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SYMMETRIC INVARIANTS RELATED TO REPRESENTATIONS OF EXCEPTIONAL SIMPLE GROUPS

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To our teacher Ernest B. Vinberg
on occasion of his 80th birthday

Abstract. We classify the finite-dimensional rational representations V of the exceptional algebraic groups G with $\mathfrak{g} = \mathsf{Lie}\, G$ such that the symmetric invariants of the semi-direct product $\mathfrak{g} \ltimes V$, where V is an Abelian ideal, form a polynomial ring.

Introduction

The ground field $\Bbbk$ is algebraically closed and of characteristic 0. In 1976, Vinberg et al. classified the irreducible representations of simple algebraic groups with polynomial rings of invariants [KPV76]. Such representations are sometimes called *coregular*. The most important class of coregular representations of reductive groups is provided by the θ-groups introduced and studied in depth by Vinberg; see [Vin76]. Since then the classification of coregular representations of semi-simple groups has attracted much attention. The *reducible* coregular representations of *simple* groups have been classified independently by Schwarz [Sch78] and Adamovich–Golovina [AG79], while the *irreducible* coregular representations of *semi-simple* groups are classified by Littelmann [Lit89].

For several decades, only rings of invariants of representations of *reductive* groups were considered. However, invariants of non-reductive groups are also very important in Representation Theory. Let S be an algebraic group with $\mathfrak{s} = \mathsf{Lie}\, S$. The invariants of S in the symmetric algebra $\mathcal{S}(\mathfrak{s})$ of $\mathfrak{s}$ (= *symmetric invariants* of $\mathfrak{s}$ or S) help us to understand the coadjoint action $(S : \mathfrak{s}^*)$ and in particular, coadjoint orbits, as well as representation theory of S. Several classes of non-reductive groups S such that $\mathcal{S}(\mathfrak{s})^S$ is a polynomial ring have been found recently; see e. g. [Jos07, PPY07, Pan07b, Yak]. A quest for this type of groups continues. Hopefully, one can find interesting properties of S and its representations under the assumption that the ring $\mathcal{S}(\mathfrak{s})^S$ is polynomial.

A natural class of non-reductive groups, which is still tractable, is given by a semi-direct product construction; see §2 for details. In [Yak], the following problem has been proposed: to classify all representations V of simple algebraic groups G such that the ring of symmetric invariants of the semi-direct product $\mathfrak{q} = \mathfrak{g} \ltimes V$ is polynomial (in other words, the coadjoint representation of $\mathfrak{q}$ is coregular). It is easily seen that if $\mathfrak{q}$ has this property, then $\Bbbk[V^*]^G$ is a polynomial ring, too. Therefore, the suitable representations (G, V) have to be extracted from the lists of [Sch78, AG79]. Some natural representations of $G = \mathrm{SL}_n$ are studied in [Yak]. Those considerations imply that the SL_n-case is very difficult. For this reason, we take here the other end and classify such representations

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(G,V) for the *exceptional* algebraic groups G. In a forthcoming article, we provide such a classification for the representations of the orthogonal and symplectic groups. To a great extent, our classification results rely on the theory developed by the second author in [Yak17].

Notation. Let S act on an irreducible affine variety X. Then $\Bbbk[X]^S$ is the algebra of S-invariant regular functions on X and $\Bbbk(X)^S$ is the field of S-invariant rational functions. If $\Bbbk[X]^S$ is finitely generated, then $X/\!/S := \mathsf{Spec}\,\Bbbk[X]^S$. Whenever $\Bbbk[X]^S$ is a graded polynomial ring, the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants.* If V is an S-module and $v \in V$, then $\mathfrak{s}_v = \{\zeta \in \mathfrak{s} \mid \zeta \cdot v = 0\}$ is the *stabiliser* of v in $\mathfrak{s}$ and $S_v = \{s \in S \mid s \cdot v = v\}$ is the *isotropy group* of v in S.

In explicit examples of §3 and in Table 1, we identify the representations V of semi-simple groups with their highest weights, using the *multiplicative* notation and the Vinberg–Onishchik numbering of the fundamental weights [VO88]. For instance, if $\varpi_1, \dots, \varpi_n$ are the fundamental weights of a simple algebraic group G, then $V = \varpi_i^2 + 2\varpi_j$ stands for the direct sum of three simple G-modules, with highest weights $2\varpi_i$ (once) and ϖ_j (twice). If $H \subset G$ is semi-simple and we are describing the restriction of V to H (i.e., $V|_H$), then the fundamental weights of H are denoted by $\widetilde{\varpi}_i$. Write '$\mathbb{1}$' for the trivial one-dimensional representation.

