Труды Москов. Матем. Общ.
Том 77 (2016), вып. 2

Trans. Moscow Math. Soc.
2016, Pages 149–175
http://dx.doi.org/10.1090/mosc/257
Article electronically published on November 28, 2016

THE MOSCOW MATHEMATICAL SOCIETY AND METRIC GEOMETRY: FROM PETERSON TO CONTEMPORARY RESEARCH

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Abstract. We describe the history of the development of geometric studies related to the work of the Moscow Mathematical Society from its early years to the present day. The main focus is on papers on "geometry in the large".

Introductory remarks

When the Board of Governors of the Moscow Mathematical Society (MMS) suggested that I write a paper of a historical nature with the provisional title "Geometry in the MMS", for quite some time I found it difficult to choose a more specific subject area and a title to go with it.

First, to say that any paper is related to the theme of geometric studies is in itself very ambiguous—the term "geometry" has a very broad interpretation. To illustrate this let me recall a meeting on the topic "Geometry in contemporary mathematics" which was organized on the initiative of V. M. Tikhomirov in March of 2002 for participants in the mathematics section of the Moscow House of Scientists. Four invited speakers gave their interpretations of the given theme: È. B. Vinberg, A. T. Fomenko, I. F. Sharygin, who sent his talk in written form, and me; in addition, the organizer himself summed up the proceedings. In describing what were, in their opinions, the main achievements, goals and problems of this science, all the speakers described different, almost disjoint aspects of geometry. Therefore, upon mature reflection, I decided to narrow the vast topic of geometry to a more concrete area, more familiar to me, which for many years was identified as "geometry in the large", although in recent years the term "metric geometry" is used more and more. It is not very precise[1] but is more convenient to write[2] and it sounds better. As far as I can judge, an analogue of this term, namely "the metric theory of surfaces", was first used in the title of the collection of papers [1]. It was used as an exact equivalent to the term geometry "in the large".

Subsequently this usage became almost universally accepted. Courses of lectures and books appeared using the term "metric geometry" and expounding precisely what interested the geometers who thought of themselves as studying geometry "in the large", conferences were held which were devoted to metric geometry of surfaces and polyhedra, and so on. Of course, we understand that although the areas this term is applied to has widened as the classical subjects involved in research into geometry "in the large" have expanded, nevertheless in the context of this paper this term does not encompass all that can be classified as metric geometry in the full sense of this term.

2010 *Mathematics Subject Classification.* Primary 01A55, 01A60, 01A61, 01A74, 53-03.

Key words and phrases. Moscow Mathematical Society, journal "Matematicheskiĭ Sbornik", Presidents-geometers, characteristic results and surveys on geometric studies, "geometry on the whole".

[1]However, the term "geometry in the large" is also fairly vague: later we shall have an opportunity to speak about *three* of its themes.

[2]We only need to know that the original term did not even have a non-unique written form: the words "in the large" were usually put in inverted commas, but sometimes they weren't!

Second, the history of the foundation of the Moscow Mathematical Society and its work is inseparably connected with Moscow University. To sort out accurately which mathematicians working in Moscow University were members of the MMS, and which formally were not, would be an incredibly laborious task. Therefore I decided not to focus too much on whether or not particular authors were members of the MMS. (The criterion I chose came from the following arguments: the main results mentioned in the paper were published in the journals "Matematicheskiĭ Sbornik" and "Uspekhi Matematicheskikh Nauk", and their connection with the MMS is well known.) As for other areas of geometry, the history of research into these, mainly at Moscow University, can be reconstructed in part from the material in the papers [2–4]; recent results can be found in [5] and in the collection of papers [6][3]. I looked through hundreds of issues of journals, but there is no way I can discuss papers covering all the directions geometry has taken. Nevertheless I have decided to strive at least to mention, either in the text or the bibliography, as many geometers as possible from past generations who have made a noticeable mark on their favourite science and thus pay homage to their blessed memory by showing a proper respect for their names.

1. The structure of geometry "in the large"

The topics which papers in this area of geometry deal with are conventionally thought of as consisting of four directions of study.

1. Manifolds with a metric, in particular, manifolds with a Riemannian metric or Riemannian spaces. In these, the geometry is invisible, that is, it is speculative, all the geometric or characteristic quantities in it have to be calculated, and visualization in it is very conditional. In other words, this is a logically possible geometry. It is not for nothing that Lobachevskiĭ called his geometry *imaginary*.

2. (Isometric) realizations of abstract metrics as metrics of surfaces in spaces. This is called *isometric immersion* of a given abstract metric in a given space. Usually, immersions in Euclidean spaces are the most interesting; in particular, it is interesting to immerse two-dimensional metrics into three-dimensional space.

3. If a metric is isometrically immersed as a metric of a surface or is itself given from the start as a metric of some surface, then the question arises whether this immersion is unique. These questions are dealt with by the *theory of bendings* or, more generally, the theory of *isometric transformations* of a given surface.

4. *The study of various properties* of a surface depending on its metric, external structure, topological and other characteristics.

We can represent everything we have said above in the form of the following scheme.

Manifold with a metric
(This is a seed, in which geometry is in a potential state.)
⇓
Isometric immersion, that is, a surface with a given metric
(This is a plant, in which the potential geometry of the seed develops "visually".)
⇓

[3]To those with a real love of the history of research in geometry I can recommend the paper [7], which has a detailed description by Veniamin Fedorovich Kagan (1869–1953); the work of Soviet geometers in the middle of the 1930s as discussed in [8], and still earlier papers described in [9] and [10]. Unfortunately, the additional issue of the 35th volume of Matsbornik in 1928, which was devoted to successes in mathematics over the first 10 years of Soviet Statehood, which was mentioned in the interesting paper [11] by Lazar' Aronovich Lyusternik (1899–1981), does not contain information about papers on geometry. More general information on mathematical activity in the USSR over the same period of years can be found in [12], and one can read about papers on geometry in the USSR over a 30-year period in [13].

Bendings and isometric transformations of a surface
(This is a set of plants that can be obtained from one seed, or a “harvest”.)
$\Downarrow$
Various properties of surfaces
(These are characteristics of the products from the harvested plants.)

As a rule, isometric realization of metrized manifolds reduces to solving certain differential equations and so the second and third items in our scheme are questions about the *existence* and *uniqueness* of solutions of differential equations.

2. A brief excursion into history

Geometry “in the large” (Geometrie im Grossen) as a name and as a new area of research appeared in papers by German geometers from the end of the 19th to the beginning of the 20th century, but the first results and the first problems posed in it go back to the 18th century. In particular, Euler [14] established a criterion for the local isometry of surfaces (that is, the existence of a one-to-one correspondence between points of two given surfaces under which near a given point the corresponding curves on them have the same length) and conjectured that a closed surface cannot be deformed while preserving the lengths of all curves. (This conjecture, known as *Euler's conjecture on the rigidity of compact surfaces*, has been neither proved nor refuted to this day.) Legendre [15], for different reasons, looked at a similar question for polyhedra and stated a conjecture that two convex polyhedra that are made up in the same way from corresponding congruent faces can be obtained from each other by rotations and translations with the possible addition of a mirror reflection, and proved this in the case of certain concrete simple polyhedra. Cauchy, generalizing Legendre's methods, proved his conjecture for any convex polyhedra [16]. Thus, the Legendre–Cauchy theorem can be regarded as a confirmation of Euler's conjecture in the class of convex polyhedra.

In the first half of the 19th century, apart from the Legendre–Cauchy theorem, two further fundamental results were obtained in metric geometry.

1) Lobachevskiĭ established the possibility that a geometry exists in which Euclid's postulate on parallel lines does not hold (1826).

2) Gauss showed that the curvature of a surface is determined by its metric alone and, consequently, it does not change under bendings of the surface (1828).

Literally on the eve of the formation of the Moscow Mathematical Society, two events occurred that were important for the development of metric geometry: in 1862 in St. Petersburg the two-volume edition of Euler's works “Opera postuma” appeared. These papers, which had not been published in his lifetime, included Euler's paper mentioned above [14]. In addition, Gauss's letter to Schumacher was published in 1865. In the letter Gauss gave an enthusiastic response to Lobachevskiĭ's work on imaginary geometry. We would suggest that both these events must have influenced the research of those active geometers who were among the founders of the MMS.

