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ON THE EXISTENCE OF A GLOBAL SOLUTION OF THE MODIFIED NAVIER–STOKES EQUATIONS

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Abstract. We prove global existence theorems for initial–boundary value problems for the modified Navier–Stokes equations used when modeling ocean dynamic processes. First, the case of distinct vertical and horizontal viscosities for the Navier–Stokes equations is considered. Then a result due to Ladyzhenskaya for the modified Navier–Stokes equations is improved, whereby the elliptic operator is strengthened with respect to the horizontal variables alone and only for the horizontal momentum equations. Finally, the global existence and uniqueness of a solution is proved for the primitive equations describing the large-scale ocean dynamics.

Introduction

Let $\mathbf{u} = (u_1, u_2, u_3)$ be the velocity vector, and let p be the hydrostatic pressure function. The Navier–Stokes system describing a viscous incompressible flow in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ has the form [4]

$$\begin{gathered} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}, \\ \operatorname{div}\mathbf{u} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}^0(x), \quad \operatorname{div}\mathbf{u}^0 = 0, \\ \mathbf{u}|_{\partial\Omega\times[0,T]} = \mathbf{0}. \end{gathered} \tag{1}$$

The problem on the existence of a global solution for these equations remains open yet. Namely, it is unknown so far whether they have a unique solution in $\mathbf{H}^1(\Omega)\times C[0,T]$ for any $\nu > 0$ and $T > 0$ and any sufficiently smooth initial condition $\mathbf{u}^0$ and right-hand side $\mathbf{f}$. Note that the existence of a solution in the class $\mathbf{H}^1(Q_T)$, where $Q_T = \Omega \times [0,T]$, implies that the solution norm $\|\nabla\mathbf{u}\|_{L_2(\Omega)}$ is time continuous, that is, that the solution belongs to the class $\mathbf{H}^1(\Omega)\times C[0,T]$.

Nevertheless, papers proving the existence and uniqueness of a strong solution for problems like (1) under some restrictions (smallness of the domain with respect to one of the variables, restrictions on the nonlinear terms, etc.) emerge almost every year (e.g., see [21, 12, 15]). The present paper deals with equations of the Navier–Stokes type arising when modeling atmosphere and ocean dynamics. For these problems, we prove the existence and uniqueness of a global solution.

1. Case of a variable viscosity coefficient

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We denote the independent variables by $x = (x_1, x_2, x_3)$ or (x, y, z) and introduce the following norms and operators on the space

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of vector functions $\mathbf{f} = (f_1, f_2)$:

$$\|\mathbf{f}\|^2 = \sum_{i=1}^{2} \int_\Omega f_i^2(x)\,dx = \int_\Omega |\mathbf{f}|^2\,dx, \quad \|\mathbf{f}_x\|^2 = \sum_{i=1}^{2}\sum_{j=1}^{3} \int_\Omega \left(\frac{\partial f_i}{\partial x_j}\right)^2 dx,$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad \partial_{x_i} = \frac{\partial}{\partial x_i}, \quad \|\cdot\|_q = \|\cdot\|_{L_q},$$

$$\nabla = (\partial_x, \partial_y), \quad \operatorname{div}\mathbf{f} = \partial_x f_1 + \partial_y f_2, \quad |\nabla \mathbf{f}|^2 = \sum_{i,\,j=1}^{2} \left(\frac{\partial f_i}{\partial x_j}\right)^2,$$

$$|g|_{q,\,E_x}^q = \int_{-\infty}^{\infty} |g|^q\,dx, \quad |g|_{q,\,E_{xy}}^q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g|^q\,dx\,dy.$$

The case of distinct horizontal and vertical viscosities is typical of ocean dynamic problems [6]. In this case, it is expedient to separate the horizontal and vertical velocity components. Namely, let $\mathbf{u} = (u_1, u_2)$ be the horizontal projection of the velocity, and let w be the vertical component. Then problem (1) becomes

$$\begin{gathered} \mathbf{u}_t - \nu\Delta\mathbf{u} - \mu\partial_z^2\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\mathbf{u}_z = \mathbf{f}, \\ w_t - \nu\Delta w - \mu\partial_z^2 w + p_z + (\mathbf{u}\cdot\nabla)w + ww_z = g, \\ \operatorname{div}\mathbf{u} + w_z = 0, \\ (\mathbf{u}, w)(x, 0) = (\mathbf{u}^0, w^0)(x), \quad \operatorname{div}\mathbf{u}^0 + \partial_z w^0 = 0, \\ (\mathbf{u}, w)|_{\partial\Omega\times[0,T]} = \mathbf{0}. \end{gathered} \tag{1.1}$$

For simplicity, we consider the case of $\mathbf{f} = \mathbf{0}$ and $g = 0$ in what follows. The following claim holds.

Theorem 1. *For any sufficiently smooth initial condition* $(\mathbf{u}^0, w^0)$*, any* $\nu > 0$*, and any time interval* $[0, T]$*, there exists a* $\mu > 0$ *such that problem* (1.1) *has a global solution, that is, a vector function* $\mathbf{u} \in \mathbf{H}^1(Q_T)$ *that satisfies* (1.1) *in the weak sense, has norm* $\|\mathbf{u}_x\|$ *continuous in* t *on* $[0, T]$*, and admits the estimate*

$$\|\mathbf{u}_t(t)\|^2 + \|w_t(t)\|^2 \le \|\mathbf{u}_t(0)\|^2 + \|w_t(0)\|^2 \quad \forall t > 0. \tag{1.2}$$

Proof. Let us use the Ladyzhenskaya inequality [4]

$$\|f\|_4^4 \le c_1 \|f_{x_1}\| \cdot \|f_{x_2}\| \cdot \|f_{x_3}\| \cdot \|f\|, \qquad f \in H_0^1(\Omega). \tag{1.3}$$

Assume that the desired solution exists. Let us obtain an a priori estimate. To this end, we take the inner product of the first equation in (1.1) by $\mathbf{u}$ in $\mathbf{L}_2$ and of the second equation by w in L_2. By adding the results, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|^2 + \|w\|^2) + \nu(\|\nabla\mathbf{u}\|^2 + \|\nabla w\|^2) + \mu(\|\mathbf{u}_z\|^2 + \|w_z\|^2) = 0; \tag{1.4}$$

the integration of (1.4) with respect to t gives

$$\|\mathbf{u}(t)\|^2 + \|w(t)\|^2 \le \|\mathbf{u}^0\|^2 + \|w^0\|^2 \equiv M^2;$$

i.e.,

$$\max_t(\|\mathbf{u}(t)\|^2 + \|w(t)\|^2) \le \|\mathbf{u}^0\|^2 + \|w^0\|^2 \equiv M^2. \tag{1.5}$$

Since equation (1.4) implies that

$$\nu(\|\nabla\mathbf{u}\|^2 + \|\nabla w\|^2) + \mu(\|\mathbf{u}_z\|^2 + \|w_z\|^2) = -(\mathbf{u}_t, \mathbf{u}) - (w_t, w) \le \|\mathbf{u}_t\| \cdot \|\mathbf{u}\| + \|w_t\| \cdot \|w\|,$$

we use this relation and (1.5) to obtain the estimate

$$\nu(\|\nabla\mathbf{u}\|^2 + \|\nabla w\|^2) + \mu(\|\mathbf{u}_z\|^2 + \|w_z\|^2) \le M(\|\mathbf{u}_t\| + \|w_t\|). \tag{1.6}$$

Let us differentiate (1.1) with respect to t,

$$\begin{gathered}\mathbf{u}_{tt} - \nu\Delta\mathbf{u}_t - \mu\partial_z^2\mathbf{u}_t + \nabla p_t + (\mathbf{u}_t\cdot\nabla)\mathbf{u} + w_t\mathbf{u}_z + (\mathbf{u}\cdot\nabla)\mathbf{u}_t + w_t\mathbf{u}_{tz} = \mathbf{0},\\ w_{tt} - \nu\Delta w_t - \mu\partial_z^2 w_t + p_{tz} + (\mathbf{u}_t\cdot\nabla)w + w_t w_z + (\mathbf{u}\cdot\nabla)w_t + w w_{tz} = 0,\\ \operatorname{div}\mathbf{u}_t + w_{tz} = 0.\end{gathered}\tag{1.7}$$

The inner product of (1.7) by $(\mathbf{u}_t, w_t)$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|w_t\|^2) + \nu(\|\nabla\mathbf{u}_t\|^2 + \|\nabla w_t\|^2) + \mu(\|\mathbf{u}_{tz}\|^2 + \|w_{tz}\|^2)&\\ + ((\mathbf{u}_t\cdot\nabla)\mathbf{u} + w_t\mathbf{u}_z, \mathbf{u}_t) + ((\mathbf{u}_t\cdot\nabla)w + w_t w_z, w_t) &= 0.\end{aligned}\tag{1.8}$$

Let us estimate the inner products in (1.8). We have

$$\begin{aligned}&\big|((\mathbf{u}_t\cdot\nabla)\mathbf{u} + w_t\mathbf{u}_z, \mathbf{u}_t) + ((\mathbf{u}_t\cdot\nabla)w + w_t w_z, w_t)\big|\\ &\qquad = \big|(u_{jt}\mathbf{u}, \partial_{x_j}\mathbf{u}_t) + (w_t\mathbf{u}, \mathbf{u}_{tz}) + (u_{jt}w, \partial_{x_j}w_t) + (w_t w, w_{tz})\big|;\end{aligned}$$

here and in what follows, summation over repeated indices in a product is assumed.

Let us use the Hölder inequality and the estimates (1.3), (1.5), and (1.6). We have

$$\begin{aligned}\big|(u_{jt}\mathbf{u}, \partial_{x_j}\mathbf{u}_t)\big| &\le \|\mathbf{u}_{tx}\|\cdot\|\mathbf{u}_t\|_4\cdot\|\mathbf{u}\|_4 \le c\|\mathbf{u}_{tx}\|^{7/4}\cdot\|\mathbf{u}_t\|^{1/4}\cdot\|\nabla\mathbf{u}\|^{1/2}\cdot\|\mathbf{u}_z\|^{1/4}\\ &\le \varepsilon\|\mathbf{u}_{tx}\|^2 + \frac{c}{\varepsilon^7}\|\nabla\mathbf{u}\|^4\cdot\|\mathbf{u}_z\|^2\cdot\|\mathbf{u}_t\|^2 \le \varepsilon\|\mathbf{u}_{tx}\|^2 + \frac{c}{\varepsilon^7\nu^2\mu}(\|\mathbf{u}_t\|^2 + \|w_t\|^2)^{5/2},\end{aligned}$$

where c depends on M.

The remaining inner products can be estimated in a similar way. By taking an appropriate ε, we obtain

$$\begin{aligned}&\big|((\mathbf{u}_t\cdot\nabla)\mathbf{u} + w_t\mathbf{u}_z, \mathbf{u}_t) + ((\mathbf{u}_t\cdot\nabla)w + w_t w_z, w_t)\big|\\ &\qquad \le \frac{\nu}{2}(\|\mathbf{u}_{tx}\|^2 + \|w_{tx}\|^2) + \frac{cM^3}{\nu^9\mu}(\|\mathbf{u}_t^2\| + \|w_t\|^2)^{5/2}.\end{aligned}\tag{1.9}$$

The substitution of (1.9) into (1.8) yields

$$\begin{aligned}&\frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|w_t\|^2) + \nu(\|\nabla\mathbf{u}_t\|^2 + \|\nabla w_t\|^2)\\ &\qquad + \mu(\|\mathbf{u}_{tz}\|^2 + \|w_{tz}\|^2) - \frac{cM^3}{\nu^9\mu}(\|\mathbf{u}_t\|^2 + \|w_t\|^2)^{5/2} \le 0,\end{aligned}$$

whence it follows that

$$\begin{aligned}&\frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|w_t\|^2) + \nu(\|\mathbf{u}_{tx}\|^2 + \|w_{tx}\|^2)\\ &\qquad + \left(\mu - \frac{cM^3}{\nu^9\mu}(\|\mathbf{u}_t\|^2 + \|w_t\|^2)^{3/2}\right)(\|\mathbf{u}_{tz}\|^2 + \|w_{tz}\|^2) \le 0.\end{aligned}\tag{1.10}$$

The norm $\|\mathbf{u}_t(0)\|$ can be estimated from above by the norm $\|(\mathbf{u}, w)\|_{\mathbf{H}^2}$. Then it follows from (1.10) that, for any $\nu > 0$ and arbitrary $\|\mathbf{u}_t(0)\| + \|w_t(0)\|$, depending on the norm $\|\mathbf{u}_0\|_{\mathbf{H}^2} + \|w_0\|_{H^2}$ of the initial condition, there exists a $\mu > 0$ such that

$$\mu - \frac{cM^3}{\nu^9\mu}(\|\mathbf{u}_t(0)\| + \|w_t(0)\|) \ge 0.$$

Now we conclude from (1.10) that the norm $\|\mathbf{u}_t(t)\|$ satisfies the inequality

$$\|\mathbf{u}_t(t)\| \le \|\mathbf{u}_t(0)\| \quad \forall t > 0.\tag{1.11}$$

The existence and uniqueness of a global solution can be proved with the help of (1.11) in exactly the same way as in [4]. □

2. Strengthening of Ladyzhenskaya's modification

Ladyzhenskaya [5] suggested a modification of the Navier–Stokes equations that permits proving the existence of a global solution of (1). Namely, the following problem with a stronger linear operator is considered instead of problem (1):

$$\begin{aligned} &\mathbf{u}_t - \nu\Delta\mathbf{u} - \nu\partial_z^2\mathbf{u} - \nu\varepsilon\big(\operatorname{div}(D(\mathbf{u},w)\nabla\mathbf{u}) + \partial_z(D(\mathbf{u},w)\partial_z\mathbf{u})\big) \\ &\qquad + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\mathbf{u}_z = \mathbf{f}, \\ &w_t - \nu\Delta w - \nu\partial_z^2 w - \nu\varepsilon\big(\operatorname{div}(D(\mathbf{u},w)\nabla w) + \partial_z(D(\mathbf{u},w)\partial_z w)\big) \\ &\qquad + p_z + (\mathbf{u}\cdot\nabla)w + ww_z = g, \\ &\operatorname{div}\mathbf{u} + w_z = 0, \quad (\mathbf{u},w)(x,0) = (\mathbf{u}_0,w_0)(x), \\ &\operatorname{div}\mathbf{u}_0 + \partial_z w_0 = 0, \quad (\mathbf{u},w)|_{\partial\Omega\times[0,T]} = 0, \end{aligned} \tag{2.1}$$

where

$$D(\mathbf{u},w) = |\nabla\mathbf{u}|^2 + |\partial_z\mathbf{u}|^2. \tag{2.2}$$

Ocean dynamics uses a different modification [6] (the Smagorinsky model), which is as follows. One takes (2.1) for the modification of (1) but uses $D(\mathbf{u},w) = |\nabla\mathbf{u}|^2$ instead of (2.2), omits the term $\partial_z\,(D(\mathbf{u},w)\partial_z\mathbf{u})$ in the first equation in (2.1), and leaves the equation for w unchanged. Thus, only the equations pertaining to the horizontal velocity components are changed in the Navier–Stokes equations, and the linear operator is strengthened with respect to the horizontal variables alone. Then the problem acquires the form

$$\begin{aligned} &\mathbf{u}_t - \nu\Delta\mathbf{u} - \nu\partial_z^2\mathbf{u} - \nu\varepsilon\operatorname{div}(|\nabla\mathbf{u}|^2\nabla\mathbf{u}) + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\mathbf{u}_z = \mathbf{f}, \\ &w_t - \nu\Delta w - \nu\partial_z^2 w + p_z + (\mathbf{u}\cdot\nabla)w + ww_z = g, \\ &\operatorname{div}\mathbf{u} + w_z = 0, \quad (\mathbf{u},w)(x,0) = (\mathbf{u}_0,w_0)(x), \\ &\operatorname{div}\mathbf{u}_0 + \partial_z w_0 = 0, \quad (\mathbf{u},w)|_{\partial\Omega\times[0,T]} = 0. \end{aligned} \tag{2.3}$$

To obtain a priori estimates for problem (2.3), we need the following lemmas.

