Труды Московск. Матем. Общ.
Том 77 (2016), вып. 1

Trans. Moscow Math. Soc.
2016, Pages 107–126
http://dx.doi.org/10.1090/mosc/254
Article electronically published on November 28, 2016

ON SALIKHOV'S INTEGRAL

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ABSTRACT. We state a new interpolation problem, which we solve using Salikhov's integral. This was previously used in the theory of Diophantine approximations. We study the asymptotic behaviour of orthogonal polynomials related to this problem.

§ 1. Approximations of logarithms

1. A classical example of Padé approximants, that is, rational approximants with free poles [1], is given by approximations of the logarithm

$$f(z) = \int_0^1 \frac{dx}{z-x} = \ln \frac{1}{1-1/z},$$

namely,

$$R_n(z) = Q_n(z)f(z) - P_n(z) = O\Big(\frac{1}{z^{n+1}}\Big) \quad \text{as} \quad z \to \infty, \tag{1.1}$$

where Q_n is a nonzero polynomial of degree n and P_n is the corresponding polynomial of the second kind, $n \in \mathbb{Z}_+$.

Here, Q_n are Legendre polynomials, which are orthogonal on the segment $[0,1]$ with respect to the classical Lebesgue measure:

$$\int_0^1 Q_n(x)x^m\,dx = 0, \quad m = 0,\dots,n-1.$$

These polynomials satisfy Rodrigues' formula

$$Q_n(x) = \frac{1}{n!}\Big(\frac{d}{dx}\Big)^n \{x^n(x-1)^n\}, \quad n \in \mathbb{Z}_+.$$

We fixed some standard normalization of the Legendre polynomials. Then the functions of the second kind (1.1) satisfy the integral representation

$$R_n(z) = \int_0^1 \frac{x^n(1-x)^n}{(z-x)^{n+1}}\,dx, \quad n \in \mathbb{Z}_+.$$

For example, applying this construction at the point $z = 2$ we obtain the following estimate of the irrationality measure of the number $\ln 2$:

$$\Big|\ln 2 - \frac{p}{q}\Big| > \frac{c}{q^\mu}, \quad \text{where } \mu = 1 + \frac{2\ln(\sqrt{2}+1)+1}{2\ln(\sqrt{2}+1)-1} = 4{,}622\dots$$

and c is a positive constant that can be effectively calculated. Here, p is any integer and q is any positive integer. Similar results are obtained for large rational z. For constructing Diophantine approximations of the numbers $\ln 3$ or $\ln 5$, for example, this construction is not sufficient.

2010 *Mathematics Subject Classification.* Primary 30E10; Secondary 30C85, 33C47, 11J82.
Key words and phrases. Approximations of logarithms, Hermite–Padé approximations, saddle-point method, algebraic functions and Riemann surfaces, equilibrium logarithmic potentials.

Problems in the theory of numbers gave rise to many new analytic constructions for the approximation of logarithms. Mahler [2] studied joint approximations of powers of the logarithm f, f^2, f^3, … and obtained estimates for the transcendence measure of the logarithms of any rational numbers, and later of algebraic numbers. Fel'dman [3] studied joint approximations of the first type and Nikishin [4] of the second type for certain logarithms.

2. Consider approximations of the second type. Let A be an arbitrary set of $r+1$ different points on the complex plane, where $r \in \mathbb{N}$. We define the logarithmic functions

$$f_{a,b}(z) = \int_a^b \frac{dx}{z-x} = \ln \frac{z-a}{z-b}, \quad a, b \in \mathsf{A},\ a \neq b.$$

Then the approximations have the form

$$R_n^{a,b}(z) = Q_n(z) f_{a,b}(z) - P_n^{a,b}(z) = O\Big(\frac{1}{z^{n+1}}\Big) \quad \text{as} \quad z \to \infty, \tag{1.2}$$

where $a, b \in \mathsf{A}$, $a \neq b$. Here, Q_n is a nonzero polynomial of degree rn and the $P_n^{a,b}$ are the corresponding polynomials of the second kind, $n \in \mathbb{Z}_+$.

The polynomials Q_n satisfy the joint orthogonality conditions:

$$\int_a^b Q_n(x) x^m \, dx = 0, \quad m = 0, \dots, n-1,\ a, b \in \mathsf{A},\ a \neq b. \tag{1.3}$$

We can reduce the equations in relations (1.2) and (1.3) to rn independent equations by fixing the point a arbitrarily, for example. The other equations will hold by the Cauchy integral theorem. The polynomials Q_n satisfy Rodrigues' formula

$$Q_n(x) = \frac{(-1)^n}{n!} \Big(\frac{d}{dx}\Big)^n \prod_{\alpha \in \mathsf{A}} (x-\alpha)^n, \quad n \in \mathbb{Z}_+.$$

For functions of the second kind we have the integral representations

$$R_n^{a,b}(z) = \int_a^b \frac{\prod_{\alpha \in \mathsf{A}} (x-\alpha)^n}{(z-x)^{n+1}} \, dx, \quad a, b \in \mathsf{A},\ a \neq b.$$

In [5] we considered a further generalization of this construction, which includes, in particular, all polynomials of the functions $f_{a,b}$.

3. In [6] Hata considered similar approximations, but with an additional effect of the so-called collision of charges. Following this paper, we confine ourselves to the case of three functions, that is, when the set A consists of three points. The approximations have the form

$$R_n^{a,b}(z) = Q_n(z) f_{a,b}(z) - \frac{P_n^{a,b}(z)}{(z-a)^n (z-b)^n} = O\Big(\frac{1}{z^{3n+1}}\Big) \quad \text{as} \quad z \to \infty,$$

where $a, b \in \mathsf{A}$, $a \neq b$. Here, Q_n is a polynomial of degree $3n$ and the $P_n^{a,b}$ are the corresponding polynomials of the second kind, $n \in \mathbb{Z}_+$.

The polynomials Q_n satisfy the following joint orthogonality relations with variable weight:

$$\int_a^b Q_n(x) x^m (x-a)^n (x-b)^n \, dx = 0, \quad m = 0, \dots, n-1,$$

where $a, b \in \mathsf{A}$, $a \neq b$. These polynomials satisfy Rodrigues' formula:

$$Q_n(x) = \frac{(-1)^n}{(3n)!} \Big(\frac{d}{dx}\Big)^{3n} \prod_{\alpha \in \mathsf{A}} (x-\alpha)^{2n}, \quad n \in \mathbb{Z}_+.$$

For functions of the second kind we have the integral representations

$$R_n^{a,b}(z)=\int_a^b\frac{\prod_{\alpha\in\mathsf{A}}(x-\alpha)^{2n}}{(z-x)^{3n+1}}\,dx,\qquad a,b\in\mathsf{A},\ a\neq b.$$

Using this construction, Hata obtained the following estimates of the irrationality measure of the numbers $\ln 3$ and π:

$$\left|\ln 3-\frac{p}{q}\right|>\frac{c}{q^{11,101\dots}}\quad\text{and}\quad\left|\pi-\frac{p}{q}\right|>\frac{c}{q^{8,016\dots}}.$$

4. If we make the change of variable $z=1/(1-x)$ in problem (1.1), we arrive at the equivalent problem of approximating the logarithm $g(x)=\ln(1/x)$, namely,

$$R_n(x)=Q_n(x)g(x)-P_n(x)=O((1-x)^{2n+1})\quad\text{as}\quad x\to 1,$$

where $Q_n\not\equiv 0$ and P_n are polynomials of degree $n\in\mathbb{Z}_+$. The polynomials Q_n satisfy the orthogonality relations

$$\int_{-\infty}^0\frac{Q_n(x)x^m\,dx}{(1-x)^{2n+1}}=0,\qquad m=0,\dots,n-1.$$

These polynomials satisfy the Hadamard composition formula:

$$Q_n(x)=(1-x)^n\circ(1-x)^n=\sum_{k=0}^n\binom{n}{k}\binom{n}{k}x^k,\qquad n\in\mathbb{Z}_+.$$

