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SOME PROBLEMS CONCERNING THE SOLVABILITY OF THE NONLINEAR STATIONARY BOLTZMANN EQUATION IN THE FRAMEWORK OF THE BGK MODEL

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Abstract. In the framework of the BGK (Bhatnagar–Gross–Krook) model, we derive a system of nonlinear integral equations for the macroscopic variables both in a finite plane channel Π_r of thickness r ($r < +\infty$) and in the subspace Π_∞ ($r = +\infty$) from the nonlinear integro-differential Boltzmann equation. Solvability problems are discussed and solution methods are suggested for these systems of nonlinear integral equations. Theorems on the existence of bounded positive solutions are proved and two-sided estimates of these solutions are obtained for the resulting nonlinear integral equations of the Urysohn type describing the temperature (Theorems 1 and 3). A theorem on the existence of a unique solution in the space $L_1[0, r]$ is proved for the linear integral equations describing the velocity and density. Integral estimates for the solutions are obtained (see Theorem 2 and the Corollary).

The nonlinear system of integral equations in the subspace obtained for the macroscopic variables in the framework of the nonlinear BGK model of the Boltzmann equation is shown to have no bounded solutions with finite limit at infinity other than a constant solution.

The solution of the linear problem obtained by linearizing the corresponding nonlinear system is proved to be $O(x)$ as $x \to +\infty$ (Theorem 3).

Introduction

The complicated structure of the collision integral is well known to cause the main difficulties when solving problems involving the Boltzmann equation. That is why there are various models devised for the collision term. Among these, the BGK (Bhatnagar–Gross–Krook) model [1, 2, 3] is distinguished, in which the collision integral is replaced with the model

$$\mathfrak{J}(f) = f_0^{\mathrm{loc}} - f, \tag{0.1}$$

where f_0^{loc} is the local Maxwell distribution function and f is the desired distribution function.

The BGK model has a number of advantages over the other models, and at the same time it has some shortcomings, most of which can be rectified.

The main advantage of the BGK model is that every specific problem can be reduced to a system of nonlinear integral equations for the macroscopic variables (the density $\rho(x)$, the temperature $T(x)$, and the mass-averaged velocity $u(x)$). These equations are essentially nonlinear and have a more complicated structure than the quadratic nonlinearity of the true collision integral in the Boltzmann equation. The linearized form of the BGK model, which permits exactly solving a number of important problems in kinetic

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theory and comparing the results with experimental data, is another advantage of the model.

Rarefied gas flow problems in a plane channel Π_r of thickness $r \leq +\infty$ are of special importance in the kinetic theory of gases. No exact solution of the Boltzmann equation has been obtained so far for the nonlinear problem on the gas flow and heat transfer between two parallel infinite plates moving with respect to each other (the Couette problem). This problem can be reduced to a system of nonlinear integral equations on the interval $[0, r]$ for the density $\rho(x)$, velocity $u(x)$, and temperature $T(x)$. This system is essentially nonlinear and can only be solved numerically, without a rigorous mathematical justification, in the general case [2, 4].

The gas flow problem in the half-space bounded by a rigid plane wall (the temperature jump problem) is another nonlinear problem. It can be reduced to a nonlinear system of integral equations on the half-line ($r = +\infty$) for the macroscopic variables ρ, u, and T. This system is fairly complicated. On the one hand, the corresponding nonlinear operators are not monotone; on the other hand, they are not compact, which strongly complicates the construction of positive fixed points of these operators. That is why the analysis of the Boltzmann equation in the framework of the BGK model is usually accompanied with an approximate linearization of the nonlinear integro-differential equation. In the linearized BGK model, this system is reduced to a two-dimensional system of integral equations of the Wiener–Hopf type with alternating symmetric matrix kernel. The latter system is a singular (nonelliptic) system of integral equations with a noninvertible matrix integral operator whose symbol has a fourth-order degeneration at zero. Hence the linear temperature jump problem is harder than similar gas kinetic problems described by a linear scalar convolution equation (e.g., the Kramers problem).

There are numerous papers dealing with the linear temperature jump problem (see [5, 6, 7, 8, 9, 10] and the references therein). These papers use and develop various analytical solution approaches to prove the existence of a positive solution with the asymptotics $O(x)$ as $x \to +\infty$ (where x is the distance from the wall).

The present paper discusses solvability problems and suggests solution methods for the above-mentioned nonlinear and linear systems on a finite interval as well as in the half-space.

First, we derive nonlinear integral equations for the macroscopic variables in a finite plane channel Π_r ($r < +\infty$) and in the half-space Π_∞ ($r = +\infty$) from the model Boltzmann equation. Theorems on the existence of bounded positive solutions are proved and two-sided (upper and lower) bounds for these solutions are obtained for the nonlinear integral equations of the Urysohn type for the temperature (Theorems 1 and 3). A theorem on the existence of a unique solution in the space $L_1[0, r]$ is proved for the linear integral equations describing the velocity and density, and integral estimates for the solutions are obtained (see Theorem 2 and the Corollary).

The nonlinear system of integral equations in the subspace obtained for the macroscopic variables in the framework of the nonlinear BGK model of the Boltzmann equation is shown to have no bounded solutions with finite limit at infinity other than a constant solution.

The solution of the linear problem obtained by linearizing the corresponding nonlinear system is proved to have linear growth at infinity (Theorem 3).

1. Derivation of the main equations for the case of $r < +\infty$

Let the channel Π_r bounded by two rigid plane parallel infinite plates occupying the planes $x = 0$ and $x = r$ be filled with a gas. The plates move at the velocities $\pm\omega$ in the direction perpendicular to the axis OX, and the gas flows at the velocity

$\vec{u}(x) = (0, \bar{u}(x), 0)$ in the positive direction of the axis OY. Let $f(x, \vec{s})$ be the distribution function of the particles with velocity $\vec{s} = (s_1, s_2, s_3)$. By the symmetry assumption, the distribution function is independent of y and z.

The steady-state nonlinear BGK Boltzmann equation with a constant collision frequency has the form [1, 2, 3]

$$s_1 \frac{\partial f(x, \vec{s})}{\partial x} = f_0^{\text{loc}}(x, \vec{s}) - f(x, \vec{s}) \equiv \mathfrak{J}(f). \tag{1.1}$$

Here

$$f_0^{\text{loc}}(x, \vec{s}) = \frac{\rho(x) e^{-(\vec{s}-\vec{u})^2/T(x)}}{(\pi T(x))^{3/2}} \tag{1.2}$$

is the local Maxwell distribution function, and $\rho(x)$, $T(x)$, and $\vec{u}(x)$ are the gas density, temperature, and macroscopic velocity, respectively, which can be expressed via moments of the distribution function,

$$\rho(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, \vec{s})\, d^3 s, \tag{1.3}$$

$$\rho(x)\vec{u}(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vec{s} f(x, \vec{s})\, d^3 s, \tag{1.4}$$

$$\rho(x)T(x) = \frac{2}{3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\vec{s} - \vec{u})^2 f(x, \vec{s})\, d^3 s. \tag{1.5}$$

The solution of Eq. (1.1) is sought in the class of functions $f(x, \vec{s})$ defined on $[0, r] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ such that their x-derivatives are bounded and the integrals (1.3)–(1.5) uniformly converge and define continuous functions of x on the interval $[0, r]$. We introduce the following particle velocity distribution functions of $\vec{s} = (s_1, s_2, s_3)$:

$$f^+(x, \vec{s}) = \begin{cases} f(x, s_1, s_2, s_3) & \text{if } s_1 \geq 0, \\ 0 & \text{if } s_1 < 0, \end{cases} \qquad f^-(x, \vec{s}) = \begin{cases} f(x, -s_1, s_2, s_3) & \text{if } s_1 \geq 0, \\ 0 & \text{if } s_1 < 0. \end{cases}$$

The following molecule–plate interaction model is usually considered in the kinetic theory of gases. It is assumed that the ε-fraction of all molecules is reflected (in the plate-fixed frame) with the Maxwell distribution, and the $(1 - \varepsilon)$-fraction is mirror reflected:

$$f^+(0, \vec{s}) = (1 - \varepsilon) f^-(0, -s_1, s_2, s_3) + \frac{\varepsilon c_+}{(\pi T_+)^{3/2}}\, e^{-\frac{s_1^2 + (s_2 + \omega)^2 + s_3^2}{T_+}},$$

$$f^-(r, \vec{s}) = (1 - \varepsilon) f^+(r, -s_1, s_2, s_3) + \frac{\varepsilon c_-}{(\pi T_-)^{3/2}}\, e^{-\frac{s_1^2 + (s_2 - \omega)^2 + s_3^2}{T_-}}.$$

For simplicity, we assume that the particles reflect from the walls ($x = 0$ and $x = r$) diffusely ($\varepsilon = 1$) with the Maxwell distribution.

