \mu_1(x)\}^{1/4} = 2 \text{Re}\left\{\left(-\mu_1(x)\left(\frac{\sigma(1 + i\gamma^{-1})}{\mu_1(x)} + 1\right)\right)^{1/4}\right\}$$

$$= 2 \text{Re}\left\{\left(-\mu_1(x)\left(\frac{\gamma(1 + o(1))}{\mu_1(x)} + 1\right)\right)^{1/4}\right\} = 2 \text{Re}\{(-\mu_1(x))^{1/4}(1 + o(1))\}. $$

Since $|\mu_1(x)| \to \infty$ as $x \to \infty$, there are two possible cases:

1) if $\mu_1(x) \to -\infty$, then $-\mu_1(x) > 0$; consequently, $(-\mu_1(x))^{1/4}$ is a real number and $\text{Re}\{(-\mu_1(x))^{1/4}\}$ does not change sign for large $x$;

2) if $\mu_1(x) \to +\infty$, then $-\mu_1(x) < 0$; consequently,

$$(-\mu_1(x))^{1/4} = |\mu_1(x)|^{1/4} \left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right),$$
and then
\[ 2 \text{Re} \left\{ |\mu_1(x)|^{1/4} \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \right\} = \sqrt{2} |\mu_1(x)|^{1/4}, \]
and this expression also does not change sign for large \( x \).

Since the integral of \( |\Theta(x,\lambda)| \) converges on \((0, \infty)\), we see that Lemma 1 on p. 288 of [2] can be applied to our system.

Returning to the vector \( y \) by the inverse changes of variables, we obtain the required asymptotic formulae. Theorem 1.1 is proved. \( \square \)

The asymptotic formulae in Theorem 1.1 let us deduce a result on the deficiency indices of the operator \( L_0 \).

**Theorem 1.2.** Suppose that all the conditions of Theorem 1.1 hold and \( \mu_i(x) \to -\infty \) as \( x \to \infty \). Then the deficiency indices of the operator \( L_0 \) are equal to \((4,4)\).

Theorem 1.2 follows from the fact that for large \( x \) four of the exponent coefficients in the asymptotic formulae of Theorem 1.1 are positive, while the other four are negative.

We use Carleman’s method [7] to study the asymptotics of the spectrum of a selfadjoint extension of the minimal differential operator \( L_0 \).

It is known (see [2]) that the resolvent of any real selfadjoint extension \( L_0 \) of the operator \( L_0 \) is an integral operator whose kernel \( K_0(x, \eta, \lambda) \) is defined by the formula

\[
K_0(x, \eta, \lambda) = \begin{cases} 
\sum_{k=1}^{2} Y_k(x) H_k(\eta) - \sum_{k=3}^{4} Y_k(x) V_k(\eta), & x < \eta, \\
\sum_{k=1}^{2} Y_k(x)(H_k(\eta) + V_k(\eta)), & x > \eta,
\end{cases}
\]

where

\[
\begin{align*}
Y_1(x) &= (y_3, y_4), \\
Y_2(x) &= (y_5, y_6), \\
Y_3(x) &= (y_1, y_2), \\
Y_4(x) &= (y_7, y_8),
\end{align*}
\]

\[ H_k(\eta) = \sum_{i=1}^{2} A_{k,i} Y_i(\eta), \]

the matrices \( A_{k,i} \) are defined by the formulae in § 24 of [2], while the \( V_j(x) \) are matrix solutions of the system

\[ \sum_{j=1}^{4} V_j(x) Y_j^k(x) = \delta_{k,3}, \quad k = 0, 1, 2, 3. \]

Let

\[ v(x, \lambda) = \int_{0}^{x} m(t, \lambda) dt. \]

