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BOUNDARY-PRESERVING MAPPINGS OF A MANIFOLD WITH INTERMINGLING BASINS OF COMPONENTS OF THE ATTRACTOR, ONE OF WHICH IS OPEN

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Abstract. We construct an open set of C^2-diffeomorphisms which preserve the boundary of some manifold, and which have the following property: for each mapping, the basin of attraction of one component of the attractor is open and everywhere dense, but the basin of attraction of the second component is nowhere dense, though its measure is positive.

§ 1. Introduction

1.1. **Background and the result.** One of the main problems in the theory of dynamical systems is the study of attractors. There are several different definitions of attractors, but all the known cases where the attractors differ are atypical. The prevailing hypothesis on the properties of attractors of metrically typical dynamical systems was given by Palis [8, §2.7].

The definition of attractor given in [8] includes many requirements, and not every diffeomorphism has an attractor satisfying them. Below, we give Milnor's definition of an attractor, which exists for any homeomorphism of a metric measure space, onto or into itself.

Definition 1. The *Milnor attractor* of a homeomorphism of a metric measure space is the smallest closed set (by inclusion) which contains the ω-limit sets of almost all points.

A *component* of a Milnor attractor is a closed invariant subset of the attractor which is indecomposable (that is, it contains a dense orbit) and has no other points of the attractor in any neighbourhood.

A *quasicomponent* of a Milnor attractor is a closed invariant indecomposable subset of the attractor which cannot be represented as a union of closed indecomposable invariant subsets.

The *basin of attraction* of a component (quasicomponent) of an attractor is the set of points whose ω-limits belong to this component (quasicomponent).

Ittai Kan [6] constructed an example of an endomorphism with intermingled basins of attraction of the components of the Milnor attractor; both basins are metrically dense, that is, they intersect any ball on a set of positive measure.

An open set of C^2-smooth diffeomorphisms with the same properties of basins was constructed in [5] and [12].

The goal of these papers was to prove the following theorem.

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Theorem 1. *In the class of boundary-preserving C^2-diffeomorphisms of* $\mathbb{T}^2 \times [0,1]$, *there exists an open set of mappings whose connected components of the boundary of the manifold are quasicomponents of the Milnor attractor, for which the basin of the quasicomponent* $\mathbb{T}^2 \times \{1\}$ *is open and everywhere dense, and the basin of the quasicomponent* $\mathbb{T}^2 \times \{0\}$ *has positive measure.*

A mapping with the stated properties was first noted as an inverse to one of the mappings considered in the construction of a Milnor attractor of positive Lebesgue measure in [3].

1.2. **Scheme of the proof.** Throughout this paper, we consider boundary-preserving C^2-diffeomorphisms of the product of the 2-torus and an interval, $\mathbb{T}^2 \times [0,1]$.

Definition 2. A diffeomorphism

$$F\colon B \times M \to B \times M, \quad (b,x) \mapsto (Ab, f(b,x))$$

is called a *fibre bundle*. The set B is the *base*, and M is the *fibre*. The mapping A is a diffeomorphism of B into itself, and the family of "fibred" mappings $f(b,\cdot)\colon M \to M$ depends on the point b in B. Sometimes, for notational convenience, we shall denote a mapping f over a base point b by f_b.

In Section 2.1, we construct an example of a fibre bundle with base $\mathbb{T}^2$ and fibre $[0,1]$, for which the assertions of Theorem 1 on basins hold true.

In Section 3, we show that any diffeomorphism obtained from this construction by a sufficiently small perturbation (in the class of C^2-smooth diffeomorphisms), has the required properties. The technique of the proof is based upon properties of hyperbolicity (Hirsch–Pugh–Shub [2]), namely, on the fact that for small perturbations of a fibre bundle in the class of diffeomorphisms, compact central fibres "survive".