Lemma 1. *Let $v \in H_0^1[0,l]$. Then*

$$\max_x v^2(x) \le 2\|v_x\|\cdot\|v\|. \tag{2.4}$$

Proof. Let us extend v by zero to the entire axis. Then

$$v^2(x) = 2\int_{-\infty}^{x} v_x(x)v(x)\,dx \le 2\|v_x\|\cdot\|v\|.$$

The proof of the lemma is complete. □

Lemma 2. *Assume that $f \in H_0^1(\Omega)$ and $f_x, f_y \in L_4(\Omega)$, $\Omega \in \mathbb{R}^3$. Then*

$$\|f\|_5^5 \le \frac{25}{2}\|f_x\|_4\cdot\|f_y\|_4\cdot\|f_z\|\cdot\|f\|^2. \tag{2.5}$$

Proof. The idea of the proof is similar to that of the proof of the estimate (1.3) (see [4]) and is based on an integral representation of the function. Let us extend f by zero to the entire $\mathbb{R}^3$. Then

$$|f(x,y,z)|^5 = \frac{25}{4}\int_{-\infty}^{x}\sqrt{|f(x,y,z)|}\,f(x,y,z)\,f_x(x,y,z)\,dx \times \int_{-\infty}^{y}\sqrt{|f(x,y,z)|}\,f(x,y,z)\,f_y(x,y,z)\,dy.$$

By using the Hölder inequality, we obtain

$$|f(x,y,z)|^5 \le \frac{25}{4} |f|_{E_x}^{3/2} \cdot |f_x|_{4,\,E_x} |f|_{E_y}^{3/2} \cdot |f_y|_{4,\,E_y}.$$

Let us integrate this relation over $\mathbb{R}^3$ and apply the Hölder inequality. As a result, we obtain

$$\begin{aligned}\int_{E_z}\int_{E_{xy}} |f|^5\,dx\,dy\,dz &\le \frac{25}{4}\int_{E_z}\left(\int_{E_y} |f|_{E_x}^{3/2}\cdot|f_x|_{4,\,E_x}dy\int_{E_x}|f|_{E_y}^{3/2}\cdot|f_y|_{4,\,E_y}dx\right)dz\\ &\le \frac{25}{4}\int_{E_z}|f|_{E_{xy}}^{3}\cdot|f_x|_{4,\,E_{xy}}\cdot|f_y|_{4,\,E_{xy}}dz\\ &\le \frac{25}{4}\max_z |f|_{E_{xy}}^2\int_{E_z}|f|_{E_{xy}}\cdot|f_x|_{4,\,E_{xy}}\cdot|f_y|_{4,\,E_{xy}}dz\\ &\le \frac{25}{2}\|f_x\|_4\cdot\|f_y\|_4\cdot\|f_z\|\cdot\|f\|^2\,dz \qquad \text{(by Lemma 1)},\end{aligned}$$

as desired. □

Corollary 1. *Since*

$$abc \le 0{,}5a^2b^2 + 0{,}5c^2 \le 0{,}25(a^4+b^4) + 0{,}5c^2, \quad a,b,c\ge 0,$$

it follows from (2.5) *that*

$$\|f\|_5^5 \le \frac{25}{8}(\|f_x\|_4^4 + \|f_y\|_4^4 + 2\|f_z\|^2)\|f\|^2. \tag{2.6}$$

Let us obtain an a priori estimate for the solution of problem (2.3). Take the inner product of (2.3) by $(\mathbf{u}, w)$,

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|^2 + \|w\|^2) + \nu(\|\mathbf{u}_x\|^2 + \|w_x\|^2) + \varepsilon\nu\|\nabla\mathbf{u}\|_4^4 = (\mathbf{f},\mathbf{u}) + (g,w). \tag{2.7}$$

By estimating the right-hand side of (2.7) and by integrating with respect to t from 0 to T, we obtain

$$\begin{aligned}\max_{0\le t\le T}\big(\|\mathbf{u}(t)\|^2 + \|w(t)\|^2\big) + \nu\int_0^T\big(\|\mathbf{u}_x\|^2 + \|w_x\|^2 + \varepsilon\|\nabla\mathbf{u}\|_4^4\big)\,dt\\ \le c_1\left(\|\mathbf{u}_0\|^2 + \|w_0\|^2 + \frac{1}{\nu}\int_0^T(\|\mathbf{f}\|_{-1}^2 + \|g\|_{-1}^2)\,dt\right).\end{aligned} \tag{2.8}$$

It follows from (2.7) that

$$\nu(\|\mathbf{u}_x\|^2 + \|w_x\|^2) + \varepsilon\nu\|\nabla\mathbf{u}\|_4^4 = (\mathbf{f},\mathbf{u}) + (g,w) - (\mathbf{u}_t,\mathbf{u}) - (w_t,w),$$

whence we have

$$\|\mathbf{u}_x\|^2 + \|w_x\|^2 + \varepsilon\|\nabla\mathbf{u}\|_4^4 \le c_2\big(\|\mathbf{u}_t\| + \|w_t\| + \|\mathbf{f}\|_{-1}^2 + \|g\|_{-1}^2\big). \tag{2.9}$$

Let us differentiate (2.3) with respect to t,

$$\begin{gathered}\mathbf{u}_{tt} - \nu\operatorname{div}((1+\varepsilon|\nabla\mathbf{u}|^2)\nabla\mathbf{u}_t) - \nu\varepsilon\operatorname{div}((|\nabla\mathbf{u}|^2)_t\nabla\mathbf{u}) - \nu\partial_z^2\mathbf{u}_t\\ +\nabla p_t + (\mathbf{u}\cdot\nabla)\mathbf{u}_t + (\mathbf{u}_t\cdot\nabla)\mathbf{u} + w\mathbf{u}_{tz} + w_t\mathbf{u}_z = \mathbf{f}_t,\\ w_{tt} - \nu\Delta w_t - \nu\partial_z^2 w_t + p_{tz} + (\mathbf{u}\cdot\nabla)w_t + (\mathbf{u}_t\cdot\nabla)w + ww_{tz} + w_tw_z = g_t,\\ \operatorname{div}\mathbf{u}_t + w_{tz} = 0, \quad (\mathbf{u}_t, w_t)|_{\partial\Omega\times[0,T]} = 0.\end{gathered} \tag{2.10}$$

Take the inner product of (2.10) by $(\mathbf{u}_t, w_t)$,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \frac{1}{2}\frac{d}{dt}\|w_t\|^2 + \nu\|\mathbf{u}_{tx}\|^2 + \nu\|w_{tx}\|^2 + \varepsilon\nu\int_\Omega (\nabla\mathbf{u})^2(\nabla\mathbf{u}_t)^2\,dx + \frac{\varepsilon\nu}{2}\left\|((\nabla\mathbf{u})^2)_t\right\|^2 + (u_{kt}\mathbf{u}_{x_k}, \mathbf{u}_t) + (w_t\mathbf{u}_z, \mathbf{u}_t) + (u_{kt}w_{x_k}, w_t) + (w_t w_z, w_t) = (\mathbf{f}_t, \mathbf{u}_t) + (g_t, w_t). \tag{2.11}$$

To estimate the inner products in (2.11), we use the inequalities (e.g., see [4])

$$\begin{aligned} \|v\|_{10/3} &\le (48)^{1/10}\|v_x\|^{3/5}\cdot\|v\|^{2/5},\\ \|v\|_3 &\le (48)^{1/2}\|v_x\|^{1/2}\cdot\|v\|^{1/2},\\ \|v\|_{8/3} &\le (48)^{1/16}\|v_x\|^{3/8}\cdot\|v\|^{5/8}, \end{aligned} \tag{2.12}$$

which hold for functions in $H_0^1(\Omega)$, $\Omega\in\mathbb{R}^3$. Then

$$|I_1| = |(u_{kt}\mathbf{u}_{x_k}, \mathbf{u}_t)| \le c\|\nabla\mathbf{u}\|_3\cdot\|\mathbf{u}_t\|_3^2 \le c\|\nabla\mathbf{u}\|_4\cdot\|\mathbf{u}_{tx}\|\cdot\|\mathbf{u}_t\| \le \delta\|\mathbf{u}_{tx}\|^2 + \frac{c}{\delta}\|\nabla\mathbf{u}\|_4^2\cdot\|\mathbf{u}_t\|^2.$$

Let us estimate the second inner product. Integration by parts with the use of the continuity equation gives

$$I_2 = (w_t\mathbf{u}_z, \mathbf{u}_t) = (\operatorname{div}\mathbf{u}_t\mathbf{u}, \mathbf{u}_t) - (w_t\mathbf{u}, \mathbf{u}_{tz}) = I_2' + I_2''.$$

Next,

$$\begin{aligned} |I_2'| &= |(\operatorname{div}\mathbf{u}_t\mathbf{u}, \mathbf{u}_t)|\\ &\le \|\mathbf{u}_{tx}\|\cdot\|\mathbf{u}\|_5\cdot\|\mathbf{u}_t\|_{10/3} \qquad \text{(by Hölder inequality with exponents 2, 5, and 10/3)}\\ &\le (48)^{1/10}\|\mathbf{u}_{tx}\|^{8/5}\cdot\|\mathbf{u}\|_5\cdot\|\mathbf{u}_t\|^{2/5} \qquad \text{(by (2.12))}\\ &\le \delta\|\mathbf{u}_{tx}\|^2 + c_\delta\|\mathbf{u}\|_5^5\cdot\|\mathbf{u}_t\|^2 \qquad \text{(by Young inequality)}\\ &\le \delta\|\mathbf{u}_{tx}\|^2 + c_\delta(\|\nabla\mathbf{u}\|_4^4 + \|\mathbf{u}_z\|^2)\|\mathbf{u}_t\|^2 \qquad \text{(by (2.6))}. \end{aligned}$$

Likewise,

$$\begin{aligned} |I_2''| &= |(w_t\mathbf{u}, \mathbf{u}_{tz})|\\ &\le c\|\mathbf{u}_{tz}\|\cdot\|\mathbf{u}\|_5\cdot\|w_t\|_{10/3} \qquad \text{(by Hölder inequality with exponents 10/3, 5, and 2)}\\ &\le c\|\mathbf{u}_{tz}\|\cdot\|\mathbf{u}\|_5\cdot\|w_{tx}\|^{3/5}\cdot\|w_t\|^{2/5}\\ &\le \delta\|\mathbf{u}_{tz}\|^2 + c_\delta\|\mathbf{u}\|_5^2\cdot\|w_{tx}\|^{6/5}\cdot\|w_t\|^{4/5} \qquad \text{(by (2.12))}\\ &\le \delta\|\mathbf{u}_{tz}\|^2 + \delta\|w_{tx}\|^2 + c_\delta\|\mathbf{u}\|_5^5\cdot\|w_t\|^2\\ &\qquad\qquad \text{(by Young inequality with exponents 5/3 and 5/2)}\\ &\le \delta\|\mathbf{u}_{tz}\|^2 + \delta\|w_{tx}\|^2 + c_\delta(\|\nabla\mathbf{u}\|_4^4 + \|\mathbf{u}_z\|^2)\|w_t\|^2 \qquad \text{(by (2.6))}. \end{aligned}$$

To estimate the remaining inner products, we need the following lemma.

Lemma 3. *Let* $(\mathbf{u}, w)$ *be a solution of* (2.3). *Then one has the estimate*

$$\max_z |w|_{4,\,E_{xy}} \le c\|\nabla\mathbf{u}\|_4. \tag{2.13}$$

Proof. Let us extend w and $\mathbf{u}$ by zero to the entire $\mathbb{R}^3$. Then, by using the third equation in (2.3) (the continuity equation) and the Minkowski inequality, we obtain

$$|w|_{4,\,E_{xy}} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{z}\operatorname{div}\mathbf{u}\,dz\right)^4 dx\,dy\right)^{1/4} \le \int_{-\infty}^{\infty}|\nabla\mathbf{u}|_{4,\,E_{xy}}dz \le c\|\nabla\mathbf{u}\|_4,$$

as desired. □

Let us estimate I_3. By integrating by parts, we obtain

$I_3 = (u_{kt} w_{x_k}, w_t) = -(\operatorname{div} \mathbf{u}_t w, w_t) - (u_{kt} w, w_{tx_k}) = (w_{tz} w, w_t) - (u_{kt} w, w_{tx_k}) = I_3' + I_3''$.

