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NECESSARY AND SUFFICIENT CONDITIONS FOR THE TOPOLOGICAL CONJUGACY OF 3-DIFFEOMORPHISMS WITH HETEROCLINIC TANGENCIES

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Abstract. In this paper we consider a class of three-dimensional diffeomorphisms that differ from gradient-like systems through the presence of heteroclinic tangencies. It is well known that such cascades are not structurally stable. However, here we find a complete system of topological invariants for a certain meaningful class of such diffeomorphisms.

§ 1. Introduction and statement of the results

Gradient-like flows are a classical object in regular dynamics. These are just the structurally stable flows generated by a vector field of the gradient of some Morse function (see, for example, [21]). On surfaces, such simple dynamics implies that the topological equivalence class[1] of a gradient-like flow is completely determined by the mutual position of the saddle separatrices. This fact was established in the classical papers by Andronov and Pontryagin [1] and Leontovich and Maĭer [10] for flows in a bounded part of the plane and was generalized to arbitrary surfaces by Peixoto [20], who expressed the information about the behaviour of separatrices using a distinguishing graph and proved that for a gradient-like flow the isomorphism class of its distinguishing graph is a complete topological invariant.

If the gradient flow generated on a surface by a Morse function is not structurally stable, then it has a pair of saddle equilibrium positions where the stable separatrix of one coincides with the unstable separatrix of the other. Such a violation of the condition of transversality of the intersection of the invariant manifolds of fixed points implies that in any C^1-neighbourhood of such a flow there exists a continuum of pairwise nonconjugate flows; this was first discovered by Palis [18]. A complete topological invariant of a nonrough gradient flow on a surface is in principle no longer combinatorial (like the distinguishing graph in the rough case) but necessarily has an analytic constituent describing a neighbourhood of the flow.

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[1]Flows f^t and g^t defined on an n-manifold M^n are said to be *topologically equivalent* if there exists a homeomorphism $h\colon M^n \to M^n$ taking the trajectories of the flow f^t to trajectories of the flow g^t preserving the orientation of motion along the trajectories.

Morse–Smale diffeomorphisms (structurally stable diffeomorphisms with a finite nonwandering set) without heteroclinic points[2] give a discrete analogue of gradient-like flows. If a Morse–Smale diffeomorphism is defined on a surface and the invariant manifolds of its different saddle points are disjoint, then, as in the continuous case, its topological conjugacy class is determined by a graph similar to the Peixoto graph. This was proved by Bezdenezhnykh and Grines in [2] and [3]. Palis [18] also observed that the tangency of invariant manifolds of saddle points of a cascade along at least one orbit implies that the system is nonrough and, moreover, that a continuum of topologically nonconjugate diffeomorphisms exists in any C^1-neighbourhood of the system (*modulus of topological conjugacy*). If, in some neighbourhood of the diffeomorphism, it is possible to describe the set of equivalence classes by using finitely many parameters, then the *diffeomorphism is said to have finitely many moduli of topological conjugacy.* The term "modulus of topological conjugacy" was used in the papers of Shil'nikov and Gonchenko and corresponds to the term "modulus of stability", which is used in the Western literature.

Palis's paper led to a number of papers (see, for example, [11, 12, 13]) in which the structure of a neighbourhood of such a diffeomorphism was studied. In particular, in [13] necessary and sufficient conditions were found for a diffeomorphism of an orientable surface to have finitely many moduli of topological conjugacy describing all the topological conjugacy classes that belong to some neighbourhood of this diffeomorphism, and in [12] diffeomorphisms of n-dimensional manifolds with one orbit of one-sided heteroclinic tangency were considered and a classification of diffeomorphisms in a neighbourhood was given. Furthermore, a necessary condition for topological conjugacy of diffeomorphisms of n-dimensional manifolds containing one orbit of one-sided heteroclinic tangency was proved in [17]. In [9] a necessary condition was proved for the topological conjugacy of diffeomorphisms defined on manifolds of dimension 3 which have finitely many orbits of heteroclinic tangency of two-dimensional invariant manifolds.

Mitryakova and Pochinka [15, 16] obtained a complete topological classification of diffeomorphisms on a surface that have a finite hyperbolic nonwandering set and are such that the saddle invariant manifolds of any two of its saddle points intersect (possibly, are tangent) over finitely many orbits. In this case, a complete topological invariant is a scheme consisting of finitely many tori with a set of smooth closed curves, where a real number is assigned to the point of tangency of every pair (the modulus of topological conjugacy).

In this paper necessary and sufficient conditions for topological conjugacy are obtained for diffeomorphisms of class Ψ that are defined on smooth three-dimensional closed orientable manifolds M^3 and are such that any diffeomorphism $f \in \Psi$ has the following properties:

1) the nonwandering set Ω_f of the diffeomorphism f consists of a finite number of hyperbolic points;
2) for different saddle points $p, q \in \Omega_f$ the intersection $W^s_p \cap W^u_q$ is not empty only in the case where $\dim W^s_p = \dim W^u_q = 2$; in addition, it is transversal everywhere, except for, possibly, one orbit of nondegenerate one-sided tangency;[3]

[2]A diffeomorphism of a manifold is called a *Morse–Smale diffeomorphism* if its nonwandering set consists of finitely many hyperbolic periodic points whose invariant manifolds intersect transversally. If for different saddle periodic points p, q of a Morse–Smale diffeomorphism the intersection $W^s_p \cap W^u_q$ is nonempty, then it is an infinite set. Furthermore, if $\dim W^s_p + \dim W^u_q = n$, then every point belonging to $W^s_p \cap W^u_q$ is called a *heteroclinic point*, and if $\dim W^s_p + \dim W^u_q > n$, then every connected component of $W^s_p \cap W^u_q$ is called a *heteroclinic component.* A Morse–Smale diffeomorphism is said to be *gradient-like* if its nonwandering set does not contain heteroclinic points.

[3]Let N_1, N_2 be two-dimensional submanifolds of a manifold M^3. A point $x \in N_1 \cap N_2$ is called a point of *nondegenerate one-sided tangency* if there exists a chart (U_x, φ_x) of the manifold M^3, where

3) the saddle points of the diffeomorphism f have C^2-linearizing neighbourhoods (see Definition 2.1 below).

The phase portrait of a diffeomorphism in the class Ψ is shown in Figure 1. Note that when there are no heteroclinic points of tangency a diffeomorphism of class Ψ is gradient-like. A complete topological classification of gradient-like diffeomorphisms on 3-manifolds was obtained in [5] (see also Part 5 of the book [8]).

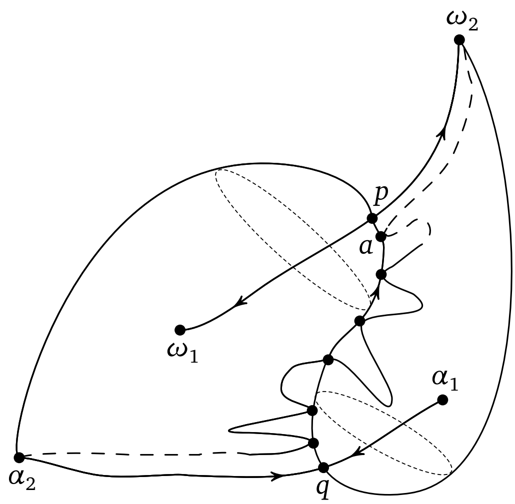

FIGURE 1. The phase portrait of a diffeomorphism in the class Ψ

In order to state our result, we introduce the following notation for a diffeomorphism $f \in \Psi$.

For $i \in \{0,1,2,3\}$, let Ω_i denote the subset of Ω_f consisting of the points p such that $\dim W^u_p = i$. We set $A_f = W^u_{\Omega_0 \cup \Omega_1}$, $R_f = W^s_{\Omega_2 \cup \Omega_3}$, $V_f = M^3 \setminus (A_f \cup R_f)$, and $\widehat{V}_f = V_f/f$. It follows from [7] that the sets A_f, R_f, V_f, and $\widehat{V}_f$ are connected, $\widehat{V}_f$ is a smooth closed 3-manifold, and the natural projection $p_f \colon V_f \to \widehat{V}_f$ is a covering which induces an epimorphism $\eta_f \colon \pi_1(\widehat{V}_f) \to \mathbb{Z}$ acting as follows. Let $\widehat{c}$ be a loop in $\widehat{V}_f$ such that $\widehat{c}(0) = \widehat{c}(1) = \widehat{x}$. By the monodromy theorem there exists a loop c in V_f starting at the point x (where $c(0) = x$) which is a lifting of the path $\widehat{c}$, and there exists an element $k \in \mathbb{Z}$ such that $c(1) = f^k(x)$. Then the map $\eta_f \colon \pi_1(\widehat{V}_f) \to \mathbb{Z}$ takes $[\widehat{c}]$ to k.

We set

$$\widehat{\mathbb{W}}^s_f = \bigcup_{p \in \Omega_1} \widehat{W}^s_p \quad \text{and} \quad \widehat{\mathbb{W}}^u_f = \bigcup_{p \in \Omega_2} \widehat{W}^u_p.$$

Every connected component $\widehat{W}^\delta_p$, $\delta \in \{s,u\}$, of the set $\widehat{\mathbb{W}}^\delta_f$ is an η_f-essential two-dimensional torus or an η_f-essential Klein bottle on the manifold $\widehat{V}_f$ in the following sense. Let $j \colon \widehat{W}^\delta_p \to \widehat{V}_f$ be an inclusion, and $j_* \colon \pi_1(\widehat{W}^\delta_p) \to \pi_1(\widehat{V}_f)$ the induced homomorphism; then

$$\eta_f\bigl(j_*(\pi_1(\widehat{W}^\delta_p))\bigr) \neq \{0\}.$$

Property 2 of the class Ψ implies that connected components $\widehat{W}^s_p \subset \widehat{\mathbb{W}}^s_f$ and $\widehat{W}^u_p \subset \widehat{\mathbb{W}}^u_f$ either are disjoint, or intersect transversally, or intersect nontransversally, where the

$U_x \subset M^3$ is an open neighbourhood of the point x and $\varphi_x \colon U_x \to \mathbb{R}^3$ is a C^2-diffeomorphism such that $\varphi_x(x) = (0,0,0)$, $\varphi_x(N_1 \cap U_x) = \{(x,y,z) \in \mathbb{R}^3 \colon z = 0\}$, $\varphi_x(N_2 \cap U_x) = \{(x,y,z) \in \mathbb{R}^3 \colon z = x^2 + y^2\}$.

condition of transversality at the intersection is violated at exactly one point, which is a point of nondegenerate one-sided tangency.

Let $\mathscr{A}$ denote the set of heteroclinic tangency points. For any point $a \in \mathscr{A}$ let σ_a^s and σ_a^u denote the saddle points such that a belongs to the intersection of the invariant manifolds $W^s_{\sigma_a^s}$ and $W^u_{\sigma_a^u}$. Let μ_a (λ_a) denote the eigenvalue of the point σ_a^s (σ_a^u) whose absolute value is greater than (less than) 1. We set $\widehat{\mathscr{A}} = p_f(\mathscr{A})$. For $\widehat{a} \in \widehat{\mathscr{A}}$ we set $\Theta_{\widehat{a}} = \dfrac{\ln|\mu_a|}{\ln|\lambda_a|}$. Note that $\Theta_{\widehat{a}}$ is independent of the choice of the point in the set $p_f^{-1}(\widehat{a})$. We set $\widehat{C}_f = \{\Theta_{\widehat{a}}, \widehat{a} \in \widehat{\mathscr{A}}\}$.

Definition 1.1. The tuple $S_f = (\widehat{V}_f, \eta_f, \widehat{\mathbb{W}}^s_f, \widehat{\mathbb{W}}^u_f, \widehat{C}_f)$ is called the scheme of a diffeomorphism $f \in \Psi$.