Theorem 2. *Suppose that a fibre bundle F over an Anosov diffeomorphism of the torus, A, satisfies the condition of* dominated splitting, *that is,*

$$\max_b \operatorname{Lip} f_b < \mu, \quad \max_b \operatorname{Lip} f_b^{-1} < \frac{1}{\lambda},$$

where $\lambda < 1 < \mu$ are the constants of contraction and expansion of A.

Then any of its small (in C^1) perturbations is semiconjugate to a mapping into the base: there exists a projection $p\colon X \to B = \mathbb{T}^2$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{A} & B \end{array}$$

is commutative, and the fibre $M_b = p^{-1}(b)$ depends continuously on b and is C^1-smooth.

The family $\{M_b \mid b \in B\}$ forms a "central foliation", every fibre of which is the graph of a function $\beta_b\colon M \to B$. Theorem 2 asserts only that β_b depends continuously on b.

Theorem 3 (Gorodetskii [11])**.** *Under the hypotheses of Theorem* 2, *β_b is Hölder continuous as a function of b.*

Theorem 4 (Ilyashenko–Negut [4])**.** *The Hölder exponent in Theorem* 3 *tends to* 1 *when the C^1-norm of the perturbation of F tends to zero.*

An overview of Hölder foliations may be found, for example, in [9].

The most interesting proof in §3 is that the points attracted to $\mathbb{T}^2 \times \{0\}$ have positive measure. We integrate the lengths of segments of "curved" fibres, whose points are attracted to the specified boundary component, over the base. The fact that the measure of the set of relevant points in the base is positive follows from a lemma of Falconer's [1], Theorem 4 in [4], a special ergodic theorem [14], and the possibility of "repeated integration" of the length of segments of a "curved" foliation, which follows from theorems due to Anosov and also Pesin [13]. Pesin's Theorem is based on the existence of a central-stable foliation. The statements of these theorems are given in §3.

§ 2. An example of a fibre bundle

2.1. **Good fibre bundles.** We consider a boundary-preserving C^2-smooth fibre bundle $\mathscr{F}$ over an Anosov diffeomorphism A on the 2-torus. The fibre is the unit interval. Put $X=\mathbb{T}^2\times[0,1]$,

$$\mathscr{F}\colon X\to X,\quad (b,x)\mapsto (Ab,f_b(x)).$$

Such a fibre bundle is called *good* if it satisfies the following conditions.

1. There exists a point a in A of period m such that

(a) the composition of fibred mappings along its orbit shifts all the points in the interval $(0,1)$ upwards:

$$f_{A^{m-1}a}\circ\dots\circ f_a(x)>x\quad \forall x\in(0,1); \tag{1}$$

(b) the point 0 is repulsive for this composition:

$$(f_{A^{m-1}a})'(0)\cdot\ldots\cdot(f_a)'(0)>1. \tag{2}$$

2. All fibred mappings on the interval $(1/2,1)$ *shift points upwards*:

$$\forall b\in\mathbb{T}^2\ \forall x\in\left(\frac{1}{2},1\right): f_b(x)>x \text{ and } (f_b)'(1)<1. \tag{3}$$

The latter condition guarantees that for a small perturbation of F the neighbourhood $\mathbb{T}^2\times(1/2,1]$ is, as before, attracted to the upper boundary component $\mathbb{T}^2\times\{1\}$.

3. The boundary component $\mathbb{T}^2\times\{0\}$ attracts on average:

$$\int_{\mathbb{T}^2}\ln(f_b)'(0)\,db<0. \tag{4}$$

4. For any b, a fibred mapping is close to the identity in the following sense:

$$\|(f_b)'-1\|_C<\frac{1}{4}, \tag{5}$$

and the coefficients of contraction and expansion of A are sufficiently far from 1 for $\mathscr{F}$, as a fibre bundle over A, to satisfy the condition of dominated splitting.

This completes the description of good fibre bundles.

We denote by B_0 and B_1 the sets of points which tend to the boundary components $\mathbb{T}^2\times\{0\}$ and $\mathbb{T}^2\times\{1\}$ as $n\to\infty$, respectively.