3. Geometric research in the MMS before the 1940s

It is well known that Lobachevskiĭ's ideas did not find any support or understanding in Russia during his lifetime. The publication [17] of Gauss's correspondence with Schumacher, which appeared in one of the first issues of the journal “Matematicheskiĭ Sbornik” founded by the Moscow Mathematical Society, and of the translation of Lobachevskiĭ's brochure “Geometrische Untersuchungen zur Theorie der Parallellinien” with an accompanying introductory paper by Alekseĭ Vasil'evich Letnikov (1837–1888), a founder-member of the MMS (who concealed his identity, using the abbreviation “A. L-v”) which gave a positive appraisal of the "remarkable but little-known works" of Lobachevskiĭ was, in the words of V. F. Kagan,[4] "the first, though tentative, favourable comment" in Russia on his work.[5] This "caution" was probably due to the fact that, contemporaneously with Lobachevskiĭ, the academician V. Ya. Bunyakovskiĭ was also looking at the problem of parallels but with the intention of proving that Euclid's axiom was valid. The paper [17] also mentions Bunyakovskiĭ's works as achievements in this area. Moreover, the very next issue of the journal contains the translation [18] of two notes of Bertrand's (who characterized the studies of Lobachevskiĭ as a "caprice of abuse by logic") with an explanation and defence of a "proof" of the fifth postulate given by a certain Cartan. It is very curious how Bertrand complained about the critics of "proofs" of the fifth postulate: he claimed they stopped examining such proofs as soon as they encountered arguments of the type "it is obvious that...". Apparently, the authority of Bunyakovskiĭ was such that in 1872 his paper [19] was published with a "proof" of the 5th postulate on the basis of a really fanciful definition of a straight line. (A straight line is an unbounded line on which there are no points that are in any respect different from its other points; for example, a parabola cannot be regarded as a straight line, since the vertex of a parabola is different from its other points by the fact that at this point the line has the greatest value of the curvature.)

Subsequently the problem of parallels as such was no longer discussed in the journal,[6] but for the 100th anniversary of Lobachevskiĭ's birth the paper [20] was published, which gave a very detailed biography and confirmed his priority in the discovery of non-Euclidean geometry, and the paper [21] also supported Lobachevskiĭ's geometry. In this connection we take the liberty of making the following remark: it seems to us that the most amazing thing in this whole story of the *discovery of non-Euclidean geometry* is that although the difference between the formulae in spherical geometry (in particular, for the sum of the angles in a triangle) and the formulae in Euclidean planimetry had been known since antiquity, the recognition that the geometry on a sphere is an *example of non-Euclidean planimetry* (which differs from the Euclidean model, in particular, because of the property that there *does not exist* a straight line passing through a given point parallel to a given straight line) only came in the 1870s, in papers by Klein [22] and other authors after Riemann's dissertation [23] had been published. (The Russian translations of these two papers can be found in [24]; by the way, note that Lobachevskiĭ's geometry only came to be recognized after the discovery that its formulae coincide, at least locally, with the formulae for the intrinsic geometry of a pseudosphere). It was also Klein who pointed out that Riemannian geometry admits realization in two forms, as spherical geometry and as elliptic geometry, and the latter term, according to [25], is also due to Klein.

In general, a heightened interest in geometry among Moscow mathematicians of that era is confirmed, in particular, by the fact that a complete translation of the book by M. Chasles, "History of geometry", was published in "Matematicheskiĭ Sbornik", as a supplement in 19 issues from the 5th to the 10th volumes; this is something which was not done for any work in other areas of mathematics.

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[4] I would like to express my gratitude to A. V. Kostin, who informed me about this statement by V. F. Kagan.

[5] We do not speak of recognition: the author only writes that "Such bold and original ideas, being developed rigorously and consistently, in a purely geometric spirit, — obviously have a right to attention".

[6] We are writing about the history of the MMS, but its pre-revolutionary activity is almost completely reflected in the papers in "Matematicheskiĭ Sbornik", since any paper was accepted for publication only after a report was made on it at a meeting of the MMS, after which it was assigned to one of the members of the MMS, who had to present a detailed review.

After the discovery by Gauss that the curvature of a surface depends only on its metric, the first papers appeared, devoted to studying the possible shapes of surfaces with the same metric: Minding (1838), Codazzi (1856), and Bour (1862)). Euler's posthumous works, published in 1862, contained a paper on isometric surfaces [14] written by his student Mikhail Evseevich Golovin (1756–1790). Apart from the "negative" part (the unproved assertion that a closed surface is not flexible), the paper also contained a positive programme, namely, to determine the bendings for nonclosed surfaces, although this problem for a hemisphere, for example, was characterized as a "most difficult problem" by Euler. In the same paper Euler says that the second surface into which the first given surface must be bent has to be sought in a certain special way.

The person best prepared to take on this programme would, of course, have been Karl Mikhaĭlovich Peterson (1828–1883), one of the founders of the MMS, who was already well known for his contributions in differential geometry. However, he was only later recognized as one of those who had discovered the basic system of equations of the theory of surfaces — the Peterson–Mainardi–Codazzi equations. It was he who expounded, in the very first issue of "Matematicheskiĭ Sbornik" [26], a concrete idea for finding the bendings of a given surface. He suggested that the desired isometric deformations should be restricted by imposing certain additional conditions, making it possible to write out the equations of such bendings, which leads to the possibility of finding the required deformations in terms of the solutions of these equations. His idea was that, apart from the natural requirement that surfaces which are candidates for being isometric should be homeomorphic (which Peterson called the "relation" property), it is also necessary to introduce the property of "affinity consisting in the existence of a connection defined by three or more equations between the coordinates or variables (meaning the intrinsic coordinates of points on the surfaces — *I. S.*) of both surfaces".

The "affinity" condition can of course be interpreted very widely, but in the case when we study the condition for two surfaces to be isometric there are already three equations for the coordinates of their points — these are equalities between the three coefficients of the first quadratic forms of the surfaces under consideration. Peterson shows that for two isometric surfaces one can always choose common intrinsic coordinates (u, v) in such a way that the parametric lines $u = \text{const}$, $v = \text{const}$ are conjugate on both surfaces. Peterson calls such lines a *base of bending*. If these lines retain the property of being a base of bending for a *continuous* family of isometric surfaces, then they are called a *principal base* of bending.

It is interesting and important to note that in this problem Peterson again found himself, if not ahead of, then at least on a par with the leading European geometers. One of them, O. Bonnet, published a long paper in 1865–67 [27] in which he proposed to search for isometric surfaces under the condition that the principal curvatures were preserved or, equivalently, that the mean curvature was preserved. It is interesting to remark that, as it turns out, bendings "à la Peterson" and bendings "à la Bonnet" have one common property: namely, if there are *three* isometric surfaces with a common base à la Peterson, then there are also bendings over a principal base [26]; similarly for bendings "à la Bonnet" if there are *three* isometric surfaces with the same mean curvature, then there is also a continuous family of such surfaces. (Apparently, this property had been known for a long time, but the first proof known to us was given in [28]; see a more detailed proof in [29].) There are probably grounds for trying to connect these two approaches to choosing "candidates" for finding pairs of isometric surfaces.

Peterson not only indicated ways of studying the problem of bending surfaces but also used his method to obtain some explicit families of bendings for many types of surface; in particular, he wrote out in explicit form one class of bendings for all surfaces of the

2nd order [30]. The origins of this result still baffle us today, since up until now, by and large only existence theorems or approximate methods for finding isometric deformations of surfaces had been known, and mainly for sufficiently small deformations. (Even at the time B. K. Mlodzeevskiĭ conjectured [31] that apparently "Peterson had more general methods for solving the problem of bending of surfaces, but these are not known to us".) To illustrate what we have said above, following [31] we write out the equations of a family of surfaces

$$x = r\cos\varphi\cos u, \quad y = r\sin\varphi\cos u, \qquad (1)$$
$$z = \int \sqrt{a^2(1-t^2)\sin^2 u + c^2\cos^2 u}\, du,$$

where

$$r = \sqrt{a^2t^2-(a^2-b^2)\sin^2 v}, \quad \varphi = \int \frac{a\sqrt{t^2(a^2\sin^2 v + b^2\cos^2 v)-(a^2-b^2)\sin^2 v}}{a^2t^2-(a^2-b^2)\sin^2 v}\, dv,$$

which represent a bending of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(The paper [30] itself only gives the initial and final equations of the surfaces in the family, for the values of the parameter $t = 1$ and $t = 0$, respectively.)