Definition 1.2. The schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in \Psi$ are said to be equivalent if there exists a homeomorphism $\widehat{\varphi}\colon \widehat{V}_f \to \widehat{V}_{f'}$ with the following properties:

1) $\eta_f = \eta_{f'}\widehat{\varphi}_*$;
2) $\widehat{\varphi}(\widehat{\mathbb{W}}^s_f) = \widehat{\mathbb{W}}^s_{f'}$ and $\widehat{\varphi}(\widehat{\mathbb{W}}^u_f) = \widehat{\mathbb{W}}^u_{f'}$;
3) $\Theta_{\widehat{a}} = \Theta_{\widehat{\varphi}(\widehat{a})}$ for $\Theta_{\widehat{a}} \in \widehat{C}_f$.

Figure 2 depicts a three-dimensional annulus with touching cylinders embedded in it. Identifying the boundary spheres of the annulus results in a manifold $\mathbb{S}^2 \times \mathbb{S}^1$. Gluing together the boundary circles of the cylinders produces touching tori. This shows the geometric constituent of the scheme of the diffeomorphism $f \in \Psi$, whose phase portrait is depicted in Figure 1. In this case the manifold $\widehat{V}_f$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, $\widehat{\mathbb{W}}^s_f = \widehat{W}^s_p$ and $\widehat{\mathbb{W}}^u_f = \widehat{W}^u_q$.

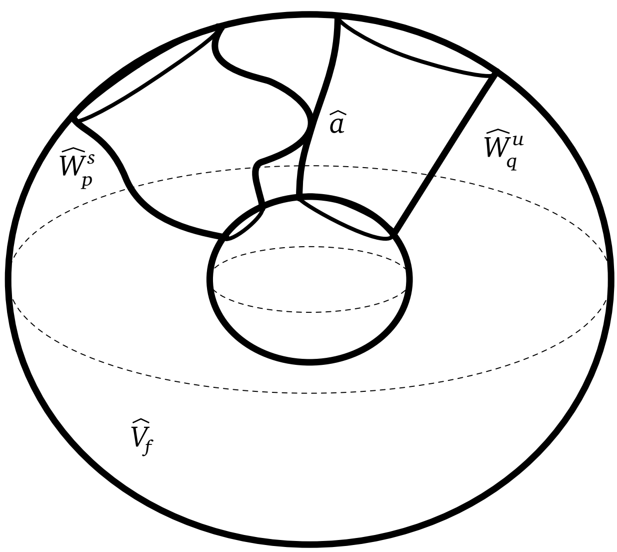

FIGURE 2. The geometric constituent of the scheme of the diffeomorphism $f \in \Psi$ whose phase portrait is depicted in Figure 1

The main result of this paper is the following theorem.

Theorem. *Diffeomorphisms $f, f' \in \Psi$ are topologically conjugate if and only if the schemes S_f and $S_{f'}$ are equivalent.*

§ 2. Linearizing neighbourhood

Suppose that a diffeomorphism f belongs to the class Ψ and σ is one of its saddle points, which has period m_σ and a two-dimensional stable manifold. Let $J_\sigma\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear diffeomorphism defined by the Jordan form of the linear part of the diffeomorphism f^{m_σ} in a neighbourhood of the point σ. The point $O(0,0,0)$ is a saddle point of the diffeomorphism J_σ and has a J_σ-invariant neighbourhood $\mathscr{U}_{J_\sigma}$. Furthermore, $J_\sigma\colon \mathbb{R}^3 \to \mathbb{R}^3$ and the J_σ-invariant neighbourhood $\mathscr{U}_{J_\sigma}$ of the saddle point $O(0,0,0)$ of J_σ have one of the following three forms.

1) $J_\sigma(x_1,x_2,x_3) = (\lambda_1 x_1, \lambda_2 x_2, \mu x_3)$, where $0<|\lambda_1|,|\lambda_2|<1$ and $|\mu|>1$;

$$\mathscr{U}_{J_\sigma} = \Bigl\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon \bigl(|x_1|\cdot|x_3|^{-\log_{|\mu|}|\lambda_1|}\bigr)^2 + \bigl(|x_2|\cdot|x_3|^{-\log_{|\mu|}|\lambda_2|}\bigr)^2 < 1\Bigr\}$$

(see Figure 3).

2) $J_\sigma(x_1,x_2,x_3) = (\lambda x_1 + x_2, \lambda x_2, \mu x_3)$, where $0<|\lambda|<1$ and $\mu>1$;

$$\mathscr{U}_{J_\sigma} = \biggl\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon (x_2|x_3|^{-\log_\mu|\lambda|})^2 + \biggl(x_1|x_3|^{-\log_\mu|\lambda|} - \frac{x_2\ln|x_3|}{|\lambda|\ln\mu}\cdot|x_3|^{-\log_\mu|\lambda|}\biggr)^2 < 1\biggr\}.$$

3) $J_\sigma(x_1,x_2,x_3) = \bigl(\rho(x_1\cos\varphi - x_2\sin\varphi),\ \rho(x_1\sin\varphi + x_2\cos\varphi),\ \mu x_3\bigr)$ where $0<\rho<1$, $-\pi<\varphi\geq\pi$ and $\mu>1$;

$$\mathscr{U}_{J_\sigma} = \Bigl\{(x_1,x_2,x_3)\in\mathbb{R}^3\colon \sqrt{x_1^2+x_2^2}\,|x_3|^{-\log_\mu\rho} < 1\Bigr\}.$$

Definition 2.1. An f^{m_σ}-invariant neighbourhood $\mathscr{U}_\sigma$ of the saddle point σ is said to be C^2-linearizing if there exists a C^2-diffeomorphism $\psi_\sigma\colon \mathscr{U}_\sigma \to \mathscr{U}_{J_\sigma}$ conjugating the diffeomorphism $f^{m_\sigma}|_{\mathscr{U}_\sigma}$ with the diffeomorphism $J_\sigma|_{\mathscr{U}_{J_\sigma}}$.

A C^2-linearizing neighbourhood of a hyperbolic saddle point is constructed by applying the technique given in [9, Lemma 2] if the diffeomorphism is at least C^7-smooth and there are no resonances up to and including sixth order[4] (see [4, Ch. 6, § 5] or [22, Theorem 3.20]).

In the neighbourhood $\mathscr{U}_{J_\sigma}$ we define a pair of transversal foliations $(\mathscr{F}^2,\mathscr{F}^1)$ as follows:

• the leaves of the foliation $\mathscr{F}^2$ are the level sets of the function $(x_1,x_2,x_3)\mapsto x_3$ in $\mathscr{U}_{J_\sigma}$;

• the leaves of the foliation $\mathscr{F}^1$ are the level sets of the function $(x_1,x_2,x_3)\mapsto(x_1,x_2)$ in $\mathscr{U}_{J_\sigma}$.

By means of the C^2-diffeomorphism ψ_σ^{-1} the foliations $\mathscr{F}^2$, $\mathscr{F}^1$ induce f^{m_σ}-invariant foliations $\mathscr{F}^2_\sigma$, $\mathscr{F}^1_\sigma$ on the linearizing neighbourhood $\mathscr{U}_\sigma$, which are called *linearizing foliations*. If $m_\sigma>1$, then by constructing a C^2-linearizing neighbourhood $\mathscr{U}_\sigma$ with a diffeomorphism $\psi_\sigma\colon \mathscr{U}_\sigma\to\mathscr{U}_{J_\sigma}$, for any $k=0,\dots,m_\sigma-1$ we obtain a linearizing neighbourhood $U_{f^k(\sigma)} = f^k(U_\sigma)$ with the diffeomorphism $\psi_{f^k(\sigma)} = \psi_\sigma f^{-k}\colon \mathscr{U}_{f^k(\sigma)} \to \mathscr{U}_{J_{f^k(\sigma)}}$ and linearizing foliations $\mathscr{F}^2_{f^k(\sigma)}$, $\mathscr{F}^1_{f^k(\sigma)}$. Throughout what follows we assume that linearizing neighbourhoods of saddle points of the same orbit are chosen in this compatible way.

[4]Let p be a fixed hyperbolic point of a diffeomorphism $f\colon M^n\to M^n$, and let $\rho_1,\dots,\rho_n$ be the eigenvalues of the Jacobi matrix D_pf. We say that a *resonance of order* $m\geq 2$ takes place at the point p if there exist nonnegative numbers $m_1,\dots,m_n$ such that $m=\sum_{k=1}^n m_k$ and the relation $\rho_j = \rho_1^{m_1}\rho_2^{m_2}\dots\rho_n^{m_n}$ holds for some $j\in\{1,\dots,n\}$.

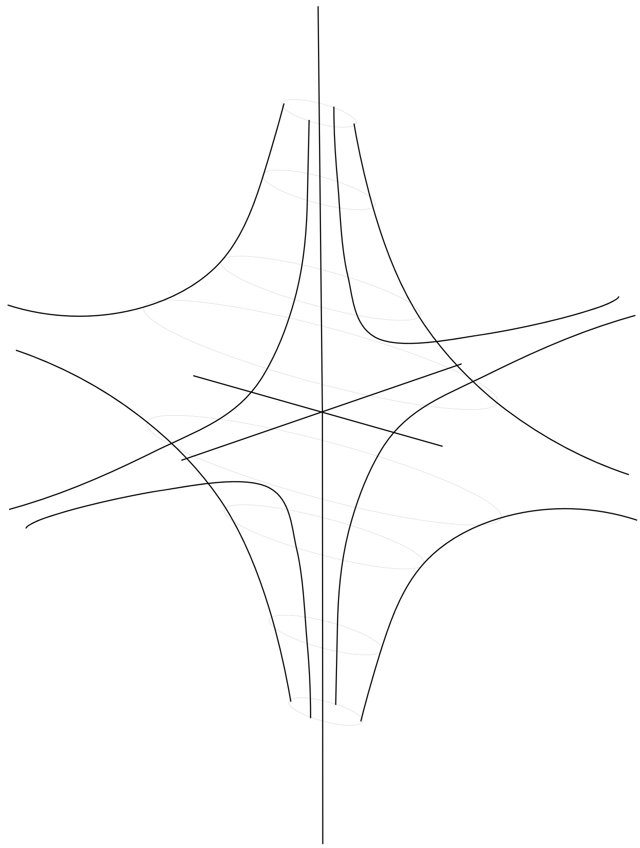

FIGURE 3. A linearizing neighbourhood $\mathscr{U}_{J_\sigma}$ for $J_\sigma(x_1, x_2, x_3) = (\lambda_1 x_1, \lambda_2 x_2, \mu x_3)$

By passing to the diffeomorphism f^{-1} we define the C^2-linearizing neighbourhood with the linearizing foliations $\mathscr{F}^2_\sigma$, $\mathscr{F}^1_\sigma$ for a saddle point σ with a two-dimensional unstable manifold. For any point $x \in \mathscr{U}_\sigma$ let $\mathscr{F}^2_{\sigma,x}$, $\mathscr{F}^1_{\sigma,x}$ denote the leaf of the foliation $\mathscr{F}^2_\sigma$, $\mathscr{F}^1_\sigma$ passing through the point x.