The function of the second kind is defined by Mellin's convolution:

$$R_n(x)=(1-x)^n\star(1-x)^n=\int_x^1\left(1-\frac{x}{t}\right)^n(1-t)^n\frac{dt}{t},\qquad n\in\mathbb{Z}_+.$$

In [7] the joint approximation of the logarithms

$$g_a(x)=\ln\frac{a}{x},\qquad a\in\mathsf{A},$$

was considered, where A is an arbitrary set of r distinct nonzero complex numbers, and r is a positive integer. Namely,

$$R_n^a(x)=Q_n(x)g_a(x)-P_n^a(x)=O((a-x)^{rn+n+1})\quad\text{as}\quad x\to a,\quad\text{where}\quad a\in\mathsf{A},$$

where $Q_n\not\equiv 0$ and P_n^a, $a\in\mathsf{A}$, are polynomials of degree rn. These polynomials Q_n satisfy the joint orthogonality relations

$$\int_\infty^0\frac{Q_n(x)x^m\,dx}{(a-x)^{rn+n+1}}=0,\qquad m=0,\dots,n-1,\ a\in\mathsf{A}.$$

Here we also have the Hadamard composition

$$Q_n(x)=(1-x)^{rn}\circ\prod_{\alpha\in\mathsf{A}}(\alpha-x)^n,\qquad n\in\mathbb{Z}_+,$$

and Mellin's convolution

$$R_n^a(x)=\int_x^a\left(1-\frac{x}{t}\right)^{rn}\prod_{\alpha\in\mathsf{A}}(\alpha-t)^n\frac{dt}{t},\qquad a\in\mathsf{A},\ n\in\mathbb{Z}_+.$$

As did Hata's construction, this construction makes it possible to estimate the irrationality measures of the numbers $\ln 3$ and π. If we set $\mathsf{A}=\{4/3,3/2\}$ and $x=1$, then we obtain the estimate

$$\left|\ln 3-\frac{p}{q}\right|>\frac{c}{q^{20}}.$$

If we take $\mathsf{A} = \{1+i, (1+i)/2\}$ and $x = 1$, then we have the inequality

$$\left|\pi - \frac{p}{q}\right| > \frac{c}{q^{25}}.$$

These estimates are worse than Hata's, but they are obtained without applying Rukhadze's method [8], which makes it possible to factor-out common prime factors in the numerator and denominator of numerical rational approximations.

5. The examples given above show that to construct good Diophantine approximations we need ever more complicated analytic constructions of Hermite–Padé type. Therefore, in papers on number theory, an approach was formulated in which the starting point of study is not an interpolation problem of Hermite–Padé type but rather an integral representation of functions of the second kind. By making a good choice of the parameters in the integrand it is possible to obtain good approximations. Zudilin's survey [9] is written in this spirit. Results we have not mentioned in this section can be found there.

Salikhov [10] obtained the best estimate of the irrationality measure of the number $\ln 3$ up to now, namely,

$$\left|\ln 3 - \frac{p}{q}\right| > \frac{c}{q^{5,125}}.$$

To do this he used the integral

$$\int_{35}^{\alpha} \frac{(x-28)^n(x-30)^n(x-35)^{2n}(x-40)^n(x-42)^n}{x^{2n+1}(70-x)^{2n+1}}\,dx, \tag{1.4}$$

where $\alpha = 40$ or 42.

In this paper we are interested in the question of which interpolation problem this integral solves, and we are also interested in the question of the asymptotic behaviour of the corresponding approximants in the whole of the complex plane. This problem has a number of interesting features. In order to better understand them, in § 2 we consider a simple model example reflecting these features. Then in § 3 we study Salikhov's integral.

§ 2. Example

1. In order to better understand the approximation problem that is solved by Salikhov's integral, we first consider a simple example.

We define polynomials $Q_n(x)$ by the following Rodrigues' formula:

$$\frac{Q_n(x)}{x^{2n}} = C_n\left(\frac{d}{dx}\frac{1}{x}\right)^{2n}\{x^{2n}(x^2-1)^n\}, \quad n \in \mathbb{Z}_+, \tag{2.1}$$

where C_n is a normalizing constant. Then Q_n is an even polynomial of degree $2n$. In what follows it is convenient to choose the normalizing constant to be equal to

$$C_n = \frac{1}{(4n)!!}.$$

We let

$$f(z) = \int_{-1}^{+1} \frac{dx}{z-x} = \ln\frac{z+1}{z-1}$$

denote the principal branch of the logarithm in a neighbourhood of infinity. We set

$$S_n(z) = B_n(z)f(z) - E_n(z),$$

where $B_n(z) = z^{2n}(z^2-1)^n$ and E_n is the polynomial part of the series $B_n f$, so that

$$S_n(z) = O\left(\frac{1}{z}\right) \quad \text{as} \quad z \to \infty.$$

We define a function of the second kind by the formula

$$R_n(z) = C_n\Big(\frac{d}{dz}\frac{1}{z}\Big)^{2n} S_n(z) = C_n\Big(\frac{d}{dz}\frac{1}{z}\Big)^{2n}\{B_n(z)f(z)\}.$$

Then

$$R_n(z) = \frac{\widetilde{R}_n(z)}{z^{2n}} = \frac{1}{z^{2n}}\Big\{Q_n(z)f(z) - \frac{P_n(z)}{(1-z^2)^n}\Big\}, \tag{2.2}$$

where P_n is some polynomial of the second kind. Since

$$R_n(z) = O\Big(\frac{1}{z^{4n+1}}\Big) \quad \text{as} \quad z \to \infty,$$

it follows that $P_n(z)$ is the polynomial part of the series $(1-z^2)^n Q_n(z) f(z)$. On the other hand,

$$S_n(z) = \int_{-1}^{+1} \frac{B_n(x)}{z-x}\,dx = \int_0^1 B_n(x)\frac{2z}{z^2-x^2}\,dx.$$

Consequently,

$$R_n(z) = 2z\int_0^1 \frac{B_n(x)}{(z^2-x^2)^{2n+1}}\,dx. \tag{2.3}$$

We denote the principal branch of the logarithm in a neighbourhood of zero by

$$f^*(z) = \ln\frac{1+z}{1-z}.$$

We set $S_n^*(z) = B_n(z)f^*(z)$. This is an odd function, which is holomorphic at zero. Consequently, the function

$$R_n^*(z) = C_n\Big(\frac{d}{dz}\frac{1}{z}\Big)^{2n} S_n^*(z)$$

is also holomorphic at zero. It has the form

$$R_n^*(z) = \frac{\widetilde{R}_n^*(z)}{z^{2n}} = \frac{1}{z^{2n}}\Big\{Q_n(z)f^*(z) - \frac{P_n(z)}{(1-z^2)^n}\Big\},$$

where Q_n and P_n are the same polynomials as in formula (2.2). In a similar way, for the functions of the second kind R_n^* we obtain the integral representation

$$R_n^*(z) = 2z\int_0^1 \frac{(1-x^2)^n}{(1-x^2z^2)^{2n+1}}\,dx. \tag{2.4}$$

We pose the following problem.

Problem 2.1. For every nonnegative integer n, find a nonzero polynomial Q_n of degree at most $2n$ such that for some polynomial P_n the following interpolation conditions hold:

$$\widetilde{R}_n(z) = Q_n(z)f(z) - \frac{P_n(z)}{(1-z^2)^n} = O\Big(\frac{1}{z^{2n+1}}\Big) \quad \text{as} \quad z \to \infty,$$
$$\widetilde{R}_n^*(z) = Q_n(z)f^*(z) - \frac{P_n(z)}{(1-z^2)^n} = O(z^{2n+1}) \quad \text{as} \quad z \to 0.$$

We have proved the following result.

Proposition 2.1. *The polynomials Q_n defined by formula* (2.1) *are solutions of Problem* 2.1.

2. We will now analyse the asymptotic behaviour of the quantities $R_n(z)$ as $n \to \infty$. We apply the saddle-point method. We set $\zeta = z^2$ and $t = x^2$. In accordance with (2.3) consider the function

$$\Phi(t, \zeta) = \frac{t(t-1)}{(\zeta - t)^2}, \tag{2.5}$$

which depends on the variable t and the parameter ζ. This function has one critical point,

$$t_*(\zeta) = \frac{\zeta}{2\zeta - 1}.$$

The corresponding critical value is equal to

$$\Phi_*(\zeta) = \Phi(t_*(\zeta), \zeta) = \frac{1}{4\zeta(1-\zeta)}.$$

We analyse the lemniscate $|4z^2(1-z^2)| = 1$ depicted in Figure 2.1.