We now set \( \eta = x \) and calculate the trace of the resulting matrix:

\[ \text{tr} K_0(x, x; \lambda) = \frac{-16 + 16i + (69i - 5)(e^{2i \nu(x)} + ie^{-2i \nu(x)}) - (544\sqrt{2} + 629i \sqrt{2})e^{(-1+i) \nu(x)}}{32 m^{3/4}(x)}, \]

for \( \lambda \in \Gamma \) as \( \lambda \to \infty \) uniformly in \( x, 0 < x < \infty \). Integrating this equation we obtain

\[ \int_{0}^{\infty} K_0(x, x; \lambda) \, dx \sim \frac{i - 1}{2} \int_{0}^{\infty} \frac{dx}{m^{3/4}(x, \lambda)}, \quad \lambda \in \Gamma, \; \lambda \to \infty. \]

We put

\[ \rho(\lambda) = \int_{0}^{\infty} K_0(x, x; \lambda) \, dx \sim \frac{i - 1}{2} \int_{0}^{\infty} \frac{dx}{m^{3/4}(x, \lambda)}. \]

One can show that the function \( \rho(\lambda) \) is an \( R \)-function [5].
We consider the function \( \varsigma(t) \):

\[
\varsigma(t) = \begin{cases}
\frac{1}{\pi} \lim_{\tau \to +0} \int_0^t \Im \{\rho(\sigma + i\tau)\} \, d\sigma \\
= \frac{2^{5/4}}{\pi} \int_0^\infty \left( (2t - (\mu_1 + \mu_2))^{1/4} - ((-\mu_1 + \mu_2))^{1/4} \right) \, dx, \quad t > 0; \\
\frac{2^{1/4}}{\pi} \int_0^t d\sigma \int_{2\sigma > \mu_1 + \mu_2} \frac{dx}{(\mu_1(x) + \mu_2(x) - 2\sigma)^{3/4}}, \quad t < 0.
\end{cases}
\]

Then by the Stieltjes inversion formula we have

\[
\rho(\lambda) = \int_0^\infty \frac{d\varsigma(t)}{t - \lambda}.
\]

It follows from the asymptotic formulae from Theorem 1.1 that

\[
\int_0^\infty \int_0^\infty |K_0(x, \eta, \lambda)|^2 \, dx \, d\eta < \infty.
\]

The latter means that the spectrum of the operator \( L_u \) is discrete and consists of two sets of eigenvalues \( \lambda_u^+ \to \infty, \lambda_u^- \to -\infty \).

We now introduce the distribution function \( N(\lambda) \) of the eigenvalues of the operator \( L_u \):

\[
N(\lambda) = \begin{cases}
\sum_{\lambda_u < \lambda} 1, \quad \lambda > 0; \\
- \sum_{\lambda_u > \lambda} 1, \quad \lambda > 0.
\end{cases}
\]

Next, applying the two-sided Tauberian theorem \[6\], we arrive at the following assertion.

**Theorem 1.3.** Suppose that all the conditions of Theorem 1.1 hold, together with the Tauberian conditions

\[
A_1 \leq \left| \frac{\varsigma(t)(-t)}{\varsigma(t)(t)} \right| \leq A_2, \quad \alpha \varsigma(t) \leq k \varsigma(t)' \leq \beta \varsigma(t)(t), \quad 0 < \alpha < \beta < 1,
\]

for large \(|t|\). Then the function \( N(\lambda) \), the number of eigenvalues of the operator \( L_0 \) not exceeding \( \lambda \), satisfies asymptotic formulae \( N(\lambda) \sim \varsigma(t)(\lambda) \) as \( \lambda \to \pm \infty \).

The proof of this theorem is similar to the proof of Theorem 3 in \[7\].

§ 2. **Spectral properties of the Sturm–Liouville operator in a space of vector-functions**

1. We consider the differential expression

\[
ly = -y'' + A(x)y, \quad 0 < x < \infty,
\]

where \( y(x) = (y_1(x), y_2(x), \ldots, y_n(x)) \) is a vector-function, \( A(x) \) is a Hermitian matrix of order \( n \) with elements \( a_{ij}(x) \in C^2[0, \infty) \). We let \( L^2[0, \infty) \) denote the complex Hilbert space of \( n \)-dimensional vector-functions with the scalar product

\[
(y, z) = \sum_k \int_0^\infty y_k(t)z_k(t) \, dt,
\]

and let \( L_0 \) denote the minimal operator generated by the expression (2.1) in \( L^2_n[0, \infty) \).