Let $a_1, \dots, a_k \in \mathscr{A}$ be points that belong to pairwise different orbits such that their orbits comprise the whole set $\mathscr{A}$. By condition 2) in the description of the class Ψ, the point a_l, $l \in \{1, \dots, k\}$, is a point of nondegenerate one-sided tangency of the manifolds $W^s_{\sigma^s_{a_l}, x}$ and $W^u_{\sigma^u_{a_l}, x}$, which are leaves of the C^2-smooth two-dimensional foliations $\mathscr{F}^2_{\sigma^s_{a_l}}$ and $\mathscr{F}^2_{\sigma^u_{a_l}}$. Then for any $l \in \{1, \dots, k\}$ there exist a neighbourhood $\mathscr{U}_{a_l} \subset (\mathscr{U}_{\sigma^s_{a_l}} \cap \mathscr{U}_{\sigma^u_{a_l}})$, a C^1-curve $\ell_{a_l} \subset \mathscr{U}_{a_l}$, and a two-dimensional foliation B_{a_l} such that

1) the foliations $\mathscr{F}^2_{\sigma^s_{a_l}}$ and $\mathscr{F}^2_{\sigma^u_{a_l}}$ are transversal at every point of the set $\mathscr{U}_{a_l} \setminus \ell_{a_l}$, and the leaves $\mathscr{F}^2_{\sigma^s_{a_l}, x}$ and $\mathscr{F}^2_{\sigma^u_{a_l}, x}$ have nondegenerate one-sided tangency at every point $x \in \ell_{a_l}$;

2) the leaves of the foliations $F^2_{\sigma^s_{a_l}}$ and $F^2_{\sigma^u_{a_l}}$ are transversal to the foliation B_{a_l} on $\mathscr{U}_{a_l} \setminus \ell_{a_l}$, and there exists a homeomorphism $\psi_{a_l} : U_{a_l} \to \mathbb{R}^3$ such that

$$\psi_{a_l}(\ell_{a_l} \cap U_{a_l}) = Ox_3, \quad B_{a_l} = \psi^{-1}_{a_l}(B),$$

where

$$B = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus Ox_3 \colon x_2 = kx_1,\ k \in \mathbb{R}\big\} \cup \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus Ox_3 \colon x_1 = 0\big\};$$

3) $f^k(\mathscr{U}_{a_l}) \cap \mathscr{U}_{a_l} = \varnothing$ for any $k \in (\mathbb{Z}\backslash\{0\})$ (see Figure 4).
We set $L_{a_l} = \bigcup_{n\in\mathbb{Z}} f^n(\ell_{a_l})$, $L_{\mathscr{A}} = L_{a_1} \cup \dots \cup L_{a_k}$, and $\mathscr{U}_{\mathscr{A}} = \mathscr{U}_{a_1} \cup \dots \cup \mathscr{U}_{a_k}$.

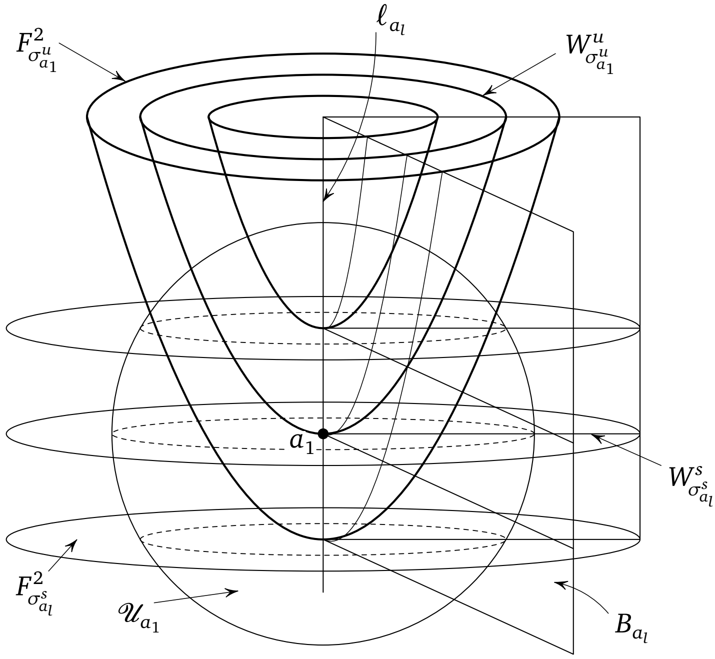

FIGURE 4. Foliations in a neighbourhood of a heteroclinic tangency point

§ 3. A compatible system of neighbourhoods

In this section we prove the existence of a compatible system of neighbourhoods of saddle points of any diffeomorphism in the class Ψ in the sense of the following definition.

Definition 3.1. Let $f \in \Psi$. A system $\{U_\sigma \subset \mathscr{U}_\sigma,\ \sigma \in (\Omega_1 \cup \Omega_2)\}$ of neighbourhoods of all saddle points is said to be compatible if their union is f-invariant and every neighbourhood U_σ is equipped with a pair of f^{m_σ}-invariant foliations F^2_σ, F^1_σ with the following properties:

1) the foliation F^2_σ is equal to $\mathscr{F}^2_\sigma \cap U_\sigma$;
2) the foliation F^1_σ is a one-dimensional foliation that is transversal to the foliation F^2_σ on $U_\sigma\backslash L_{\mathscr{A}}$, contains a one-dimensional manifold of the saddle point σ as a leaf, and has singularities on the set $U_\sigma \cap L_{\mathscr{A}}$;
3) if $W^s_{\sigma_1} \cap W^u_{\sigma_2} = \varnothing$ for distinct saddle points σ_1 and σ_2, then $U_{\sigma_1} \cap U_{\sigma_2} = \varnothing$;
4) if $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$, then for any point $x \in (U_{\sigma_1} \cap U_{\sigma_2}\backslash L_{\mathscr{A}})$ and a leaf $F^1_{\sigma_1,x}$ ($F^1_{\sigma_2,x}$) of the foliation $F^1_{\sigma_1}$ ($F^1_{\sigma_2}$) passing through the point x, the condition $(F^1_{\sigma_1,x} \cap U_{\sigma_2}) \subset F^2_{\sigma_2,x}$ holds ($(F^1_{\sigma_2,x} \cap U_{\sigma_1}) \subset F^2_{\sigma_1,x}$, respectively);
5) any point a_l, $l \in \{1,\dots,k\}$, has a neighbourhood $U_{a_l} \subset (U_{\sigma^s_{a_l}} \cap U_{\sigma^u_{a_l}})$ in which every connected component of the intersection of a leaf of the foliation $F^2_{\sigma^s_{a_l}}$ ($F^2_{\sigma^u_{a_l}}$) with a leaf of the foliation B_{a_l} is contained in a leaf of the foliation $F^1_{\sigma^u_{a_l},a_l}$ ($F^1_{\sigma^s_{a_l},a_l}$, respectively).

Theorem 3.1. *For any diffeomorphism $f \in \Psi$ there exists a compatible system of neighbourhoods.*

Proof. We split the construction of a compatible system of neighbourhoods into steps.

Step 1. For any point $\sigma_1 \in \Omega_1$ we set $\widehat{\mathscr{U}_{\sigma_1}} = p_f(\mathscr{U}_{\sigma_1})$ and $\widehat{\mathscr{F}^2_{\sigma_1}} = p_f(\mathscr{F}^2_{\sigma_1})$. For any point $\sigma_2 \in \Omega_2$ we set $\widehat{\mathscr{U}_{\sigma_2}} = p_f(\mathscr{U}_{\sigma_2})$ and $\widehat{\mathscr{F}^2_{\sigma_2}} = p_f(\mathscr{F}^2_{\sigma_2})$. We set $\widehat{\mathscr{A}} = p_f(\mathscr{A})$. Every connected component of the sets $\widehat{\mathbb{W}}^u_f$ and $\widehat{\mathbb{W}}^s_f$ is a smooth compact surface. Since the intersection of the sets $\widehat{\mathbb{W}}^u_f$ and $\widehat{\mathbb{W}}^s_f$ is transversal everywhere apart from points in the set $\widehat{\mathscr{A}}$, the set $\widehat{H} = \widehat{\mathbb{W}}^u_f \cap \widehat{\mathbb{W}}^s_f \backslash \widehat{\mathscr{A}}$ consists of a finite number of simple closed curves $\widehat{\gamma}_1, \dots, \widehat{\gamma}_r$, which are the projections of all the heteroclinic curves of the diffeomorphism f.

Any curve $\widehat{\gamma}_l$, $l = 1, \dots, r$, is contained in the intersection $\widehat{W}^s_{\sigma_1(l)} \cap \widehat{W}^u_{\sigma_2(l)}$ for some saddle points $\sigma_1(l) \in \Omega_1$ and $\sigma_2(l) \in \Omega_2$ (note that $\sigma_1(l_1)$ may coincide with $\sigma_1(l_2)$, just as the saddle $\sigma_2(l_1)$ may coincide with the saddle $\sigma_2(l_2)$ for distinct l_1, l_2). Since the curve $\widehat{\gamma}_l$ is compact, there exists a tubular neighbourhood $\mathscr{U}_{\widehat{\gamma}_l} \subset (\widehat{\mathscr{U}_{\sigma_1(l)}} \cap \widehat{\mathscr{U}_{\sigma_2(l)}})$ with a foliation $\widehat{G}_{\widehat{\gamma}_l} = \{\widehat{d}^l_x,\, x \in \widehat{\gamma}_l\}$ consisting of discs transversal to the leaves of the foliations $\widehat{\mathscr{F}^2_{\sigma_1(l)}}$ and $\widehat{\mathscr{F}^2_{\sigma_2(l)}}$ such that the connected components of the intersection of the leaves of the foliations $\widehat{\mathscr{F}^2_{\sigma_1(l)}}$, $\widehat{\mathscr{F}^2_{\sigma_2(l)}}$ with the leaves of the foliation $\widehat{G}_{\widehat{\gamma}_l}$ form C^1-foliations $\widehat{F}^1_{\sigma_2(l),\widehat{\gamma}_l}$, $\widehat{F}^1_{\sigma_1(l),\widehat{\gamma}_l}$ on $\mathscr{U}_{\widehat{\gamma}_l}$ consisting of one-dimensional open arcs. Note that this fact will guarantee that properties 2) and 4) in Definition 3.1 hold in a neighbourhood of heteroclinic curves in what follows.

Let $U_{\widehat{\gamma}_l} \subset \mathscr{U}_{\widehat{\gamma}_l}$ be a neighbourhood of the curve $\widehat{\gamma}_l$ for which the projection

$$\pi_{\sigma_2(l),\widehat{\gamma}_l} \colon U_{\widehat{\gamma}_l} \to \widehat{W}^u_{\sigma_2(l)} \quad (\,\pi_{\sigma_1(l),\widehat{\gamma}_l} \colon U_{\widehat{\gamma}_l} \to \widehat{W}^s_{\sigma_1(l)}),$$

along the leaves of the foliation $\widehat{F}^1_{\sigma_2(l),\widehat{\gamma}_l}$ ($\widehat{F}^1_{\sigma_1(l),\widehat{\gamma}_l}$), is well defined. We set $U_{\widehat{H}} = U_{\widehat{\gamma}_1} \cup \dots \cup U_{\widehat{\gamma}_r}$. Let $\widehat{G}$ denote the two-dimensional C^1-foliation on $U(\widehat{H})$ composed of the discs of the foliations $\widehat{G}_{\widehat{\gamma}_1}, \dots, \widehat{G}_{\widehat{\gamma}_r}$, let $\widehat{F}^1_{\Omega_2,\,\widehat{H}}$ ($\widehat{F}^1_{\Omega_1,\,\widehat{H}}$) denote the one-dimensional C^1-foliation composed of the leaves of the foliations $\widehat{F}^1_{\sigma_2(1),\widehat{\gamma}_1}, \dots, \widehat{F}^1_{\sigma_2(r),\widehat{\gamma}_r}$ ($\widehat{F}^1_{\sigma_1(1),\widehat{\gamma}_1}, \dots, \widehat{F}^1_{\sigma_1(r),\widehat{\gamma}_r}$), and let

$$\pi_{\Omega_2,\,\widehat{H}} \colon U_{\widehat{H}} \to \widehat{\mathbb{W}}^u_f \quad (\pi_{\Omega_1,\,\widehat{H}} \colon U_{\widehat{H}} \to \widehat{\mathbb{W}}^s_f)$$

denote the projection along the leaves of the foliation $\widehat{F}^1_{\Omega_2,\,\widehat{H}}$ ($\widehat{F}^1_{\Omega_1,\,\widehat{H}}$).