Theorem 5. *For a good fibre bundle, B_1 is open and dense, and the Lebesgue measure of B_0 is positive.*

In other words, the boundary $\mathbb{T}^2\times\{0,1\}$ belongs to the Milnor attractor (since the mapping in the base is transitive) and the basin of the upper component is open and dense.

Remark. A good fibre bundle forms an open set in the class of boundary-preserving smooth fibre bundles.

We prove the three assertions of Theorem 5 in turn.

2.2. **The basin B_1 is open.** A point belongs to the basin of the upper boundary component if it belongs to some preimage of the absorbing domain $\mathbb{T}^2 \times (1/2, 1]$.

$$B_1 = \bigcup_{n \geq 0} \mathscr{F}^{-n}\{\mathbb{T}^2 \times (1/2, 1]\};$$

therefore B_1 is open, since it is a countable union of open sets.

2.3. **The basin B_1 is dense.** We will construct a dense subset of B_1. In the base $\mathbb{T}^2$ we consider the stable manifold $w^s(a)$ of the periodic point a under the action of A, and its saturation by fibres $\{b \times [0, 1] \mid b \in w^s(a)\}$. We also consider the action of $\mathscr{F}^m$. Starting from a sufficiently large number of iterations, the image of each fibre of the given set belongs to a small neighbourhood of the fibre $a \times [0, 1]$, which is fixed for $\mathscr{F}^m$. By properties 1 and 2, in the C^1-topology the restriction of $\mathscr{F}^m$ to the fixed fibre is far from a mapping with fixed points in the interior of the interval; therefore the restriction of $\mathscr{F}^m$ to a fibre sufficiently close to the fixed fibre shifts the points of the interval upwards. Hence, after a sufficient number of iterations $\mathscr{F}^m$, the interior points of any fibre of the set we have constructed end up in a small neighbourhood of the upper boundary component. By property 2, this neighbourhood belongs to B_1.

Thus, B_1 is open and dense, so any subset of its complement is nowhere dense. Consequently, B_0 is nowhere dense.

2.4. **The measure of B_0 is positive.** To begin with, we construct a set of positive measure, and then show that it belongs to the basin of the lower boundary component. We take $\delta > 0$ such that

$$\int_{\mathbb{T}^2} \ln \max_{0 \leq x \leq \delta} (f_b)'(x)\, db < 0.$$

Such a $\delta > 0$ exists by the smoothness of $\mathscr{F}$: the integral on the left-hand side of the inequality depends continuously on δ and (by condition (4)) is negative for $\delta = 0$. Setting $g(b) = \max_{0 \leq x \leq \delta} (f_b)'(x)$, we obtain

$$\int_{\mathbb{T}^2} \ln g(b)\, db < 0. \tag{6}$$

By property 4, $\ln g$ is measurable; hence, by the ergodic theorem, the time averages of $\ln g$ are close to spatial for almost all b, namely: for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for any $n \geq N(\varepsilon)$

$$\left| \frac{1}{n}\big(\ln g(b) + \ln g(Ab) + \cdots + \ln g(A^{n-1}b)\big) - \int_{\mathbb{T}^2} \ln g(b)\, db \right| < \varepsilon. \tag{7}$$

We choose

$$\varepsilon = \frac{1}{2}\left| \int_{\mathbb{T}^2} \ln g(b)\, db \right|$$

and decompose almost all points of the base into a countable number of classes: the k-th class contains the points for which $N(\varepsilon) = k$. We consider an interval $[0, \delta_k]$ in the fibre over each point in the k-th class which is small enough that $(\max_{\mathbb{T}^2} g(b))^k \cdot \delta_k < \delta$.

Thus, in the fibre over each point in the k-th class we take an interval $[0, \delta_k]$ and consider the union of such intervals over the points in all such classes. The measure of the set we have constructed is positive, since the measure of at least one of the classes is positive—if not, we have divided almost all points in the base into a countable number of sets of measure zero. We will show that the set we have constructed lies in B_0.