Further,

$$\begin{aligned}
|I_3''| = |(u_{kt} w, w_{tx_k})| &\le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u_{kt} w w_{tx_k}| \, dx \, dy \right) dz \\
&\le \int_{-\infty}^{\infty} |w|_{4,\, E_{xy}} \cdot |\mathbf{u}|_{4,\, E_{xy}} \cdot |w_{tx}|_{2,\, E_{xy}} dz \\
&\le c\|\nabla \mathbf{u}\|_4 \int_{-\infty}^{\infty} |\mathbf{u}_t|_{4,\, E_{xy}} \cdot |w_{tx}|_{2,\, E_{xy}} dz \qquad \text{(by (2.13))} \\
&\le c\|\nabla \mathbf{u}\|_4 \int_{-\infty}^{\infty} |\nabla \mathbf{u}_t|_{2,\, E_{xy}}^{1/2} \cdot |\mathbf{u}_t|_{2,\, E_{xy}}^{1/2} \cdot |w_{tx}|_{2,\, E_{xy}} dz \\
&\le c\|\nabla \mathbf{u}\|_4 \cdot \|\nabla \mathbf{u}_t\|^{1/2} \cdot \|\mathbf{u}_t\|^{1/2} \cdot \|w_{tx}\| \le \delta \|w_{tx}\|^2 + c_\delta \|\nabla \mathbf{u}\|_4^2 \cdot \|\nabla \mathbf{u}_t\| \cdot \|\mathbf{u}_t\| \\
&\le \delta \|w_{tx}\|^2 + \delta \|\nabla \mathbf{u}_t\|^2 + c_\delta \|\nabla \mathbf{u}\|_4^4 \cdot \|\mathbf{u}_t\|^2, \\
|I_3'| = |(w_t, w_z, w_t)| = |(\operatorname{div} \mathbf{u}, w_t^2)| &\le \|\nabla \mathbf{u}\|_4 \cdot \|w_t\|_{8/3}^2 \\
&\le c\|\nabla \mathbf{u}\|_4 \cdot \|w_{tx}\|^{3/4} \cdot \|w_t\|^{5/4} \le \delta \|w_{tx}\|^2 + c_\delta \|\nabla \mathbf{u}\|_4^{8/5} \cdot \|w_t\|^2.
\end{aligned}$$

Since $|I_4| = |I_3'|$, it follows that the same estimate holds for I_4.

By substituting the resulting inequalities into (2.11) with an appropriate δ and by estimating the right-hand side, we obtain

$$\begin{aligned}
\text{(2.14)} \quad & \frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|w_t\|^2) + \nu \|\mathbf{u}_{tx}\|^2 + \nu \|w_{tx}\|^2 + \varepsilon\nu \int_\Omega (\nabla \mathbf{u})^2 (\nabla \mathbf{u}_t)^2 \, dx \\
&+ \frac{\varepsilon\nu}{2} \big\| [(\nabla \mathbf{u})^2]_t \big\|^2 \le c\big(\|\mathbf{f}_t\|_{-1}^2 + \|g_t\|_{-1}^2 + (\|\nabla \mathbf{u}\|_4^4 + \|w_x\|^2)(\|\mathbf{u}_t\|^2 + \|w_t\|^2)\big).
\end{aligned}$$

It follows by the Gronwall inequality from (2.14) that

$$\begin{aligned}
& \max_{0 \le t \le T} \big(\|\mathbf{u}_t(t)\|^2 + \|w_t(t)\|^2\big) + \int_0^T (\|\mathbf{u}_{tx}\|^2 + \|w_{tx}\|^2) \, dt \\
\text{(2.15)} \qquad & \le \left(\|\mathbf{u}_t(0)\|^2 + \|w_t(0)\|^2 + \int_0^T (\|\mathbf{f}_t\|_{-1}^2 + \|g_t\|_{-1}^2) \, dt \right) \\
& \qquad \times \exp\left(c \int_0^T (\|\nabla \mathbf{u}\|_4^4 + \|w_x\|^2) \, dt \right).
\end{aligned}$$

The subsequent argument is similar to that in [5]. We introduce the space $\mathbf{V}$ of solenoidal vector functions $(\mathbf{u}, w)$ (vector functions satisfying the continuity equation) vanishing on $\partial\Omega \times [0, T]$ with the norm

$$\|(\mathbf{u}, w)\|_V = \left(\int_0^T \big(\|\mathbf{u}_x\|^2 + \|w_x\|^2 + \|\mathbf{u}_t\|^2 + \|w_t\|^2\big) \, dt \right)^{1/2} + \left(\int_0^T \|\nabla \mathbf{u}\|_4^4 \, dt \right)^{1/4}$$

and define a solution of problem (2.3) as a vector function $(\mathbf{u}, w) \in \mathbf{V}$ that coincides with $(\mathbf{u}^0, w^0)$ for $t = 0$ and satisfies the integral identity

$$\begin{aligned}
\text{(2.16)} \quad & \int_0^T \Big((\mathbf{u}_t, \mathbf{v}) + \nu(\mathbf{u}_x, \mathbf{v}_x) + \nu\varepsilon(|\nabla \mathbf{u}|^2 \nabla \mathbf{u}, \nabla \mathbf{v}) + \nu(w_x, h_x) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) \\
& \qquad + ((\mathbf{u} \cdot \nabla) w, h) + (w w_z, h) - (\mathbf{f}, \mathbf{v}) - (g, h) \Big) dt = 0 \quad \forall (\mathbf{v}, h) \in \mathbf{V}.
\end{aligned}$$

One can use a standard technique based on the Galerkin method and the estimate (2.15) to prove that there exists a unique solution of (2.3) and that the norm $\|\mathbf{u}_x\|+\|w_x\|$ is continuous in time. Thus, we have proved the following theorem.

Theorem 2. *For each initial condition*

$$\mathbf{u}^0 \in \mathbf{H}^2 \cap \mathbf{H}_0^1, \quad w^0 \in H^2 \cap H_0^1, \quad \operatorname{div} \mathbf{u}^0 + w_z^0 = 0,$$

any right-hand sides $\mathbf{f}$ *and* g *such that*

$$\int_0^T (\|\mathbf{f}_t\|_{-1}^2 + \|g_t\|_{-1}^2)\, dt < \infty,$$

any $\nu > 0$ *and* $\varepsilon > 0$*, and an arbitrary time interval* $[0, T]$*, there exists a global solution of* (2.3)*; i.e., there exists a unique solution* $(\mathbf{u}, w) \in V$ *of problem* (2.3) *such that* (2.16) *holds, and the norm* $\|\mathbf{u}_x\|^2 + \|w_x\|^2$ *is continuous in time on* $[0, T]$. □

In the closing of this section, we mention the paper [2], where problem (2.1) is considered with

$$D(\mathbf{u}, w) = |\nabla \mathbf{u}|^q + |\partial_z \mathbf{u}|^q,$$

where q is some number less than 2; this strengthens Ladyzhenskaya's result as well.

3. Ocean dynamic equations

In what follows, we study an initial–boundary value problem for the system of ocean dynamic equations obtained from the Navier–Stokes equations under the assumption that the domain height is small compared with the horizontal dimensions (the hydrostatic approximation). This system is supplemented with nonlinear equations for temperature and salinity. The resulting system is known as primitive equations. For this initial–boundary value problem, a local existence theorem (that is, a theorem stating the existence of a solution on a small time interval) was proved in [18, 19]. We also note the papers [21, 20], where the existence of a global solution is proved under the assumption that the domain is small in the vertical variable. In [10], the existence of a global solution is proved in a cylinder whose base is a polygon.[1] The technique used there is different from that considered in the present paper.

In what follows, we prove the global existence of a solution of the ocean dynamic equations without any restrictions on the domain size, the viscosity coefficient, the norms of the initial conditions and right-hand sides, and the time interval (see [3, 14, 16]).

Let us proceed to the statement of the problem. Let Ω be a cylinder in $\mathbb{R}^3$ of the form

$$\Omega = \{x = (x_1, x_2, x_3) \colon (x_1, x_2) \in \Omega', \, x_3 \in [0, l]\},$$

where Ω' is a bounded two-dimensional domain whose boundary is formed by finitely many smooth arcs meeting at nonzero angles. Let us represent $\partial\Omega$ in the form $\partial\Omega = S_1 \cup S$, $S = \partial\Omega' \times [0, l]$; i.e., S_1 consists of the lower and upper bases of the cylinder, and S is the lateral surface.

Just as before, let $\mathbf{u} = (u_1, u_2)$ be the vector of horizontal velocity components, and let w be the vertical component. From now on, we assume that the indices i and j run over the values 1 and 2 and the index k varies from 1 to 3. Sometimes we denote the vertical

[1] The paper actually deals with a cylinder whose base is a domain with piecewise smooth boundary, but in this case one should have estimated the boundary integral; for reason unknown, this integral is treated in the paper as if it were zero, which is only true for a polygonal boundary.

velocity component by u_3. The independent variables will be denoted by $x = (x_1, x_2, x_3)$ or by (x, y, z). We also use the following additional notation:

$$\partial_k v = \frac{\partial v}{\partial x_k}, \quad \nabla v = (\partial_1 v, \partial_2 v), \quad \operatorname{div} \mathbf{u} = \partial_1 u_1 + \partial_2 u_2, \quad |v|_q = \left(\int_{\Omega'} |v|^q \, dx_1 \, dx_2 \right)^{1/q}.$$

As usual, the letter c with or without indices is used in inequalities to denote constants independent of the functions occurring in these inequalities but depending on the domain geometry, the time interval, and the norms of the right-hand sides and the initial conditions. Distinct constants may be denoted by one and the same letter as long as this does not lead to a misunderstanding.

The system of equations (known as the *primitive equations*) describing the large-scale ocean dynamics in the Cartesian coordinate system has the form [6, 18]

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + l\mathbf{u} + \nabla p + u_j \mathbf{u}_{x_j} + w\mathbf{u}_{x_3} = \mathbf{0}, \tag{3.1}$$

$$\frac{\partial p}{\partial x_3} = -g\rho, \tag{3.2}$$

$$\operatorname{div} \mathbf{u} + w_{x_3} = 0, \tag{3.3}$$

$$\rho_t - \nu_1 \Delta\rho + u_j \rho_{x_j} + w\rho_{x_3} = 0, \tag{3.4}$$

where ν and ν_1 are positive constants. Equations (3.1)–(3.4) must be supplemented with the initial and boundary conditions

$$\begin{gathered} \mathbf{u} \cdot \mathbf{n} = \frac{\partial \mathbf{u}}{\partial n} \times \mathbf{n} = 0 \text{ on } S, \quad \frac{\partial \mathbf{u}}{\partial n} = \mathbf{0} \text{ on } S_1, \quad w = 0 \text{ on } S_1, \\ \frac{\partial \rho}{\partial n} = 0 \text{ on } \partial\Omega, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \int_0^l \operatorname{div} \mathbf{u}_0 \, dz = 0, \quad \rho(x, 0) = \rho_0(x). \end{gathered} \tag{3.5}$$

Here and in what follows, summation over repeated indices in a product is assumed, g is the acceleration due to gravity, $\mathbf{n}$ is the unit outward normal vector on S, $\partial/\partial n$ is the normal derivative, and $\mathbf{a} \times \mathbf{b} = a_1 b_2 - a_2 b_1$. The operator $l\mathbf{u}$ (the Coriolis force) has the form

$$l\mathbf{u} = \omega(u_2, -u_1), \quad \omega = \text{const}.$$

It is easily seen that l is skew-symmetric; i.e., $(l\mathbf{u}, \mathbf{u}) = 0$.

To simplify the exposition, we assume that the right-hand sides of (3.1) and (3.4) are zero. It follows from the subsequent exposition that the whole argument remains the same for the case of more complicated linear operators and nonzero right-hand sides. In contrast to the above-mentioned papers, we deal with the case of Cartesian coordinates, but all the results can readily be transferred to the case of spherical coordinates [1].

Since the existence of a local solution has already been proved in [18] for problem (3.1)–(3.5), we see that, to prove the existence of a global solution, it suffices to have an appropriate a priori estimate. We split the derivation of such an estimate into several stages.

4. A priori estimates for the density and pressure

4.1. Estimates of the norms $\max_t \|\rho\|_4$ and $\max_t \|\mathbf{u}\|$.

Lemma 4. *Any solution of problem* (3.1)–(3.5) *satisfies the estimate*

$$\max_{0 \le t \le T} \|\rho(t)\|_4^4 + \nu_1 \int_0^T \|\rho\rho_x\|^2 \, dt \le c\|\rho_0\|_4^4. \tag{4.1}$$

Proof. Assume that there exists a sufficiently smooth solution. Take the inner product of (3.4) by ρ^3 in $L_2(\Omega)$. Then we obtain

$$\frac{1}{4}\frac{d}{dt}\|\rho\|_4^4 + \frac{3}{4}\big\|(\sqrt{\nu_1}\rho^2)_x\big\|^2 = 0.$$

By integrating with respect to t, we arrive at (4.1). The proof of the lemma is complete. □

Relations (3.2) and (4.1) imply the inequality

$$\max_{0\le t\le T} \|\partial_3 p(t)\|_4 \le c\|\rho_0\|_4. \tag{4.2}$$

Now let us find an a priori estimate for $\|\mathbf{u}\|$.

Lemma 5. *Any solution of problem* (3.1)–(3.5) *satisfies the estimate*

$$\max_{0\le t\le T} \|\mathbf{u}(t)\|^2 + \nu\int_0^T \|\mathbf{u}_x\|^2\,dt \le c(\|\mathbf{u}_0\|^2 + \|\rho_0\|_4^2) \equiv c. \tag{4.3}$$

Proof. Take the inner product of (3.1) by $\mathbf{u}$ in $\mathbf{L}_2(\Omega)$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\mathbf{u}_x\|^2 - (p, \operatorname{div}\mathbf{u}) = 0. \tag{4.4}$$

Let us estimate the inner product in (4.4). It follows from the continuity equation (3.3), the boundary condition $w = 0$ on S_1, and embedding theorems that

$$\|u_3\| \le c\|\partial_3 u_3\| \le c\|\operatorname{div}\mathbf{u}\| \le c\|\mathbf{u}_{x'}\|. \tag{4.5}$$

Then

$$|(p, \operatorname{div}\mathbf{u})| = (\text{by (2.3)}) = |(p, \partial_3 u_3)| = |(\partial_3 p, u_3)| \le \|\partial_3 p\|\cdot\|u_3\|$$
$$\le (\text{by (3.5)}) \le c\|\partial_3 p\|\cdot\|\mathbf{u}_{x'}\| \le \frac{\nu}{2}\|\mathbf{u}_{x'}\|^2 + \frac{c}{\nu}\|\partial_3 p\|^2.$$

This inequality, together with (4.4), gives

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \frac{\nu}{2}\|\mathbf{u}_x\|^2 \le \frac{c}{\nu}\|\partial_3 p\|^2.$$

By integrating the last inequality with respect to t and by taking into account (4.2), we obtain (4.3). The proof of the lemma is complete. □

Corollary 2. *The vertical component of the velocity vector satisfies the estimate*

$$\int_0^T (\|u_3\|^2 + \|\partial_3 u_3\|^2)\,dt \le c. \tag{4.6}$$

Proof. Inequality (4.6) readily follows from (4.5) and (4.2). □

4.2. **Estimate of the norm $\|p\|_4$.** The estimate of the norm of the hydrostatic pressure function plays a key role in the proof of the existence of a global solution. In what follows, we obtain an a priori estimate for $\|p\|_4$ via $\|\mathbf{u}\mathbf{u}_x\|$.