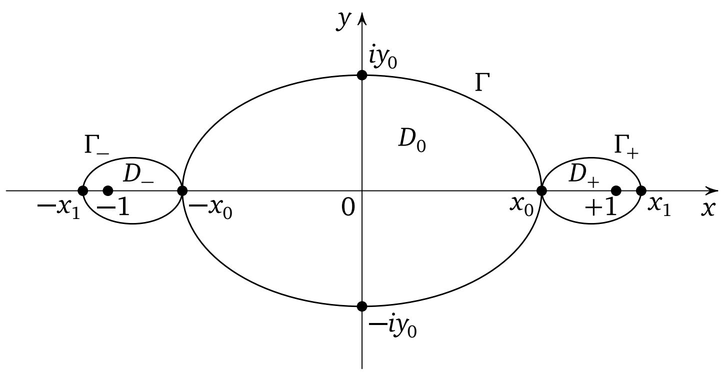

FIGURE 2.1

This curve passes through the points $\pm x_0$, $\pm x_1$, $\pm iy_0$, where

$$x_0 = \frac{1}{\sqrt{2}}, \quad x_1 = \sqrt{\frac{\sqrt{2}+1}{2}}, \quad y_0 = \sqrt{\frac{\sqrt{2}-1}{2}}.$$

The lemniscate divides the plane into four connected components. We denote the component containing the point zero by D_0, and its boundary by Γ. The components containing the points ± 1 are denoted by $D_\pm$, and their boundaries by $\Gamma_\pm$, respectively. In the unbounded component we have

$$|4z^2(1-z^2)| > 1,$$

whilst in the bounded components

$$|4z^2(1-z^2)| < 1.$$

Let D_∞ denote the domain lying outside this curve.

It is easy to see that the curve connecting the points $t = 0$ and $t = 1$ can be drawn through the critical point t_* in such a way that the function $|\Phi|$ attains its maximum at this point. Hence we have the following.

Proposition 2.2. *The following holds uniformly within the entire complex plane*:

$$\lim_{n\to\infty} |R_n(z)|^{1/n} = \frac{1}{|4z^2(1-z^2)|}. \tag{2.6}$$

If $z \in (-1, +1)$, then in formula (2.6) we take the limit values of the function $R_n(z)$ either in the upper or in the lower half-plane.

We now analyse the asymptotic behaviour of the quantities $R_n^*(z)$ as $n \to \infty$. We apply the saddle-point method to the integral (2.4). Consider the function

$$\Phi(t, \zeta) = \frac{1-t}{(1-\zeta t)^2}. \tag{2.7}$$

It has one critical point,

$$t_*(\zeta) = \frac{2\zeta - 1}{\zeta}.$$

The corresponding critical value is again equal to $\Phi_*(\zeta)$.

If $z \in D_\pm$, then a contour of steepest descent connecting the points $t = 0$ and $t = 1$ can be drawn through the critical point t_*. In all the other cases, these points can be connected by a contour on which $|\Phi|$ attains its maximum, which is equal to 1, at $t = 0$. Hence we have the following.

Proposition 2.3. *The following holds uniformly within the domains indicated*:

$$\lim_{n\to\infty} |R_n^*(z)|^{1/n} = \begin{cases} \dfrac{1}{|4z^2(1-z^2)|} & \text{if } z \in D_\pm, \\ 1 & \text{otherwise.} \end{cases}$$

Finally we analyse the asymptotic behaviour of the polynomials $Q_n(z)$ as $n \to \infty$. This will let us answer the question we are mainly interested in, namely the limit distribution of the zeros of these polynomials.

We observe that

$$Q_n(z) = \pm\frac{1}{\pi i}\big(\widetilde{R}_n^*(z) - \widetilde{R}_n(z)\big),$$

where the choice of the sign $\pm$ depends on whether the point z is in the upper or lower half-plane. It follows from (2.3) and (2.4) that we have the integral representation

$$\frac{Q_n(z)}{z^{2n}} = \pm\frac{2z}{\pi i}\int_0^\infty \frac{(1-x^2)^n}{(1-x^2z^2)^{2n+1}}\,dx.$$

Thus, to study the asymptotics of this integral we must again use the function (2.7). But now the contour of integration is any curve connecting the points zero and infinity that does not go around the poles of the function under the integral. If $z \in D_0$, then we can draw such a contour on which the maximum of $|\Phi|$ is attained at a critical point, and if $z \in D_\infty$, then it is attained at zero. In these domains, there are no zeros of the polynomials Q_n in the limit. If $z \in \Gamma$, then both points contribute to the asymptotics of the integral. The curve Γ is the limit set of the distribution of the zeros of the polynomials Q_n. We have proved the following.

Proposition 2.4. *The following holds uniformly within the domains indicated*:

$$\lim_{n\to\infty} |Q_n(z)|^{1/n} = \begin{cases} |z|^2 & \text{if } z \in D_\infty, \\ \dfrac{1}{4|1-z^2|} & \text{if } z \in D_0. \end{cases}$$

3. We denote the set of zeros of the polynomial Q_n by $\mathsf{Z}_n = \mathsf{Z}(Q_n)$ and the measure counting the zeros of this polynomial by

$$\lambda_n = \lambda(Q_n) = \frac{1}{n}\sum_{\xi\in\mathsf{Z}_n} \delta_\xi.$$

Here, δ_ξ is the unit measure placed at the point ξ, that is, the Dirac delta-function. If the limit $\lambda_n \underset{*}{\rightarrow} \lambda$ in the weak-$*$ topology of the dual space exists, then the measure λ is called the limit measure of the distribution of zeros of the polynomials $\{Q_n\}$. Let

$$V^\lambda(z) = \int \ln \frac{1}{|z-t|} d\lambda(t), \quad z \in \mathbb{C},$$

denote the logarithmic potential of the measure λ. As a corollary of Proposition 2.4 we obtain the following.

Proposition 2.5. *The limit measure λ of the distribution of the zeros of the polynomials $\{Q_n\}$ exists, and its logarithmic potential is equal to*

$$V^\lambda(z) = \begin{cases} 2\ln \dfrac{1}{|z|} & \text{if } z \in \bar{D}_\infty, \\ \ln|4(1-z^2)| & \text{if } z \in \bar{D}_0. \end{cases}$$

Corollary 2.1. *The measure λ is the balayage of the measure $2\delta_0$ onto the curve Γ.*

Corollary 2.2. *The measure λ is a unique solution of the following problem of equilibrium in an external field*:

$$V^\lambda + V^{\delta_{+1}} + V^{\delta_{-1}} = w \quad \text{on } \Gamma, \tag{2.8}$$

under the conditions that the total variation of this measure is equal to $\|\lambda\| = 2$ and its support satisfies $\mathsf{S}_\lambda \subset \Gamma$. Here, w is some equilibrium constant; *in our case it is equal to* $\ln 4$.

Finally we can draw the following conclusion.

Proposition 2.6. *Let γ be an arbitrary closed Jordan curve separating the point zero from the points ± 1, and let λ^γ be a solution of the equilibrium problem* (2.8) *on γ. Then the curve Γ is the only curve among the curves γ on which $\lambda = \lambda^\Gamma$ is the balayage of the measure $2\delta_0$.*

We have used results from logarithmic potential theory which can be found in [11] and have been further developed in the papers of Gonchar, Rakhmanov and Sorokin [12].

§ 3. Salikhov's integral

1. Making the change of variable $x \to 35(x+1)$ in the integral (1.4) and introducing parameters a and b instead of the numbers $1/7$ and $1/5$, and also introducing the variable z, we write down Salikhov's integral (1.4) in the following general form:

$$R_n^\alpha(z) = 2z \int_0^\alpha \frac{x^{2n}(x^2-a^2)^n(x^2-b^2)^n}{(z^2-x^2)^{2n+1}}\, dx, \quad \alpha \in \{a,b\}. \tag{3.1}$$

The parameters a and b are arbitrary distinct nonzero points on the complex plane.

