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LOCAL DYNAMICS OF TWO-COMPONENT SINGULARLY PERTURBED PARABOLIC SYSTEMS

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Abstract. We consider the local dynamics in a neighbourhood of a stationary state of a two-component system of parabolic equations with periodic boundary conditions. In the critical cases we construct families of special equations—quasinormal forms whose solutions in principle give asymptotic solutions, up to the residual, of the original singularly perturbed system.

Introduction

Consider a nonlinear two-component system of parabolic equations

$$\frac{\partial u}{\partial t} = (D_0 + \varepsilon D_1)\frac{\partial^2 u}{\partial x^2} + (A_0 + \mu A_1)u + F(u,u) + \Phi(u) \tag{1}$$

with periodic boundary conditions

$$u(t, x + 2\pi) \equiv u(t,x). \tag{2}$$

Here, $u = (u_1, u_2) \in \mathbb{R}^2$ and the positive parameters ε and μ are sufficiently small, that is, $0 < \varepsilon, \mu \ll 1$. For all sufficiently small ε the eigenvalues of the matrix $D(\varepsilon) = D_0 + \varepsilon D_1$ only have positive real parts. We further assume that the nonlinear vector function $F(u,u)$ is linear in every argument (thus, $F(0,u) = F(u,0) = 0$), and at zero the function $\Phi(u)$ is of order of smallness at least three. We shall investigate the behaviour as $t \to \infty$ of all the solutions of the boundary value problem (1), (2) with initial conditions lying in a sufficiently small neighbourhood in $C_{[0,2\pi]}(\mathbb{R}^2)$ of the zero equilibrium state (independent of ε and μ).

Together with (1), (2), we also study the local dynamics of another system of two equations,

$$\frac{\partial u}{\partial t} = (D_0 + \varepsilon D_1)\frac{\partial^2 u}{\partial x^2} + (A_0 + \mu A_1)u + F(u, \frac{\partial u}{\partial x}) + \Phi(u), \tag{3}$$

with the same periodic boundary conditions (2). This system differs from (1) only in the fact that the argument of the nonlinear vector function F involves the expression $\partial u/\partial x$. We shall compare the local dynamics of these two boundary value problems.

When both eigenvalues of the principal part of the diffusion matrix D_0 have positive real parts, the problem stated above is well researched (see, for example, [1]). The method of local invariant integral manifolds [1, 2] and the method of normal forms [3] make it possible to reduce (1), (2) and (3), (2) to a certain special system of nonlinear ordinary differential equations, the dynamics of which completely determines the behaviour of all

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solutions of (1), (2) and (3), (2) with sufficiently small (in the norm of $C_{[0,2\pi]}(\mathbb{R}^2)$) initial conditions.

In [4] the problem stated above was studied under the condition that the matrix D_0 has one zero and one positive eigenvalue, and in [5, 6, 7, 8] when D_0 is a zero matrix. Here, it is no longer possible to use the methods of invariant manifolds and normal forms. In [5, 6, 7] a special asymptotic method of constructing a family of nonlinear boundary value problems was developed; the nonlocal dynamics of these determines the behaviour of solutions of the original boundary value problems for small ε and μ. We point out that in these situations the problems (1), (2) and (3), (2) are singularly perturbed.

In this paper we assume that

$$D_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus the original boundary value problems are singularly perturbed. Some results have been published in [9].

Using a transformation close to the identity, the matrix $D(\varepsilon)$ can be reduced to the form

$$D(\varepsilon) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 \\ d_1 & d_2 \end{pmatrix} + O(\varepsilon^2)$$

(see [10]).

We assume that the nondegeneracy conditions $d_1 \neq 0$, $d_2 \neq 0$ hold, and therefore,

$$d_1 < 0, \quad d_2 > 0. \tag{4}$$

An important role in the study of the local dynamics is played by the location of the roots of the characteristic equation (of the boundary value problems linearized at zero)

$$\det\left(A_0 - n^2 \begin{pmatrix} 0 & 1 \\ \varepsilon d_1 & \varepsilon d_2 \end{pmatrix} - \lambda I \right) = 0, \quad n = 0, \pm 1, \pm 2, \dots. \tag{5}$$

Let

$$A_0 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix}, \quad \Delta = \det A_0, \quad a_0 = a_1 + a_4.$$

Equation (5) can be represented in the form

$$\lambda^2 - [a_0 - \varepsilon d_2 n^2]\lambda + \Delta + n^2[a_3 + \varepsilon(d_1 a_2 - d_2 a_1) - \varepsilon d_1 n^2] = 0. \tag{6}$$

When (6) has a root whose real part is positive and is separated away from zero as $\varepsilon \to 0$, the problem stated above is nonlocal: in a small neighbourhood of the zero equilibrium state the boundary value problems (1), (2) and (3), (2) do not have an attractor. However, if all the roots of (6) have negative real parts and are separated from zero as $\varepsilon \to 0$, the problem stated above is trivial: in some sufficiently small neighbourhood of the zero equilibrium state all solutions tend to zero for sufficiently small ε and μ. In what follows we consider the critical cases when there are no roots of (6) whose real parts are positive and separated from zero as $\varepsilon \to 0$, and there are roots whose real parts are asymptotically small (as $\varepsilon \to 0$). Hence it is necessary that the inequalities

$$a_3 \geq 0, \quad \Delta \geq 0, \quad a_0 \leq 0 \tag{7}$$

hold, and that some of them are equalities. We single out the situations we have to investigate. First we consider the cases when only one of the inequalities in (7) becomes an equality.

1^*. Suppose that $a_3 = 0$, $\Delta > 0$, $a_0 < 0$. Under these conditions all the roots of (6) have negative real parts and are separated from the imaginary axis as $\varepsilon \to 0$. Thus this case does not need to be studied.