Here we should point out that, of course, an ellipsoid as a convex surface does not bend and is in fact uniquely determined by its metric. However, up until the beginning of the 1930s in differential geometry both the coordinates of points on a surface and its intrinsic coordinates were regarded as complex variables and accordingly were used as imaginary asymptotic lines on surfaces of positive curvature, imaginary minimal surfaces, spheres of zero and imaginary radius, and so on. (For example, probably few modern geometers would be able to write out the equation of a sphere in terms of its imaginary rectilinear generatrices, while this was considered to be common knowledge; see [58, p. 39]). This makes it very difficult to understand and interpret results from that period; the results sometimes do not hold in real domains, as Cartan noted (see [32]). Peterson understood this, and in his studies he indicated conditions under which his results were true in real domains. Thus, the real bendings in (1) relate only to certain domains on an ellipsoid, but at the same time they give an idea of how bendings can be extended to the complex domain in such a way that the whole ellipsoid is deformed as an object in complex space with its metric preserved. This is an interesting and, as far as we know, as yet unexplored problem.

To continue our discussion of Peterson's research, we note that even though he published his results in a separate book [33] in German, for a long time these results were not known in Europe. They were rediscovered there much later, by some very eminent scientists (Darboux, Lie, Bianchi, and others). Only in the middle of the 1890s did his papers attract attention in the West, and then they were quickly translated into French and published. The translator had to invent French expressions for some of the mathematical terms which were first introduced by Peterson in Russian. (For example, the term "principal base" was translated, revealing its mathematical meaning as "réseau conjugué persistant", that is, literally as "persistent conjugate network".) Therefore we can say that not only did Peterson come up with the idea of searching for bendings of surfaces with certain prescribed properties of the desired isometric deformations simultaneously with Bonnet, but also in the concrete results he obtained along the way, he preceded the achievements of Western geometers by several decades. There is a good description of this in [31] and in the detailed analysis of Peterson's contribution to differential geometry in [34].

We note that not only was Peterson a founder member of the MMS but right up until his death he participated actively in its work; he gave 18 talks at its meetings over the 15 years of his membership. Respect for him and his influence on the study of geometry in Russia was such that 20 years after his death at a meeting of the MMS on the 18th of January, 1905 the question of publishing his works was discussed, and at the meeting of the 15th of February, 1905 the decision was made to do so. Unfortunately, this decision could not be implemented. The same suggestion was discussed again later, for example in the above paper [34]. The author of this survey is in agreement with the opinion expressed there.

Peterson did not have direct students, but his studies in the theory of bendings over a principal base and on differential geometry in general remained prominent as a topic in the papers published by Moscow geometers over almost 75 years. Dissertations were defended on this topic (B. K. Mlodzeevskiĭ (1886), S. S. Byushgens (1917), S. P. Finikov (1917), S. V. Bakhvalov (1940), S. D. Rossinskiĭ (1943), et al.) and the papers [35–57] were devoted to this topic.)

We have cited only those papers published in "Matematicheskiĭ Sbornik". Bearing in mind that from 1866 to 1935 a little more than 100 papers were published in this journal on all aspects of geometry, and that after this the MMS ceased to be its publisher, we find that about a quarter of all the papers on geometry were devoted to bendings of surfaces over a principal base. (In addition, a considerable number of papers were devoted to the theory of congruences; see the bibliography in [58] and [59].) Furthermore, we must take account of the fact that a considerable number of papers by Moscow geometers on these topics appeared in foreign journals. These papers can be found in the bibliography given in [60], which has about 150 titles. (The bibliography in [60] contains five foreign publications on this topic by D. F. Egorov alone.) In addition, a large part of the bibliography in this book is comprised of papers by foreign authors, which gives us the right to say that for many geometers Peterson formulated a programme of research lasting several decades.[7]

Boleslav Kornelievich Mlodzeevskiĭ (1858–1923) and Dmitiĭ Fedorovich Egorov (1869–1931), who both in their time were Presidents of the MMS, were prominent representatives of the Moscow school of metric geometry. Being very cultured and with high moral standards, they greatly facilitated the establishment of the MMS and the way its authority grew in the scientific community. They were personally active in giving talks at its meetings. (It suffices to say that Mlodzeevskiĭ gave 66 talks at meetings of the MMS, and Egorov gave 41.) Mlodzeevskiĭ was known as an active member of the MMS even before he became President because, as secretary of the MMS since 1892, at a meeting of the MMS on 20 September, 1905 he proposed that officers of the MMS be elected not indefinitely but for a certain term. This was adopted at the next meeting. Egorov, as President, secured a change to the Charter of the MMS so that the journal "Matematicheskiĭ Sbornik" could publish papers not only in Russian (as was the case under the first Charter) but also in the main European languages, which immediately heightened its prestige.

[7]But we must note that in the MMS they were also interested in other topics in the theory of surfaces. For example, the papers by Nikolaĭ Borisovich Delone (1856–1931) [61] and Leonid Kuz'mich Lakhtin (1863–1927) [62], both members of the MMS, considered the question of representing one-sided surfaces, in particular the Möbius band, by explicit equations, and in [61] the problem was first posed of finding a parametric representation of the Möbius band with a locally Euclidean metric and one version of its solution was given. This was many decades prior to the Western geometers Sadovsky, Wunderlich and G. Schwarz.

Even in his Magister dissertation [63] Mlodzeevskiĭ showed a broad knowledge of contemporary geometric research, and this dissertation of his, apart from its serious scientific content, was, effectively the first survey paper on the bending of surfaces. In it he also revealed the significance of Peterson's papers and subsequently he repeatedly returned to explaining and advancing Peterson's ideas and results; see the papers [31,36, 37]. He was the first in Russia to write a paper on infinitesimal bendings [38]. In it, in modern terminology, he studied the invariance of the property of flexibility of a surface under certain transformations of space, including affine transformations as a special case. In his doctoral dissertation "On manifolds of many dimensions" (1890), he also considered the then new topic of multi-dimensional Riemannian spaces, establishing some differential invariants of Riemannian metrics. (In the two-dimensional case, a similar problem was considered in § 94 of the book [64].) A more detailed biography of Mlodzeevskiĭ and a description of his contribution to geometry can be found in [65] and [66].

A younger contemporary and a colleague of Mlodzeevskiĭ's, Egorov is better known for his results in the theory of functions of a real variable (see [67]), but he made his mark in differential geometry through his dissertation [68] (also published in [69, p. 33–284]), in which he introduced a new class of so-called *potential* surfaces, which were included in Darboux's monograph [70] (Book III, Chapters VIII and IX) as E-surfaces.[8] It is worth looking at this paper of his in more detail, since it has proved to be in great demand in the most recent research in differential geometry, related to problems in mathematical physics and field theory. Egorov says that a surface is potential if orthogonal curvilinear coordinates (u, v) can be introduced on it in which the metric of the surface has the form

$$ds^2 = \frac{\partial\omega}{\partial u}\, du^2 + \frac{\partial\omega}{\partial v}\, dv^2$$

with some function $\omega(u, v)$, which he calls a main function and, in the kinematic interpretation, connects with the potential of the velocity field of a stationary flow of a compressible fluid on the surface. In the spatial case, he calls a triorthogonal system of coordinates (u, v, w) a system of *potential type* if the (planar) metric of the space can be represented in the form

$$ds^2 = \frac{\partial\omega}{\partial u}\, du^2 + \frac{\partial\omega}{\partial v}\, dv^2 + \frac{\partial\omega}{\partial w}\, dw^2$$

in it, with a so-called main function $\omega(u, v, w)$. Obviously, the coordinate surfaces in this system are potential surfaces,[9] and Egorov established many interesting and important properties for them. Potential surfaces and systems of orthogonal coordinates of potential type (these and certain analogues of them became known as *Egorov* coordinates) started being actively used after the paper by S. P. Novikov and B. A. Dubrovin [75]. (Novikov was President of the MMS from 1985–1996 and is now an Honorary President.) In it they showed that a quasilinear system of hydrodynamical type for n unknowns $U^i(x, t)$

$$\frac{\partial U^i}{\partial t} = \sum V^i_k \frac{\partial U^k}{\partial x}, \quad 1 \le i \le n, \tag{2}$$

is Hamiltonian if the matrix (V^i_k) can be represented in a certain special form involving the Hamiltonian and some tensor g^{ij} defining a planar metric form ds^2 in $\mathbb{R}^n$. Conditions

[8] A very detailed description of his mathematical and, in particular, his geometric results can be found in [71]; see also [72]. For a biography and his spiritually-philosophical views see [73].