We now find out which interpolation problem is solved by the integrals (3.1). We denote the principal branch of the logarithm in a neighbourhood of infinity by

$$f_\alpha(z) = \ln \frac{z+\alpha}{z-\alpha},$$

and the principal branch of the logarithm in a neighbourhood of zero by

$$f_\alpha^*(z) = \ln \frac{\alpha+z}{\alpha-z}.$$

We pose the following problem.

Problem 3.1. For every nonnegative integer n, it is required to find a nonzero polynomial Q_n of degree at most $4n$ such that for some polynomials of the second kind P_n^a and P_n^b the following interpolation conditions hold: for $\alpha \in \{a, b\}$,

$$\widetilde{R}_n^{\alpha}(z) = Q_n(z) f_\alpha(z) - \frac{P_n^\alpha(z)}{(\alpha^2 - z^2)^n} = O\Big(\frac{1}{z^{2n+1}}\Big) \quad \text{as} \quad z \to \infty,$$
$$\widetilde{R}_n^{*\alpha}(z) = Q_n(z) f_\alpha^*(z) - \frac{P_n^\alpha(z)}{(\alpha^2 - z^2)^n} = O(z^{2n+1}) \quad \text{as} \quad z \to 0.$$

Just as in § 2 we can easily obtain the following.

Proposition 3.1. 1°. *A polynomial Q_n that is a solution of Problem* 3.1 *can be defined by Rodrigues' formula*

$$\frac{Q_n(x)}{x^{2n}} = C_n \Big(\frac{d}{dx}\frac{1}{x}\Big)^{2n} \{x^{2n}(x^2 - a^2)^n (x^2 - b^2)^n\}, \tag{3.2}$$

where $C_n = 1/(4n)!!$ is a normalizing constant.

2°. *For functions of the second kind $\widetilde{R}_n^\alpha$ the integral representation*

$$\widetilde{R}_n^\alpha(z) = z^{2n} R_n^\alpha(z), \quad \alpha \in \{a, b\},$$

holds, where R_n^α is Salikhov's integral defined by formula (3.1).

2. In this subsection we state the main result of the paper, namely, we describe the limit measure of the distribution of zeros of Salikhov's polynomials. Here, we confine ourselves to real values of the parameters a and b: $0 < a < b$.

We carry out the following construction. Let a_* denote an arbitrary point on the interval $(0, a)$. We draw an arbitrary simple closed analytic curve Γ through the points $\pm a_*$ going around the origin. We also draw simple analytic arcs Γ_+ and Γ_- as shown in Figures 3.1 and 3.2. We assume that this system of cuts is symmetric with respect to the coordinate axes. If $a < b/\sqrt{2}$ we use Figure 3.1, and if $a > b/\sqrt{2}$ we use Figure 3.2. If $a = b/\sqrt{2}$ the curves $\Gamma_\pm$ pass through infinity.

As above, we denote by δ_{x_0} the unit measure at a point x_0. Let

$$\phi_a = V^{\delta_{+a}} + V^{\delta_{-a}} \quad \text{and} \quad \phi_b = V^{\delta_{+b}} + V^{\delta_{-b}}$$

be the external fields created by unit charges located at the points $+a, -a$ and $+b, -b$, respectively.

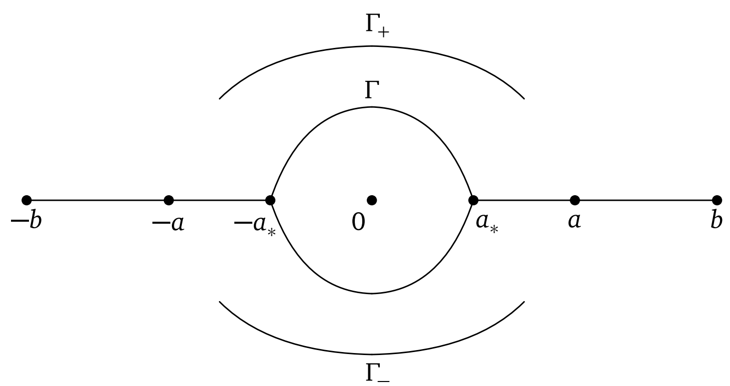

FIGURE 3.1

We pose the following equilibrium problem in logarithmic potential theory.

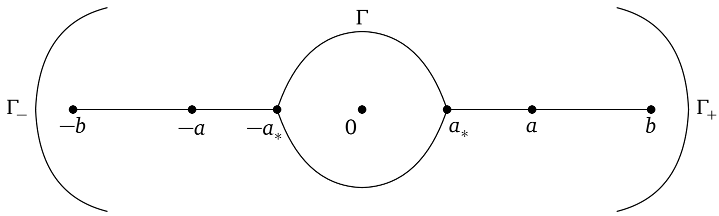

FIGURE 3.2

Problem 3.2. Find three measures λ_Γ, λ_Δ, λ_* satisfying the following conditions:

1) The support of the measure λ_Γ is contained in the curve Γ, and its total variation is equal to 2: $\mathsf{S}(\lambda_\Gamma) \subset \Gamma$, $\|\lambda_\Gamma\| = 2$.

2) The support of the measure λ_Δ is contained in the segments $\Delta_b = [-b, -a_*] \cup [a_*, b]$, and its total variation is equal to 2: $\mathsf{S}(\lambda_\Delta) \subset \Delta_b$, $\|\lambda_\Delta\| = 2$.

3) The support of the measure λ_* is contained in the curves $\Gamma_* = \Gamma_+ \cup \Gamma_-$, and its total variation is equal to 2: $\mathsf{S}(\lambda_*) \subset \Gamma_*$, $\|\lambda_*\| = 2$.

4) The following equilibrium conditions hold:

a) On the set $\Delta_b \cup \Gamma$ the measure $\lambda = \lambda_\Gamma + \lambda_\Delta$ satisfies

$$W_\Delta = V^{\lambda_\Gamma} + 2V^{\lambda_\Delta} - V^{\lambda_*} + \phi_b \begin{cases} = w_\Delta & \text{on } \mathsf{S}(\lambda), \\ \geq w_\Delta & \text{on } \Delta_b \cup \Gamma, \end{cases} \tag{3.3}$$

where w_Δ is some equilibrium constant.

b) On the set Γ_* the measure λ_* satisfies

$$W_* = 2V^{\lambda_*} - V^{\lambda} - \phi_b + \phi_a \begin{cases} = w_* & \text{on } \mathsf{S}(\lambda_*), \\ \geq w_* & \text{on } \Gamma_*, \end{cases} \tag{3.4}$$

where w_* is some equilibrium constant.

This is the Nikishin equilibrium problem with external fields. Logarithmic potential theory implies the following result.

Proposition 3.2. *Problem* 3.2 *has a unique solution.*

Next we vary the curves Γ (together with the point a_*) and Γ_*. Namely, we pose the following problem.

Problem 3.3. Find curves Γ and $\Gamma_\pm$ such that

1) the measure λ_Γ is the balayage of the measure $2\delta_0$ onto the curve Γ;

2) the support of the measure λ_Δ has the form $\mathsf{S}(\lambda_\Delta) = [-b_*, -a_*] \cup$, where $a < b_* < b$;

3) the support of the measure λ_* is the set $\Gamma_* = \Gamma_+ \cup \Gamma_-$: $\mathsf{S}(\lambda_*) = \Gamma_*$, and in addition the curves $\Gamma_\pm$ have the S-property, that is, the normal derivatives of the function W_* taken on the opposite sides at points of these curves coincide.

Proposition 3.3. *Problem* 3.3 *has a unique solution.*

We will not prove that Problem 3.3 has a unique solution here. We state the main result of the paper.

Theorem. *The limit measure* λ *of the distribution of the zeros of Salikhov's polynomials exists and is a solution of Problems* 3.2 *and* 3.3.

We shall not study the asymptotic behaviour of functions of the second kind in this paper. We simply note that the measure λ_* is the limit measure of the distribution of the zeros of the functions R_n^b, that is, of the so-called additional interpolation points.