2^*. Suppose that $\Delta = 0$, $a_3 > 0$, $a_0 < 0$. Under these conditions one root of (6) is zero, and all the other roots have negative real parts and are separated from zero as $\varepsilon \to 0$. In this case we apply the method of invariant manifolds and the method of normal forms. Using these methods we reduce the boundary value problem (1), (2) to a scalar (real) first order equation (on a one-dimensional invariant integral manifold) of the form

$$\frac{d\xi}{d\tau} = a\xi + b\xi^2 \quad (\tau = \mu t). \tag{8}$$

We can write the values of the coefficients a and b. Suppose that $A_0 h = 0$, $A_0^* g = 0$, and $(h, g) = 1$. Then

$$a = (A_1 h, g), \quad b = (F(h, h), g).$$

(In the case of the boundary value problem (3), (2) we obtain that $b = 0$, thus (8) is a linear equation.) The solutions of equation (8) and of the original boundary value problems (1), (2) and (3), (2) are connected by the formula

$$u(t, x) = \mu\xi(\mu t)h + o(\mu^2).$$

3^*. Let $a_0 = 0$, $a_3 > 0$, $\Delta > 0$. In this situation the real parts of an infinite set of roots of (6) tend to 0 as $\varepsilon \to 0$. Thus an 'infinite-dimensional' critical case is realized. We apply the formalism of the method of normal forms. Let h_n be an eigenvector corresponding to the purely imaginary eigenvalue $i(n^2 a_3 + \Delta)^{1/2}$ of the matrix $A_0 - n^2 D_0$:

$$h_n = (i(n^2 a_3 + \Delta)^{1/2} - a_4, a_3).$$

We look at the formal series

$$u(t, x) = \sum_{n=-\infty}^{\infty} (\xi_n(t, x) \exp\bigl(i[(a_3 n^2 + \Delta)^{1/2} t + nx]\bigr) h_n \\ + \bar{\xi}(t, x) \exp\bigl(-i[(a_3 n^2 + \Delta)^{1/2} t + nx]\bigr) \bar{h}_n) + \cdots$$

in which $\xi_n(t)$ is a sufficiently small complex amplitude slowly changing with time. Substituting this expression into (1), (2) (or (3), (2)) and collecting the coefficients of like powers (normalizing the variable ξ appropriately and introducing some slow time instead of t (see, for example, [6])), we can obtain an infinite system of ordinary differential equations for determining the amplitude ξ_n. In contrast to the situations considered in [11, 5, 6, 7, 8], this system does not simplify, nor can it be written in a compact form. We do not study this case here.

There are also three situations where exactly two of the three inequalities (7) turn into equality. One of these, when $a_0 = a_3 = 0$ and $\Delta > 0$, is trivial: all the roots of (6) for all n have negative real parts separated from zero. The main aim of this paper is to analyze the other two cases. In §§ 1 and 2 we consider the case where $a_3 = \Delta = 0$ and $a_0 < 0$, and in § 3 we assume that $\Delta = a_0 = 0$, $a_3 > 0$.

Note that the 'relationship' between the small parameters ε and μ is important. In what follows it is convenient to assume that

$$\mu = c\varepsilon^\alpha \tag{9}$$

for some positive constants c and α.

We define a parameter c_0, which will be needed in what follows, by the rule $c_0 = c$ if $\alpha = 1$ and $c_0 = 0$ if $\alpha > 1$.

§ 1. Quasinormal forms under the conditions $a_3 = \Delta = 0,\ a_0 < 0$

Under the conditions

$$a_3 = \Delta = 0, \quad a_0 < 0, \tag{10}$$

the characteristic equation (6) has the set of roots

$$\lambda_n(\varepsilon) = \varepsilon\lambda_{n1} + O(\varepsilon^2),$$

where $n = 0, \pm 1, \pm 2, \dots$ and

$$\lambda_{n1} = [-n^2(d_1a_2 + d_2a_1) + d_1n^4]a_0{}^{-1}.$$

All the other roots of (6) have negative real parts and are separated from zero as $\varepsilon \to 0$. It follows from the equation $\Delta = \alpha_1\alpha_4 = 0$ that either $a_1 = 0$ or $a_4 = 0$. These two cases are considered separately. The constructions for $\alpha \geq 1$ and for $1/2 < \alpha < 1$ are substantially different; therefore they need to be looked at separately. We introduce some more notation. Since $F(u, v)$ is linear in each argument, this vector function can be represented in the form

$$F(u, v) = \begin{pmatrix} (\alpha_1u_1 + \alpha_2u_2)(\alpha_3v_1 + \alpha_4v_2) \\ (\beta_1u_1 + \beta_2u_2)(\beta_3v_1 + \beta_4v_2) \end{pmatrix}.$$

1.1. Suppose that $\alpha \geq 1$, conditions (10) hold, and

$$a_1 = 0, \quad a_4 \neq 0. \tag{11}$$

Consequently, $a_0 = a_4 < 0$. The matrices $A_0 - n^2D_0$ for all n have zero eigenvalue with eigenvector $e_1 = (1, 0)$. Let $b_n = (a_4, n^2 - a_2)$ denote an eigenvector of the matrix $(A_0 - n^2D_0)^*$ corresponding to the zero eigenvalue.

We introduce into consideration the formal series

$$u = \varepsilon \sum_{m=-\infty}^{\infty} \xi_m(\tau)e_1e^{imx} + \varepsilon^2u_2(\tau, x) + \cdots, \tag{12}$$

where $\tau = \varepsilon t$. We substitute (12) into (1), (2) and into (3), (2) and collect the coefficients of like powers of ε. Then at the second step, from the conditions for the resulting equation to be solvable for with respect to $u_2(\tau, x)$, we arrive at an infinite system of equations with respect to $\xi_m(\tau)$. We note the important fact that this system can be written in the form of a scalar parabolic boundary value problem with respect to $\xi(\tau, x) = \sum_{m=-\infty}^{\infty} \xi_m(\tau)e^{imx}$:

a) in the case of the boundary value problem (1), (2) we have

$$\begin{gathered} a_4\frac{\partial\xi}{\partial\tau} = -d_1\frac{\partial^4\xi}{\partial x^4} + (-d_1a_2 + c_0a_{21})\frac{\partial^2\xi}{\partial x^2} + c_0(a_{11}a_4 - a_{21}a_2)\xi \\ + (a_4\alpha_1\alpha_3 - a_2\beta_1\beta_3)\xi^2 - \beta_1\beta_3\frac{\partial^2}{\partial x^2}(\xi^2), \\ \xi(\tau, x + 2\pi) = \xi(\tau, x); \end{gathered} \tag{13}$$

b) in the case of the boundary value problem (3), (2) we have

$$\begin{gathered} a_4\frac{\partial\xi}{\partial\tau} = -d_1\frac{\partial^4\xi}{\partial x^4} + (-d_1a_2 + c_0a_{21})\frac{\partial^2\xi}{\partial x^2} + c_0(a_{11}a_4 - a_{21}a_2)\xi \\ + (a_4\alpha_1\alpha_3 - a_2\beta_1\beta_3)\xi\frac{\partial\xi}{\partial x} - \frac{1}{2}\beta_1\beta_3\frac{\partial^3}{\partial x^3}(\xi^2), \\ \xi(\tau, x + 2\pi) = \xi(\tau, x). \end{gathered} \tag{14}$$