1. Preliminaries on the coadjoint representations

Let S be an affine algebraic group with Lie algebra $\mathfrak{s}$. The symmetric algebra $\mathcal{S}(\mathfrak{s})$ over $\Bbbk$ is identified with the graded algebra of polynomial functions on $\mathfrak{s}^*$ and we also write $\Bbbk[\mathfrak{s}^*]$ for it. The *index of* $\mathfrak{s}$, $\mathsf{ind}\,\mathfrak{s}$, is the minimal codimension of S-orbits in $\mathfrak{s}^*$. Equivalently, $\mathsf{ind}\,\mathfrak{s} = \min_{\xi\in\mathfrak{q}^*} \dim \mathfrak{s}_\xi$. By Rosenlicht's theorem [VP89, 2.3], one also has $\mathsf{ind}\,\mathfrak{s} = \text{tr.}\deg \Bbbk(\mathfrak{s}^*)^S$. The "magic number" associated with $\mathfrak{s}$ is $b(\mathfrak{s}) = (\dim\mathfrak{s} + \mathsf{ind}\,\mathfrak{s})/2$. Since the coadjoint orbits are even-dimensional, the magic number is an integer. If $\mathfrak{s}$ is reductive, then $\mathsf{ind}\,\mathfrak{s} = \mathrm{rk}\,\mathfrak{s}$ and $b(\mathfrak{s})$ equals the dimension of a Borel subalgebra. The Poisson bracket $\{\ ,\ \}$ in $\Bbbk[\mathfrak{s}^*]$ is defined on the elements of degree 1 (i.e., on $\mathfrak{s}$) by $\{x,y\} := [x,y]$. The *centre* of the Poisson algebra $\mathcal{S}(\mathfrak{s})$ is $\mathcal{S}(\mathfrak{s})^{\mathfrak{s}} = \{F \in \mathcal{S}(\mathfrak{s}) \mid \{F,x\} = 0\ \forall x \in \mathfrak{s}\}$. If S^o is the identity component of S, then $\mathcal{S}(\mathfrak{s})^{\mathfrak{s}} = \mathcal{S}(\mathfrak{s})^{S^o}$.

The set of S-*regular* elements of $\mathfrak{s}^*$ is $\mathfrak{s}^*_{\mathsf{reg}} = \{\eta \in \mathfrak{s}^* \mid \dim S\cdot\eta \ge \dim S\cdot\eta' \text{ for all } \eta' \in \mathfrak{s}^*\}$. We say that $\mathfrak{s}$ has the *codim*-2 property if $\mathrm{codim}(\mathfrak{s}^* \setminus \mathfrak{s}^*_{\mathsf{reg}}) \ge 2$. The following useful result appears in [Pan07b, Theorem 1.2]:

(♦) *Suppose that S is connected, $\mathfrak{s}$ has the* codim-2 *property, and there are homogeneous algebraically independent $f_1, \dots, f_l \in \Bbbk[\mathfrak{s}^*]^S$ such that $l = \mathsf{ind}\,\mathfrak{s}$ and $\sum_{i=1}^{l} \deg f_i = b(\mathfrak{s})$. Then $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[f_1, \dots, f_l]$ and $(df_1)_\xi, \dots, (df_l)_\xi$ are linearly independent if and only if $\xi \in \mathfrak{s}^*_{\mathsf{reg}}$.*

More generally, one can define the set of S-regular elements for any S-action on an irreducible variety X; that is, $X_{\mathsf{reg}} = \{x \in X \mid \dim S\cdot x \ge \dim S\cdot x' \text{ for all } x' \in X\}$.

We say that the action $(S : X)$ has a generic stabiliser, if there exists a dense open subset $\Omega \subset X$ such that all stabilisers $\mathfrak{s}_x$, $x \in \Omega$, are S-conjugate. Then *any* subalgebra $\mathfrak{s}_x$, $x \in \Omega$, is called a *generic stabiliser* (= g.s.). The points of Ω are said to be S-generic (or, just generic if the group is clear from the context). Likewise, one defines a *generic isotropy group* (= g.i.g.), which is a subgroup of S. By [Ric72, §4], $(S : X)$ has a generic stabiliser if and only if it has a generic isotropy group. It is also shown therein that g.i.g. always exists if S is reductive and X is smooth. If H is a generic isotropy group for $(S : X)$ and $\mathfrak{h} = \mathsf{Lie}\,H$, then we write $H = \text{g.i.g.}(S : X)$ and $\mathfrak{h} = \text{g.s.}(S : X)$.

A systematic treatment of generic stabilisers in the context of reductive group actions can be found in [VP89, §7].

Note that if a generic stabiliser for $(S : X)$ exists, then any S-generic point is S-regular, but not vice versa.

Recall that $f \in \Bbbk[X]$ is called a *semi-invariant* of S if $S \cdot f \subset \Bbbk f$. A semi-invariant f is *proper* if $f \notin \Bbbk[X]^S$.

2. On the coadjoint representations of semi-direct products

In this section, we gather some results on the coadjoint representation that are specific for semi-direct products. In particular, we recall the necessary theory from [Yak17].

Let $G \subset GL(V)$ be a connected algebraic group with $\mathfrak{g} = \mathsf{Lie}\, G$. The vector space $\mathfrak{g} \oplus V$ has a natural structure of Lie algebra, the *semi-direct product of* $\mathfrak{g}$ *and* V. Explicitly, if $x, x' \in \mathfrak{g}$ and $v, v' \in V$, then

$$[(x, v), (x', v')] = ([x, x'], x \cdot v' - x' \cdot v).$$

This Lie algebra is denoted by $\mathfrak{s} = \mathfrak{g} \ltimes V$, and $V \simeq \{(0, v) \mid v \in V\}$ is an abelian ideal of $\mathfrak{s}$. The corresponding connected algebraic group S is the semi-direct product of G and the commutative unipotent group $\exp(V) \simeq V$. The group S can be identified with $G \times V$, the product being given by

$$(s, v)(s', v') = (ss', (s')^{-1} \cdot v + v'), \quad \text{where} \quad s, s' \in G.$$

In particular, $(s, v)^{-1} = (s^{-1}, -s \cdot v)$. Then $\exp(V)$ can be identified with

$$1 \ltimes V := \{(1, v) \mid v \in V\} \subset G \ltimes V.$$

If G is reductive, then the subgroup $1 \ltimes V$ is the unipotent radical of S, also denoted by $R_u(S)$.

