Труды Москов. Матем. Общ.
Том 77 (2016), вып. 2

Trans. Moscow Math. Soc.
2016, Pages 203–246
http://dx.doi.org/10.1090/mosc/259
Article electronically published on November 28, 2016

INTEGRABLE SYSTEMS, SHUFFLE ALGEBRAS, AND BETHE EQUATIONS

BORIS L. FEIGIN

Abstract. We speak about the part of integrable system theory dealing with conformal theory and W-algebras (ordinary and deformed). Some new approaches to finding Bethe equations that describe the spectrum of Hamiltonians of these quantum integrable systems are developed. The derivation of the Bethe equations is based on the technique of shuffle algebras arising in quantum group theory.

1. Introduction

The present paper deals with the theory of quantum integrable systems. The topic is fairly broad and quite involved. As far as classical integrable systems are concerned, mathematicians, as well as physicists, have been studying them for a very long time, for more than 150 years at a minimum.

There were periods of acutely rising interest in the subject, when it was deemed fashionable and was studied by thousands of researchers. Then the wave would subside, other things would become fashionable, but integrable system theory would not die altogether; the studies were continued. Later, a new surge of interest would occur, marking the beginning of a new cycle.

In most general terms, integrability is synonymous to exact solvability. One should not exaggerate this "solvability", although the solutions of Liouville integrable systems can be expressed via theta functions under some additional conditions. Further, integrable Hamiltonian systems have integrals of motion. (Some variables are conserved as the system undergoes evolution.) This is an attractive feature, because it means that the system evolution satisfies a number of laws and is not just random.

Although we no longer believe in general laws and the predestined development of the Universe so firmly as we used to, the interest in solvable systems is rising rather than falling. No matter what people do or what they think of, there is a desire to see some structure susceptible to study in the minutest detail.

The Ising model is a textbook example of such a structure. Here one speaks of quantum rather than classical systems or of problems arising from statistical physics. This field, as well as many other ones, is very complicated technically and conceptually. Quantum mechanics is a bunch of empirical facts discovered experimentally and an intertwinement of complex mathematical constructions that extend each other and are interrelated in the most complicated way. A human feels small and weak near this infinite world of entities; hence the desire to single out a part (if only infinitely small) that reflects the overall picture and, most importantly, is at the same time within the "operational field".

2010 *Mathematics Subject Classification.* Primary 17B37; Secondary 17B69, 17B80.

Key words and phrases. Integrable system, shuffle algebra, Bethe equations, affine Lie algebra, center at the critical level, Drinfeld–Sokolov reduction, quantum group.

This research was supported by the Russian Science Foundation (grant no. 16-11-10316).

This means that the researcher deals with something pretty specific rather than with the frightening infinity.

The Ising model is a good example just in this sense. Why has scrutinization of this fairly simple structure given rise to so many new, completely unexpected ideas? This, of course, is utterly mystical. One could hardly predict the scale of consequences following Onsager's solution of the two-dimensional Ising model. These consequences include the Yang–Baxter equations (and hence quantum group theory), two-dimensional conformal field theory, and everything that ensued from it. Note that we are far from exhausting the list. Why the Ising model? No one knows whether this is due to the beauty of the model or to anything else.

One can vaguely compare this with a Chinese (or better Japanese) garden. Such a garden is something very beautiful; it is separated from the outside world and reflects it at the same time. The gardener creates orderliness and beauty in a confined space having in mind the secret idea that orderliness and beauty in the garden would somehow affect the chaos in the outside world.

Let us say a few words about the development of integrable system theory in Russia. The present paper only reflects a small part of what was going on. Unavoidably, our notes on the history of the topic are extremely incomplete. We cannot mention all the key researchers; nor can we explain what happened—or present a timeline showing the results and the corresponding sequence of events. We only say some general words, outline the trend of studies leading to the papers by Bazhanov, Luk'yanov, and Zamolodchikov dealing with the quantum Korteweg–de Vries equation, and describe what happened afterwards.

Integrable systems have been understood as solvable systems of differential equations as early as the nineteenth century. In field theory, systems of differential equations arise when seeking minima (extremals) of Lagrangians. Knowing the structure of the set of extremals is very important in field theory, and Yang–Mills theory has proved to be integrable in this sense; one can say very much about the solution set of the Yang–Mills equations. The very intensive activity in this field started from the closed-form solutions (instantons) found by Belavin–Polyakov–Schwarz–Tyupkin [22].

Zakharov and some other mathematical physicists developed a method for constructing and solving systems of nonlinear partial differential equations. It is known as the inverse problem method (see the book [7]). This method was applied to a very broad class of equations, in particular, the Yang–Mills equation. The best-known application is to systems of the Korteweg–de Vries type. This is a large class of equations, whose study has led to the discovery of the famous soliton solutions. Systems of the Korteweg–de Vries type are completely integrable Hamiltonian systems, and hence their solutions can be expressed via theta functions. These equations were studied by quite a few Moscow mathematicians, including S. P. Novikov, I. M. Krichever, I. M. Gelfand, Yu. I. Manin, and many, many others.

One can associate the solutions of completely integrable systems with manifolds of extrema of Lagrangians and try to construct the corresponding quantum field theory. From the Hamiltonian viewpoint, this is the problem of quantization of a Hamiltonian system. The conventional scheme is that the Hamiltonians should turn into operators acting on some (quantum) space. The classical Hamiltonians commute with respect to the Poisson bracket. After the quantization, one should obtain a "large" commutative algebra together with a representation where this algebra acts.

The two-dimensional Ising model is a lattice integrable system. This means that the partition function is the trace of some operator that is an element of the "large" commutative algebra (i.e., of the quantum integrable system). The Leningrad school (Faddeev and others) developed the quantum inverse problem method [15]. They invented a method (which is not fully rigorous) for the quantization of classical Hamiltonians of the Korteweg–de Vries type. One idea is to approximate the quantum integrable system by two-dimensional lattice systems, namely, by integrable systems like the Ising model. The Leningrad school's approach permitted obtaining two-dimensional lattice systems in a natural way rather than by guesswork. The algebraic structures discovered in Leningrad later evolved into quantum groups.

It is already clear from the preceding that field theory and integrable system theory are closely related. Nevertheless, these are distinct sciences, each of which has its own philosophy and its own view of the same things.

Quantum field theory, like a majority of equally complicated disciplines, was in need of solvable examples. Such field theory models (solvable, or integrable) were sought for consciously, and this way two-dimensional conformal field theories (Belavin–Polyakov–Zamolodchikov [23]), as well as their integrable deformations, were found.

Conformal field theories are related in at least two ways to integrable models of other types. First, they describe statistical systems at the phase transition points. For example, if one takes the Ising model on the two-dimensional lattice and passes to the thermodynamic limit by refining the lattice, then one can discover a minimal conformal field theory of the type $(3,4)$ (see [23]). Hypothetically, this is true for all lattice models; one can (sometimes) verify this for a solvable lattice model.

On the other hand, one can obtain conformal field theories by a quantization procedure. More precisely, take a Poisson manifold, say, the one arising when studying the integrable Hamiltonian Korteweg–de Vries system. The quantization of this manifold gives the universal enveloping algebra of the Virasoro algebra.

Quantization is also possible for more complicated Hamiltonian systems, and one can use this to obtain quite a few various conformal field theories. For example, the Wess–Zumino theories can be obtained this way. Gelfand–Dikii Hamiltonian systems or the more general Drinfeld–Sokolov systems lead to conformal field theories corresponding to W-algebras. (See [1, 5] for classical Hamiltonian systems and [9, 28] for quantum Hamiltonian systems.)

It turned out later that quantum groups can be viewed as symmetries of conformal field theories; this point of view is very close to the original approach of Faddeev's school.

More precisely, conformal field theories include the simplest ones, known as free theories. Finite-dimensional quantum groups act on such theories by symmetries, and the invariants of these actions are more complicated conformal field theories. In this manner, one obtains conformal field theories related to W-algebras (Luk'yanov–Fateev [9]). Part of generators of finite-dimensional quantum groups are called screenings.

Infinite-dimensional quantum groups, say, affine ones, can act on conformal field theories as well. An invariant subalgebra of an affine quantum group is no longer the operator algebra corresponding to a conformal theory; it is an object of different nature, and it is still unclear so far what it is. The paper [30] gives a definition of an invariant part with respect to an affine quantum group; this is neither a conformal theory nor a field theory at all, but simply an algebra. This algebra was computed in [32]. For example, if the free theory is generated by a single free field and the affine quantum group is $U_q(\widehat{\mathfrak{sl}}_2)$, then the invariant subalgebra is commutative and consists of quantum Korteweg–de Vries Hamiltonians; they are called local integrals of motion. Bazhanov, Luk'yanov, and Zamolodchikov [19] constructed a different system of integrals of motion;

they commute with the quantum Hamiltonians and are called nonlocal integrals of motion. The meaning of these integrals is as follows. We have already mentioned that the affine quantum group $U_q(\widehat{\mathfrak{sl}}_2)$ acts on the free theory. This algebra contains the "center at the critical level" $\mathcal{Z}_q(\widehat{\mathfrak{sl}}_2)$; this center acts on the representations of the conformal algebra corresponding to the free theory, and its image is the algebra of nonlocal integrals of motion. All of this pertains to any quantum affine algebras acting on conformal theories.

Note also that the main goal in [19] is to deform the conformal theory into a massive theory. The affine quantum group is a natural part of the related structure. In a massive theory, one can compute correlation functions, but unfortunately there are few rigorous mathematical results in this direction.

The method due to Bazhanov, Luk'yanov, and Zamolodchikov permits one to construct large commutative subalgebras of vertex algebras. Then there arises a problem of finding the eigenvalues and eigenvectors for the action of a commutative algebra on representations. This is a difficult problem, and rigorous mathematical approaches to this problem have been lacking for a long time. (The problem was considered from the physical viewpoint in [20].)

The eigenvalues were found in quite a few examples in numerous nontrivial models by solving a remarkable system, which is called the system of Bethe equations. Further, a method for finding the eigenvectors was suggested, which is called the algebraic Bethe ansatz (e.g., see [17]). It was only natural to try acting in the same manner for the case of the quantum Korteweg–de Vries equation. It turns out that this is possible in several distinct ways [36, 37].

There exists yet another class of "solvable" models, namely, topological field theories. Topological theories arose in the usual way as models in which many things can be computed in closed form. However, they proved to be so interesting from both mathematical and physical viewpoints that their rapid development followed, and nowadays they are a very broad area of knowledge. Integrable systems, both classical and quantum, arise very naturally in topological theories. It has turned out that the generating function of correlators in two-dimensional field theories is the tau function of the classical integrable system.

Quantum integrable systems arise in at least two distinct ways. In one approach, they appear as multiplication operators in quantum cohomology or quantum K-theory (e.g., see [49]). In some cases, the commutative algebras thus obtained give quantized Drinfeld–Sokolov systems as well as many other systems. In another approach, quantum integrable systems occur in a two-dimensional topological theory when studying so-called double ramification cycles (see [24]). A majority of integrable systems arising in topological theories are new; i.e., they cannot be identified with any already known systems.

Topological theories produce not only integrable systems themselves but also generalizations of conformal algebras. It has been known for a fairly long time that vertex algebras can be deformed in some cases. For example, the deformed Virasoro algebra gives a new object, the so-called elliptic Virasoro algebra; there also exist elliptic W-algebras (see [32, 43, 58]).

It is these algebras that arise when studying instanton manifolds. A mathematical language for describing the resulting objects is yet to be found, but a large class of examples can be described with the help of quantum toroidal algebras [35]. These algebras are natural generalizations of finite-dimensional and affine quantum groups. Lattice systems of the XXZ or XYZ type can be solved with the use of quantum affine algebras or their elliptic versions. Systems of Bethe equations are a traditional object in this theory. The generalization of the theory to toroidal algebras permits using the Bethe method to find the eigenvalues of systems generalizing the quantum Korteweg–de Vries system.

The paper is organized as follows. Section 2 presents quantum and classical W-algebras. To each finite-dimensional semisimple algebra, there corresponds a current algebra $\widehat{\mathfrak{g}}$, to which, in turn, there corresponds a conformal algebra, more precisely, a family of conformal algebras $U_k(\widehat{\mathfrak{g}})$ depending on the value of the central charge. A broad class of vertex algebras can be obtained from $U_k(\widehat{\mathfrak{g}})$ by a reduction procedure; in particular, W_k is obtained by the Drinfeld–Sokolov reduction. The family W_k has two classical limits, one as $k \to \infty$ and the other as $k \to -h^\vee$, where $h^\vee$ is the dual Coxeter number. If we replace the algebra $\mathfrak{g}$ by its Langlands dual, then we obtain the same family of W-algebras up to a change of the variable k: the limit as $k \to \infty$ for one family is isomorphic to the limit as $k \to -h^\vee$ for the other family. The limit as $k \to -h^\vee$ is a commutative algebra, which is isomorphic to the center of the affine algebra at the critical level. Numerous integrable systems arise as the image of the center under some representation, and the center structure is very important when studying such systems. In the same section, we discuss the deformed W-algebra obtained by reduction of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. In the case of $\widehat{\mathfrak{gl}}_n$ (ordinary as well as q-deformed), the family of W-algebras can be analytically continued with respect to n. The resulting algebra W depend on q and also on the parameter q^n. The operation $\mathfrak{gl}_n \oplus \mathfrak{gl}_m \to \mathfrak{gl}_{n+m}$ permits defining comultiplication on the family of W-algebras and obtaining a quantum group, which arises in a different language as the quantum theory group $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$. (From now on, we place two dots over a symbol instead of the double hat.)

Integrable systems are studied in Section 3. We are most interested in systems related to toroidal algebras, and we explain how to construct "large" commutative algebras acting on their representations. Further, we explain how the standard method for finding the spectrum can be extended to toroidal algebras.

Section 4 deals with shuffle algebras. Such algebras are attractive owing to their simple construction. They can be used to construct representations of affine quantum algebras and toroidal algebras in the most natural way. In addition, the language of shuffle algebras is very close to that used in the theory of vertex algebras. Sometimes, shuffle algebras admit finding commutative subalgebras in closed form, i.e., constructing quantum integrable systems.

In the last section, we use shuffle algebras to construct integrable systems. The construction has the specific feature that the Bethe equations are obtained in the most natural way. The integrable systems constructed here depend on a function parameter. Hypothetically, it is such systems that arise in topological theories (see [24]). The following (unaccomplished) task is to compare our systems with those arising when studying quantum K-theories and in similar other problems.

For various reasons, a number of things have been omitted from the paper. This is largely due to the limited volume of the paper and the complicated structure of the subject. We refer the reader to the books [13, 14] for information about integrable systems. Bethe equations were studied by Vigman and, from the combinatorial point of view, by Anatolii Kirillov. Note also an important series of papers by Varchenko, Mukhin, Tarasov, and Shekhtman. Bethe equations arose when studying the semiclassical limit of integral representations of solutions of the Knizhnik–Zamolodchikov equations [57]. There are quite a few remarkable results about Bethe equations, integrable systems of the XXZ type, and Gaudin systems. We cannot even list these results; just note that the only known proof of completeness of the system of Bethe vectors can be found in [51, 52].

Bethe equations have occurred multiple times in connection with the asymptotics of integrals. Quite recently, Nekrasov and Shatashvili used them when studying higher-dimensional topological field theories [53, 54].

2. Quantum and classical W-algebras

2.1. **Center of affine Kac–Moody algebras at the critical level.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[[t, t^{-1}] \oplus \mathbb{C}K$ be the corresponding current algebra, which is a central extension of the algebra of Laurent series with coefficients in $\mathfrak{g}$; here K is a central element. The category $\mathcal{O}$ of representations of $\widehat{\mathfrak{g}}$ consists of representations on V such that, for each vector $v \in V$, the subspace $U(\widehat{\mathfrak{b}}^+)v$ is finite-dimensional, where $\mathfrak{b}^+$ is the Borel subalgebra of $\mathfrak{g}$ and $\widehat{\mathfrak{b}}^+$ is the Borel subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{b}}^+ = \mathfrak{b}^+ \oplus \mathfrak{g} \otimes \mathbb{C}[[t]]t$. By $\mathcal{O}_k$, $k \in \mathbb{C}$, we denote the subcategory of $\mathcal{O}$ formed by the representations in which the central element K acts as the multiplication by k.

Let $\bar{U}_k(\widehat{\mathfrak{g}})$ be the completion of the algebra $U(\widehat{\mathfrak{g}})/(K-k)U(\widehat{\mathfrak{g}})$. It consists of infinite linear combinations of elements of $U(\widehat{\mathfrak{g}})/(K-k)U(\widehat{\mathfrak{g}})$ defining an operator on an arbitrary representation V in $\mathcal{O}_k$. In addition,we require that the degree in $\widehat{\mathfrak{g}}$ of these infinite linear combinations be bounded (see [42] for more detail).

The algebra $\bar{U}_k(\widehat{\mathfrak{g}})$ has a nontrivial center only if $k = -h^\vee$, where $h^\vee$ is the dual Coxeter number [31]. This value of k is called the *critical level*; the center of the algebra $\bar{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ is called the *center at the critical level* and will be denoted by $\mathcal{Z}(\widehat{\mathfrak{g}})$. The dual Coxeter number for the algebra $\mathfrak{g} = \mathfrak{sl}_n$ is $h^\vee = n$.

The algebra $\mathcal{Z}(\widehat{\mathfrak{g}})$ is a Poisson algebra; i.e., it can be equipped with a bracket compatible with multiplication. This is a special case of the following construction. Given a flat family of algebras $\mathcal{A}(h)$, $h \in \mathbb{C}$, let $\mathcal{Z}$ be the center of $\mathcal{A}(0)$. Then one can define a bracket on $\mathcal{Z}$ as follows. Consider two elements $z_1, z_2 \in \mathcal{Z}$, $z_i(h) \in \mathcal{A}(h)$, $z_i(0) = z_i$, and set $\{z_1, z_2\} = [z_1(h), z_2(h)]/h$. That is, we extend z_1 and z_2 to families of elements $z_i(h)$, consider the commutator of these families, and take the first term with respect to h. (The zeroth term is zero, because $[z_1, z_2] = 0$.) One can readily show that the bracket $\{z_1, z_2\}$ is independent of the choice of the continuations $z_1(h)$ and $z_2(h)$. In our case, the family of algebras is $\bar{U}_k(\widehat{\mathfrak{g}})$.

Thus, the center $\mathcal{Z}(\widehat{\mathfrak{g}})$ consists of functions on the Poisson manifold defined as the spectrum of the ring $\mathcal{Z}(\widehat{\mathfrak{g}})$. This manifold is called the oper manifold on the punctured disk; see Section 3.2. The algebra $\mathcal{Z}(\widehat{\mathfrak{g}})$ itself is isomorphic to the classical W-algebra associated with the Langlands dual Lie algebra $\mathfrak{g}^L$.

If $\mathfrak{g} = \mathfrak{gl}_n$, then the oper manifold is the manifold of nth-order differential operators $\partial^n + q_1\partial^{n-1} + \cdots + q_n$, where the q_i are Laurent series in t. The bracket on the oper manifold is the Gelfand-Dikii bracket (see [31]). If $\mathfrak{g} = \mathfrak{sl}_2$, then the oper manifold is the dual space of the Virasoro algebra with the Kostant–Kirillov bracket.

The quantum W-algebra $W(\mathfrak{g})$ associated with $\mathfrak{g}$ is the quantization of the oper manifold viewed as a Poisson manifold. This quantization can be carried out explicitly, which will be done in the next subsection.

2.2. **Classical and quantum Hamiltonian reductions.** Let $\mathfrak{a}$ be a Lie algebra, let A be the group corresponding to $\mathfrak{a}$, and let M be a Poisson manifold equipped with a Hamiltonian action of $\mathfrak{a}$; i.e., a moment map $\mu\colon M \to \mathfrak{a}^*$ is given. The map μ is a morphism in the category of Poisson manifolds, which defines a Poisson algebra homomorphism $S(\mathfrak{a}) \to \mathcal{O}(M)$, where $\mathcal{O}(M)$ is the algebra of functions on M. The Hamiltonian reduction $M/\!/\mathfrak{a}$ is the quotient space $\mu^{-1}(0)/A$. More generally, a Hamiltonian reduction is the quotient $\mu^{-1}(\chi)/A$, where $\chi \in \mathfrak{a}^*$ is a character of the algebra $\mathfrak{a}$.

The orbit manifold A on $\mu^{-1}(0)$ may behave badly, but in the most general case one can define a dg-Poisson supermanifold, which it is expedient to take for the definition of $M//\mathfrak{a}$. Namely, consider the Poisson supermanifold $\mathfrak{a}\oplus\mathfrak{a}^*$, where $\mathfrak{a}$ and $\mathfrak{a}^*$ are odd spaces and the bracket is flat. On linear functions on $\mathfrak{a}\oplus\mathfrak{a}^*$, the bracket has the form

$$\{\alpha_1+\beta_1,\alpha_2+\beta_2\}=\beta_2(\alpha_1)+\beta_1(\alpha_2),\quad \alpha_i\in\mathfrak{a},\ \beta_i\in\mathfrak{a}^*.$$

We define a function Q on the supermanifold $M\times\{\mathfrak{a}\oplus\mathfrak{a}^*\}$ by the formula

$$Q(m,\alpha,\beta)=\mu(m)(\alpha)+C,$$

where $m\in M$, $\alpha\in\mathfrak{a}$, $\beta\in\mathfrak{a}^*$, and $C\in\Lambda^2(\mathfrak{a}^*)\otimes\mathfrak{a}$ is the structure constant tensor of $\mathfrak{a}$ viewed as a function on $\mathfrak{a}\oplus\mathfrak{a}^*$. One can verify that $\{Q,Q\}=0$. The cohomology of the operator $f\mapsto\{Q,f\}$ forms a Poisson algebra, which coincides with the algebra of functions on $M//\mathfrak{a}$ if the group A freely acts on $\mu^{-1}(0)$.