Theorem 1. *Suppose that $\alpha \geq 1$ and that conditions* (10), (11) *hold. Suppose that the boundary value problem* (13) (*the boundary value problem* (14)) *has a solution $\xi_0(\tau, x)$ that is bounded as $\tau \to \infty$. Then the boundary value problem* (1), (2) (*the boundary value problem* (3), (2)) *has an asymptotic solution $u_0(t, x, \varepsilon)$, up to the residual, such that*

$$u_0(t, x, \varepsilon) = \varepsilon\xi_0(\varepsilon t, x)e_1 + O(\varepsilon^2).$$

1.2. Here we assume that conditions (10), (11) hold and that the parameter α satisfies the inequality

$$\frac{1}{2} < \alpha < 1. \tag{15}$$

We fix an arbitrary value $z \neq 0$ and denote by $\theta = \theta(z, \varepsilon)$ the number in the half-open interval $[0, 1)$ that complements the quantity $z\varepsilon^{(\alpha-1)/2}$ to an integer. Here the role of the formal series (12) is played by the formal series

$$u = \varepsilon^{\gamma_1} \sum_{m=-\infty}^{\infty} \xi_m(\tau)e_1e^{imy} + \varepsilon^{\gamma_2}u_2(\tau, y) + \cdots , \tag{16}$$

where $\tau = \varepsilon^{2\alpha-1}t$, $y = (z\varepsilon^{(\alpha-1)/2} + \theta)x$, and the positive parameters γ_1, γ_2 ($\gamma_1 < \gamma_2$) are defined separately for equations (1) and (3):

a) for the boundary value problem (1), (2) we have $\gamma_1 = \alpha$, $\gamma_2 = 3\alpha - 1$, and for determining

$$\xi(\tau, y) = \sum_{m=-\infty}^{\infty} \xi_m(\tau)e^{imy}$$

we arrive at the equation

$$a_4\frac{\partial\xi}{\partial\tau} = z^2\frac{\partial^2}{\partial y^2}\left(-z^2d_1\frac{\partial^2\xi}{\partial y^2} + c_1a_{21}\xi - \beta_1\beta_3\xi^2\right), \quad \xi(\tau, y + 2\pi) = \xi(\tau, y); \tag{17}$$

b) for the boundary value problem (3), (2) we have $\gamma_1 = \frac{1}{2}(\alpha + 1)$, $\gamma_2 = \frac{1}{2}(3\alpha - 1)$, and for finding $\xi(\tau, y)$ we obtain the equation

$$\begin{gathered} a_4\frac{\partial\xi}{\partial\tau} = z^2\frac{\partial^2}{\partial y^2}\left(-z^2d_1\frac{\partial^2\xi}{\partial y^2} + c_1a_{21}\xi - \frac{1}{2}\beta_1\beta_3z\frac{\partial}{\partial y}\xi^2\right), \\ \xi(\tau, y + 2\pi) = \xi(\tau, y). \end{gathered} \tag{18}$$

An analogue of Theorem 1 in this case takes the following form.

Theorem 2. *Suppose that conditions* (10), (11), *and* (15) *hold. Suppose that the boundary value problem* (17) (*or* (18)) *has a solution $\xi_0(\tau, y)$ which is bounded as $\tau \to \infty$. Then the boundary value problem* (1), (2) (*or* (3), (2)) *has an asymptotic solution $u_0(t, x, \varepsilon)$, up to the residual, such that*

$$u_0(t, x, \varepsilon) = \varepsilon^{\gamma_1}\xi\bigl(\varepsilon^{2\alpha-1}t, (z\varepsilon^{(\alpha-1)/2} + \theta)x\bigr) + O(\varepsilon^{\gamma_2}).$$

We draw the reader's attention to the fact that inequality $\gamma_2 > \gamma_1$, which is necessary in order to look at the structures in the form of the formal series (12), (15), holds only under the condition $\alpha > 1/2$.

1.3. Suppose that $\alpha \geq 1$, conditions (10) hold, and that

$$a_4 = 0, \quad a_1 \neq 0 \quad (a_1 = a_0 < 0). \tag{19}$$

The matrices $A_0 - n^2D_0$ have an eigenvector $h_n = (n^2 - a_2, a_1)$ corresponding to the zero eigenvalue, and $e_2 = (0, 1)$ is an analogous eigenvector of the matrix $(A_0 - n^2D_0)^*$.

Consider the formal series

$$u = \varepsilon \sum_{m=-\infty}^{\infty} \xi_m(\tau) h_m e^{imx} + \varepsilon^2 u_2(\tau, x) + \cdots , \tag{20}$$

where $\tau = \varepsilon t$. Substituting this into (1), (2) and (3), (2) and performing some standard operations we arrive at equations with respect to $\xi(\tau, x) = \sum_{m=-\infty}^{\infty} \xi_m(\tau) e^{imx}$:

a) in the case of the boundary value problem (1), (2),

$$\begin{gathered} a_1 \frac{\partial \xi}{\partial \tau} = -d_1 \frac{\partial^4 \xi}{\partial x^4} + (-d_1 a_2 + c_0 a_{21}) \frac{\partial^2 \xi}{\partial x^2} + c_0(a_1 a_{22} - a_2 a_{21}) \xi \\ + \beta_1 \beta_3 \left(\frac{\partial^2 \xi}{\partial x^2} + a_2 \xi \right)^2 + \beta_2 \beta_4 a_1^2 \xi^2 - (\alpha_1 \alpha_4 + \alpha_2 \alpha_3) a_1 \xi \left(\frac{\partial^2 \xi}{\partial x^2} + a_2 \xi \right), \\ \xi(\tau, x) = \xi(\tau, x + 2\pi); \end{gathered} \tag{21}$$

b) in the case of the boundary value problem (3), (2),

$$\begin{gathered} a_1 \frac{\partial \xi}{\partial \tau} = -d_1 \frac{\partial^4 \xi}{\partial x^4} + (-d_1 a_2 + c_0 a_{21}) \frac{\partial^2 \xi}{\partial x^2} + c_0(a_1 a_{22} - a_2 a_{21}) \xi \\ + \beta_1 \beta_3 \left(\frac{\partial^2 \xi}{\partial x^2} + a_2 \xi \right) \left(\frac{\partial^3 \xi}{\partial x^3} + a_2 \frac{\partial \xi}{\partial x} \right) + \beta_2 \beta_4 a_1^2 \xi \frac{\partial \xi}{\partial x} \\ - a_1 \beta_1 \beta_4 \left(\frac{\partial^2 \xi}{\partial x^2} + a_2 \xi \right) \frac{\partial \xi}{\partial x} - a_2 \beta_1 \beta_3 \xi \left(\frac{\partial^3 \xi}{\partial x^3} + a_2 \frac{\partial \xi}{\partial x} \right), \\ \xi(\tau, x) = \xi(\tau, x + 2\pi). \end{gathered} \tag{22}$$