There is a general formula for the index of $\mathfrak{s} = \mathfrak{g} \ltimes V$, which is due to M. Raïs [Raï78]. Namely, there is a dense open subset $\Omega \subset V^*$ such that

$$\mathsf{ind}\, \mathfrak{s} = \text{tr. deg}\, \Bbbk(V^*)^G + \mathsf{ind}\, \mathfrak{g}_\xi$$

for any $\xi \in \Omega$. In particular, if a generic stabiliser for $(G : V^*)$ exists, then one can take $\mathfrak{g}_\xi$ to be a generic stabiliser.

Remark 2.1. There are some useful observations related to the symmetric invariants of the semi-direct product $\mathfrak{s} = \mathfrak{g} \ltimes V$:

(i) The decomposition $\mathfrak{s}^* = \mathfrak{g}^* \oplus V^*$ yields a bi-grading of $\Bbbk[\mathfrak{s}^*]^S$ [Pan07b, Theorem 2.3(i)]. If $\boldsymbol{H}$ is a bi-homogenous S-invariant, then $\deg_{\mathfrak{g}} \boldsymbol{H}$ and $\deg_V \boldsymbol{H}$ stand for the corresponding degrees.

(ii) The algebra $\Bbbk[V^*]^G$ is contained in $\Bbbk[\mathfrak{s}^*]^S$. Moreover, a minimal generating system for $\Bbbk[V^*]^G$ is a part of a minimal generating system of $\Bbbk[\mathfrak{s}^*]^S$ [Pan07b, Sect. 2 (A)]. Therefore, if $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring, then so is $\Bbbk[V^*]^G$.

Proposition 2.2 (Prop. 3.11 in [Yak17]). *Let G be a connected algebraic group acting on a finite-dimensional vector space V. Suppose that G has no proper semi-invariants in $\Bbbk[\mathfrak{s}^*]^{1 \ltimes V}$ and $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring in $\mathsf{ind}\, \mathfrak{s}$ variables. For generic $\xi \in V^*$, we then have*

- *the restriction map $\psi\colon \Bbbk[\mathfrak{s}^*]^S \to \Bbbk[\mathfrak{g}^* \times \{\xi\}]^{G_\xi \ltimes V} \simeq \mathcal{S}(\mathfrak{g}_\xi)^{G_\xi}$ is surjective;*
- *$\mathcal{S}(\mathfrak{g}_\xi)^{G_\xi}$ coincides with $\mathcal{S}(\mathfrak{g}_\xi)^{\mathfrak{g}_\xi}$;*
- *$\mathcal{S}(\mathfrak{g}_\xi)^{G_\xi}$ is a polynomial ring in $\mathsf{ind}\, \mathfrak{g}_\xi$ variables.*

Note that G is not assumed to be reductive and G_ξ is not assumed to be connected in the above proposition! We mention also that there are isomorphisms $\Bbbk[\mathfrak{g}^*\times\{\xi\}]^{G_\xi\ltimes V} \simeq \Bbbk[\mathfrak{g}_x]^{G_x} \simeq \mathcal{S}(\mathfrak{g}_\xi)^{G_\xi}$ for any $\xi\in V^*$; see [Yak17, Lemma 2.5].

From now on, G is supposed to be reductive. The action $(G:V)$ is said to be *stable* if the union of closed G-orbits is dense in V. Then a generic stabiliser $\mathsf{g.s.}(G:V)$ is necessarily reductive.

Consider the following assumptions on G and V:

($\diamondsuit$) The action of $(G:V^*)$ is stable, $\Bbbk[V^*]^G$ is a polynomial ring, $\Bbbk[\mathfrak{g}_\xi^*]^{G_\xi}$ is a polynomial ring for generic $\xi\in V^*$, and G has no proper semi-invariants in $\Bbbk[V^*]$.

The following result of the second author was excluded from the final text of [Yak17].

Theorem 2.3. *Suppose that G and V satisfy condition ($\diamondsuit$) and $V^*/\!\!/G = \mathbb{A}^1$, i.e., $\Bbbk[V^*]^G = \Bbbk[F]$ for some homogeneous F. Let H be a generic isotropy group for $(G:V^*)$ and $\mathfrak{h} = \mathsf{Lie}\, H$. Assume further that $D = \{x\in V^* \mid F(x)=0\}$ contains an open G-orbit, say $G\cdot y$, $\mathsf{ind}\,\mathfrak{g}_y = \mathsf{ind}\,\mathfrak{h} =: \ell$, and $\mathcal{S}(\mathfrak{g}_y)^{G_y}$ is a polynomial ring in ℓ variables with the same degrees of generators as $\mathcal{S}(\mathfrak{h})^H$. Then $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring in $\mathsf{ind}\,\mathfrak{s} = \ell+1$ variables.*

Proof. If $\ell=0$, then $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[F]$ and we have nothing to do. Assume that $\ell\ge 1$. Let $\{\boldsymbol{H}_i \mid 1\le i\le \ell\}$ be bi-homogeneous S-invariants chosen as in [Yak17, Lemma 3.5(i)]. Assume that $\deg_{\mathfrak{g}} \boldsymbol{H}_i \le \deg_{\mathfrak{g}} \boldsymbol{H}_j$ if $i<j$. We will show that these polynomials can be modified in such a way that the new set satisfies the conditions of [Yak17, Lemma 3.5(ii)] and therefore freely generates $\Bbbk[\mathfrak{s}^*]^S$ over $\Bbbk[F]$.