Step 2. For a point $\widehat{a}_l \in \widehat{\mathscr{A}}, l = 1, \dots, k$, we set $\mathscr{U}_{\widehat{a}_l} = p_f(\mathscr{U}_{a_l})$, $\ell_{\widehat{a}_l} = p_f(\ell_{a_l})$, and $L_{\widehat{\mathscr{A}}} = p_f(L_{\mathscr{A}})$. The point $\widehat{a}_l$ is a point of nondegenerate one-sided tangency of the surfaces $\widehat{W}^u_{\sigma^u_{a_l}}$ and $\widehat{W}^s_{\sigma^s_{a_l}}$, which are leaves of the two-dimensional C^2-foliations $\widehat{\mathscr{F}^2_{\sigma^s_{a_l}}}$ and $\widehat{\mathscr{F}^2_{\sigma^u_{a_l}}}$ that are transversal everywhere, except for points belonging to the C^1-curve $\ell_{\widehat{a}_l}$. We set $\widehat{B}_{a_l} = p_f(B_{a_l})$. Then the connected components of the intersection of the leaves of the foliations $\widehat{\mathscr{F}^2_{\sigma^s_{a_l}}}$, $\widehat{\mathscr{F}^2_{\sigma^u_{a_l}}}$ with the leaves of the foliation $\widehat{B}_{\widehat{a}_l} = \psi^{-1}_{\widehat{a}_l}(B)$ form C^1-foliations $\widehat{F}^1_{\sigma^u_{a_l}\,\widehat{a}_l}$ and $\widehat{F}^1_{\sigma^s_{a_l},\,\widehat{a}_l}$ on $\mathscr{U}_{\widehat{a}_l} \backslash \ell_{\widehat{a}_l}$ consisting of one-dimensional open arcs.

Let $K^u_{\widehat{a}_l} \subset \mathscr{U}_{\widehat{a}_l}$ ($K^s_{\widehat{a}_l} \subset \mathscr{U}_{\widehat{a}_l}$) be the union of the leaves of the foliation $\widehat{F}^1_{\sigma^s_{a_l},\,\widehat{a}_l}$ ($\widehat{F}^1_{\sigma^u_{a_l},\,\widehat{a}_l}$) that have nonempty intersection with $\widehat{W}^s_{\sigma^s_{a_l}}$ ($\widehat{W}^u_{\sigma^u_{a_l}}$, respectively) (see Figure 5). Then on the set $K^u_{\widehat{a}_l} \subset \mathscr{U}_{\widehat{a}_l}$ ($K^s_{\widehat{a}_l} \subset \mathscr{U}_{\widehat{a}_l}$), the projection

$$\pi_{\sigma^s_{a_l},\,\widehat{a}_l} \colon K^u_{\widehat{a}_l} \to \widehat{W}^s_{\sigma^s_{a_l}} \quad (\pi_{\sigma^u_{a_l},\,\widehat{a}_l} \colon K^s_{\widehat{a}_l} \to \widehat{W}^u_{\sigma^u_{a_l}})$$

along the leaves of the foliation $\widehat{F}^1_{\sigma^s_{a_l},\,\widehat{a}_l}$ ($\widehat{F}^1_{\sigma^u_{a_l},\,\widehat{a}_l}$) is well defined. We choose a connected component of the intersection of a leaf of the foliation $\widehat{\mathscr{F}^2_{\sigma^u_{a_l}}}$ ($\widehat{\mathscr{F}^2_{\sigma^s_{a_l}}}$) with the interior of

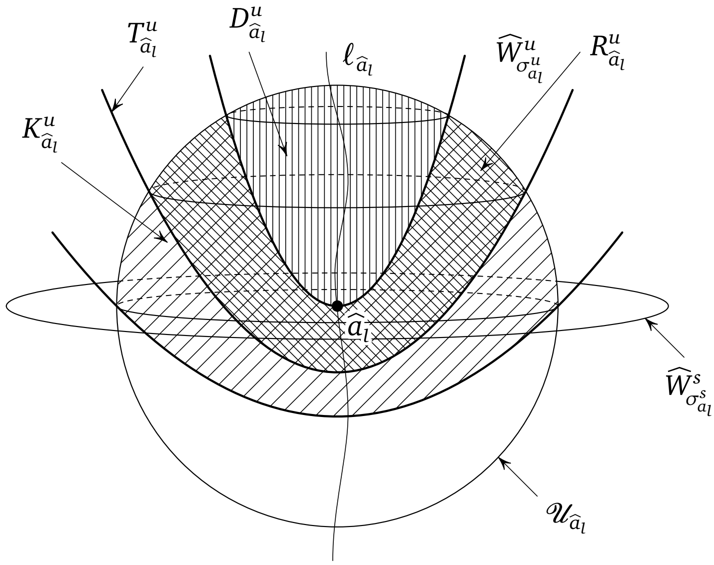

FIGURE 5. The neighbourhood $\mathscr{U}_{\widehat{a}_l}$

the set $K^u_{\widehat{a}_l}$ ($K^s_{\widehat{a}_l}$), and denote it by $T^u_{\widehat{a}_l}$ ($T^s_{\widehat{a}_l}$). Let $R^u_{\widehat{a}_l}$ ($R^s_{\widehat{a}_l}$) denote the closure of the connected component of the set $K^u_{\widehat{a}_l} \backslash cl\ T^u_{\widehat{a}_l}$ ($K^s_{\widehat{a}_l} \backslash cl\ T^s_{\widehat{a}_l}$) that has nonempty intersection with $\widehat{W}^u_{\sigma^u_{a_l}}$ ($\widehat{W}^s_{\sigma^s_{a_l}}$, respectively) (the set with the double hatching in Figure 5). Let $D^u_{\widehat{a}_l}$ ($D^s_{\widehat{a}_l}$) denote the closure of the connected component of the set $\mathscr{U}_{\widehat{a}_l} \backslash \widehat{W}^u_{\sigma^u_{a_l}}$ ($\mathscr{U}_{\widehat{a}_l} \backslash \widehat{W}^s_{\sigma^s_{a_l}}$) the interior of which is disjoint from $\widehat{W}^s_{\sigma^s_{a_l}}$ ($\widehat{W}^u_{\sigma^u_{a_l}}$, respectively) (the set with the vertical hatching in Figure 5).

We set $K^u_{\widehat{\mathscr{A}}} = K^u_{\widehat{a}_1} \cup \dots \cup K^u_{\widehat{a}_k}$, $R^u_{\widehat{\mathscr{A}}} = R^u_{\widehat{a}_1} \cup \dots \cup R^u_{\widehat{a}_k}$, $D^u_{\widehat{\mathscr{A}}} = D^u_{\widehat{a}_1} \cup \dots \cup D^u_{\widehat{a}_k}$, $K^s_{\widehat{\mathscr{A}}} = K^s_{\widehat{a}_1} \cup \dots \cup K^s_{\widehat{a}_k}$, $R^s_{\widehat{\mathscr{A}}} = R^s_{\widehat{a}_1} \cup \dots \cup R^s_{\widehat{a}_k}$, and $D^s_{\widehat{\mathscr{A}}} = D^s_{\widehat{a}_1} \cup \dots \cup D^s_{\widehat{a}_k}$. Let $\widehat{B}$ denote the two-dimensional C^1-foliation on $\mathscr{U}_{\widehat{\mathscr{A}}} \backslash L_{\widehat{\mathscr{A}}}$ composed of the leaves of the foliations $\widehat{B}_{\widehat{a}_1}, \dots, \widehat{B}_{\widehat{a}_k}$, let $\widehat{F}^1_{\Omega_2, \widehat{\mathscr{A}}}$ (respectively, $\widehat{F}^1_{\Omega_1, \widehat{\mathscr{A}}}$) denote the one-dimensional C^1-foliation composed of the leaves of the foliations $\widehat{F}^1_{\sigma^u_{a_1}, \widehat{a}_1}, \dots, \widehat{F}^1_{\sigma^u_{a_r}, \widehat{a}_r}$ ($\widehat{F}^1_{\sigma^s_{a_1}, \widehat{a}_1}, \dots, \widehat{F}^1_{\sigma^s_{a_r}, \widehat{a}_r}$), and let $\pi_{\Omega_2, \widehat{\mathscr{A}}} \colon K^u_{\widehat{\mathscr{A}}} \to \widehat{\mathbb{W}}^u_f$ ($\pi_{\Omega_1, \widehat{\mathscr{A}}} \colon K^s_{\widehat{\mathscr{A}}} \to \widehat{\mathbb{W}}^s_f$) denote the projection along the leaves of the foliation $\widehat{F}^1_{\Omega_2, \widehat{\mathscr{A}}}$ ($\widehat{F}^1_{\Omega_1, \widehat{\mathscr{A}}}$).

Step 3. We construct pairwise disjoint f-invariant neighbourhoods U_σ, $\sigma \in \Omega_2$, equipped with one-dimensional f-invariant C^1-foliations F^1_σ that have properties 2) and 4) from Definition 3.1 in some neighbourhood of heteroclinic curves and heteroclinic tangency points.

The set $\widehat{W}^u_\sigma$ is a smooth submanifold of the manifold $\widehat{V}_f$ homeomorphic to a two-dimensional torus or the Klein bottle. Then there exists a tubular neighbourhood $\widehat{N}^u_\sigma$ of the manifold $\widehat{W}^u_\sigma$ with the projection $\pi_\sigma \colon \widehat{N}^u_\sigma \to \widehat{W}^u_\sigma$ along the one-dimensional leaves of the foliation $\{\widehat{I}_{\sigma, \widehat{x}},\ \widehat{x} \in \widehat{W}^u_\sigma\}$. We set $P_\sigma = \widehat{N}^u_\sigma \cap U_{\widehat{H}}$, $Q_\sigma = \widehat{N}^u_\sigma \cap K^u_{\widehat{\mathscr{A}}}$, and $Q'_\sigma = \widehat{N}^u_\sigma \cap R^u_{\widehat{\mathscr{A}}}$. Let $\pi'_\sigma \colon P_\sigma \cup Q_\sigma \to \widehat{W}^u_\sigma$ denote the map coinciding with $\pi_{\Omega_2, \widehat{H}}$ on P_σ and coinciding with $\pi_{\Omega_2, \widehat{\mathscr{A}}}$ on Q_σ. Let $P'_\sigma \subset P_\sigma$ be a neighbourhood of the set $\widehat{W}^u_\sigma \cap \widehat{H}$ such that the set $P_\sigma \backslash P'_\sigma$ is fibred by the leaves of the foliation $\widehat{F}^1_{\Omega_2, \widehat{H}}$. Let $\phi \colon \widehat{N}^u_\sigma \backslash D^u_{\widehat{\mathscr{A}}} \to [0, 1]$ be a smooth function with support on $P_\sigma \cup Q_\sigma$ that is equal to 1 on $P'_\sigma \cup Q'_\sigma$. Since the surface $\widehat{W}^u_\sigma$ has the structure of a Riemannian manifold $\mathbb{R}^2/G$, where G is a group of isometries acting freely and discontinuously,[5] it follows that the following formula gives a well-defined C^1-retraction $q_\sigma\colon \widehat{N}^u_\sigma \backslash D^u_{\widehat{\mathscr{A}}} \to \widehat{W}^u_\sigma$:

$$q_\sigma(x) = (1-\phi(x))\pi_\sigma(x) + \phi(x)\pi'_\sigma(x).$$

Since $q_\sigma(x) = x$ for $x \in \widehat{W}^u_\sigma$, there exists a neighbourhood $\widetilde{U}^u_\sigma \subset \widehat{N}^u_\sigma$ of the manifold $\widehat{W}^u_\sigma$ with the one-dimensional foliation $\widetilde{F}^1_\sigma$ composed of the arcs $\{(q_\sigma)^{-1}(x) \cap \widetilde{U}^u_\sigma,\ x \in \widehat{W}^u_\sigma \backslash Q'_\sigma\}$, the arcs $L_{\widehat{\mathscr{A}}} \cap \widetilde{U}^u_\sigma$, and on the remaining set composed of the leaves of the foliation $\widehat{F}^1_{\Omega_2,\widehat{\mathscr{A}}} \cap \widetilde{U}^u_\sigma$. We set $S_\sigma = \widetilde{U}^u_\sigma \cap (P'_\sigma \cup Q'_\sigma \cup D^u_{\widehat{\mathscr{A}}})$ and $S_{\Omega_2} = \bigcup_{\sigma\in\Omega_2} S_\sigma$.