Theorem 3. *Suppose that $\alpha \geq 1$ and that* (10) *and* (19) *hold. Suppose that the boundary value problem* (21) (*or* (22)) *has a bounded solution $\xi_0(\tau, x)$ as $\tau \to \infty$. Then the boundary value problem* (1), (2) (*or* (3), (2)) *has an asymptotic solution $u_0(t, x, \varepsilon)$, up to the residual, such that*

$$u_0(t, x, \varepsilon) = \varepsilon \begin{pmatrix} -\frac{\partial^2 \xi_0(\tau, x)}{\partial x^2} - a_2 \xi_0(\tau, x) \\ a_1 \xi_0(\tau, x) \end{pmatrix} + O(\varepsilon^2), \quad \tau = \varepsilon t.$$

1.4. Now suppose that together with conditions (10) and (19), the inequality

$$\frac{1}{2} < \alpha < 1$$

holds. The series

$$u = \varepsilon^{\gamma_1} \sum_{m=-\infty}^{\infty} \xi_m(\tau) h_m e^{imy} + \varepsilon^{\gamma_2} u_2(\tau, y) + \cdots$$

is analogous to the formal series (20) where $\tau = \varepsilon^{2\alpha - 1} t$, and differs from (20) only by the normalizing factors $\varepsilon^{\gamma_1}, \varepsilon^{\gamma_2}$ and by the fact that here the argument x is replaced by $y = (z\varepsilon^{(\alpha-1)/2} + \theta)x$. As above, $z \neq 0$ is arbitrarily fixed, while $\theta = \theta(z, \varepsilon) \in [0; 1)$ is such that $z\varepsilon^{(\alpha-1)/2} + \theta$ is an integer. Substituting this series into (1), (2) and into (3), (2), and choosing the values γ_1, γ_2 in the normalizing factors $\varepsilon^{\gamma_1}, \varepsilon^{\gamma_2}$ correspondingly, we arrive at an equation for determining a function $\xi(\tau, y) = \sum_{m=-\infty}^{\infty} \xi_m(\tau) e^{imy}$ that is 2π-periodic in the argument y:

a) for the boundary value problem (1), (2) we have $\gamma_1 = 1$, $\gamma_2 = 2\alpha$ and

$$a_1 \frac{\partial \xi}{\partial \tau} = -z^4 d_1 \frac{\partial^4 \xi}{\partial y^4} - c a_{21} z^2 \frac{\partial^2 \xi}{\partial y^2} + \beta_1 \beta_3 z^4 \left(\frac{\partial^2 \xi}{\partial y^2} \right)^2, \quad \xi(\tau, x) = \xi(\tau, x + 2\pi); \tag{23}$$

b) for the boundary value problem (3), (2) we have $\gamma_1 = \frac{1}{2}(3-\alpha)$, $\gamma_2 = \frac{1}{2}(3\alpha+1)$, and

$$a_1\frac{\partial\xi}{\partial\tau} = -z^4 d_1\frac{\partial^4\xi}{\partial y^4} - ca_{21}z^2\frac{\partial^2\xi}{\partial y^2} + \beta_1\beta_3 z^5\frac{\partial^2\xi}{\partial y^2}\frac{\partial^3\xi}{\partial y^3}, \quad \xi(\tau,x) = \xi(\tau,x+2\pi). \tag{24}$$

As above, every solution of (23) and (24) determines the principal part of asymptotic solutions, up to the residual, of the original boundary value problems by the formula

$$u(t,x,\varepsilon) = \varepsilon^{\gamma_1}\begin{pmatrix} c - z^2\frac{\partial^2\xi(\tau,y)}{\partial y^2} - a_2\xi(\tau,y) \\ a_1\xi(\tau,y)\end{pmatrix} + O(\varepsilon^{\gamma_2}),$$
$$\tau = \varepsilon^{2\alpha-1}t, \quad y = (z\varepsilon^{(\alpha-1)/2} + \theta)x.$$

1.5. **Normalization in quasilinear parabolic systems.** We consider a quasilinear parabolic boundary value problem

$$\frac{\partial u}{\partial t} = (D_0 + \varepsilon D_1)\frac{\partial^2 u}{\partial x^2} + A_0 u + \varepsilon\Phi(u), \quad u(t,x+2\pi) = u(t,x), \tag{25}$$

where $\Phi(u)$ is some sufficiently smooth nonlinear vector function. Suppose that conditions (10), (11) hold. An analogue of the basic formula (16) for (25) has the form

$$u = \xi(\tau,x) + \varepsilon u_2(\tau,x) + \cdots, \quad \tau = \varepsilon t.$$

Substituting this expression into (25) and performing standard operations, we arrive at the boundary value problem

$$a_4\frac{\partial\xi}{\partial\tau} = -d_1\frac{\partial^4\xi}{\partial x^4} - d_1 a_2\frac{\partial^2\xi}{\partial x^2} + f_1(\xi) - \frac{\partial^2}{\partial x^2}(f_2(\xi)), \quad \xi(\tau,x+2\pi) = \xi(\tau,x) \tag{26}$$

to determine $\xi(\tau,x)$. Here,

$$f_1 = \left(\begin{pmatrix} a_4 \\ -a_2\end{pmatrix}, \Phi(e_1\xi)\right), \quad f_2 = \left(\begin{pmatrix} 0 \\ 1\end{pmatrix}, \Phi(e_1\xi)\right).$$