Notice that F is an irreducible polynomial, because ($\diamondsuit$) includes also the absence of proper semi-invariants. Thereby $\Bbbk[D]^G = \Bbbk$ and a non-trivial relation over $\Bbbk[D]^G$ among $\widetilde{\boldsymbol{H}}_i = \boldsymbol{H}_i|_{\mathfrak{g}^*\times D}$ gives also a non-trivial relation among $\tilde{\mathbf{h}}_i = \boldsymbol{H}_i|_{\mathfrak{g}^*\times\{y\}}$. Recall that $\tilde{\mathbf{h}}_i\in\mathcal{S}(\mathfrak{g}_y)^{G_y}$ by [Yak17, Lemma 2.5]. Assume that $\tilde{\mathbf{h}}_1,\dots,\tilde{\mathbf{h}}_j$ are algebraically independent if $j=d$ and dependent for $j=d+1$. Then $\tilde{\mathbf{h}}_{d+1}$ is not among the generators of $\mathcal{S}(\mathfrak{g}_y)^{G_y}$ and it can be expressed as a polynomial $R(\tilde{\mathbf{h}}_1,\dots,\tilde{\mathbf{h}}_d)$. Then also

$$\widetilde{\boldsymbol{H}}_{d+1} - R(\widetilde{\boldsymbol{H}}_1,\dots,\widetilde{\boldsymbol{H}}_d) = 0$$

and we can replace $\boldsymbol{H}_{d+1}$ by the bi-homogeneous part of $\frac{1}{F}(\boldsymbol{H}_{d+1} - R(\boldsymbol{H}_1,\dots,\boldsymbol{H}_d))$ of bi-degree $(\deg_{\mathfrak{g}}\boldsymbol{H}_{d+1}, \deg_V \boldsymbol{H}_{d+1} - \deg F)$. Clearly, $\sum_i \deg\boldsymbol{H}_i$ is decreasing and therefore the process will end up at some stage and bring a new set $\{\boldsymbol{H}_i\}$ satisfying the conditions of [Yak17, Lemma 3.5(ii)]. $\square$

Remark 2.4. Although Theorem 2.3 asserts that $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring under certain conditions, it does not say anything about the (bi-)degrees of the generators $\boldsymbol{H}_i$. Finding these degrees is not an easy task. Let us say a few words about it.

Suppose that G is semi-simple, $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring, and $\mathfrak{s}$ has the *codim*-2 property. As is mentioned above,

$$\mathsf{ind}\,\mathfrak{s} = \text{tr. deg}\,\Bbbk(V^*)^G + \mathsf{ind}\,\mathfrak{g}_\xi$$

with $\xi\in V^*$ generic. Here

$$\text{tr. deg}\,\Bbbk(V^*)^G = \dim V^*/\!\!/G$$

and

$$\Bbbk[\mathfrak{s}^*]^S = \Bbbk[\boldsymbol{H}_1,\dots,\boldsymbol{H}_\ell, F_1,\dots,F_r],$$

where $\ell = \mathsf{ind}\,\mathfrak{g}_\xi$, $r = \dim V^*/\!\!/G$, and all generators are bi-homogeneous. The generators F_j are elements of $\mathcal{S}(V)$. The $\mathfrak{g}$-degrees of the polynomials $\boldsymbol{H}_i$ are the degrees of basic invariants in $\mathcal{S}(\mathfrak{g}_\xi)^{G_\xi}$. Furthermore,

$$\sum_{i=1}^{\ell} \deg_V \boldsymbol{H}_i + \sum_{j=1}^{r} \deg F_j = \dim V;$$

see [Yak, Section 2] for a detailed explanation. Thus, the only open problem is how to determine the V-degrees of the $\boldsymbol{H}_i$. In particular, the problem simplifies considerably, if ℓ is small.

3. The classification and table

In this section, G is an *exceptional* algebraic group, i.e., G is a simple algebraic group of one of the types $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2$. We classify the (finite-dimensional rational) representations $(G : V)$ such that the symmetric invariants of $\mathfrak{s} = \mathfrak{g} \ltimes V$ form a polynomial ring. This will be referred to as property (FA) for $\mathfrak{s}$. We also say that $\mathfrak{s}$ (or just the action $(G : V)$) is *good* (resp. *bad*), if (FA) is (resp. is not) satisfied for $\mathfrak{s}$.

To distinguish exceptional groups and Lie algebras, we write, say, $\mathbf{E}_7$ for the group and $\mathcal{E}_7$ for the respective algebra; while the corresponding Dynkin type is referred to as E_7.

Example 3.1. If G is arbitrary semi-simple, then $\mathfrak{g} \ltimes \mathfrak{g}$ always has (FA) [Tak71]. Therefore we exclude the adjoint representations from our further consideration.