We perform similar constructions with similar notation for points of the set Ω_1. For points $\sigma \in \Omega_2$ ($\sigma \in \Omega_1$) we choose neighbourhoods $\widehat{U}^u_\sigma \subset \widetilde{U}^u_\sigma$ ($\widehat{U}^s_\sigma \subset \widetilde{U}^s_\sigma$) such that the union of these neighbourhoods $\widehat{U}^u_{\Omega_2} = \bigcup_{\sigma\in\Omega_2} \widehat{U}^u_\sigma$ ($\widehat{U}^s_{\Omega_1} = \bigcup_{\sigma\in\Omega_1} \widehat{U}^s_\sigma$, respectively) has the property that $\widehat{U}^u_{\Omega_2} \cap \widehat{U}^s_{\Omega_1} \subset (S_{\Omega_2} \cap S_{\Omega_1})$.

For $\sigma \in \Omega_2$ ($\sigma \in \Omega_1$) we set $\widehat{F}^1_\sigma = \widetilde{F}^1_\sigma \cap \widehat{U}^u_\sigma$ ($\widehat{F}^1_\sigma = \widetilde{F}^1_\sigma \cap \widehat{U}^s_\sigma$), and $U_{\mathscr{O}_\sigma} = p_f^{-1}(\widehat{U}^u_\sigma) \cup W^s_{\mathscr{O}_\sigma}$ ($U_{\mathscr{O}_\sigma} = p_f^{-1}(\widehat{U}^s_\sigma) \cup W^u_{\mathscr{O}_\sigma}$), where $\mathscr{O}_\sigma$ is the orbit of the point σ. Let $F^1_{\mathscr{O}_\sigma}$ denote the foliation on $U_{\mathscr{O}_\sigma}$ consisting of the connected components of the pre-images under p_f of the leaves of the foliation $\widehat{F}^1_\sigma$ and the stable (unstable) manifolds $W^s_{\mathscr{O}_\sigma}$ ($W^u_{\mathscr{O}_\sigma}$, respectively). Let U_σ be a connected component of the set $U_{\mathscr{O}_\sigma}$. Then the restriction F^1_σ of the foliation $F^1_{\mathscr{O}_\sigma}$ to U_σ is the one we are looking for. □

§ 4. Necessary and sufficient conditions for topological conjugacy

Theorem 4.1. *Diffeomorphisms $f, f' \in \Psi$ are topologically conjugate if and only if the schemes S_f and $S_{f'}$ are equivalent.*

Proof. The necessity for the geometric constituents of the schemes to be homeomorphic is proved as in [6, Lemma 1.3.6], and that it is necessary that the analytic parameters of the schemes be equal is proved in [9, Theorem 1]. We now prove the sufficiency of the conditions of the theorem, that is, we construct a homeomorphism conjugating the diffeomorphisms f and f' in the class Ψ (the scheme of construction is close to the construction of a conjugating homeomorphism for diffeomorphisms of surfaces with the same modulus of stability found in [14]).

Suppose that the schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in \Psi$ are equivalent. Then by Definition 1.2 there exists a homeomorphism $\widehat{\varphi}\colon \widehat{V}_f \to \widehat{V}_{f'}$ such that

1) $\eta_f = \eta_{f'}\widehat{\varphi}_*$;

2) $\widehat{\varphi}(\widehat{\mathbb{W}}^s_f) = \widehat{\mathbb{W}}^s_{f'}$ and $\widehat{\varphi}(\widehat{\mathbb{W}}^u_f) = \widehat{\mathbb{W}}^u_{f'}$;

3) $\Theta_{\widehat{a}} = \Theta_{\widehat{\varphi}(\widehat{a})}$ for $\Theta_{\widehat{a}} \in \widehat{C}_f$.

In constructing a conjugating homeomorphism we make key use of the existence of a compatible system of neighbourhoods for a Morse–Smale diffeomorphism (see Definition 3.1). We divide the construction of a homeomorphism $h\colon M^3 \to M^3$ conjugating the diffeomorphisms f, f' into steps.

In Step 1 we prove the existence of a lifting $\varphi\colon V_f \to V_{f'}$ of a homeomorphism $\widehat{\varphi}$ conjugating the diffeomorphisms $f|_{V_f}$ and $f'|_{V_{f'}}$, which is uniquely extended to the set $\Omega_1 \cup \Omega_2$.

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[5] If $\widehat{W}^u_\sigma$ is a torus, then we find that G is a group of diffeomorphisms of the plane $\mathbb{R}^2$ with generators $a(x_1,x_2) = (x_1+1, x_2)$ and $b(x_1,x_2) = (x_1, x_2+1)$. If $\widehat{W}^u_\sigma$ is a Klein bottle, then G is a group of diffeomorphisms of the plane $\mathbb{R}^2$ with generators $c(x_1,x_2) = (x_1+1, x_2)$ and $d(x_1,x_2) = (1-x_1, x_2+1/2)$.

In Step 2 we construct a homeomorphism $\varphi^u_{\Omega_1}\colon W^u_{\Omega_1}\to W^u_{\Omega'_1}$ conjugating the diffeomorphisms $f|_{W^u_{\Omega_1}}$ and $f'|_{W^u_{\Omega'_1}}$. Here, the map $\varphi^u_{\Omega_1}$ is composed of the maps $\varphi^u_\sigma\colon W^u_\sigma\to W^u_{\sigma'}$, $\sigma\in\Omega_1$, which are uniquely defined by the eigenvalues μ_σ of the map J_σ (eigenvalues $\mu_{\sigma'}$ of $J_{\sigma'}$) that are greater than 1 in absolute value.

In Step 3 we construct a homeomorphism $\varphi^s_{\Omega_2}\colon W^s_{\Omega_2}\to W^s_{\Omega'_2}$ conjugating the diffeomorphisms $f|_{W^s_{\Omega_2}}$ and $f'|_{W^s_{\Omega'_2}}$. Here, the map $\varphi^s_{\Omega_1}$ is composed of the maps $\varphi^s_\sigma\colon W^s_\sigma\to W^s_{\sigma'}$, $\sigma\in\Omega_2$, which are uniquely defined by the eigenvalues λ_σ $(\lambda_{\sigma'})$ of the map J_σ $(J_{\sigma'})$ that are less than 1 in absolute value, and by the map of passing from one set of linearizing coordinates to another in a neighbourhood of a heteroclinic tangency point.

In Step 4, using the map $\varphi^u_{\Omega_1}$, we modify the homeomorphism φ to a homeomorphism $\varphi_0\colon V_f\to V_{f'}$ conjugating the diffeomorphisms $f|_{V_f}$ and $f'|_{V_{f'}}$ that coincides with φ outside a neighbourhood $V^2_{\mathscr{A}}$ of the set of heteroclinic tangency points and takes compatible leaves of the diffeomorphism f to compatible leaves of the diffeomorphism f' in a neighbourhood $V^1_{\mathscr{A}}\subset V^2_{\mathscr{A}}$ of the set of heteroclinic tangency points.

In Step 5 we modify the homeomorphism φ_0 to a homeomorphism $\varphi_1\colon V_f\to V_{f'}$ that conjugates the diffeomorphisms $f|_{V_f}$ and $f'|_{V_{f'}}$, takes compatible two-dimensional stable foliations of the diffeomorphism f in the neighbourhood $W^u_{\Omega_1}$ to compatible two-dimensional stable foliations of the diffeomorphism f' in the neighbourhood $W^u_{\Omega'_1}$, and can be continuously extended to the set $W^u_{\Omega_1}$ by a homeomorphism $\varphi^u_{\Omega_1}$.

In Step 6 the homeomorphism $\widehat{\varphi}_1$ is modified to a homeomorphism $\varphi_2\colon V_f\to V_{f'}$ that conjugates the diffeomorphisms $f|_{V_f}$ and $f'|_{V_{f'}}$, takes compatible two-dimensional foliations of the diffeomorphism f in the neighbourhood $W^s_{\Omega_2}\cup W^u_{\Omega_1}$ to compatible two-dimensional foliations of the diffeomorphism f' in the neighbourhood $W^s_{\Omega'_2}\cup W^u_{\Omega'_1}$, and can be continuously extended to the set $W^u_{\Omega_1}$ by the homeomorphism $\varphi^u_{\Omega_1}$, and to the set $W^s_{\Omega_2}$ by the homeomorphism $\varphi^s_{\Omega_2}$. The homeomorphism φ_2 thus constructed can be uniquely extended on the set of nodal points to a homeomorphism h, which is the one required.

Step 1. It follows from condition 1) in Definition 1.2 concerning the equivalence of schemes that there exists a homeomorphism $\varphi\colon V_f\to V_{f'}$ conjugating the restriction of the diffeomorphism f to V_f with the restriction of the diffeomorphism f' to $V_{f'}$ such that $\widehat{\varphi}=p_{_{f'}}\varphi p^{-1}_{_f}$ (see, for example, [6, Proposition 1.2.4]). Thus, we have a conjugating homeomorphism on the set $M^3\backslash(W^u_{\Omega_0\cup\Omega_1}\cup W^s_{\Omega_2\cup\Omega_3})$.

Due to condition 2) in Definition 1.2, for any point $\sigma\in\Omega_1$ $(\sigma\in\Omega_2)$ there exists a point $\sigma'\in\Omega'_1$ $(\sigma'\in\Omega'_2)$ such that $\varphi(W^s_\sigma\backslash\sigma)=W^s_{\sigma'}\backslash\sigma'$ $(\varphi(W^u_\sigma\backslash\sigma)=W^u_{\sigma'}\backslash\sigma')$. We extend the homeomorphism φ to the set $\Omega_1\cup\Omega_2$ by setting $\varphi(\sigma)=\sigma'$ for $\sigma\in(\Omega_1\cup\Omega_2)$.

Step 2. We define a conjugating homeomorphism $\varphi^u\colon W^u_{\Omega_1}\to W^u_{\Omega'_1}$.

Let $\sigma\in\Omega_1$ and $\sigma'=\varphi(\sigma)$, and let ψ_σ and $\psi_{\sigma'}$ be linearizing diffeomorphisms on linearizing neighbourhoods $\mathscr{U}_\sigma$ and $\mathscr{U}_{\sigma'}$ (see Definition 2.1). For any point $w\in W^u_\sigma$ $(w'\in W^u_{\sigma'})$ the point $\psi_\sigma(w)$ $(\psi_{\sigma'}(w'))$ has coordinates $(0,0,w_3)$ $((0,0,w'_3))$. We set $r=\dfrac{\ln|\mu_{\sigma'}|}{\ln|\mu_\sigma|}$, where μ_σ is the eigenvalue of the map J_σ ($\mu_{\sigma'}$ is the eigenvalue of $J_{\sigma'}$) that is greater than 1 in absolute value. We define a homeomorphism $\varphi^u_\sigma\colon W^u_\sigma\to W^u_{\sigma'}$ by the formula $\varphi^u_\sigma(w)=w'$, where $|w'_3|=|w_3|^r$, and by the following condition: if w belongs to a connected component E of the set $U_\sigma\backslash W^s_\sigma$, then w' belongs to the connected component E' of the set $U_{\sigma'}\backslash W^s_{\sigma'}$ such that $\varphi(E\backslash W^u_\sigma)\cap E'\neq\varnothing$.