By Lagrange's Theorem on finite increments, $f_b(x) \leq g(b)x$, for $x \leq \delta$. Therefore, so long as the x-coordinate of the image of the point does not exceed δ,

$$f_{A^{n-1}b} \circ \ldots \circ f_b(x) \leq g(A^{n-1}b) \cdot \ldots \cdot g(b)x. \tag{8}$$

Assertion. *For a point (b,x), where b is in the k-th class, and $x \le \delta_k$, the estimate* (8) *holds for all n.*

Proof by induction on n. Base step for the induction: for $n \le k$, the estimate holds by the choice of δ_k. Induction step: if the estimate holds for some $n \ge k$, then it also holds for $n+1$. To prove this we apply $f_{A^n b}$ to both sides of the inequality:

$$f_{A^n b} \circ f_{A^{n-1} b} \circ \dots \circ f_b(x) \le f_{A^n b} \circ g(A^{n-1}b) \cdot \ldots \cdot g(b)x.$$

It remains to apply Lagrange's Theorem to the right-hand side, and to do this it is necessary to show that the argument of $f_{A_n b}$ on the right-hand side of the last inequality does not exceed δ. By the definition of the k-th class, for any $n \ge k$ we have:

$$\frac{1}{n}\bigl(\ln g(b) + \ln g(Ab) + \dots + \ln g(A^{n-1}b)\bigr) < \int_{\mathbb{T}^2} \ln g(b)\, db + \varepsilon.$$

From (6) and by the choice of ε, it follows that the right-hand side of the inequality is negative. Thus, for a base point b in the k-th class, we have, for any $n \ge k$,

$$\begin{aligned}&\ln\bigl((A^{n-1}b) \cdot \ldots \cdot g(b)x\bigr)\\ &\qquad = \ln g(b) + \ln g(Ab) + \dots + \ln g(A^{n-1}b) + \ln x < -cn + \ln x < \ln x.\end{aligned}$$

It remains to note that $x \le \delta_k < \delta$. The assertion is proved. □

We apply (8): for a point (b,x), where b is in the k-th class, and $x \le \delta_k$,

$$\begin{aligned}\ln x(\mathscr{F}^n(b,x)) &= \ln(f_{A^{n-1}b} \circ \dots \circ f_b(x)) \le \ln\bigl(g(A^{n-1}b) \cdot \ldots \cdot g(b) \cdot x\bigr)\\ &= \ln x + \ln g(b) + \ln g(Ab) + \ldots + \ln g(A^{n-1}b) \to -\infty,\end{aligned}$$

so (b,x) is attracted to $\mathbb{T}^2 \times \{0\}$.

The first theorem is proved. □

Remark. From (7) and the proof, we see that there are no points from B_0 in a fibre over a point b only if the upper limit of the time averages of g deviates from the spatial average by more than ε. For almost all points b belonging to one stable manifold of the action of A in the base, the upper limit of the time averages in (7) is identical, since, after a sufficiently large number of iterations, the images of any two points from one stable manifold are found to be sufficiently close and, after dividing by n, the sum of the initial segment of the time average series is sufficiently small.

§ 3. Perturbation of a fibre bundle

We consider a C^2-smooth diffeomorphism $\mathscr{G}$, sufficiently close to $\mathscr{F}$, which preserves the boundary of $\mathbb{T}^2 \times [0,1]$. By Theorems 2, 3 and 4, $\mathscr{G}$ has an invariant foliation $\{p^{-1}b \mid b \in \mathbb{T}^2\}$; moreover, the fibres tend to the vertical when the norm of the perturbation of the fibre bundle tends to zero. We denote the basin of the lower boundary component for $\mathscr{G}$ by $B_0(\mathscr{G})$. and use the analogous notation for the upper boundary component.

3.1. **The basin of the upper boundary component.** We will prove that $B_1(\mathscr{G})$ is open.