Under conditions (10), (19), an analogue of (20) is given by the expression

$$u = \begin{pmatrix} \frac{\partial^2\xi}{\partial x^2} + a_2\xi \\ -a_1\xi\end{pmatrix} + \varepsilon u_2(\tau,x) + \cdots, \quad \tau = \varepsilon t,$$

and the corresponding quasinormal form is

$$a_1\frac{\partial\xi}{\partial\tau} = d_1\frac{\partial^4\xi}{\partial x^4} + (d_1a_2 + d_2a_1)\frac{\partial^2\xi}{\partial x^2} + \left(\Phi\left(\begin{pmatrix} \frac{\partial^2\xi}{\partial\tau^2} + a_2\xi \\ -a_1\xi\end{pmatrix}\right), \begin{pmatrix} 0 \\ 1\end{pmatrix}\right),$$
$$\xi(\tau,x+2\pi) = \xi(\tau,x). \tag{27}$$

Remark. Consider the boundary value problem

$$\frac{\partial u}{\partial t} = (D_0 + \varepsilon D_1)\frac{\partial^2 u}{\partial x^2} + A_0 u + \varepsilon^\alpha\Phi(u), \quad u(t,x+2\pi) = u(t,x),$$

where $1/2 < \alpha < 1$. When conditions (10) and (11) hold ($a_4 \neq 0$), using the constructions of (16) we arrive at the equation

$$a_4\frac{\partial\xi}{\partial\tau} = -z^4 d_1\frac{\partial^4\xi}{\partial y^4} - z^2\frac{\partial^2}{\partial y^2}(f_2(\xi)), \quad \xi(\tau,x+2\pi) = \xi(\tau,x), \tag{28}$$

where

$$\tau = \varepsilon^{2\alpha-1}t, \quad y = (z^{(1-\alpha)/2} + \theta)x, \quad u = \xi(\tau,y) + \varepsilon u_2(\tau,y) + \cdots.$$

Thus we obtain a generalization of the result of § 1.2.

If, however, conditions (10) and (19) hold ($a_1 \neq 0$), then the situation is much more difficult. To investigate an analogue of the quasinormal form (28), we need more complete information on the nonlinear vector function $\Phi(u)$, namely its asymptotic behaviour as one or another component of its argument increases unboundedly.

§ 2. Quasinormal forms describing the interaction of several mode groups

Here we assume that the condition $1/2 < \alpha < 1$ holds. All the quasinormal forms constructed in § 1 under this condition describe the behaviour of solutions that are formed for every fixed z at the modes numbered

$$n(z,\varepsilon) = (z\varepsilon^{(\alpha-1)/2} + \theta)m \quad (m = 0, \pm1, \pm2, \dots). \tag{29}$$

We consider the question of constructing quasinormal forms describing the interaction of several mode groups of the form (29) for various values of z. We introduce some notation. We fix an arbitrary number $k > 1$, and let $z_1, \dots, z_k$ be arbitrary fixed real nonzero parameters. Let

$$\theta_j = \theta_j(z_j, \varepsilon) \quad (j = 1, \dots, k)$$

denote the values in the half-open interval $[0, 1)$ that complement the expressions $z_j\varepsilon^{(\alpha-1)/2}$ to integers, respectively. We then set

$$y_j = (z_j\varepsilon^{(\alpha-1)/2} + \theta_j)x \quad (j = 1, 2, \dots, k)$$

and look at the formal series

$$u = \varepsilon^{\gamma_1} \sum_{m_1,\dots,m_k=-\infty}^{\infty} \xi_{m_1,\dots,m_k}(\tau) g_{m_1,\dots,m_k} e^{i(m_1y_1+\dots+m_ky_k)} + \varepsilon^{\gamma_2} u_2(\tau, y_1, \dots, y_k) + \cdots, \tag{30}$$

where $g_{m_1,\dots,m_k} = e_1$ if condition (11) holds, and

$$g_{m_1,\dots,m_k} = ((m_1 + \dots + m_k)^2 - a_2, a_1)$$

under condition (19). Expression (30) shows that we consider solutions of the original boundary value problems formed at the modes with numbers

$$(z_j\varepsilon^{(\alpha-1)/2} + \theta_j)m \quad (m = 0, \pm1, \pm2, \dots;\ j = 1, \dots, k).$$

Substituting (30) into (1), (2) and into (3), (2) and performing standard operations we arrive at analogues of equations (17), (18), (23), and (24) for the variable

$$\xi(\tau, y_1, \dots, y_k) = \sum_{m_1,\dots,m_k=-\infty}^{\infty} \xi_{m_1,\dots,m_k}(\tau) e^{i(m_1y_1+\dots+m_ky_k)}.$$

The only difference is that the operator $z\frac{\partial}{\partial y}$ in the above equations is replaced by the operator $z_1\frac{\partial}{\partial y_1} + \dots + z_k\frac{\partial}{\partial y_k}$. We present for example the corresponding analogue of equation (24):

$$a_1\frac{\partial \xi}{\partial \tau} = -d_1\left(z_1\frac{\partial}{\partial y_1} + \dots + z_k\frac{\partial}{\partial y_k}\right)^4\xi - ca_{21}\left(z_1\frac{\partial}{\partial y_1} + \dots + z_k\frac{\partial}{\partial y_k}\right)^2\xi + \beta_1\beta_3\left(\left(z_1\frac{\partial}{\partial y_1} + \dots + z_k\frac{\partial}{\partial y_k}\right)^2\xi\right)\left(\left(z_1\frac{\partial}{\partial y_1} + \dots + z_k\frac{\partial}{\partial y_k}\right)^3\xi\right)$$

with the boundary conditions

$$\xi(\tau, y_1, y_2, \dots, y_k) = \xi(\tau, y_1 + 2\pi, y_2, \dots, y_k) = \dots = \xi(\tau, y_1, \dots, y_{k-1}, y_k + 2\pi)$$

that are 2π-periodic in every space variable $y_1, y_2, \dots, y_k$.