3. Before proceeding to the proof of the theorem, we consider the limit cases when Problem 3.1 is degenerate.

Let $a = 0$. We can assume without loss of generality that $b = 1$. Then Rodrigues' formula (3.2) takes the form

$$Q_n^\star(x) = C_n\Big(\frac{d}{dx}\frac{1}{x}\Big)^{2n}\{x^{4n}(x^2-1)^n\}, \tag{3.5}$$

where $Q_n(x) = x^{2n}Q_n^\star(x)$ and $Q_n^\star$ is a polynomial of degree $2n$. These polynomials solve the following interpolation problem:

$$R_n(z) = Q_n^\star(z)f(z) - \frac{P_n(z)}{(1-z^2)^n} = O\Big(\frac{1}{z^{4n+1}}\Big) \quad \text{as} \quad z\to\infty.$$

Thus, the polynomials $Q_n^\star$ satisfy the orthogonality relations with variable weight,

$$\int_{-1}^{+1} Q_n^\star(x)x^m(1-x^2)^n\,dx = 0, \quad m = 0,\dots,2n-1.$$

This is a generalization of the Legendre polynomials of Hata type (see § 1.3). It follows from the orthogonality relations that all zeros of the polynomial $Q_n^\star$ are simple and belong to the interval $(-1,+1)$. Then logarithmic potential theory implies the following.

Proposition 3.4. *There exists a limit measure λ of the distribution of the zeros of the polynomials $Q_n^\star$. This is a unique measure of magnitude 2 with support on the segment $[-1,+1]$ and where its logarithmic potential satisfies equilibrium conditions in the external field created by unit positive charges located at the points ±1, namely,*

$$W = 2V^\lambda + V^{\delta_{+1}} + V^{\delta_{-1}} \begin{cases} = w & \text{on } \mathsf{S}(\lambda), \\ \ge w & \text{on } [-1,+1], \end{cases} \tag{3.6}$$

where w is some equilibrium constant, and the support of the measure λ is some segment $[-b_*,+b_*]$ contained strictly inside the segment $[-1,+1]$.

Remark. Thus, when passing to the limit as $a\to0$ in Problem 3.1, the point a_* tends to 0, while the curve Γ disappears.

We find an explicit form of the measure λ; namely, we prove the following.

Proposition 3.5. *The limit measure λ has the following density*:

$$\lambda'(x) = \frac{1}{\pi}\frac{\sqrt{8-9x^2}}{1-x^2}, \quad -b_*\le x\le +b_*, \tag{3.7}$$

where $b_* = 2\sqrt{2}/3$.

The graph of the density is depicted in Figure 3.3.

Proof. In order to find an explicit form of the measure λ, we make some calculations following the saddle-point method. We set $t = x^2$ and $\zeta = z^2$. In accordance with Rodrigues' formula (3.5), studying the asymptotic behaviour of the polynomials $Q_n^\star(z)$ as $n\to\infty$ reduces to studying the critical values of the function

$$\Phi(t,\zeta) = \frac{t^2(t-1)}{(\zeta-t)^2}.$$

The equation for the critical points, namely, $\frac{\partial}{\partial t}\ln\Phi = 0$, gives rise to the quadratic equation

$$t^2 - 3\zeta t + 2\zeta = 0, \quad \text{where } \zeta = z^2.$$

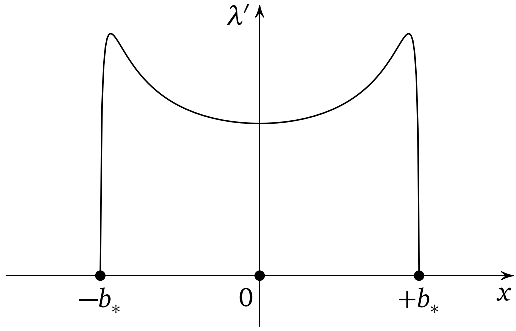

FIGURE 3.3

Its discriminant is equal to $z^2(9z^2-8)$. The algebraic function $t(z)$ has two branch points of second order at $z=\pm b_*$, where $b_* = 2\sqrt{2}/3$. Outside the segment $[-b_*,+b_*]$ we isolate two single-valued branches of this function,

$$t_\pm(z) = \frac{1}{2}\bigl(3z^2 \pm z\sqrt{9z^2-8}\bigr),$$

where $\sqrt{9z^2-8} \sim 3z$ as $z\to\infty$. Corresponding to these we have two single-valued branches $\Phi_\pm$ of the algebraic function $\Phi(z) = \Phi(t(z), z^2)$. As $z\to\infty$ we obtain

$$\Phi_+(z) \sim \frac{27}{4}z^2, \quad \Phi_-(z) \sim -\frac{4}{27}\frac{1}{z^4}.$$

By using the saddle-point method we obtain the asymptotics

$$\lim_{n\to\infty} |Q_n^\star(z)|^{1/n} = |\Phi_+(z)|, \quad z\in\mathbb{C}\setminus[-b_*,+b_*].$$

Consequently,

$$V^\lambda = -\ln|\Phi_+| + \ln\frac{27}{4}.$$

The Cauchy transform (Markov function) of the measure λ, that is, the integral

$$\widehat{\lambda}(z) = \int \frac{d\lambda(x)}{z-x},$$

is calculated using the formula

$$\widehat{\lambda}(z) = -\frac{\partial}{\partial z}V^\lambda(z) = \frac{d}{dz}\ln\Phi_+(z) = \frac{\partial}{\partial z}\ln\Phi(t,z^2)\big|_{t=t_+(z)} = \frac{4z}{t_+(z)-z^2} = \frac{8}{z+\sqrt{9z^2-8}}.$$

We find the density of the measure λ by Sokhotskiĭ's formula

$$\lambda'(x) = \frac{1}{\pi}\Im\widehat{\lambda}(x - i\cdot 0), \quad -b_* < x < +b_*.$$

Hence, (3.7) follows.

We now verify the equilibrium condition. We have

$$W = -2\ln|\Phi_+| + 2\ln\frac{27}{4} + \ln\frac{1}{|1-z|} + \ln\frac{1}{|1+z|} = w - \ln\left|\frac{((1-z^2)\Phi_+\Phi_-)\cdot\Phi_+}{\Phi_-}\right|,$$

where $w = 2\ln\frac{27}{4}$. By Vièta's theorem,

$$\Phi_+(z)\Phi_-(z)(1-z^2) = 1.$$

On the segment $[-b_*,+b_*]$, the functions Φ_+ and Φ_- take complex-conjugate values, and the ratio of their absolute values is equal to 1. Consequently, $W(x) = w$, $x\in[-b_*,+b_*]$.

The fact that the function W is increasing on the interval $(b_*,1)$ is verified by a direct calculation of the derivative.

Proposition 3.5 is proved. □

4. Now we consider the second limit case when Problem 3.1 degenerates. Let $a=b=1$. Then Rodrigues' formula (3.2) takes the form

$$\frac{Q_n(x)}{x^{2n}}=C_n\Big(\frac{d}{dx}\frac{1}{x}\Big)^{2n}\{x^{2n}(x^2-1)^{2n}\},\quad n\in\mathbb{Z}_+.$$

The polynomial Q_n has degree $4n$. This polynomial solves the interpolation problem

$$\begin{aligned}\widetilde{R}_n(z)&=Q_n(z)f(z)-P_n(z)=O\Big(\frac{1}{z^{2n+1}}\Big)\quad\text{as}\quad z\to\infty,\\ \widetilde{R}^*_n(z)&=Q_n(z)f^*(z)-P_n(z)=O(z^{2n+1})\quad\text{as}\quad z\to0.\end{aligned}\tag{3.8}$$

Condition (3.8) means that the polynomials Q_n satisfy the orthogonality relations

$$\int_{-1}^{+1}Q_n(x)x^m\,dx=0,\quad m=0,\dots,2n-1.$$

Consequently, at least $2n$ zeros of this polynomial belong to the interval $(-1,+1)$.