It is well known \[2\] that studying the spectral properties of the operator \( L_0 \) and selfadjoint extensions of it is closely connected with studying the asymptotic behaviour of a
fundamental system of solutions (FSS) of the equation
\[ (2.2) \quad ly = \lambda y, \quad x \to \infty. \]

The papers \[3\] are devoted to studying the spectral properties of a singular Sturm–Liouville operator in a space of vector-functions. In \[3\], for a very general situation (vector-functions with values in an infinite-dimensional Hilbert space), the spectral properties of the operator are described in terms of the eigenvalues of the matrix \(A(x)\). In \[8\], for \(n = 2\) and a positive definite matrix \(A(x)\), it was shown that the spectral properties of the operator are determined not only by the behaviour at infinity of the eigenvalues of the matrix \(A(x)\) but also by the rotational velocity of the eigenvectors of this matrix. The case of a negative definite matrix \(A(x)\), \(n = 2\), was considered in \[7\]. Here, we look at the asymptotic behaviour as \(x \to \infty\) of an FSS of the system (2.2), both in the case where the asymptotics of solutions is determined by the behaviour of the eigenvalues of the matrix \(A(x)\) at infinity and in the case where the asymptotics of solutions is determined in terms of the rotational matrix of the eigenvectors of the matrix \(A(x)\).

2. We consider the matrix
\[ A(x) = \Lambda_0(x) + G(x), \]
where \(\Lambda_0(x)\) is a diagonal matrix with elements \(\lambda_0^i(x)\). Let the elements of the matrix \(G(x)\) be \(g_{ij}(x)\). We set
\[ \|G(x)\| = \max_{i,j} |g_{ij}(x)|, \quad d(x) = \max_{k,i,j,k\neq i} \left| \frac{g_{ij}}{\lambda_0^j(x) - \lambda_0^i(x)} \right|. \]

**Theorem 2.1.** Suppose that there exists a positive integer \(n\) such that
1) \(\|G(x)\|d^n(x) \in L[x_0, \infty);\)
2) \(\left( \frac{g_{ij}}{\lambda_0^j(x) - \lambda_0^i(x)} \right)' \in L[x_0, \infty), \quad i \neq k.\)

Then there exists a matrix \(T(x)\) such that
a) \(T^{-1}(x)(\Lambda_0(x) + G(x))T(x) = \Lambda(x) + D(x), \) where \(\Lambda(x)\) is a diagonal matrix and \(\int_{x_0}^{\infty} \|D(t)\| dt < \infty;\)
b) \(\lim_{x \to \infty} T(x) = I;\)
c) \(\int_{x_0}^{\infty} \|T^{-1}(t)T'(t)\| dt < \infty.\)

**Proof.** Instead of \(\Lambda_0(x) + G(x)\) we consider the matrix \(\Lambda_0(x) + \varepsilon G(x)\) and seek diagonal matrices \(\Lambda_i(x)\) and matrices \(T_i(x)\) such that
\[ (2.3) \quad (\Lambda_0(x) + \varepsilon G(x)) \sum_{i=0}^{\infty} T_i(x) \varepsilon^i = \sum_{i=0}^{\infty} T_i(x) \varepsilon^i \sum_{i=0}^{\infty} \Lambda_i(x) \varepsilon^i. \]

Equating coefficients of the same powers of \(\varepsilon\) in (2.3) we obtain
\[ (2.4) \quad \Lambda_0(x) T_n(x) - T_n(x) \Lambda_0(x) = \sum_{k=0}^{n-1} T_k(x) \Lambda_{n-k}(x) - G(x) T_{n-1}(x). \]