It can be verified directly that the homeomorphism φ^u_σ conjugates the diffeomorphisms $f^{m_\sigma}|_{W^u_\sigma}$ and $f'^{m_{\sigma'}}|_{W^u_{\sigma'}}$ (see the proof of a similar fact in Step 2 of the proof of Theorem 1 in [14], for example). Since $\widehat{\varphi}(\widehat{W}^s_\sigma)=\widehat{W}^s_{\sigma'}$ and $\eta_f=\eta_{f'}\widehat{\varphi}_*$, we find that the inclusions $j_{\widehat{W}^s_\sigma}\colon \widehat{W}^s_\sigma\to\widehat{V}_f$ and $j'_{\widehat{W}^s_{\sigma'}}\colon \widehat{W}^s_{\sigma'}\to\widehat{V}_{f'}$ satisfy

$$\eta_f\big(j_{\widehat{W}^s_\sigma *}(\pi_1(\widehat{W}^s_\sigma))\big)=\eta_{f'}\big(j'_{\widehat{W}^s_{\sigma'}*}(\pi_1(\widehat{W}^s_{\sigma'}))\big).$$

On the other hand,

$$\eta_f\big(j_{\widehat{W}^s_\sigma *}(\pi_1(\widehat{W}^s_\sigma))\big)=m_\sigma\mathbb{Z}\quad\text{and}\quad \eta_{f'}\big(j'_{\widehat{W}^s_{\sigma'}*}(\pi_1(\widehat{W}^s_{\sigma'}))\big)=m_{\sigma'}\mathbb{Z},$$

and so $m_\sigma=m_{\sigma'}$. Then the homeomorphism

$$\varphi^u_{f^k(\sigma)}=f'^k\varphi^u_\sigma f^{-k}\colon W^u_{f^k(\sigma)}\to W^u_{f'^k(\sigma')}$$

conjugates the diffeomorphisms $f^{m_\sigma}|_{W^u_{f^k(\sigma)}}$ and $f'^{\,f'^m_{\sigma'}}|_{W^u_{f'^k(\sigma')}}$ for every $k=0,\dots,m_\sigma$.

By performing similar constructions for every periodic orbit of the set Ω_1, we obtain the desired homeomorphism $\varphi^u\colon W^u_{\Omega_1}\to W^u_{\Omega'_1}$.

Step 3. We define a conjugating homeomorphism $\varphi^s\colon W^s_{\Omega_2}\to W^s_{\Omega'_2}$.

Let $\sigma\in\Omega_2$, $\sigma'=\varphi(\sigma)$ and let ψ_σ, $\psi_{\sigma'}$ be linearizing diffeomorphisms (see Definition 2.1). We set $\beta=1$ if W^u_σ does not contain heteroclinic tangency points, and define β as follows if there is a heteroclinic tangency point $a\in W^u_\sigma$. We set

$$g_a=\psi_{\sigma^u_a}(\psi_{\sigma^s_a}|_{U_a})^{-1}\colon \psi_{\sigma^s_a}(U_a)\to\psi_{\sigma^u_a}(U_a)$$

and write down the map g_a in coordinate form:

$$g_a(x_1,x_2,x_3)=\big(\xi_a(x_1,x_2,x_3),\eta_a(x_1,x_2,x_3),\zeta_a(x_1,x_2,x_3)\big).$$

We set $\rho=\dfrac{\ln|\lambda_{\sigma'}|}{\ln|\lambda_\sigma|}$, where λ_σ is the eigenvalue of the map J_σ and $\lambda_{\sigma'}$ is the eigenvalue of $J_{\sigma'}$ that is less than 1 in absolute value. We set

$$\beta=\frac{\dfrac{\partial\zeta'_a}{\partial x_3}(a')}{\left(\dfrac{\partial\zeta_a}{\partial x_3}(a)\right)^{\rho}},\quad\text{where } a'=\varphi(a).$$

For any point $w\in W^s_\sigma$ ($w'\in W^s_{\sigma'}$) the point $\psi_\sigma(w)$ ($\psi_{\sigma'}(w')$) has coordinates $(0,0,w_3)$ ($(0,0,w'_3)$). We define a homeomorphism $\varphi^s_\sigma\colon W^s_\sigma\to W^s_{\sigma'}$ by the formula $\varphi^s_\sigma(w)=w'$, where $|w'_3|=|\beta|\cdot|w_3|^\rho$, and by the following condition: if w belongs to a connected component E of the set $U_\sigma\backslash W^u_\sigma$, then w' belongs to the connected component E' of the set $U_{\sigma'}\backslash W^u_{\sigma'}$ such that $\varphi(E\backslash W^s_\sigma)\cap E'\neq\varnothing$. It can be verified directly that the homeomorphism

$$\varphi^s_{f^k(\sigma)}=f'^k\varphi^s_\sigma f^{-k}\colon W^s_{f^k(\sigma)}\to W^s_{f'^k(\sigma')}$$

conjugates the diffeomorphisms $f^{m_\sigma}|_{W^s_{f^k(\sigma)}}$ and $f'^{m_{\sigma'}}|_{W^s_{f'^k(\sigma')}}$ for every $k=0,\dots,m_\sigma$ (see the proof of a similar fact in Step 3 of the proof of Theorem 1 in [14], for example).

By performing similar constructions for every periodic orbit of the set Ω_2, we obtain a sought-for homeomorphism $\varphi^s\colon W^s_{\Omega_2}\to W^s_{\Omega'_2}$.

Step 4. We modify the homeomorphism φ in a neighbourhood of the set $\mathscr{A}$.

To do this we set $a'_l=\varphi(a_l)$. Then, on some arc $\tilde{\ell}_{a_l}\subset\ell_{a_l}$ containing the point a_l, there is a homeomorphism onto the image $\varphi_{\ell_{a_l}}:\tilde{\ell}_{a_l}\to\ell_{a'_l}$ defined by the formula $\varphi_{\ell_{a_l}}(z)=z'$, where we associate the point $z'=F^2_{\sigma^s_{a'_l},\varphi^u(y)}\cap\ell_{a_l}$ with a point $z\in\tilde{\ell}_{a_l}$ that is the point of intersection of the leaf $F^2_{\sigma^s_{a_l},y}$, $y \in W^u_{\sigma^s_{a_l}}$, and the arc $\tilde{\ell}_{a_l}$. We choose a smooth two-dimensional disc $d_l \subset (U_{a_l} \cap W^s_{\sigma^s_{a_l}})$ ($d'_l \subset (U_{a'_l} \cap W^s_{\sigma^s_{a'_l}})$) containing the point a_l (a'_l) such that its boundary $c_l = \partial d_l$ ($c'_l = \partial d'_l$) intersects every leaf of the foliation B_{a_l} ($B_{a'_l}$). We choose an orientation on the curve c_l (c'_l) which corresponds to moving around in such a way that the disc d_l (d'_l) remains on the left and denote a homeomorphism preserving the chosen orientation by $\varphi_{c_l}\colon c_l \to c'_l$. We let $B_{a_l,x}$ ($B_{a'_l,x'}$) denote the leaf of the foliation B_{a_l} ($B_{a'_l}$) passing through the point $x \in c_l$ ($x' \in c'_l$).

We observe that for any point z (z') in some neighbourhood of the point a_l (a'_l) there exists a unique triple of points (z_1, z_2, z_3) ((z'_1, z'_2, z'_3)) such that

$$z_1, z_2 \in \ell_{a_l} \quad (z'_1, z'_2 \in \ell_{a'_l}), \quad z_3 \in c_l \quad (z'_3 \in c'_l),$$

and

$$z = F^2_{\sigma^s_{a_l},z_1} \cap F^2_{\sigma^u_{a_l},z_2} \cap B_{a_l,z_3} \quad (z' = F^2_{\sigma^s_{a'_l},z'_1} \cap F^2_{\sigma^u_{a'_l},z'_2} \cap B_{a'_l,z'_3}).$$

Then in some neighbourhood of the point a_l, a homeomorphism onto the image φ_{a_l} is well defined that associates with a point z a point z' such that $z'_1 = \varphi_{\ell_{a_l}}(z_1)$, $z'_2 = \varphi_{\ell_{a_l}}(z_2)$, $z'_3 = \varphi_{c_l}(z_3)$. We choose 3-balls $V^1_{a_l}, V^2_{a_l}$ with the following properties:

1) $V^1_{a_l} \subset \operatorname{int} V^2_{a_l}$ and $f^k(V^2_{a_l}) \cap V^2_{a_l} = \varnothing$ for any $k \in (\mathbb{Z}\setminus\{0\})$;
2) the spheres $\Sigma^1_{a_l} = \partial V^1_{a_l}$ and $\Sigma^2_{a_l} = \partial V^2_{a_l}$ intersect every manifold $W^s_{\sigma^s_{a_l}}$ and $W^u_{\sigma^u_{a_l}}$ in one circle;
3) there is a well-defined homeomorphism φ_{a_l} on the set $V^1_{a_l}$ and $\varphi_{a_l}(V^1_{a_l}) \subset \operatorname{int} \varphi(V^2_{a_l})$.

We construct a homeomorphism onto the image $\varphi_{V^2_{a_l}}$ that coincides with φ_{a_l} on $V^1_{a_l}$, coincides with φ on Σ^2, and is such that $\varphi_{V^2_{a_l}}(W^s_{\sigma^s_{a_l}} \cap V^2_{a_l}) \subset W^s_{\sigma^s_{a'_l}}$ and $\varphi_{V^2_{a_l}}(W^u_{\sigma^u_{a_l}} \cap V^2_{a_l}) \subset W^u_{\sigma^u_{a'_l}}$. To do this it is sufficient to construct a homeomorphism $\phi_{V^2_{a_l}}\colon V^2_{a_l} \to V^2_{a_l}$ that coincides with $\phi_{V^1_{a_l}} = \varphi^{-1}\varphi_{a_l}$ on $V^1_{a_l}$, coincides with id on Σ^2, and satisfies the conditions $\phi_{V^2_{a_l}}(W^s_{\sigma^s_{a_l}} \cap V^2_{a_l}) = W^s_{\sigma^s_{a_l}} \cap V^2_{a_l}$ and $\varphi_{V^2_{a_l}}(W^u_{\sigma^u_{a_l}} \cap V^2_{a_l}) = W^u_{\sigma^u_{a_l}} \cap V^2_{a_l}$. Then $\varphi_{V^2_{a_l}} = \varphi\phi_{V^2_{a_l}}$ is the required map.