We now analyse the critical values of the function

$$\Phi(t,\zeta)=\frac{t(t-1)^2}{(t-\zeta)^2},\quad\zeta=z^2.\tag{3.9}$$

The equation on critical points reduces to the quadratic equation

$$t^2+(1-3z^2)t+z^2=0.$$

Its discriminant is equal to $(1-z^2)(1-9z^2)$. The zeros of the discriminant, that is, the points $z=\pm1$ and $z=\pm1/3$, are second order branch points of the algebraic functions $t(z)$ and $\Phi(z)=\Phi(t(z),z^2)$. The function Φ satisfies the quadratic equation

$$4z^2\Phi^2+(1+18z^2-27z^4)\Phi+4=0.$$

Outside the segments $\Delta=[-1,-1/3]\cup[+1/3,+1]$ two single-valued branches, $t_\pm(z)$ and $\Phi_\pm(z)$ respectively, are isolated such that

$$\Phi_+(z)\sim\frac{27}{4}z^2\quad\text{and}\quad\Phi_-(z)\sim\frac{4}{27}\frac{1}{z^4}\quad\text{as } z\to\infty.$$

The Riemann surface for these functions is a torus.

The set of points z on the complex plane for which

$$|\Phi_+(z)|=|\Phi_-(z)|$$

consists of the segments Δ and some curve, as shown in Figure 3.4.

Here,

$$y_*=\frac{\sqrt{2\sqrt3-3}}{3},\quad x_*=\frac{\sqrt{2\sqrt3+3}}{3},\quad x_0=\frac{1}{3}.$$

Let Γ denote the Jordan component of this curve going around the origin.

Proposition 3.6. *The following asymptotic formula holds uniformly inside the domain situated outside the curve* Γ *with cuts along the segments* Δ:

$$\lim_{n\to\infty}|Q_n(z)|^{1/n}=|z^2\Phi_+(z)|.\tag{3.10}$$

The following asymptotic formula holds uniformly inside the domain bounded by the curve Γ:

$$\lim_{n\to\infty}|Q_n(z)|^{1/n}=|z^2\Phi_-(z)|.\tag{3.11}$$

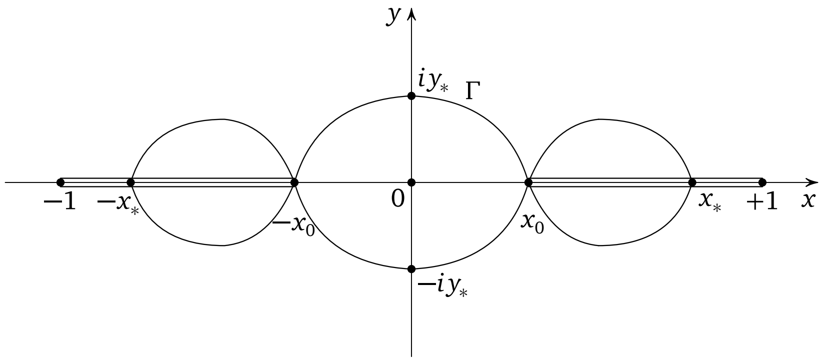

FIGURE 3.4

This assertion follows easily from an analysis of the arrangement of the critical points.

Proposition 3.7. *There exists a measure λ—a limit measure of the distribution of the zeros of the polynomials Q_n. Furthermore, $\lambda = \lambda_\Gamma + \lambda_\Delta$, where λ_Γ is a measure of magnitude 2, whose support is the curve Γ, and where λ_Δ is a measure of magnitude 2, whose support consists of the segments Δ. The measure λ_Δ has the density*

$$\lambda'_\Delta(x) = \frac{1}{\pi|x|}\sqrt{\frac{9x^2-1}{1-x^2}}, \quad \frac{1}{3} < |x| < 1. \tag{3.12}$$

The measure λ_Γ is the balayage of the measure $2\delta_0$ onto the curve Γ.

Figure 3.5 depicts the graph of the density of the measure λ_Δ.

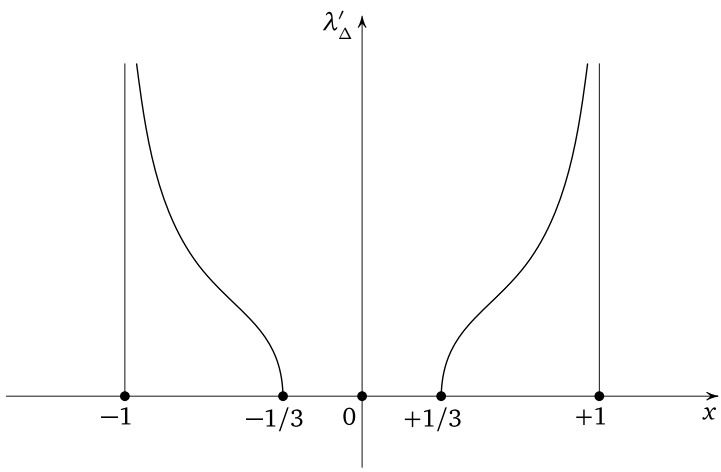

FIGURE 3.5

Proposition 3.8. *The measure λ is a unique solution of the following equilibrium problem:*

$$W = 2V^{\lambda_\Delta} + V^{\lambda_\Gamma} \begin{cases} \le w & \text{on } \mathsf{S}(\lambda), \\ \ge w & \text{on } \Delta \cup \Gamma, \end{cases} \tag{3.13}$$

where w is some equilibrium constant. Furthermore, the curve Γ can be found from the condition $V^{\lambda_\Gamma} = 2V^{\delta_0}$ outside Γ, and $w = 2\ln\frac{27}{4}$.

Proof of Propositions 3.7 *and* 3.8. It follows from (3.10) and (3.11) that the measure λ exists and a formula for its logarithmic potential is

$$V^{\lambda} = \begin{cases} -2\ln|z| - \ln|\Phi_+| + \ln\frac{27}{4} & \text{outside } \Gamma, \\ -2\ln|z| - \ln|\Phi_-| + \ln\frac{27}{4} & \text{inside } \Gamma. \end{cases}$$

In particular,

$$V^{\lambda_\Delta} = -\ln|\Phi_+| + \ln\frac{27}{4} \quad \text{in } \mathbb{C}$$

and

$$V^{\lambda_\Gamma} = \begin{cases} -2\ln|z| & \text{outside } \Gamma, \\ \ \ 2\ln|\Phi_+| & \text{inside } \Gamma. \end{cases}$$

We now verify the equilibrium condition (3.13). By Vièta's theorem,

$$\Phi_+\Phi_- = \frac{1}{z^2}, \quad z \in \mathbb{C}.$$

Consequently, on $\Gamma \cup \Delta$ we have

$$W = 2\ln\frac{27}{4} - \ln|\Phi_+^2 z^2| = w - \ln\left|\frac{\Phi_+^2}{\Phi_+\Phi_-}\right| = w - \ln\left|\frac{\Phi_+}{\Phi_-}\right| = w.$$

We now find the Markov function of the measure λ_Δ:

$$\begin{aligned} \widehat{\lambda}_\Delta(z) = -\frac{\partial}{\partial z}V^{\lambda_\Delta}(z) = \frac{d}{dz}\ln\Phi_+(z) = \frac{\partial}{\partial\zeta}\ln\Phi\Big|_{\substack{\zeta=z^2\\ t=t_+}} \\ = \frac{8z}{z^2 - 1 + \sqrt{(z^2-1)(9z^2-1)}} = \frac{1}{z}\left\{\sqrt{\frac{9z^2-1}{z^2-1}} - 1\right\}. \end{aligned}$$

We find the density of the measure by Sokhotskiĭ's formula,

$$\lambda'_\Delta(x) = \frac{1}{\pi}\Im\widehat{\lambda}_\Delta(x - i\cdot 0), \quad x \in \Delta.$$

Hence (3.12) follows, which is what was required to be proven. □

5. We now prove the main theorem. We will not go into the details of the proof, but by referring to the arguments of subsections 3 and 4, we conclude that to find the asymptotics of the polynomials Q_n it is required to study the Riemann surface of some algebraic function $\Phi(z)$.