Theorem 3. *Let $v = \mathbf{u}^2$. Then any solution of problem* (3.1)–(3.5) *satisfies the estimate*

$$\|p\|_4 \le c\big((\|v_x\|^{1/2} + \|v\|^{1/2} + 1)\|v\|^{1/2} + 1\big). \tag{4.7}$$

Prior to proving the theorem, we need some estimates that hold for functions in the Sobolev space $H^1(\Omega)$. Let $H_0^1 \subset H^1$ be the subspace of functions vanishing on the boundary (in the sense of traces).

Lemma 6. *Let $f \in H^1(\Omega)$, where Ω is a Lipschitz domain in $\mathbb{R}^3$. Then*

$$\|f\|_4 \le c_1\big(\|f\| + \|\nabla f\|^{3/4}\cdot\|f\|^{1/4}\big). \tag{4.8}$$

Proof. If $g \in H_0^1$, then [4]

$$\|g\|_4 \le c\|g_x\|^{3/4} \cdot \|g\|^{1/4}. \tag{4.9}$$

Let $\Omega \subset G$. Then f can be extended from Ω to G with Sobolev class and norm preserved [8] in such a way that the extension vanishes on the boundary of G. Namely, there exists a function $\tilde f \in H_0^1(G)$ such that

$$\|\tilde f\|_{W_2^1(G)} \le c\|f\|_{W_2^1(\Omega)}.$$

It follows from the proof of the existence of $\tilde f$ that the norm in L_4 is preserved as well,

$$\|\tilde f\|_{L_4(G)} \le c\|f\|_{L_4(\Omega)}.$$

Then one has the chain of inequalities

$$\|f\|_{L_4(\Omega)} \le c\|\tilde f\|_{L_4(G)} \le c\|\nabla \tilde f\|_G^{3/4} \cdot \|\tilde f\|_G^{1/4}$$
$$\le c\|f\|_\Omega^{1/4}(\|f\|_\Omega^2 + \|\nabla f\|_\Omega^2)^{3/8} \le c\big(\|f\| + \|\nabla f\|^{3/4} \cdot \|f\|^{1/4}\big).$$

This completes the proof of the lemma. □

In a similar way, one can obtain the corresponding estimate in the two-dimensional case. Namely, the following claim is true.

Lemma 7. *In the two-dimensional case, one has the estimate*

$$\|f\|_4 \le c_1(\|f\| + \|\nabla f\|^{1/2} \cdot \|f\|^{1/2}). \tag{4.10}$$

Proof of Theorem 3. Let us represent p in the form $p = p_1 + p_2$, where

$$p_1(x,y) = \int_0^l p(x,y,z)\,dz.$$

Then p_2 satisfies $\int_0^l p_2\,dz = 0$ and is a z-antiderivative of p_z. Since p for given t is determined by equations (3.1) and (3.2) up to an additive constant, we can choose this constant in such a way that $(p_1^3, 1) = 0$. This representation, together with inequality (4.2), implies the following estimate for the norm of p_2:

$$\max_{0\le t\le T} \|p_2\|_4 \le c\|\partial_3 p_2\|_4 \le c. \tag{4.11}$$

Let us estimate the norm of p_1. Let $q = q(x,y)$ be the solution of the following boundary value problem in the domain Ω':

$$\Delta_2 q = p_1^3, \quad \frac{\partial q}{\partial n}\Big|_{\partial\Omega'} = 0, \quad (q,1)_{\Omega'} = 0,$$

where Δ_2 is the Laplace operator with respect to the variables x and y. Then the solution of this problem satisfies the inequality

$$\|q_{x_i x_j}\|_{4/3} \le c\|p_1\|_4^3. \tag{4.12}$$

Let

$$\mathbf{h}(x,y) = (h_1, h_2) = \int_0^l \mathbf{u}\,dz.$$

Let us represent $\mathbf{u}$ in the form $\mathbf{u} = \mathbf{h} + \mathbf{r}$. Then

$$\int_0^l \mathbf{r}\,dz = \mathbf{0}.$$

The continuity equation for $\mathbf{h}$ implies the relation

$$\operatorname{div}\mathbf{h} = \frac{1}{l}\int_0^l (\operatorname{div}\mathbf{h} + \operatorname{div}\mathbf{r})\,dz = \frac{1}{l}\int_0^l \operatorname{div}\mathbf{u}\,dz = -\frac{1}{l}\int_0^l \frac{\partial w}{\partial z}\,dz = 0,$$

and $\mathbf{h}$ satisfies the boundary conditions (see (3.5))

$$\mathbf{h}\cdot\mathbf{n} = \left(\frac{\partial \mathbf{h}}{\partial n}\right)\times\mathbf{n} = 0$$

on the lateral surface S of the cylinder. Then $\mathbf{h}$ can be represented (e.g., see [4]) in the form $\mathbf{h} = \operatorname{rot}\psi$, where $\operatorname{rot}\psi = (\psi_y, -\psi_x)$ satisfies the boundary conditions

$$\operatorname{rot}\psi\cdot\mathbf{n} = \left(\frac{\partial}{\partial n}\operatorname{rot}\psi\right)\times\mathbf{n} = 0 \quad \text{on } \partial\Omega'.$$

The first of these conditions gives $\partial\psi/\partial\tau = 0$.

Let us estimate the inner products occurring in the multiplication of (3.1) by ∇q. By integrating by parts and by using the above relations, we obtain

$$(\mathbf{u}_t, \nabla q) = -(\operatorname{div}\mathbf{u}_t, q) = (w_{tz}, q) = 0 \tag{4.13}$$

and

$$(\partial_3^2\mathbf{u}, \nabla q) = 0 \tag{4.14}$$

by (3.5).

Further,

$$\begin{aligned}
-(\Delta_2\mathbf{u}, \nabla q) &= -(\Delta_2(\operatorname{rot}\psi + \mathbf{r}), \nabla q)\\
&= -(\operatorname{rot}\Delta_2\psi, \nabla q) \qquad \left(\text{because } \textstyle\int_0^l \mathbf{r}\,dz = \mathbf{0}\right)\\
&= (\nabla\Delta_2\psi, \operatorname{rot} q)\\
&= \int_{\partial\Omega'}\Delta_2\psi(q_y n_1 - q_x n_2)\,ds - (\Delta_2\psi, \operatorname{div}\operatorname{rot} q) = \int_{\partial\Omega'}\Delta_2\psi\frac{\partial q}{\partial\tau}\,ds \equiv I.
\end{aligned}$$

Let us estimate the boundary integral I. The differentiation of the first boundary condition in (3.5) in the tangential direction gives

$$\begin{aligned}
\frac{\partial}{\partial\tau}(\mathbf{h}\cdot\mathbf{n}) &= (\partial_1 h_1\, n_1 + h_1\,\partial_1 n_1)n_2 - (\partial_2 h_1\, n_1 + h_1\,\partial_2 n_1)n_1\\
&\quad + (\partial_1 h_2\, n_2 + h_2\,\partial_1 n_2)n_2 - (\partial_2 h_2\, n_2 + h_2\,\partial_2 n_2)n_1 = J + K = 0,
\end{aligned}$$

where

$$\begin{aligned}
J &= (\partial_1 h_1 - \partial_2 h_2)n_1 n_2 - \partial_2 h_1\, n_1^2 + \partial_1 h_2\, n_2^2 = 2\psi_{xy}n_1n_2 - \psi_{yy}n_1^2 - \psi_{xx}n_2^2,\\
K &= h_1\,\partial_1 n_1\, n_2 - h_1\,\partial_2 n_1\, n_1 + h_2\,\partial_1 n_2\, n_2 - h_2\,\partial_2 n_2\, n_1.
\end{aligned}$$

From the second boundary condition, we obtain

$$\begin{aligned}
\frac{\partial\mathbf{h}}{\partial n}\times\mathbf{n} &= (\partial_1 h_1\, n_1 + \partial_2 h_1\, n_2)n_2 - (\partial_1 h_2\, n_1 + \partial_2 h_2\, n_2)n_1\\
&= 2\psi_{xy}n_1n_2 + \psi_{yy}n_2^2 + \psi_{xx}n_1^2\\
&= \Delta_2\psi + 2\psi_{xy}n_1n_2 - \psi_{yy}n_1^2 - \psi_{xx}n_2^2 = \Delta_2\psi + J = 0.
\end{aligned}$$

Thus,

$$\Delta_2\psi = K \quad \text{on } \partial\Omega'. \tag{4.15}$$

The function ψ of two variables satisfies the estimate [7]

$$\|\psi_{x_ix_j}\|_{4/3,\,\Omega'} \le c\|\Delta\psi\|_{4/3,\,\Omega'}. \tag{4.16}$$

Recall that $|\cdot|$ stands for the norm of a function of the three variables (x, y, z) with respect to the variables (x, y). Then

$$
\begin{aligned}
|I| = \left| \int_{\partial\Omega'} \Delta\psi \frac{\partial q}{\partial \tau}\, ds \right| &= \left| \int_{\partial\Omega'} K \frac{\partial q}{\partial \tau}\, ds \right| \qquad \text{(by (4.15))} \\
&\le c\|\mathbf{h}\|_{4,\partial\Omega'} \cdot \|q_x\|_{4/3,\,\partial\Omega'} \\
&\le c\left(\int_{\partial\Omega'} \mathbf{h}^4\, ds \right)^{1/4} (\|q_{xx}\|_{4/3,\,\Omega'} + \|q_x\|_{4/3,\,\Omega'}) \\
&= c\left(\int_{\partial\Omega'} \left(\frac{1}{l} \int_0^l \mathbf{u}\, dz \right)^4 ds \right)^{1/4} (\|q_{xx}\|_{4/3,\,\Omega'} + \|q_x\|_{4/3,\,\Omega'}).
\end{aligned}
$$

By continuing the preceding inequality, we obtain

$$
\begin{aligned}
|I| &\le c\left(\int_{\partial\Omega'} \left(\frac{1}{l} \int_0^l \mathbf{u}\, dz \right)^4 ds \right)^{1/4} (\|q_{xx}\|_{4/3,\,\Omega'} + \|q_x\|_{4/3,\,\Omega'}) \\
&\le c \int_0^l \left(\int_{\partial\Omega'} v^2\, ds \right)^{1/4} dz\, \|p_1\|^3_{4,\,\Omega'} \qquad \text{(by (4.12) and the Minkowski inequality)} \\
&\le c \int_0^l (|v_{x'}|^{1/2} + |v|^{1/2})\, dz\, \|p_1\|_4^3 \\
&\le c(\|v_x\|^{1/2} + \|v\|^{1/2})\|p_1\|_4^3.
\end{aligned}
$$

Thus,

$$
|I| \le c(\|v_x\|^{1/2} + \|v\|^{1/2})\|p_1\|_4^3. \tag{4.17}
$$

Let us continue the estimate of the norm of p. Take the inner product of (3.1) by ∇q. In view of (4.13)–(4.15), we obtain

$$
\|p_1\|_4^4 + \nu I + (u_k \partial_k u_j, \partial_j \Delta_2^{-1} p_1^3) + (l\mathbf{u}, \nabla \Delta_2^{-1} p_1^3) = 0; \tag{4.18}
$$

here Δ_2^{-1} is the inverse operator of Δ_2 with the Neumann boundary conditions. Recall once more that summation over repeated indices in products is assumed with j varying from 1 to 2 and k varying from 1 to 3, and $u_3 = w$.

Inequality (4.10) has the form

$$
|f|_4 \le c(|\nabla f|^{1/2} + |f|^{1/2})|f|^{1/2}. \tag{4.19}
$$

Let us estimate the inner products in (4.18). To this end, we use the Hölder inequality with exponents 4 and 4/3. We have

(4.20)

$$
\left|(l\mathbf{u}, \nabla \Delta_2^{-1} p_1^3)\right| \le c\|\mathbf{u}\|_4 \cdot \|\nabla \Delta_2^{-1} p_1^3\|_{4/3} \le c\|\mathbf{u}\|_4 \left(\int_\Omega |p_1|^4\, dx \right)^{3/4} = c\|v\|^{1/2} \cdot \|p_1\|_4^3.
$$

Let us estimate the remaining inner product in (4.18). A substantial role is played here by the fact that p_1 only depends on two spatial variables. By using the Hölder inequality, we obtain

(4.21)

$$\begin{aligned}\left|(u_iu_j, \partial_i\partial_j\Delta_2^{-1}p_1^3)\right| &\le \int_0^l\int_{\Omega'} \underbrace{|u_iu_j|}_{4}\cdot\underbrace{|\partial_i\partial_j\Delta_2^{-1}p_1^3|}_{4/3}\,dx_1\,dx_2\,dz\\ &\le c\int_0^l |v|_4\cdot|p_1|_4^3\,dz \le c|p_1|_4^3\int_0^l(|\nabla v|^{1/2}+|v|^{1/2})|v|^{1/2}\,dz\\ &\le c|p_1|_4^3\left[\left(\int_0^l|\nabla v|^2\,dz\right)^{1/4}\left(\int_0^l \underbrace{|v|^{2/3}}_{3}\,dz\right)^{3/4}+\int_0^l|v|_{\Omega'}\,dz\right]\\ &\le c\|p_1\|_4^3(\|v_x\|^{1/2}+\|v\|^{1/2})\|v\|^{1/2}.\end{aligned}$$

The estimates (4.20), (4.21), and (4.18) imply the inequality

$$\|p_1\|_4 \le c(\|v_x\|^{1/2}+\|v\|^{1/2}+1)\|v\|^{1/2}, \tag{4.22}$$

and (4.11) and (4.22), in turn, imply (4.7). The proof of the theorem is complete. □

5. Estimate of norms of the velocity vector

5.1. Estimate of the norms $\max_t\|\mathbf{u}\|_4$ **and** $\|\mathbf{u}\mathbf{u}_x\|$**.** In this section, we use the preceding inequalities to obtain estimates for the norms $\max_t\|\mathbf{u}\|_4$ and $\|\mathbf{u}\mathbf{u}_x\|$.