If $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring, then so is $\Bbbk[V^*]^G$ (Remark 2.1(ii)). For this reason, we have to examine all representations of G with polynomial rings of invariants. If $(G : V)$ is a representation of an exceptional algebraic group such that $V \neq \mathfrak{g}$ and $\Bbbk[V]^G$ is a polynomial ring, then V or V^* is contained in Table 1. This can be extracted from the classifications in [AG79] or [Sch78]. Furthermore, the algebras $\Bbbk[V]^G$ and $\Bbbk[V^*]^G$ (as well as $\mathcal{S}(\mathfrak{g} \ltimes V)^{G \ltimes V}$ and $\mathcal{S}(\mathfrak{g} \ltimes V^*)^{G \ltimes V^*}$) are isomorphic, so that it is enough to keep track of only V or V^*. As in [AG79, Ela72, Yak17], we use the Vinberg–Onishchik numbering of fundamental weights ϖ_i. Here H is a generic isotropy group for $(G : V^*)$ and column (FA) refers to the presence of property that $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring, where $\mathfrak{s} = \mathfrak{g} \ltimes V$. We write $q(V/\!/G)$ for the sum of degrees of the basic invariants in $\Bbbk[V^*]^G$.

Table 1. Representations of the exceptional groups with polynomial ring $\Bbbk[V^*]^G$.

№	G	V	$\dim V$	$\dim V^*/\!/G$	$q(V/\!/G)$	H	ind $\mathfrak{s}$	(FA)	Ref.
1a	$\mathbf{G}_2$	ϖ_1	7	1	2	$\mathbf{A}_2$	3	+	Eq. (3.1)
1b		$2\varpi_1$	14	3	6	$\mathbf{A}_1$	4	+	[Yak17, Ex. 4.8]
1c		$3\varpi_1$	21	7	15	$\{1\}$	7	+	Example 3.3
2a	$\mathbf{F}_4$	ϖ_1	26	2	5	$\mathbf{D}_4$	6	–	Example 3.4
2b		$2\varpi_1$	52	8	22	$\mathbf{A}_2$	10	+	Eq. (3.1)
3a	$\mathbf{E}_6$	ϖ_1	27	1	3	$\mathbf{F}_4$	5	+	Example 3.2
3b		$\varpi_1 + \varpi_5$	54	4	12	$\mathbf{D}_4$	8	–	Example 3.7
3c		$2\varpi_1$	54	4	12	$\mathbf{D}_4$	8	–	Example 3.7
3d		$3\varpi_1$	81	11	36	$\mathbf{A}_2$	13	+	Theorem 3.11
3e		$2\varpi_1 + \varpi_5$	81	11	36	$\mathbf{A}_2$	13	+	Theorem 3.11
4a	$\mathbf{E}_7$	ϖ_1	56	1	4	$\mathbf{E}_6$	7	–	Theorem 3.12
4b		$2\varpi_1$	112	7	28	$\mathbf{D}_4$	11	–	Example 3.8

In Table 1, the group H is always reductive. Since G is semi-simple, this implies that the action $(G : V^*)$ is stable in all cases; see [VP89, Theorem 7.15]. The fact that G is semi-simple means also that G, as well as $G \ltimes V$, has only trivial characters and therefore has no proper semi-invariants.

We provide below necessary explanations.

Example 3.2. Consider item 3a in the table. Here $V^*/\!\!/G = \mathbb{A}^1$, i.e., $\Bbbk[V^*]^G = \Bbbk[F]$ for some homogeneous F. The divisor $D = \{\xi \in V^* \mid F(\xi) = 0\}$ contains a dense G-orbit, say $G\cdot\eta$, whose stabiliser is the semi-direct product $\mathfrak{g}_\eta = \mathfrak{so}_9 \ltimes \varpi_4$. A generic isotropy group for $(G : V^*)$ is the exceptional group $\mathbf{F}_4$ and $\mathfrak{g}_\eta$ is a $\mathbb{Z}_2$-contraction of $\mathcal{F}_4$. By [Pan07b, Theorem 4.7], $\mathcal{S}(\mathfrak{g}_\eta)^{G_\eta}$ is a polynomial ring whose degrees of basic invariants are the same as those for $\mathcal{F}_4$. Therefore, using Theorem 2.3, we obtain that $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring.

Example 3.3. If $\mathfrak{h} = 0$, then $\Bbbk[\mathfrak{s}^*]^S \simeq \Bbbk[V^*]^G$ [Pan07a, Theorem 6.4]; cf. [Yak17, Example 3.1]. Therefore item 1c is a good case.

Example 3.4. The semi-direct product in 2a is a $\mathbb{Z}_2$-contraction of $\mathcal{E}_6$. This is one of the four bad $\mathbb{Z}_2$-contractions of simple Lie algebras, i.e., $\Bbbk[\mathfrak{s}^*]^S$ is not a polynomial ring here; see [Yak17, Section 6.1].

If G is semi-simple, V is a reducible G-module, say $V = V_1 \oplus V_2$, then there is a trick that allows us to relate the polynomiality problem for the symmetric invariants of $\mathfrak{s} = \mathfrak{g} \ltimes V$ to a smaller semi-direct product. The precise statement is as follows.