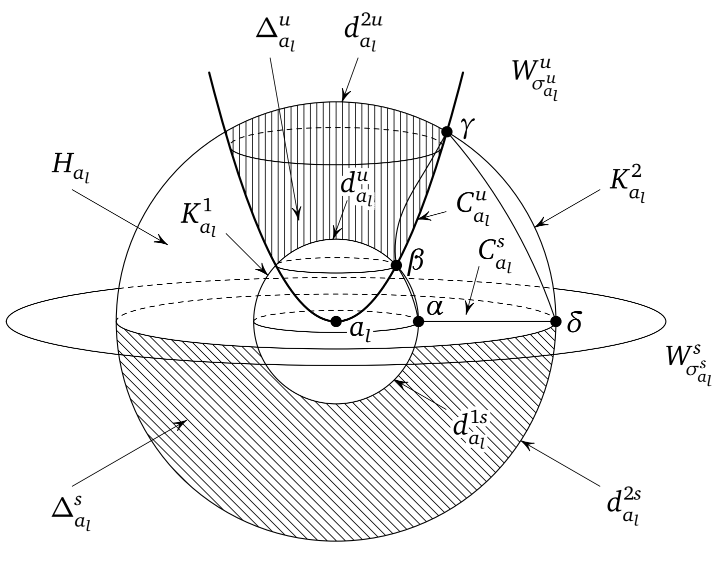

FIGURE 6. Three-dimensional annulus G_{a_l}

We set $\widetilde{V}^1_{a_l} = \phi_{V^1_{a_l}}(V^1_{a_l})$, $\widetilde{\Sigma}^1_{a_l} = \partial\widetilde{V}^1_{a_l}$, $G_{a_l} = cl\ (V^2_{a_l}\setminus V^1_{a_l})$ and $\widetilde{G}_{a_l} = cl\ (V^2_{a_l}\setminus\widetilde{V}^1_{a_l})$. We observe that the set G_{a_l} ($\widetilde{G}_{a_l}$) is a three-dimensional annulus and the two-dimensional cylinders $C^u_{a_l} = G_{a_l}\cap W^u_{\sigma^u_{a_l}}$ and $C^s_{a_l} = G_{a_l}\cap W^s_{\sigma^u_{a_l}}$ ($\widetilde{C}^u_{a_l} = \widetilde{G}_{a_l}\cap W^u_{\sigma^u_{a_l}}$ and $\widetilde{C}^s_{a_l} = \widetilde{G}_{a_l}\cap W^s_{\sigma^s_{a_l}}$, respectively) divide it into three parts, $\Delta^u_{a_l}, \Delta^s_{a_l}, H_{a_l}$ ($\widetilde{\Delta}^u_{a_l}, \widetilde{\Delta}^s_{a_l}, \widetilde{H}_{a_l}$). Here $\Delta^u_{a_l}$, which is the set with the vertical hatching in Figure 6, is the three-dimensional ball bounded by the sphere $S^u_{a_l}$ composed of the annulus $C^u_{a_l}$ and the discs $d^{1u}_{a_l}\subset\Sigma^1_{a_l}$, $d^{2u}_{a_l}\subset\Sigma^2_{a_l}$ ($\widetilde{\Delta}^u_{a_l}$ is the three-dimensional ball bounded by the sphere $\widetilde{S}^u_{a_l}$ composed of the annulus $\widetilde{C}^u_{a_l}$ and the discs $\widetilde{d}^{1u}_{a_l}\subset\widetilde{\Sigma}^1_{a_l}$, $d^{2u}_{a_l}\subset\Sigma^2_{a_l}$); $\Delta^s_{a_l}$, which is the set with the slanting hatching in Figure 6, is the three-dimensional ball bounded by the sphere $S^s_{a_l}$ composed of the annulus $C^s_{a_l}$ and the discs $d^{1s}_{a_l}\subset\Sigma^1_{a_l}$ and $d^{2s}_{a_l}\subset\Sigma^2_{a_l}$ ($\widetilde{\Delta}^s_{a_l}$ is the three-dimensional ball bounded by the sphere $\widetilde{S}^s_{a_l}$ composed of the annulus $\widetilde{C}^s_{a_l}$ and the discs $\widetilde{d}^{1s}_{a_l}\subset\widetilde{\Sigma}^1_{a_l}$, $d^{2s}_{a_l}\subset\Sigma^2_{a_l}$); H_{a_l} is the solid torus bounded by the torus T_{a_l} composed of the cylinders $C^u_{a_l}$, $C^s_{a_l}$ and the annuli $k^1_{a_l}\subset\Sigma^1_{a_l}$ and $k^2_{a_l}\subset\Sigma^2_{a_l}$ ($\widetilde{H}_{a_l}$ is the solid torus bounded by the torus $\widetilde{T}_{a_l}$ composed of the cylinders $\widetilde{C}^u_{a_l}$, $\widetilde{C}^s_{a_l}$ and the annuli $\widetilde{k}^1_{a_l}\subset\Sigma^1_{a_l}$ and $k^2_{a_l}\subset\Sigma^2_{a_l}$); see Figure 6.

By construction, $\widetilde{d}^{1s}_{a_l} = \phi_{V^1_{a_l}}(d^{1s}_{a_l})$, $\widetilde{d}^{1u}_{a_l} = \phi_{V^1_{a_l}}(d^{1u}_{a_l})$ and $\widetilde{k}^1_{a_l} = \phi_{V^1_{a_l}}(k^1_{a_l})$. The choice of the homeomorphism φ_{c_l} implies that there exists a homeomorphism $\phi_{C^u_{a_l}}: C^u_{a_l}\to\widetilde{C}^u_{a_l}$ coinciding with $\phi_{V^1_{a_l}}$ on $C^u_{a_l}\cap\Sigma^1_{a_l}$ and equal to the identity on $C^u_{a_l}\cap\Sigma^2_{a_l}$. Then on the sphere $S^u_{a_l}$ we have constructed a homeomorphism $\phi_{S^u_{a_l}}: S^u_{a_l}\to\widetilde{S}^u_{a_l}$ coinciding with $\phi_{C^u_{a_l}}$ on $C^u_{a_l}$, with $\phi_{V^1_{a_l}}$ on $d^{1u}_{a_l}$ and equal to the identity on $d^{2u}_{a_l}$. Let $\phi_{\Delta^u_{a_l}}: D^u_{a_l}\to\widetilde{\Delta}^u_{a_l}$ denote an extension of the homeomorphism $\phi_{S^u_{a_l}}$ to the ball $\Delta^u_{a_l}$. We choose a meridian $c\subset T_{a_l}$ of the solid torus H_{a_l} (that is, a closed curve contractible on H_{a_l} but not contractible on T_{a_l}) in such a way that it consists of closed arcs $[\alpha,\beta]\subset\Sigma^1_{a_l}$, $[\beta,\gamma]\subset C^u_{a_l}$, $[\gamma,\delta]\subset\Sigma^2_{a_l}$ and $[\delta,\alpha]\subset C^s_{a_l}$ (see Figure 6). We set $[\widetilde{\alpha},\widetilde{\beta}] = \phi_{V^1_{a_l}}([\alpha,\beta])$ and $[\widetilde{\beta},\gamma] = \phi_{C^u_{a_l}}([\beta,\gamma])$. We choose a curve $[\delta,\widetilde{\alpha}]\subset\widetilde{C}^s_{a_l}$ such that the closed curve $\tilde{c}\subset\widetilde{T}_{a_l}$ composed of the closed arcs $[\widetilde{\alpha},\widetilde{\beta}]\subset\widetilde{\Sigma}^1_{a_l}$, $[\widetilde{\beta},\gamma]\subset\widetilde{C}^u_{a_l}$, $[\gamma,\delta]\subset\Sigma^2_{a_l}$ and $[\delta,\widetilde{\alpha}]\subset\widetilde{C}^s_{a_l}$ is a meridian of the solid torus $\widetilde{H}_{a_l}$. Since $\widetilde{C}^s_{a_l}\setminus[\delta,\widetilde{\alpha}]$ is a two-dimensional disc, the choice of the homeomorphism φ_{c_l} implies that there exists a homeomorphism $\phi_{C^s_{a_l}}: C^s_{a_l}\to\widetilde{C}^s_{a_l}$ that coincides with $\phi_{V^1_{a_l}}$ on $C^s_{a_l}\cap\Sigma^1_{a_l}$, is equal to the identity on $C^s_{a_l}\cap\Sigma^2_{a_l}$, and is such that $\phi_{C^s_{a_l}}([\delta,\alpha]) = [\delta,\widetilde{\alpha}]$. Then on the torus T_{a_l} we have constructed a homeomorphism $\phi_{T_{a_l}}: T_{a_l}\to\widetilde{T}_{a_l}$ that coincides with $\phi_{C^u_{a_l}}$ on $C^u_{a_l}$, with $\phi_{C^s_{a_l}}$ on $C^s_{a_l}$, with $\phi_{V^1_{a_l}}$ on $k^1_{a_l}$, and is equal to the identity on $k^2_{a_l}$. Since the map $\phi_{T_{a_l}}$ takes a meridian of the torus H_{a_l} to a meridian of the torus $\widetilde{T}_{a_l}$, this map can be extended to a homeomorphism $\phi_{H_{a_l}}: H_{a_l}\to\widetilde{H}_{a_l}$. This automatically constructs a homeomorphism $\phi_{S^s_{a_l}}: S^s_{a_l}\to\widetilde{S}^s_{a_l}$ on the sphere $S^s_{a_l}$ that coincides with $\phi_{C^s_{a_l}}$ on $C^s_{a_l}$, with $\phi_{V^1_{a_l}}$ on $d^{1s}_{a_l}$, and is equal to the identity on $d^{2s}_{a_l}$. Let $\phi_{\Delta^s_{a_l}}: \Delta^s_{a_l}\to\widetilde{\Delta}^s_{a_l}$ denote an extension of the homeomorphism $\phi_{S^s_{a_l}}$ to the ball $\Delta^s_{a_l}$. Finally, the homeomorphism we require, $\phi_{V^2_{a_l}}: V^2_{a_l}\to V^2_{a_l}$, coincides with $\phi_{V^1_{a_l}}$ on $V^1_{a_l}$, with $\phi_{\Delta^s_{a_l}}$ on $\Delta^s_{a_l}$, with $\phi_{H_{a_l}}$ on H_{a_l} and with $\phi_{\Delta^u_{a_l}}$ on $\Delta^u_{a_l}$.

We set $V^2_{\mathscr{A}} = V^2_{a_1}\cup\cdots\cup V^2_{a_k}$ and let $\varphi_{V^2_{\mathscr{A}}}$ denote the homeomorphism composed of the homeomorphisms $\varphi\phi_{V^2_{a_1}},\dots,\varphi\phi_{V^2_{a_k}}$. Let $\varphi_0\colon V_f\to V_{f'}$ denote the homeomorphism that coincides with $f'^n\varphi_{V^2_{\mathscr{A}}}f^{-n}$ on $f^n(V^2_{\mathscr{A}})$, $n\in\mathbb{Z}$, and coincides with φ outside $\bigcup_{n\in\mathbb{Z}}f^n(V^2_{\mathscr{A}})$.

We observe that for any point y (y') in the set $V^1_{a_l}\backslash a_l$ ($\varphi_0(V^1_{a_l})\backslash a'_l$) there exists a unique pair of points (y_s, y_u) ((y'_s, y'_u)) such that

$$y_s \in W^s_{\sigma^s_{a_l}}, \quad y_u \in W^u_{\sigma^s_{a_l}} \quad \big(y'_s \in W^s_{\sigma^s_{a'_l}}, \quad y'_u \in W^u_{\sigma^s_{a'_l}}\big)$$

and

$$y = F^2_{\sigma^s_{a_l},\, y_u} \cap F^1_{\sigma^s_{a_l},\, y_s} \quad \big(y' = F^2_{\sigma^s_{a'_l},\, y'_u} \cap F^1_{\sigma^s_{a'_l},\, y'_s}\big).$$

Then $\varphi_0(y) = y'$, where $y'_s = \varphi_0(y_s)$ and $y'_u = \varphi^u(y_u)$. At the same time, there exists a unique pair of points $(\widetilde{y}_s, \widetilde{y}_u)$ for the point y ($(\widetilde{y}'_s, \widetilde{y}'_u)$ for y') such that

$$\widetilde{y}_s \in W^s_{\sigma^u_{a_l}}, \quad \widetilde{y}_u \in W^u_{\sigma^u_{a_l}} \quad \big(\widetilde{y}'_s \in W^s_{\sigma^u_{a'_l}}, \quad \widetilde{y}'_u \in W^u_{\sigma^u_{a'_l}}\big)$$

and

$$y = F^2_{\sigma^u_{a_l},\, \widetilde{y}_s} \cap F^1_{\sigma^u_{a_l},\, \widetilde{y}_u} \quad \big(y' = F^2_{\sigma^u_{a'_l},\, \widetilde{y}'_s} \cap F^1_{\sigma^u_{a'_l},\, \widetilde{y}'_u}\big).$$

Then $\varphi_0(y) = y'$, where $\widetilde{y}'_s = \varphi^s(\widetilde{y}_s)$ and $\widetilde{y}'_u = \varphi_0(\widetilde{y}_u)$.

Step 5. We modify the homeomorphism φ_1 on the set $U_{\Omega_1} = \bigcup_{\sigma\in\Omega_1} U_\sigma$.

By construction, for every curve $\ell_{a_l} \cap V^1_{a_l}$ there exists a neighbourhood $N^u_{a_l} \subset V^1_{a_l}$ composed of one-dimensional compatible leaves of the diffeomorphism f. Property 5) in Definition 3.1 and the construction of the homeomorphism φ_0 imply that the set $N^u_{a'_l} = \varphi_0(N^u_{a_l})$ is a neighbourhood of the curve $\ell_{a'_l} \cap \varphi_0(V^1_{a_l})$ composed of one-dimensional compatible leaves of the diffeomorphism f'. We set $N^u_{\mathscr{A}} = N^u_{a_1} \cup \dots \cup N^u_{a_k}$ and $N^u_{\mathscr{A}'} = N^u_{a'_1} \cup \dots \cup N^u_{a'_k}$.