Lemma 8. *The norms* $\max_t\|\mathbf{u}\|_4$ *and* $\int_0^T\|\mathbf{u}\mathbf{u}_x\|^2\,dt$ *satisfy the estimate*

$$\max_{0\le t\le T}\|\mathbf{u}\|_4^4+\nu\int_0^T\|\mathbf{u}\mathbf{u}_x\|^2\,dt\le c, \tag{5.1}$$

where the constant c *depends on the initial data of the problem.*

Proof. Let $\mathbf{u}^2 = v$, as before. Take the inner product of (3.1) by $\mathbf{u}^2\mathbf{u}$ in $\mathbf{L}_2$. We obtain

$$\frac{1}{4}\frac{d}{dt}\|\mathbf{u}(t)\|_4^4-\nu(\Delta\mathbf{u},\mathbf{u}^2\mathbf{u})+(\nabla p,\mathbf{u}^2\mathbf{u})+(u_j\mathbf{u}_{x_j},\mathbf{u}^2\mathbf{u})+(u_3\mathbf{u}_z,\mathbf{u}^2\mathbf{u})=0. \tag{5.2}$$

Let us transform the inner products in (5.2). Note that any solenoidal smooth vector functions $\mathbf{u}$ and $\mathbf{v}$ whose normal components vanish on the boundary satisfy the relation

$$\begin{aligned}(u_k\mathbf{u}_{x_k},\mathbf{u}^2\mathbf{u}) &= (u_k\partial_k u_j,\mathbf{u}^2u_j)=\frac{1}{2}(u_k\partial_k\mathbf{u}^2,\mathbf{u}^2)\\ &=\frac{1}{4}\big(u_k,((\mathbf{u}^2)^2)_{x_k}\big)=-\frac{1}{4}(\operatorname{div}\mathbf{u},v^2)+\int_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})v^2\,ds=0.\end{aligned}$$

Further,

$$\begin{aligned}-(\Delta\mathbf{u},\mathbf{u}^2\mathbf{u}) &= -(\partial_k^2u_i,\mathbf{u}^2u_i)=(\partial_ku_i,\partial_k(vu_i))\\ &=(|\nabla\mathbf{u}|^2,v)+\frac{1}{2}(\partial_kv,\partial_kv)=\int_\Omega v\,|\nabla\mathbf{u}|^2\,dx+\frac{1}{2}\|v_x\|^2;\end{aligned}$$

hence (5.2) can be rewritten as

$$\frac{1}{4}\frac{d}{dt}\|v(t)\|^2+\nu\int_\Omega v|\nabla\mathbf{u}|^2\,dx+\frac{\nu}{2}\|v_x\|^2+(\nabla p,\mathbf{u}^2\mathbf{u})=0. \tag{5.3}$$

By using inequalities (4.11) and (4.22) for the norms of p_1 and p_2, we estimate the inner product in (5.3) as follows:

$$\begin{aligned}|(\nabla p,\mathbf{u}^2\mathbf{u})| &= |(p,\operatorname{div}(\mathbf{u}^2\mathbf{u}))|\le|(p\mathbf{u},\nabla\mathbf{u}^2)|+|(p\mathbf{u}^2,\operatorname{div}\mathbf{u})|\\ &\le c(|p|v,|\mathbf{u}_x|)\le c\big((|p_1|v,|\mathbf{u}_x|)+(|p_2|v,|\mathbf{u}_x|)\big).\end{aligned} \tag{5.4}$$

We separately estimate each inner product on the right-hand side in (5.4),

$$
\begin{aligned}
(|p_1|v, |\mathbf{u}_x|) &\le \int_0^l |p_1|_4 \cdot |v|_4 \cdot |\mathbf{u}_x|\, dz \\
&= |p_1|_4 \int_0^l |v|_4 \cdot |\mathbf{u}_x|\, dz \\
&\le c|p_1|_4 \int_0^l (|\nabla v|^{1/2} + |v|^{1/2})|v|^{1/2} \cdot |\mathbf{u}_x|\, dz \qquad \text{(by (4.19))} \\
&\le c|p_1|_4 \left(\int_0^l (|\nabla v| + |v|)|v|\, dz \right)^{1/2} \left(\int_0^l |\mathbf{u}_x|^2\, dz \right)^{1/2} \\
&\le c|p_1|_4 (\|\nabla v\|^{1/2} + \|v\|^{1/2})\|v\|^{1/2} \cdot \|\mathbf{u}_x\| \\
&= c\|p_1\|_4 (\|v_{x'}\|^{1/2} + \|v\|^{1/2})\|v\|^{1/2} \cdot \|\mathbf{u}_x\| \\
&\le c(\|v_{x'}\| + \|v\| + 1)\|v\| \cdot \|\mathbf{u}_x\| \qquad \text{(by (4.22))} \\
&\le \frac{\nu}{4}\|v_x\|^2 + \frac{c}{\nu}\|v\|^2 \cdot \|\mathbf{u}_x\|^2 + c(\|v\|^2 + 1).
\end{aligned} \tag{5.5}
$$

Then, by using (4.11) and the Young inequality, we obtain

$$
\begin{aligned}
(|p_2|v, |\mathbf{u}_x|) &\le \|p_2\|_4 \cdot \|v\|_4 \cdot \|\mathbf{u}_x\| \\
&\le c(\|v_x\|^{3/4} + \|v\|^{3/4})\|v\|^{1/4} \cdot \|\mathbf{u}_x\| \le \frac{\nu}{4}\|v_x\|^2 + c\|v\|^2 + c\|\mathbf{u}_x\|^2.
\end{aligned} \tag{5.6}
$$

Let us substitute (5.5) and (4.6) into (5.2),

$$
\frac{d}{dt}\|v(t)\|^2 + \nu \int_\Omega \mathbf{u}^2 |\mathbf{u}_x|^2\, dx \le c(\|v\|^2 + 1)(\|\mathbf{u}_x\|^2 + 1). \tag{5.7}
$$

Then the Gronwall inequality, together with (5.7), implies the estimate

$$
\max_{0 \le t \le T} \|\mathbf{v}(t)\| = \max_{0 \le t \le T} \|\mathbf{u}(t)\|_4^2 \le c. \tag{5.8}
$$

In turn, we obtain, from (5.7) and (5.8),

$$
\int_0^T \int_\Omega \mathbf{u}^2 |\mathbf{u}_x|^2\, dx\, dt \le c. \tag{5.9}
$$

The estimates (5.8) and (5.9) give (5.1), as desired. □

Corollary 3. *The norm of the third component of the velocity vector satisfies the estimate*

$$
\int_0^T \int_\Omega u_j^2 (w_z)^2\, dx\, dt \le c, \quad j = 1, 2. \tag{5.10}
$$

Proof. Since $\|u_j w_z\| = \|u_j(\partial_1 u_1 + \partial_2 u_2)\|$, we see that the desired estimate follows from (5.9). □

Corollary 4. *The norm of the hydrostatic pressure function satisfies the inequality*

$$
\int_0^T \|p\|_4^4\, dt \le c. \tag{5.11}
$$

Proof. From (4.11), (4.22), (5.8), and (5.9), we obtain the chain of inequalities

$$
\|p\|_4 \le \|p_1\|_4 + \|p_2\|_4 \le c(\|\mathbf{u}\mathbf{u}_x\|^{1/2} + \|\mathbf{u}\|_4 + 1)\|\mathbf{u}\|_4 + c \le c(\|\mathbf{u}\mathbf{u}_x\|^{1/2} + 1).
$$

This, together with (5.9), implies (5.11). □

By summing all of these inequalities, we obtain

$$\max_{0\le t\le T}(\|\mathbf{u}\|_4^4+\|\rho\|_4^4)+\int_0^T\left(\|\rho_x\|^2+\|\rho\rho_x\|^2+\|\mathbf{u}\mathbf{u}_x\|^2+\|\mathbf{u}w_z\|^2+\|p\|_4^4\right)dt\le c. \tag{5.12}$$

5.2. **Estimate of the norm** $\max_t\|\mathbf{u}_z\|$**.** Note that, in contrast to the case of the Navier–Stokes equations, the estimate (5.12) is not sufficient for the existence of a global solution. That is, we need additional estimates. In this subsection, we prove the boundedness of $\max_{0\le t\le T}\|\mathbf{u}_z\|_4$. To this end, we differentiate (3.1) with respect to z. Note that this is possible, because the domain is a cylinder whose axis coincides with Oz. We write $\mathbf{u}_z=\mathbf{v}=(v_1,v_2)$ and $w_z=v_3$; then we have

$$\begin{gathered}\mathbf{v}_t-\nu\Delta\mathbf{v}+l\mathbf{v}+\nabla p_z+v_k\mathbf{u}_{x_k}+u_k\mathbf{v}_{x_k}=\mathbf{0},\\ \frac{\partial p}{\partial z}=-g\rho,\quad \operatorname{div}\mathbf{v}+v_{3z}=0,\\ \rho_{zt}-\nu_1\Delta\rho_z+v_k\rho_{x_k}+u_k\rho_{zx_k}=0.\end{gathered}\tag{5.13}$$

By using the same technique as above for the derivation of a priori estimates, we obtain an estimate for the norm $\max_t\|\mathbf{u}_z\|$.

Lemma 9. *The norm* $\max_t\|\mathbf{u}\|$ *satisfies the estimate*

$$\max_{0\le t\le T}\|\mathbf{v}(t)\|^2+\nu\int_0^T\|\mathbf{v}_x\|^2\,dt\le c.\tag{5.14}$$

Proof. Take the inner product of the first equation in (5.13) by $\mathbf{v}$ in $\mathbf{L}_2$,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2+\nu\|\mathbf{v}_x\|^2-(p_z,\operatorname{div}\mathbf{v}+v_{3z})+(v_k\mathbf{u}_{x_k},\mathbf{v})+(u_k\mathbf{v}_{x_k},\mathbf{v})=0.\tag{5.15}$$

By integrating by parts, we see that the last inner product in (5.15) is zero. Let us estimate the remaining two inner products. By using (4.2), we obtain

$$|(p_z,\operatorname{div}\mathbf{v})|\le\|p_z\|\cdot\|\operatorname{div}\mathbf{v}\|\le c\|\operatorname{div}\mathbf{v}\|\le\frac{\nu}{4}\|\mathbf{v}_x\|^2+c.$$

Further,

$$\begin{aligned}|(v_k\mathbf{u}_{x_k},\mathbf{v})|&=|(v_k\mathbf{u},\mathbf{v}_{x_k})|\le|(v_j\mathbf{u},\mathbf{v}_{x_j})|+|(v_3\mathbf{u},\mathbf{v}_{x_3})|\\&\le c\|\mathbf{u}\|_4\cdot\|\mathbf{v}\|_4\cdot\|\mathbf{v}_x\|+|(\operatorname{div}\mathbf{u}\cdot\mathbf{u},\mathbf{v}_{x_3})|\\&\le c\|\mathbf{v}_x\|^{7/4}\cdot\|\mathbf{v}\|^{1/4}+\|\operatorname{div}\mathbf{u}\cdot\mathbf{u}\|\cdot\|\mathbf{v}_x\|\qquad\text{(by (4.8) and (5.1))}\\&\le\frac{\nu}{4}\|\mathbf{v}_x\|^2+c\|\mathbf{v}\|^2+c\|\operatorname{div}\mathbf{u}\cdot\mathbf{u}\|^2.\end{aligned}$$

Consequently, it follows from (5.15) that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2+\frac{\nu}{2}\|\mathbf{v}_x\|^2\le c\|\mathbf{v}\|^2+c\|\operatorname{div}\mathbf{u}\cdot\mathbf{u}\|^2.\tag{5.16}$$

Relations (5.16) and (5.12) and the Gronwall inequality imply (5.14), as desired. □

5.3. **Estimate of the norm** $\max_t\|\mathbf{u}_z\|_4$**.** Now we are in a position to establish an estimate for the norm $\max_t\|\mathbf{v}\|_4$.