Hence we conclude that
\[ T_0 = I, \quad \Lambda_1(x) = 0, \quad (T_1(x))_{ij} = \frac{g_{ij}}{\lambda_0^j(x) - \lambda_0^i(x)}, \quad (T_1(x))_{ii} = 0, \]
and for $n \geq 2$ we have

$$
\Lambda_n(x) = \text{diag}(G(x)T_{n-1}(x)),
$$

$$(T_n(x))_{ij} = \frac{1}{\lambda_i^0(x) - \lambda_j^0(x)} \left( \sum_{k=0}^{n-1} T_k(x)\Lambda_{n-k}(x) - G(x)T_{n-1}(x) \right), \quad i \neq j,$$

$$
(T_n(x))_{ii} = D_{ii} = 1 \text{ in (2.6) we obtain the assertion of the theorem.}
$$

We now estimate the norms of the matrices $T_n(x)$ and $\Lambda_n(x)$. We observe that the elements of the matrix $T_n(x)$ are sums of products of $n$ elements of the form

$$
g_{ij} = \frac{\lambda_i^0(x) - \lambda_j^0(x)}{\lambda_i^0(x) - \lambda_k^0(x)};$$

hence,

$$
\|T_n(x)\| \leq a_n d^n(x), \quad \|\Lambda_n(x)\| \leq b_n \|G(x)\| d^{n-1}(x),
$$

where $a_n$ and $b_n$ are some constants.

Next, from the formulae for the elements of the matrix $T(x)$ we obtain

$$
\|T'_n(x)\| \leq c_n d^{n-1}(x) \max_{k, i, j, k \neq j} \left| \frac{g_{kj}}{\lambda_k^0(x) - \lambda_i^0(x)} \right|.
$$

Therefore by condition 2 of Theorem 2.1 we have

$$
\int_{x_0}^{\infty} \|T'_n(t)\| \, dx < \infty.
$$

It follows from relation (2.4) that

$$
(A_0(x) + \varepsilon G(x)) \sum_{k=0}^{n} T_k(x) \varepsilon^k
$$

$$
= \sum_{k=0}^{n} T_k(x) \varepsilon^k \sum_{k=0}^{n} \Lambda_k(x) \varepsilon^k - \varepsilon^{n+1} G(x)T_n(x) + \sum_{j=1}^{n} \varepsilon^{n+j} \sum_{k=j}^{n} T_k(x) \Lambda_{n+j+k}(x).
$$

We introduce the following notation:

$$
T(\varepsilon) = \sum_{k=0}^{n} T_k(x) \varepsilon^k, \quad \Lambda(\varepsilon) = \sum_{k=0}^{n} \Lambda_k(x) \varepsilon^k,
$$

$$
F(\varepsilon) = \sum_{j=1}^{n} \varepsilon^{n+j} \sum_{k=j}^{n} T_k(x) \Lambda_{n+j-k}(x) - \varepsilon^{n+1} G(x)T_n(x), \quad D(\varepsilon) = T^{-1}(\varepsilon) F(\varepsilon).
$$

Since $\lim_{x \to \infty} T_n(x) = 0, T_0 = I$, it follows from (2.5) that

$$
\|D(\varepsilon)\| \leq \text{const} \|G(x)\| d^n(x).
$$

Consequently,

$$
(2.6) \quad T^{-1}(\varepsilon, x)(\Lambda_0(x) + \varepsilon G(x))T(\varepsilon, x) = \Lambda(\varepsilon) + D(\varepsilon, x),
$$

where $D(\varepsilon) \in L[x_0, \infty)$,

$$
\lim_{x \to \infty} T(\varepsilon, x) = I, \quad \int_{x_0}^{\infty} \|T^{-1}(\varepsilon) T'(\varepsilon, x)\| \, dx < \infty.
$$

Setting $\varepsilon = 1$ in (2.6) we obtain the assertion of the theorem. \qed
Corollary 2.1. The elements \( \lambda_k(x) \) of the matrix \( \Lambda(x) \) satisfy the following estimate:

\[
\int_{x_0}^{\infty} |\lambda_i(x) - \tilde{\lambda}_i(x)| < \infty,
\]

where \( \tilde{\lambda}_i(x) \) is the corresponding eigenvalue of the matrix \( \Lambda_0(x) + G(x) \).