A solution $\xi(\tau, y_1, y_2, \dots, y_k)$ of this problem determines the principal part of an asymptotic solution of (3), (2), up to the residual, by the formula

$$u(\tau, x, \varepsilon) = \varepsilon^{\gamma_1} \begin{pmatrix} -\left(z_1 \frac{\partial}{\partial y_1} + \dots + z_k \frac{\partial}{\partial y_k}\right)^2 \xi(\tau, y_1, \dots, y_k) - a_2 \xi(\tau, y_1, \dots, y_k) \\ a_1 \xi(\tau, y_1, \dots, y_k) \end{pmatrix} + O(\varepsilon^{\gamma_2}),$$
$$\tau = \varepsilon^{2\alpha-1} t, \quad y_j = (z_j \varepsilon^{(2\alpha-1)/2} + \theta_j) x.$$

§ 3. Quasinormal forms under the conditions $a_3 > 0,\ a_0 = a_1 + a_4 = 0,\ \Delta = 0$

When

$$a_3 > 0, \quad a_0 = a_1 + a_4 = 0, \quad \Delta = 0 \tag{31}$$

the characteristic equation (6) takes the form

$$\lambda^2 + \varepsilon d_2 n^2 \lambda + a_3 n^2 + \varepsilon n^2 [-d_1 n^2 + d_1 a_2 - d_2 a_1] = 0, \quad n = 0, \pm 1, \pm 2, \dots.$$

This equation has the roots

$$\lambda_n = i\sqrt{a_3} n + \varepsilon \lambda_{n1} + O(\varepsilon^2),$$

where

$$\lambda_{n1} = \frac{1}{2}\left[-\frac{1}{2} n^2 d_2 + i(2\sqrt{a_3})^{-1} n(-d_1 n^2 + d_1 a_2 - d_2 a_1)\right].$$

An eigenvector $h_n = (i\sqrt{a_3} n + a_1, a_3)$ corresponds to the eigenvalue $i\sqrt{a_3} n$ of the matrix $A_0 - n^2 D_0$. Let $g_n = (a_3, -i\sqrt{a_3} n - a_1)$ denote an eigenvector of the matrix $(A_0 - n^2 D_0)^*$ corresponding to the eigenvalue $-i\sqrt{a_3} n$. Note that the linear boundary value problem

$$\frac{\partial u}{\partial t} = D_0 \frac{\partial^2 u}{\partial x^2} + A_0 u, \quad u(t, x + 2\pi) \equiv u(t, x),$$

has the family of periodic solutions

$$h_n e^{in(\sqrt{a_3} t + x)} \quad (n = 0, \pm 1, \pm 2, \dots).$$

The matrix A_0 of this system of equations has a double root at zero. The boundary value problems (1), (2) and (3), (2) obviously have a two-dimensional invariant integral manifold of solutions which are independent of the space variable x. On this manifold this boundary value problem is written in the form of the system of two ordinary differential equations

$$\dot{v} = (A_0 + \varepsilon A_1) v + F(v, v).$$

The local dynamics of such systems is fairly complicated. It was studied in detail in [12]. Thus, in the case under consideration infinitely many roots of the characteristic equation (6) tend to the imaginary axis as $\varepsilon \to 0$. Thus an infinite-dimensional critical case is realized in the problem of the stability of the zero equilibrium state of the boundary value problems (1), (2) and (3), (2). To study the local dynamics of these boundary value problems for small ε we apply the same scheme of analysis as in § 1.

3.1. Suppose that $\alpha \geq 1$. We introduce the formal series

$$u = \varepsilon\Bigl[\eta(t) + \sum_{m=-\infty,\, m\neq 0}^{\infty} (\xi_m(t)) h_m e^{im(\sqrt{a_3}t+x)}\Bigr] + \varepsilon^2 u_2(t, \sqrt{a_3}t + x) + \cdots . \tag{32}$$

Here, $\eta = (\eta_1(t), \eta_2(t)) \in \mathbb{R}^2$. We set

$$s = \sqrt{a_3}t + x \quad \text{and} \quad \xi(t,s) = \sum_{m=-\infty,\, m\neq 0}^{\infty} \xi_m(t) e^{ims}.$$

We note that the first summand is equivalent to the expression

$$\varepsilon \begin{pmatrix} \eta_1(t) + \sqrt{a_3}\frac{\partial \xi}{\partial s} + a_1\xi \\ \eta_2(t) + a_3\xi \end{pmatrix}. \tag{33}$$

We substitute (32) into (1), (2) and into (3), (2) (taking account of (33)), and by performing standard operations we arrive at a system of equations for determining $\eta(t)$ and a function $\xi(t,s)$ that is 2π-periodic in s:

a) in the case of the boundary value problem (1), (2) we have

$$\begin{aligned} 2\sqrt{a_3^3}\,\frac{\partial^2 \xi}{\partial t \partial s} &= d_1 a_3 \frac{\partial^4 \xi}{\partial s^4} + \sqrt{a_3^3}\, d_2 \frac{\partial^3 \xi}{\partial s^3} + a_3(d_2 a_1 - d_1 a_2 + c_0 a_{21})\frac{\partial^2 \xi}{\partial s^2} \\ &\quad - c_0\sqrt{a_3^3}\, a_{22}\frac{\partial \xi}{\partial s} - c_0 a_1(a_1 a_{21} + a_3 a_{22})\xi \\ &\quad + \varepsilon\bigl[a_3 P(\xi,\eta,\bar\alpha) - a_1 P(\xi,\eta,\bar\beta) + \sqrt{a_3^3}\,\frac{\partial}{\partial s}P(\xi,\eta,\bar\beta) \\ &\qquad - M\bigl(a_3 P(\xi,\eta,\bar\alpha) - a_1 P(\xi,\eta,\bar\beta)\bigr)\bigr], \end{aligned} \tag{34}$$

$$\xi(t, s+2\pi) \equiv \xi(t,s), \quad M(\xi) = 0, \tag{35}$$

$$\frac{\partial \eta}{\partial t} = (A_0 + c\varepsilon^{\alpha} A_1)\eta + \varepsilon\left(F(\eta,\eta) + f_1 M\left(\left(\frac{\partial \xi}{\partial s}\right)^2\right) + f_2 M(\xi^2)\right). \tag{36}$$