Proposition 3.5. *With $\mathfrak{s} = \mathfrak{g} \ltimes (V_1 \oplus V_2)$ as above, let H be a generic isotropy group for $(G : V_1^*)$ and $\mathfrak{h} = \mathsf{Lie}\,H$. If $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring, then so is $\Bbbk[\tilde{\mathfrak{q}}^*]^{\widetilde{Q}}$, where $\tilde{\mathfrak{q}} = \mathfrak{h} \ltimes (V_2|_H)$.*

Proof. Consider the (non-reductive) semi-direct product $\mathfrak{q} = \mathfrak{g} \ltimes V_2$. Then $\mathfrak{s} = \mathfrak{q} \ltimes V_1$. It is assumed that the unipotent radical of Q, $1 \ltimes V_2$, acts trivially on V_1. If $\xi \in V_1^*$ is generic for $(G : V_1^*)$, then it is also generic for $(Q : V_1^*)$. If $G_\xi = H$, then the corresponding isotropy group in Q is $Q_\xi = H \ltimes V_2$ and $\mathfrak{q}_\xi = \mathfrak{h} \ltimes V_2 \simeq \tilde{\mathfrak{q}}$, where V_2 is considered as H-module. Since G is semi-simple, all hypotheses of Proposition 2.2 are satisfied for Q in place of G and $V = V_1$. Therefore $\mathcal{S}(\mathfrak{q}_\xi)^{Q_\xi} = \mathcal{S}(\mathfrak{q}_\xi)^{\mathfrak{q}_\xi}$ is a polynomial ring. □

Remark 3.6. It is easily seen that the passage from $(G : V = V_1 \oplus V_2)$ to $(H : V_2)$, where $H = \mathsf{g.i.g.}(G : V_1)$, preserves generic isotropy groups. For generic stabilizers, this appears already in [Ela72, §3].

One can use Proposition 3.5 as a tool for proving that $\Bbbk[\mathfrak{s}^*]^S$ is not a polynomial ring.

Example 3.7. In case 3b, we take $Q = \mathbf{E}_6 \ltimes \varpi_1$ and $V_1 = \varpi_5$. Then

$$\mathfrak{s} = (\mathcal{E}_6 \ltimes \varpi_1) \ltimes \varpi_5 \simeq \mathcal{E}_6 \ltimes (\varpi_1 + \varpi_5).$$

If $\xi \in V_1^* = \varpi_1$ is generic, then $\mathfrak{g}_\xi \simeq \mathcal{F}_4$ and $\varpi_1|_{\mathbf{F}_4} = \widetilde{\varpi}_1 + \mathbb{1}$ [Ela72]. Therefore, $\mathfrak{q}_\xi$ is isomorphic to the semi-direct product related to item 2a, modulo a one-dimensional centre. Therefore, $\mathcal{S}(\mathfrak{q}_\xi)^{Q_\xi}$ is not a polynomial ring (see Example 3.4), and we conclude, using Proposition 3.5, that $\Bbbk[\mathfrak{s}^*]^S$ is not a polynomial ring, too.

In case 3c, we take the same Q and $V_1 = \varpi_1$. The rest is more or less the same, since $\mathbf{F}_4 \ltimes (\varpi_1|_{\mathbf{F}_4})$ is again a generic isotropy subgroup for $(Q : V_1^*)$.

Example 3.8. In case 4b, we take $Q = \mathbf{E}_7 \ltimes \varpi_1$ and $V_1 = \varpi_1$. Note that ϖ_1 is a symplectic $\mathbf{E}_7$-module. If $\xi \in V_1^*$ is generic, then the corresponding stabiliser in $\mathcal{E}_7$ is isomorphic to $\mathcal{E}_6$ [Har71, Ela72]. Thereby $\mathfrak{q}_\xi = \mathcal{E}_6 \ltimes (\varpi_1|_{\mathcal{E}_6}) = \mathcal{E}_6 \ltimes (\widetilde{\varpi}_1 + \widetilde{\varpi}_5 + 2\mathbb{1})$ [Ela72]. Hence $\mathfrak{q}_\xi$ represents item 3b (modulo a two-dimensional centre) and we have already demonstrated in the previous example that here $\Bbbk[\mathfrak{q}_\xi^*]^{Q_\xi}$ is not a polynomial ring!

The output of Examples 3.7 and 3.8 is that there is the tree of reductions to a "root" bad semi-direct product:

$$\begin{tikzcd}
 & (\mathbf{E}_6, 2\varpi_1) \arrow[dr] & \\
(\mathbf{E}_7, 2\varpi_1) \arrow[r] & (\mathbf{E}_6, \varpi_1+\varpi_5) \arrow[r] & \boxed{(\mathbf{F}_4, \varpi_1)}
\end{tikzcd},$$

and therefore all these items represent bad semi-direct products. Using Proposition 3.5 in a similar fashion, one obtains another tree of reductions:

$$\begin{tikzcd}
(\mathbf{E}_6, 2\varpi_1+\varpi_5) \arrow[dr] & & & \\
(\mathbf{E}_6, 3\varpi_1) \arrow[r] & (\mathbf{F}_4, 2\varpi_1) \arrow[r] & (\mathbf{D}_4, \varpi_1+\varpi_3+\varpi_4) \arrow[r] & (\mathbf{G}_2, \varpi_1).
\end{tikzcd} \tag{3.1}$$

Some details for the passage from $\mathcal{E}_6$ to $\mathfrak{so}_8$ and then to $\mathcal{G}_2$ are given below in the proof of Theorem 3.11. Note that the representations occurring in (3.1) have one and the same generic isotropy group, namely SL_3. As we will shortly see, tree (3.1) actually consists of good cases. Here our strategy is to prove that both "crown" $\mathbf{E}_6$-cases are good. To this end, we need some properties of the representation $(\mathbf{G}_2, \varpi_1)$ related to the "root" case.

Lemma 3.9. *Let v_1 be a highest weight vector in the $\mathbf{G}_2$-module ϖ_1 and $Q := (\mathbf{G}_2)_{v_1}$ the respective isotropy group. Then* (i) *$\mathfrak{q} = \mathsf{Lie}\, Q$ has the codim-2 property and* (ii) *the coadjoint representation of Q has a polynomial ring of invariants whose degrees of basic invariants are* $2, 3$.