We supplement Eq. (1.1) with boundary conditions on the upper and lower walls, which have the following form in the case of purely diffuse reflection:

$$f^+(0, \vec{s}) = \frac{e^{-(\vec{s} + \vec{\omega})^2/T_+}}{(\pi T_+)^{3/2}} c_+, \tag{1.6}$$

$$f^-(r, \vec{s}) = \frac{e^{-(\vec{s} - \vec{\omega})^2/T_-}}{(\pi T_-)^{3/2}} c_-. \tag{1.7}$$

It is assumed in the boundary conditions (1.6) and (1.7) that the particles undergo purely diffuse reflection at the walls, i.e., they are first completely absorbed and then isotropically re-emitted. Here T_+ and T_- are the temperatures of the molecules reflected at the upper and lower walls, and the free parameters c_+ and c_- represent the number of particles reflected at the upper and lower walls, respectively. They can be determined if one specifies the gas rarefaction (the mean gas density) between the plates and uses the no-flow condition. One has

$$c_+ = \sqrt{\frac{T_-}{T_+}} c_- .$$

In what follows, we assume that the reflected molecule temperatures T_+ and T_- and one of the parameters c_+ and c_- determining the density are given.

It follows from (1.1) that

$$f^+(x, \vec{s}) = C^+(\vec{s}) e^{-x/s_1} + \int_0^x e^{-(x-t)/s_1} f_0^{\text{loc}}(t, \vec{s}) \frac{dt}{s_1}, \tag{1.8}$$

$$f^-(x, \vec{s}) = C^-(\vec{s}) e^{-(r-x)/s_1} + \int_x^r e^{-(t-x)/s_1} f_0^{\text{loc}}(t, \vec{s}) \frac{dt}{s_1}. \tag{1.9}$$

Here the functions $C^+(\vec{s})$ and $C^-(\vec{s})$ are determined from the boundary conditions (1.6) and (1.7).

In view of (1.8) and (1.9), the boundary value problem (1.1)–(1.7) can be reduced to the following system of nonlinear integral equations [2]:

$$\rho(x) = \gamma_1(h_1(x) + h_2(r-x)) + \int_0^r \Gamma(x, t, T(t)) \rho(t)\, dt, \tag{1.10}$$

$$\rho(x) u(x) = \gamma_1(h_2(r-x) - h_1(x)) + \int_0^r \Gamma(x, t, T(t)) \rho(t) u(t)\, dt, \tag{1.11}$$

$$\rho(x) T(x) = \gamma_1(g_1(x) + g_2(r-x)) + \int_0^r W(x, t, T(t)) \rho(t) T(t)\, dt, \qquad x \in [0, r]. \tag{1.12}$$

Here we use the following notation:

$$\begin{aligned} h_1(x) &= \frac{1}{\sqrt{\pi} T_+} \int_0^\infty e^{-xs} e^{-1/(s^2 T_+)} \frac{ds}{s^2}, \\ h_2(x) &= \frac{1}{\sqrt{\pi} T_-} \int_0^\infty e^{-xs} e^{-1/(s^2 T_-)} \frac{ds}{s^2}, \end{aligned} \tag{1.13}$$

$$\begin{aligned} g_1(x) &= \frac{2}{3\sqrt{\pi}} \int_0^\infty e^{-xs} e^{-1/(s^2 T_+)} \left(\frac{1}{s^2 T_+} + 1 + \frac{(w+u)^2}{T_+} \right) \frac{ds}{s^2}, \\ g_2(x) &= \frac{2}{3\sqrt{\pi}} \int_0^\infty e^{-xs} e^{-1/(s^2 T_-)} \left(\frac{1}{s^2 T_-} + 1 + \frac{(w-u)^2}{T_-} \right) \frac{ds}{s^2}, \end{aligned} \tag{1.14}$$

$$W(x, t, T) = \frac{2}{3\sqrt{\pi T(t)}} \int_0^\infty e^{-|x-t|s} e^{-1/(s^2 T(t))} \left(\frac{1}{s^2 T(t)} + 1 \right) \frac{ds}{s}, \tag{1.15}$$

$$\Gamma(x, t, T) = \frac{1}{\sqrt{\pi T(t)}} \int_0^\infty e^{-|x-t|s} e^{-1/(s^2 T(t))} \frac{ds}{s}, \tag{1.16}$$

$$u(x) = \frac{\bar{u}(x)}{\omega}, \qquad \gamma_1 = \frac{c_+}{\sqrt{T_-}} = \frac{c_-}{\sqrt{T_+}}. \tag{1.17}$$

A *solution* of system (1.10)–(1.12) is understood as a triple (ρ, u, T) of continuous functions defined on the interval $[0, r]$ and satisfying system (1.10)–(1.12).

2. Solvability of the nonlinear system of integral equations (1.10)–(1.12)

System (1.10)–(1.12) cannot be solved exactly in the general case. Note that if $T_+ = T_- \equiv \tau$, then $c_+ = c_- \equiv \alpha$, $\bar{u}(x) = 0$ ($\omega = 0$), and the triple $(\rho, u, T) = (\alpha, 0, \tau)$ satisfies system (1.10)–(1.12). Here $\alpha = (c_+ + c_-)/2$ is the mean gas density.

Note also that the system in question is weakly coupled in the following sense: for any given form of the function $T(x)$, Eqs. (1.10) and (1.11) split into independent linear integral equations for $\rho(x)$ and $\rho(x)u(x)$, and for any given forms of the functions $\rho(x)$ and $u(x)$, Eq. (1.12) becomes a nonlinear scalar Urysohn integral equation for $\rho(x)T(x)$.

These specific features of system (1.10)–(1.12) suggest that the following scheme can be used to solve it.

First, consider Eq. (1.12). To obtain the first approximation to the temperature, we replace the function $\rho(x)$ in (1.12) by its mean value $\rho = \alpha$. Then Eq. (1.12) becomes the following nonlinear integral equation of the Urysohn type for the temperature function $T(x)$:

$$T(x) = \gamma(g_1(x) + g_2(r - x)) + \int_0^r \bar{W}(x, t, T(t))\, dt, \quad x \in [0, r], \tag{2.1}$$

where

$$\bar{W}(x, t, T) = \frac{2}{3\sqrt{\pi}} \sqrt{T(t)} \int_0^\infty e^{-|x-t|s} e^{-1/(s^2T)} \left(\frac{1}{s^2T} + 1\right) \frac{ds}{s}, \tag{2.2}$$

g_1 and g_2 are given by (1.14) with $u = 0$ ($w = 0$), and

$$\gamma = \frac{\gamma_1}{\alpha} = \frac{2\sqrt{T^+T^-}}{\sqrt{T_+} + \sqrt{T_-}}.$$

Once the function $T(x)$ has been found, Eqs. (1.10) and (1.11) become independent linear scalar equations of identical structure.

In what follows, we solve Eq. (2.1) and then consider Eqs. (1.10) and (1.11), where $T(x)$ is a solution of Eq. (2.1).

We prove a theorem stating that the nonlinear equation (2.1) has a positive bounded solution and find two-sided bounds for this solution. It is important that the proof provides a constructive scheme for finding the solution. For a given form of the function $T(x)$, we prove theorems stating the existence and uniqueness of solutions of Eqs. (1.10) and (1.11) in the space $L_1[0, r]$ and obtain integral estimates for these solutions.

Consider Eq. (2.1).

Theorem 1. *Let $W(x, t, T)$, $g_1(x)$, and $g_2(x)$ be given by* (2.2) *and* (1.14)*. Then the nonlinear equation* (2.1) *has a positive bounded solution. This solution satisfies the two-sided inequality*

$$g(x) \le T(x) \le c_0 \equiv \max(t_{01}^2, t_{02}^2),$$

where t_{01} and t_{02} are the positive roots of the algebraic equations

$$t^4 - \gamma t^3 - T_+^2 = 0, \qquad t^4 - \gamma t^3 - T_-^2 = 0, \tag{2.3}$$

respectively, and

$$g(x) = \gamma(g_1(x) + g_2(r - x)), \quad x \in [0, r]. \tag{2.4}$$

Proof. Consider the function

$$\xi(t) = t^4 - \gamma t^3 - \tau^2, \quad t \in R^+ \equiv [0, +\infty),$$

where τ is a given parameter ($\tau = T_+$ or $\tau = T_-$).