Proof. It follows from (2.6) that the eigenvalues of the matrices \( \Lambda_0(x) + \varepsilon G(x) \) and \( \Lambda(x) + D(\varepsilon, x) \) coincide. Therefore, by Gershgorin’s theorem \[4\], \( |\lambda_i(x) - \tilde{\lambda}_i(x)| < \|D\| \), and since

\[
\int_{x_0}^{\infty} \|D(\varepsilon, x)\| dx < \infty,
\]

we obtain

\[
\int_{x_0}^{\infty} |\lambda_i(x) - \tilde{\lambda}_i(x)| < \infty.
\]

\[\Box\]

3. Now suppose that \( \lambda_i(x), i = 1, \ldots, n \), are the eigenvalues of the matrix \( A(x), \Lambda(x) = \text{diag}(\lambda_1(x), \ldots, \lambda_n(x)) \), \( U(x) \) is a unitary matrix reducing the matrix \( A(x) \) to diagonal form, \( Q(x) = U^*(x)U' \) has elements \( q_{ij}(x) \), and \( \bar{q}_i(x) \) are the columns of the matrix \( Q(x) \). We introduce the matrix pencil

\[
(2.7) \quad F(p, x, \varepsilon) = (Q(x) + pI^2) - \Lambda(x) + \lambda I + (pI + 2Q(x))\mu^{-1}(x, \lambda)\mu'(x, \lambda),
\]

where \( \mu(x, \lambda) = (\Lambda(x) - \lambda I)^{1/2} \). We denote the eigenvalues of the pencil \( (2.7) \) by \( p_k(x, \lambda) \).

In this subsection we shall analyse the behaviour of an FSS of (2.1) as \( x \to \infty \) under the assumption that the matrix \( Q(x) \) is, in some sense, subordinate to the matrix \( \Lambda(x) \). We next introduce functions that characterize the degree of subordination of the matrix \( Q(x) \) to the matrix \( \Lambda(x) \):

\[
f(x) = \max_{i,j,k, i \neq k} \left( \frac{|q_{ij}(x)|}{\mu_i(x, \lambda)} + \frac{|\bar{q}_i(x), \bar{q}_j(x)|}{\mu(x, \lambda)} \right) \cdot \frac{1}{|\mu_i(x, \lambda) \pm \mu_k(x, \lambda)|},
\]

where \( \langle \bar{q}_i(x), \bar{q}_j(x) \rangle = (Q^2(x))_{ij} \), and

\[
\sigma(x) = \max_{j \neq k} \left( \left| \left( \frac{q_{ij}(x)}{\mu_i(x, \lambda) \pm \mu_k(x, \lambda)} \right) \right| + \left| \left( \frac{\bar{q}_i(x), \bar{q}_j(x)}{\mu_i(x, \lambda) \pm \mu_k(x, \lambda)} \right) \right| \right).
\]

Theorem 2.2. Suppose that \( \lim_{x \to \infty} f(x) = 0 \) and there exists a positive integer \( n \in \mathbb{N} \) such that

a) \( \int_{x_0}^{\infty} (\|Q(t)\| + \|Q^2(t)\| \mu^{-1}(t, \lambda)) f^n(t) dt < \infty; \)

b) \( \int_{x_0}^{\infty} |q_{i,j}(t)| \mu^{-1}(t, \lambda) dt < \infty; \)

c) \( \int_{x_0}^{\infty} \sigma(t) dt < \infty; \)

d) \( \text{Re}(p_k(x, \lambda) - p_j(x, \lambda)) \neq 0 \) for \( x > x_0, k \neq j \).