Proof. A generic isotropy group for $(\mathbf{G}_2 : \varpi_1)$ is connected and isomorphic to SL_3 [Har71] and $\mathfrak{q}$ is a contraction of $\mathfrak{sl}_3$ (see [VGO90, Ch. 7, §2] for Lie algebra contractions). We also have $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}$, where $\mathfrak{l} = \mathfrak{sl}_2$ and the nilpotent radical $\mathfrak{n}$ is a 5-dimensional $\mathbb{Z}$-graded non-abelian Lie algebra of the form

$$\mathfrak{n}(1) \oplus \mathfrak{n}(2) \oplus \mathfrak{n}(3) = \Bbbk^2_I \oplus \Bbbk \oplus \Bbbk^2_{II}.$$

Here $\Bbbk^2_I$ and $\Bbbk^2_{II}$ are standard $\mathfrak{sl}_2$-modules and $\Bbbk$ is the trivial $\mathfrak{sl}_2$-module. Let $\{a_1, b_1\}$ be a basis for $\Bbbk^2_I$, $\{a_2, b_2\}$ a basis for $\Bbbk^2_{II}$, and $\{u\}$ a basis for $\Bbbk$. Without loss of generality, we may assume that $[a_1, b_1] = u$, $[a_1, u] = a_2$, and $[b_1, u] = b_2$.

(i) Since $\mathfrak{q}$ is a contraction of $\mathfrak{sl}_3$ and $\mathsf{ind}\, \mathfrak{sl}_3 = 2$, we have $\mathsf{ind}\, \mathfrak{q} \geq 2$. On the other hand, if $0 \neq \xi \in \mathfrak{n}(3)^* \subset \mathfrak{n}^* \subset \mathfrak{q}^*$, then $\dim \mathfrak{q}_\xi = 2$. Hence $\mathsf{ind}\, \mathfrak{q} = 2$ and $\mathfrak{n}(3)^* \setminus \{0\} \subset \mathfrak{q}^*_{\mathsf{reg}}$. Since $\dim \mathfrak{n}(3) = 2$, the last property readily implies that $\mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathsf{reg}}$ cannot contain divisors.

(ii) It is easily seen that

$$\mathbf{h}_1 = 2a_1 b_2 - 2b_1 a_2 + u^2 \in \mathcal{S}(\mathfrak{n}) \subset \mathcal{S}(\mathfrak{q})$$

is a $\mathfrak{q}$-invariant. There is also another invariant of degree three. Let $\{e, h, f\}$ be a standard basis of $\mathfrak{sl}_2$ (i.e., $[e, f] = h, [h, e] = 2e, [h, f] = -2f$). We assume that $[e, a_1] = 0$ and $[f, a_1] = b_1$, which implies $[h, a_1] = a_1$ and $[h, b_1] = -b_1$. Then

$$\mathbf{h}_2 = b_2^2 e + a_2 b_2 h - a_2^2 f + u(a_1 b_2 - a_2 b_1) + \frac{1}{3} u^3$$

is an $\mathfrak{sl}_2$-invariant and in addition, the following Poisson bracket can be computed:

$$\{a_1, \mathbf{h}_2\} = a_2 b_2 (-a_1) - a_2^2 (-b_1) + a_2 (a_1 b_2 - a_2 b_1) - u a_2 u + u^2 a_2 = 0.$$

Since $\mathfrak{l} = \mathfrak{sl}_2$ and a_1 generate $\mathfrak{q}$ as Lie algebra, $\mathbf{h}_2$ is also a $\mathfrak{q}$-invariant. The polynomials $\mathbf{h}_1$ and $\mathbf{h}_2$ are algebraically independent, because $\mathbf{h}_1 \in \mathcal{S}(\mathfrak{n})$ and $\mathbf{h}_2 \notin \mathcal{S}(\mathfrak{n})$. By (i), $\mathfrak{q}$ has the *codim*-2 property. Since $\dim\mathfrak{q} = 8$, $\mathsf{ind}\,\mathfrak{q} = 2$, and $b(\mathfrak{q}) = 5$, we have $\deg \mathbf{h}_1 + \deg \mathbf{h}_2 = b(\mathfrak{q})$. Therefore, $\mathbf{h}_1$ and $\mathbf{h}_2$ freely generate $\mathcal{S}(\mathfrak{q})^Q$; see ($\blacklozenge$) in §1. $\square$

Remark 3.10. In this case, $(\mathbf{G}_2)_{v_1}$ is a so-called *truncated parabolic subgroup.* Symmetric invariants of truncated (bi-)parabolics were intensively studied by Fauquant-Millet and Joseph; see e.g., [FMJ08, Jos07]. Let $\mathfrak{r}_{\mathsf{tr}}$ be a truncated (bi-)parabolic in type A or C. Then $\mathcal{S}(\mathfrak{r}_{\mathsf{tr}})^{\mathfrak{r}_{\mathsf{tr}}}$ is a polynomial ring in $\mathsf{ind}\,\mathfrak{r}_{\mathsf{tr}}$ homogeneous generators and the sum of their degrees is equal to $b(\mathfrak{r}_{\mathsf{tr}})$. The same properties hold for many truncated (bi-)parabolics in other types [Jos07]. It is very probable that a sufficient condition of [Jos07] is satisfied for $(\mathcal{G}_2)_{v_1}$. However, we prefer to keep the explicit construction of generators.