[9] Generally speaking, this paper of Egorov overlaps considerably with the work of the French mathematicians; in particular, Darboux praises Egorov's results as a remarkable description of surfaces which correspond to his hopes for the answer to a problem which he first studied back in 1866; Egorov himself, in [74], which was produced as a complement to [68] and as a remark on parallel similar results due to Fouché, wrote that "the orthogonal systems under consideration (by Egorov — *I. S.*) can be defined by the fact they *admit a continuous group of Combescure transformations*".

for this system to be integrable were found in [76]; these included the requirement that the metric form ds^2 be orthogonal, and this led to the need for Egorov's results and generalizations of them to the multi-dimensional case. This was one of the first reasons why interest in his papers arose. Subsequently, a number of papers appeared in which this geometric approach was used to solve more complicated systems of equations of hydrodynamical type and to pose new problems; see, for example, the survey [77], the papers [78–81] (which include a detailed description of the connection between these systems and the flexibility of surfaces), [82–85], and the paper [86] which has a good historical survey.

But Egorov's main achievement is his contribution to establishing and raising the standard of mathematical education at Moscow State University and supervising a large number of students who subsequently brought fame to Moscow University; these include Nikolaĭ Nikolaevich Luzin (1883–1950), the founder of the famous Luzitaniya. In geometry, among the students supervised jointly by Mlodzeevskiĭ and Egorov, Sergeĭ Sergeevich Byushgens (1882–1963) and Sergeĭ Pavlovich Finikov (1883–1964) became well-known scientists, and both were active members of the MMS. (In particular, Finikov was the librarian and secretary of the MMS for many years, and in 1953, at the meeting of the MMS devoted to the 70th anniversary of his birth, he was elected an honorary member of the MMS.) Their biographies and details of their scientific achievements can be found in [87] and [88], respectively, and also in [73]. Finikov's scientific works are also described in detail in V. T. Bazylev's paper [89].

4. Metric geometry in the MMS since the 1940s

The second half of the 150-year history of the MMS began with significant changes not only in the MMS itself and in the fate of the journal "Matematicheskiĭ Sbornik", but also in the areas of research in geometry. The fact is that classical differential geometry was essentially local, and furthermore, investigations were carried out under the assumption that the points under consideration were of a single type (that is, they were points in general position), and that the objects under consideration were of a more formally algebraic rather than a descriptive geometric nature. But results with an explicit geometric content were already appearing such as, for example, the Lyusternik–Shnirel'man theorem (1930) on the existence of three closed geodesics on an arbitrary ovaloid and even, more abstractly, on any two-dimensional manifold of positive curvature homeomorphic to a sphere, Cohn-Vossen's theorem (1936) on the rigidity of an ovaloid (these are examples of results for surfaces in the large), theorems on isometric immersions of multi-dimensional metrics and bendings of multi-dimensional surfaces (examples of results in multi-dimensional geometry due to C. Burstin, 1931).[10]

In the last issues of the old series of "Matematicheskiĭ Sbornik" geometers, such as Sergeĭ Vladimirovich Bakhvalov (1898–1963), Nikolaĭ Vladimirovich Efimov (1910–1982), Pëtr Konstantinovich Rashevskiĭ (1907–1983) and others, who subsequently became well known, succeeded in publishing work on their new areas of research. The paper [92] considers an example of how the new approaches can influence earlier problems. It looks at a method of expounding the topics of bending over a principal base in [93, § 86], different from the universally accepted one (in [60], say), where, starting from the Efimov's papers, bases of bending are explained very graphically as directions of extremal bending, which

[10]Tselestin Leonovich Burstin (1888–1938), an academician of the Academy of Sciences of Belorussian SSR who was a victim of political repression in 1937, published his papers in Matsbornik in German [90, 91]; therefore his first name and patronymic are unknown to a wide audience.

are uniquely defined everywhere except for the so-called points of congruence, at which the normal curvatures of isometric surfaces coincide in all directions.[11]

Although new trends were gradually displacing the interest in the study of bendings over a principal base which had been cultivated for several decades, a truly dramatic end to this direction of geometry came with Luzin's paper [94],[12] in which he proved that the existence of bending over a principal base is an extremely rare phenomenon in the sense that almost no analytic surface has a principal base. We reproduce word-for-word his blunt comments about papers on this topic:

"All this forces us to regard the property that surfaces "have a principal base" as a very special property which occurs only in exceptional cases. In our opinion, there is no comparison between the network of principal bending and the network of curvature lines or asymptotic lines, or the family of geodesic lines and so on. The latter are present on every analytic surface, and their investigation quite rightly forms chapters in classical differential geometry, while the phenomenon of a principal base is exceptional, and the study of this rare phenomenon, which rightfully deserves the title 'Peterson's phenomenon', can comprise not a chapter but merely a section in differential geometry".

At the same time, Luzin, probably wishing to soften the excessive harshness of his words, writes in a footnote that this is merely "the personal opinion of the author, who is by no means a classical geometer", and he further pays tribute to earlier studies on this topic, noting that "the Peterson phenomenon is deep and therefore in the future it will attract the deserved interest of geometers and analysts" and he indicates its unexpected and deep connection with other parts of mathematics. Note that when Luzin questioned the insufficient generality of the topic of bending over a principal base this was apparently founded on his shrewd observation that all the results on bendings over a principal base known at that time related to concrete cases of lines that were principal bases; these lines were either conical, cylindrical, planar or spiral, and so on. But all this relates to bendings over a principal base, that is, to *continuous* isometric deformations, rather than to bendings over an ordinary base. A similar situation is also true for isometric deformations which preserve the mean curvature. In any case, as we already mentioned at the beginning of our survey, the question of the connection between bendings "à la Peterson" and "à la Bonnet" remains unexplored. There are grounds for thinking that in fact an investigation of this subject could have been found in the works of geometers of that period.

Bendings and infinitesimal bendings of surfaces in a modern formulation were already attracting the attention of young geometers even before Luzin's paper appeared. In particular, in 1936, in the first volume of the new series of "Matematicheskiĭ Sbornik", Aleksandr Danilovich Aleksandrov (1912–1999), who subsequently became an honorary member of the MMS, published his paper [95]. In it he was the first to consider the question of infinitesimal bendings in the class of general convex surfaces (that is, without an a priori assumption of smoothness on the surface and on the bending field) and he proved that general closed convex surfaces of revolution are rigid. A long survey article by Cohn-Vossen [96] on bendings, with a proof of the rigidity of C^3-smooth ovaloids also appeared in the first issue of the new journal "Uspekhi Matematicheskikh Nauk", and the author was made a member of the editorial board of this journal.

[11] For comparison we give the definition of conjugate directions in [26]: "Two systems of curves intersecting on a surface are said to be conjugate if the tangent plane of the surface constructed with respect to a curve of one system intersects in two consecutive positions in tangent curves of the other system". And yet another version, no less convoluted, to explain conjugacy is given. We can only be amazed that people could work with these definitions!

[12] He published a shorter paper on this topic: a brief exposition appeared in Doklady Akad. Nauk SSSR in 1938.

At the end of the 1930s Efimov (also an active member of the MMS and on its Board of Governors for many years) started to study the same topic and immediately obtained world-class results by showing that in a neighbourhood of a point of flattening, surfaces can even be locally rigid. We should emphasise that the second key feature of the term geometry "in the large" shows up in these papers. The first key feature, of course, consists in contrasting the words "locally" and "in the large". As we have already pointed out above, almost all research in geometry in the 19th century was devoted to studying the properties of surfaces in a small neighbourhood of a given point (to confirm this, just browse through Darboux's multivolume opus "Lectures on the theory of surfaces" recently translated into Russian), while geometry "in the large" considers surfaces in their entire expanse, and, as a result, the topology of the surface is important in obtaining results, as is its behaviour near the border if one exists.