Then the system (2.2) has an FSS for which the following asymptotic formula holds as \( x \to \infty: \)

\[
y_k(t, \lambda) = U(x)\mu^{-1}(x, \lambda) \exp \left\{ \int_{x_0}^{\infty} p_k(t, \lambda) dt \right\} E_k,
\]

where the \( E_k \) are the basis vectors in \( \mathbb{R}^n \).
Proof. We set $y = Uz$ in (2.2) and multiply on the left by $U^*$. We obtain

$$z'' - 2Qz' + (\Lambda - Q'^2 - Q' - \lambda I)z = 0.$$  

Setting $v_1 = z$, $v_2 = z'$ we pass from the system (2.8) to the system of first-order equations in a space of vector-functions of dimension $2n$:

$$(2.9) \quad \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right)' = \left( \begin{array}{cc} 0 & I \\ \Lambda - Q'^2 - Q' - \lambda I & 2Q \end{array} \right) \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right).$$

We set $F = TV$ in (2.9), where

$$T = \begin{pmatrix} \mu^{-1} & \mu^{-1} \\ I & -I \end{pmatrix}.$$  

Then

$$(2.10) \quad F' = (\Lambda_0 + G + B)F,$$

where

$$\Lambda_0 = \begin{pmatrix} \mu(x, \lambda) & 0 \\ 0 & -\mu(x, \lambda) \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} Q'(x)\mu^{-1}(x, \lambda) & Q'(x)\mu^{-1}(x, \lambda) \\ -Q'(x)\mu^{-1}(x, \lambda) & -Q'(x)\mu^{-1}(x, \lambda) \end{pmatrix},$$

$$G = \frac{1}{2} \begin{pmatrix} Q^2(x) - \mu'(x, \lambda) & Q^2(x) - \mu'(x, \lambda) \\ -Q^2(x) - \mu'(x, \lambda) & -Q^2(x) + \mu'(x, \lambda) \end{pmatrix} \mu^{-1}(x, \lambda) + \frac{1}{2} \begin{pmatrix} -Q(x) & Q(x) \\ Q(x) & -Q(x) \end{pmatrix}.$$  

By conditions a), c), and d), we can apply Theorem 2.1 to the matrix $\Lambda_0 + G$, and so there exists a matrix $\Theta(x)$ such that

$$\lim_{x \to \infty} \Theta(x) = 0, \quad \int_{x_0}^{\infty} \|(I + \Theta(t))^{-1}\Theta'(t)\| \, dt < \infty,$$

$$(I + \Theta(x))^{-1}(\Lambda_0 + G)(I + \Theta(x)) = \Psi + D, \quad \int_{x_0}^{\infty} \|D(t)\| \, dt < \infty,$$

where $\Psi(x, \lambda)$ is a diagonal matrix with elements $\psi_i(x, \lambda)$. Setting $F = (I + \Theta)W$ in (2.10) we obtain the system of equations

$$(2.11) \quad W' = (\Psi + D + (I + \Theta)^{-1}\Theta' + (I + \Theta)^{-1}B(I + \Theta)) W.$$  

By condition b) of the theorem we have

$$\int_{x_0}^{\infty} \|(I + \Theta(t))^{-1}B(I + \Theta(t))\| \, dt < \infty.$$

All the conditions of the theorem on $L$-diagonal systems [2] hold for the system (2.11). Consequently, the system (2.11) has a fundamental system of solutions

$$W_k(x, \lambda) = \exp\left\{ \int_{x_0}^{x} \psi_k(t, \lambda) \, dt \right\} \cdot \bar{E}_k(I + o(1)).$$

Taking the changes of variables into account we obtain the asymptotic formula for an FSS of the system (2.11):

$$y_k(x, \lambda) = U(x)\mu^{-1}(x, \lambda) \exp\left\{ \int_{x_0}^{x} \psi_k(t, \lambda) \, dt \right\} \cdot \bar{E}_k(I + o(1)).$$