The Drinfeld–Sokolov Hamiltonian reduction can be obtained as follows. Take the dual space of the Lie algebra $\widehat{\mathfrak{g}}$ with the Kostant–Kirillov bracket for the manifold M; then $\widehat{\mathfrak{g}}^*$ can be identified with the space of connections on the formal punctured disk, that is, with the space of operators of the form $\alpha\partial/\partial z+A(z)dz$, where $A(z)\in\mathfrak{g}\otimes\mathbb{C}[[z,z^{-1}]]$, $\alpha\in\mathbb{C}$. The parameter α can be zero. Given α, the set $\widehat{\mathfrak{g}}^*_\alpha=\{\alpha\partial/\partial z+A(z)dz\}$ of operators is a Poisson submanifold. Let $\mathfrak{n}$ be a maximal nilpotent subalgebra of $\mathfrak{g}$, let $\widehat{\mathfrak{n}}=\mathfrak{n}\otimes\mathbb{C}[[z,z^{-1}]]$ be the algebra of currents ranging in $\mathfrak{n}$, let $\widehat{N}$ be the corresponding group, and let $\chi\colon\widehat{\mathfrak{n}}\to\mathbb{C}$ be a character of $\widehat{\mathfrak{n}}$, which we assume to be generic. The action of $\widehat{\mathfrak{n}}$ on $\widehat{\mathfrak{g}}^*$ is Hamiltonian. The Drinfeld–Sokolov reduction is the quotient $\mu^{-1}(\chi)/\widehat{N}$, where $\mu\colon\widehat{\mathfrak{g}}^*_\alpha\to\widehat{\mathfrak{n}}^*$ is the moment map.

This construction can be presented in a different way. Let $P(\chi)$ be the zero-dimensional Poisson manifold consisting of a single point and equipped with a Hamiltonian action of $\widehat{\mathfrak{n}}$ via the moment map $P(\chi)\to\widehat{\mathfrak{n}}^*$, which can be described in terms of the dual map $\chi\colon\widehat{\mathfrak{n}}\to\mathbb{C}$, where χ is a generic character of the algebra $\widehat{\mathfrak{n}}$. Then the Drinfeld–Sokolov reduction is $\widehat{\mathfrak{g}}^*_\alpha\times P(\chi)//\widehat{\mathfrak{n}}$, $\alpha\neq 0$.

The following theorem was proved in [29].

Theorem 2.1. *Let $\mathfrak{g}^L$ be the Langlands dual Lie algebra of the Lie algebra $\mathfrak{g}$. Then the Poisson algebra $\mathcal{Z}(\widehat{\mathfrak{g}})$ is isomorphic to the algebra of functions on $\widehat{\mathfrak{g}}^*_\alpha\times P(\chi)//\widehat{\mathfrak{n}}$.*

This theorem plays a substantial role in the geometric approach to Langlands duality; e.g., see [21].

Remark 2.1. It is desirable to quantize the center of the universal enveloping algebra $U(\widehat{\mathfrak{g}})$ at the critical level explicitly. We would like the following construction to exist. Let A be an associative algebra, let $C\subset A$ be a commutative subalgebra, and let $A_\hbar$ be a formal deformation of A; one has $A_\hbar=A\otimes\mathbb{C}[[\hbar]]$ as vector spaces. In our case, C is the center at the critical level. The deformation along $\hbar$ defines a Poisson structure, and it is desirable to have an explicit quantization method for this structure in terms of the family $A_\hbar$. Unfortunately, it is unclear how to do this. Hence to quantize the center at the critical level one uses Hamiltonian reduction, which admits explicit quantization.

Let again M be a Poisson manifold equipped with an action of a Lie algebra $\mathfrak{a}$. A quantization of M is a deformation of the function algebra $\mathcal{O}(M)$, and a deformation of the pair $(M,\mathfrak{a})$ is a family of algebras $\mathcal{O}_\hbar(M)$ such that for each $\hbar$ a homomorphism $U(\mathfrak{a})\to\mathcal{O}_\hbar(M)$ is given that tends to $S(\mathfrak{a})\to\mathcal{O}(M)$ as $\hbar\to 0$. In our case, the Poisson manifold $\widehat{\mathfrak{g}}^*$ with the action of $\widehat{\mathfrak{n}}$ can be quantized in a natural way.

Quantum Hamiltonian reduction is applied to a pair consisting of an associative algebra B and a homomorphism $\rho\colon U(\mathfrak{a})\to B$. Let $\Lambda^*(\mathfrak{a}^*)$ be the Chevalley complex of the Lie algebra $\mathfrak{a}$, and let $\mathrm{Clif}(\mathfrak{a}\oplus\mathfrak{a}^*)$ be the algebra of differential operators on $\Lambda^*(\mathfrak{a}^*)$, that is, the Clifford algebra with generators $\mathfrak{a} \oplus \mathfrak{a}^*$. The differential on the Chevalley complex is a first-order differential operator, that is, an element $c \in \mathrm{Clif}(\mathfrak{a} \oplus \mathfrak{a}^*)$; the algebra $\mathrm{Clif}(\mathfrak{a} \oplus \mathfrak{a}^*)$ is equipped with the natural $\mathbb{Z}_2$-grading, c is an odd element, and $[c, c] = 2c^2 = 0$. Now consider the tensor product $B \otimes \mathrm{Clif}(\mathfrak{a} \oplus \mathfrak{a}^*)$ and the element $Q \in B \otimes \mathrm{Clif}(\mathfrak{a})$, $Q = \theta(I) + 1 \otimes c$, where I is an element of $\mathfrak{a} \otimes \mathfrak{a}^*$ (the standard pairing) and θ is the map of $\mathfrak{a} \otimes \mathfrak{a}^*$ into $B \otimes \mathrm{Clif}(\mathfrak{a} \oplus \mathfrak{a}^*)$ taking $\mathfrak{a}$ into B by the homomorphism ρ and taking $\mathfrak{a}^*$ into $\mathfrak{a}^* \subset \mathrm{Clif}(\mathfrak{a} \oplus \mathfrak{a}^*)$. The square of Q is zero, and the operation $f \mapsto [Q, f]$ defines the structure of an associative dg-algebra on $B \otimes \mathrm{Clif}(\mathfrak{a} \oplus \mathfrak{a}^*)$. We will refer to the pair $(B \otimes \mathrm{Clif}(\mathfrak{a} \oplus \mathfrak{a}^*), Q)$ as the quantum Hamiltonian reduction.

Note that the quantum Hamiltonian reduction is a quantization of the classical Hamiltonian reduction. Namely, we construct a Poisson dg-manifold along the "classical" ways, and then we quantize it and obtain an associative dg-algebra.

Now let us apply all of this to the map $\rho_\chi \colon \bar{U}(\widehat{\mathfrak{n}}) \to \bar{U}(\widehat{\mathfrak{g}})$, where ρ_χ is the embedding $i \colon \bar{U}(\widehat{\mathfrak{n}}) \hookrightarrow \bar{U}(\widehat{\mathfrak{g}})$ twisted by the character χ; i.e., $\rho_\chi(u) = \chi(u)i(u)$. The algebras $\widehat{\mathfrak{n}}$ and $\widehat{\mathfrak{g}}$ are infinite-dimensional, and we deal with completions of their universal enveloping algebras; thus, we have to clarify what we mean by the tensor product $\bar{U}(\widehat{\mathfrak{g}}) \otimes \mathrm{Clif}(\widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{n}}^*)$. The algebra $\mathrm{Clif}(\widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{n}}^*)$ has an irreducible representation Λ generated by a vector v such that $(\mathfrak{n} \otimes \mathbb{C}[[t]])v = 0$ and $(\mathfrak{n}^* \otimes t\mathbb{C}[[t]]dt)v = 0$. The category $\mathcal{O}$ of representations of $\mathrm{Clif}(\widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{n}}^*)$ is semisimple, and Λ is the only irreducible representation. Now we can define the category $\mathcal{O}$ of representations of the algebra $U(\widehat{\mathfrak{g}}) \otimes \mathrm{Clif}(\widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{n}}^*)$. The algebra $U(\widehat{\mathfrak{g}}) \otimes \mathrm{Clif}(\widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{n}}^*)$ is a Virasoro algebra and has the corresponding vacuum representation in the category $\mathcal{O}$. The differential Q acts on this representation, and the resulting dg-vertex algebra is called the quantum Hamiltonian reduction. We denote this algebra by $H(\widehat{\mathfrak{n}}, \chi, \widehat{\mathfrak{g}})$. Note that this algebra is graded; the grading on the Clifford algebra is given by the formulas $\deg \widehat{\mathfrak{n}}^* = 1$ and $\deg \widehat{\mathfrak{n}} = -1$.

The center $\mathcal{Z}(\widehat{\mathfrak{g}})$ is quantized with the help of quantum Hamiltonian reduction as follows. First, one shows that, for a generic character χ, the reduction of the algebra $\bar{U}(\widehat{\mathfrak{g}})$ at the critical level is isomorphic to the center, and then one proves that the family of reductions $\bar{U}_k(\widehat{\mathfrak{g}})$, where k lies in a formal neighborhood of the critical value, is flat.

The construction of Hamiltonian reduction can be applied both to the finite-dimensional algebra $\mathfrak{g}$ and to a generic character χ of the algebra $\mathfrak{n}$. It is well known that the algebra $H(\mathfrak{g}, \chi, \mathfrak{n})$ is isomorphic to the center of $U(\mathfrak{g})$. In the case of an affine algebra $\widehat{\mathfrak{g}}$, the algebra $H(\widehat{\mathfrak{g}}, \chi, \widehat{\mathfrak{n}})$ is noncommutative. More precisely, this is a family of algebras depending on the parameter k. This algebra becomes commutative for $k = \infty$ and $k = -h^\vee$. The limit as $k \to \infty$ is said to be classical, because the algebra $\bar{U}_k(\widehat{\mathfrak{g}})$ tends in this limit to the space of functions on $\widehat{\mathfrak{g}}$. The limit as $k \to -h^\vee$ is quantum, but it turns out that it coincides with the classical limit for the Langlands dual algebra. It was shown in [29] that the reduction of the algebra $U_k(\widehat{\mathfrak{g}})$ is isomorphic to the reduction of $U_{k^L}(\widehat{\mathfrak{g}}^L)$ if $(k + h^\vee)(k^L + h^{\vee L}) = 1$.

Let us present one more statement, whose proof has not been published yet.

Proposition 2.1. *There exists a vertex algebra W_k containing two subalgebras $\widehat{\mathfrak{n}}$ and $\widehat{\mathfrak{n}}^L$ such that the reduction of W_k by the character $\widehat{\mathfrak{n}}$ is isomorphic to $U_{k^L}(\widehat{\mathfrak{g}}^L)$ and the reduction with respect to the character $\widehat{\mathfrak{n}}^L$ is isomorphic to $U_k(\widehat{\mathfrak{g}})$.*

Remark 2.2. The quantum Hamiltonian reduction is a special case of a construction that can naturally be called the generalized Hecke algebra. Let $\mathcal{A}$ be an algebra, and let $\mathcal{B} \subset \mathcal{A}$ be a nonunital subalgebra. Consider the induced module $I(\mathcal{B}, \mathcal{A}) = \mathcal{A}/(\mathcal{B} \cdot \mathcal{A})$.

Recall a classical example. Let $\mathcal{A} = \mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)]$ be the group algebra of the group $\mathrm{GL}_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field, let B be the Borel subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$, and let $\mathcal{B} \subset \mathbb{C}[B]$ be the kernel of the augmentation $\mathbb{C}[B] \to \mathbb{C}$. Then the algebra $\mathrm{Hom}_{\mathcal{A}}(I(\mathcal{B}, \mathcal{A}), I(\mathcal{B}, \mathcal{A}))$ is the ordinary Hecke algebra.

In the general case, a *Hecke algebra* can be defined as the endomorphism algebra of the module $I(\mathcal{B},\mathcal{A})$ or, in a better way, as the dg-algebra computing the functors $\mathrm{RHom}_{\mathcal{A}}(I(\mathcal{B},\mathcal{A}),I(\mathcal{B},\mathcal{A}))$. In the corresponding derived category, this algebra is uniquely defined.

Now let $\mathcal{A}=\bar{U}(\widehat{\mathfrak{g}})$, and let $\mathcal{B}$ be the kernel of the homomorphism $\chi\colon \bar{U}\to\mathbb{C}$. Then the quantum Drinfeld–Sokolov reduction is quasi-isomorphic to the algebra

$$\mathrm{RHom}_{\mathcal{A}}(I(\mathcal{B},\mathcal{A}),I(\mathcal{B},\mathcal{A})).$$

Here we cannot give a rigorous definition of the dg-algebra RHom itself and the module $I(\mathcal{B},\mathcal{A})$ in this case. This construction is technically rather complicated, and unfortunately it is not described anywhere. Note only that the module $I(\mathcal{B},\mathcal{A})$ is obtained by semi-infinite induction.

Remark 2.3. The definition of W-algebras was suggested by Fateev and Luk'yanov [9]. Their definition is based on the technique of screening operators. Let us present some details. Let $\mathfrak{h}$ be a finite-dimensional vector space with nondegenerate inner product $\langle\cdot,\cdot\rangle$, and let $\widehat{\mathfrak{h}}$ be the Heisenberg algebra, that is, the Lie algebra $\mathfrak{h}\otimes\mathbb{C}[[t,t^{-1}]]\oplus\mathbb{C}K$, where K is a central element. Then a W-algebra in the sense of Fateev–Luk'yanov is the subalgebra of $\bar{U}(\widehat{\mathfrak{h}})$ formed by the elements commuting with screening operators.

The screening construction due to Fateev and Luk'yanov can be restated as follows in the language of Hamiltonian reductions. The differential on the dg-algebra $H(\widehat{\mathfrak{n}},\chi,\widehat{\mathfrak{g}})$ is a sum of two terms, $Q=Q_0+Q_1(\chi)$, where Q_0 is the differential in the complex $H(\widehat{\mathfrak{n}},0,\widehat{\mathfrak{g}})$ (corresponding to the zero character) and the term Q_1 can be understood as a perturbation. The deformation of the differential defines a spectral sequence. The first term of this spectral sequence is the homology of Q_0. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ normalizing $\mathfrak{n}$, then the enveloping algebra of its loop algebra $\bar{U}(\widehat{\mathfrak{h}})$ normalizes $\widehat{\mathfrak{n}}$, and hence $\bar{U}(\widehat{\mathfrak{h}})$ is contained in the zero cohomology of $H(\widehat{\mathfrak{n}},0,\widehat{\mathfrak{g}})$. Let H_i be the ith graded cohomology component (where i may be negative); their direct sum is an algebra, and hence all the H_i are bimodules over the Heisenberg algebra. The differential on the first term of the spectral sequence is a derivation of the algebra $\bigoplus H_i$; in particular, we obtain a derivation $S(\chi)\colon \bar{U}(\widehat{\mathfrak{h}})\to H_1$. The map $S(\chi)$ is a screening operator (or, depending on the language used, the commutator with the screening current). The kernel of $S(\chi)$ is just the W-algebra in the sense of Fateev–Luk'yanov. One can show that this kernel coincides with the cohomology of the complex $H(\widehat{\mathfrak{n}},\chi,\widehat{\mathfrak{g}})$.

Remark 2.4. The Fateev–Luk'yanov construction is very general from the algebraic viewpoint. Let $\mathcal{A}$ be an algebra. In the Fateev–Luk'yanov case, $\mathcal{A}$ is the universal enveloping algebra of the Heisenberg algebra. Let R be an $\mathcal{A}$-bimodule, and let $Q\colon\mathcal{A}\to R$ be a derivation; the kernel of Q is a subalgebra of $\mathcal{A}$. An interesting example is given by the lattice version of the Fateev–Luk'yanov construction. Let $\mathcal{D}=\mathbb{C}[x_i,x_i^{-1}]$, $i\in\mathbb{Z}_{\geq 0}$, be the algebra of skew Laurent polynomials, $x_ix_j=qx_jx_i$, $i<j$; then $\mathcal{D}=\bigoplus\mathcal{D}_n$ is a graded algebra (graded by degree in the x_i). The algebra $\mathcal{A}$ is just $\mathcal{D}_0$, and R is just $\mathcal{D}_1$. The derivation $Q\colon\mathcal{D}_0\to\mathcal{D}_1$ is given by the formula $Q(f)=[\Sigma,f]$, where $\Sigma=\Sigma x_i$. Note that Σ is an infinite sum; i.e., Σ does not lie in $\mathcal{D}_1$, but the commutator $[\Sigma,f]$ takes finite sums to finite ones.

The kernel of Q is the lattice version of the Virasoro algebra (according to Volkov [60]). However, for the kernel to be nontrivial, one should extend the algebra $\mathcal{D}_0$ by admitting not only Laurent polynomials of finite segments $x_i,x_{i+1},\dots,x_{i+n}$ but also more general functions. In this connection, one has the following interesting and important problem. Can the kernel of Q be obtained as the quantum Hamiltonian reduction of some algebra, so that Q arises as the differential in the spectral sequence related to a deformation of the differential?

2.3. $W(\mathfrak{gl}_n)$, analytic continuation with respect to n, and the Lie algebra $W(\mathfrak{gl}_\lambda)$ as a quantum group. First, recall some general notions of quantum group theory. We do not give the full definition; everything can be found in the landmark papers by Drinfeld and Jimbo [2, 46]. We only discuss some facts important to us.

A quantum group is, first of all, a bialgebra U, that is, an algebra equipped with a coassociative comultiplication $\Delta\colon U \to U \otimes U$. The algebra U is sometimes called the quantum universal enveloping algebra, and the dual space U^* equipped with the multiplication $\Delta^*\colon U^* \otimes U^* \to U^*$ is called the algebra of functions on the quantum group. The algebra U is equipped not only with the comultiplication but also with an augmentation $\varepsilon\colon U \to \mathbb{C}$ and an antiautomorphism $S\colon U \to U$, which is an analog of the operation $g \mapsto g^{-1}$ on an ordinary group.

There are quite a few known examples of quantum groups. Drinfeld systematically studied quantum groups that arise as deformations of ordinary groups. Namely, let $\mathfrak{g}$ be a Lie algebra, let $U(\mathfrak{g})$ be its enveloping algebra, and let $U_h(\mathfrak{g})$ be a flat deformation. In other words, we can assume that the space $U(\mathfrak{g})$ is equipped with a multiplication and a comultiplication depending on h such that in the limit as $h \to 0$ they become the multiplication and comultiplication on $U(\mathfrak{g})$. Drinfeld showed that the first term of such a deformation defines a Lie bialgebra structure on $\mathfrak{g}$. The Lie bialgebra structure on $\mathfrak{g}$ defines a so-called Manin triple. A Manin triple is a Lie algebra $\mathfrak{a}$ equipped with an invariant nondegenerate quadratic form and a decomposition $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, where the Lie subalgebras $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are isotropic with respect to the form. Drinfeld showed that each Manin triple defines a bialgebra structure on $\mathfrak{a}_+$ as well as on $\mathfrak{a}_-$, and vice versa: if $\mathfrak{g}$ is a Lie bialgebra, then one can define a Manin triple structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ by setting $\mathfrak{g} = \mathfrak{a}_+$ and $\mathfrak{g}^* = \mathfrak{a}_-$.

Drinfeld, and later Etingof and Kazhdan [27], showed that for each Lie bialgebra one can construct a formal deformation $U_h(\mathfrak{g})$; in addition, this deformation is canonical up to some group action. More precisely, the Etingof–Kazhdan theorem guarantees the existence of a quantum Lie bialgebra only for a finite-dimensional bialgebra $\mathfrak{g}$. There is no guarantee if $\mathfrak{g}$ is infinite-dimensional, but usually one can still quantize it.

Further, Drinfeld defined the notion of double for each quantum group (see [2]). The double $\mathcal{D}(U)$ is a quantum group naturally acting on the algebra U. The double can be viewed as the group of symmetries of U preserving some structure. The algebra $\mathcal{D}(U)$ contains two subalgebras U and U^*, and the multiplication map $U \otimes U^* \to \mathcal{D}(U)$ is an isomorphism of vector spaces. If $\mathfrak{g}$ is a Lie bialgebra, then the universal enveloping algebra $U(\mathfrak{g}\oplus\mathfrak{g}^*)$ can be included in the family $U_h(\mathfrak{g}\oplus\mathfrak{g}^*)$, where $U_h(\mathfrak{g}\oplus\mathfrak{g}^*)$ is the double of the quantum group $U_h(\mathfrak{g})$.

The category of representations of the double is a symmetric tensor category. This means that for two arbitrary representations V_1 and V_2 of the algebra $\mathcal{D}(U)$ there exists an intertwining operator $R\colon V_1 \otimes V_2 \to V_2 \otimes V_1$, which can be obtained as the product of the transposition and the action on $V_1 \otimes V_2$ of some element of $\mathcal{D}(U) \otimes \mathcal{D}(U)$, which is called the universal R-matrix.

Thus, to construct quantum groups, one should take a Manin triple. The best-studied example is given by Kac–Moody algebras; in this example, there exists a quantization even though the algebra is infinite-dimensional.

Another example is given by the Lie algebra PD of pseudodifferential operators on the circle [47]. The elements of this algebra are the expressions

$$f = \sum_{i=-\infty}^{N} f_i(z)\partial^i,$$

where $N \in \mathbb{Z}$, $\partial = \partial/\partial z$, and $f_i(z)$ is a function on the circle; one can also consider the case in which $f_i(z)$ is a formal Laurent series. The bracket on the algebra PD is the usual commutator $[f, g] = f \cdot g - g \cdot f$.

The Manin triple is defined as follows. There exists an analog of the trace on the algebra PD; it has the form

$$\mathrm{Tr}\colon \mathrm{PD} \to \mathbb{C}, \quad \mathrm{Tr}(f) = \int_{S^1} f_{-1}(x)\, dx,$$

and possesses the property

$$\mathrm{Tr}(f \cdot g) = \mathrm{Tr}(g \cdot f), \quad f, g \in \mathrm{PD}.$$

An invariant form (an analog of the Killing form) is defined by the formula $\langle f, g\rangle = \mathrm{Tr}(fg)$. The algebra PD splits into the direct sum of two subalgebras PD_+ and PD_-, where

$$\mathrm{PD}_+ = \Big\{ \sum_{i \geq 0} f_i \partial^i \Big\}$$

is the subalgebra of differential operators on S^1 and

$$\mathrm{PD}_- = \Big\{ \sum_{i < 0} f_i \partial^i \Big\}$$

is the subalgebra of "purely integral" operators on S^1. If the $f_i(x)$ are Laurent series, there exists a different triangular decomposition, in which z and ∂ are interchanged by the automorphism acting by $\partial \mapsto z$ and $z \mapsto -\partial$.

The quantization of the Lie algebra PD gives the quantum group known in conformal field theory as W_λ, $\lambda \in \mathbb{C}$.