Moreover, in the 19th century geometry looked at a surface in a neighbourhood of not any given point but in a neighbourhood of a point in general position, while geometry "in the large" does not leave any point out of consideration, including all points without exception. This is its second key feature. (By the way, in a neighbourhood of any point of flattening the whole of the preceding theory of bendings over a principal base fails to work, confirming how true Luzin's criticism was.) As a result, in accordance with these ideas which emerged in Germany and were apparently communicated to him by Cohn-Vossen, Efimov started to study the following problem: in a neighbourhood of a point in general position, any surface (at least, in the analytic class) is flexible; what happens in a neighbourhood of a point of flattening? He proved that there exist analytic surfaces that are rigid in any arbitrarily small neighbourhood of a point of flattening.[13] These results, which comprised the contents of his doctoral dissertation (1940), were published in a series of papers [98–100] and were subsequently published as a monograph [101], which was given the Lobachevskiĭ Prize of the Academy of Sciences of the USSR in 1950. It is important to point out that when Efimov started his research he was in the position of playing "catch up". By 1938 several papers by Hopf and Schilt had already been published in Germany on this topic. In parallel with these papers, Efimov published a key paper [102] in "Uspekhi Mat. Nauk" which was fundamental, and to this day serves as an introduction to the theory of bendings and infinitesimal bendings, giving proofs of many of the theorems[14] and posing numerous problems, among which some are still unsolved.

The reader has possibly already noticed that, when speaking about Efimov's results (and also those of Aleksandrov and Cohn-Vossen), we indicate in which smoothness class they are obtained. In this, the third key property of geometry "in the large" manifests itself: making the smoothness class in which one or other result is true more precise. Often, what is true in one smoothness class turns out to be false in another. (The work of geometers in previous generations did not pay any attention to this, but Luzin makes it quite clear that he is talking about analytic surfaces.) I remember how happy Efimov was when, in 1973 in a joint paper with Z. D. Usmanov, he managed to extend a result in a paper of his published 25 years before [103] on the existence of locally rigid surfaces in the class C^∞ to the case of surfaces of analytic class. (Rigidity of a surface means that it does not admit infinitesimal bendings of the 1st order.) In connection with the fact that smoothness conditions play an essential role in what follows, we consider this question in a separate subsection. But before we go on we make the following remark.

[13] Of course, there also exist cases when a surface is flexible in a neighbourhood of a point of flattening; see, for example, [97].

[14] In particular, in it Efimov presented his own proof of Herglotz's formula, which he knew about only through a description in a review journal; Efimov's proof was valid in a wider smoothness class than was the author's version.

Remark. We would like to emphasize once again that we are mainly discussing the results of Moscow geometers in the theory of isometric immersions and isometric deformations of surfaces (that is, we are looking at parts 2 and 3 of the scheme we gave at the start of this paper describing the directions which papers in metric geometry took), but along with these investigations, in the MMS many other aspects of geometry were investigated intensively and successfully. This can be verified by simply looking at the lists of talks at the meetings of the MMS over the periods 1946–1960 and 1961–1986. The talks were published in "Uspekhi Mat. Nauk" (1961), **16**:6, and "Uspekhi Mat. Nauk" (1987), **42**:1. A substantial contribution to the popularization of geometry was made by the publication of the 5th volume of the Encyclopedia of Elementary Mathematics [104], which was devoted to geometry. The authors of articles in this were, in particular, the members of the MMS V. G. Boltyanskiĭ, Vadim Arsen'evich Efremovich (1903–1989), Boris Abramovich Rosenfel'd (1917–2008), Isaak Moiseevich Yaglom (1921–1988), some of whom also gave talks at meetings of the MMS. Many results due to members of the MMS and other mathematicians are reflected in survey papers published in the geometric volumes of the VINITI series "Contemporary problems in mathematics. Fundamental directions" Geometry-1 (1988), the paper [106], Geometry-2 (1988), the papers [105] and [107], Geometry-3 (1989), the papers [108, 122, 141], and Geometry-4 (1989), the papers [109] (its author Yu. G. Reshetnyak is an honorary member of the MMS) and [110]. Among publications close to the subject of this survey, we should mention the papers byI. K. Babenko [111, 112] (shortest geodesics and systolic geometry, which can be related to the first direction of research in our scheme of the structure of geometry "in the large"), A. T. Fomenko [113], A. O. Ivanov and A. A. Tuzhilin [114] (calculus of variations, which can be related to the fourth direction in the scheme). A new topic in the theory of isometric immersions which we would like to mention is immersions with additional algebraic structures on a tangent bundle. These are immersions of the so-called Frobenius manifolds, which were introduced to geometry by B. A. Dubrovin at the beginning of the 1990s in connection with a search for methods for integrating various differential equations and which are defined as manifolds with a planar Euclidean or pseudo-Euclidean metric and with the structure of a Frobenius algebra on the tangent spaces with some rather strong additional requirements on the operations. In [115] O. I. Mokhov considered the question of isometric realization of such manifolds in pseudo-Euclidean spaces in the class of k-potential submanifolds, which he introduced.

4a. Smoothness problems in the theory of isometric immersions and bendings. We begin with the statement of an important theorem, which explains many results and closes many problems involving smoothness questions in the theory of surfaces.

The first observation. In their studies at the beginning of the 1950s, the American mathematicians Hartman and Winter discovered that geodesic lines in an abstractly defined metric of smoothness class C^1 and on a C^2-smooth surface that also has by calculation a metric of class C^1, certain properties (for example, uniqueness, extendability, etc.) are very different: the metric of the surface behaves as if it is C^2-smooth, although direct calculations (for example, for surfaces defined in the explicit form $z = f(x, y)$) give the metric one less degree of smoothness than the smoothness of the surface. They called this phenomenon and some other similar phenomena "the differentiability paradox".

The second observation. In the same period, a kind of race developed between Soviet (A. D. Aleksandrov, A. V. Pogorelov, et al.) and foreign geometers (L. Nirenberg, E. Heinz, et al.) to establish the smoothness class of the solution of Weyl's problem, which consisted in proving the existence of a convex surface that is isometric to a metric with positive curvature abstractly defined on a sphere. In all these and other theorems on immersions, the regularity of the surface of the immersion was turning out to be no higher than the regularity of the metric being immersed, although it would seem that in the best theorems the immersion must have smoothness class greater by 1 than the order of smoothness of the given metric. All these questions were eliminated by the following theorem, proved in [116].

Theorem 1. *Suppose that a k-dimensional, $2 \leq k \leq n-1$, surface S of smoothness class $C^{m,\alpha}$ ($m \geq 2, 0 < \alpha < 1$) is situated in an n-dimensional Riemannian space of smoothness at least $C^{m,\alpha}$. Then, in its intrinsic metric, S is a k-dimensional Riemannian manifold of smoothness $C^{m,\alpha}$.*

In other words, there exists an atlas on S in which the metric of the surface S has smoothness of class $C^{m,\alpha}$. Therefore there is no differentiability paradox; the metric does indeed have the same smoothness as the surface itself.[15]

Corollary. *A Riemannian manifold of exact smoothness class $C^{m,\alpha}$ cannot be isometrically immersed in Euclidean space as a surface of smoothness higher than $C^{m,\alpha}$.*

We note that the assertion that there does not exist an isometric immersion of a C^r-smooth ($r \geq 2$) metric as the metric of a C^{r+1}-smooth surface was also stated in [117] using arguments of general position. An analogue of Theorem 1 was proved again in 1981 by the American mathematicians D. M. De Turk and J. J. Kazdan.

Next, the author proved [118] that if in the Weyl problem the metric that is being immersed is of smoothness class $C^{m,\alpha}$, then an isometric immersion of it in $\mathbb{R}^3$ has smoothness of the same class. By the preceding corollary we obtain that this result, in the aspect of smoothness, is a best-possible refinement of A. V. Pogorelov's well-known theorems on the order of smoothness of convex surfaces with a smooth metric.