Theorem 4. *The norm* $\|\mathbf{u}_z\|_4$ *satisfies the inequality*

$$\max_{0\le t\le T}\|\mathbf{u}_z(t)\|_4^4+\nu\int_0^T\int_\Omega\left(\mathbf{u}_z^2(\mathbf{u}_{zx})^2+\frac{1}{2}[(\mathbf{u}_z^2)_x]^2\right)dx\,dt\le c.\tag{5.17}$$

Proof. Take the inner product of the first equation in (5.13) by $\mathbf{v}^2\mathbf{v}$. By integrating by parts, we obtain

$$(5.18)\quad \frac{1}{4}\frac{d}{dt}\|\mathbf{v}\|_4^4 + \nu\int_\Omega \left(\mathbf{v}^2(\mathbf{v}_x)^2 + \frac{1}{2}[(\mathbf{v}^2)_x]^2\right)dx \\ + (l\mathbf{v}, \mathbf{v}^2\mathbf{v}) - (p_z, \operatorname{div}(\mathbf{v}^2\mathbf{v})) + (v_k\mathbf{u}_{x_k}, \mathbf{v}^2\mathbf{v}) = 0.$$

Let us estimate the inner products in (5.18). The estimate of the first of them gives

$$|(l\mathbf{v}, \mathbf{v}^2\mathbf{v})| \le c\|\mathbf{v}\|_4^4.$$

The second inner product can be estimated as follows:

$$I_1 = |(p_z, \operatorname{div}(\mathbf{v}^2\mathbf{v}))| \le |(p_z, \mathbf{v}^2\operatorname{div}\mathbf{v})| + |(p_z, \mathbf{v}\nabla\mathbf{v}^2)|.$$

Let us estimate the first term on the right-hand side by using the Hölder inequality with exponents 4, 2, and 4 for the functions p_z, $|\mathbf{v}|\operatorname{div}\mathbf{v}$, and $|\mathbf{v}|$, respectively. We obtain

$$|(p_z, \mathbf{v}^2\operatorname{div}\mathbf{v})| \le \|p_z\|_4\cdot\|\operatorname{div}\mathbf{v}\cdot\mathbf{v}\|\cdot\|\mathbf{v}\|_4 \\ \le c\|\operatorname{div}\mathbf{v}\cdot\mathbf{v}\|\cdot\|\mathbf{v}\|_4 \le \varepsilon\int_\Omega \mathbf{v}^2(\mathbf{v}_x)^2\,dx + c_\varepsilon\|\mathbf{v}\|_4^2.$$

The second term on the right-hand side in the inequality for I_1 can be estimated in a similar way. Thus, this inequality implies the estimate

$$(5.19)\qquad |(p_z, \operatorname{div}(\mathbf{v}^2\mathbf{v}))| \le \varepsilon\int_\Omega \mathbf{v}^2(\mathbf{v}_x)^2\,dx + c_\varepsilon(\|\mathbf{v}_x\|^2 + \|\mathbf{v}\|^2).$$

Let us estimate the last inner product on the right-hand side in (5.18). We have

$$I_2 = |(v_k\mathbf{u}_{x_k}, \mathbf{v}^2\mathbf{v})| \le |(v_j\mathbf{u}_{x_j}, \mathbf{v}^2\mathbf{v})| + |(v_3\mathbf{u}_{x_3}, \mathbf{v}^2\mathbf{v})| \equiv I_2' + I_2''.$$

The first inner product can be estimated as follows. First, we integrate by parts in I_2' and estimate the absolute value of the sum of inner products by the sum of their absolute values,

$$I_2' = |(v_j\mathbf{u}_{x_j}, \mathbf{v}^2\mathbf{v})| \le |(\operatorname{div}\mathbf{v}\cdot\mathbf{v}, \mathbf{v}^2\mathbf{u})| + |(v_j\mathbf{u}, \mathbf{v}(\mathbf{v}^2)_{x_j})| + |(v_j\mathbf{u}, \mathbf{v}^2\mathbf{v}_{x_j})|.$$

The estimate of all the inner products on the right-hand side can be obtained in one and the same way. Let us estimate, say, the third inner product. To this end, we use the Hölder inequality with exponents 2, 4, and 4 for the functions $v_j\mathbf{v}_{x_j}$, $\mathbf{u}$, and $\mathbf{v}^2$, respectively. We obtain

$$I_2' = |(v_j\mathbf{u}, \mathbf{v}^2\mathbf{v}_{x_j})| \le c\|\mathbf{v}\mathbf{v}_x\|\cdot\|\mathbf{u}\|_4\left(\int_\Omega |\mathbf{v}|^8\,dx\right)^{1/4}.$$

Since the norm $\|\mathbf{u}\|_4$ is uniformly bounded with respect to t, and since the estimate $\|f\|_4 \le c\|f_x\|^{3/4}\cdot\|f\|^{1/4}$ holds for any function f of three variables vanishing on part of the boundary, we obtain

$$I_2' \le c\|\mathbf{v}\mathbf{v}_x\|\cdot\|(\mathbf{v}^2)_x\|^{3/4}\cdot\|\mathbf{v}^2\|^{1/4}.$$

By applying the Young inequality with exponents 2, 8/3, and 8, respectively, we obtain

$$(5.20)\qquad I_2' \le \varepsilon\int_\Omega \mathbf{v}^2(\mathbf{v}_x)^2\,dx + \varepsilon\int_\Omega [(\mathbf{v}^2)_x]^2\,dx + c_\varepsilon\|\mathbf{v}\|_4^4.$$

Let us estimate I_2''. We have

$$I_2'' = |(v_3\mathbf{u}_{x_3}, \mathbf{v}^2\mathbf{v})| = |(\operatorname{div}\mathbf{u}\cdot\mathbf{v}, \mathbf{v}^2\mathbf{v})| = |(\mathbf{u}\cdot\mathbf{v}^2, \nabla\mathbf{v}^2)|$$
$$\le \|\mathbf{u}\|_4\cdot\|\mathbf{v}^2\|_4\cdot\|\nabla\mathbf{v}^2\| \le c\|(\mathbf{v}^2)_x\|^{3/4}\cdot\|\mathbf{v}^2\|^{1/4}\cdot\|\nabla\mathbf{v}^2\| \tag{5.21}$$
$$\le \varepsilon\int_\Omega \mathbf{v}^2(\mathbf{v}_x)^2\,dx + \varepsilon\int_\Omega [(\mathbf{v}^2)_x]^2\,dx + c_\varepsilon\|\mathbf{v}\|_4^4.$$

Thus it follows from (5.18)–(5.21) that

$$\frac{1}{4}\frac{d}{dt}\|\mathbf{v}\|_4^4 + c_1\nu\int_\Omega\left(\mathbf{v}^2(\mathbf{v}_x)^2 + \frac{1}{2}[(\mathbf{v}^2)_x]^2\right)dx \le c_2\big(\|\mathbf{v}_x\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{v}\|_4^4\big).$$

We use the Gronwall inequality and (5.17) and see that this relation yields

$$\max_{0\le t\le T}\|\mathbf{v}(t)\|_4^4 + c_3\nu\int_0^T\int_\Omega\left(\mathbf{v}^2(\mathbf{v}_x)^2 + \frac{1}{2}[(\mathbf{v}^2)_x]^2\right)dx\,dt \le c_4. \tag{5.22}$$

Thus, we have established the boundedness of the norm $\|\mathbf{v}(t)\|_4$ for $t\in[0,T]$, which completes the proof. □

5.4. **Estimate of the norm** $\max_t\|\rho_z\|$**.** In this subsection, we estimate the norm $\max_t\|\rho_z\|$.

Lemma 10. *One has the estimate*

$$\max_{0\le t\le T}\|\rho_z\|^2 + \int_0^T\|\rho_{zx}\|^2\,dt \le c. \tag{5.23}$$

Proof. Take the inner product of the equation for ρ_z in (5.13) by ρ_z. We write $\rho_z = r$; then

$$\frac{1}{2}\frac{d}{dt}\|r\|^2 + \nu_1\|r_x\|^2 + (v_k\rho_{x_k}, r) = 0. \tag{5.24}$$

We estimate the inner product in (5.24) by using inequalities (4.1), (4.9) and (5.22),

$$(v_k\rho_{x_k}, r) = (v_j\partial_j\rho, r) - (\operatorname{div}\mathbf{u}\cdot\partial_3\rho, r) \equiv I_1 + I_2,$$
$$|I_1| = \big|(\operatorname{div}\mathbf{v}, \rho r) + (\rho\mathbf{v}, \nabla r)\big| \le \|\operatorname{div}\mathbf{v}\|\cdot\|\rho\|_4\cdot\|r\|_4 + \|\rho\|_4\cdot\|\mathbf{v}\|_4\cdot\|r_x\|$$
$$\le c\|\mathbf{v}_x\|\cdot\|r_x\|^{3/4}\cdot\|r\|^{1/4} + c\|r_x\| \le \varepsilon\|r_x\|^2 + c_\varepsilon(\|\mathbf{v}_x\|^2 + \|r\|^2) + c,$$
$$|I_2| = |(\operatorname{div}\mathbf{u}, r^2)| = 2|(\mathbf{u}\cdot\nabla r, r)|$$
$$\le 2\|\mathbf{u}\|_4\cdot\|r_x\|\cdot\|r\|_4 \le c\|r_x\|^{7/4}\cdot\|r\|^{1/4} \le \varepsilon\|r_x\|^2 + c_\varepsilon\|\rho_x\|^2.$$

Then, for sufficiently small ε, from (5.24) we obtain

$$\frac{d}{dt}\|r\|^2 + c\nu_1\|r_x\|^2 \le c_\varepsilon(\|r\|^2 + \|\mathbf{v}_x\|^2 + \|\rho_x\|^2) + c,$$

whence (5.23) follows by the Gronwall inequality. The proof of the lemma is complete. □

6. Estimate of the norms $\max_t\|\mathbf{u}_t\|$ and $\max_t\|\rho_t\|$

To complete the chain of a priori inequalities, let us estimate the norm $\max_t\|\mathbf{u}_t\|$. To this end, we differentiate equations (3.1)–(3.4) with respect to t,

$$\mathbf{u}_{tt} - \nu\Delta\mathbf{u}_t + l\mathbf{u}_t + \nabla p_t + u_{kt}\mathbf{u}_{x_k} + u_k\mathbf{u}_{tx_k} = \mathbf{0},$$
$$\operatorname{div}\mathbf{u}_t = 0, \tag{6.1}$$
$$\rho_{tt} - \nu_1\Delta\rho_t + u_{kt}\rho_{x_k} + u_k\rho_{tx_k} = 0.$$

6.1. **Estimate of the norms** $\max_t \|\mathbf{u}_t\|$ **and** $\max_t \|\rho_t\|$**.**

Theorem 5. *The norm* $\mathbf{u}_t$ *satisfies the estimate*

$$\max_{0\le t\le T}(\|\mathbf{u}_t(t)\|^2+\|\rho_t(t)\|^2)+\int_0^T(\|\mathbf{u}_{tx}\|^2+\|\rho_{tx}\|^2)\,dt\le c_T(\|\mathbf{u}_t(0)\|^2+\|\rho_t(0)\|^2). \tag{6.2}$$

Proof. Take the inner product of the first equation in (6.1) by $\mathbf{u}_t$ and of the third equation in (6.1) by ρ_t. We obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2+\nu\|\mathbf{u}_{tx}\|^2+(\nabla p_t,\mathbf{u}_t)+(u_{kt}\mathbf{u}_{x_k},\mathbf{u}_t)=0,\\ &\frac{1}{2}\frac{d}{dt}\|\rho_t\|^2+\nu_1\|\rho_{tx}\|^2+(u_{kt}\rho_{x_k},\rho_t)=0.\end{aligned} \tag{6.3}$$

Let us estimate the inner product in (6.3). For the first inner product, we have

$$\begin{aligned}|(\nabla p_t,\mathbf{u}_t)|&=|(p_t,\operatorname{div}\mathbf{u}_t)|=|(p_t,w_{tz})|=|(p_{tz},w_t)|\\ &\le c\|\rho_t\|\cdot\|w_{tz}\|\le c\|\rho_t\|\cdot\|\mathbf{u}_{tx}\|\le\varepsilon\|\mathbf{u}_{tx}\|^2+c_1\|\rho_t\|^2.\end{aligned}$$

The estimate for the second inner product is slightly more complicated,

$$\begin{aligned}&|(u_{kt}\mathbf{u}_{x_k},\mathbf{u}_t)|=|(u_{kt}\mathbf{u},\mathbf{u}_{tx_k})|\le|(u_{jt}\mathbf{u},\mathbf{u}_{tx_j})|+|(w_t\mathbf{u},\mathbf{u}_{tx_3})|\equiv I_1+I_2,\\ &I_1\le\|\mathbf{u}_{tx}\|\cdot\|\mathbf{u}_t\|_4\cdot\|\mathbf{u}\|_4\le c\|\mathbf{u}_{tx}\|\cdot\|\mathbf{u}_t\|_4\le c\|\mathbf{u}_{tx}\|\big(\|\mathbf{u}_t\|+\|\mathbf{u}_{tx}\|^{3/4}\cdot\|\mathbf{u}_t\|^{1/4}\big)\\ &\qquad\le c\big(\|\mathbf{u}_{tx}\|^{7/4}\cdot\|\mathbf{u}_t\|^{1/4}+\|\mathbf{u}_{tx}\|\cdot\|\mathbf{u}_t\|\big)\le\varepsilon\|\mathbf{u}_{tx}\|^2+c\|\mathbf{u}_t\|^2,\\ &I_2=|(w_t\mathbf{u},\mathbf{u}_{tx_3})|\le|(w_{tx_3}\mathbf{u},\mathbf{u}_t)|+|(w_t\mathbf{u}_{x_3},\mathbf{u}_t)|\equiv I_2'+I_2'',\\ &I_2'=|(\operatorname{div}\mathbf{u}_t,\mathbf{u}\mathbf{u}_t)|\le\|\mathbf{u}_{tx}\|\cdot\|\mathbf{u}\|_4\cdot\|\mathbf{u}_t\|_4\le\varepsilon\|\mathbf{u}_{tx}\|^2+c\|\mathbf{u}_t\|^2,\\ &I_2''=|(w_t\mathbf{u}_{x_3},\mathbf{u}_t)|=|(w_t\mathbf{v},\mathbf{u}_t)|\le\|w_t\|\cdot\|\mathbf{v}\|_4\cdot\|\mathbf{u}_t\|_4\le c\|w_{tx_3}\|\cdot\|\mathbf{u}_t\|_4\le\varepsilon\|\mathbf{u}_{tx}\|^2+c\|\mathbf{u}_t\|^2.\end{aligned}$$

Then

$$\begin{aligned}&|(u_{kt}\rho_{x_k},\rho_t)|=|(u_{kt}\rho,\rho_{tx_k})|\le|(u_{jt}\rho,\rho_{tx_j})|+|(w_t\rho,\rho_{tx_3})|,\\ &|(u_{jt}\rho,\rho_{tx_j})|\le c\|\mathbf{u}_t\|_4\cdot\|\rho\|_4\cdot\|\rho_{tx}\|\le c\|\mathbf{u}_t\|_4\cdot\|\rho_{tx}\|\\ &\qquad\le\varepsilon\|\rho_{tx}\|^2+c\|\mathbf{u}_t\|_4^2\le\varepsilon\|\rho_{tx}\|^2+\varepsilon\|\mathbf{u}_{tx}\|^2+c\|\mathbf{u}_t\|^2,\\ &|(w_t\rho,\rho_{tx_3})|\le|(w_{tz}\rho,\rho_t)|+|(w_t\rho_z,\rho_t)|\equiv I_3+I_4,\\ &I_3\le\|w_{tz}\|\cdot\|\rho\|_4\cdot\|\rho_t\|_4\le c\|\mathbf{u}_{tx}\|\cdot\|\rho_t\|_4\le\varepsilon\|\mathbf{u}_{tx}\|^2+c\|\rho_t\|_4^2\\ &\qquad\le\varepsilon\|\mathbf{u}_{tx}\|^2+\varepsilon\|\rho_{tx}\|^2+c\|\rho_t\|^2,\\ &I_4=|(u_{3t}\rho_{x_3},\rho_t)|=\left|\int_0^l\int_{\Omega'}\left(\int_0^{x_3}\operatorname{div}\mathbf{u}_t\,dz\right)\rho_{x_3}\rho_t\,dx_1\,dx_2\,dx_3\right|\\ &\qquad\le\int_0^l\left(\int_{\Omega'}\int_0^l|\operatorname{div}\mathbf{u}_t|\,dz\,\cdot|\rho_{x_3}|\cdot|\rho_t|\,dx_1\,dx_2\right)dx_3.\end{aligned}$$