In the preceding, we have constructed a family of reductions $H(\widehat{\mathfrak{n}}, \chi, \widehat{\mathfrak{gl}}_n) = W_n$, where W_n is a vertex algebra (see [42]). Recall that a vertex algebra can be specified by its vacuum representation. The algebra $\widehat{\mathfrak{gl}}_n$ defines a vertex algebra as well; the vacuum representation of the latter is the parabolic Verma module Vac_k, that is, the module induced from the one-dimensional representation of the subalgebra $\mathfrak{gl}_n \otimes \mathbb{C}[[t]] \oplus \mathbb{C}K$ which is trivial on $\mathfrak{gl}_n \otimes \mathbb{C}[[t]]$ and takes the central element K to the constant k.

The vacuum representation of the reduction can be obtained as follows. We have already mentioned that the reduction is the dg-algebra $\bar{U}_k(\widehat{\mathfrak{gl}}_n) \otimes \mathrm{Clif}(\widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{n}}^*)$ with differential Q, where $\mathrm{Clif}(\widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{n}}^*)$ is the vertex algebra whose vacuum representation is the representation Λ of the Clifford algebra. Hence $(\bar{U}_k(\widehat{\mathfrak{gl}}_n) \otimes \mathrm{Clif}(\widehat{\mathfrak{n}} \oplus \widehat{\mathfrak{n}}^*), Q)$ is the dg-vertex algebra whose vacuum representation is $\mathrm{Vac}_k \otimes \Lambda$ and whose differential is Q acting on this space. The cohomology of Q is the semi-infinite cohomology (see [18]) of the algebra $\widehat{\mathfrak{n}}$ with coefficients in the module $\mathrm{Vac}_k \otimes \mathbf{1}_\chi$, where $\mathbf{1}_\chi$ is the space of the one-dimensional representation χ. For generic k, the cohomology $H^{\infty/2+i}(\widehat{\mathfrak{n}}, \mathrm{Vac}_k \otimes \mathbf{1}_\chi)$ is easy to compute. It turns out that this cohomology is only nontrivial in dimension $i = 0$, and $H^{\infty/2}(\widehat{\mathfrak{n}}, \mathrm{Vac}_k \otimes \mathbf{1}_\chi)$ is a graded vector space whose character has the form $\prod_{j=1}^{n}(1-q^j)^{-j} \prod_{j=n+1}^{\infty}(1-q^j)^{-n}$. The character stabilizes as $n \to \infty$ and turns into the MacMahon product $\prod_{j=1}^{\infty}(1-q^j)^{-j}$. Further, the spaces $H^{\infty/2}(\widehat{\mathfrak{n}}, \mathrm{Vac}_k \otimes \mathbf{1}_\chi)$ have a basis such that the coefficients of the operator product of the corresponding operators depend on the parameter n algebraically. We omit technical details, and the result is as follows:

Proposition 2.2. *There exists a family of vertex algebras $W_n(k)$ depending on two complex parameters n and k. The character of the vacuum representation of this algebra is* *given by* $\prod_{j=1}^{\infty}(1-q^j)^{-j}$. *For integer* n, *this algebra has an ideal such that the corresponding quotient algebra is a vertex algebra isomorphic to the Hamiltonian reduction* $\bar{U}_k(\widehat{\mathfrak{gl}}_n)$.

The embedding $\widehat{\mathfrak{gl}}_n \oplus \widehat{\mathfrak{gl}}_m \subset \widehat{\mathfrak{gl}}_{n+m}$ results in the embedding $W_{n+m}(k) \subset W_n(k) \oplus W_m(k)$. To see this, note the following. The character χ of the maximal nilpotent subalgebra in $\widehat{\mathfrak{gl}}_{n+m}$ is a sequence of characters $\chi_{1,2}, \chi_{2,3}, \dots, \chi_{n,n+1}, \chi_{n+1,n+2}, \dots, \chi_{n+m-1,n+m}$, where $\chi_{i,i+1}$ corresponds to the ith simple root in $\mathfrak{gl}_{n+m}$. Let $\chi(\varepsilon)$ be the sequence $\chi_{1,2}, \dots, \chi_{n-1,n}, \varepsilon\chi_{n,n+1}, \chi_{n+1,n+2}, \dots, \chi_{n+m-1,n+m}$, where $\varepsilon \in \mathbb{C}$. For $\varepsilon = 0$, there occur nontrivial cohomology components in positive degrees, and the zero degree cohomology component (for generic k) is $W_n \otimes W_m$. The deformation with respect to ε, just as above, determines the differential, and its kernel is a subalgebra isomorphic to W_{n+m} in $W_n \otimes W_m$. This map extends to complex values of n and m, and one obtains a comultiplication $W_{n+m} \to W_n \otimes W_m$ (for generic k).

Proposition 2.3. *The family of vertex algebras* $W_n(k)$ *quantizes the bialgebra* PD$_+ \subset$ PD. *In particular, the limit of the algebras* $W_n(k)$ *as* $n \to \infty$ *is the vertex algebra corresponding to the Lie algebra of differential operators on* S^1.

Remark 2.5. It is a technically challenging problem to verify these propositions. Although it is clear how to proceed, there is no rigorous mathematical proof in the literature. Apparently the simplest way is to consider difference W-algebras, where the verification is dramatically easier, and then pass to the limit.

2.4. **Difference W-algebras.** Now let us replace differential operators with difference operators in the preceding constructions. Namely, let $\mathfrak{D} = \mathbb{C}[X, X^{-1}, \delta, \delta^{-1}]$ be the algebra of functions on the quantum torus, that is, the algebra with two generators X and δ and the relations $X\delta = q\delta X$, $q \in \mathbb{C}^*$. For $q = 1$, the algebra $\mathbb{C}[X, X^{-1}, \delta, \delta^{-1}]$ is the algebra of functions on $\mathbb{C}^* \times \mathbb{C}^*$; hence the name. There exists an analog of trace on the algebra $\mathfrak{D}$. It is given by the map Tr defined by $\mathrm{Tr}\big(\sum a_{ij} Z^i \delta^j\big) = a_{00}$; clearly, $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$. The quadratic form $\langle f, g\rangle = \mathrm{Tr}(f \cdot g)$ is nondegenerate. The algebra $\mathfrak{D}$ is isomorphic to the algebra of difference operators, because $\mathfrak{D}$ has the representation on the space $\mathbb{C}[x, x^{-1}]$ of Laurent polynomials in which X acts as the operator of multiplication by x and δ acts as the shift $x \mapsto qx$. We define the following triangular decomposition on the quotient by the constants: $\mathfrak{D}/1 \cdot \mathbb{C} = \mathfrak{D}_+ \oplus \mathfrak{D}_-$, where $\mathfrak{D}_+ = \langle x^i\delta^j, j > 0; x^i, i > 0\rangle$ and $\mathfrak{D}_- = \langle x^i\delta^j, j < 0; x^i, i < 0\rangle$.

A natural generalization is given by the algebra $M_n \otimes \mathbb{C}[X, X^{-1}, \delta, \delta^{-1}]$, where M_n is the algebra of $n \times n$ matrices. This algebra has the trace $\mathrm{Tr}\big(\sum m_{ij} Z^i \delta^j\big) = \mathrm{Tr}\, m_{00}$, where $m_{ij} \in M_n$, as well. There exist many triangular decompositions, and we do not present them here. After quantization, we obtain the quantum toroidal algebra, which is denoted by $U_{\vec{q}}(\ddot{\mathfrak{gl}}_n)$, where $\vec{q} = (q_1, q_2, q_3)$ are three parameters related by the formula $q_1q_2q_3 = 1$. Let us describe in more detail what happens in the case of $\mathfrak{gl}_1$.

The Lie algebra $\mathfrak{D}$ has the Cartan decomposition $\mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where

$$\mathfrak{n}_+ = \mathbb{C}[x, x^{-1}, D]D, \quad \mathfrak{h} = \mathbb{C}[x, x^{-1}], \quad \mathfrak{n}_- = \mathbb{C}[x, x^{-1}, D^{-1}]D^{-1}.$$

Let $e_i = x^iD$, $f_i = x^iD^{-1}$, and $h_i = x^i$ be the generators of the algebras $\mathfrak{n}_+$, $\mathfrak{n}_-$, and $\mathfrak{h}$, respectively. The relations have the form

$$\begin{aligned} &[h_i, h_j] = 0, && [h_i, e_j] = (1 - q^i)e_{i+j}, \\ &[h_i, f_j] = (1 - q^{-i})f_{i+j}, && [e_i, f_j] = (q^{-i} - q^j)h_{i+j}, \end{aligned} \tag{1}$$

$$\begin{aligned} &\frac{1}{q^j - q^i}[e_i, e_j] = \frac{1}{q^{j'} - q^{i'}}[e_{i'}, e_{j'}], \\ &\frac{1}{q^j - q^i}[f_i, f_j] = \frac{1}{q^{j'} - q^{i'}}[f_{i'}, f_{j'}] \quad \text{for } i + j = i' + j', \end{aligned} \tag{2}$$

$$[e_0, [e_1, e_{-1}]] = 0, \quad [f_0, [f_1, f_{-1}]] = 0. \tag{3}$$

Relations (2) and (3) are called Serre relations. It was proved in [50, 59] that the algebra $\mathfrak{D}$ is generated by the elements e_i, h_i, and f_i and relations (1)–(3).

The last claim can be stated in a different form as follows. Consider the algebra $\mathfrak{L}$ with generators e_i, h_i,and f_i and relations (1). Then $\mathfrak{L}$ is graded by $\deg e_i = 1$, $\deg h_i = 0$, and $\deg f_i = -1$, and one has $\mathfrak{L} = \mathfrak{L}_+ \oplus \mathfrak{h} \oplus \mathfrak{L}_-$, where $\mathfrak{L}_+$ is the free algebra generated by the f_i and $\mathfrak{L}_-$ is the free algebra generated by the e_i. The algebra $\mathfrak{D}$ is the quotient of $\mathfrak{L}$ by the Serre relations (2) and (3). Thus, we have a slightly generalized version of the Kac–Moody Lie algebra; i.e., we take the generators e_i, h_i, f_i, impose the commutation relations, and take the quotient by the maximal ideal in $\mathfrak{L}_+$ and $\mathfrak{L}_-$.

The quantum deformation goes along similar lines; the formulas are given below in Definition 3.2. Note that this definition describes a deformation of a larger algebra. First, the algebra $\mathbb{C}[X, X^{-1}, \delta, \delta^{-1}]$ has a two-dimensional universal central extension. One cocycle, ψ, has the form $\psi(x^i\delta^j, \delta^{-j}x^{-i}) = i$ and is zero otherwise. The second (orthogonal) cocycle $\psi^\perp$ has the form $\psi^\perp(x^i\delta^j, \delta^{-j}x^{-i}) = j$ and is zero otherwise. Further, two outer derivations D and $D^\perp$, $[D, x^i\delta^j] = ix^i\delta^j$, $[D^\perp, x^i\delta^j] = jx^i\delta^j$, are added to the algebra. (The derivation $D^\perp$ is omitted in Definition 3.2.)

The relation between the toroidal algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ and W-algebras is established with the use of the bosonization construction. Namely, the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ has the so-called *Fock representations*. The precise formulas are given below in Definition 3.3. The name of the module (as well as the term "bosonization") is due to the fact that the representation space is the Fock representation of the Heisenberg algebra. It was shown in [39] that the image of the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ in representations that are tensor products of Fock representations of one and the same type is a q-deformed W-algebra. In [33], such algebras are called elliptic W-algebras.

3. Integrable systems associated with Lie algebras

The classical integrable systems associated with the Lie algebra $\mathfrak{g}$ are subalgebras, commutative with respect to the Kostant–Kirillov bracket, in the algebra of functions on the dual space $\mathfrak{g}^*$. In addition, one usually requires that the subalgebra is a maximal commutative subalgebra. The quantization of such integrable systems gives commutative algebras (in an ideal scenario, maximal as well) in the universal enveloping algebra $U(\mathfrak{g})$. There are no regular quantization procedures for commutative Poisson subalgebras. A well-known procedure applies to an algebra that is obtained by the argument shift method. This procedure is as follows. From the algebra $\mathfrak{g}$, one proceeds to the loop algebra $\widehat{\mathfrak{g}}$. The latter algebra has a large center at the critical level, and it can be mapped into $U(\mathfrak{g})$; the image is the desired commutative subalgebra.

Some important classical integrable systems (the Korteweg–de Vries equation, the nonlinear Schrödinger equation, etc.) can be obtained by the argument shift method on the loop algebra $\widehat{\mathfrak{g}}$. To quantize them, it is natural to try to proceed to the double loop algebra $\ddot{\mathfrak{g}}$. The preceding argument does not apply to the algebra $\ddot{\mathfrak{g}}$ word for word. However, if one takes the q-deformation of the toroidal algebra rather than the toroidal algebra itself, then some version of the preceding method proves to work.

3.1. **Classical systems: The argument shift method.** Let $\mathfrak{a}$ be a Lie algebra, let $\mathfrak{a}^*$ be the dual space with the standard Kostant–Kirillov bracket, and let $\mathcal{Z}[\mathfrak{a}^*]$ be the center of this Poisson algebra. The argument shift method is a simple and hence very common method for constructing a commutative subalgebra of the function algebra $\mathbb{C}[\mathfrak{a}^*]$.

Proposition 3.1 ([10, 11]). *Let $a \in \mathfrak{a}^*$, let $f_1, f_2 \in \mathbb{C}[\mathfrak{a}^*]^{\mathfrak{a}}$ be functions constant on the orbits, and let $\lambda_1, \lambda_2 \in \mathbb{C}$. Then the two functions $f_1^a(\lambda_1; x) = f_1(x + \lambda_1 a)$ and $f_2^a(\lambda_2; x) = f_2(x + \lambda_2 a)$, $x \in \mathfrak{a}^*$, commute.*

The argument shift method permits constructing nontrivial integrable systems (e.g., generalized tops) if $\mathfrak{a}$ is finite-dimensional, but the most interesting things happen if $\mathfrak{a}$ is an affine Lie algebra. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, let G be the corresponding Lie group, and let $\widehat{\mathfrak{g}}$ be the central extension of $\mathfrak{g}$-valued currents on the circle. The dual space $\widehat{\mathfrak{g}}^*$ is the space of operators of the form

$$\alpha \frac{\partial}{\partial \varphi} + A(\varphi),$$

where $A(\varphi)\, d\varphi$ is a $\mathfrak{g}$-valued smooth 1-form on the circle and $\alpha \in \mathbb{C}$. The hyperplane $\alpha = 1$ is a Poisson submanifold in $\widehat{\mathfrak{g}}^*$, whose elements are $\mathfrak{g}$-valued connections on the circle. The functions in $\mathcal{Z}[\mathfrak{a}^*]$ are constant on the orbits with respect to the adjoint action, that is, with respect to gauge transformations of the connections. To construct such functions, take a conjugation-invariant function F on G and define a function F^M on the space of connections by the formula

$$F_M\left(\frac{\partial}{\partial \varphi} + A(\varphi)\right) = F(M),$$

where M is the monodromy matrix of the connection

$$\frac{\partial}{\partial \varphi} + A(\varphi).$$

The argument shift method produces the following results in this case. Let $c = c(\varphi)\, d\varphi$ be a $\mathfrak{g}$-valued 1-form on the circle, and let $\lambda \in \mathbb{C}$. We assign a function $F_M^c(\lambda)$ to the pair (F, λ) by the formula

$$F_M^c\left(\lambda; \frac{\partial}{\partial \varphi} + A(\varphi)\right) = F_M\left(\frac{\partial}{\partial \varphi} + A(\varphi) + \lambda c\right).$$

All such function for various F and λ (but for fixed c) commute.

Define functions I_k by the formula

$$\log F_M^c(\lambda) = \sum I_k \lambda^{-k}.$$

They are local; i.e., they are integrals of differential polynomials of $A(\varphi)$. Hence they are called local integrals of motion. The coefficients of the expansion at $\lambda = 0$ are called nonlocal integrals of motion.

Let $\mathfrak{a}$ be a Lie algebra, let $\mathfrak{b}$ be a subalgebra, let χ be a character of $\mathfrak{b}$ (that is, a one-point orbit in $\mathfrak{b}^*$), and let $\rho\colon \mathfrak{a}^* \to \mathfrak{b}^*$ be the moment map (which is the adjoint of the embedding of $\mathfrak{b}$ in $\mathfrak{a}$). Let $a \in \mathfrak{a}^*$ be an element lying in $\rho^{-1}(\chi)$. Then one can use the argument shift method to construct integrable systems on the reduction $\rho^{-1}(\chi)/B$, where B is the group corresponding to the Lie algebra $\mathfrak{b}$. The functions on $\mathfrak{a}^*$, which we obtain by shifting central functions, can be restricted to the preimage $\rho^{-1}(\chi)$, and then they descend to the quotient. This way one can obtain integrable systems in classical W-algebras; these systems have been constructed as reductions in the preceding section.

There are at least two ways to obtain the famous Korteweg–de Vries Hamiltonian system by the argument shift method. Let Vir be the Virasoro algebra, that is, the central extension of the algebra of vector fields on the circle. The dual space Vir^* is traditionally identified with the space of Schrödinger operators

$$\alpha\frac{\partial^2}{\partial^2\varphi}+q(\varphi)$$

on the circle. Just as before, the condition $\alpha=1$ specifies a Poisson submanifold, whose points are operators defining projective structures on the circle [8]. The Kostant–Kirillov bracket restricted to the hypersurface $\alpha=1$ is known as the second Hamiltonian structure in Korteweg–de Vries theory.

Define the quadratic differential $q_0(\varphi)(d\varphi)^2$. Let $\lambda\in\mathbb{C}$. Define a function $T(\lambda)$ on the space of Schrödinger operators by setting

$$T\biggl(\lambda;\frac{\partial^2}{\partial^2\varphi}+q(\varphi)\biggr)=\operatorname{Tr}\biggl(M\biggl(\frac{\partial^2}{\partial^2\varphi}+q(\varphi)+\lambda q_0(\varphi)\biggr)\biggr),$$

where M is the monodromy matrix of a second-order equation. The functions $T(\lambda)$ with various λ (and fixed q_0) commute. The standard Korteweg–de Vries Hamiltonians are obtained for $q_0=1$ from the expansion of $\log T(\lambda)$ as $\lambda\to\infty$. The first nontrivial term of the expansion is proportional to the function $I_1(q)=\int q$, and the second term is proportional to the function $I_2(q)=\int q^2$; here I_2 is the Hamiltonian defining the Korteweg–de Vries flow.

The second way to obtain the Korteweg–de Vries system is as follows. First, we realize the dual space of the Virasoro algebra as the reduction of $\widehat{\mathfrak{sl}}_2$ by the character χ of the algebra $\widehat{\mathfrak{n}}$, where $\widehat{\mathfrak{n}}$ consists of the matrices

$$\begin{pmatrix}0 & f(\varphi)\\ 0 & 0\end{pmatrix}$$

and the character χ acts by the formula

$$f(\varphi)\mapsto\int f(\varphi)\,d\varphi.$$

Take an element $c\in\rho^{-1}(\chi)$; then the shift of the argument by c defines Hamiltonians on the Hamiltonian reduction; these are the same Hamiltonians as in the preceding paragraph.

In forthcoming subsections, we explain how to quantize commutative algebras obtained by the argument shift method. First, we do this for the finite-dimensional case, and then we discuss affine algebras, the Virasoro algebra, and W-algebras. To quantize the argument shift method for the algebra $\mathfrak{a}$, we use the center of the current algebra $\widehat{\mathfrak{a}}$, which is only nontrivial at the critical level in the case of finite-dimensional semisimple algebras.

3.2. **Center at the critical level and opers.** By $\widehat{\mathfrak{g}}$ we denote the central extension of the algebra $\mathfrak{g}[[t,t^{-1}]]$ of Laurent series with coefficients in $\mathfrak{g}$. (We have slightly changed notation compared with the preceding subsection, where the coefficients were arbitrary smooth functions on the circle.) It has already been mentioned in the preceding section that the center of the algebra $U(\widehat{\mathfrak{g}})$ at the critical level is formed by functions on the so-called oper manifold, which was described in terms of the Drinfeld–Sokolov reduction. We denote this center by $\mathcal{Z}(\widehat{\mathfrak{g}})$.

The oper sheaf $Op_{\mathfrak{g}}$ on a complex curve C is defined in [21] . For the case of the algebra $\mathfrak{sl}_2$, the oper sheaf is the classical geometric structure that takes each open set to the set of projective structures on it. The spectrum of the center of $U(\widehat{\mathfrak{sl}}_2)$ at the critical level is the manifold of projective structures on the punctured disk. Recall that the sheaf of projective structures on C is defined as follows: its sections over an open set are embeddings of this open set in $\mathbb{CP}^1$. Two such embeddings are identified if they differ by a linear-fractional transformation of $\mathbb{CP}^1$. The sheaf $Op_{\mathfrak{g}}$ for a semisimple algebra $\mathfrak{g}$ is defined in a similar way. Namely, let F be the flag manifold of the group G; it bears the so-called Grothendieck distribution, which is defined as the field of subspaces $V_p \subset T_p$, where $p \in F$. Each point p defines a Borel subalgebra $\mathfrak{b}_p \subset \mathfrak{g}$, one has $T_p \simeq \mathfrak{g}/\mathfrak{b}_p$, and the distribution space $V_p \subset T_p$ is defined as the annihilator of the radical $\mathfrak{n}_p \subset \mathfrak{b}_p$. The distribution V_p is completely nonintegrable. The sheaf $Op_{\mathfrak{g}}$, or the sheaf of generalized projective structures on the curve C, is defined as follows: its sections over an open set are immersions (maps whose differential is an embedding) of this open set in F which are integral with respect to the Grothendieck distribution. One identifies two such immersions if they can be obtained from one another by the action of G. The sections $Op_{\mathfrak{g}}(D^\times)$ of this sheaf over the punctured disk are the Hamiltonian reduction (the Drinfeld–Sokolov reduction) of the space $\widehat{\mathfrak{g}}^*$.

Remark 3.1. Opers on the nonpunctured formal disk are defined merely as maps into the flag space. Opers on the punctured disk cannot be defined in this manner. If we live in an analytic category, i.e., if the disk is "small" rather than formal, then the definition works. The procedure is different for the formal disk (see the book [21]).