Here we are using the notation

$$M(\varphi(x)) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x)\,dx,$$

$$P(\xi,\eta,\nu) = p_1(\nu)\left(\frac{\partial \xi}{\partial s}\right)^2 + p_2(\nu)\xi^2 + p_3(\nu)\xi\frac{\partial \xi}{\partial s} + p_4(\nu)\eta_1\xi + p_6(\nu)\eta_1\frac{\partial \xi}{\partial s} + p_7(\nu)\eta_2\frac{\partial \xi}{\partial s},$$

$$\begin{aligned} p_1(\nu) &= a_3\nu_1\nu_2, \\ p_2(\nu) &= a_1^2\nu_1\nu_3 + a_1 a_3 p_0(\nu) + a_3^2\nu_2\nu_4, \\ p_3(\nu) &= 2a_1\sqrt{a_3}\,\nu_1\nu_3 + p_0(\nu)\sqrt{a_3^3}, \\ p_0(\nu) &= \nu_1\nu_4 + \nu_2\nu_3, \\ p_4(\nu) &= \nu_1(\nu_3 a_1 + \nu_4 a_3) + \nu_3(\nu_1 a_1 + \nu_2 a_3), \\ p_5(\nu) &= 2\sqrt{a_3}\,\nu_1\nu_3, \\ p_6(\nu) &= \nu_2(\nu_3 a_1 + \nu_4 a_3) + \nu_4(\nu_1 a_1 + \nu_2 a_3), \\ p_7(\nu) &= \sqrt{a_3}\, p_0(\nu), \end{aligned}$$

$$f_1 = a_3\begin{pmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{pmatrix}, \quad f_2 = (a_1 + a_3)\begin{pmatrix} \alpha_1\alpha_3 a_1 + \alpha_1\alpha_4 a_3 \\ \beta_1\beta_3 a_1 + \beta_1\beta_4 a_3 \end{pmatrix},$$

where $\bar\alpha$ and $\bar\beta$ denote the tuples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$, respectively;

b) in the case of the boundary value problem (3), (2) we have

$$\begin{aligned}(37)\qquad 2\sqrt{a_3^3}\,\frac{\partial^2\xi}{\partial t\partial s} &= d_1a_3\frac{\partial^4\xi}{\partial s^4} + \sqrt{a_3^3}\,d_2\frac{\partial^3\xi}{\partial s^3} + a_3(d_2a_1 - d_1a_2 + c_0a_{21})\frac{\partial^2\xi}{\partial s^2}\\ &\quad - c_0\sqrt{a_3^3}\,a_{22}\frac{\partial\xi}{\partial s} - c_0a_1(a_1a_{21} + a_3a_{22})\xi\\ &\quad + \varepsilon\big[a_3Q(\xi,\eta,\bar{\alpha}) - a_1Q(\xi,\eta,\bar{\beta}) + \sqrt{a_3}\,\frac{\partial}{\partial s}Q(\xi,\eta,\bar{\beta})\\ &\qquad - M\big(a_3Q(\xi,\eta,\bar{\alpha}) - a_1Q(\xi,\eta,\bar{\beta})\big)\big],\end{aligned}$$

$$(38)\qquad \xi(t,s+2\pi)\equiv\xi(t,s),\quad M(\xi)=0,$$

$$(39)\qquad \frac{\partial\eta}{\partial t} = (A_0 + c\varepsilon^{\alpha}A_1)\eta + \varepsilon gM\left(\left(\frac{\partial\xi}{\partial s}\right)^2\right),$$

where we have used the notation

$$\begin{aligned}Q(\xi,\eta,\nu) &= q_1(\nu)\frac{\partial^2\xi}{\partial s^2}\frac{\partial\xi}{\partial s} + q_2(\nu)\frac{\partial^2\xi}{\partial s^2}\xi + q_3(\nu)\left(\frac{\partial\xi}{\partial s}\right)^2 + q_4(\nu)\frac{\partial\xi}{\partial s}\xi + q_5(\nu)\eta_1\frac{\partial^2\xi}{\partial s^2}\\ &\quad + q_6(\nu)\eta_2\frac{\partial^2\xi}{\partial s^2} + q_7(\nu)\eta_1\frac{\partial\xi}{\partial s} + q_8(\nu)\eta_2\frac{\partial\xi}{\partial s},\end{aligned}$$

$$\begin{aligned}q_1(\nu) &= a_3\nu_1\nu_3,\\ q_2(\nu) &= \sqrt{a_3}\,\nu_3(\nu_1a_1 + \nu_2a_3),\\ q_3(\nu) &= \sqrt{a_3}\,\nu_1(\nu_3a_1 + \nu_4a_3),\\ q_4(\nu) &= (\nu_1a_1 + \nu_2a_3)(\nu_3a_1 + \nu_4a_3),\\ q_5(\nu) &= \sqrt{a_3}\,\nu_1\nu_3,\\ q_6(\nu) &= \sqrt{a_3}\,\nu_2\nu_3,\\ q_7(\nu) &= \nu_1(\nu_3a_1 + \nu_4a_3),\\ g &= \sqrt{a_3^3}\begin{pmatrix}\alpha_1\alpha_4 - \alpha_2\alpha_3\\ \beta_1\beta_4 - \beta_2\beta_3\end{pmatrix}.\end{aligned}$$

Theorem 4. *Suppose that* $\alpha \geq 1$, *that conditions* (31) *hold, and let* $(\eta_0(t), \xi_0(t,s))$ *be a bounded solution of the boundary value problem* (35)–(37) (*or* (38)–(40)), *as* $\tau \to \infty$. *Then the boundary value problem* (1), (2) (*or* (3), (2)) *has an asymptotic solution* $u_0(t,x,\varepsilon)$, *up to the residual, such that*

$$u_0(t,x,\varepsilon) = \varepsilon\eta_0(t) + \varepsilon\begin{pmatrix}a_1\xi_0(t,\sqrt{a_3}t + x) + \sqrt{a_3}\frac{\partial}{\partial s}\xi_0(t,\sqrt{a_3}t + x)\\ a_3\xi_0(t,\sqrt{a_3}t + x)\end{pmatrix} + O(\varepsilon^2).$$

3.2. Suppose that $1/2 < \alpha < 1$ and the nondegeneracy condition $a_2 \neq 0$ holds. Here we will confine ourselves to giving only the final result. In the case of the boundary value problem (1), (2), the quasinormal form is the boundary value problem

$$\sqrt{a_3^3}\,z\frac{\partial^2\xi}{\partial\tau\partial s}=d_1a_3z^4\frac{\partial^4\xi}{\partial s^4}+ca_3a_{21}z^2\frac{\partial^2\xi}{\partial s^2}+\sqrt{a_3}\,\beta_1\beta_3z^3\frac{\partial}{\partial s}\left(\frac{\partial\xi}{\partial s}\right)^2 + a_3z^2(2\beta_1\beta_3\eta_1+p_0(\beta)\eta_2)\frac{\partial^2\xi}{\partial s^2}, \tag{40}$$

$$\xi(\tau,s+2\pi)\equiv\xi(\tau,s),\quad M(\xi)=0, \tag{41}$$

$$\varepsilon^{\gamma_0}\frac{d\eta}{d\tau}=(A_0+c\varepsilon^\alpha A_1)\eta+\varepsilon^\alpha\left[F(\eta,\eta)+d_1M\left(\left(\frac{\partial\xi}{\partial s}\right)^2\right)\right]. \tag{42}$$