Let $\sigma \in \Omega_1$, $\sigma' = \varphi(\sigma)$ and let U_σ, $U_{\sigma'}$ be neighbourhoods in a compatible system. We set $N^u_\sigma = U_\sigma \cap \bigcup_{n\in\mathbb{Z}} f^n(N^u_{\mathscr{A}})$ ($N^u_{\sigma'} = U_{\sigma'} \cap \bigcup_{n\in\mathbb{Z}} f^n(N^u_{\mathscr{A}'})$). We observe that for any point y (y') in the set $\widetilde{U}_\sigma = U_\sigma \backslash N^u_\sigma$ ($\widetilde{U}_{\sigma'} = U_{\sigma'} \backslash N^u_{\sigma'}$) there exists a unique pair of points (y_s, y_u) such that $y_s \in W^s_\sigma$ and $y_u \in W^u_\sigma$ and $y = F^1_{\sigma, y_s} \cap F^2_{\sigma, y_u}$ ((y'_s, y'_u) such that $y'_s \in W^s_{\sigma'}$ and $y'_u \in W^u_{\sigma'}$ and $y' = F^1_{\sigma', y'_s} \cap F^2_{\sigma', y'_u}$). Then there exists an f^{m_σ}-invariant neighbourhood $V_\sigma \subset U_\sigma$ of the point σ such that on the set $\widetilde{V}_\sigma = V_\sigma \backslash N^u_\sigma$ a homeomorphism onto the image $\varphi_{\widetilde{V}_\sigma} : \widetilde{V}_\sigma \to U_{\sigma'}$ is well defined that associates with a point $y \in \widetilde{V}_\sigma$ a point y' such that $y'_s = \varphi_0(y_s)$ and $y'_u = \varphi^u(y_u)$. We assume without loss of generality that the set V_σ is chosen in such a way that $\varphi_{\widetilde{V}_\sigma}(\widetilde{V}_\sigma) \subset \varphi_0(\widetilde{U}_\sigma)$.

We define a topological embedding $\phi_\sigma \colon \widetilde{V}_\sigma \backslash W^u_\sigma \to \widetilde{U}_\sigma$ by the formula $\phi_\sigma = \varphi_0^{-1}\varphi_{\widetilde{V}_\sigma}$. We set $Z_\sigma = U_\sigma \cap W^u_{\Omega_2}$. The properties of a compatible system of neighbourhoods imply that the set Z_σ consists of one-dimensional leaves of the foliation F^1_σ. By construction, the topological embedding ϕ_σ is the identity on the set ∂N^u_σ and $\phi_\sigma(Z_\sigma) \subset W^u_{\Omega_2}$. Then, by Lemma 4.3.2 and Corollary 4.3.2 in the book [8], there exists a homeomorphism $\Phi_\sigma \colon U_\sigma \backslash W^u_\sigma \to U_\sigma \backslash W^u_\sigma$ commuting with the diffeomorphism $f^{m_\sigma}|_{U_\sigma \backslash W^u_\sigma}$, coinciding with ϕ_σ on $\widetilde{V}_\sigma \backslash W^u_\sigma$, equal to the identity on $N^u_\sigma \cup \partial U_\sigma$, and such that $\Phi_\sigma(U_\sigma \cap W^u_{\Omega_2}) = U_\sigma \cap W^u_{\Omega_2}$. We define a topological embedding $\varphi_\sigma \colon U_\sigma \backslash W^u_\sigma \to U_{\sigma'} \backslash W^u_{\sigma'}$ by the formula $\varphi_\sigma = \varphi_0 \phi_\sigma$. By construction, the homeomorphism φ_σ can be continuously extended to W^u_σ by a homeomorphism φ^u_σ.

For every $k = 0, \dots, m_\sigma$ we define a homeomorphism onto the image by the formula $\varphi_{f^k(\sigma)} = f'^k \varphi_\sigma f^{-k} \colon U_{f^k(\sigma)} \backslash W^u_{f^k(\sigma)} \to U_{f'^k(\sigma')} \backslash W^u_{f'^k(\sigma')}$. Let φ_{Ω_1} denote the map composed of φ_σ, $\sigma \in \Omega_1$. Then the desired homeomorphism $\varphi_1 \colon V_f \to V_{f'}$ coincides with φ_{Ω_1} on $U_{\Omega_1} \backslash W^u_{\Omega_1}$ and with φ_0 outside U_{Ω_1}.

Step 6. We modify the homeomorphism φ_1 on the set $U_{\Omega_2} = \bigcup_{\sigma\in\Omega_2} U_\sigma$. The equation $\Theta_{a_l} = \Theta_{a'_l}$ for $a_l \in \mathscr{A}$ implies the relation

$$\frac{\ln|\mu_{\sigma^s_{a'_l}}|}{\ln|\mu_{\sigma^s_{a_l}}|} = \frac{\ln|\lambda_{\sigma^u_{a'_l}}|}{\ln|\lambda_{\sigma^u_{a_l}}|}.$$

Then given a point $z \in \ell_{a_l}$ that is the intersection point of the leaf $F^2_{\sigma^u_{a_l},y}$, $y \in W^s_{\sigma^u_{a_l}}$ and the arc ℓ_{a_l}, the map $\varphi_{\ell_{a_l}}: \ell_{a_l} \to \ell_{a'_l}$ associates z with the point $z' = F^2_{\sigma'^u_{a'_l},\varphi^s(y)} \cap \ell_{a'_l}$ (for a proof of a similar fact see Step 4 of the proof of Theorem 1 in [14]).

By construction, for every curve $\ell_{a_l} \cap V^1_{a_l}$ there exists a neighbourhood $N^s_{a_l} \subset V^1_{a_l}$ composed of one-dimensional stable compatible leaves of the diffeomorphism f. It follows from property 5) in Definition 3.1 and from the construction of the homeomorphism φ_1 that the set $N^s_{a'_l} = \varphi_1(N^s_{a_l})$ is a neighbourhood of the curve $\ell_{a'_l} \cap \varphi_1(V^1_{a_l})$ composed of one-dimensional stable compatible leaves of the diffeomorphism f'. We set $N^s_{\mathscr{A}} = N^s_{a_1} \cup\dots\cup N^u_{s_k}$ and $N^u_{\mathscr{A}'} = N^s_{a'_1} \cup\dots\cup N^s_{a'_k}$.

Let $\sigma \in \Omega_2$, $\sigma' = \varphi(\sigma)$, and let U_σ, $U_{\sigma'}$ be neighbourhoods in a compatible system. We set $N^s_\sigma = U_\sigma \cap \bigcup_{n\in\mathbb{Z}} f^n(N^s_{\mathscr{A}})$ and $N^s_{\sigma'} = U_{\sigma'} \cap \bigcup_{n\in\mathbb{Z}} f^n(N^s_{\mathscr{A}'})$. We observe that for any point y in the set $\widetilde{U}_\sigma = U_\sigma \backslash N^s_\sigma$ (y' in $\widetilde{U}_{\sigma'} = U_{\sigma'} \backslash N^s_{\sigma'}$) there exists a unique pair of points (y_s, y_u) ((y'_s, y'_u)) such that

$$y_s \in W^s_\sigma, \quad y_u \in W^u_\sigma \quad \big(y'_s \in W^s_{\sigma'}, \quad y'_u \in W^u_{\sigma'}\big)$$

and

$$y = F^2_{\sigma,y_s} \cap F^1_{\sigma,y_u} \quad (y' = F^2_{\sigma',y'_s} \cap F^1_{\sigma',y'_u}, \text{ respectively}).$$

Then there exists an f^{m_σ}-invariant neighbourhood $V_\sigma \subset U_\sigma$ of the point σ such that a homeomorphism onto the image $\varphi_{V_\sigma}: \widetilde{V}_\sigma \to U_{\sigma'}$ is well defined on the set $\widetilde{V}_\sigma = V_\sigma \backslash N^s_\sigma$ and that given a point $y \in \widetilde{V}_\sigma$ it sets up a correspondence between it and the point y' such that $y'_u = \varphi_1(y_u)$ and $y'_s = \varphi^s(y_s)$. We assume without loss of generality that the set V_σ is chosen in such a way that $\varphi_{\widetilde{V}_\sigma}(\widetilde{V}_\sigma) \subset \varphi_1(\widetilde{U}_\sigma)$.

We define a topological embedding $\phi_\sigma: \widetilde{V}_\sigma \backslash W^s_\sigma \to \widetilde{U}_\sigma$ by $\phi_\sigma = \varphi_1^{-1}\varphi_{\widetilde{V}_\sigma}$. We set $Z_\sigma = U_\sigma \cap W^s_{\Omega_1}$. It follows from the properties of a compatible system of neighbourhoods that the set Z_σ has a neighbourhood N_{Z_σ} consisting of connected components K of the intersection of two-dimensional stable compatible leaves with N_{Z_σ} fibred by one-dimensional stable leaves of the foliation F^1_σ. By construction, the topological embedding ϕ_σ is equal to the identity on the set ∂N^s_σ, and $\phi_\sigma(K) \subset F^2_{\widetilde{\sigma},x}$, $\widetilde{\sigma} \in \Omega_1$, for a two-dimensional stable leaf $F^2_{\widetilde{\sigma},x}$ such that $K \subset F^2_{\widetilde{\sigma},x}$. Then, by Lemma 4.3.2 and Corollary 4.3.2 in [8], there exists a homeomorphism $\Phi_\sigma: U_\sigma \backslash W^s_\sigma \to U_\sigma \backslash W^s_\sigma$ commuting with the diffeomorphism $f^{m_\sigma}|_{U_\sigma \backslash W^u_\sigma}$, coinciding with ϕ_σ on $\widetilde{V}_\sigma \backslash W^s_\sigma$, equal to the identity on $N^s_\sigma \cup \partial U_\sigma$, and such that $\Phi_\sigma(U_\sigma \cap K) = K$. We define a topological embedding $\varphi_\sigma: U_\sigma \backslash W^s_\sigma \to U_{\sigma'} \backslash W^s_{\sigma'}$ by $\varphi_\sigma = \varphi_1\phi_\sigma$. By construction the homeomorphism φ_σ can be continuously extended to W^s_σ by a homeomorphism φ^s_σ.

For every $k = 0, \dots, m_\sigma$ we define a homeomorphism onto the image by the formula

$$\varphi_{f^k(\sigma)} = f'^k \varphi_\sigma f^{-k}: U_{f^k(\sigma)} \backslash W^s_{f^k(\sigma)} \to U_{f'^k(\sigma')} \backslash W^s_{f'^k(\sigma')}.$$

Let φ_{Ω_2} denote the map composed of φ_σ, $\sigma \in \Omega_2$. Then the homeomorphism we require, $\varphi_2: V_f \to V_{f'}$, coincides with φ_{Ω_2} on $U_{\Omega_2} \backslash W^s_{\Omega_1}$ and with φ_1 outside U_{Ω_2}.

Finally, we extend the homeomorphism φ_2 to the homeomorphism $h: M^3 \to M^3$ we are looking for as follows: we set $h|_{W^u_{\Omega_1}} = \varphi^u|_{W^u_{\Omega_1}}$ and $h|_{W^s_{\Omega_2}} = \varphi^s|_{W^s_{\Omega_2}}$, and for any point $\omega \in \Omega_0$ we set $h(\omega) = \omega' \in \Omega'_0$ where $\varphi_3(W^s_\omega \setminus \omega) = W^s_{\omega'} \setminus \omega'$ (for any point $\alpha \in \Omega_3$ set $h(\alpha) = \alpha' \in \Omega'_3$, where $\varphi_3(W^u_\alpha \setminus \alpha) = W^u_{\alpha'} \setminus \alpha'$). □

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