The center of the algebra $U(\widehat{\mathfrak{g}})$ at the critical level is formed by functions on the set $Op_{\mathfrak{g}^L}(D^\times)$ of sections of the oper sheaf for the Langlands dual algebra $\mathfrak{g}^L$ over the punctured disk.

Monodromy is well defined for an oper on the curve C (i.e., for a global section of the sheaf $Op_{\mathfrak{g}}(C)$). Let $p \in C$, let U be an open neighborhood of the point p, and let γ be a closed curve beginning and ending at p. Then one can extend the projective structure along γ and obtain a different but equivalent projective structure on U. The resulting map $\pi_1(C) \to G$ is called the monodromy of the oper.

An oper is a structure resembling a connection on the manifold, and it can alternatively be described in terms of some differential operator. This language permits one to define opers on $D^\times$ as differential operators with pole singularities. This description looks most explicit for the case in which $\mathfrak{g} = \mathfrak{sl}_n$. An oper for $\mathfrak{sl}_2$ can be described as follows. Let $K^{1/2}$ be the square root of the canonical bundle on the curve C; then the sheaf $Op_{\mathfrak{sl}_2}$ is the sheaf of differential operators acting from $K^{-1/2}$ into $K^{3/2}$ with leading term $\partial^2/\partial z^2$ and with zero coefficient of $\partial/\partial z$. For the case of $\mathfrak{sl}_n$, these are nth-order differential operators from $K^{(1-n)/2}$ into $K^{(1+n)/2}$ with leading term $\partial^n/\partial z^n$ and with zero subleading term. Given a differential operator, the projective structure is constructed as follows: take a point $p \in C$ and consider the kernel of this operator in a neighborhood of p. Then the point p defines a complete flag in this space, the mth subspace of the flag being formed by the sections vanishing to the order $n - m$ at p. One can readily verify that the corresponding map of a small neighborhood of p into F is an integral curve of the Grothendieck distribution.

We also need the center $\mathcal{Z}_q(\widehat{\mathfrak{g}})$ of the quantum algebra $U_q(\widehat{\mathfrak{g}})$. Just as before, there exists a large center at the critical level. The spectrum of this center is called the quantum (or difference) oper manifold. However, this center has a relatively explicit construction in the quantum case [3, 56]. Being applied to the algebra $U_q(\widehat{\mathfrak{g}})$, this construction gives a homomorphism of the Grothendieck ring of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ into the center $\mathcal{Z}_q(\widehat{\mathfrak{g}})$ at the critical level. This homomorphism is actually an isomorphism. Just as in the q-deformed case, the center of $U_q(\widehat{\mathfrak{g}})$ at the critical level is a Poisson algebra. The paper [43] studies this Poisson structure, and the papers [33, 44] construct its quantization.

Example 3.1. Let $\mathfrak{g} = \mathfrak{sl}_2$. The center of $U_q(\widehat{\mathfrak{sl}}_2)$ at the critical level is generated by the elements $T(z)$, $z \in \mathbb{C}^*$, corresponding to two-dimensional irreducible representations of $U_q(\widehat{\mathfrak{sl}}_2)$. The Poisson structure is defined on these generators by the formula

$$\{T(z), T(w)\} = (q - q^{-1})(f(w/z)T(z)T(w) + \delta(w/q^2 z) - \delta(q^2 w/z)),$$

where

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n \quad \text{and} \quad f(x) = \sum_{n\in\mathbb{Z}} \frac{q^n - q^{-n}}{q^n + q^{-n}} x^n.$$

By analogy with the preceding, the spectrum of the center of $U_q(\widehat{\mathfrak{g}})$ at the critical level for the algebra $\mathfrak{g} = \mathfrak{sl}_n$ can be identified with the manifold of difference operators

$$\mathbb{D}^n + T_1(z)\mathbb{D}^{n-1} + \cdots + T_{n-1}(z)\mathbb{D} + 1,$$

where $\mathbb{D}$ is the shift by q^2 in the variable z.

3.3. **Systems of Gaudin type and quantization of the argument shift method.** We use the term "systems of Gaudin type" for quantum integrable systems whose construction uses the center of the universal enveloping algebra $U(\widehat{\mathfrak{g}})$ at the critical level.

The technique of using the center for constructing integrable systems is very simple. First, consider the case of a finite-dimensional rather than affine algebra. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, let G be the corresponding Lie group, let $\mathcal{Z}$ be the center of the universal enveloping algebra $U(\mathfrak{g})$, and let $\mathrm{D}(G)$ be the algebra of differential operators on G. Consider a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, and let N_+, H, and N_- be the corresponding subgroups. The group G contains the open set $O = N_+ \cdot H \cdot N_-$; let $\mathrm{D}(O)$ be the algebra of differential operators on it. The Lie algebra $\mathfrak{g}$ has both the left and the right action on the space of smooth functions on G, and hence there exists a map $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to \mathrm{D}(G) \to \mathrm{D}(O)$. The images of the subalgebras $\mathcal{Z} \otimes 1$ and $1 \otimes \mathcal{Z}$ are one and the same subalgebra of $\mathrm{D}(G)$ formed by the operators commuting with the right and left actions. Thus, there exists a homomorphism $\mathcal{Z} \simeq \mathcal{Z} \otimes 1 \to \mathrm{D}(O) \simeq \mathrm{D}(N_+) \otimes \mathrm{D}(H) \otimes \mathrm{D}(N_-)$.

Let χ_+ and χ_- be characters of the Lie algebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$, respectively, and let $\chi = (\chi_+, \chi_-)$ be a character of $\mathfrak{n}_+ \oplus \mathfrak{n}_-$. The algebra $U(\mathfrak{n}_+) \otimes U(\mathfrak{n}_-)$ is taken into $\mathrm{D}(O)$ ($\mathfrak{n}_+$ acts on the left, and $\mathfrak{n}_-$ acts on the right), and we can construct the quantum Hamiltonian reduction $\mathrm{D}(O)$ as soon as the kernel of the character $\chi\colon U(\mathfrak{n}_+ \oplus \mathfrak{n}_-) \to \mathbb{C}$ is known. The result of reduction can naturally be identified with the algebra $\mathrm{D}(H)$. We have constructed a map $\mathcal{Z} \to \mathrm{D}(H)$, depending on χ_+ and χ_-. If χ_+ and χ_- are generic, then this is the quantum integrable system known as the Toda system, and the system obtained for $\mathfrak{g} = \mathfrak{sl}_n$ is sometimes called the open Toda system.

In the affine case, one can proceed in a similar way. The algebra $\widehat{\mathfrak{g}}$ has the Cartan decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_-$; take some characters χ_+ and χ_- of the algebras $\widehat{\mathfrak{n}}_+$ and $\widehat{\mathfrak{n}}_-$, respectively, and reproduce the preceding construction. Then we obtain a map of the center $\mathcal{Z}(\widehat{\mathfrak{g}})$ at the critical level into the algebra of differential operators on H [26]. The image of this map is called the affine Toda system, and for the case of $\mathfrak{g} = \mathfrak{sl}_n$ it is called the closed Toda system.

The Hitchin system is a Gaudin type system as well, which we will now discuss. The group $\widehat{G}$ is a central extension of the group of maps of the formal punctured disk into G. Let C be a complex curve, let ξ be a G-bundle, and let p be a point on C. Consider the automorphism group of the sheaf ξ on a formal punctured neighborhood of p; let $\widehat{G}(\xi, p)$ be its central extension. This group contains two subgroups; one of them consist of bundle automorphisms over a formal neighborhood of p and will be denoted by $\widehat{G}_+$. The other consists of homomorphisms of ξ over $C \setminus p$ and will be denoted by $\widehat{G}_-$. If

ξ is a stable bundle, then the space $\widehat{G}_+\backslash\widehat{G}(p,\xi)/\widehat{G}_-$ of double cosets is a line bundle over the moduli space of G-bundles on C; we denote this moduli space by $\mathcal{M}(C,G)$. The center at the critical level is mapped into the space of differential operators on the double quotient, i.e., into the algebra of twisted differential operators on $\mathcal{M}(C,G)$. These twisted differential operators act from $K^{1/2}$ into itself, where $K^{1/2}$ is the square root of the canonical bundle on $\mathcal{M}(C,G)$. The image of the center is called the Hitchin system [21].

The above-constructed map of the center of the algebra $U(\widehat{\mathfrak{g}}(p,\xi))$ at the critical level into the algebra $\mathcal{M}(C,G)$ of twisted differential operators is surjective. In other words, this algebra of differential operators is isomorphic to the algebra of functions on some submanifold of the oper manifold $Op_{\mathfrak{g}^L}(D^\times)$ on the punctured disk. This submanifold consists of opers that can be extended both into the interior and into the exterior of the disk and hence are globally defined regular opers in $Op_{\mathfrak{g}^L}(C)$.

This construction is a starting point of what is now called the Langlands geometrical program. The Langlands correspondence (or the higher reciprocity law) assigns a D-module over the moduli space $\mathcal{M}(C,G)$ of G-bundles to each representation of the fundamental group of the curve in G^L. The above-described construction takes each $o\in Op_{\mathfrak{g}^L}(C)$ to a D-module over $\mathcal{M}(C,G)$, namely, to the system of differential equations saying that the operator corresponding to a central element z acts as the multiplication by the number $z(o)$. The oper o defines a monodromy, that is, a representation of the fundamental group, and hence we have a construction that assigns a D-module to such a representation.

A close but slightly different way to use the center of the universal enveloping algebra is as follows. It is none other than a quantum version of the Adler–Kostant scheme. Let $\mathfrak{a}$ be a Lie algebra (finite-dimensional for simplicity), let $\mathfrak{a}=\mathfrak{a}_+\oplus\mathfrak{a}_-$, where $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are two subalgebras, and let A, A_+, and A_- be the corresponding groups. The product $A_+\cdot A_-$ is an open subset of A. The center $\mathcal{Z}$ of the universal enveloping algebra $U(\mathfrak{a})$ is mapped into differential operators $\mathrm{D}(A_+\times A_-)=\mathrm{D}(A_+)\otimes\mathrm{D}(A_-)$ commuting with the left action of A_+ and the right action of A_-. Hence it is mapped into what is generated by the right action of A_+ and the left action of A_-, that is, into the algebra $U(\mathfrak{a}_+)\otimes U(\mathfrak{a}_-)$.

Consider the composition $\mathcal{Z}\to U(\mathfrak{a}_+)\otimes U(\mathfrak{a}_-)\to U(\mathfrak{a}_-)$, where the second map is the tensor product of the augmentation $U(\mathfrak{a}_+)$ by the identity map. The resulting commutative subalgebra of $U(\mathfrak{a}_-)$ can be described in slightly different words. Consider the $\mathfrak{a}$-module M induced from the trivial representation of $\mathfrak{a}_+$. Then M is a free rank 1 left $U(\mathfrak{a}_-)$-module. The action of the center $\mathcal{Z}$ commutes with the left action and hence can be expressed via the right action. Thus, for each $z\in\mathcal{Z}$ one can construct the element $U(\mathfrak{a}_-)$, and the elements thus obtained commute. It is this argument, which is independent of the existence of the groups A, A_-, and A_+, that will be used in what follows.

Following [12], we apply this to the case in which $\mathfrak{a}=\widehat{\mathfrak{g}}$, $\mathfrak{a}_-=\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]$, and $\mathfrak{a}_+=\mathfrak{g}\otimes\mathbb{C}[[t]]\oplus K\mathbb{C}$. Then the center $\mathcal{Z}(\widehat{\mathfrak{g}})$ at the critical level gives a commutative subalgebra in $U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])$. To construct a quantization of the subalgebra obtained by the argument shift method, let us construct a homomorphism of $U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])$ into $U(\mathfrak{g})$. Take a $\chi\in\mathfrak{g}^*$ and a $u\in\mathbb{C}$; then the image of $x(t)=\sum_{r>0}x_rt^{-r}$ is $\chi(x_1)\sum x_ru^r$. The image of the center under the composition

$$\mathcal{Z}(\widehat{\mathfrak{g}})\to U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])\to U(\mathfrak{g})$$

is independent of the choice of u but depends on χ and is a quantization of the algebra obtained by the argument shift method (the shift by $\chi\in\mathfrak{g}^*$).

One can use this construction to obtain the Gaudin model (e.g., see [16]). To this end, take another decomposition of $\mathfrak{a} = \widehat{\mathfrak{g}}$ with $\mathfrak{a}_- = \mathfrak{g}\otimes\mathbb{C}[t^{-1}]$ and $\mathfrak{a}_+ = \mathfrak{g}\otimes t\mathbb{C}[[t]]\oplus K\mathbb{C}$. Then the image of $\mathcal{Z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra in $U(\mathfrak{g}\otimes\mathbb{C}[t^{-1}])$. The set of evaluation homomorphisms $t^{-1}\to u_1,\dots,u_n$ gives a homomorphism $U(\mathfrak{g}\otimes\mathbb{C}[t^{-1}])\to U(\mathfrak{g})^{\otimes n}$. The images of elements of the algebra $\mathcal{Z}(\widehat{\mathfrak{g}})$ under the composition of these maps are called the Hamiltonians of the Gaudin system; in particular, the image contains the ordinary Gaudin operators

$$D_i = \sum_{j\neq i}\frac{\Omega_{ij}}{u_i-u_j}, \quad \text{where} \quad \Omega_{ij} = \sum_k 1\otimes\cdots\otimes a_k\otimes\cdots\otimes a_k\otimes\cdots\otimes 1;$$

here the nonunit elements are in the (i,j)th positions, and a_k is the basis in $\mathfrak{g}$ orthonormal with respect to the Killing form.

This can be generalized to the case of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. More precisely, there is no known generalization of Hitchin systems to the case of $U_q(\widehat{\mathfrak{g}})$, but the construction involving the subalgebras $\mathfrak{a}_+$ and $\mathfrak{a}_-$ can be generalized.

We again take $\widehat{\mathfrak{g}}$ for $\mathfrak{a}$. To avoid dealing with the q-deformation of the decomposition $\mathfrak{a}_+\oplus\mathfrak{a}_-$ used above, take $\mathfrak{a}_+ = \mathbb{C}K\oplus\widehat{\mathfrak{n}}^+$ and $\mathfrak{a}_- = \mathfrak{h}\oplus\widehat{\mathfrak{n}}_-$. Consider the universal Verma module M at the critical level, that is, the module induced from the one-dimensional representation of $U_q(\mathbb{C}K\oplus\widehat{\mathfrak{n}}^+)$ on which $U_q(\widehat{\mathfrak{n}}_+)$ acts by augmentation and K acts by minus the Coxeter number $q^{-h^\vee}$. Then M is a rank 1 free regular $U_q(\mathfrak{h}\oplus\widehat{\mathfrak{n}}_-)$-module. The action of the algebra $\mathcal{Z}_q(\widehat{\mathfrak{g}})$ on M commutes with this left action and hence can be expressed via the right action. Thus, we have obtained a map of $\mathcal{Z}(\widehat{\mathfrak{g}})$ into a commutative subalgebra in $U_q(\mathfrak{h}\oplus\widehat{\mathfrak{n}}_-)$.

The algebra $U_q(\mathfrak{h}\oplus\widehat{\mathfrak{n}}^-)$ has finite-dimensional representations $W(u)$ obtained by the restriction of finite-dimensional evaluation representations of the algebra $U_q(\widehat{\mathfrak{g}})$; here $u\in\mathbb{C}$ is the evaluation point. The above-constructed commutative algebra acting in the tensor product $W_1(u_1)\otimes\cdots\otimes W_n(u_n)$ gives the Hamiltonians of the XXZ-model. (One obtains the completely standard XXZ model if $\mathfrak{g}=\mathfrak{sl}_2$ and the W_i are two-dimensional.)

The ordinary Gaudin model is the degenerate form of the XXZ model as $q\to 1$.

3.4. **Commutative subalgebras of toroidal algebras.** Toroidal algebras are quantum groups as well. They are Drinfeld doubles, and one might expect that the construction providing the center at the critical level would also work for these algebras. However, this is not the case. This happens because for the ordinary affine algebras, which are doubles as well, the center is obtained at the critical value of the central element. If we formally try to compute the critical value for toroidal algebras, then we obtain infinity. However, the construction of a commutative algebra even for $U_q(\widehat{\mathfrak{g}})$ can be carried out without using the center at the critical level.

Let U be a quantum group; take the algebra U^* and assume that it contains a group element c, that is, an element such that $\Delta(c)=c\otimes c$. The functionals on U^* vanishing on c-commutators, that is, on elements of the form $xy-c^{-1}ycx$, where $x,y\in U^*$, will be called c-characters. If there exists a representation R of U^* on which the trace can be evaluated, then it defines a c-character by the formula $T_R(c;x)=\operatorname{Tr}R(xc)$. The space of functionals on $U^*/\langle\{xy-c^{-1}ycx\}\rangle$ is a subalgebra of U. Let R_1 and R_2 be two representations of U^* such that $R_1\otimes R_2$ and $R_2\otimes R_1$ have the same Jordan–Hölder series. Then the elements $T_{R_1}(c)$ and $T_{R_2}(c)$ commute. This condition may fail for arbitrarily chosen R_1 and R_2, but if we assume that R_1 and R_2 are representations of the Drinfeld double $\mathcal{D}(U)$, then $R_1\otimes R_2$ and $R_2\otimes R_1$ are related by the intertwining operator (the R-matrix), which is an isomorphism in many cases, which means that the products with distinct orders are isomorphic.

Remark 3.2. Actually, it suffices that only one of the representations R_1 and R_2 be a representation of the double, because the universal R-matrix lies in $U \otimes U^* \subset \mathcal{D}(U) \otimes \mathcal{D}(U)$.

We obtain a map of the Grothendieck ring of the double $\mathcal{D}(U)$ (consisting of representations on which the trace can be evaluated) into U; the range of this map is a commutative algebra.

One can slightly modify this construction to avoid using representations of the algebra U^* altogether. To this end, recall the notion of the L-operator. Consider some set S of representations of U. Assume that one can define the R-matrix on this set; in other words, for two arbitrary representations V and W in this set, there exists an intertwining operator $R\colon V \otimes W \to W \otimes V$, and we assume in addition that the Yang–Baxter equation is satisfied. Fix some representation $W \in S$; then the R-matrix defines an operator $R\colon \operatorname{End}(V) \to \operatorname{End}(W)$. Hence the algebra whose generators are $\bigoplus_{V \in S} \operatorname{End}(V)$ acts on W. The operator corresponding to $a \in \operatorname{End}(V)$ is denoted by $L(a)$ and is called the *L-operator*. These L-operators satisfy quadratic relations known as the LLR-relations.

Proposition 3.2. *Assume that the algebra U contains a group element $\bar{c}$; by $\pi_v\colon U \to \operatorname{End}(V)$ we denote the representation homomorphism. Then the operators $L(\pi_{V_1}(\bar{c}))$ and $L(\pi_{V_2}(\bar{c}))$ commute for any $V_1, V_2 \in S$.*

Thus, the commutative algebra constructed on the basis of the group element $\bar{c} \in U$ acts on a representation $W \in S$.

The two constructions presented above are very different from each other: one of them uses U^*, and the other uses representations of U; one produces elements of U, and the other only gives elements acting on some representations of U. However, in many examples they often give elements of one and the same subalgebra. Both constructions apply to affine as well as toroidal algebras. Let us analyze two examples, $U_q(\widehat{\mathfrak{sl}}_2)$ and $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$.

Definition 3.1 (e.g., see [6])**.** The quantum group $U_q(\widehat{\mathfrak{sl}}_2)$ is the algebra with generators E_0, E_1, $K_0^{\pm 1}$, $K_1^{\pm 1}$, $D^{\pm 1}$, F_0, and F_1 and the relations

$$\begin{gathered} DE_0D^{-1} = qE_0, \quad DF_0D^{-1} = q^{-1}F_0, \\ DxD^{-1} = x, \quad x = E_1, K_0^{\pm 1}, K_1^{\pm 1}, F_1, \\ K_iE_iK_i^{-1} = q^2E_i, \quad K_iF_iK_i^{-1} = q^{-2}F_i, \\ K_iE_jK_i^{-1} = q^{-2}E_j, \quad K_iF_jK_i^{-1} = q^2F_i \quad (i \neq j), \\ [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ E_i^3E_j - [3]_qE_i^2E_jE_i + [3]_qE_iE_jE_i^2 - E_jE_i^3 = 0 \quad (i \neq j), \\ F_i^3F_j - [3]_qF_i^2F_jF_i + [3]_qF_iF_jF_i^2 - F_jF_i^3 = 0 \quad (i \neq j). \end{gathered}$$

The comultiplication Δ, the antipode S, and the augmentation ε in the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ are defined by the formulas

$$\begin{gathered} \Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = f_i \otimes K_i^{-1} + 1 \otimes F_i, \\ \Delta(x) = x \otimes x, \quad x = D, K_i^{\pm 1} \quad (i = 0, 1), \\ S(K_i) = K_i^{-1}, \quad S(D) = D^{-1}, \quad S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \\ \varepsilon(K_i) = \varepsilon(D) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0. \end{gathered}$$

By $U_q(\widehat{\mathfrak{sl}}_2)_+$ we denote the Hopf subalgebra generated by E_0, E_1, K_0, and K_1. Then $U_q(\widehat{\mathfrak{sl}}_2)_+^*$ is isomorphic to the subalgebra $U_q(\widehat{\mathfrak{sl}}_2)_-$ of $U_q(\widehat{\mathfrak{sl}}_2)$ generated by F_0, F_1, K_0, and D.

The algebra $U_q(\widehat{\mathfrak{sl}}_2)_+$ has a family of large commutative subalgebras, that is, of quantum integrable systems. These systems are numbered by one complex parameter p. The system corresponding to the parameter p contains a linear combination of E_0E_1 and E_1E_0 with coefficient depending on p. To construct this integrable system by the first method, consider two-dimensional representations of the algebra $U_q(\widehat{\mathfrak{sl}}_2)_-$. We denote these representations by $\mathbb{C}^2(u)$, where u is known as the evaluation parameter. The representation $\mathbb{C}^2(u)$ defines a homomorphism of $U_q(\widehat{\mathfrak{sl}}_2)_-$ into the matrix algebra $\operatorname{End}(\mathbb{C}^2(u))$. The dual map embeds $\operatorname{End}(\mathbb{C}^2(u))^*$ in the completion of $U_q(\widehat{\mathfrak{sl}}_2)_+$. This map defines four elements $a(u)$, $b(u)$, $c(u)$, and $d(u)$ in the completion of $U_q(\widehat{\mathfrak{sl}}_2)_+$ as the images of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The elements $a(u)$, $b(u)$, $c(u)$, and $d(u)$ satisfy the above-mentioned LLR relations, which implies the existence of a family of commutative algebras. Fix the parameters α and β; then the operators $t(u) = \alpha a(u) + \beta d(u)$ with various u commute. This integrable system is obtained by the first construction for $c = K_0^{\alpha} K_1^{-\beta}$. It is convenient to set $p = \beta/\alpha$.