Here,

$$\tau=\varepsilon^{\gamma_0}t,\quad s=(z\varepsilon^{(\alpha-1)/2}+\theta)(\sqrt{a_3}t+x),$$

the number $z\neq 0$ is fixed arbitrarily, and $\theta=\theta(z,\varepsilon)\in[0,1)$ complements the expression $z\varepsilon^{(\alpha-1)/2}$ to an integer. A connection between the solutions of the problem (40)–(42) and of the original system (1), (2) is established by the formula

$$u(t,x,\varepsilon)=\varepsilon^\alpha\eta(\varepsilon t)+\varepsilon^{\gamma_1}\begin{pmatrix}a_3z\frac{\partial\xi(\varepsilon t,s)}{\partial s}+a_1\xi(\varepsilon t,s)\\ a_3\xi(\varepsilon t,s)\end{pmatrix}+O(\varepsilon^{\gamma_2}), \tag{43}$$

and γ_0, γ_1, and γ_2 satisfy the equations

$$\gamma_0=\frac{1}{2}(3\alpha-1),\quad \gamma_1=\frac{1}{2}(\alpha+1),\quad \gamma_2=\frac{1}{2}(3\alpha-1).$$

In the case of the boundary value problem (3), (2), the quasinormal form is the boundary value problem

$$2\sqrt{a_3^3}\,z\frac{\partial^2\xi}{\partial\tau\partial s}=d_1a_3z^4\frac{\partial^4\xi}{\partial s^4}+ca_3a_{21}z^2\frac{\partial^2\xi}{\partial s^2}+\sqrt{a_3}\,\beta_1\beta_3z^4\frac{\partial}{\partial s}\left(\frac{\partial\xi}{\partial s}\left(\frac{\partial^2\xi}{\partial s^2}\right)\right) + \sqrt{a_3}\,[q_5(\beta)\eta_1+q_6(\beta)\eta_2]\frac{\partial^3\xi}{\partial s^3}, \tag{44}$$

$$\xi(\tau,s+2\pi)\equiv\xi(\tau,s),\quad M(\xi)=0, \tag{45}$$

$$\varepsilon^{\gamma_0}\frac{d\eta}{d\tau}=A_0\eta+\varepsilon^\alpha\left[cA_1\eta+F(\eta,\eta)+gM\left(\left(\frac{\partial\xi}{\partial s}\right)\right)^2\right]. \tag{46}$$

The parameters γ_0, γ_1, and γ_2 in formula (43) satisfy the equations

$$\gamma_0=\frac{1}{2}(3\alpha+1),\quad \gamma_1=1,\quad \gamma_2=2\alpha.$$

3.3. In § 3, under the condition $1/2<\alpha<1$, we have looked at the interaction of mode (solution) groups with the modes

$$(z_j\varepsilon^{(\alpha-1)/2}+\theta_j)m\quad (j=1,\dots,k;\ m=0,\pm1,\pm2,\dots).$$

In connection with the situation considered in this section, on the same lines as in § 2 we arrive at boundary value problems which differ from (40), (42), (44), (46), respectively, only by the fact that the operator $z\frac{\partial}{\partial s}$ in them is replaced by the more complicated operator

$$z_1\frac{\partial}{\partial s_1}+z_2\frac{\partial}{\partial s_2}+\cdots+z_k\frac{\partial}{\partial s_k},$$

and for s_j we have the formula

$$s_j=(z_j\varepsilon^{(\alpha-1)/2}+\theta_j)(\sqrt{a_3}t+x).$$

Conclusions

1. The quasinormal forms (QNFs) constructed above are fairly complicated. It may seem that it is much more effective to use the original boundary value problems to do the analysis, for example, for numerical analysis. But this is not the case. As we have shown above, the principal terms of solutions which are asymptotic up to the residual are expressed precisely in terms of solutions of QNFs.

Next, by contrast with the original equations, QNFs do not contain large and small parameters, thus ensuring singular dependence of solutions on them. For example, solutions of the boundary value problems (1), (2) and (3), (2) containing components oscillating rapidly with respect to both the space and the time variable are reconstructed from solutions of QNFs. Numerical analysis of (1), (2) and (3), (2) in these situations would be very difficult.

2. Solutions of QNFs may have attractors with a fairly complicated structure. Furthermore, the phenomenon of multi-stability turns out to be typical in many cases. Indeed, many of the QNFs constructed above (for $1/2 < \alpha < 1$) contain an arbitrary real parameter z or, as shown in § 2, whole groups of sets of arbitrary parameters. To every fixed parameter (set of parameters) there may correspond its own attractor in QNF. This means that a fairly rich set of attractors may co-exist simultaneously in the original boundary value problems. Of course, it should be pointed out that above we only discussed constructing asymptotic solutions of (1), (2) and of (3), (2), up to the residual. Apparently, for many situations there exist exact solutions that are asymptotically close to the ones constructed above. In some situations this can be rigorously substantiated and more exact asymptotic expansions can be constructed. It is especially interesting to note that as the so-called supercriticality coefficient, that is, the coefficient $c\varepsilon^{\alpha}$, increases, a sharp increase in the number of stationary regimes can be observed.

3. By comparing the characteristics of asymptotic solutions, up to the residual, of the boundary value problems (1), (2) and (3), (2), we arrive at the conclusion that when they are supercritical in ε the amplitudes of the oscillations are approximately the same, but for $1/2 < \alpha < 1$ the amplitudes in the boundary value problem (3), (2) are substantially smaller (with respect to the order of magnitude) than the corresponding amplitudes of stationary regimes in (1), (2).

4. The method developed above also admits an extension to more complicated systems of equations and to equations with other boundary conditions (for example, with boundary conditions of Neumann or Dirichlet type). Effects which are essentially new can appear [13] when we consider systems of equations in a two-dimensional spatial domain of definition (replacing the operator $D_0\frac{\partial^2}{\partial x^2}$ by the operator $D_{01}\frac{\partial^2}{\partial x_1^2} + D_{02}\frac{\partial^2}{\partial x_2^2}$; $(x_1, x_2) \in \Omega \subset \mathbb{R}^2$) and with the corresponding boundary conditions.

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