By condition (3), there exists a small neighbourhood $\mathbb{T}^2 \times [1-\delta, 1]$ of the upper boundary component, in which all fibred mappings f_b attract points to 1 faster than some linear contraction. Therefore, for a diffeomorphism $\mathscr{G}$ sufficiently close to $\mathscr{F}$, the whole domain $\mathbb{T}^2 \times (1-\delta/2, 1]$ is attracted to the upper boundary component. The basin $B_1(\mathscr{G})$ is a countable union of preimages of this open domain, and is therefore open.

We will prove that $B_1(\mathscr{G})$ is dense.

Because Anosov diffeomorphisms are coarse, the restrictions of $\mathscr{F}$ and $\mathscr{G}$ to the boundary are conjugate. We recall that the restriction of $\mathscr{F}$ to the boundary has a periodic point of period m. Therefore, the restriction of $\mathscr{G}^m$ to the upper boundary component is a fixed point $\tilde{a}$, close to the corresponding fixed point a of $\mathscr{F}^m$. We consider a neighbourhood $U_\varepsilon(a)$ in the base and the closed cylinder $C = Cl(U_\varepsilon(a)) \times [0,1]$. We also consider the "boundary cylinder" $\widetilde{C} = \{p^{-1}(b) \mid b \in U_{\varepsilon/2}(\tilde{a})\}$. We reduce the norm of the perturbation sufficiently so that $\widetilde{C} \subset C$.

For small ε, the restriction of $\mathscr{F}$ to C is separated in C^1 from mappings with fixed points in the interior of C. Therefore, for a perturbation of sufficiently small norm, the restriction of $\mathscr{G}$ to the interior of $\widetilde{C}$ also has no fixed points.

To prove that B_1 is dense, it remains to note that after a sufficient number of iterations $\mathscr{G}^m$, any point from the dense set $\{p^{-1}b \mid b \in w^s(\tilde{a})\}$ appears in $\bigl\{p^{-1}b \mid b \in w^s(\tilde{a}) \bigcap U_{\varepsilon/2}(\tilde{a})\bigr\}$. From the preceding paragraph, it follows that all points from the interior of the latter set are attracted in future time to the upper boundary component. Thus we have proved that B_1 is dense.

3.2. **The measure of $B_0(\mathscr{G})$ is positive.** We need the *special ergodic theorem*:

Theorem 6 (Saltykov, 2011). *Let $\varphi \in C(\mathbb{T}^2)$, and let A be an Anosov diffeomorphism of the 2-torus. Let $L(b)$ be the set of limit points of the sequence of Birkhoff means of φ under the action of A,*

$$S(\varphi) = \int_{\mathbb{T}^2} \varphi(b)\, db.$$

We let

$$K_\varkappa = \{b \mid L(b) \setminus [S - \varkappa, S + \varkappa] \neq \emptyset\}$$

be the set of points of the torus over which the time average deviates from the spatial average by more than $\varkappa$.

Then $\dim_H K_\varkappa < 2$, for all $\varkappa > 0$.

A stronger special ergodic theorem was proved in [7].

Definition 3. In §2.4, we defined the function g and chose $\varepsilon > 0$. For $\mathscr{F}$, a set of points in the base which has deviation $\varkappa$ (Theorem 6) for g no smaller than ε is called *bad*. For $\mathscr{G}$, the image of a bad set under the action of conjugation of the base is termed *bad*. The complement of a bad set in the base is called *good*.

3.2.1. *The structure of the bad set.* According to the remarks at the end of section 2.4, a bad set for $\mathscr{F}$ contains all those points in the base whose fibres contain no points of $B_0(\mathscr{F})$. In addition, together with each of its points, a bad set for $\mathscr{F}$ also contains the whole of its stable manifold. The conjugations of the restrictions of $\mathscr{F}$ and $\mathscr{G}$ to the boundary translate bad sets and stable foliations of the mappings in the base into each other.