Let $\mathcal{XZ}_p$ be the algebra generated by $t(u)$. More precisely, let us expand $t(u)$ in the series

$$\log t(0)^{-1} t(u) = \sum I_r(p) u^{-r},$$

and let $\mathcal{XZ}_p$ be the algebra generated by $I_1(p), I_2(p), \dots$.

The spectrum of the algebra $\mathcal{XZ}_p$ is formed by functions on the manifold of formal difference opers on the punctured disk $D^\times$ with regular singularities. This interpretation is related to the fact that this commutative subalgebra can be constructed with the use of the center $\mathcal{Z}_q(\widehat{\mathfrak{g}})$ at the critical level; see Section 3.2. It follows from this relation that, acting on the tensor product $W_1(u_1) \otimes \cdots \otimes W_n(u_n)$ of evaluation representations, the algebra $\mathcal{XZ}_p$ gives the Hamiltonians of the XXZ model.

In addition, let us present a description of the algebra $U_q(\widehat{\mathfrak{sl}}_2)$; it is called the new Drinfeld realization [4]. We will need it in Section 3.6, while here it is mainly used to draw an analogy with the definition of toroidal algebra given after it.

Proposition 3.3. *The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ has the set of generators e_i, f_i, h_r, $i \in \mathbb{Z}$, $r \in \mathbb{Z} \setminus \{0\}$, $C^{\pm 1}$, $\psi_0^{\pm 1}$, and $D^{\pm 1}$. To write out the relations, it is convenient to introduce the currents*

$$e(z) = \sum_{k \in \mathbb{Z}} e_k z^{-k}, \qquad f(z) = \sum_{k \in \mathbb{Z}} f_k z^{-k},$$

$$\psi^{\pm}(z) = \sum_{r=0}^{\infty} \psi^{\pm}_{\pm r} z^{\mp r} = \psi_0^{\pm 1} \exp\biggl(\sum_{r=1}^{\infty} (q - q^{-1}) h_{\pm r} z^{\mp r}\biggr).$$

Then the relations have the following form:

$$C \text{ is central,}$$
$$De(z) = e(qz)D, \qquad Df(z) = f(qz)D, \qquad D\psi^{\pm}(z) = \psi^{\pm}(qz)D,$$
$$\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z), \qquad \frac{z-q^2w}{z-w}\psi^-(z)\psi^+(w) = \frac{w-q^2z}{w-z}\psi^+(w)\psi^-(z),$$
$$(z-q^2w)\psi^{\pm}(C^{(-1\mp 1)/2}z)e(w) + (w-q^2z)e(w)\psi^{\pm}(C^{(-1\mp 1)/2}z) = 0,$$
$$(w-q^2z)\psi^{\pm}(C^{(-1\pm 1)/2}z)f(w) + (z-w^2w)f(w)\psi^{\pm}(C^{(-1\pm 1)/2}z) = 0,$$
$$(z-q^2w)e(z)e(w) + (w-q^2z)e(w)e(z) = 0,$$
$$(z-q^{-2}w)f(z)f(w) + (w-q^{-2}z)f(w)f(z) = 0,$$
$$[e(z), f(w)] = \frac{1}{(q-q^{-1})zw}\left(\delta\left(\frac{Cw}{z}\right)\psi^+(w) - \delta\left(\frac{Cz}{w}\right)\psi^-(z)\right).$$

Definition 3.2. The quantum toroidal algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ is an algebra with generators e_i, f_i, h_r, $i \in \mathbb{Z}$, $r \in \mathbb{Z} \setminus \{0\}$, $C^{\pm 1}$, $(C^{\perp})^{\pm 1}$, and $D^{\pm 1}$ and depends on three parameters q_1, q_2, q_3 such that $q_1q_2q_3 = 1$. To write out the relations, it is convenient to introduce the currents

$$e(z) = \sum_{k\in\mathbb{Z}} e_k z^{-k}, \quad f(z) = \sum_{k\in\mathbb{Z}} f_k z^{-k},$$
$$\psi^{\pm}(z) = \sum_{r=0}^{\infty} \psi^{\pm}_{\pm r} z^{\mp r} = (C^{\perp})^{\mp 1}\exp\left(\sum_{r=1}^{\infty} \kappa_r h_{\pm r} z^{\mp r}\right).$$

Then the full set of relations has the following form:

$$C,\ C^{\perp} \text{ are central,}$$
$$De(z) = e(qz)D, \quad Df(z) = f(qz)D, \quad D\psi^{\pm}(z) = \psi^{\pm}(qz)D,$$
$$\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z),$$
$$\frac{g(C^{-1}z, w)}{g(Cz, w)}\psi^-(z)\psi^+(w) = \frac{g(w, C^{-1}z)}{g(w, Cz)}\psi^+(w)\psi^-(z),$$
$$g(z,w)\psi^{\pm}(C^{(-1\mp 1)/2}z)e(w) + g(w,z)e(w)\psi^{\pm}(C^{(-1\mp 1)/2}z) = 0,$$
$$g(w,z)\psi^{\pm}(C^{(-1\pm 1)/2}z)f(w) + g(z,w)f(w)\psi^{\pm}(C^{(-1\pm 1)/2}z) = 0,$$
$$[e(z), f(w)] = \frac{1}{\kappa_1}\left(\delta\left(\frac{Cw}{z}\right)\psi^+(w) - \delta\left(\frac{Cz}{w}\right)\psi^-(z)\right),$$
$$g(z,w)e(z)e(w) + g(w,z)e(w)e(z) = 0,$$
$$g(w,z)f(z)f(w) + g(z,w)f(w)f(z) = 0,$$
$$\operatorname*{Sym}_{z_1,z_2,z_3} z_2z_3^{-1}[e(z_1), [e(z_2), e(z_3)]] = 0,$$
$$\operatorname*{Sym}_{z_1,z_2,z_3} z_2z_3^{-1}[f(z_1), [f(z_2), f(z_3)]] = 0,$$

where we have used the following notation:

$$g(z,w) = (z-q_1w)(z-q_2w)(z-q_3w),$$
$$\kappa_r = (1-q_1^r)(1-q_2^r)(1-q_3^r), \quad q = q_2^{1/2}.$$

The comultiplication Δ and the augmentation ε are defined by the formula

$$\Delta(e_n) = \sum_{j\geq 0} e_{n-j}\otimes \psi_j^+ C^n + 1\otimes e_n, \quad \Delta(f_n) = f_n\otimes 1 + \sum_{j\geq 0}\psi_{-j}^- C^n\otimes f_{n+j},$$
$$\Delta h_r = h_r\otimes 1 + C^{-r}\otimes h_r, \quad \Delta h_{-r} = h_{-r}\otimes C^r + 1\otimes h_{-r},$$
$$\Delta X = X\otimes X \quad (x = C, C^\perp, D),$$
$$\varepsilon(e_n) = \varepsilon(f_n) = 0, \quad \varepsilon(h_{\pm r}) = 0, \quad \varepsilon(X) = 1 \quad (x = C, C^\perp).$$

In addition, the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ has the antipode, the formula for which can be found, say, in [39].

Definition 3.3. Consider the Fock representation of the Heisenberg algebra $\mathcal{F}$,

$$[h_r, h_s] = \frac{q^r - q^{-r}}{r\kappa_r}\delta_{r+s,0} \quad (r, s\in\mathbb{Z}\setminus\{0\}),$$

spanned by the highest vector $|\varnothing\rangle$,

$$h_r|\varnothing\rangle = 0 \quad \text{for } r > 0.$$

For each $u\in\mathbb{C}^\times$, one can equip the space $\mathcal{F}$ with the structure of a module of generators of $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ by the formulas $C^\perp = 1$, $C = q_i^{1/2}$, $i = 1, 2, 3$, and

$$e(z) = \frac{1-q_i}{\kappa_1} u \exp\biggl(\sum_{r=1}^\infty \frac{\kappa_r}{1-q_i^r} h_{-r} z^r\biggr) \exp\biggl(\sum_{r=1}^\infty \frac{q_i^{r/2}\kappa_r}{1-q_i^r} h_r z^{-r}\biggr),$$
$$f(z) = \frac{1-q_i^{-1}}{\kappa_1} u^{-1} \exp\biggl(-\sum_{r=1}^\infty \frac{q_i^{r/2}\kappa_r}{1-q_i^r} h_{-r} z^r\biggr) \exp\biggl(-\sum_{r=1}^\infty \frac{q_i^r\kappa_r}{1-q_i^r} h_r z^{-r}\biggr).$$

The resulting module is denoted by $\mathcal{F}_i(u)$ and is called the *Fock module* over $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$.

The algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)_+$ is defined as the subalgebra of $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ generated by e_{r-1}, h_r, f_r, and C, where $r\geq 1$. Then $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)_+^*$ is the subalgebra in $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ generated by e_{-r}, h_{-r}, f_{-r+1}, and D, where $r\geq 1$; this happens because $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ is the Drinfeld double of $U_{\vec{q}}(\ddot{\mathfrak{gl}}_{1,+})$ (up to the fact that we also need to add the element $D^\perp$). Let us formally introduce the element d by the formula $D = q^d$ and take $c = p^d$, $p\in\mathbb{C}$, for a group element in $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)_+^*$. We use the first construction to define the operators $T_{\mathcal{F}_i(u)}(p)$. We have obtained a commutative algebra depending on u; let us expand these operators in u^{-1},

$$T_{\mathcal{F}_i(u)}(c) = \sum_{r\geq 0} T_{\mathcal{F}_i,r}(c) u^{-r},$$

and define operators $I_r(p)$ by the formula

$$\log(T_{\mathcal{F}_i,0}(c)^{-1} T_{\mathcal{F}_i(u)}(c)) = \sum_{r\geq 0} I_r(p) u^{-r}.$$

Since the $T_{\mathcal{F}_i(u)}(p)$ with various u commute with each other, we find that $[I_r(p), I_{r'}(p)] = 0$. For the first operator $I_1(p)$, one can write out a closed-form expression.

Proposition 3.4 ([36])**.** *Let $\tilde{p} = pC^{-1}$. Define a new current $e(z,p)$ by the formula*

$$e(z,p) = e(z)\prod_{j=1}^\infty \psi^+(\tilde{p}^{-j} q_i^{-1/2} z),$$

and let $e_0(p)$ be the coefficient of z^0 in this current. Then the operator $I_1(p)$ is proportional to $e_0(p)$.

This commutative algebra permits constructing integrable systems in the representations $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)_+$. For example, consider the tensor product $\mathcal{F}_i(u_1)\otimes\cdots\otimes\mathcal{F}_i(u_n)$ of n Fock modules. It was shown in [39] that the image of the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)_+$ in $\mathrm{End}(\mathcal{F}_i(u_1)\otimes\cdots\otimes\mathcal{F}_i(u_n))$ coincides with the q-deformed W-algebra corresponding to $\mathfrak{gl}_n$. In the limit as $q_1, q_2, q_3, p \to 1$, we obtain an integrable system, which is called the $\mathfrak{gl}_n$-intermediate long wave equation and is a generalization of the quantum Korteweg–de Vries equation. This integrable system was studied, e.g., in [55] and [48].

A similar construction can be carried out for the toroidal algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_n)_+$. One obtains integrable systems depending on n parameters. The toroidal algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_n)_+$ has the evaluation homomorphism $U_{\vec{q}}(\ddot{\mathfrak{gl}}_n)_+ \to U_q(\widehat{\mathfrak{gl}}_n)$, and the image of the commutative algebra is the quantization of the algebra constructed by the argument shift method in $U(\widehat{\mathfrak{gl}}_n)$ (for a special choice of the shift). One also has the homomorphism $U_{\vec{q}}(\ddot{\mathfrak{gl}}_n)_+ \to U_q(\widehat{\mathfrak{gl}}_n)^{\otimes m}$ depending on the evaluation parameters $u_1, \dots, u_m$. Its image is an affine version of the Gaudin model; it was studied in the paper [34], where the notion of affine oper was introduced. The construction of an integrable system in that paper differs from the one given above, but one can show that the integrable system is obtained in the limit as $q_i \to 1$.

3.5. **Bethe equations and opers.** In Section 3.3, we have constructed a homomorphism of the center $\mathcal{Z}(\widehat{\mathfrak{g}})$ into the universal enveloping algebra $U(\mathfrak{g}\otimes\mathbb{C}[z^{-1}])$. (In this subsection, we use the variable z instead of t.) The kernel of this homomorphism is an ideal defining some submanifold $\mathcal{M}$ in the spectrum of $\mathcal{Z}(\widehat{\mathfrak{g}})$, that is, in the oper manifold $Op_{\mathfrak{g}^L}(D^\times)$. This manifold can be described as the manifold of opers on $(D^\times)$ with regular singularities. For the case of $\mathfrak{sl}_2$, the manifold $\mathcal{M}$ consists of the operators

$$\frac{\partial^2}{\partial z^2} + q(z), \quad \text{where} \quad q(z) = \sum_{i=-2}^{\infty} a_{-i} z^i.$$

The case of general $\mathfrak{g}$ was discussed in [21].

Given a set of points $u_1, \dots, u_n \in \mathbb{C}^\times$, one has the evaluation homomorphism $U(\mathfrak{g}\otimes\mathbb{C}[t^{-1}]) \to U(\mathfrak{g})^{\otimes n}$ at these points. As was already mentioned in Section 3.3, the image of $\mathcal{Z}(\widehat{\mathfrak{g}})$ under this homomorphism is the commutative Gaudin algebra $\mathcal{Z}(\vec{u})$. The kernel of the homomorphism is an ideal in $\mathcal{Z}(\widehat{\mathfrak{g}})$ defining a submanifold in $\mathcal{M}(\vec{u}) \subset \mathcal{M}$, and $\mathcal{M}(\vec{u})$ is the manifold of opers on $\mathbb{CP}^1 - \{u_1, \dots, u_n\}$ with regular singularities at the points u_i. For the case of $\mathfrak{g} = \mathfrak{sl}_2$, the manifold $\mathcal{M}(\vec{u})$ consists of the operators

$$\frac{\partial^2}{\partial z^2} + q(z), \quad \text{where} \quad q(z) = \sum \left(\frac{\alpha_{i,2}}{(z-u_i)^2} + \frac{\alpha_{i,1}}{z-u_i} \right).$$

Fix the set of weights $\lambda_1, \dots, \lambda_n$ of the algebra $\mathfrak{g}$. Then the Gaudin algebra acts on the tensor product $M_{\lambda_1}\otimes\cdots\otimes M_{\lambda_n}$ of Verma modules, and we wish to find the eigenvalues. It follows from the preceding that to each eigenvector v there corresponds a functional on $\mathcal{Z}(\vec{u})$, that is, a point $o_v \in \mathcal{M}(\vec{u})$.

Let us describe the answer for the case of $\mathfrak{sl}_2$. The oper $o_v = \partial^2/\partial z^2 + q(z)$ corresponding to an eigenvector should satisfy two conditions. First, the coefficients $\alpha_{i,2}$ of the second-order poles of $q(z)$ should be $-\lambda_i(\lambda_i+2)/4$. Second, the oper o_v should admit a representation in the form of the product

$$\left(\frac{\partial}{\partial z} + a(z)\right)\left(\frac{\partial}{\partial z} - a(z)\right),$$

where $a(z)dz$ is a rational 1-form. The change of variables $q(z) = -a^2(z) - a'(z)$ is called the Miura transformation.

We see that the function $a(z)$ should have the form

$$\sum_{i=1}^{n} \frac{-\lambda_i}{2(z-u_i)} + \sum_{j=1}^{m} \frac{1}{z-\omega_j}$$

for some points ω_j, the second-order poles of the function

$$q(z) = -a'(z) - a^2(z)$$

cancel out, and the cancellation of first-order poles is a system of equations on the points ω_j,

$$\sum_{i=1}^{n} \frac{\lambda_i}{\omega_k - u_i} - \sum_{j \neq k} \frac{2}{\omega_k - \omega_j} = 0. \tag{4}$$

There are m of these equations. They are called Bethe equations. The fact that the operator $\partial^2/\partial z^2 + q(z)$ factorizes into a product means that the monodromy matrix of the corresponding equation can be reduced to a triangular form, because it has a solution $f(z)$ such that $f'(z)/f(z) = a(z)$. The function $f(z)$ itself should have the form

$$\prod (z-u_i)^{-\lambda_i/2} \cdot \prod (z-w_j).$$

Let us outline the proof of this claim. First, the center of $U(\mathfrak{sl}_2)^{\otimes n}$ lies in the Gaudin algebra $\mathcal{Z}(\vec{u})$ and consists of functions depending on $\alpha_{i,2}$ alone. The center of $U(\mathfrak{sl}_2)^{\otimes n}$ acts on the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$ of Verma modules by the character. As a result, the leading terms at the poles of $q(z)$ at the points u_i are given by the expressions $-\lambda_i(\lambda_i+2)/4$, that is, coincide with the Casimir values of $\mathfrak{sl}_2$ on M_{λ_i}.

The second condition (saying that $q(z)$ lies in the range of the Miura transformation) follows from the fact that the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$ has a cyclic vector that is an eigenvector with respect to the subalgebra $\mathfrak{b}^+ \otimes \mathbb{C}[z^{-1}]$.

These two conditions mean that the spectrum of the Gaudin algebra is the intersection of two submanifolds of M. The first of them is the range of the Miura transformation, and the second is the manifold of opers with regular singularities at the points u_i and with $\alpha_{i,2} = -\lambda_i(\lambda_i+2)/4$.

Let the numbers λ_i and u_i be generic. Then the Bethe equations (4) describe the spectrum of the Gaudin algebra in the following sense. The diagonal algebra $\mathfrak{sl}_2^{\Delta}$ commutes with the Gaudin algebra, and hence it suffices to describe the spectrum of $\mathcal{Z}(\vec{u})$ on the space of highest vectors. The solutions of the Bethe equations (4) are in a one-to-one correspondence with the eigenvectors $\mathcal{Z}(\vec{u})$ that are highest vectors of $\mathfrak{sl}_2^{\Delta}$ with highest weight $\sum \lambda_i - 2m$, where m is the number of ω_j.

A similar description of the eigenvalues of $\mathcal{Z}(\vec{u})$ holds for any algebra $\mathfrak{g}$. Namely, the corresponding opers should lie in the image of the Miura transformation of a rational 1-form and be regular at the points u_i, whose singularity type is determined by the highest weights λ_i. We do not clarify what all these words about regular singularities mean (see [21]), but we will now define the Miura transformation.

The algebra $\mathfrak{g}\otimes\mathbb{C}[z^{-1}]$ has the subalgebra $\mathfrak{b}^+\otimes\mathbb{C}[z^{-1}]$, where $\mathfrak{b}^+$ is the Borel subalgebra of $\mathfrak{g}$. Let θ be a character of the algebra $\mathfrak{h} \otimes \mathbb{C}[z^{-1}]$; it defines a character of the algebra $\mathfrak{b}^+ \otimes \mathbb{C}[z^{-1}]$, which will also be denoted by θ. Let M_θ be the $\mathfrak{g} \otimes \mathbb{C}[z^{-1}]$-module induced from the one-dimensional representation θ of the algebra $\mathfrak{b}^+ \otimes \mathbb{C}[z^{-1}]$. The highest vector in the representation M_θ is an eigenvector for $\mathcal{Z}(\widehat{\mathfrak{g}})$, and hence we have a map of the character space of the algebra $\mathfrak{h} \otimes \mathbb{C}[z^{-1}]$ into the oper manifold.

The set of characters θ can be defined as follows. The dual space $\mathfrak{h} \otimes \mathbb{C}[z^{-1}]$ is isomorphic to the space Ω of forms $\mathfrak{h} \otimes z^{-1}\mathbb{C}[\![z]\!]dz$, where the pairing is by evaluation of the Killing form and the residue. The preceding construction gives a map $\Omega \to \mathcal{M}$, which is called the Miura transformation. To describe the spectrum of $\mathcal{Z}(\vec{u})$, we need the image of the Miura transformation of 1-forms that are rational functions of z.

The case of $\mathfrak{g} = \mathfrak{sl}_n$ is very similar to that of $\mathfrak{sl}_2$. In this case, the Miura transformation has the form

$$\frac{\partial^n}{\partial z^n} + q_2(z)\frac{\partial^{n-2}}{\partial z^n} + \cdots + q_n(z) = \left(\frac{\partial}{\partial z} + a_1(z)\right) \cdot \cdots \cdot \left(\frac{\partial}{\partial z} + a_n(z)\right),$$

where $a_1(z) + \cdots + a_n(z) = 0$.

3.6. **Bethe equations and difference opers.** In this subsection, we study the spectrum of commutative algebras arising from quantum groups (including toroidal algebras). By analogy with the preceding subsection, the spectrum will be described by the Bethe equations, which will be interpreted with the use of difference opers. Let us examine the simplest case of $U_q(\widehat{\mathfrak{sl}}_2)$.

It was shown in Section 3.4 that the algebra $U_q(\widehat{\mathfrak{sl}}_2)_+$ has the commutative subalgebra $\mathcal{XZ}_p$ generated by the operators $t(u)$, $u \in \mathbb{C}^\times$. There exists an evaluation homomorphism $U_q(\widehat{\mathfrak{sl}}_2)_+ \to U_q(\mathfrak{sl}_2)^{\otimes n}$, which depends on n points $u_1, \dots, u_n \in \mathbb{C}^\times$. For each $U_q(\mathfrak{sl}_2)$, consider the Verma module M_λ with highest weight q^λ. Then the evaluation homomorphism defines an action of the algebra $U_q(\widehat{\mathfrak{sl}}_2)_+$ and hence of $\mathcal{XZ}_p$ on the tensor product $M_{\vec{\lambda}}(\vec{u}) = M_{\lambda_1}(u_1)\otimes\cdots\otimes M_{\lambda_n}(u_n)$. The main result concerning the XXZ-model is that the eigenvectors of the algebra $\mathcal{XZ}_p$ are in a one-to-one correspondence with the solutions of the Bethe equations.