Set

$$\int_0^l|\operatorname{div}\mathbf{u}_t|\,dz=q(x,y).$$

Then for the estimate of I_4 we have

$$\begin{aligned}I_4&\le\int_0^l\left(\int_{\Omega'}q|\rho_{x_3}|\cdot|\rho_t|\,dx_1\,dx_2\right)dx_3\le\int_0^l|q|\cdot|\rho_z|_4\cdot|\rho_t|_4\,dz\\ &\le c|q|\int_0^l(|\rho_{zx}|^{1/2}+|\rho_z|^{1/2})|\rho_z|^{1/2}\,(|\rho_{tx}|^{1/2}+|\rho_t|^{1/2})|\rho_t|^{1/2}\,dz.\end{aligned}$$

By applying the Cauchy–Schwarz inequality and by taking into account the relation

$$|q|^2 = \int_{\Omega'} q^2\,dx_1\,dx_2 = \int_{\Omega'}\left(\int_0^l |\operatorname{div}\mathbf{u}_t|\,dz\right)^2 dx_1\,dx_2$$
$$\le \int_{\Omega'}\int_0^l |\operatorname{div}\mathbf{u}_t|^2\,dz\,dx_1\,dx_2 \le c\|\mathbf{u}_{tx}\|^2,$$

we obtain

$$\begin{aligned} I_4 &\le c\|\mathbf{u}_{tx}\|\big(\|\rho_{zx}\|^{1/2}\cdot\|\rho_z\|^{1/2}\cdot\|\rho_{tx}\|^{1/2}\cdot\|\rho_t\|^{1/2} + \|\rho_z\|\cdot\|\rho_{tx}\|^{1/2}\cdot\|\rho_t\|^{1/2} \\ &\qquad + \|\rho_t\|\cdot\|\rho_{zx}\|^{1/2}\cdot\|\rho_z\|^{1/2} + \|\rho_z\|\cdot\|\rho_t\|\big) \\ &\le c\|\mathbf{u}_{tx}\|\big(\|\rho_{zx}\|^{1/2}\cdot\|\rho_{tx}\|^{1/2}\cdot\|\rho_t\|^{1/2} + \|\rho_{tx}\|^{1/2}\cdot\|\rho_t\|^{1/2} + \|\rho_t\|\cdot\|\rho_{zx}\|^{1/2} + \|\rho_t\|\big) \\ &\le \varepsilon\|\mathbf{u}_{tx}\|^2 + \varepsilon\|\rho_{tx}\|^2 + c(\|\rho_{zx}\|^2+1)\|\rho_t\|^2. \end{aligned}$$

We take an appropriate ε and substitute these estimates into (6.3). By summing the resulting inequalities, we obtain

$$\begin{aligned} \frac{d}{dt}(\|\mathbf{u}_t\|^2 + \|\rho_t\|^2) + c_1\nu\|\mathbf{u}_{tx}\|^2 + c_2\nu_1\|\rho_{tx}\|^2 \\ - c_3\|\rho_t\|^2 - c_4\|\mathbf{u}_t\|^2 - c_5(\|\rho_{zx}\|^2+1)\|\rho_t\|^2 \le 0. \end{aligned} \tag{6.4}$$

By integrating (6.4) with respect to t from 0 to $t \le T$ and by using the Gronwall inequality, we arrive at (6.2). The proof of the theorem is complete. □

6.2. **Estimate of the norms $\|\mathbf{u}_t(0)\|$ and $\|\rho_t(0)\|$.** Note that the right-hand side of (6.2) depends on some values that are not input data of the problem. Hence we need to estimate the right-hand side of (6.2) via the norms of the input data. Equation (3.4) for the density gives

$$\|\rho_t(0)\| \le c(\|\rho_0\|_{H^2} + \|\mathbf{u}_0\|_{H^2}). \tag{6.5}$$

Take the inner product of (3.1) by $\mathbf{u}_t$. We obtain

$$\|\mathbf{u}_t\|^2 = -\nu(\Delta\mathbf{u},\mathbf{u}_t) - (l\mathbf{u},\mathbf{u}_t) - (\nabla' p,\mathbf{u}_t) - (u_k\mathbf{u}_{x_k},\mathbf{u}_t) \equiv I_1 + I_2 + I_3 + I_4 = 0. \tag{6.6}$$

Let us estimate the inner products in (6.6),

$$\begin{aligned} |I_1| &\le \nu\|\mathbf{u}\|_{H^2}\cdot\|\mathbf{u}_t\|, \quad |I_2| \le c\|\mathbf{u}\|\cdot\|\mathbf{u}_t\|, \\ I_3 &= -(\nabla p_1,\mathbf{u}_t) - (\nabla p_2,\mathbf{u}_t) \equiv I_3' + I_3'', \\ I_3' &= (p_1,\operatorname{div}\mathbf{u}_t) = (p_1, w_{tz}) = 0, \\ |I_3''| &\le c\|p_{2zx}\|\cdot\|\mathbf{u}_t\| \le c\|\rho_x\|\cdot\|\mathbf{u}_t\|, \\ I_4 &= (u_j\mathbf{u}_{x_j},\mathbf{u}_t) + (w\mathbf{u}_{x_3},\mathbf{u}_t) \equiv I_4' + I_4'', \\ |I_4'| &\le \left(\int_\Omega \mathbf{u}^2\mathbf{u}_x^2\,dx\right)^{1/2}\|\mathbf{u}_t\| \le c\|\mathbf{u}\|_{H^2}^2\cdot\|\mathbf{u}_t\|, \\ |I_4''| &\le \left(\int_\Omega w^2\mathbf{u}_{x_3}^2\,dx\right)^{1/2}\|\mathbf{u}_t\| \le \|w\|_6\cdot\|\mathbf{u}_x\|_3\cdot\|\mathbf{u}_t\| \\ &\le c\|w\|_{H^1}\cdot\|\mathbf{u}\|_{H^2}\cdot\|\mathbf{u}_t\| \le c\|\mathbf{u}\|_{H^2}^2\cdot\|\mathbf{u}_t\|. \end{aligned}$$

In view of these inequalities, from (6.6) we obtain

$$\|\mathbf{u}_t(0)\| \le c\big(\|\mathbf{u}_0\|_{W_2^2}^2 + \|\rho_0\|_{W_2^2}\big). \tag{6.7}$$

Thus, inequalities (6.2), (6.5), and (6.7) imply the estimate

$$(6.8)\quad \max_{0\le t\le T}\big(\|\mathbf{u}_t(t)\|^2+\|\rho_t(t)\|^2\big)+\int_0^T\big(\|\mathbf{u}_{tx}\|^2+\|\rho_{tx}\|^2\big)dt\le c_T\big(\|\mathbf{u}_0\|^4_{W_2^2}+\|\rho_0\|^2_{W_2^2}\big).$$

It follows from (6.8) that one can obtain a stronger estimate for the norm $\|\rho_x\|$. Namely, the relation obtained by taking the inner product of (3.4) by ρ implies the inequality

$$\nu_1\|\rho_x\|^2=-(\rho_t,\rho)\le\|\rho_t\|\cdot\|\rho\|\le c\|\rho_t\|\le c,$$

whence, in view of (6.8), one has

$$(6.9)\qquad \max_{0\le t\le T}\|\rho_x\|\le c.$$

It is easily seen that the estimate (6.8) implies the continuity of the norm $\|\mathbf{u}_x\|$ in t on $[0,T]$. Indeed, the norm $\|\mathbf{u}_x\|$ treated as a function of t lies in $H^1[0,T]$. Since the embedding $H^1[0,T]\subset C[0,T]$, $\|\cdot\|_C\le c\|\cdot\|_{H^1}$, holds in the one-dimensional case, we see that the desired assertion follows from (6.8).

Since

$$w(x,t)=-\int_0^{x_3}\operatorname{div}\mathbf{u}(t,x_1,x_2,z)\,dz,$$

it follows, in particular, that the norm $\|u_3\|_{L_2(\Omega)}$ is continuous in $t\subset[0,T]$ as well.

By summing all inequalities obtained above, we arrive at the definitive a priori estimate

$$(6.10)\quad \max_{0\le t\le T}\big(\|\rho_x\|+\|\mathbf{u}_x\|+\|w\|+\|\mathbf{u}_z\|_4+\|\mathbf{u}_t\|+\|\rho_t\|\big)$$
$$+\int_0^T\big(\|\rho_{zx}\|^2+\|\rho_{tx}\|^2+\|\mathbf{u}_x\|^2+\|\mathbf{u}\mathbf{u}_x\|^2+\|\mathbf{u}_{zx}\|^2+\|\mathbf{u}_z\mathbf{u}_{zx}\|^2+\|\mathbf{u}_{tx}\|^2\big)dt\le c_T,$$

where c_T depends on the norm of the initial condition, the domain shape, the viscosity coefficients, and the time interval T.

7. Existence and uniqueness of the global solution

To define a solution, we need the spaces $\mathbf{V}$ and R. Here $\mathbf{V}$ is the space of vector functions $\mathbf{v}=(v_1,v_2)$ that belong to $\mathbf{H}^1(Q_T)$ and satisfy the boundary conditions (3.5) and the conditions

$$\mathbf{v}_z\in\mathbf{H}^1(Q_T)\quad\text{and}\quad\int_0^l\operatorname{div}\mathbf{v}(x,y,z,t)\,dz=0;$$

R is the space of functions $r\in H^1(Q_T)$ such that $r_z\in H^1(Q_T)$.

7.1. **Definition of a solution.** By taking the inner product of equation (3.4) for the density by $r\in R$ and by integrating with respect to t from 0 to t, we obtain, after integration by parts,

$$(7.1)\qquad \int_{Q_t}(-\rho r_t+\nu_1\rho_x r_x-u_k\rho r_{x_k})\,dx\,dt+\int_\Omega\rho r|_{t=t}\,dx-\int_\Omega\rho_0 r|_{t=0}\,dx=0.$$

Further, we take the inner product of (3.1) by $\mathbf{v}\in\mathbf{V}$ and integrate with respect to t, thus obtaining

$$(7.2)\quad \int_{Q_t}(-\mathbf{u}\mathbf{v}_t+\nu\mathbf{u}_x\mathbf{v}_x+l\mathbf{u}\mathbf{v}+\nabla p\cdot\mathbf{v}+u_k\mathbf{u}_{x_k}\mathbf{v})\,dx\,dt+\int_\Omega\mathbf{u}\mathbf{v}|_{t=t}\,dx-\int_\Omega\mathbf{u}_0\mathbf{v}|_{t=0}\,dx=0;$$

here $u_3=w$ is determined via $\mathbf{u}$ from the relations $\operatorname{div}\mathbf{u}+w_z=0$, $w(x,y,0,t)=0$.

Let us transform the terms in (7.2) containing the vertical velocity component and the hydrostatic pressure function,

$$(\nabla p, \mathbf{v}) = -(p, \operatorname{div} \mathbf{v}) = (p, \partial_3 v_3) = -(p_z, v_3) = g(\rho, v_3) = -g\left(\rho, \int_0^{x_3} \operatorname{div} \mathbf{v}\, dz\right),$$

$$(u_k \mathbf{u}_{x_k}, \mathbf{v}) = -(u_k \mathbf{u}, \mathbf{v}_{x_k}) = -(u_j \mathbf{u}, \mathbf{v}_{x_j}) + \left(\int_0^{x_3} \operatorname{div} \mathbf{u}\, dz\, \mathbf{u}, \mathbf{v}_{x_3}\right).$$

Then (7.2) becomes

(7.3)
$$\int_{Q_t}\left(-\mathbf{u}\mathbf{v}_t + \nu \mathbf{u}_x \mathbf{v}_x + l\mathbf{u}\mathbf{v} - g\rho \int_0^{x_3} \operatorname{div} \mathbf{v}\, dz - u_j \mathbf{u}\mathbf{v}_{x_j} + \int_0^{x_3} \operatorname{div} \mathbf{u}\, dz\, \mathbf{u}\mathbf{v}_{x_3}\right) dx\, dt$$
$$+ \int_\Omega \mathbf{u}\mathbf{v}|_{t=t}\, dx - \int_\Omega \mathbf{u}_0 \mathbf{v}|_{t=0}\, dx = 0.$$

Definition. A *solution* of problem (3.1)–(3.5) is a pair of functions $\mathbf{u} \in \mathbf{V}$ and $\rho \in R$ such that relations (7.1) and (7.3) are satisfied for any $\mathbf{v} \in \mathbf{V}$, $r \in R$, and $t \in [0, T]$.