We now define this function. Consider the rational function

$$\Phi(t;\zeta;a,b) = \frac{t(t-a^2)(t-b^2)}{(\zeta - t)^2}.$$

This is a function of t and depends on the parameter $\zeta = z^2$ and two additional fixed parameters a and b. We write down the equation for the critical points of this function. After some calculations we obtain the cubic equation

$$t^3 - 3z^2t^2 + (2(a^2+b^2)z^2 - a^2b^2)t - a^2b^2z^2 = 0. \tag{3.14}$$

A solution of equation (3.14) is given by a three-valued algebraic function $t(z)$. Correspondingly, a three-valued algebraic function emerges:

$$\Phi(z) = \Phi(t(z); z^2; a, b).$$

The algebraic functions $t(z)$ and $\Phi(z)$ have the same Riemann surface. The function Φ satisfies the equation

$$4z^2(z^2-a^2)(z^2-b^2)\Phi^3+\bigl(-27z^8+36(a^2+b^2)z^6-2(4a^4+23a^2b^2+4b^4)z^4 \\ +8a^2b^2(a^2+b^2)z^2+a^4b^4\bigr)\Phi^2+\bigl(-2(2a^6-3a^4b^2-3a^2b^4+2b^6)z^4 \\ +\bigl(4a^2b^2(a^4-4a^2b^2+b^4)z^2+2a^4b^4(a^2+b^2)\bigr)\bigr)\Phi+a^4b^4(b^2-a^2)^2=0.$$

We calculate the discriminant of the cubic equation (3.14):

$$Q(z)=36(a^4-a^2b^2+b^4)z^8-8(4a^6+3a^4b^2+3a^2b^4+4b^6)z^6 \\ +24a^2b^2(2a^4+a^2b^2+2b^4)z^4-24a^4b^4(a^2+b^2)z^2+4a^6b^6.$$

We analyse the roots of the polynomial $Q(z)$. We have

$$Q(b)=4b^6(b^2-a^2)^3>0,$$
$$Q(a)=4a^6(a^2-b^2)^3<0,$$
$$Q(0)=4a^6b^6>0.$$

Consequently, on the real axis the polynomial $Q(z)$ has at least four roots, $\pm a_*$ and $\pm b_*$, such that $0<a_*<a<b_*<b$. It is easy to show that the biquadratic trinomial $(d/dz^2)^2Q(z)$ is positive on the real axis. Consequently, the polynomial $Q(z)$ has no real roots apart from $\pm a_*$ and $\pm b_*$. The remaining four roots of this polynomial are complex. They are located symmetrically with respect to the coordinate axes. We denote the two complex-conjugate roots in the right half-plane by z_+^+ and z_-^+, and the two complex-conjugate roots in the left half-plane by z_+^- and z_-^-.

It is easy to see that all eight roots of the discriminant $Q(z)$ are branch points of second order for the algebraic functions $t(z)$ and $\Phi(z)$. We make cuts on the complex plane along the segments $\Delta_+=[a_*,b_*]$ and $\Delta_-=[-b_*,-a_*]$ and along some arcs $\Gamma_\pm$, as shown in Figures 3.1 and 3.2. In what follows we discuss only the first case. The second case is similar. A precise definition of the curves $\Gamma_\pm$ will be given later.

Outside this system of cuts, we distinguish three single-valued branches of the function $t(z)$ and three single-valued branches of the function $\Phi(z)$, respectively. We identify these branches by analyzing their asymptotic behaviour as $z\to\infty$. One of the roots of equation (3.14), which is denoted by $t_\infty(z)$, tends to infinity, namely, $t_\infty(z)\sim 3z^2$ as $z\to\infty$. The two other roots tend to constants: $t_\pm(z)\to\theta_\pm$ as $z\to\infty$, where θ_+ and θ_- are roots of the quadratic equation

$$3\theta^2-2(a^2+b^2)\theta+a^2b^2=0,$$

namely,

$$\theta_\pm=\frac{1}{3}\bigl((a^2+b^2)\pm\sqrt{a^4-a^2b^2+b^4}\bigr).$$

We now find out at which points each of the branches has a singularity. From equation (3.14) we express $\zeta=z^2$ in terms of t. We obtain the rational function

$$\zeta=\frac{t(t^2-a^2b^2)}{3t^2-2(a^2+b^2)t+a^2b^2}.$$

The numerator vanishes at the points $t=0$ and $t=\pm ab$. The denominator vanishes at the points $t=\theta_\pm$. Furthermore, $\theta_-<ab<\theta_+$. By analysing the graph of this function, we obtain the following graphs of the functions t_∞ and $t_\pm$ on the real axis (see Figure 3.6).

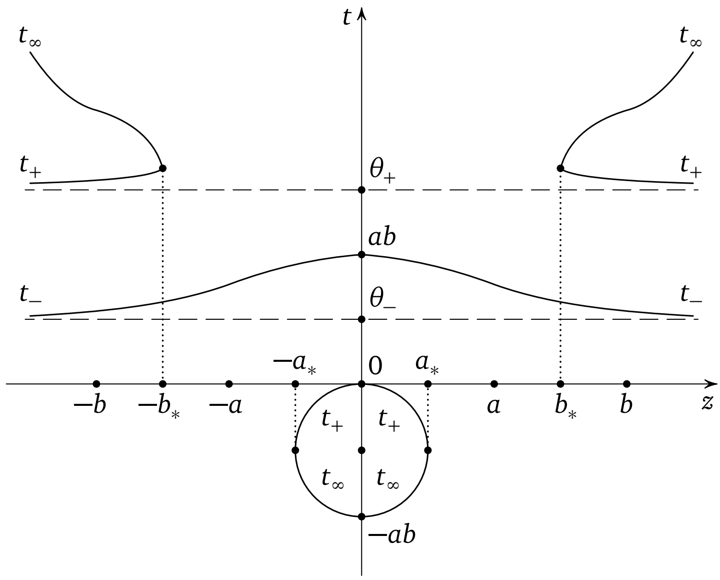

Figure 3.6

We see that the functions t_∞ and t_+ have second order branch points at $z = \pm a_*$ and $z = \pm b_*$, while the function t_- is holomorphic at these points. Consequently, the functions t_+ and t_- have second order branch points at $z^\pm_\pm$, while the function t_∞ is holomorphic at these points. We construct a Riemann surface by gluing together the branches indicated (see Figure 3.7).

This surface has genus $g = 2$.

We continue our analysis of the function $\Phi(z)$. As $z \to \infty$ we have

$$\Phi_\infty(z) \sim \frac{27}{4} z^2, \quad \Phi_+(z) \sim \frac{c_+}{z^4}, \quad \Phi_-(z) \sim \frac{c_-}{z^4},$$

where $c_\pm = \theta_\pm(\theta_\pm - a^2)(\theta_\pm - b^2)$. We observe that $\theta_- < a^2 < \theta_+ < b^2$. Consequently, $c_+ < 0$, $c_- > 0$. We draw the graph of the function Φ on the real axis (see Figure 3.8).