The system of Bethe equations has the form

$$p\prod_{i=1}^{n} q^{\lambda_i}\frac{w_k - u_i q^{-\lambda_i}}{w_k - u_i q^{\lambda_i}} = q^{-n}\prod_{j\neq k}\frac{w_k - w_j q^{-2}}{w_k - w_j q^2}. \tag{5}$$

This is a system of m equations for the unknowns $w_1, \dots, w_m$. Just as with the Gaudin model, the solutions of this system are in a one-to-one correspondence with the eigenvectors v that are highest vectors with respect to the diagonal action of $U_q(\mathfrak{sl}_2)$ with highest weight $q^{\sum \lambda_i - 2m}$.

The eigenvalue problem can be interpreted in terms of difference opers; let us explain how to do this.

By analogy with the preceding case, let us introduce the notion of sheaf $Opd_{\mathfrak{sl}_2}$ of difference opers. It is natural to define difference opers on open sets invariant with respect to the shift $z \mapsto q^2 z$. The sections of the sheaf $Opd_{\mathfrak{sl}_2}$ over an open set U invariant with respect to the shift by q^2 are difference operators of the form $\mathbb{D}^2 + T(z)\mathbb{D} + 1$, where $T(z) \in \mathbb{C}[U]$ is a regular function on U.

The Miura transformation plays an important role in the Gaudin model when describing the opers corresponding to eigenvectors. In the difference case, the Miura transformation has the form

$$\mathbb{D}^2 + T(z)\mathbb{D} + 1 = (\mathbb{D} + \Lambda(z))(\mathbb{D} + \Lambda^{-1}(z)).$$

Thus, the Miura transformation is the map $\mathbb{C}[U] \to Opd_{\mathfrak{sl}_2}(U)$ that takes $\Lambda(z)$ to $T(z) = \Lambda(z) + \Lambda^{-1}(q^2 z)$.

In the case of $\widehat{\mathfrak{sl}}_2$, the Miura transformation arises when computing the spectrum of the commutative algebra $\mathcal{Z}(u)$ on the highest vector of the module M_θ. A similar construction can be carried out for the case of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$. The algebra $U_q(\widehat{\mathfrak{sl}}_2)_+$ contains the subalgebra $U_q(\mathfrak{b}_+ \otimes \mathbb{C}[z])$ generated by e_r and ψ_r^+, $r > 0$. Here e_i and ψ_i^+ are part of the Drinfeld generators of the algebra $U_q(\widehat{\mathfrak{sl}}_2)$; see Proposition 3.3.

By M_θ we denote the representation $U_q(\widehat{\mathfrak{sl}}_2)_+$ induced from the one-dimensional representation $U_q(\mathfrak{b}_+ \otimes \mathbb{C}[z])$, where θ is a character of the commutative algebra generated by ψ_r^+. Denote the space of these characters by Ω. An element of the commutative algebra $\mathcal{XZ}_p$ acts on the highest vector of the representation M_θ by the character, and we obtain a map of Ω into the space of difference opers $Opd_{\mathfrak{sl}_2}(D^\times)$.

Recall that $t(u) = \alpha a(u) + \beta d(u)$. The elements $a(u)$ and $d(u)$ themselves multiply the highest vector M_θ by a number as well, and hence $a(u)$ and $d(u)$ define functions on Ω. Denote these functions by $\lambda_1(u)$ and $\lambda_2(u)$; they are related to each other by $\lambda_1(uq^2)\lambda_2(u) = 1$.

The algebra of functions on Ω is the algebra of polynomials in infinitely many variables. There are various ways to choose generators in this algebra; for example, for these one can take the coefficients of the power series expansion of $\lambda_1(u)$ in u^{-1}. This permits identifying Ω with functions $\Lambda(z)$ in a formal neighborhood of zero: the function that takes $\Lambda(z)$ to the rth coefficient in the expansion at zero corresponds to the rth coefficient in the series expansion of $\lambda_1(u)$ in u^{-1}.

We have a map of Ω into $Opd_{\mathfrak{sl}_2}(D^\times)$. The coordinates on the first space are $\Lambda(z)$, the coordinates on the second space are $T(z)$, and the Miura transformation is the expression of the map in these coordinates.

The oper $\mathbb{D}^2 + T(z)\mathbb{D} + 1$ corresponding to an eigenvector of the XXZ system has three properties. First, the formal series $T(z)$ extends to be a function meromorphic on the entire complex plane and holomorphic in a neighborhood of infinity. Second, the difference equation $\mathbb{D}^2 f + T\mathbb{D} f + 1 = 0$ should have meromorphic solutions. Third, the oper has regular singularities.

The second property means that $\mathbb{D}^2 + T(z)\mathbb{D} + 1$ lies in the image of the Miura transformation, because it can be represented in the form $(\mathbb{D} + \Lambda)(\mathbb{D} + \Lambda^{-1})$.

Let us briefly explain the third property: $T(z)$ is regular in a neighborhood of zero, and hence the equation $\mathbb{D}^2 f + T\mathbb{D} f + 1 = 0$ has solutions regular in a neighborhood of zero. In the ordinary (nondifference) case, the oper $\partial^2 + q(z)$ is regular at a point c if the equation $\partial^2 f + qf = 0$ has solutions of the form $(z - c)^\lambda u(z)$ in a neighborhood of c, where $u(z)$ is a function regular at the point c. The difference counterpart of the power-law function $(z - c)^\lambda$ is the infinite product

$$\prod_{k\geq 0} \frac{z - aq^{4k}}{z - bq^{4k}} = \psi(a, b, z).$$

A difference oper is regular if there exists a solution of the form

$$\prod_{j=1}^{m} \psi(a_i, b_i, z) \cdot u(z)$$

in a neighborhood of zero, where $u(z)$ is a regular function. Clearly, the analog of singular points is given by the orbits $\{a_i q^{4k}\}$ and $\{b_i q^{4k}\}$.

Details can be found in [45].

3.7. **Opers and the representation ring.** We have already noted that the notion of oper exists for affine algebras; the algebra of functions on opers and the representation ring are isomorphic algebras.

In the case of toroidal algebras, the notion of oper is lacking, and the construction uses the representation ring of the Borel subalgebra. For the case in which opers still exist, that is, for the case of affine algebras, the representation ring of the Borel algebra is the algebra of functions on the manifold of opers on the formal disk.

The algebra $U_q(\widehat{\mathfrak{sl}}_2)_+$ has a family of two-dimensional representations $\mathbb{C}^2(u)$, and the corresponding functions on $Opd_{\mathfrak{sl}_2}(D)$ take $T(z)$ to the value at the point u. (This also makes sense on the formal disk, provided that one deals with formal series.) The algebra $U_q(\widehat{\mathfrak{sl}}_2)_+$ has a family of representations $Q(u)$, $u \in \mathbb{C}$. The representation $Q(u)$ is irreducible for $u \neq 0$. The modules $Q(u)$ are defined as the representations induced from a one-dimensional subrepresentation of the subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)_+$ generated by $\{e_r, r > 1; \psi_r^+, f_r, r \geq 0\}$. (Recall that $U_q(\widehat{\mathfrak{sl}}_2)_+$ is generated by the same Drinfeld generators and the additional generator e_1.) The one-dimensional character of the subalgebra is given by the following formulas on the generators: $e_r \mapsto 0$, $r > 2$; $f_r \mapsto 0$, $r \geq 0$; $\psi_0^+ \mapsto 1$, $\psi_1^+ \mapsto u$, and $\psi_r^+ \mapsto 0$, $r > 2$.

The representation $Q(u)$ defines an element of the Grothendieck ring; this element will be denoted by the same letter.

Proposition 3.5. *The relation*

$$Q(uq^2) + T(u)Q(u) + Q(uq^{-2}) = 0$$

holds in the Grothendieck ring.

We see that Q is a solution of the equation $\mathbb{D} + T + \mathbb{D}^{-1} = 0$, which, up to the multiplication by $\mathbb{D}$, is exactly a difference oper.

One can use Proposition 3.5 to prove the following. Let v be an eigenvector of the integrable system, and let $q(v)$ and $t(v)$ be the eigenvalues of the operators $Q(v)$ and $T(v)$ on v. Then

$$A(v)q(vq^2) + t(v)q(v) + B(v)q(vq^{-2}) = 0,$$

where $A(v)$ and $B(v)$ are some functions depending on the representation. They occur in the equation, because, when identifying elements $T(v)$ of the algebra of functions on opers with operators in the integrable system, there arise normalization constants depending on α, β, and the highest weight of the representation. We obtain related details for brevity.

The representation $Q(u)$ is singled out by the following property. The commutative algebra $\{\psi_i\}$ acts on the representation space, and the support of the spectrum of this algebra is a singleton. In the case of $\mathfrak{sl}_2$, which we consider now, this means that $Q(u)$ has a basis of eigenvectors on which the operators $\{\psi_i\}$ act in one and the same way.

The Bethe equations are equivalent to the requirement that the operator corresponding to $Q(u)$ acts on the representation polynomially with respect to u up to a factor.

This scheme is applied in [37] to the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$. For the commutative algebra one takes the algebra generated by $\psi_i^\perp$, where the $\psi_i^\perp$ differ from the generators ψ_i in Definition 3.2 by a "rotation by 90°"; see [36]. These elements $\psi_i^\perp$ are related to the K_i introduced in Proposition 4.4 in terms of a shuffle algebra.

One can single out the subclass of representations of the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)_+$ consisting of "finite" representations, for which the support of the spectrum of the algebra $\{\psi_i^\perp\}$ is a finite set. The modules themselves in this class are infinite-dimensional, and the category of such representations is an analog of the $U_q(\widehat{\mathfrak{sl}}_2)_+$-module category generated by $\mathbb{C}^2(u)$ and $Q(u)$. Finite representations of the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)_+$ cannot be extended to representations of the entire algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$.

Proposition 3.6. *The Grothendieck ring of finite $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)_+$-modules contains representations $Q(u)$ and $M(u)$ such that the support of the spectrum of $\{\psi_i^\perp\}$ consists of a single point on $Q(u)$ and of two points on $M(u)$ and the following relation holds in the Grothendieck ring*:

$$Q(uq_1)Q(uq_2)Q(uq_3) + Q(u)M(u) + Q(uq_1^{-1})Q(uq_2^{-1})Q(uq_3^{-1}) = 0. \quad (6)$$

The Bethe equations for $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ follow from this relation.

Remark 3.3. Clearly, equations of the form (6) on the punctured disk are the counterpart of the oper manifold for the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$. It would be desirable to understand the meaning of the function algebra on this manifold. Possibly, this is the center of some algebra that is unknown yet.

4. Quantum groups and shuffle algebras (a survey of some constructions)

4.1. **Construction of a bialgebra from a commutation matrix.** Let (A_{ij}) be an arbitrary square $n \times n$ matrix with nonzero entries, let L be the free algebra with n generators $e_1, \dots, e_n$, and let $L^\vee \cong \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \ltimes L$ be the semidirect product of L by the algebra of Laurent polynomials, $x_i e_j = A_{ij} e_j x_i$. In view of this relation, the matrix A is called a commutation matrix.

One can equip the algebra $L^\vee$ with the structure of a bialgebra by defining a coassociative comultiplication $\Delta L^\vee \to L^\vee \otimes L^\vee$ according to the formulas

$$\Delta(x_i) = x_i \otimes x_i, \quad \Delta(e_i) = e_i \otimes 1 + x_i \otimes e_i.$$

The bialgebra $L^\vee$ is naturally graded by the lattice $\Gamma = \mathbb{Z}^n$; here $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has zero degree, and the degree of the generator e_i, $1 \le i \le n$, is the ith basis element of Γ.

An ideal J in $L^\vee$ is called a *Hopf ideal* if $\Delta(J) \subset J \otimes L^\vee + L^\vee \otimes J$. Let $I^\vee(A)$ be a maximal homogeneous ideal in $L^\vee$ that does not meet the space of generators, that is, the space

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \oplus \left(\bigoplus_i e_i \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \right).$$

We denote the quotient algebra $L^\vee / I^\vee(A)$ by $B(A)$; then $B(A)$ is a Hopf bialgebra and the quantum universal enveloping algebra of the Borel subalgebra in the Drinfeld–Jimbo quantum universal enveloping algebra corresponding to the matrix A. The system of generators of the ideal $I^\vee(A)$ is called the *Serre relations*.

Example 4.1. Let the matrix (A_{ij}) satisfy the unitarity condition $A_{ij} A_{ji} = 1$. Then the expressions $e_i e_j - A_{ij} e_j e_i$ lie in $I^\vee(A)$. That is, the subalgebra in $B(A)$ generated by $\{e_i\}$ is a quotient of the algebra of skew polynomials. There can be additional Serre relations as well. If $A_{ii} = -1$, then e_i^2 is a Serre relation.

With the matrix (A_{ij}), one can also associate the quantum Kac–Moody bialgebra [2, 46]. This algebra is defined by the generators e_i, x_i, y_i, and f_i and the following relations:

$$x_i, y_i \text{ commute with each other}, \quad [e_i, f_j] = \delta_{ij}(x_i - y_j),$$
$$x_i e_j = A_{ij} e_j x_i, \quad y_i e_j = A_{ji} e_j y_i, \quad A_{ij} x_i f_j = f_j x_i, \quad A_{ji} y_i f_j = f_j y_i.$$

The comultiplication, the antipode, and the counit are given by the formulas

$$\Delta(x_i) = x_i \otimes x_i, \quad \Delta(y_i) = y_i \otimes y_i,$$
$$\Delta(e_i) = e_i \otimes 1 + x_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes y_i + 1 \otimes f_i,$$
$$S(x_i) = y_i, \quad S(y_i) = x_i, \quad S(e_i) = -y_i e_i, \quad S(f_i) = -f_i,$$
$$\varepsilon(x_i) = \varepsilon(y_i) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0.$$

Further, one should take the quotients by the ideals $I^\vee(A)$ in the subalgebra $B_+(A)$ generated by x_i and e_i and the subalgebra $B_-(A)$ generated by y_i and f_i. The resulting bialgebra will be denoted by $U(A)$. The algebra $U(A)$ is the Drinfeld double of the "half" $B_+(A)$ of itself.

This definition is stated in a form suitable for any matrix (A_{ij}), which may contain zeros. The elements $x_i y_i \in U(A)$ are central. If we take the quotient by them, then we obtain an algebra that contains the algebra $B(A)$ as a subalgebra.

Further, recall the notion of adjoint action. Let U be a bialgebra with antipode, and let S be an antiautomorphism satisfying some additional conditions (see [2]). In our case, $S(x_i) = x_i^{-1}$ and $S(e_i) = -x_i e_i$. The map

$$\widetilde{\Delta}\colon U \xrightarrow{\Delta} U \otimes U \xrightarrow{1\otimes S} U \otimes U^o$$

is an algebra homomorphism. Let C be an algebra, and let $U \to C$ be a homomorphism. Then C is a $(C \otimes C^o)$-module, and the homomorphism $\widetilde{\Delta}$ defines the adjoint action of U on C.

4.2. **"Twisted" bialgebra.** In this subsection, we restate the construction in Section 4.1 in a slightly different language. Consider the category of Γ-graded algebras, and let $\Omega\colon \Gamma \times \Gamma \to \mathbb{C}^*$ be a bilinear form with matrix A on the group Γ. Let $C \otimes_q D$ be the twisted tensor product of two Γ-graded algebras C and D. This is the tensor product of C and D as Γ-graded vector spaces equipped with the multiplication

$$(c_1 \otimes d_1) \cdot (c_2 \otimes d_2) = \Omega(\deg d_1, \deg c_2) \cdot (c_1 c_2 \otimes d_1 d_2).$$

Here c_i and d_i are homogeneous elements in C and D, respectively, and deg is the degree of a homogeneous element.

A twisted bialgebra is a Γ-graded algebra B equipped with a coassociative comultiplication $\Delta B \to B \otimes_q B$.

The free algebra L with n generators $e_1, \dots, e_n$ is Γ-graded, and one can define a twisted comultiplication $L \to L \otimes_q L$ by the formula $e_i \to e_i \otimes 1 + 1 \otimes e_i$. One can consider the quotient of L by a maximal homogeneous Hopf ideal $I(A)$ that does not meet the space spanned by the generators e_i. The quotient algebra $L/\bar{I}(A)$ is the quantum enveloping algebra of the maximal nilpotent subalgebra in the (twisted) Drinfeld–Jimbo algebra. The generators of $I(A)$ are the same Serre relations defined in Section 4.1. In the general case, the description of the ideal $I(A)$ is a hard unsolved problem.

For nontrivial Serre relations to occur, the matrix (A_{ij}) must satisfy "resonance conditions". Namely, it is necessary that there exist indices $i_1, i_2, \dots, i_r$, $1 \le i_\alpha \le n$, such that

$$A_{i_1 i_2} \cdot A_{i_2 i_3} \cdot \dots \cdot A_{i_{r-1} i_r} \cdot A_{i_r i_1} = 1.$$

This condition is also known as the *wheel condition.* The simplest and hence most common Serre relation arises if the resonance relation has the form $A_{11}^k A_{12} A_{21} = 1$. Then the Serre relation has the familiar form

$$[\cdots[[e_2, e_1]_q, e_1]_q \cdots, e_1]_q = 0,$$

where the commutator is taken $k+1$ times. Here $[c,d]_q$ is the q-commutator, which is given by

$$[c,d]_q = cd - \Omega(\deg c, \deg d) \cdot dc$$

for homogeneous elements c and d.

4.3. **Bimodules over twisted bialgebras, the Heisenberg double, and its representations.** Let a commutation matrix (A_{ij}), $1 \le i,j \le n$, be a submatrix of an $(n+1)\times(n+1)$ commutation matrix (A'_{ij}); i.e., $A_{ij} = A'_{ij}$, $1 \le i,j \le n$. Let $e_1, \dots, e_{n+1}$ be the generators of the bialgebra $L(A')$. It is natural to view $L(A')$ as an extension of $L(A)$. We introduce a $\mathbb{Z}$-grading $\overline{\deg}$ on $L(A')$ by setting $\overline{\deg}(e_i) = 0$, $1 \le i \le n$, and $\overline{\deg}(e_{n+1}) = 1$. Then $L(A)$ is the degree zero component. Let $M(A')$ be the degree 1 component. Clearly, $M(A')$ is an $L(A)$-bimodule. There is an additional structure on $M(A')$ arising from comultiplication.

Namely, consider the components $\Delta_1 \colon M(A') \to M(A') \otimes L(A)$ and $\Delta_2 \colon M(A') \to L(A) \otimes M(A')$ of comultiplication. (Here the tensor product is taken in the category of Γ-graded spaces.) The maps Δ_1 and Δ_2 are $L(A)$-bimodule homomorphisms, $M(A')$ and $L(A)$ bear the structure of a bimodule, and the structure of a bimodule on the tensor product is defined via comultiplication. Further, Δ_1 and Δ_2 define the structure of an $L(A)$-bicomodule on $M(A')$.

Recall that $I(A)$ and $I(A')$ are maximal Hopf ideals in $L(A)$ and $L(A')$, respectively. Denote the quotient $M(A')/(M(A') \cap I(A'))$ by $R(A')$. Clearly, $R(A')$ is an $N(A)$-bimodule with an additional structure.

Let us formalize this construction. It is more convenient to do this in the category of nontwisted bialgebras.

Assume that U is a bialgebra and we wish to describe graded bialgebras of the form $U \oplus R \oplus \cdots$, where U lies in degree zero and R lies in degree 1. We use coassociativity and find that R is a U-bimodule and there exists bimodule homomorphisms $\Delta_1 \colon R \to R \otimes U$ and $\Delta_2 \colon R \to U \otimes R$; they define the structure of a U^*-bimodule on R. Thus, R is equipped with the actions of $U \otimes U^o$ and $U^* \otimes (U^*)^o$. These two actions make R a module over the algebra that is called the Heisenberg double of $U \otimes U^o$ and is denoted by $H(U \otimes U^o)$. The algebra $H(U \otimes U^o)$ can be described as follows: $H(U \otimes U^o)$ is generated by the two subalgebras $U \otimes U^o$ and $U^* \otimes (U^*)^o$. The relations between the two commuting subalgebras U and U^o and the two commuting subalgebras U^* and $(U^*)^o$ express the fact that Δ_1 and Δ_2 are homomorphisms. Here we do not present explicit formulas for these homomorphisms. Modules over $H(U \otimes U^o)$ are also called Heisenberg bimodules.

One can take tensor products of representations of the Heisenberg double $H(U \otimes U^o)$. Let R_1 and R_2 be two $H(U \otimes U^o)$-modules; one can restrict them to $U \otimes U^o$ and take the tensor product of the resulting $(U \otimes U^o)$-bimodules. The product $R_1 \otimes R_2$ still bears a right action of U^* and a left action of U^*, and we again obtain the structure of an $H(U \otimes U^o)$-module.

Proposition 4.1. *Let U be a bialgebra, and let R be a representation of the Heisenberg double $H(U \otimes U^o)$. Then there exists a graded bialgebra $U \oplus R_1 \oplus R_2 \oplus \cdots = L(U, R)$, where $R_n = R \otimes \cdots \otimes R$ (n times) and the tensor product is taken in the category of $H(U \otimes U^o)$-modules.*

The subalgebra $\bigoplus_n R_n$ is a free tensor algebra, and comultiplication is encoded in the structure of an R_1-module, which gives two maps $\Delta_1 \colon R \to R \otimes U$ and $\Delta_2 \colon R \to U \otimes R$. These maps extend to the entire algebra by multiplicativity.

The bialgebra $L(U, R)$ contains a maximal homogeneous Hopf ideal $I(U, R)$ that does not meet $U \oplus R_1$. The quotient $L(U, R)/I(U, R)$ will be denoted by $B(U, R)$.

Example 4.2. The algebra U of functions on a group G is a Hopf algebra with commutative multiplication. Then $H(U \otimes U^o)$ is the semidirect product $\mathbb{C}(G \times G^o)^*$ of the group algebra of $G \times G^o$ by the algebra $\mathbb{C}[G \times G^o]$ of functions on $G \times G^o$. The group $G \times G^o$ acts on the set $G \times G^o$ as follows: the pair (g, h) takes (x_1, x_2) to (gx_1h, gx_2h). For this action, the diagonal is an orbit. Hence the Heisenberg double has a class of representations "concentrated on the diagonal". This means that the support of the restriction of the representation to the commutative subalgebra of functions on $G \times G$ is concentrated on the diagonal. The category of such representations is equivalent to the category of representations of the group G.