7.2. **Uniqueness of the solution.** Let us prove the uniqueness of the solution of problem (3.1)–(3.5). Assume that there exist two solutions $(\mathbf{u}, p, \rho_1)$ and $(\mathbf{v}, q, \rho_2)$. Then their difference $(\mathbf{w}, r, \rho) = (\mathbf{u}, p, \rho_1) - (\mathbf{v}, q, \rho_2)$ satisfies the equations

$$\begin{gathered}
\mathbf{w}_t - \nu \Delta \mathbf{w} + l\mathbf{w} + \nabla' r + u_k \mathbf{w}_{x_k} + w_k \mathbf{v}_{x_k} = \mathbf{0}, \quad \frac{\partial r}{\partial x_3} = -g\rho, \\
\operatorname{div} \mathbf{w} = 0, \quad \rho_t - \operatorname{div}(\nu_1 \nabla \rho) + u_k \rho_{x_k} + w_k \rho_{2x_k} = 0, \\
\mathbf{w} \cdot \mathbf{n} = \partial_2 w_1 - \partial_1 w_2 = 0 \text{ on } S, \\
\frac{\partial \mathbf{w}}{\partial n} = \mathbf{0}, \ w_3 = 0 \text{ on } S_1, \quad \frac{\partial \rho}{\partial n} = 0 \text{ on } \partial\Omega, \\
\mathbf{w}(x, 0) = \mathbf{0}, \quad \rho(x, 0) = 0.
\end{gathered} \tag{7.4}$$

Take the inner product of the equation for ρ in (7.4) by ρ,

$$\frac{1}{2}\frac{d}{dt}\|\rho\|^2 + \|\sqrt{\nu_1}\rho_x\|^2 - (w_k \rho_2, \rho_{x_k}) = 0. \tag{7.5}$$

Let us estimate the inner product in (7.5). We have

$$\begin{aligned}
|(w_j \rho_2, \rho_{x_j})| &\le \|\mathbf{w}\|_4 \cdot \|\rho_2\|_4 \cdot \|\rho_x\| \le c\|\rho_x\| \cdot \|\mathbf{w}\|_4 \\
&\le \varepsilon\|\rho_x\|^2 + c_\varepsilon \|\mathbf{w}\|_4^2 \le \varepsilon\|\rho_x\|^2 + \varepsilon\|\mathbf{w}_x\|^2 + c_\varepsilon\|\mathbf{w}\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
|(w_3 \rho_2, \rho_{x_3})| &\le |(\operatorname{div} \mathbf{w} \rho_2, \rho)| + |(w_3 \rho_{2z}, \rho)|, \\
|(\operatorname{div} \mathbf{w} \rho_2, \rho)| &\le \|\operatorname{div} \mathbf{w}\| \cdot \|\rho_2\|_4 \cdot \|\rho\|_4 \le \varepsilon\|\operatorname{div} \mathbf{w}\|^2 + c_\varepsilon\|\rho\|_4^2 \\
&\le \varepsilon\|\operatorname{div} \mathbf{w}\|^2 + \varepsilon\|\rho_x\|^2 + c_\varepsilon\|\rho\|^2, \\
I = |(w_3 \rho_{2z}, \rho)| &= \left| \int_0^l \int_{\Omega'} \left(\int_0^{x_3} \operatorname{div} \mathbf{w}\, dz\right) \rho_{2x_3} \rho\, dx_1\, dx_2\, dx_3 \right| \\
&\le \int_0^l \left(\int_{\Omega'} \int_0^l |\operatorname{div} \mathbf{w}|\, dz\, |\rho_{2x_3}| \cdot |\rho|\, dx_1\, dx_2\right) dx_3.
\end{aligned}$$

Set $h = h(x,y) = \int_0^l |\operatorname{div} \mathbf{w}|\, dz$. Then

$$I \le \int_0^l \left(\int_{\Omega'} h\, |\rho_{2z}| \cdot |\rho|\, dx_1\, dx_2 \right) dz \le \int_0^l |h| \cdot |\rho_{2z}|_4 \cdot |\rho|_4\, dz$$
$$\le c|h| \int_0^l |\rho_{2zx}|^{1/2} \cdot |\rho_{2z}|^{1/2} \cdot |\rho_x|^{1/2} \cdot |\rho|^{1/2}\, dz.$$

Since

$$|h|^2 = \int_{\Omega'} h^2\, dx_1\, dx_2 \le \int_{\Omega'} \left(\int_0^l |\operatorname{div} \mathbf{w}|\, dz \right)^2 dx_1\, dx_2$$
$$\le \int_{\Omega'} \int_0^l |\operatorname{div} \mathbf{w}|^2\, dz\, dx_1\, dx_2 \le c\|\mathbf{w}_x\|^2,$$

we find that

$$I \le c\|\mathbf{w}_x\| \cdot \|\rho_{2zx}\|^{1/2} \cdot \|\rho_{2z}\|^{1/2} \cdot \|\rho_x\|^{1/2} \cdot \|\rho\|^{1/2}$$
$$\le c\|\mathbf{w}_x\| \cdot \|\rho_{2zx}\|^{1/2} \cdot \|\rho_x\|^{1/2} \cdot \|\rho\|^{1/2} \le \varepsilon \|\mathbf{w}_x\|^2 + c\|\rho_{2zx}\| \cdot \|\rho_x\| \cdot \|\rho\|$$
$$\le \varepsilon\|\mathbf{w}_x\|^2 + \varepsilon\|\rho_x\|^2 + c\|\rho_{2zx}\|^2 \cdot \|\rho\|^2.$$

By taking a sufficiently small ε, from (7.5) we obtain

$$\frac{d}{dt}\|\rho\|^2 + c_1\|\sqrt{\nu_1}\rho_x\|^2 - c\|\rho_{2zx}\|^2 \cdot \|\rho\|^2 - \varepsilon\|\mathbf{w}_x\|^2 \le 0. \tag{7.6}$$

Take the inner product of the first equation in (7.4) by $\mathbf{w}$,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \nu\|\mathbf{w}_x\|^2 + (\nabla r, \mathbf{w}) + (w_k \mathbf{v}_{x_k}, \mathbf{w}) = 0. \tag{7.7}$$

By transforming the inner product in (7.7), we obtain

$$I_1 = (\nabla r, \mathbf{w}) = -(r, \operatorname{div} \mathbf{w}) = (r, w_{3z}) = -(r_z, w_3),$$

whence it follows that

$$|I_1| \le c\|\rho\| \cdot \|\mathbf{w}_x\| \le \frac{\nu}{4}\|\mathbf{w}_x\|^2 + c\|\rho\|^2.$$

Then

$$I_2 = (w_k \mathbf{v}_{x_k}, \mathbf{w}) = (w_k \mathbf{v}, \mathbf{w}_{x_k}) = (w_j \mathbf{v}, \mathbf{w}_{x_j}) + (w_3 \mathbf{v}, \mathbf{w}_{x_3}) = I_2' + I_2'',$$
$$|I_2'| \le \|\mathbf{w}_x\| \cdot \|\mathbf{w}\|_4 \cdot \|\mathbf{v}\|_4 \le c\|\mathbf{w}_x\|(\|\mathbf{w}_x\|^{3/4} + \|\mathbf{w}\|^{3/4})\|\mathbf{w}\|^{1/4} \le \frac{\nu}{8}\|\mathbf{w}_x\|^2 + c\|\mathbf{w}\|^2,$$
$$I_2'' = -(w_{3x_3}, \mathbf{v} \cdot \mathbf{w}) - (w_3 \mathbf{v}_{x_3}, \mathbf{w}).$$

Here the first inner product has been estimated as

$$|(w_{3x_3}, \mathbf{v} \cdot \mathbf{w})| \le \|\mathbf{w}_x\| \cdot \|\mathbf{v}\|_4 \cdot \|\mathbf{w}\|_4$$
$$\le c\|\mathbf{w}_x\|(\|\mathbf{w}_x\|^{3/4} + \|\mathbf{w}\|^{3/4})\|\mathbf{w}\|^{1/4} \le \varepsilon\|\mathbf{w}_x\|^2 + c_\varepsilon\|\mathbf{w}\|^2;$$

and the second, as

$$|(w_3 \mathbf{v}_{x_3}, \mathbf{w})| \le \|\mathbf{v}_{x_3}\|_4 \cdot \|\mathbf{w}\|_4 \cdot \|w_3\|$$
$$\le c(\|\mathbf{w}_x\|^{3/4} + \|\mathbf{w}\|^{3/4})\|\mathbf{w}\|^{1/4} \cdot \|\operatorname{div} \mathbf{w}\| \le \varepsilon\|\mathbf{w}_x\|^2 + c_\varepsilon\|\mathbf{w}\|^2.$$

By taking an appropriate ε and by using the above inequalities, from (7.7) we obtain

$$\frac{d}{dt}\|\mathbf{w}\|^2 + c_1\|\mathbf{w}_x\|^2 - c_2\|\mathbf{w}\|^2 - c_3\|\rho\|^2 \le 0. \tag{7.8}$$

Let us add (7.8) and (7.5),

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \|\rho\|^2) + c_4\|\mathbf{w}_x\|^2 + c_5\|\rho_x\|^2 - c_6(\|\mathbf{w}\|^2 + \|\rho\|^2) \le 0,$$
$$\mathbf{w}(0) = \mathbf{0}, \quad \rho(0) = 0.$$

Now an application of the Gronwall inequality gives $\|\mathbf{w}(t)\| = 0$, $\|\rho(t)\| = 0$. Consequently, problem (3.1)–(3.5) can have at most one solution.

It follows from the continuity equation and the boundary conditions for w_3 that $w_3 = 0$.

8. Existence of a solution

The results in [20] and [13] imply the existence of a strong solution on some time interval $[0, t_*]$. It is easily seen that $t_* = T$. Indeed, assume the contrary. Then $\|\mathbf{u}_x(t)\| \to \infty$ as $t \to t_* < T$. But this is impossible in view of the estimate (6.10).

Thus, we have proved the following theorem

Theorem 6. *Assume that* $\mathbf{u}_0 \in \mathbf{H}^2(\Omega)$ *and* $\rho_0 \in H^2(\Omega)$ *satisfy the boundary conditions* (3.5) *and* $\operatorname{div}\mathbf{u}_0 + \partial_z w_0 = 0$. *Then, for any* $\nu, \nu_1 > 0$ *and an arbitrary* $T > 0$, *problem* (3.1)–(3.5) *in* Q_T *has a unique solution* $\mathbf{u} \in \mathbf{V}$, $\rho \in R$ *such that* $\mathbf{u}^2$, $\mathbf{u}_z^2$, $\mathbf{u}_x$, $\mathbf{u}_{zx}^2$, $\mathbf{u}_t$, $\mathbf{u}_{tx} \in \mathbf{L}_2(Q_T)$ *and the norm* $\|\mathbf{u}_x\|$ *is continuous in* t.

In closing, let us make some remarks concerning our results and their possible generalizations. As was already noted, the theorem on the existence of a global solution of the ocean dynamic equations was proved here in a Cartesian coordinate system. The same technique was used in [1] to prove the existence of a global solution of the ocean dynamic equations on a manifold with or without boundary. Further, the case of variable bottom is important from the viewpoint of applications. It turns out that in this case one should slightly change the statement of the problem. Namely, one should change the nonlinear terms containing small-norm terms, which permits proving the existence of a global solution in this case. Note the paper [12], where the existence of a global solution was proved by a similar technique for the atmosphere dynamic equations. This case has the important distinction that it contains a nonstationary boundary condition, which necessitates using additional technical tricks. Thus, the extension of results to more general problems shows the efficiency of the technique suggested here for the derivation of a priori estimates.

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References

[1] A. V. Drutsa, *Existence "in the large" of a solution to a system of equations of large-scale ocean dynamics on a manifold*, Mat. Sb. **202** (2011), no. 10, 55–86; English transl., Sb. Math. **202** (2011), no. 9–10, 1463–1492. MR2895550

[2] V. V. Zhikov, *On an approach to the solvability of generalized Navier–Stokes equations*, Funktsional. Anal. i Prilozhen. **43** (2009), no. 3, 33–53; English transl., Funct. Anal. Appl. **43** (2009), no. 3, 190–207. MR2583638

[3] G. M. Kobel'kov, *Global existence of a solution to the ocean dynamics equations*, Dokl. Akad. Nauk **407** (2006), no. 4, 457–459; English transl., Dokl. Math. **73** (2006), no. 2, 296–298. MR2348674

[4] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961; Nauka, Moscow, 1970; English transl., Gordon and Breach, New York, 1969. MR0254401

[5] O. A. Ladyzhenskaya, *Modifications of the Navier–Stokes equations for large gradients of the velocities*, Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7** (1968) 126–154. (Russian) MR0241832

[6] G. I. Marchuk and A. S. Sarkisyan, eds., *Mathematical models of circulation in the ocean*, Nauka, Novosibirsk, 1980. (Russian) MR86A05

[7] S. A. Nazarov and B. A. Plamenevsky, *Elliptic problems in domains with piecewise smooth boundaries*, Nauka, Moscow, 1991; English transl., De Gruyter, Berlin, 1994. MR1283387

[8] S. M. Nikol'skii, *Approximation of functions of several variables and imbedding theorems*, Nauka, Moscow, 1977, 2nd rev. aug. ed.; English transl. of 1st ed., Springer, Berlin, 1975. MR0374877

[9] V. M. Ipatova, V. I. Agoshkov, G. M. Kobelkov, and V. B. Zalesny, *Theory of solvability of boundary value problems and data assimilation problems for ocean dynamics equations*, Russ. J. Numer. Anal. Math. Model. **25** (2010), no. 6, 511–534. MR2748256

[10] C. Cao and E. S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. Math. **166** (2007), no. 1, 245–267. MR2342696

[11] A. V. Drutsa, *Existence "in the large" of a solution to primitive equations in a domain with uneven bottom*, Russ. J. Numer. Anal. Math. Model. **24** (2009), no. 6, 515–542. MR2604497

[12] A. V. Gorshkov, *Uniqueness of a solution to the problem of atmosphere dynamics*, J. Math. Sci. **167** (2010), no. 3, 340–357. MR2839025

[13] F. Guillén-González, N. Masmoudi, and M. A. Rodríguez-Bellido, *Anisotropic estimates and strong solutions of the primitive equations*, Diff. Int. Equat. **14** (2001), no. 11, 1381–1408. MR1859612

[14] G. M. Kobelkov, *Existence of a solution "in the large" for the 3D large scale ocean dynamics equations*, C. R. Acad. Sci. Paris. Ser. II **343** (2006), no. 4, 283–286. MR2245395

[15] G. M. Kobelkov, *Modifications of the Navier–Stokes equations*, Russ. J. Numer. Anal. Math. Model. **30** (2015), no. 2, 87–94. MR3334106

[16] G. M. Kobelkov, *Existence of a solution "in the large" for ocean dynamics equations*, J. Math. Fluid Mech. **9** (2007), no. 4, 588–610. MR2374160

[17] R. Lewandowski, *Analyse mathématique en océanographie*, Masson, Paris, 1997. MR3468702

[18] J. L. Lions, R. Temam, and S. Wang, *On the equations of the large-scale ocean*, Nonlinearity **5** (1992), 1007–1053. MR1187737

[19] J. L. Lions, R. Temam, and S. Wang, *New formulations of the primitive equations of the atmosphere and applications*, Nonlinearity **5** (1992), 237–288. MR1158375

[20] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, in *Handbook mathematical fluid dynamics*, vol. 3, Elsevier, Amsterdam, 2004, pp. 535–658. MR2099038

[21] C. Hu, R. Temam, and M. Ziane, *The primitive equations on the large scale ocean under the small depth hypothesis*, Discrete Contin. Dyn. Syst. **9** (2003), no. 1, 97–131. MR1951315

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