Note that $\xi(0) = -\tau^2$, $\xi'(t) = 4t^3 - 3\gamma t^2 \ge 0$ if $t \in [3\gamma/4, \infty)$, and $\xi'(t) \le 0$ if $t \in [0, 3\gamma/4]$. Further, note that $\xi(\gamma) < 0$ and $\lim_{t\to+\infty} \xi(t) = +\infty$. Hence there exists a unique point $t_0 > \gamma$ such that $\xi(t_0) = 0$, and $\xi(t) > 0$ for $t > t_0$. One has

$$t_0^2 > \tau. \tag{2.5}$$

Indeed, the last inequality readily follows from the relation

$$0 = t_0^4 - \gamma t_0^3 - \tau^2 < t_0^4 - \tau^2. \tag{2.6}$$

Let t_{01} and t_{02} be the positive roots of the equations

$$t_0^4 - \gamma t_0^3 - T_+^2 = 0, \qquad t_0^4 - \gamma t_0^3 - T_-^2 = 0, \tag{2.7}$$

respectively. Set

$$c_0 \equiv \max(t_{01}^2, t_{02}^2). \tag{2.8}$$

Consider the following iterations for Eq. (2.1):

$$\begin{gathered} T_{n+1}(x) = \gamma(g_1(x) + g_2(r-x)) + \int_0^r \bar{W}(x,t,T_n(t))\,dt, \\ T_0(x) = c_0, \quad x \in [0,r], \quad n = 0,1,2,\dots. \end{gathered} \tag{2.9}$$

First, let us prove that $T_1(x) \le T_0(x)$. Indeed, in view of the representation (2.2) of $\bar{W}(x,t,z)$, it follows from (2.9) that

$$\begin{aligned} T_1(x) &= \gamma(g_1(x) + g_2(r-x)) + \int_0^r \bar{W}(x,t,c_0)\,dt \\ &= \gamma(g_1(x) + g_2(r-x)) + \int_0^\infty \bar{W}(x,t,c_0)\,dt - \int_r^\infty \bar{W}(x,t,c_0)\,dt \\ &= \gamma g_1(x) + c_0 - I(x) + \gamma g_2(r-x) - \int_r^\infty \bar{W}(x,t,c_0)\,dt, \quad x \in [0,r], \end{aligned} \tag{2.10}$$

where

$$I(x) = \frac{2}{3\sqrt{\pi}}\sqrt{c_0}\int_0^\infty e^{-xs} e^{-1/(s^2 c_0)}\left(\frac{1}{s^2 c_0} + 1\right)\frac{ds}{s^2}. \tag{2.11}$$

It readily follows from (2.10) that if the conditions

$$\gamma g_2(r-x) \le \int_r^\infty \bar{W}(x,t,c_0)\,dt, \qquad x \in [0,r], \tag{2.12}$$

$$\gamma g_1(x) \le I(x), \qquad x \in [0,r], \tag{2.13}$$

are satisfied simultaneously, then $T_1(x) \le T_0(x)$. Let us prove that conditions (2.12) and (2.13) hold. First, we prove (2.12).

Since $x \in [0,r]$ and $t \ge r$ in the integral (2.12), we have

$$\begin{aligned} \int_r^\infty \bar{W}(x,t,c_0)\,dt &= \frac{2}{3\sqrt{\pi}}\sqrt{c_0}\int_0^\infty \left(\int_r^\infty e^{-(t-x)s}dt\right) e^{-1/(s^2 c_0)}\left(\frac{1}{s^2 c_0} + 1\right)\frac{ds}{s} \\ &= \frac{2}{3\sqrt{\pi}}\sqrt{c_0}\int_0^\infty e^{-(r-x)s} e^{-1/(s^2 c_0)}\left(\frac{1}{s^2 c_0} + 1\right)\frac{ds}{s^2}. \end{aligned} \tag{2.14}$$

We will see that

$$\begin{aligned} &\frac{2}{3\sqrt{\pi}}\sqrt{c_0}\int_0^\infty e^{-(r-x)s} e^{-1/(s^2 c_0)}\left(\frac{1}{s^2 c_0} + 1\right)\frac{ds}{s^2} \\ &\qquad \ge \frac{2}{3\sqrt{\pi}}\gamma\int_0^\infty e^{-(r-x)s} e^{-1/(s^2 T_-)}\left(\frac{1}{s^2 T_-} + 1\right)\frac{ds}{s^2}, \quad x \in [0,r]. \end{aligned} \tag{2.15}$$

To this end, it suffices to verify that

$$\sqrt{\frac{1}{c_0}}\, e^{-1/(s^2 c_0)}\left(c_0 + \frac{1}{s^2}\right) \ge \gamma e^{-1/(s^2 T_-)}\left(\frac{1}{s^2 T_-} + 1\right). \tag{2.16}$$

Consider the function

$$\varphi(s^2) = \gamma\sqrt{c_0}\, e^{\frac{1}{s^2}\left(\frac{1}{c_0} - \frac{1}{T_-}\right)}\left(\frac{1}{s^2 T_-} + 1\right), \quad s^2 \in R^+. \tag{2.17}$$

Note that $s_0^2 = (c_0 - T_-)/T_-^2$ is the unique point of maximum of φ. Consequently,

$$\varphi(s^2) \le \varphi(s_0^2) = \gamma\sqrt{c_0}\left(\frac{T_-}{c_0 - T_-} + 1\right)e^{-(1/c_0)T_-}. \tag{2.18}$$

We apply the well-known inequality $e^{-x} \le 1/(1+x)$, $x \ge 0$, and obtain

$$\varphi(s^2) \le \frac{\gamma c_0^2\sqrt{c_0}}{c_0^2 - T_-^2} \tag{2.19}$$

from (2.18). Note that

$$\frac{\gamma c_0^2\sqrt{c_0}}{c_0^2 - T_-^2} \le c_0 \tag{2.20}$$

for $c_0 = \max(t_{01}^2, t_{02}^2)$. Indeed, since the function $\xi_2(t) \equiv t^4 - \gamma t^3 - T_-^2$ is increasing with respect to t on $[t_{02}, +\infty)$, we have

$$\xi_2(\sqrt{c_0}) \ge \xi_2(t_{02}) = 0 \tag{2.21}$$

or

$$c_0^2 - T_-^2 \ge \gamma c_0\sqrt{c_0}, \tag{2.22}$$

because $c_0 \ge t_{02}^2 > T_-$ (see (2.5) and (2.8)). Now (2.20) follows from (2.22).

By combining inequalities (2.19) and (2.20), we arrive at the inequality

$$\varphi(s^2) \le c_0 + \frac{1}{s^2}. \tag{2.23}$$

Now (2.17) and (2.23) imply (2.16).

Thus, we have proved (2.12).

A similar argument can be used to verify inequality (2.13), since $c_0 = \max(t_{01}^2, t_{02}^2) \ge t_{01}^2 > T_+$. We find from (2.12) and (2.13) that $T_1(x) \le T_0(x)$. Assume that $T_n(x) \le T_{n-1}(x)$ for some positive integer n. Since $\bar{W}(x,t,z)$ is increasing with respect to z, it follows from (2.9) that

$$T_{n+1}(x) \le \gamma(g_1(x) + g_2(r-x)) + \int_0^r \bar{W}(x,t,T_{n-1}(t))\,dt. \tag{2.24}$$

Now let us prove that the sequence $\{T_n(x)\}_{n=0}^{\infty}$ is bounded below by the function

$$g(x) = \gamma(g_1(x) + g_2(r-x)). \tag{2.25}$$

First, let us show that

$$T_0(x) \ge g(x), \quad x \in [0,r]. \tag{2.26}$$

Since $t_{01}^2 > T_+$ and $t_{02}^2 > T_-$, we have

$$\begin{aligned} T_0(x) = c_0 &= \max(t_{01}^2, t_{02}^2) \ge \max(T_+, T_-) \ge \sqrt{T_+T_-} \\ &= \gamma\frac{\sqrt{T_+} + \sqrt{T_-}}{2} \ge \gamma(g_1(x) + g_2(r-x)) \equiv g(x), \end{aligned} \tag{2.27}$$

because

$$\begin{gathered} g_1(x) \le g_1(0) = \frac{\sqrt{T_+}}{2}, \qquad g_2(r-x) \le g_2(0) = \frac{\sqrt{T_-}}{2}, \\ \gamma = \frac{2\sqrt{T_+T_-}}{\sqrt{T_+} + \sqrt{T_-}}, \qquad x \in [0,r]. \end{gathered} \tag{2.28}$$

Assuming that $T_n(x) \geq g(x)$ for some $n \in N$ and taking into account the fact that the function $\bar{W}(x,t,z)$ is nonnegative and monotone increasing with respect to z, we find from (2.9) that

$$T_{n+1}(x) \geq \gamma(g_1(x) + g_2(r-x)) + \int_0^r \bar{W}(x,t,g(t))\,dt \geq \gamma(g_1(x) + g_2(r-x)) \equiv g(x).$$

Thus, the sequence $\{T_n(x)\}_{n=0}^{\infty}$ has a pointwise limit as $n \to +\infty$. By the Beppo Levi theorem, this limit satisfies the original equation (2.1) and the two-sided inequality

$$g(x) \leq T(x) \leq c_0 \equiv \max(t_{01}^2, t_{02}^2), \tag{2.29}$$

where t_{01} and t_{02} are the positive roots of the characteristic equation (2.7) with T_+ and T_-, respectively.