The situation is different in the case of metrics and surfaces of negative curvature: the smoothness of a surface is not automatically connected with the smoothness of its metric (for example, one can construct non-analytic surfaces of constant negative curvature, that is, with an analytic metric). But Èmil' Renol'dovich Rozendorn (1936–2012) in [119] showed that if on a surface of negative curvature there is an arc with certain special properties, then, under rather weak conditions on the a priori smoothness of the surface, its smoothness in fact turns out to be only 1 less than the smoothness class of the metric; this result was published in "Trudy Moskov. Mat. Ob-va" (Proc. MMS).

The following series of results relates to the existence of surfaces that are locally rigid in classes of deformations with little smoothness. As mentioned above, the existence of surfaces that are locally rigid in the class of analytic deformations was first proved by Efimov in 1948 in [103]. In [120] surfaces of revolution of class C^∞ were found that are locally rigid in the class of deformations of smoothness C^1. This fact was of fundamental significance: according to Nash and Kuiper's well-known results, all surfaces are flexible even globally in the class of C^1-smooth surfaces, while the examples found in [120] show that this is not the case in the linear approximation; therefore one should expect that deformations of Nash–Kuiper type cannot be smooth with respect to the deformation parameter. (However, the question of the non-existence of isometric deformations in the class of C^1-smooth surfaces that are C^1-smooth with respect to the parameter remains open until now; it is also not known whether they belong to a Hölder class with respect to the parameter for some exponent $\alpha > 0$.) In [121] it was shown that there exist convex surfaces that are locally rigid in the class of C^∞-smooth deformations, but this result, although interesting, is not quite adequate, because the field of infinitesimal bendings is required to have greater smoothness than the smoothness of the surfaces themselves.

[15]In classes of exact smoothness C^m, $m \geq 2$, owe should actually be talking about preserving the smoothness in Sobolev classes.

(This is similar to the fact that a solution of a differential equation is required to be of greater smoothness than the smoothness of the coefficients of the equation allows in the general case.) Further details about local questions in the theory of bendings and infinitesimal bendings are given in [122], while modern achievements and problems in this area of studies are reflected in the survey paper [123]. Among the unsolved theoretical problems, we would like to highlight the open question of a correct definition of infinitesimal bendings of higher orders; see the papers [122, pp. 206–207], [124, pp. 137–138], and [125] concerning this.

4b. Bendings and infinitesimal bendings of surfaces. Above we wrote about the local rigidity of surfaces, and in this subsection we present results on global rigidity/flexibility. The topic of infinitesimal bendings was extensively studied by Èduard Genrikhovich Poznyak (1923–1993), who published his first paper on infinitesimal bendings in "Uspekhi Mat. Nauk" in 1947.[16] Among his achievements in this area we must mention an example of a closed surface of revolution with a complete set of nontrivial harmonics [126] and an example of a closed surface with flexibility of the 2nd order [127]. Further advances are related to first, a proof of the rigidity of 2nd order of troughs of revolution under really minimal conditions on the smoothness of a meridian of the surface and the class of deformations [128], which makes it possible to establish rigidity of the 2nd order for a wide class of tori of revolution, and second, reducing the study of analytic infinitesimal bendings of the 1st and 2nd orders for surfaces of revolution of genus 0 with flattenings at the poles to solving a certain system of Diophantine equations (see [129–131]). Establishing rigidity of the 1st or 2nd order for a surface is not only interesting in its own right but also because such rigidity implies the inflexibility of the surface in the class of analytic with deformations with respect to the parameter. Thus, all the surfaces of revolution whose rigidity we have discussed in the preceding assertions satisfy Euler's conjecture on their inflexibility. As for whether this conjecture holds or not in the general case, we think that an important step towards proving its validity in the class of analytic surfaces and analytic deformations is the reduction of this problem in [132] to considering a certain Beltrami system on a compact manifold. Moreover, one hope for constructing a counterexample in the class of nonanalytic surfaces which are analytic with respect to parameter deformations is given by the method indicated in [133] for the construction of a surface of revolution that is flexible for a fixed parallel.

A general survey of the available papers on bendings and infinitesimal bendings of surfaces in three-dimensional space is given in [124] and [134]. We will not dwell on the results and problems in the multi-dimensional theory of bendings. We merely note that this topic was not unknown to the MMS; for example, at the meeting on 25 March, 1952 Nikolaĭ Nikolaevich Yanenko (1921–1984) gave a talk devoted to infinitesimal bendings of multi-dimensional surfaces (and the first paper on this topic in Russia–USSR [91] was published in "Matematicheskiĭ Sbornik" in 1931 by Ts. L. Burstin, who was mentioned above). See [135] for a discussion of the modern state of the multi-dimensional topic of bendings and infinitesimal bendings together with a discussion of various generalizations of the basic Gauss–Peterson–Mainardi–Codazzi equations in the theory of surfaces.

4c. The problem of isometric immersiblity for metrics with negative curvature. Finally, we will begin the second part of our discussion. At the start of the paper we described our view of how the subject matter of geometry "in the large" is structured. The question of the isometric realization of a given metric as the metric of

[16]È. G.'s very first scientific publication was in "Matematicheskiĭ Sbornik" in 1945 and was written under the supervision of Mark Yakovlevich Vygodskiĭ (1898–1985), whose book "Differential geometry" he illustrated with beautiful pictures.

some surface was first considered in Russian mathematics in 1931 in [90], in which the author "extended and completed in a certain direction... Janet's paper" on embedding n-dimensional metrics into $N = C_n^2$-dimensional Euclidean space.[17] But after Aleksandrov's papers from in 1942, in which he proved that any convex metric on a sphere can be realized as the metric of some convex surface,[18] the question of isometric immersions became very popular and many papers appeared concerning the existence or non-existence of such immersions, on their smoothness properties, on uniqueness, embedding, and so on. Here, a distinctive division of interests emerged: while in Leningrad and Khar'kov they studied mainly metrics of positive curvature, in Moscow most attention was devoted to metrics and surfaces of negative curvature. (Of course, this did not apply to *all* the published papers. In reality research in the various directions of geometry "in the large" went on everywhere; in particular, in Novosibirsk there was a strong school looking at the geometry of Riemannian spaces.) Of course, a decisive factor was the fact that the leader of the Moscow school of geometry "in the large", N. V. Efimov, had long been interested in the problem of the existence in $\mathbb{R}^3$ of a complete surface with negative curvature that is nonzero. One origin of this problem was in Hilbert's theorem stating that there is no isometric immersion of the Lobachevskiĭ plane into $\mathbb{R}^3$ in the form of a regular surface. To cap Efimov's many years of effort, he proved the celebrated theorem that on any complete C^2-smooth surface in three-dimensional space the supremum of the Gauss curvature is non-negative (see [136, 137]). Efimov was awarded the Lenin Prize for this and received an invitation to give an hour-long plenary talk at the International Congress of Mathematicians in Moscow. A description of this and his other results in the theory of surfaces of negative curvature is to be found in [138]. There is also a book devoted to his life and scientific-pedagogical activity [139].

The closest associate of Nikolaĭ Vladimirovich in the topic of surfaces of negative curvature was his student Rozendorn, who was masterly in constructing subtle examples of surfaces with particular necessary properties, on the boundary between the possible and the impossible. At the beginning of his scientific career he constructed an example of a complete bounded surface of nonpositive curvature that is C^2-smooth everywhere except at several isolated points at which the surface is $C^{1,1}$-smooth and the limit of the Gaussian curvature exists [140], thus partially refuting Hadamard's conjecture that such a surface was impossible under the condition that its curvature be negative. (A "pure" counterexample to Hadamard's conjecture was constructed much later, and in the form of a complete minimal surface (N. S. Nadirashvili, 1996).) He further constructed examples showing that Efimov's theorem, with respect to the requirement of regularity of the surface, cannot be strengthened even to the class of $C^{1,1}$-smoothness, that is, this theorem was obtained for surfaces with the minimal possible smoothness. Other important and interesting achievements of Rozendorn's can be seen in his survey paper [141].

While Efimov studied questions on the non-immersiblity of metrics of negative curvature, È. G. Poznyak[19] headed a group of geometers studying the possibility of immersions of domains with a metric of negative curvature. His first big success consisted in proving the existence of an isometric immersion in $\mathbb{R}^3$ of any closed geodesic disc taken in a domain with metric of variable negative curvature [143]. A series of papers then followed, in which he and his students obtained isometric immersions of ever wider domains with

[17]Note that at that time there was no established Russian-language terminology in the theory of immersions, and in the Russian abstract of his paper the author wrote about the "*containment* problem".