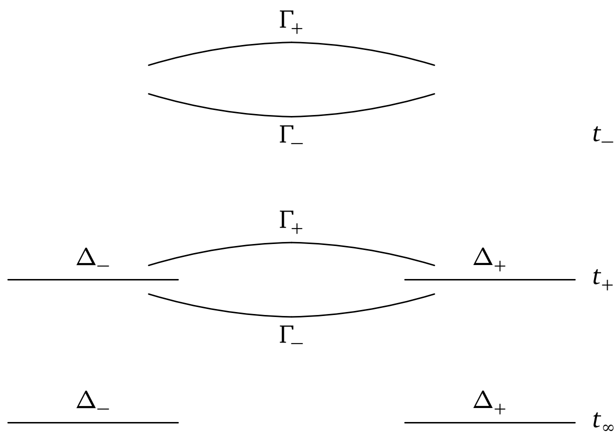

Figure 3.7

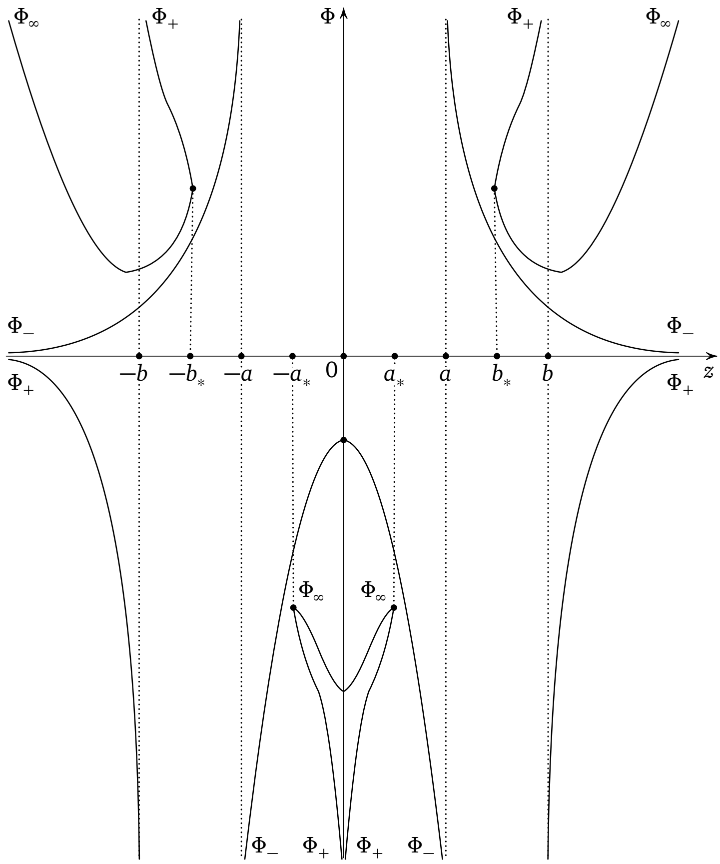

FIGURE 3.8

We now indicate the divisor of the function Φ. At the point $z = \infty$ the function Φ_∞ has a second order pole, while the functions Φ_+ and Φ_- have fourth order zeros. The function Φ_+ has a second order pole at $z = 0$, and first order poles at $z = \pm b$. The function Φ_- has first order poles at $z = \pm a$. We calculate the genus of the Riemann surface using the Riemann–Hurwitz formula

$$g = \frac{1}{2}B + 1 - N,$$

where B is the total ramification of the surface and N is the number of sheets. Since $B = 8$, $N = 3$, it follows, as already noted above, that $g = 2$.

6. We complete the proof of the theorem. Let Γ denote the closed Jordan curve contained in the set $\{z \in \mathbb{C}\colon |\Phi_\infty(z)| = |\Phi_+(z)|\}$ that passes through the points $z = \pm a_*$ and goes around the origin. Uniformly inside the domain situated outside Γ with the cuts $\Delta = \Delta_+ \cup \Delta_-$, we have

$$\lim_{n\to\infty} |Q_n(z)|^{1/n} = |z|^2 \cdot |\Phi_\infty(z)|.$$

Uniformly inside the domain bounded by the curve Γ, we have

$$\lim_{n\to\infty} |Q_n(z)|^{1/n} = |z|^2 \cdot |\Phi_+(z)|.$$

Consequently, the measure λ exists—the limit measure of the distribution of the zeros of the polynomials Q_n and its logarithmic potential is equal to

$$V^{\lambda}(z) = \begin{cases} 2V^{\delta_0}(z) - \ln|\Phi_\infty(z)| + w_0 & \text{outside } \Gamma, \\ 2V^{\delta_0}(z) - \ln|\Phi_+(z)| + w_0 & \text{inside } \Gamma, \end{cases}$$

where $w_0 = \ln(27/4)$. Let λ_Γ denote the balayage of the measure $2\delta_0$ onto the curve Γ. This is a measure of magnitude 2 with support Γ. Then $\lambda = \lambda_\Gamma + \lambda_\Delta$, where λ_Δ is a measure of magnitude 2 with support Δ, and its logarithmic potential is

$$V^{\lambda_\Delta}(z) = -\ln|\Phi_\infty(z)| + w_0 \quad \text{in } \mathbb{C}.$$

We take the Jordan arcs contained in the set $\{z \in \mathbb{C}\colon |\Phi_+(z)| = |\Phi_-(z)|\}$ as the curves Γ_+ and Γ_- .

If $a^2/b^2 < 1/2$, then these curves are situated as shown in Figure 3.1, and if $a^2/b^2 > 1/2$, as in Figure 3.2. The reason is as follows. The set $\Gamma_* = \Gamma_+ \cup \Gamma_-$ is the limit set of the zeros of the functions of the second kind, R_n^b. In the first case the inequality $|c_-| < |c_+|$ holds, that is, $|\Phi_-| < |\Phi_+|$ in some neighbourhood of infinity. In the second case the reverse inequalities hold. Therefore different critical values make the main contribution to the asymptotics of the integral R_n^b in different cases, and as a consequence the picture showing the distribution of the zeros changes.

On the set Γ_* we define a measure λ_* of magnitude 2 such that

$$V^{\lambda_*}(z) = \ln|\Phi_-(z)| - \phi_a(z) - w_1,$$

where $w_1 = \ln|c_-|$. By Vièta's theorem,

$$\ln|\Phi_\infty\Phi_+\Phi_-| = -\ln 4 + 2V^{\delta_0} + \phi_a + \phi_b.$$

Consider the following combination of potentials:

$$\begin{aligned} W_\Delta &= V^{\lambda_\Gamma} + 2V^{\lambda_\Delta} - V^{\lambda_*} + \phi_b \\ &= 2V^{\delta_0} + 2(-\ln|\Phi_\infty| + w_0) - (\ln|\Phi_-| - \phi_a - w_1) + \phi_b \\ &= -\ln|\Phi_\infty^2\Phi_- z^2(z^2-a^2)(z^2-b^2)| + (2w_0 + w_1) = \ln\left|\frac{\Phi_+}{\Phi_\infty}\right| + w_\Delta, \end{aligned}$$

where $w_\Delta = 2w_0 + w_1 + \ln 4$. By construction, $|\Phi_\infty| = |\Phi_+|$ on the cuts Δ and on the curve Γ. Consequently, $W_\Delta = w_\Delta$ on $\mathsf{S}(\lambda)$. The inequality $W_\Delta \geq w_\Delta$ on $[-b, -b_*] \cup [b_*, b]$ is verified by examining the derivative of the function W_Δ directly.

We now consider the following combination of potentials:

$$\begin{aligned} W_* &= 2V^{\lambda_*} - V^{\lambda} - \phi_b + \phi_a \\ &= 2(\ln|\Phi_-| - \phi_a - w_1) - (2V^{\delta_0} - \ln|\Phi_\infty| + w_0) - \phi_b + \phi_a \\ &= \ln|\Phi_-^2\Phi_\infty z^2(z^2-a^2)(z^2-b^2)| + (-2w_1 - w_0) = \ln\left|\frac{\Phi_-}{\Phi_+}\right| + w_*, \end{aligned}$$

where $w_* = -2w_1 - w_0 - \ln 4$. By construction, $|\Phi_-| = |\Phi_+|$ on the curves Γ_+ and Γ_-. Consequently, $W_* = w_*$ on Γ_*. At the same time, this equation means that the curves Γ_+ and Γ_- have the S-property.

In conclusion we return to the limit cases considered in subsections 3 and 4. If $a \to 0$, then $a_* \to 0$, $b_* \to \frac{2\sqrt{2}}{3}b$. The curve Γ and the curves $\Gamma_\pm$ contract to zero, $\lambda_\Gamma \to 2\delta_0$, $\lambda_* \to 2\delta_0$. The equilibrium condition (3.4) disappears. The equilibrium condition (3.3) is transformed into (3.6).

If $a \to b$, then $a_* \to b/3$, $b_* \to b$. The curves Γ_+ and Γ_- contract to the points $+b$ and $-b$, respectively, and $V^{\lambda_*} \to \phi_b$. The equilibrium condition (3.4) again disappears, and the equilibrium condition (3.3) is transformed into (3.13).

The theorem is proved. □

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Translated by E. KHUKHRO

Originally published in Russian