The proof of the theorem is complete. □

Remark 1. Since $g(x) \geq \delta_0(r)$, where

$$\delta_0(r) = \int_0^\infty e^{-rs}(G_+(s) + G_-(s))\,ds, \qquad G_\pm(s) = \frac{2\gamma}{3\sqrt{\pi}} e^{-1/(s^2T_\pm)}\left(\frac{1}{s^2T_\pm} + 1\right)\frac{1}{s^2},$$

it follows from (2.29) that, for each finite r, the solution $T(x)$ of Eq. (2.1) is bounded below by $\delta_0(r)$ and above by c_0; i.e.,

$$\delta_0(r) \leq T(x) \leq c_0, \qquad x \in [0,r].$$

Remark 2. In the special case where the walls have the same temperature $T_+ = T_- = \tau$, $\gamma \equiv 1$, and $t_{01} = t_{02} \equiv t_0$, one should take t_0^2 for c_0, where t_0 is the positive root of the equation $t^4 - t^3 - \tau^2 = 0$.

Next, we proceed to Eq. (1.10) under the assumption that the function $T(x)$ in (1.16) is a given positive bounded function,

$$\delta_0(r) \leq T(x) \leq c_0.$$

Consider the following iterations for Eq. (1.10):

$$\begin{gathered} \rho_{n+1}(x) = \gamma_1(h_1(x) + h_2(r-x)) + \int_0^r \Gamma(x,t,T(t))\rho_n(t)\,dt, \quad x \in [0,r], \\ \rho_0(x) = 0, \quad n = 0,1,2,\ldots. \end{gathered} \tag{2.30}$$

One can readily verify by induction that

$$\rho_n(x) \text{ increases with respect to } n, \tag{2.31}$$

$$\rho_n \in L_1[0,r], \qquad n = 0,1,2,\ldots. \tag{2.32}$$

It follows from (2.30) in view of (2.31) and (2.32) that

$$\begin{aligned} \int_0^r \rho_{n+1}(x)\,dx &= \int_0^r h(x)\,dx + \int_0^r \int_0^r \Gamma(x,t,T(t))\rho_n(t)\,dt\,dx \\ &\leq \int_0^r h(x)\,dx + \int_0^r \rho_{n+1}(t) \int_0^r \Gamma(x,t,T(t))\,dx\,dt \\ &= \int_0^r h(x)\,dx + \int_0^r \rho_{n+1}(t)(1-\gamma_r(t))\,dt, \end{aligned} \tag{2.33}$$

where $h(x) = \gamma_1(h_1(x) + h_2(r-x))$ and

$$\gamma_r(t) = \frac{1}{\sqrt{\pi T(t)}} \int_0^\infty e^{-1/(s^2T(t))}(e^{-ts} + e^{-(r-t)s})\frac{ds}{s^2}, \qquad t \in [0,r]. \tag{2.34}$$

Inequality (2.33) implies that

$$\int_0^r \gamma_r(t)\rho_{n+1}(t)\,dt \le \int_0^r h(t)\,dt. \tag{2.35}$$

Since $\gamma_r(t) \ge 0$, it follows from (2.35) and (2.31) that there exists a pointwise limit of the sequence $\{\rho_n(x)\}_{n=0}^{\infty}$, $\lim_{n\to\infty}\rho_n(x) = \rho(x)$, and the limit function satisfies Eq. (1.10) and the inequalities

$$\begin{aligned} \rho(x) &\ge \gamma_1(h_1(x) + h_2(r-x)) \\ &\ge \frac{\gamma_1}{\sqrt{\pi}}\int_0^\infty e^{-rs}\left(\frac{1}{T_+}e^{-1/(s^2T_+)} + \frac{1}{T_-}e^{-1/(s^2T_-)}\right)\frac{ds}{s^2}, \\ \int_0^r \gamma_r(t)\rho(t)\,dt &\le \int_0^r h(x)\,dx, \qquad x\in[0,r]. \end{aligned} \tag{2.36}$$

Now since

$$0 < \delta_0(r) \le T(t) \le c_0, \qquad t \in [0,r], \tag{2.37}$$

we find from (2.34) that

$$\begin{aligned} \gamma_r(t) &\ge \frac{1}{\sqrt{\pi}c_0}\int_0^\infty e^{-1/(s^2\delta_0)}(e^{-ts} + e^{-(r-t)s})\frac{ds}{s^2} \\ &\ge \frac{2}{\sqrt{\pi}c_0}\int_0^\infty e^{-1/(s^2\delta_0)}e^{-rs/2}\frac{ds}{s^2} \equiv \eta_r > 0, \end{aligned} \tag{2.38}$$

because the minimum value of the function $\zeta(t) \equiv e^{-ts} + e^{-(r-t)s}$ on the interval $[0,r]$ is equal to $2e^{-rs/2}$ for each s. In view of (2.38), we find from the second inequality in (2.36) that

$$\int_0^r \rho(t)\,dt \le \frac{1}{\eta_r}\int_0^r h(t)\,dt. \tag{2.39}$$

Let us prove the uniqueness of the solution of Eq. (1.10) in the space $L_1[0,r]$. Suppose the contrary: Eq. (1.10) has two distinct solutions in $L_1[0,r]$, i.e., $\rho^1, \rho^2 \in L_1[0,r]$, and

$$\rho^j(x) = \gamma_1(h_1(x) + h_2(r-x)) + \int_0^r \Gamma(x,t,T(t))\rho^j(t)\,dt, \quad j = 1,2.$$

Then for their difference we obtain the estimate

$$\begin{aligned} \int_0^r |\rho^1(x) - \rho^2(x)|\,dx &\le \int_0^r\int_0^r \Gamma(x,t,T(t))|\rho^1(t) - \rho^2(t)|\,dt\,dx \\ &= \int_0^r |\rho^1(t) - \rho^2(t)|\cdot(1 - \gamma_r(t))\,dt \end{aligned}$$

or

$$\int_0^r |\rho^1(t) - \rho^2(t)|\cdot\gamma_r(t)\,dt \le 0. \tag{2.40}$$

Since $\gamma_r(t) \ge \eta_r > 0$, it follows from (2.40) that $\rho^1(x) = \rho^2(x)$ a.e. on $[0,r]$. Thus, the following theorem holds.

Theorem 2. *Let $T(x)$ be a solution of Eq.* (2.1) *with property* (2.37). *Then Eq.* (1.10) *has a unique positive solution in the space $L_1[0,r]$, and this solution satisfies inequalities* (2.36) *and* (2.39).

Corollary. *Equation* (1.11) *has a unique solution in* $L_1[0,r]$, *and this solution satisfies the following integral estimate*:

$$\int_0^r h_2(x)\,dx - \frac{1}{\eta_r}\int_0^r h_1(x)\,dx \le \frac{1}{\gamma_1}\int_0^r \rho(x)u(x)\,dx \le \frac{1}{\eta_r}\int_0^r h_2(x)\,dx - \int_0^r h_1(x)\,dx. \tag{2.41}$$

Indeed, by linearity, the solution of Eq. (1.11) can be represented in the form

$$\rho(x)u(x) = F_2(x) - F_1(x), \tag{2.42}$$

where the $F_i(x)$, $i = 1, 2$, satisfy the equations

$$F_1(x) = \gamma_1 h_1(x) + \int_0^r \Gamma(x,t,T(t))F_1(t)\,dt, \tag{2.43}$$

$$F_2(x) = \gamma_1 h_2(x) + \int_0^r \Gamma(x,t,T(t))F_2(t)\,dt. \tag{2.44}$$

It follows from Theorem 2 that

$$\int_0^r F_k(x)\,dx \le \frac{\gamma_1}{\eta_r}\int_0^r h_k(x)\,dx, \quad k = 1, 2, \tag{2.45}$$

$$\int_0^r F_k(x)\,dx \ge \gamma_1 \int_0^r h_k(x)\,dx, \quad k = 1, 2. \tag{2.46}$$

We integrate both parts of (2.42) with respect to x from 0 to r, take into account (2.45) and (2.46), and arrive at the integral estimate (2.41).

Having constructed the solution of Eq. (1.10), we proceed to solving the nonlinear equation (1.12) in the second approximation for given continuous functions $\rho(x)$ and $u(x)$ on $[0,r]$.