Assertion. *In any fibre over a good set of $\mathscr{G}$, there are points in $B_0(\mathscr{G})$.*

The proof of the assertion repeats the arguments in §2.4, but now in place of f_b we must consider a mapping acting from a "curved" fibre into the "curve". The fibred mapping into the distorted fibres is close to the original; therefore the corresponding time averages over the points in the good set do not deviate strongly from the spatial average.

By analogy with the proof in Section 2.4, to prove that the measure of $B_0(\mathscr{G})$ is positive it is enough to use Fubini's Theorem to integrate the lengths of the segments of the fibres which grow from good points in the lower boundary component $\mathbb{T}^2 \times \{0\}$ and belong to $B_0(\mathscr{G})$. However, first of all, the conjugation might translate bad points into a set of full measure. Second, the central-stable foliation must be absolutely continuous, so that Fubini's Theorem can be applied. To overcome the first difficulty, that is, to prove that the measure of a bad set for $\mathscr{G}$ is equal to zero, we estimate its Hausdorff dimension and show that this is less than full.

3.2.2. *The Hausdorff dimension of a bad set.*

Theorem 7 (Falconer's Lemma [1]). *Let $h\colon X \to Y$ be Hölder continuous with exponent α, and let the Hausdorff dimension of the set $D \subset X$ be equal to d. Then*

$$\dim_H hD \leq \frac{d}{\alpha}.$$

From Saltykov's Theorem and the remarks in Section 2.4, it follows that the Hausdorff dimension of a bad set for $\mathscr{F}$ is less than full. From Falconer's Lemma and Theorem 4, it follows that when the C^1-norm of a perturbation tends to zero, the dimension of a bad set for $\mathscr{G}$ tends to the dimension of a bad set for $\mathscr{F}$. Therefore, for a perturbation of sufficiently small norm, the Hausdorff dimension of a bad set for $\mathscr{G}$ is less than full. In particular, the measure of a bad set for $\mathscr{G}$ is zero.

3.2.3. *The basin $B_0(\mathscr{G})$ has positive measure.* Now, in the "curved" fibre over each point of a good set for $\mathscr{G}$, there are points from $B_0(\mathscr{G})$. A good set, like a bad one, is a union of stable manifolds of the mapping in the base.

The concept of the absolute continuity of a foliation was introduced in [10, § 5]: a foliation is absolutely continuous if the holonomy mapping along the foliation maps a set of measure zero to a set of measure zero. In the same paper, the following theorem was proved.

Theorem 8 (Anosov). *The foliation on the stable manifold is absolutely continuous for the action of a hyperbolic C^2-diffeomorphism.*

We consider the action of the restriction of $\mathscr{G}$ to the base. In the base we take an arbitrary curve γ, transversal to the stable foliation of the restriction of $\mathscr{G}$ to the base. Almost all points of γ are good—otherwise, using theorems due to Anosov and Pesin, the absolutely continuous stable foliation passing through that part of the bad points of the curve with positive measure forms a subset of a bad set for $\mathscr{G}$ which has positive measure.

From any good point on γ we produce an arc of the stable foliation for the action of the restriction of $\mathscr{G}$ to the base. All the points on this arc are good (see Section 3.2.1). From any point in a good set in the base, we produce an arc of the central fibre for the action of $\mathscr{G}$ on X. The arcs of the stable manifold in the base, together with the arcs of the "curved" fibres over them, form a subset of a central-stable foliation for the action of $\mathscr{G}$, since $\mathscr{F}$ satisfies the condition of dominated splitting.

Theorem 9 (Pesin [13, Chapter 7]). *The central-stable foliation for the action of a C^2-diffeomorphism is absolutely continuous.*

The measure of $B_0(\mathscr{G})$ can now be estimated from below. The subset of the basin we have constructed consists of two-dimensional domains of an absolutely continuous central-stable foliation which passes through almost all the points on a one-dimensional curve γ. Therefore, by the theorem of Pesin mentioned above and by Fubini's Theorem, the measure of the constructed subset is positive. The third theorem is proved. $\square$

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