Note that the algebra $B(A)$ considered in Section 4.1 is an example of such a construction. Here G is the abelian group $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (n times), and the representation $V = \langle e_1, \ldots, e_n \rangle$ is the sum of n one-dimensional representations of G.

Let U be a bialgebra, let R be a Heisenberg bimodule, and let $B(U, R)$ be the bialgebra constructed above. With $B(U, R)$ one can associate the Drinfeld double $\mathcal{D}(B(U, R))$ (recall that it is $B(U, R) \otimes B(U, R)^*$ as a coalgebra rather than as algebra), which contains the parabolic subalgebra $P(U, R)$ generated by U, U^*, and R. The $P(U, R)$-module induced from the trivial representation of U^* is the sum $U \oplus R \oplus \cdots$, and hence R has the structure of a left $\mathcal{D}(U)$-module.

By definition, there is a structure of a U-bimodule on R, where the action of $U \subset \mathcal{D}(U)$ is the left action. The action of $U^* \subset \mathcal{D}(U)$ on R is the adjoint action of U^* obtained from the left (or right) action of U^* on R.

The $P(U, R)$-module induced from the trivial representation of $\mathcal{D}(U)$ is the sum of quotients $U/IU \oplus R/(R \cdot IU) \oplus \cdots$, where IU is the kernel of the augmentation $U \to \mathbb{C}$. The correspondence $R \mapsto R/(R \cdot IU)$ takes each Heisenberg bimodule to a representation of $\mathcal{D}(U)$. There exists a functor acting in the opposite direction as well; it assigns a Heisenberg bimodule to a representation of $\mathcal{D}(U)$.

Let us return to the beginning of this subsection. If $A = (A_{ij})$ is an $n \times n$ submatrix of an $(n+1) \times (n+1)$ matrix A', then the quantum Kac–Moody algebra $U(A)$ is a subalgebra of $U(A') = \cdots \oplus R_{-1} \oplus U(A) \oplus R_1 \oplus \cdots$, where R_1 is the Heisenberg bimodule. The quotient $R_1/(R_1 \cdot IU)$ is a Verma module for $U(A)$ with highest weight determined by the matrix A'.

4.4. **Shuffle algebras.** Let S be a set, and let $\lambda\colon S \times S \to \mathbb{C}$ be a function of two variables. Define the graded algebra

$$\mathrm{SH}(\lambda) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{SH}(\lambda)_m.$$

The component $\mathrm{SH}(\lambda)_m$ consists of symmetric functions $S \times S \times \cdots \times S \to \mathbb{C}$ (m factors), and the multiplication $\mathrm{SH}(\lambda)_n \times \mathrm{SH}(\lambda)_m \to \mathrm{SH}(\lambda)_{n+m}$ is denoted by the symbol $*$ and is given by the formula

$$f * g(s_1, \ldots, s_{n+m}) = \mathrm{Sym}\left(f(s_1, \ldots, s_n) g(s_{n+1}, \ldots, s_{n+m}) \prod_{\substack{1 \leq \alpha \leq n, \\ n+1 \leq \beta \leq n+m}} \lambda(s_\alpha, s_\beta) \right). \tag{7}$$

Here Sym is the symmetrization with respect to the variables $s_1, \ldots, s_{n+m}$. The associativity of this multiplication is obvious.

One can define the skew-symmetric version $\mathrm{SH}^{\mathrm{alt}}(\lambda)$ of this algebra by considering skew-symmetric functions rather than symmetric ones and by replacing symmetrization with alternation.

Note that if S is a finite set and the function λ vanishes nowhere, then $\mathrm{SH}(\lambda)$ is the algebra of skew polynomials. To see this, it suffices to take delta functions on S, that is, functions equal to zero at all but one point of S, for the generators. One can readily verify that the delta functions generate everything and that their products taken in different orders differ by a factor.

Definition 4.1. An m-tuple of possibly repeated elements $s_1, s_2, \ldots, s_m \in S$ is said to satisfy the *wheel condition* if

$$\lambda(s_1, s_2) = 0, \quad \lambda(s_2, s_3) = 0, \quad \ldots, \quad \lambda(s_m, s_1) = 0.$$

We say that a function $F\colon S \times S \times \cdots \times S \to \mathbb{C}$ satisfies the wheel condition (or the vanishing condition) if F vanishes on any n-tuple $(s_1, \dots, s_n)$ such that the subtuple $(s_1, \dots, s_m)$ satisfies the wheel condition for some $m \le n$.

Proposition 4.2. *The algebra* $\mathrm{SH}(\lambda)$ *contains the subalgebra* $\mathrm{sh}(\lambda)$ *of functions satisfying the wheel condition.*

To prove this, it suffices to look at formula (7). If f and g satisfy the vanishing condition, then the same is true for $f * g$, because all terms in (7) vanish on the desired tuples.

Note that if S is a finite set, then the algebra $\mathrm{sh}(\lambda)$ is generated by the first component of itself.

The construction of a shuffle algebra admits a simple generalization. Let C be a commutative algebra, and let $\lambda \in C \otimes C$. Using these data, one can construct the graded algebra $\mathrm{SH}(\lambda) = \bigoplus_{m \in \mathbb{Z}_{\ge 0}} \mathrm{SH}(\lambda)_m$ whose components have the form $\mathrm{SH}(\lambda)_m = S^m(C)$ and the multiplication is given by the formula

$$f * g = \mathrm{Sym}(\lambda_{n,m} f \otimes g).$$

Here

$$f \in S^n(C), \quad g \in S^m(C), \quad f \otimes g \in S^n(C) \otimes S^m(C) \subset C^{\otimes n+m},$$
$$\lambda_{n,m} = \prod_{1 \le \alpha \le n < \beta \le n+m} \lambda_{\alpha,\beta} \in C^{\otimes (n+m)},$$

and $\lambda_{\alpha,\beta}$ is the image of λ under the map $C \otimes C \to C^{\otimes m+n}$ defined by

$$c_1 \otimes c_2 \mapsto 1 \otimes \cdots \otimes c_1 \otimes \cdots \otimes c_2 \otimes \cdots \otimes 1,$$

where c_1 is moved into position α and c_2, into position β.

Note that if S is a semisimple finite-dimensional algebra, then C consists of functions on the finite set S, λ is a function on $S \times S$, and we return to the definition at the beginning of the subsection.

Now let us return to the twisted bialgebras in Section 4.2. The bialgebra constructed there can be realized as a subalgebra of a shuffle algebra. Let A be an $n \times n$ matrix, and let $B = \mathbb{C} \oplus V$ be the quotient of the algebra $L(A)$ by the ideal generated by the quadratic part. By using comultiplication, one can obtain a homomorphism $L(A) \to B^{\otimes_q N}$ or even onto $B^{\otimes_q \infty}$. To define this infinite tensor product, we proceed as follows. Let $S = \mathbb{Z}$; we define a function λ on $S \otimes S$ by setting

$$\begin{cases} \lambda(x,y) = A_{ij} & \text{if } x < y,\ x \equiv i \bmod n,\ y \equiv j \bmod n, \\ \lambda(x,x) = 0, \\ \lambda(x,y) = 1 & \text{if } x > y. \end{cases}$$

Now consider the $\widetilde{\mathrm{SH}}(\lambda)$ of symmetric functions $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \to \mathbb{C}$ that vanish on tuples in which at least two elements coincide. In $\widetilde{\mathrm{SH}}(\lambda)$, we define the subalgebra $\widetilde{\mathrm{SH}}^{\mathrm{loc}}(\lambda)$ of locally constant functions. We say that a function $F\colon \mathbb{Z} \times \cdots \times \mathbb{Z} \to \mathbb{C}$ is locally constant if the value of f on a tuple $x_1, \dots, x_m$ of integers depends only on the residues of x_i modulo n and the mutual arrangement of the elements x_i. This means that $f(x_1, \dots, x_m) = f(x'_1, \dots, x'_m)$ whenever $x_i \equiv x'_i \bmod n$ and the differences $x_i - x_j$ and $x'_i - x'_j$ have coinciding signs for any pair of indices (i,j). Note that the algebra $\widetilde{\mathrm{SH}}^{\mathrm{loc}}(\lambda)$ has the same dimensions as the tensor algebra of the n-dimensional space. To prove this, let us describe the basis in a component. Since the function f is symmetric, it suffices to define it for $x_1 < x_2 < \cdots < x_m$. A basis element is a function that is nonzero on a tuple $\{x_i\}$ only for some given distribution of residues modulo n. It is easily seen that the resulting algebra polynomially depends on the numbers A_{ij}; i.e., the structure constants of multiplication in the basis are polynomials of A_{ij}. Clearly, there exists a homomorphism $\varphi\colon L \to \mathrm{SH}(A)$ taking e_i to the identically unit function if $x \equiv i \bmod n$ and the identically zero function if $x \not\equiv i \bmod n$.

Proposition 4.3. *The kernel of the homomorphism φ coincides with the ideal $I(A)$. In other words, $N(A)$ is the subalgebra of* $\mathrm{SH}(A)$ *generated by the component of degree* 1.

4.5. **Shuffle algebras of rational functions.** Let $\mathcal{K}$ be the algebra of functions on the curve $\mathbb{C}^*$, that is, of Laurent polynomials in one variable z. To construct a shuffle algebra, we need the function λ; it belongs to $\mathcal{K}\otimes\mathcal{K}$; i.e., λ is a function of two variables. In our examples, λ is the rational homogeneous function

$$\lambda(z_1, z_2) = \prod_{1\le i\le s} \frac{z_1 - q_i z_2}{z_1 - z_2},$$

where $\{q_i\}$ is a tuple of nonzero complex numbers. Since such a function λ has singularities, we should slightly modify the definition of the algebra $\mathrm{SH}(\lambda)$.

Definition 4.2. Let s be an odd number, and let P be a symmetric Laurent polynomial. The algebra $\mathrm{SH}(\lambda)$ consists of functions of the form

$$\frac{P(z_1, \dots, z_n)}{\prod_{i<j}(z_i - z_j)^{s-1}}.$$

Multiplication is given by the same formula (7). For even s, the algebra $\mathrm{SH}(\lambda)$ consists of functions of the form

$$\frac{P(z_1, \dots, z_n)}{\prod_{i<j}(z_i - z_j)^{s}}.$$

Clearly, the space of such functions is closed with respect to the shuffle product (7), because the degrees of poles higher than those indicated cannot occur.

Definition 4.3. The algebra $\mathrm{SH}(\lambda)$ contains the subalgebra $\mathrm{sh}(\lambda)$ of functions f vanishing at a point $(z_1, \dots, z_n)$ whenever

$$\frac{z_1}{z_2} = q_{i_1}, \quad \frac{z_2}{z_3} = q_{i_2}, \quad \dots, \quad \frac{z_{n-1}}{z_n} = q_{i_{n-1}}, \quad \frac{z_n}{z_1} = q_{i_n},$$

for some $q_{i_1}, q_{i_2}, \dots, q_{i_m}$, $1 \le i_\alpha \le s$, $m \le n$.

Clearly, for the vanishing conditions to be nontrivial, it is necessary that the parameters q_i satisfy the resonance condition $\prod q_i^{k_i} = 1$ for some $k_i \in \mathbb{Z}_{\ge 0}$.

Let B be a commutative algebra, and let $C = B \otimes \mathbb{C}[z, z^{-1}]$ be the algebra of Laurent polynomials with coefficients in B. One can construct a shuffle algebra by using a rational function $\lambda(z_1, z_2)$ ranging in $B \otimes B$. If B is an n-dimensional semisimple algebra and P_i is a basis of idempotents, then an element $\lambda \in C \otimes C$ is given by a tuple of functions

$$\{\lambda_{ij}(z_1, z_2)\}, \quad \lambda = \sum_{i,j} P_i \otimes P_j \lambda_{ij}(z_1, z_2).$$

Assume that the functions $\lambda_{ij}(z_1, z_2)$ have the form

$$\lambda_{ij}(z_1, z_2) = \frac{z_1 - q_{ij} z_2}{z_1 - z_2}, \quad q_{ij} \in \mathbb{C}.$$

Definition 4.4. The shuffle algebra $\mathrm{SH}(\{\lambda_{ij}\})$ is the $\mathbb{Z}^n$-graded algebra whose graded component of degree $a_1, \dots, a_n$, $a_i \in \mathbb{Z}_{\geq 0}$, consists of functions of the form

$$\frac{P\big(z_1^1, \dots, z_{a_1}^1; z_1^2, \dots, z_{a_2}^2; \dots; z_1^n, \dots, z_{a_n}^n\big)}{\prod (z_\alpha^i - z_\beta^j)}.$$

In this formula, P is a Laurent polynomial symmetric in the variables z_α^i for each fixed i, and the product in the denominator is taken over the pairs (i, α), (j, β) such that either $i < j$ or $i = j$ and $\alpha < \beta$. Multiplication is given by formula (7).

The algebra $\mathrm{SH}(\{\lambda_{ij}\})$ contains the subalgebra $\mathrm{sh}(\{\lambda_{ij}\})$ of functions satisfying the wheel condition; i.e., $f \in \mathrm{sh}(\{\lambda_{ij}\})$ if f vanishes on any tuple z_α^i such that there exists a sequence $\{z_{\alpha_1}^{i_1}, z_{\alpha_2}^{i_2}, \dots, z_{\alpha_m}^{i_m}\}$ in which

$$\frac{z_{\alpha_1}^{i_1}}{z_{\alpha_2}^{i_2}} = q_{i_1, i_2}, \quad \dots, \quad \frac{z_{\alpha_m}^{i_m}}{z_{\alpha_1}^{i_1}} = q_{i_m, i_1}.$$

Note that if $q_{ij} = q_{ji} = 1$, then the wheel conditions force f to have no pole on the corresponding diagonal.

Let $U_q(\widehat{\mathfrak{g}})$ be the affine quantum Kac–Moody algebra. In this algebra, one can take the so-called Drinfeld current generators [4]. In $U_q(\widehat{\mathfrak{g}})$, there exists a subalgebra that quantizes the universal enveloping algebra of currents into a nilpotent subalgebra of $\mathfrak{g}$. This subalgebra is a shuffle algebra with $q_{ij} = q^{A_{ij}}$, where (A_{ij}) is the symmetrized Cartan matrix of the algebra $\mathfrak{g}$.

One can take an arbitrary Cartan matrix, in particular, the Cartan matrix of an affine rather than finite-dimensional Lie algebra. As a result, one obtains toroidal algebras.

A natural generalization of this construction was suggested by Ding and Iohara [25]. This constructed an algebra from the function matrix $(\lambda_{ij}(z_1, z_2))$. This algebra $DI(\lambda)$ has the Cartan decomposition

$$DI(\lambda)_> \otimes DI(\lambda)_0 \otimes DI(\lambda)_<,$$

where $DI(\lambda)_>$ and $DI(\lambda)_<$ are the shuffle algebras corresponding to (λ_{ij}). The subalgebra $DI(\lambda)_0$ is commutative; its generators are traditionally denoted by $\psi_i^+(z)$ and $\psi_i^-(z)$. The algebra $DI(\lambda)_0$ is the Drinfeld double of the subalgebra of itself generated by $\psi_i^+(z)$ and $DI(\lambda)_>$. The algebras $U_q(\widehat{\mathfrak{sl}}_2)$ and $U_q(\ddot{\mathfrak{gl}}_1)$ introduced in Section 3.4 are special cases of the Ding–Iohara algebra.

In the next subsection, we discuss toroidal algebras corresponding to $\mathfrak{gl}_n$ on the basis of a slightly different construction.

4.6. **Toroidal algebras and their commutative subalgebras.** The toroidal algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ (more precisely, the universal enveloping algebra of a maximal nilpotent subalgebra in it) is the algebra $\mathrm{sh}(\lambda)$, where

$$\lambda = \frac{(z_1 - q_1 z_2)(z_1 - q_2 z_2)(z_1 - q_3 z_2)}{(z_1 - z_2)^3}, \quad q_1 q_2 q_3 = 1.$$

The toroidal algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_n)$, $n \geq 2$, is the algebra $\mathrm{sh}(\lambda_{ij})$ for some function tuple $\{\lambda_{ij}\}$. It is convenient to describe this tuple and the algebra itself as follows; we give a description for the case of $n \geq 3$.

Let $\mathcal{E}_n$ be the degenerate elliptic curve that is the union of n projective lines $\mathbb{P}^1$, $n \geq 3$. The lines $\mathbb{P}^1$ are numbered by elements of the group $\mathbb{Z}/n\mathbb{Z}$, and the point ∞ on the line $\mathbb{P}_i^1$ is glued with the point 0 on the line $\mathbb{P}_{i+1}^1$, $i \in \mathbb{Z}/n\mathbb{Z}$. Let $\mathcal{E}_n^*$ be the set of nonsingular points of $\mathcal{E}_n$, $\mathcal{E}_n^* = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$. Then $\mathcal{E}_n^*$ is a group, and the semidirect product $\Gamma_n = \mathbb{Z}/n\mathbb{Z} \ltimes \mathcal{E}_n^*$ acts on $\mathcal{E}_n$ by automorphisms. In Γ_n, take three elements $g_1 = (m_1, q_1)$, $g_2 = (m_2, q_2)$, and $g_3 = (m_3, q_3)$, $m_i \in \mathbb{Z}/n\mathbb{Z}$, $q_i \in \mathcal{E}_n^*$, such that $g_1 g_2 g_3 = 1$ and at least one of the m_i is nonzero. We also need the elements $\bar{g}_1 = (m_1, 1)$, $\bar{g}_2 = (m_2, 1)$, and $\bar{g}_3 = (m_3, 1)$. Let λ be a function on $\mathcal{E}_n \times \mathcal{E}_n$ whose divisor is $D_+ - D_-$, where $D_+ = \{(x_1, x_2) \mid \exists i\, x_2 = g_i x_1\}$ and $D_- = \{(x_1, x_2) \mid \exists i\, x_1 = \bar{g}_i\}$. Note that λ is a rational function on $\mathcal{E}_n^* \times \mathcal{E}_n^*$ with poles on D_-.

The algebra $\mathrm{sh}(\lambda)$ consists of rational symmetric functions on $\mathcal{E}_n^* \times \cdots \times \mathcal{E}_n^*$ with poles of order ≤ 1 on $\widetilde{D}$, where the divisor $\widetilde{D}$ consists of points $(x_1, \ldots, x_m)$ such that $(x_i, x_j) \in D_-$ for some i and j, $1 \leq i < j \leq m$. The wheel condition is as follows: $f(x_1, \ldots, x_m)$ vanishes if either $x_2 = g_1 x_1$ and $x_3 = g_2 x_2$ or $x_2 = g_2 x_1$ and $x_3 = g_1 x_2$, $x_i \in \mathcal{E}_m^*$. The standard definition of toroidal algebra is obtained for $m_1 = 1$, $m_2 = 0$, and $m_3 = -1$.

For $n = 2$, the definition should be modified, which we do not do here. For $n = 1$, the degenerate elliptic curve is $\mathbb{P}^1$ with 0 and ∞ glued together. The definition is exactly the same except that the functions have a second-order pole on the diagonal.

On can construct subalgebras of $\mathrm{sh}(\lambda)$ by imposing regularity conditions on the function f.

Take an n-tuple $\mu = (\mu_1, \ldots, \mu_n)$ of complex numbers. This tuple defines a line bundle ξ_μ on $\mathcal{E}_n$; the degree of this bundle is zero, and a section of ξ_μ is a tuple $f_1, \ldots, f_n$ of functions such that $f_i(0) = \mu_i f_{i+1}(\infty)$, $i \in \mathbb{Z}/n\mathbb{Z}$; the numbers $\mu_1, \ldots, \mu_n$ are coordinates on the Jacobian of the curve $\mathcal{E}_n$.

Let ξ be a vector bundle on $\mathcal{E}_n$. By using the same multiplication formula involving the function λ, one can define the structure of an algebra on the space $\oplus_n \Gamma(\xi^{\boxtimes m})^{S_m}$; here $\xi \boxtimes \cdots \boxtimes \xi$ is a vector bundle on $\mathcal{E}_n \times \cdots \times \mathcal{E}_n$, and Γ is the space of rational sections of $\xi^{\boxtimes m}$ with first-order poles on $\widetilde{D}$ satisfying the wheel condition. Now let $\xi = \xi_\mu$; we denote the corresponding algebra of sections by $\mathrm{sh}(\xi_\mu, \lambda)$. Clearly, $\mathrm{sh}(\xi_\mu, \lambda)$ is a subalgebra of $\mathrm{sh}(\lambda)$. It is commutative, and it was explicitly computed for toroidal algebras in [41]. It was shown there that $\mathrm{sh}(\xi_\mu, \lambda) = 0$ if $\mu_1 \mu_2 \cdots \mu_n \neq 1$. If $\mu_1 \mu_2 \cdots \mu_n = 1$ and otherwise the μ_i are general, then the algebra $\mathrm{sh}(\xi_\mu, \lambda)$ is only nontrivial in degrees $\mathrm{sh}(\xi_\mu, \lambda)_{m,\ldots,m}$. The algebra $\bigoplus \mathrm{sh}(\xi_\mu, \lambda)_{m,\ldots,m}$ is the algebra of polynomials with n generators in degree $(1, 1, \ldots, 1)$, n generators in degree $(2, 2, \ldots, 2)$, etc., all in all, n generators in each degree.

For the case of the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$, the formulas for the generators of the commutative algebra are simpler; they were evaluated in [38].

Proposition 4.4. *The algebra*

$$\mathrm{sh}(\lambda), \quad \lambda = \frac{(z_1 - q_1 z_2)(z_1 - q_2 z_2)(z_1 - q_3 z_2)}{(z_1 - z_2)^3}, \quad q_1 q_2 q_3 = 1,$$

has the commutative subalgebra generated by the elements K_r^1, $r \in \mathbb{Z}_{>0}$, where

$$K_r^1 = \prod_{1 \leq j < k \leq n} \frac{(z_j - q_1 z_k)(z_k - q_1 z_j)}{(z_j - z_k)^2}, \quad r \geq 2,$$

and K_1^1 is the identically unit function of one variable.

If we replace q_1 by q_2 or q_3 in the last formula, then we obtain elements of the same commutative subalgebra.