We introduce the following iterations:

$$\begin{gathered}\rho(x)T_{n+1}(x) = \gamma_1(g_1(x) + g_2(r-x)) + \int_0^r \bar{W}(x,t,T_n(t))\rho(t)\,dt,\\ x \in [0,r], \quad n = 0, 1, 2.\end{gathered} \tag{2.47}$$

For the zero approximation we take

$$T_0(x) = \frac{\gamma_1(g_1(x) + g_2(r-x))}{\rho(x)} = \frac{\gamma_1 g(x)}{\rho(x)}. \tag{2.48}$$

One can readily verify that, by virtue of the monotonicity of the function $\bar{W}(x,t,z)$ with respect to z, the sequence $T_n(x)_{n=0}^{\infty}$ is monotone increasing with respect to n,

$$T_{n+1}(x) \ge T_n(x), \quad n = 0, 1, 2, \ldots, \tag{2.49}$$

and $T_n(\cdot) \in C[0,r]$, $n = 0, 1, 2, \ldots$. By integrating Eq. (2.47) with respect to x from 0 to r, we obtain, in view of (2.49),

$$\begin{aligned}\int_0^r T_{n+1}(x)\rho(x)\,dx &\le \gamma_1 \int_0^r g(x)\,dx + \int_0^r\int_0^r \bar{W}(x,t,T_{n+1}(t))\rho(t)\,dt\,dx\\ &= C_r + \int_0^r \rho(t)T_{n+1}(t)(1 - \lambda(t, T_{n+1}(t)))\,dt,\end{aligned} \tag{2.50}$$

where

$$C_r = \gamma_1 \int_0^r g(x)\,dx, \tag{2.51}$$

$$\lambda(t,z) = \int_0^r W(x,t,z)\,dx = \frac{2}{3\sqrt{\pi z}}\int_0^\infty e^{-s^2/z}\left(\frac{s^2}{z} + 1\right)(e^{-t/s} + e^{-(r-t)/s})\,ds. \tag{2.52}$$

By (2.50), we have

$$\int_0^r \rho(t)T_{n+1}(t)\lambda(t, T_{n+1}(t))\, dt \le C_r. \tag{2.53}$$

The function $\lambda(t, z)$ is jointly continuous on the set $\Omega \equiv \mathbb{R}^+ \times [\delta_0/M, +\infty)$, where $M = \max_{x\in[0,r]} \rho(x)$. It is easily seen that $z\lambda(t, z)$ is monotone increasing with respect to z. Set

$$\mathcal{F}_n(x) = \rho(x)T_n(x)\lambda(x, T_n(x)). \tag{2.54}$$

Let us show that $\mathcal{F}_n(x)$ is increasing with respect to n. We have

$$\mathcal{F}_{n+1}(x) - \mathcal{F}_n(x) = \rho(x)\big(T_{n+1}(x)\lambda(x, T_{n+1}(x)) - T_n(x)\lambda(x, T_n(x))\big). \tag{2.55}$$

Since $T_{n+1} \ge T_n(x)$ and the function $z\lambda(t, z)$ is increasing with respect to z, we see that $\mathcal{F}_{n+1}(x) \ge \mathcal{F}_n(x)$. Thus, we arrive at the following estimate:

$$\int_0^r \mathcal{F}_n(x)\, dx \le C_r. \tag{2.56}$$

By passing to the limit and by using the Beppo Levi theorem and the continuity of $\lambda(t, z)$, we conclude that the limit function $\mathcal{F}(x) = \lim_{n\to\infty} \mathcal{F}_n(x)$ satisfies the inequality

$$\int_0^r \mathcal{F}(x)dx \le C_r. \tag{2.57}$$

Since $\lambda(t, z)$ is continuous on Ω and $z\lambda(t, z)$ is increasing with respect to z, we find from (2.49) and (2.56) that there exists a pointwise limit of the function sequence $\{T_n(x)\}_{n=0}^{\infty}$, $\lim_{n\to\infty} T_n(x) = T(x)$, and

$$\int_0^r \rho(t)T(t)\lambda(t, T(t))\, dt \le C_r.$$

Thus, we have the following theorem.

Theorem 3. *Let $\rho(\cdot) \in C[0, r]$ be the limit of the solutions of Eq.* (2.30)*. Then Eq.* (1.12) *has a positive solution. This solution satisfies the estimates*

$$T(x) \ge \frac{\delta_0}{M}, \qquad \int_0^r T(t)\lambda(t, T(t))\, dt \le \frac{C_r}{m}, \tag{2.58}$$

where $m = \min_{x\in[0,r]} \rho(x)$, $M = \max_{x\in[0,r]} \rho(x)$.

Remark 3. Having constructed the solution of Eq. (1.12), we again proceed to the analysis of the linear equation (1.10) in the second approximation, using the solution of Eq. (2.47) (the second approximation to the solution of Eq. (1.12)). This process is continued until the current and preceding solutions of Eqs. (1.10)–(1.12) become close (with any prescribed accuracy).

Numerical computations show that the difference between the third and second approximations does not exceed 10^{-5}. Heuristically, this permits us to claim that the solution thus constructed is a solution of the original system. Unfortunately, we do not have a rigorous mathematical proof of this claim yet.

3. Derivation of the main systems of nonlinear equations for the case of $r = +\infty$. Their solvability in the space of bounded functions

Let the half-space $x > 0$ filled with the gas be bounded by a plane rigid wall occupying the plane $x = 0$. The gas flows at the mass-averaged velocity $\vec{u} = (0, \bar{u}, 0)$ along the axis OY. The wall moves at the velocity $\vec{\omega} = (0, -\omega, 0)$ in the negative direction of the axis OY. We supplement Eq. (1.1) with the following boundary conditions on the wall and at infinity:

$$f^+(0, \vec{s}) = \frac{e^{-(\vec{s}+\vec{\omega})^2/T_+}}{(\pi T_+)^{3/2}} c_+, \tag{3.1}$$

$$f^-(x, \vec{s}) = O(e^{x/s_1}), \quad x \to +\infty. \tag{3.2}$$

In view of (3.1) and (3.2), it follows from (1.1) that

$$f^+(x, \vec{s}) = f^+(0, \vec{s}) e^{-x/s_1} + \int_0^x e^{-(x-t)/s_1} f_0^{\mathrm{loc}}(t, \vec{s}) \frac{dt}{s_1}, \tag{3.3}$$

$$f^-(x, \vec{s}) = \int_0^\infty e^{-(t-x)/s_1} f_0^{\mathrm{loc}}(t, \vec{s}) \frac{dt}{s_1}. \tag{3.4}$$

Then, in view of (1.3)–(1.5), one can reduce the boundary value problems (1.1), (3.1), (3.2) by standard transformations to the following nonlinear system of integral equations on the half-line:

$$\rho(x) = \gamma_1 h_1(x) + \int_0^\infty \Gamma(x, t, T(t)) \rho(t)\, dt, \tag{3.5}$$

$$\rho(x) u(x) = -\gamma_1 h_1(x) + \int_0^\infty \Gamma(x, t, T(t)) \rho(t) u(t)\, dt, \tag{3.6}$$

$$\rho(x) T(x) = \gamma_1 g_1(x) + \int_0^\infty W(x, t, T(t)) \rho(t) T(t)\, dt. \tag{3.7}$$

One can obtain system (3.5)–(3.7) from (1.10)–(1.12) by formally letting r tend to infinity.

Note that, by analogy with system (1.10)–(1.12), the resulting system is "weakly coupled", and the triple $(\rho, u, T) = (\alpha, -1, \tau)$ (or, which is the same, $(\rho, \bar{u}, T) = (\alpha, -\omega, \tau)$) satisfies system (3.5)–(3.7) with given $T_+ = \tau$. Let us proceed to studying the solvability of this system in the space of bounded functions having a finite limit at infinity. As was already mentioned in the introduction, one main difficulty arising when solving system (3.5)–(3.7) is due to the fact that the corresponding nonlinear integral operators fail to be monotone. Moreover, they are noncompact. However, we will show that this system cannot have a bounded solution with a finite limit at infinity other than the solution $(\alpha, -1, \tau)$.

From the physical viewpoint, it is quite natural to expect that one obtains a bounded solution, because the gas should tend to the equilibrium Maxwell distribution far from the wall.

If one assumes that there exists a bounded solution of system (3.5)–(3.7) with a finite limit at infinity, then this limit is necessarily a Maxwell distribution.