[18]But we can point out that this problem was not ignored by the Moscow mathematicians: another solution to it is given in [102], which was communicated by L. A. Lyusternik in 1943 in a talk at the Steklov Mathematical Institute; the author does not know whether Lyusternik himself published this proof.

[19]For his biography and scientific-pedagogical activity see [142].

metrics of negative curvature. In particular, Poznyak himself showed the immersiblity in $\mathbb{R}^3$ of any polygon on the Lobachevskiĭ plane, including with vertices on the absolute, and the papers of E. V. Shikin, D. V. Tunitskiĭ, et al. established the immersibility in $\mathbb{R}^3$ of a number of domains with a metric of negative curvature in noncompact cases; see, for example, the papers [144–146] and the surveys [141, 147–150]. The well-known connections between solutions of the sine-Gordon equation and the problem of the realization of the Lobachevskiĭ metric give rise to interesting applications of the theory of surfaces of negative curvature to certain problems in physics [151].

Multi-dimensional aspects of the local theory of immersions were the subject of talks by N. N. Yanenko at meetings of the MMS in 1952–53 (10 June 1952 and 19 May 1953), and he published the corresponding results in [152] and [153]. By the way, note that although subsequently Yanenko became better known as a specialist in applied mathematics and mechanics, both his Candidate (PhD) and Doctor (Habilitation) dissertations, done under the supervision of P. K. Rashevskiĭ, were devoted precisely to the main topics of metric geometry, namely isometric immersions and bendings. He considered mainly Riemannian metrics of class 1; consequently, the objects he studied were hypersurfaces. (The class of an n-dimensional metric ds^2 is defined as the least number q such that this metric can be locally isometrically embedded in $\mathbb{R}^{n+q}$; very complicated descriptions of metrics of class 1 were given in the papers by N. A. Rozenson.) Global questions concerning multi-dimensional isometric embeddings and immersions of Riemannian metrics were considered in [117], the results of which were also reported at a meeting of the MMS.

Note that there is an as yet unsolved problem concerning the analytic immersion and embedding of the Lobachevskiĭ plane (and generally of an n-dimensional space of constant negative curvature) into Euclidean space of minimal dimension (see the available results and bibliography in [141, p. 180–185]).

4d. Immersions of locally Euclidean metrics. As we mentioned above, immersions of metrics of sign-constant curvature were studied in great detail by the leaders of the school of geometry "in the large" themselves. But one natural and, at first glance, simple class of metrics, namely locally Euclidean metrics, remained almost unexplored. We should note at once that in the theory of immersions of metrics of constant curvature (we will call them local Euclidean (l. E.), locally hyperbolic (l. h.) and locally spherical (l. s.) metrics) there is a special feature, which is that for these metrics defined in an arbitrary domain by some metric form, we can start searching for their isometric realizations in the form of surfaces with immersions in standard complete spaces of constant curvature of the same dimension as the given metrics themselves. For example, it is natural to pose the following question for l. E. metrics defined in a two-dimensional domain: is an isometric immersion and embedding in the standard Euclidean plane possible, that is, can we immerse a two-dimensional metric in two-dimensional space in the form of some domain in the Euclidean plane.

If this domain turns out to be without self-intersections, then we obtain an isometric embedding of the given l. E. metric in the plane; that is to say, we obtain a *natural* representation of the l. E. metric in the form of the Euclidean metric of a planar domain, and the Euclidean geometry, which is hidden in the l. E. metric, develops graphically in a domain on the Euclidean plane. If an isometric realization of an l. E. metric on the Euclidean plane turns out to have self-intersections, then we obtain an immersion of the metric. Finally, it may happen that an l. E. metric defined in some two-dimensional domain does not admit an isometric immersion in the Euclidean plane. This variant is only possible for l. E. metrics defined in multiply-connected domains, and a study of this possibility showed that the intrinsic geometry of such a metric itself prompts the form of the surface in which it can be realized, namely a cylindrical or conical surface. An unexpected result consists in that for developable surfaces there is no intrinsic characteristic of the metric [154]. (Apparently, any surface that is developable can also be realized in the form of some conical surface.)

Generally, this part of the theory of immersions of an l. E. metric in the Euclidean plane is closely connected with the theory of functions of a complex variable, in particular with the theory of conformal maps, and a partial exposition of the corresponding results was given in our papers published between 1999–2009. As a by-product, some results for holomorphic functions were also obtained. For example, if a holomorphic function $f(z) \neq 0$ is defined in an open disc, then there exists a path in the disc going to the border of the disc that has finite length in the metric $ds = |f(z)| \cdot |dz|$.

We return to the question of isometric immersions of l. E. metrics. The first general result in this direction was obtained by the author in 1985.

Theorem 2 ([155, 156]). *If an l. E. metric defined in some n-connected domain D_n, $n \geq 1$, can be isometrically immersed in $\mathbb{R}^2$ with the standard Euclidean metric, then it can be isometrically embedded in $\mathbb{R}^3$.*

Corollary. *Any l. E. metric defined in a simply connected domain can be isometrically embedded in $\mathbb{R}^3$.*

Theorem 3. *Any l. E. metric defined in a doubly-connected domain can be isometrically immersed in $\mathbb{R}^3$.*

Under the hypotheses of Theorem 3, in some cases it is possible to guarantee the existence of an isometric embedding [157].

Theorem 4 ([155]). *In three-connected domains there exist l. E. metrics that do not admit an isometric immersion in $\mathbb{R}^3$ even in the class of C^1-smooth ruled surfaces.*

Apart from studying the problem of isometric immersion of l. E. metrics itself, there are also papers devoted to studying the structure of surfaces with an l. E. metric. The way the classical theory works is to start with C^2 smoothness and impose the additional condition that there are no points of flattening. In the smoothness class C^1, and even in $C^{1,\alpha}$, $\alpha < 1/7$, there may exist surfaces with an l. E. metric on which there is not even one straight line segment (Yu. F. Borisov). The existence of straight line rulings on them is obtained under certain additional restrictions established by Pogorelov back in 1956. The necessity of these conditions was proved by the author recently [158]. At the same time a complete analytic representation of ruled C^1-smooth surfaces with an l. E. metric was also given [159] . This question was posed in the 1970s, when it was actively studied by such well-known geometers as Yu. D. Burago and S. Z. Shefel'. The monograph [160] is devoted to these and other questions in the theory of l. E. metrics and to surfaces with an l. E. metric.

Some quite recent papers of M. I. Shtogrin's, devoted to the study of piecewise-smooth developable surfaces, deserve a special mention. It turns out that the lines along which the smoothness of developable surfaces is violated (but the property that the surface is locally isometric to a planar disc is preserved) cannot have an arbitrary nature. Also the structure of the surface on both sides of the edge necessarily has some symmetry [161, 162]. Furthermore, M. I. Shtogrin constructed amazing piecewise-smooth surfaces, in the large isometric to the border of Platonic polyhedra, whose smooth domains are not planar polygons but rather pieces of cylindrical surfaces; in addition these "curvilinear polyhedra" admit bendings, but with "floating" edges (that is, edges change their position on the surface in the sense of intrinsic geometry), and in the case of a tetrahedron or an octahedron it is even possible to make sure that the combinatorial disposition of the smooth cylindrical pieces on the corresponding surfaces is the same as on these Platonic polyhedra themselves [163–165].

Among the many unsolved problems in the theory of developable surfaces, one of the most interesting is an analogue of Plateau's problem, namely the question of whether there exists a surface with zero Gaussian curvature for a given boundary. The solution of this problem will have big practical applications.

5. Polyhedra

One could say that polyhedra are one of the favourite topics of specialists studying metric geometry. They are interesting in their own right, and at the same time they are a tool in solving many problems in the metric theory of smooth surfaces. Over the last three or four decades a breakthrough in the metric geometry of polyhedra consisted in the fact that effective methods were developed for investigating them and, corresponding to these, results were found which related to all polyhedra rather than just convex ones, as in most classical papers.

One of the breakthrough discoveries was the construction of an example of an embedding of a flexible polyhedron (Connelly, 1977). It is usual for a significant new result to lead to important new problems being posed, and this was the case here.