The algebra $\mathrm{sh}(\lambda)$ is a subalgebra of the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$, and hence the elements K_i^q generate a large commutative algebra acting in any representation $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$. If this is the Fock representation, then we obtain the system of Macdonald operators.

4.7. **Elliptic deformations of shuffle algebras.** One can construct a shuffle algebra starting from any set S and functions on it. Interesting algebras are obtained if one selects a subalgebra in the space of functions on S by imposing conditions on the behavior of functions.

In this subsection, we deal with the case in which S is an elliptic curve $\mathcal{E}$; $\mathcal{E}$ is a group. Let τ_1, τ_2, and τ_3 be three elements of $\mathcal{E}$ such that $\tau_1+\tau_2+\tau_3=0$. Fix the divisor $D=D_+-3D_-$ on $\mathcal{E}\times\mathcal{E}$, where

$$D_+=\{x,y\mid x-y=\tau_i,\ i=1,2,3\},\quad D_-=\{x,y\mid x=y\}.$$

The divisor D is equivalent to zero, because $\tau_1+\tau_2+\tau_3=0$, and hence there exists a function $\lambda\colon\mathcal{E}\times\mathcal{E}\to\mathbb{C}$ with this divisor. Let $\mathcal{K}_n$ be the field of rational symmetric functions on the manifold $\mathcal{E}\times\cdots\times\mathcal{E}$ (n factors). By $\mathrm{SH}(\lambda,\mathcal{E})$ we denote the algebra $\bigoplus\mathcal{K}_n$ with multiplication (7). Let T be a divisor on $\mathcal{E}$, $T=\sum m_iP_i$, $P_i\in\mathcal{E}$; by $\mathcal{K}^T\subset\mathcal{K}_1$ we denote the subspace of rational functions on $\mathcal{E}$ that are regular on $E\setminus T$ and have zeros or poles of multiplicity less than or equal to m_i at each P_i.

Let $\mathrm{sh}(T,\lambda)_n$ be the space of symmetric rational functions $f(x_1,\dots,x_n)$, $x_i\in\mathcal{E}$, such that f has poles of multiplicity ≤ 2 with respect to x_1 (that is, for fixed $x_2,\dots,x_n$) at the points $x_2,\dots,x_n$, has zeros of order $\le m_i$ at the points $\{P_i\}$, and is regular for $x_1\in\mathcal{E}\setminus\{x_i,P_j\}$. Further, the function f should satisfy the wheel condition, that is, vanish whenever $x_1-x_2=\tau_1$ and $x_2-x_3=\tau_2$ or $x_1-x_2=\tau_2$ and $x_2-x_3=\tau_1$. The space $\mathrm{sh}(T,\lambda)=\bigoplus_n\mathrm{sh}(T,\lambda)_n$ is closed with respect to the shuffle multiplication (7).

Proposition 4.5. *Let* $\dim\mathrm{sh}(T,\lambda)_1=N$, $N>1$. *Then the Hilbert series of the algebra* $\mathrm{sh}(T,\lambda)$ *is given by*

$$\sum_n\dim\mathrm{sh}(T,\lambda)_nt^n=\prod_{k\ge1}(1-kNt^k)^{-1}.$$

The space $\mathrm{sh}(T,\lambda)_1$ is isomorphic to the section space of some line bundle ν. If the degree of ν is n and $n\ge1$, then $\dim\mathrm{sh}(T,\lambda)_1=n$. If the degree of ν is zero, then two cases are possible. If ν is trivial, then $\dim\mathrm{sh}(T,\lambda)_1=1$; if ν is nontrivial, then $\dim\mathrm{sh}(T,\lambda)_1=0$.

Proposition 4.6 ([40])**.** *Let the bundle* ν *be trivial. Then the Hilbert series of the algebra* $\mathrm{sh}(T,\lambda)$ *is* $\prod_k(1-t^k)^{-1}$. *The algebra* $\mathrm{sh}(T,\lambda)$ *is commutative in this case.*

In this case, we denote the algebra $\mathrm{sh}(T,\lambda)$ by Z. By analogy with the case of a degenerate elliptic curve discussed in the preceding subsection, Z is the algebra of polynomials in the generators $K_r^i\in\mathrm{sh}(T,\lambda)_r$, $r\in\mathbb{Z}_{>0}$, where i is fixed and takes one of the values $1,2,3$. That is, we claim that the algebra Z has three natural systems of generators. Closed-form expressions for the functions K_r^i can be written out as follows. The functions $K_1^1=K_1^2=K_1^3$ are unity identically. The function K_2^i is a function on $\mathcal{E}\times\mathcal{E}$ whose divisor is $D_+^i-2D_-$, where D_- is the diagonal and $D_+^i=\{x_1,x_2;x_1-x_2=\pm\tau_i\}$. Further, $K_r^i(x_1,\dots,x_r)=\prod_{j<k}K_2^i(x_j,x_k)$.

The shuffle algebra associated with an elliptic curve and three shifts can be understood as a deformation of the algebra related to q_1, q_2, and q_3, three shifts on a degenerate elliptic curve. Fix a family of curves $\mathcal{E}(p)$, where $\mathcal{E}(0)$ is the degenerate elliptic curve. The family $\mathcal{E}(p)$ can be formally trivialized in a neighborhood of the singular fiber by choosing the coordinates (x,p) in a formal neighborhood of $\mathcal{E}(0)$. The function $\lambda(x_1,x_2,p)$ has the form

$$\lambda(x_1,x_2,p)=\frac{(x_1-q_1x_2)(x_1-q_2x_2)(x_1-q_3x_2)+p\Lambda_1(x_1,x_2)+p^2\Lambda_2(x_1,x_2)+\cdots}{(x_1-x_2)^3},$$

where the $\Lambda_i(x_1,x_2)$ are Laurent polynomials of degree 3. The specific form of the Λ_i depends on the choice of a local trivialization. The standard method is as follows. The function $(x_1 - qx_2)/(x_1 - x_2)$ is deformed into the theta function,

$$\frac{\prod_{i\geq 0}(x_1 - qp^i x_2)\prod_{i>0}(x_2 - q^{-1}p^i x_1)}{\prod_{i\geq 0}(x_1 - p^i x_2)\prod_{i>0}(x_2 - p^i x_1)} = \frac{\prod_{i\geq 0}\left(1 - \dfrac{qp^i x_2}{x_1}\right)\prod_{i>0}\left(1 - \dfrac{q^{-1}p^i x_1}{x_2}\right)}{\prod_{i\geq 0}\left(1 - \dfrac{p^i x_2}{x_1}\right)\prod_{i>0}\left(1 - \dfrac{p^i x_1}{x_2}\right)} = \frac{\theta\left(\dfrac{qx_1}{x_2}; p\right)}{\theta\left(\dfrac{x_1}{x_2}; p\right)},$$

where $\theta(x;p)$ is the standard theta function. After the p-deformation, the function

$$\frac{(x_1 - q_1 x_2)(x_1 - q_2 x_2)(x_1 - q_3 x_2)}{(x_1 - x_2)^3}$$

becomes

$$\frac{\theta\left(\dfrac{q_1 x_1}{x_2}; p\right)\theta\left(\dfrac{q_2 x_1}{x_2}; p\right)\theta\left(\dfrac{q_3 x_1}{x_2}; p\right)}{\theta\left(\dfrac{x_1}{x_2}; p\right)^3}.$$

The parameters τ_1, τ_2, and τ_3 are obtained if one passes from the multiplicative form of the theta function to the additive form.

In a similar way, the generators

$$K_r^i(x_1, \dots, x_r) = \prod \frac{(z_j - q_i z_k)(z_k - q_i z_j)}{(z_j - z_k)^2}$$

are deformed into

$$\prod \frac{\theta\left(\dfrac{q_i z_j}{z_k}; p\right)\theta\left(\dfrac{q_i^{-1} z_j}{z_k}; p\right)}{\theta\left(\dfrac{z_j}{z_k}\right)^2}.$$

The elliptic deformation in the toroidal case follows the same scheme.

Thus, the commutative subalgebra of $U_q(\ddot{\mathfrak{gl}}_1)$ can be deformed (at least formally with respect to the parameter p) to the commutative subalgebra of the deformed shuffle algebra with a different commutation function.

Remark 4.1. By analogy with the case of a degenerate elliptic curve, where the elements K_r^i act as the Macdonald operators on the Fock space, in the elliptic case we obtain a deformation of these operators. In particular, the operator K_1^i passes into the zero Fourier component of the current $e(z,p)$; see Proposition 3.4.

5. Integrable systems and shuffle algebras

In this section, we present an alternative construction of integrable systems and Bethe equations discussed in Sec. 3. Our exposition mainly follows [37]. For simplicity, we consider shuffle algebras associated with commutation function $\lambda(z,w)$, but our construction remains valid in the general case.

The shuffle algebra $\mathrm{sh}(\lambda) = \mathrm{sh}(\lambda)_+$ is the universal enveloping algebra of the nilpotent subalgebra of the corresponding Ding–Iohara algebra $DI(\lambda)$. This algebra also has the Cartan part generated by the operators $\psi^+(z)$ and $\psi^-(z)$ and the subalgebra $\mathrm{sh}(\lambda)_-$.

Earlier, we have defined the Ding–Iohara algebra $DI(\lambda)$. By the same argument as in Section 4.3, for each Heisenberg bimodule R one can construct the large quantum group $\cdots \oplus R^* \oplus DI(\lambda) \oplus R \oplus \cdots$, and hence R has the structure of a $DI(\lambda)$-bimodule.

The elements of the subalgebra $\mathrm{sh}(\lambda)_+ \subset DI(\lambda)$ act on R by left and right shuffle multiplication.

5.1. **Heisenberg bimodules for shuffle algebras.** The algebra $\mathrm{sh}(\lambda)$ is formed by functions on the symmetric powers $S = \mathbb{C}^\times$. Let us embed S in the set $S' = S \sqcup \{P\}$, where P is a point. Let $\lambda' \colon S' \times S' \to \mathbb{C}$ be the function such that

$$\lambda'(z,w) = \lambda(z,w), \quad \lambda'(z,P) = 1,$$
$$\lambda'(P,P) = 1, \quad \lambda'(P,z) = \frac{\prod_{i=1}^{l}(1 - K_i u/z)}{(1-u/z)^l},$$

where $z, w \in S$. The function $\lambda(x,y)$ is the continuous version of the commutation matrix (A_{ij}), and the addition of the point P is similar to the embedding of the matrix (A_{ij}) in the matrix (A'_{ij}) in Section 4.3.

Clearly, the nth symmetric power of S' is the union over m of mth symmetric powers of S, $0 \le m \le n$. Hence the algebra $\mathrm{sh}(\lambda')$ is graded, $\mathrm{sh}(\lambda') = \bigoplus \mathrm{sh}(\lambda')_r$, the zero-degree component has the form $\mathrm{sh}(\lambda')_0 = \mathrm{sh}(\lambda)$, and the component $\mathrm{sh}(\lambda')_1 = R$ is the Heisenberg bimodule over $\mathrm{sh}(\lambda)$, that is, a representation of the Heisenberg double $H(\mathrm{sh}(\lambda) \otimes \mathrm{sh}(\lambda)^o)$. In turn, the space R is graded, and its nth-degree component consists of symmetric rational functions of n variables satisfying the wheel condition.

The space R bears the structure of a module over the Ding–Iohara algebra $DI(\lambda)$; the construction of this action reproduces that at the end of Section 4.3 almost word for word. The subalgebra $\mathrm{sh}(\lambda)_+ \subset DI(\lambda)$ acts by left multiplication by elements of $\mathrm{sh}(\lambda)$. The elements $h_r \in DI(\lambda)$ act on the nth component of R by multiplication by $\sum z_i^r$. The quotient $R/(R \cdot \mathrm{Ish}(\lambda))$, where $\mathrm{Ish}(\lambda)$ stands for the kernel of the augmentation, is a representation of $DI(\lambda)$ as well.

Example 5.1. Let

$$\lambda(z,w) = \frac{(z-q_1w)(z-q_2w)(z-q_3w)}{(z-w)^3}, \quad \lambda'(P,z) = \frac{1-Ku/z}{1-u/z}, \tag{8}$$

where the numbers q_i satisfy $q_1q_2q_3 = 1$ and are otherwise in general position. Recall that $DI(\lambda)$ coincides in this case with the quantum toroidal algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$. Then the nth component of R consists of the functions

$$\frac{P(z_1,\dots,z_n)}{\prod(z_i-z_j)^2\prod(z_i-u)},$$

where P is a symmetric Laurent polynomial that satisfies the same wheel conditions as the elements $\mathrm{sh}(\lambda)$ and also additional conditions arising if K satisfies the resonance conditions $K = q_1^a q_2^b q_3^c$, $a,b,c \in \mathbb{Z}$.

If $K = q_1$, then the function P for $z_1 = u$ and $z_2 = q_1 u$. In this case, the quotient $R/(R \cdot \mathrm{Ish}(\lambda))$ is the Fock module, introduced in Definition 3.3, over the algebra $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$. To identify the two constructions, one also needs to use the "rotated" generators (see [36]).

5.2. **Bimodules over shuffle algebras and the construction of integrable systems.** This subsection re-exposes the construction in the preceding subsection using a different (but close) language. Let C be an algebraic curve, and let $\lambda(z,w)$ be a rational function on $C \times C$. As was already noted in Section 4, the space $\mathrm{SH}^r(\lambda) = \bigoplus_{n\ge} \mathcal{K}_n$, where $\mathcal{K}_n$ is the field of rational symmetric functions of $z_1,\dots,z_n \in C$, bears the structure of an algebra with shuffle multiplication (7).

The multiplicative group $\mathcal{K}^*$ of the field $\mathcal{K}_1$ acts on the algebra $\mathrm{SH}^r(\lambda)$ by automorphisms according to the formula

$$(g \cdot f) = g(z_1)\cdots g(z_n) f(z_1,\dots,z_n),$$

where $g \in \mathcal{K}^*$ and $f \in \mathcal{K}_n$. The algebra $\mathrm{SH}^r(\lambda)$ is a bimodule over itself; the bimodule structure can be deformed by twisting on the right by the action of an element $g \in \mathcal{K}^*$. Namely, the bimodule is defined $\mathrm{SH}^r(\lambda, g) \simeq \mathrm{SH}^r(\lambda)$ as a vector space, and the bimodule structure is defined by the formula

$$(v_1, v_2)f = v_1 \cdot f \cdot g(v_2), \quad \text{where } v_1, v_2 \in \mathrm{SH}^r(\lambda),\ f \in \mathrm{SH}^r(\lambda, g). \tag{9}$$

Let $U \subset C$ be a Zariski open set, and let $\mathcal{O}(U)$ be the algebra of regular functions on U. Let $\mathrm{sh}(U, \lambda)$ be the subalgebra of $\mathrm{SH}^r(\lambda)$ generated by $\mathcal{O}(U) \subset \mathcal{K}_1$, and let $\mathrm{sh}(U, \lambda, g)$ be the sub-bimodule of $\mathrm{SH}^r(\lambda, g)$ generated by the left and right action of $\mathrm{sh}(U, \lambda)$ on $1 \in \mathcal{K}_0$.

Example 5.2. Let $C = \mathbb{CP}^1$, let $U = \mathbb{C}^* \subset \mathbb{CP}^1$, and let

$$\lambda(z, w) = \frac{(z - q_1 w)(z - q_2 w)(z - q_3 w)}{(z - w)^3}, \quad g(z) = \frac{1 - Ku/z}{1 - u/z},$$

where K and u are nonzero complex numbers and q_1, q_2, and q_3 are the same as before. Then $\mathrm{sh}(U, \lambda)$ is the shuffle algebra $\mathrm{sh}(\lambda)$, and the bimodule $\mathrm{sh}(U, \lambda, g)$ coincides with the Heisenberg bimodule R in Example 5.1 viewed simply as a $\mathrm{sh}(\lambda)$-bimodule.

Consider the quotient $M(U, \lambda, g)$ of $\mathrm{sh}(U, \lambda, g)$ by the right action of the augmentation ideal in $\mathrm{sh}(U, \lambda)$. We have seen in the preceding subsection that the Ding–Iohara algebra acts on this quotient for the case of $U = \mathbb{C}^*$. A similar assertion holds for an arbitrary U.

The space $\mathrm{sh}(U, \lambda, g)$ is graded, $\mathrm{sh}(U, \lambda, g) = \bigoplus \mathrm{sh}(U, \lambda, g)_n$, where $\mathrm{sh}(U, \lambda, g)_n$ consists of symmetric rational functions on $U^{\times n}$. Let $\mathcal{O}_n(U)$ be the algebra of regular symmetric functions on $U^{\times n}$. The space $\mathrm{sh}(U, \lambda, g)_n$ is an $\mathcal{O}_n(U)$-module, and the image of the right action of the augmentation ideal is stable with respect to the action of the ideal $\mathcal{O}_n(U)$; hence $M(U, \lambda, g)_n$ is an $\mathcal{O}_n(U)$-module. We have constructed an integrable system on the space $M(U, \lambda, g)$; i.e., we have constructed an action of the large commutative algebra on each component of this module. We refer to this system as the $\mathcal{IM}$-system in honor of Ian Macdonald.

Let us return to the situation of Examples 5.1 and 5.2. Then $M(U, \lambda, g)$ is a representation of the Ding–Iohara algebra (which is isomorphic to $U_q(\ddot{\mathfrak{gl}}_1)$) with highest weight $g(z)$. The integrable system constructed here is a generalization of the system of Macdonald operators; these operators themselves are obtained if $M(U, \lambda, g)$ is the Fock representation.

The integrable system $\mathcal{IM}$ can be embedded in a family of an integrable system depending on additional parameters. The module itself on which the operators act depends on a rational function g, and the integrable system depends on a regular function h. Let us proceed to the construction of this family.

Let $h \in \mathcal{O}(U)$. Then we can equip the algebra $\mathrm{sh}(U, \lambda)$ with a different structure of a bimodule over itself. Namely, $\mathrm{sh}(h, U, \lambda) \simeq \mathrm{sh}(U, \lambda)$ as a vector space, and the bimodule structure is defined by the formula

$$(v_1, v_2)f = h(v_1) \cdot f \cdot v_2, \quad \text{where } v_1, v_2 \in \mathrm{sh}(U, \lambda) \text{ and } f \in \mathrm{sh}(h, U, \lambda). \tag{10}$$

It is easily seen that the module $M(U, \lambda, g)$ is isomorphic to the tensor product $\mathrm{sh}(U, g, \lambda)$ $\otimes\, \mathrm{sh}(0, U, \lambda)$, where the tensor product is taken as the product of modules over $\mathrm{sh}(U, \lambda)$ $\otimes\, \mathrm{sh}(U, \lambda)^o$. Consider the family of vector spaces

$$M(h, U, \lambda, g) = \mathrm{sh}(U, \lambda, g) \otimes \mathrm{sh}(h, U, \lambda)$$

indexed by functions $h \in \mathcal{O}(U)$. As a vector space, $M(h, U, \lambda, g)$ is the quotient of $\mathrm{sh}(U, g, \lambda)$ by the subspace generated by expressions of the form

$$\mathrm{Sym}\biggl(f(z_1, z_2, \dots, z_{n-1})v(z_n)\biggl(g(z_n)\prod_{i=1}^{n-1}\lambda(z_i, z_n) - h(z_n)\prod_{i=1}^{n-1}\lambda(z_n, z_i)\biggr)\biggr), \tag{11}$$

where $f \in \mathrm{sh}(U, g, \lambda)_{n-1}$ and $v \in \mathrm{sh}(U, \lambda)_1$.

Just as for $h = 0$, the algebra $\mathcal{O}_n(U)$ of symmetric functions acts on the nth component of the space $M(h, U, \lambda, g)$; this can be seen from formula (11). Thus, we obtain a family of integrable systems depending on g and h. In the general case, the family of spaces $M(h, U, \lambda, g)$ is not flat, but it is under certain conditions. In particular, this is true in the situation of Examples 5.1 and 5.2. More precisely, let the functions g and λ be the same as in these examples, but $U = \mathbb{C}^*\{p_1, \dots, p_n\}$ and the set of points p_i is assumed to be. Then the family $M(h, U, \lambda, g)$ is flat on the set $0 \sqcup \{h \mid h \text{ does not vanish on } U\}$.

We have defined the space $M(U, \lambda, g)$ as the quotient of $\mathrm{sh}(U, \lambda, g)$ by some subspace. Take a direct complement of this space for $h = 0$. If the family is flat, then this gives a trivialization in some neighborhood of the function $h = 0$. Thus, we have obtained a family of integrable systems on $M(u, \lambda, g)$ depending on h.

These integrable systems are independent of the choice of trivialization of the family; more precisely, they undergo a conjugation by some operator if the trivialization is changed.

In the situation of Examples 5.1 and 5.2, this integrable system coincides with the system of operators $I_r(p)$ constructed at the end of Section 3.4.

5.3. **Bethe equations.** The operators in an integrable system act by multiplication on symmetric functions. The space on which the integrable system acts is a module over the ring of symmetric functions. The eigenvalue problem is equivalent to the problem of finding the support of this module.

We see from equation (11) that the support is the set of n-tuples $(z_1, \dots, z_n)$ such that all terms in the inner parentheses vanish. To this end, the following n equations should be satisfied:

$$g(z_i)\prod_{j\neq i}\lambda(z_i, z_j) = h(z_i)\prod_{j\neq i}\lambda(z_j, z_i), \quad 1 \le i \le n.$$

Remark 5.1. The scheme for constructing integrable systems presented in this section is rather general, and in principle it applies to all shuffle algebras and even to their generalizations. For example, the shuffle algebra can be replaced by the Zamolodchikov algebra. Recall that this is the algebra generated by the system of generators $V(u)$, where V is a vector space, with the following quadratic relations: the products $V(u_1)V(u_2)$ and $V(u_2)V(u_1)$ are expressed via each other with the use of the R-matrix.

The Bethe equations arise in a natural way. The main problem is to relate the resulting integrable systems to those constructed in Sec. 3. This is an open problem. We only know that integrable systems generalizing the quantum Korteweg–de Vries equation can be constructed by two methods and that two methods for constructing the Bethe equations give one and the same result.

Acknowledgement

The author is deeply grateful to Misha Bernshtein for helping with this paper.

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National Research University Higher School of Economics, Moscow, Russia
E-mail address: borfeigin@gmail.com

Translated by V. E. NAZAIKINSKII
Originally published in Russian