Assume that there exist limits

$$\lim_{x\to\infty} \rho(x) = \rho_\infty < \infty \quad \text{and} \quad \lim_{x\to\infty} T(x) = T_\infty < \infty; \tag{3.8}$$

then one can readily verify that

$$f(\infty, \vec{s}) = f_0^{\mathrm{loc}}(\infty, \vec{s}) = \frac{\rho_\infty e^{-(\vec{s}-\vec{u}_\infty)^2/T_\infty}}{(\pi T_\infty)^{3/2}}. \tag{3.9}$$

Here $\vec{u}_\infty = (0, u_\infty, 0)$ is the value of the mass-averaged velocity at infinity.

In fact, (3.9) follows from (3.3) and (3.4) in view of the following well-known property of the convolution operation:

$$\lim_{x\to\infty}\int_0^x v(x-t)f(t)\,dt = f(\infty)\cdot\int_0^\infty v(x)\,dx, \tag{3.10}$$

$$\lim_{x\to\infty}\int_x^\infty v(t-x)f(t)\,dt = f(\infty)\cdot\int_0^\infty v(x)\,dx. \tag{3.11}$$

We will show that the gas distribution in the BGK model is a Maxwell distribution everywhere up to the wall.

We multiply both sides of (1.1) by $(\ln f+1)$ and integrate over the entire velocity space with regard to the fact that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s^k\mathfrak{J}(f)\,d^3s = 0,\quad k=0,1,2,$$

thus obtaining

$$\frac{\partial}{\partial x}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 f\ln f\,d^3s = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ln f\mathfrak{J}(f)\,d^3s \equiv G(x). \tag{3.12}$$

It is easily seen that $G(x)\le 0$ (see [1]). Indeed,

$$\begin{aligned} G(x) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ln\left(\frac{f}{f_0^{\mathrm{loc}}}\right)\mathfrak{J}(f)\,d^3s + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ln f_0^{\mathrm{loc}}\mathfrak{J}(f)\,d^3s \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_0^{\mathrm{loc}}\left(1-\frac{f}{f_0^{\mathrm{loc}}}\right)\ln\left(\frac{f}{f_0^{\mathrm{loc}}}\right)d^3s \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(1-\lambda)f_0^{\mathrm{loc}}\ln\lambda\,d^3s \le 0, \end{aligned} \tag{3.13}$$

because $1-\lambda$ and $\ln\lambda$ always have opposite signs.

The integral containing $\ln f_0^{\mathrm{loc}}$ in (3.13) is zero, because this variable is a linear combination of summational collision invariants.

Now consider the integral

$$G(x) = \frac{\partial}{\partial x}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 f\ln f\,d^3s \tag{3.14}$$

on the left-hand side in (3.12). We integrate $G(x)$ from 0 to ∞ with respect to x, take into account (3.9), and obtain

$$\begin{aligned} \int_0^\infty G(x)\,dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 f\ln f\,d^3s\Big|_{x=\infty} - \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 f\ln f\,d^3s\Big|_{x=0} \\ &= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 f\ln f\,d^3s\Big|_{x=0}. \end{aligned} \tag{3.15}$$

We use the Cercignani theorem, known as the “remarkable inequality,” and its corollary, the Darrozes–Guiraud theorem (see [1, p. 136, Eq. (4.12)]), to obtain

$$\begin{aligned} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 f\ln f\,d^3s\Big|_{x=0} &\le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 f\ln f_0\,d^3s\Big|_{x=0} \\ &= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 s^2 f\,d^3s\Big|_{x=0} = \frac{-q_1(0)}{RT_+}. \end{aligned} \tag{3.16}$$

Here R is the universal gas constant and $q_1(x)$ is the projection of the heat flux vector. In fact, $q_1(x)$ is zero. Indeed, by multiplying both sides of (1.1) by s^2 and by integrating with respect to d^3s, we obtain

$$\frac{\partial q_1(x)}{\partial x} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s^2 \mathfrak{J}(f)\, d^3s = 0, \tag{3.17}$$

whence it follows that $q_1(x) = \text{const}$. Since

$$q_1(\infty) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s_1 s^2 f\, d^3s \Big|_{x=\infty} = 0,$$

we see that $q_1(0) = 0$.

Thus, on the one hand, it follows from (3.15) that

$$\int_0^\infty G(x)\, dx \geq 0;$$

on the other hand,

$$\int_0^\infty G(x)\, dx \leq 0$$

by the Boltzmann H-theorem (see (3.13)). Consequently,

$$\int_0^\infty G(x)\, dx = 0,$$

i.e., $f_0^{\mathrm{loc}}(x, \vec{s}) = f(x, \vec{s})$. It follows from (1.1) that $f(x, \vec{s})$ is independent of x, and we arrive at the trivial solution

$$f(x, \vec{s}) = f_0^M = \frac{\rho_\infty e^{-(\vec{s}-\vec{u}_\infty)^2/T_\infty}}{(\pi T_\infty)^{3/2}}. \tag{3.18}$$

To ensure that the boundary condition is satisfied, it suffices to take $T_\infty = T_+ = \tau$. If we take α and $-\omega$ for ρ_∞ and $\bar{u}_\infty$, respectively, then we arrive at the particular solution $(\alpha, -1, \tau)$ of system (3.5)–(3.7).

Since the BGK model preserves the main properties of the true Boltzmann collision integral, we can use an argument similar to that for the case of the BGK model to obtain the same result, in the general case, even regardless of the model. In the latter case, the only difference from the BGK model is that one should assume from the very beginning that the gas obeys the Maxwell distribution far from the wall.

Thus, the nonlinear system (3.5)–(3.7) cannot have a bounded solution with a finite limit at infinity other than the constant solution $(\rho, u, T) = (\alpha, -1, \tau)$.

Unfortunately, so far we have not been able to prove the existence of a different solution of the nonlinear system (3.5)–(3.7) with different asymptotics. We tried to construct a bounded solution under various simplifying assumptions about system (3.5)–(3.7) in [11, 12]. Nevertheless, we show in the next section that the corresponding linear system of integral equations obtained by linearization of system (3.5)–(3.7) has a positive solution with the asymptotics $O(x)$ as $x \to +\infty$. From the physical viewpoint, the linear growth of temperature does not describe the problem adequately. Hence one usually solves the Navier–Stokes equations outside the boundary layer in some approximation and determines the macroscopic variables (ρ, u, T). The same macroscopic variables inside the boundary layer are determined from the solution of the Boltzmann equation, and then the two solutions are matched.

4. Linearization of the nonlinear system (3.5)–(3.7)

The functions $\rho(x)$ and $T(x)$ can be represented in the linear approximation as

$$\rho(x) = 1 + f_1(x), \quad T(x) = 1 + f_2(x), \tag{4.1}$$

where $f_1(x)$ and $f_2(x)$ are the density and temperature perturbations. We use the representations (4.1) of $\rho(x)$ and $T(x)$, linearize the functions $\Gamma(x,t,z)$ and $W(x,t,z)$ with respect to z in a neighborhood of zero, retain the first two terms of the expansion, and obtain

$$\Gamma(x,t,T) \approx \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-|x-t|/s} e^{-s^2} \Big(1 + \Big(s^2 - \frac{1}{2}\Big) f_2(t)\Big) \frac{ds}{s}, \tag{4.2}$$

$$W(x,t,T) \approx \frac{2}{3\sqrt{\pi}} \int_0^\infty e^{-|x-t|/s} e^{-s^2} \Big(s^2 + 1 + \Big(s^4 - \frac{s^2}{2} - \frac{1}{2}\Big) f_2(t)\Big) \frac{ds}{s}. \tag{4.3}$$

After some computations, the substitution of (4.2) and (4.3) into (3.5) and (3.7) results in the following system of linear integral equations for $f_1(x)$ and $f_2(x)$:

$$f_1(x) = \bar{h}(x) + \int_0^\infty K_{11}(x-t) f_1(t)\,dt + \int_0^\infty K_{12}(x-t) f_2(t)\,dt, \tag{4.4}$$

$$f_2(x) = \bar{g}(x) + \int_0^\infty K_{21}(x-t) f_1(t)\,dt + \int_0^\infty K_{22}(x-t) f_2(t)\,dt, \tag{4.5}$$

where

$$K_{ij}(x) = \int_0^\infty e^{-|x|/s} G_{ij}(s)\,ds, \quad i,j = 1,2, \tag{4.6}$$

$$\begin{aligned} \bar{h}(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-x/s} \Big(\frac{\gamma_1}{T_+} e^{-s^2/T_+} - e^{-s^2}\Big)\,ds, \\ \bar{g}(x) &= \frac{2}{3\sqrt{\pi}} \int_0^\infty e^{-x/s} \Big(e^{-s^2/T_+} \Big(\frac{\gamma_1}{T_+}\Big(s^2 - \frac{3}{2}\Big) + \gamma_1\Big) - \Big(s^2 - \frac{1}{2}\Big) e^{-s^2}\Big)\,ds, \end{aligned} \tag{4.7}$$

$$G_{11}(s) = \frac{1}{\sqrt{\pi} s} e^{-s^2}, \tag{4.8}$$

$$G_{12}(s) = \frac{1}{\sqrt{\pi} s}\Big(s^2 - \frac{1}{2}\Big) e^{-s^2}, \quad G_{21} = \frac{2}{3} G_{12}, \tag{4.9}$$

$$G_{22}(s) = \frac{2}{3\sqrt{\pi} s} e^{-s^2} \Big(\Big(s^2 - \frac{1}{2}\Big)^2 + 1\Big). \tag{4.10}$$

The solution of system (4.4)–(4.5) is sought in the space of measurable functions having linear growth at infinity.