The first question is as follows: since it turns out that there exist flexible polyhedra, and on the other hand it has been proved that almost all polyhedra are inflexible, it is natural to pose the problem of finding criteria for a polyhedron to be flexible or inflexible. Here the situation is similar to the problem of finding transcendental numbers: it is known that almost all numbers are non-algebraic, but it is difficult to produce concrete non-algebraic numbers. The same applies to flexible polyhedra: almost all polyhedra are inflexible, but it is difficult to give concrete classes of inflexible polyhedra. Among the papers on this topic we mention a series of papers by N. P. Dolbilin, M. A. Shtan'ko, and M. I. Shtogrin on the inflexiblility of quadrillage polyhedra, for example, the papers [166–168], as well as the author's theorem [169] on the inflexiblility of immersed pyramids. (Pyramids are polyhedra in any dimension that have a vertex connected by edges to all the other vertices; in three-dimensional space they can be of any topological type.) The problem of finding flexible polyhedra is not easy either. While criteria for inflexiblility produce whole classes of inflexible polyhedra, finding any flexible polyhedra is a one-off: it is not for nothing that it took 164 years from the Legendre–Cauchy theorem on the inflexiblility of convex polyhedra for the first example of a flexible polyhedron to be constructed. (This history and a description of the first examples of flexible polyhedra can be found in the brochure [170].) New examples of flexible polyhedra of high topological genus were first found by V. A. Aleksandrov (for genus $g = 1$, [171]), and then using a generalization of his method they were constructed[20] for any genus $g > 1$ (see [172]).

What is the second topic for investigation? It turns out that the volume of Connelly's flexible polyhedron, as well as the volumes of other flexible polyhedra constructed later, remains constant in the course of bending, although the polyhedron itself changes its external structure. In this connection, it was conjectured that this property is enjoyed

[20]Apart from metric questions, in the theory of polyhedra there are many other questions, for example, problems connected with their combinatorial structure, with tesselations and packings, and so on; from a historical viewpoint, we note the cycle of papers on graphical methods of spatial statics and kinematics published in "Uspekhi Mat. Nauk" (1940), no. 7, and on descriptive geometry in "Uspekhi Mat. Nauk" (1944), no. 10. In recent times, some fresh problems have arisen on these topics connected with the so-called fullerenes — convex polyhedra that have faces of only two types, pentagons and hexagons. Interest toward such polyhedra is explained by problems in stereochemistry, which studies and uses the spatial structure of molecules, in this case of molecules with carbon atoms. New methods of transforming fullerenes into one another were proposed in [173].

by all possible flexible polyhedra. This is known as *bellows conjecture*. The search for an answer to this question resulted in a generalization of Heron's formula being found. This formula expresses the area of a triangle in terms of the length of its sides; the generalisation extends it to volumes of polyhedra. Namely, the following theorem was proved.

Theorem 5 ([174, 175]). *Let* [**P**] *be the set of all isometric polyhedra in* $\mathbb{R}^3$ *with triangular faces that have the same combinatorial structure, and let* (l) *denote the set of the squares of the lengths of edges. Then there exists a polynomial*

$$Q(l,V) = V^{2N} + a_1(l)V^{2N-2} + \cdots + a_N(l)$$

such that the algebraic volume of every polyhedron in [**P**] *is a root of this polynomial, in which all the coefficients* $a_i(l)$ *are themselves polynomials in the squares of the lengths of edges of the polyhedron with numerical coefficients depending on the combinatorial type of the polyhedron*[21].

Obviously, it is an easy corollary of this theorem that the bellows conjecture holds.

The proof of the theorem is constructive, based on the theory of the elimination of unknowns in a system of polynomial equations, and it is independent of a concrete external form of the polyhedron in space. This means that the volume polynomial can be constructed from the known combinatorial scheme of the polyhedron alone. There is a second proof of the theorem, using a deeper algebraic technique, but it does not yield a method for finding the volume polynomial; it merely asserts its existence.

There is a natural question about whether a similar theorem is true in Euclidean spaces of higher dimension, and also in other spaces of constant curvature. In 2011 A. A. Gaĭfullin extended the theorem on the volume polynomial to polyhedra in four-dimensional Euclidean space [177], and later did this for polyhedra in Euclidean spaces of any dimension, thus proving that the bellows conjecture holds for them [178]. However, a volume polynomial cannot exist for polyhedra in spaces of constant nonzero curvature. Even the volume of a tetrahedron in these spaces is represented in terms of the lengths of its edges by a complicated analytic expression involving the integral of some function. As for the bellows conjecture itself, the following holds: in Lobachevskiĭ spaces of odd dimension, the conjecture is valid without any additional conditions [179], in a spherical space of any dimension there exist flexible polyhedra for which the volume changes (when $n = 3$ this was established by Aleksandrov (1997), and in an arbitrary dimension by Gaĭfullin [180]), but there is a so far unpublished result of Gaĭfullin's asserting that in any dimension in a space of constant nonzero curvature, the volume of any flexible simplicial polyhedron remains constant in the process of bending if the lengths of its edges are smaller than some number $d > 0$ depending only on the combinatorial structure of the polyhedron (and, of course, on the dimension and the value of the curvature of the space itself). Obviously, in the general case this number satisfies $d \to 0$ as, for example, the number of vertices of the polyhedron increases.

We return to the case of three-dimensional Euclidean space. As was pointed out, the volume polynomial of a polyhedron can be found based only on the knowledge of the combinatorial structure of the polyhedron. Therefore, if we are given the scheme of the combinatorial structure of a polyhedron with triangular faces, we can immediately write down the coefficients $a_i(l)$ of its volume polynomial as known polynomials in the squares of the lengths of its edges, and if we also know the lengths of the edges we can find all the roots of the volume polynomial. Thus, knowing the natural development (that is, the faces and the rules for identifying their edges), even *before constructing* the

[21]See [176] for an analogue of this theorem for the area of cyclic polygons (that is, polygons inscribed in a circle); it gives both the history and a rich bibliography.

polyhedra from their common development we already know the possible values for their volumes. Furthermore, sometimes we shall even know whether it is possible to construct a polyhedron from a given development: if the volume polynomial has no positive roots, then it is automatically impossible to construct a polyhedron.

Moreover, we can give an algorithm for constructing the polyhedra themselves from their development. The problem we have in constructing them is that it is not known beforehand what the angle between two adjacent faces should be. It turns out that for any diagonal, including every so-called small diagonal, connecting two noncommon vertices of adjacent faces, we can also compose a polynomial equation of the form

$$D(d, V) = A_0(l, V)d^{2K} + A_1(l, V)d^{2K-2} + \cdots + A_K(l, V) = 0,$$

where d is the small diagonal under consideration and $A_i(l, V)$, $0 \le i \le K$, are polynomials in the squares of the lengths of the edges, where the coefficients depend on the combinatorial structure of the polyhedron, its volume and on the choice of the small diagonal [181].

Note that the polynomials both for the volume and for the diagonals are invertible in the sense that we can declare as a variable the length of any edge, while assuming the volume or the diagonal is known; then we get a polynomial equation for finding the unknown length of the edge. Thus, the existence of a polynomial for the volume of a polyhedron makes it possible to say that a new chapter of geometry has emerged, which can be called the “solution of polyhedra” by analogy with the term the “solution of triangles”; see [182] for the details. Therefore, along with some very promising theoretical research on establishing connections between the algebra of polynomials and the geometry of polyhedra in a wide sense, an applied science has also emerged, using the possibilities of calculating the numerical characteristics of polyhedra even before they have been constructed. We can say that, in terms of these calculations, the science of polyhedra has become a finite science, at least in the same sense as chess is a finite game, since all these calculations can be realized only on supercomputers.

In conclusion I wish to thank O. I. Mokhov for some valuable comments and advice, which in many respects helped to improve the exposition of this paper and to make it more precise. In addition, I would like to thank the staff of the libraries of the Mechanics-Mathematics Faculty of the Moscow State University and of the Steklov Mathematical Institute and also A. K. Rybnikov for their help in the search for the literature I needed.

A note from the authors

We note that the reference to this paper in [60] is incomplete (the Cahier issue number is not indicated), and in [93] the number of the issue is incorrect and that of the volume is not given.

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Translated by E. KHUKHRO
Originally published in Russian