We now claim that

$$\int_{x_0}^{\infty} |\psi_k(t, \lambda) - p_k(t, \lambda)| \, dt < \infty,$$
where the \( p_k(t, \lambda) \) are the eigenvalues of the pencil \( (2.7) \). Indeed, we observe that the eigenvalues of the matrices \( \Psi \) and \( (I + \Theta)^{-1}(\Lambda_0 + G)(I + \Theta) \) coincide up to a summand in \( L_1[x_0, \infty) \). Next, the following chain of matrix similarities holds:

\[
(I + \Theta)^{-1}(\Lambda_0 + G)(I + \Theta) \sim \Lambda_0 + G \\
\sim T^{-1} \left( \begin{pmatrix} 0 & I \\ \Lambda_0 - Q^2 - \lambda I & -2Q \end{pmatrix} - T'T^{-1} \right) T \\
\sim \begin{pmatrix} -\mu'\mu^{-1} & I \\ \Lambda_0 - Q^2 - \lambda I & -2Q \end{pmatrix}.
\]

We find the eigenvalues of the last matrix:

\[
\begin{pmatrix} -\mu'\mu^{-1} & I \\ \Lambda_0 - Q^2 - \lambda I & -2Q \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = p \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

Hence, \( y_2 = (\mu'\mu^{-1} + pI)y_1 \), and so

\[
(\Lambda_0 - Q^2 - \lambda I)y_1 - (2Q + pI)(\mu'\mu^{-1} + pI)y_1 = 0.
\]

From this equation we obtain that

\[
\det (\Lambda_0 - Q^2 - \lambda I - 2Q\mu'\mu^{-1} - p\mu'\mu^{-1} - 2pQ - p^2 I) = \det ((Q + pI)^2 - \Lambda + \lambda I + (pI + 2Q)\mu'\mu^{-1}) = 0.
\]

Thus, the eigenvalues of the matrix pencil \( (2.7) \) coincide with the \( \psi_k(t, \lambda) \) up to summands which are integrable on \( [x_0, \infty) \). Theorem 2.2 is proved. \( \square \)

Example. Let \( n = 2 \),

\[
A(x) = \begin{pmatrix} ax^\alpha \cos^2 x^\gamma + bx^\beta \sin^2 x^\gamma \\ \frac{1}{2}(ax^\alpha - bx^\beta) \sin 2x^\gamma \\ \frac{1}{2}(ax^\alpha - bx^\beta) \sin 2x^\gamma \\ ax^\alpha \sin^2 x^\gamma + bx^\beta \cos^2 x^\gamma \end{pmatrix},
\]

where \( a < 0, b < 0, \alpha \leq 0, \beta > 2 \). It is easy to verify that \( \lambda_1(x) = ax^\alpha, \lambda_2(x) = bx^\beta \); the matrix reducing \( A(x) \) to diagonal form is

\[
U(x) = \begin{pmatrix} \cos x^\gamma & -\sin x^\gamma \\ \sin x^\gamma & \cos x^\gamma \end{pmatrix},
\]

and the matrix \( Q(x) \) is

\[
Q(x) = \gamma x^{\gamma - 1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

If \( \gamma < \alpha/2 + 1 \), then all the conditions of Theorem 2.2 hold. Consequently, the asymptotic formulae of this theorem hold, with

\[
p_{1,2}(x, \lambda) = \pm \sqrt{ax^\alpha - \lambda}, \quad p_{3,4} = \pm \sqrt{bx^\beta - \lambda}.
\]

It follows from these formulae that for \( \gamma < \alpha/2 + 1 \) the deficiency indices of the operator \( L_0 \) are equal to \( (3, 3) \). Furthermore, standard arguments (see [2]) show that the spectrum of any selfadjoint extension of the operator \( L_0 \) contains a discrete and a continuous part.
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