From (3.6), in view of (4.1) and (4.2), we obtain the following linear scalar Wiener–Hopf integral equation for the macroscopic velocity:

$$u(x) = -\gamma_1 h_1(x) + \int_0^\infty K_{11}(x-t) u(t)\,dt. \tag{4.11}$$

The kernel $K_{11}(x)$ satisfies the conservativeness condition, i.e.,

$$K_{11}(x) \geq 0, \qquad \int_{-\infty}^{+\infty} K_{11}(x)\,dx = 1.$$

The linearization of system (3.5)–(3.7) results in the separation of the problems for the temperature jump and the velocity jump [2].

Note that $G(s) = (G_{ij}(s))_{i,j=1,2}$ is an alternating measurable matrix function satisfying the conditions

$$G(s)s \in L_1^{2\times 2}(0,\infty), \qquad 2\int_0^\infty G(s)s\,ds = I, \tag{4.12}$$

where $I = (\delta_{ij})_{i,j=1}^2$ is the identity matrix. It follows from (4.12) that

$$K \in L_1^{2\times 2}(-\infty,+\infty), \qquad \int_{-\infty}^\infty K(x)\,dx = I. \tag{4.13}$$

We rewrite the system of equations in the form

$$f(x) = \bar{g}(x) + \int_0^\infty K(x-t)f(t)\,dt, \tag{4.14}$$

or, in operator form,

$$(J - \widehat{\mathcal{K}})f = \bar{g}. \tag{4.15}$$

Here $\bar{g} = (\bar{h}, \bar{g})^T$ and $f = (f_1, f_2)^T$ are the given and unknown column vectors, respectively, J is the identity operator, and $\widehat{\mathcal{K}} = (K_{ij})_{i,j=1,2}$ is the matrix Wiener–Hopf operator,

$$(\widehat{\mathcal{K}}f)(x) = \int_0^\infty K(x-t)f(t)\,dt, \tag{4.16}$$

$$K(x) = \int_0^\infty e^{-|x|/s}G(s)\,ds = (K_{ij}(x))_{i,j=1,2}. \tag{4.17}$$

The matrix equation (4.14) is a system of integral equations with a noninvertible matrix integral operator. We will show that the symbol of this operator has a fourth-order degeneration at zero.

Let E be one of the following Banach spaces: $L_p(0,+\infty)$, $p \geq 1$, $M(0,+\infty) = L_\infty(0,+\infty)$, $C_l(0,+\infty)$, etc. (Here $C_l(0,+\infty)$ is the space of continuous functions on $[0,\infty)$ with a finite limit $f(+\infty) = l$ at infinity.) Let $E^2 = E \times E$ be the space of column 2-vectors with components in E.

The invertibility of the operator $J - \widehat{\mathcal{K}}$ in E^2 and other important properties of this operator are determined by the symbol $I - \bar{K}(s)$, where $\bar{K}(s)$ is the Fourier transform of K,

$$\bar{K}(s) = \int_{-\infty}^\infty K(x)e^{isx}\,dx. \tag{4.18}$$

The Fourier transform is taken componentwise. According to the theory of systems of Wiener–Hopf integral equations [13], one necessary condition for the invertibility of the operator $J - \widehat{\mathcal{K}}$ in any of the spaces E^2 is the invertibility of the symbol, i.e.,

$$\det[I - \bar{K}(s)] \neq 0, \quad s \in (-\infty,+\infty). \tag{4.19}$$

However, it follows from (4.12) and (4.13) that condition (4.19) is violated at the point $s = 0$ for system (4.14); i.e., $\det[I - \bar{K}(0)] = 0$, which means that the operator $J - \widehat{\mathcal{K}}$ is not invertible in E^2. Hence system (4.14) (or (4.4), (4.5)) is not covered by the general theory of systems of Wiener–Hopf integral equations and is a special case of such systems.

We will show that system (4.14) has a solution even though the symbol $I - \bar{K}(s)$ has a fourth-order zero at the point $s = 0$.

Our approach is based on the extraction of simple noninvertible factors from the original noninvertible operator $J - \widehat{\mathcal{K}}$ and on the reduction of the original system to a new system with an integral operator that is a contraction operator. This approach is based on a factorization interpretation of the albedo shifting method [8, 9].

By (4.12), it follows from (4.17) and (4.18) that

$$I-\bar{K}(s)=\int_0^\infty \frac{p^2s^2 2pG(p)}{1+p^2s^2}\,dp. \tag{4.20}$$

Let $\beta>0$ be an arbitrary number. Then it follows from (4.20) with regard to (4.12) that

$$\begin{aligned} I-\bar{K}(s) &= \frac{s^2}{s^2+\beta^2}\int_0^\infty \frac{2p^3G(p)(s^2+\beta^2)\,dp}{1+p^2s^2} \\ &= \frac{s^2}{s^2+\beta^2}\left(2\int_0^\infty pG(p)\left(1-\frac{1-p^2\beta^2}{1+p^2s^2}\right)dp\right) \\ &= \frac{s^2}{s^2+\beta^2}\left(I-\int_0^\infty \frac{2pG(p)h(p)\,dp}{1+p^2s^2}\right)=\frac{s^2}{s^2+\beta^2}(I-\bar{T}(s)), \end{aligned} \tag{4.21}$$

where $h(p)=1-p^2\beta^2$ and

$$T(x)=\int_0^\infty e^{-|x|/s}G(s)h(s)\,ds. \tag{4.22}$$

One can readily verify that

$$\bar{T}(0)=\int_{-\infty}^\infty T(x)\,dx=\begin{pmatrix}1-a_{11}\beta^2 & -a_{12}\beta^2\\ -a_{21}\beta^2 & 1-a_{22}\beta^2\end{pmatrix}, \tag{4.23}$$

where

$$A=(a_{ij})=2\int_0^\infty G_{ij}(p)p^3\,dp<+\infty. \tag{4.24}$$

It follows from (4.24) and (4.23) in view of (4.8) and (4.9) that

$$A=(a_{ij})=\begin{pmatrix}1/2 & 1/2\\ 1/3 & 7/6\end{pmatrix}, \tag{4.25}$$

$$\det|I-\bar{T}(0)|=\frac{5}{12}\beta^4>0. \tag{4.26}$$

We introduce the following lower and upper Volterra matrix operators:

$$\begin{gathered} \widehat{\mathbf{U}}^{\pm}=\begin{pmatrix}\widehat{U}^{\pm} & 0\\ 0 & \widehat{U}^{\pm}\end{pmatrix}, \\ (\widehat{U}^{-}\mathfrak{f})(x)=\beta\int_x^\infty e^{-\beta(t-x)}\mathfrak{f}(t)\,dt,\quad (\widehat{U}^{+}\mathfrak{f})(x)=\beta\int_0^x e^{-\beta(x-t)}\mathfrak{f}(t)\,dt. \end{gathered} \tag{4.27}$$

Here $\mathfrak{f}$ is a scalar function in the Banach space E. The symbols of the operator $J-\widehat{\mathbf{U}}^{\pm}$ are given by the formulas

$$I-\bar{U}^{\pm}(s)=\left(\frac{\mp is}{\beta\mp is}\delta_{mj}\right)_{m,j=1,2}. \tag{4.28}$$

Then (4.20) can be rewritten in the form of the matrix product

$$I-\bar{K}(s)=(I-\bar{U}^{-}(s))(I-\bar{T}(s))(I-\bar{U}^{+}(s)). \tag{4.29}$$

Consider the product

$$(J-\widehat{\mathbf{U}}^{-})(J-\widehat{T})(J-\widehat{\mathbf{U}}^{+}) \tag{4.30}$$

of matrix operators, where the $\widehat{\mathbf{U}}^{\pm}$ are defined in (4.27) and $\widehat{T}$ is the Wiener–Hopf integral operator with kernel (4.22). It follows from (4.29) that the symbol of the operator $J-\widehat{\mathcal{K}}$ is equal to the product of symbols of the operators occurring in (4.29). Hence the symbol of the operator (4.30) coincides with that of the operator $J-\widehat{\mathcal{K}}$.

Consequently, we have

$$J - \widehat{\mathcal{K}} = (J - \widehat{\mathbf{U}}^-)(J - \widehat{T})(J - \widehat{\mathbf{U}}^+) \tag{4.31}$$

(equality of operators on E^2).

The operators $J - \widehat{\mathbf{U}}^{\pm}$ in (4.30) are noninvertible in E^2, because $I - \bar{U}^{\pm}(0) = 0$. We see from (4.26) that the symbol of the operator $J - \widehat{T}$ is nonzero at the point $s = 0$ for all $\beta > 0$.

The norm of a scalar Wiener–Hopf integral operator satisfies the estimate [13]

$$\|\widehat{K}\|_E \le \int_{-\infty}^{\infty} |K(x)|\, dx.$$

Let $f \in E^2$ be an arbitrary vector function. Consider the vector function $\varphi = \widehat{T} f$, where $\varphi = (\varphi_1, \varphi_2)^T$, $f = (f_1, f_2)^T$, and $\widehat{T} = (T_{ij})$ is a matrix integral operator.

Let us estimate the norm of the function in each of the spaces E^2. It is easily seen that the estimate

$$\|\varphi\|_{E^2} = \max(\lambda(\beta), \mu(\beta)) \|f\|_{E^2} \tag{4.32}$$

holds in each of these spaces, where

$$\begin{aligned} &\|\widehat{T}_{11}\|_{E\to E} + \|\widehat{T}_{21}\|_{E\to E} \le \|\widehat{T}_{11}\|_{L_1\to L_1} + \|\widehat{T}_{21}\|_{L_1\to L_1} \equiv \lambda(\beta), \\ &\|\widehat{T}_{12}\|_{E\to E} + \|\widehat{T}_{22}\|_{E\to E} \le \|\widehat{T}_{12}\|_{L_1\to L_1} + \|\widehat{T}_{22}\|_{L_1\to L_1} \equiv \mu(\beta). \end{aligned} \tag{4.33}$$

Indeed,

$$\begin{aligned} \|T\varphi\|_E^2 &= \|T_{11}\varphi_1 + T_{12}\varphi_2\|_E + \|T_{21}\varphi_1 + T_{22}\varphi_2\|_E \\ &\le (\|T_{11}\|_{E\to E} + \|T_{21}\|_{E\to E})\|\varphi_1\|_E + (\|T_{12}\|_{E\to E} + \|T_{22}\|_{E\to E})\|\varphi_2\|_E \\ &\le \max(\lambda(\beta), \mu(\beta))\|\varphi\|_E^2. \end{aligned}$$

A sufficient condition for the (matrix) integral operator $\widehat{T}$ to be a contraction operator is that there exists a $\beta > 0$ such that $\sigma = \max(\lambda(\beta), \mu(\beta)) < 1$.

From (4.33), (4.22), (4.8), and (4.9), we obtain

$$\begin{aligned} \lambda(\beta) = \frac{2}{\sqrt{\pi}} \int_0^\infty &\left| \int_0^\infty e^{-x/p} e^{-p^2} (1 - p^2\beta^2) \frac{dp}{p} \right| dx \\ &+ \frac{2}{\sqrt{\pi}} \int_0^\infty \left| \int_0^\infty e^{-x/p} \left(p^2 - \frac{1}{2} \right) e^{-p^2} (1 - p^2\beta^2) \frac{dp}{p} \right| dx, \\ \mu(\beta) = \frac{4}{3\sqrt{\pi}} \int_0^\infty &\left| \int_0^\infty e^{-x/p} \left(p^2 - \frac{1}{2} \right) e^{-p^2} (1 - p^2\beta^2) \frac{dp}{p} \right| dx \\ &+ \frac{4}{3\sqrt{\pi}} \int_0^\infty \left| \int_0^\infty e^{-x/p} \left(\left(p^2 - \frac{1}{2} \right)^2 + 1 \right) e^{-p^2} (1 - p^2\beta^2) \frac{dp}{p} \right| dx. \end{aligned}$$

One can readily verify that $\sigma = \max(\lambda(\beta), \mu(\beta)) < 1$ for $\beta \in (0, 1)$. Numerical computations show that $\sigma = 0{,}946$ for $\beta = 0{,}8$.

Now let us apply the factorization (4.31) to Eq. (4.15). This factorization reduces solving (4.15) to successively solving the following equations:

$$(J - \widehat{\mathbf{U}}^-)Q = \bar{g}, \tag{4.34}$$

$$(J - \widehat{T}^-)H = Q, \tag{4.35}$$

$$(J - \widehat{\mathbf{U}}^+)f = H. \tag{4.36}$$

By writing out the operator equation (4.34), we obtain two independent equations for the components of the vector $Q = (Q_1, Q_2)$,

$$Q_k(x) = \bar{g}_k(x) + \beta \int_x^\infty e^{-\beta(t-x)} Q_k(t)\, dt, \qquad k = 1, 2. \tag{4.37}$$

One can readily verify that the functions

$$Q_k(x) = \bar{g}_k(x) + \beta \int_x^\infty \bar{g}_k(t)\, dt, \qquad k = 1, 2,$$

satisfy Eq. (4.37). It is easily seen that $Q(x) \in L_1^{\times 2}(0, +\infty)$.

Let us solve Eq. (4.35). Since $\widehat{T}$ is a contraction in E^2 with ratio $\sigma < 1$ and $Q \in L_1^{\times 2}$, it follows that Eq. (4.35) has a solution

$$H \in L_1^{\times 2}(0, +\infty). \tag{4.38}$$

Finally, the solution of the original equation (4.36) has the form

$$f(x) = H(x) + \varphi(x), \qquad \varphi = \beta \int_0^x H(t)\, dt \in C_l^{\times 2}(0, +\infty).$$

Let us discuss how to find a nontrivial solution of the corresponding homogeneous equation (4.15) with $\bar{g}_i = 0$. First, note that Eq. (4.34) has the solution $Q = (1, 1)^T$. By solving Eq. (4.35) in a similar way, we obtain $H \in M^{\times 2}(0, +\infty)$. By solving Eq. (4.36), we obtain

$$f_0(x) = H(x) + \beta \int_0^x H(t)\, dt, \quad x \in \mathbb{R}^+. \tag{4.39}$$

Since $H \in M^{\times 2}(0, +\infty)$, we conclude from (4.39) that $f_0(x) = O(x)$ as $x \to +\infty$.

Since the general solution of Eq. (4.14) (or system (4.4), (4.5)) is the sum of solutions f_0 of the homogeneous equation and f of the nonhomogeneous equation, we arrive at the following result.

Theorem 4. *Let $\sigma < 1$. Then the system of Wiener–Hopf integral equations* (4.4)–(4.7) *has a solution with the asymptotics $f_i(x) = O(x)$ as $x \to +\infty$.*

Now consider the shift problem described by (4.11). Equation (4.11) is a conservative Wiener–Hopf integral equation well known in the literature [13, 14]. It was proved in these papers that the nonhomogeneous equation has a solution of the following structure:

$$u_1(x) = \gamma_0 + \psi_1(x) + \psi_2(x), \tag{4.40}$$

where

$$\psi_1(x) \in L_1 \cap L_\infty, \quad \psi_2(x) \in C_0, \quad \psi_i(\infty) = 0, \quad i = 1, 2, \quad \gamma_0 = \lim_{x\to\infty} u_1(x). \tag{4.41}$$

The paper [14] solves the problem on the existence of a limit γ_0 of the function $u(x)$ at infinity and provides a computation of this limit with the help of the Ambartsumyan function.

The solution of the corresponding homogeneous equation ($h_1(x) = 0$) admits the representation [13]

$$u_0(x) = \frac{\sqrt{2}}{\nu_2} x + q(x), \quad x \in \mathbb{R}^+, \tag{4.42}$$

where $q(x)$ is the well-known Hopf function and ν_2 is the second moment of the kernel $K_{11}(x)$.

By linearity, the general solution of Eq. (4.11) can be represented in the form

$$u(x) = u_0(x) + u_1(x). \tag{4.43}$$

Remark 4. It follows from (4.1), (4.42), (4.43), and Theorem 3 that $T(x) = O(x)$, $u(x) = O(x)$, and $\rho(x) = O(x)$ as $x \to +\infty$; i.e., the gas temperature, velocity, and density have linear growth far from the wall.

Remark 5. All results of the present paper can readily be generalized to the case where the mirror reflection of the particles at the wall is taken into account along with the diffuse reflection.

The development of the algorithm and the numerical solution of the problem will be considered elsewhere.

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