Theorem 3.11. *If $\mathfrak{s} = \mathcal{E}_6 \ltimes (3\varpi_1)$ or $\mathfrak{s} = \mathcal{E}_6 \ltimes (2\varpi_1 + \varpi_5)$, then $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring.*

Proof. Here $G = \mathbf{E}_6$, $\mathfrak{g} = \mathcal{E}_6$, and $V = 3\varpi_1$ or $2\varpi_1 + \varpi_5$. Accordingly, $V^* = 3\varpi_5$ or $2\varpi_5 + \varpi_1$. In both cases, $\mathsf{ind}\,\mathfrak{s} = 13$, $V^*/\!/G \simeq \mathbb{A}^{11}$, and a generic isotropy group for $(G : V^*)$ is SL_3. By [Yak17, Theorem 2.8 and Lemma 3.5(i)], there are bi-homogeneous irreducible polynomials $F_1, F_2 \in \Bbbk[\mathfrak{s}^*]^S$ such that $\deg_{\mathfrak{g}} F_1 = 2$, $\deg_{\mathfrak{g}} F_2 = 3$, and their restrictions to $\mathfrak{g}^* \times \{\xi\}$, where $\xi \in V^*$ is G-generic, are the basic invariants of SL_3. Here, for $\mathfrak{g}^* \times \{\xi\} \subset \mathfrak{s}^*$, we use the isomorphism

$$\Bbbk[\mathfrak{g}^* \times \{\xi\}]^{G_\xi \times V} \simeq \mathcal{S}(\mathfrak{g}_\xi)^{G_\xi}; \tag{3.2}$$

see [Yak17, Lemma 2.5]. By [Yak17, Lemma 3.5(ii)], F_1 and F_2 generate $\Bbbk[\mathfrak{s}^*]^S$ over $\Bbbk[V^*]^G$ if and only if the restrictions $F_1|_{\mathfrak{g}^* \times D}$ and $F_2|_{\mathfrak{g}^* \times D}$ are algebraically independent for any G-stable homogeneous divisor $D \subset V^*$. Let us prove that this is really the case.

• If there is a non-trivial relation for the restrictions of F_1 and F_2 to $\mathfrak{g}^* \times D$, then this also yields a non-trivial relation for the subsequent restrictions of F_1 and F_2 to $\mathfrak{g}^* \times \{\eta\}$, where η is G-generic in D.

• If $G_\eta \simeq \mathrm{SL}_3$, i.e., η is also G-generic in V^*, then $F_1|_{\mathfrak{g}^* \times \{\eta\}}$ and $F_2|_{\mathfrak{g}^* \times \{\eta\}}$ are non-zero elements of $\mathcal{S}(\mathfrak{sl}_3)^{\mathrm{SL}_3}$ of degrees 2 and 3, respectively. The invariants of degrees 2 and 3 in $\mathcal{S}(\mathfrak{sl}_3)^{\mathrm{SL}_3}$ are uniquely determined, up to a scalar factor, and they are algebraically independent. Hence $F_1|_{\mathfrak{g}^* \times D}$ and $F_2|_{\mathfrak{g}^* \times D}$ are algebraically independent for such divisors.

• However, it can happen that a divisor D contains no "globally" G-generic points. To circumvent this difficulty, consider three projections from V^* to its simple constituents, and their (non-trivial!) restrictions to D:

$$\begin{tikzcd} & D \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & \\ \varpi_5 & \varpi_5 & \varpi_1 \text{ or } \varpi_5. \end{tikzcd}$$

For $\eta = y_1 + y_2 + y_3 \in D \subset V^*$, we have $p_i(\eta) = y_i$. Since D is a divisor, at least two of the p_i's are dominant. Without loss of generality, we may assume that p_1 and p_2 are dominant, whereas p_3 is not and then $\overline{p_3(D)}$ is a divisor in $p_3(V^*)$. (For, if all p_i's are dominant, then D contains a globally generic point.) Therefore, we can take y_1, y_2 to be generic in the $p_i(V^*)$. Then $y_1 + y_2$ is G-generic in $2\varpi_5$, $G_{y_1+y_2} = \mathrm{Spin}_8$, and

$$\varpi_1|_{\mathrm{Spin}_8} \simeq \varpi_5|_{\mathrm{Spin}_8} \simeq \widetilde{\varpi}_1 + \widetilde{\varpi}_3 + \widetilde{\varpi}_4 + 3\mathbb{1}.$$

Here $G_\eta = (\mathrm{Spin}_8)_{y_3}$. Having obtained a Spin_8-stable divisor $\overline{p_3(D)}$ in the Spin_8-module $\widetilde{\varpi}_1 + \widetilde{\varpi}_3 + \widetilde{\varpi}_4 + 3\mathbb{1}$, we consider $\widetilde{V} := \widetilde{\varpi}_1 + \widetilde{\varpi}_3 + \widetilde{\varpi}_4$ and $\widetilde{D} := \overline{p_3(D)} \cap \widetilde{V}$. If $\widetilde{D} = \widetilde{V}$, then $(\mathrm{Spin}_8)_{y_3} = \mathrm{SL}_3$ for some y_3, i.e., again $G_\eta = \mathrm{SL}_3$.

• If $\widetilde{D}$ is a divisor in $\widetilde{V}$, then we can play the same game with Spin_8 and $\widetilde{D}$. Let $y_3 = x_1 + x_3 + x_4 \in \widetilde{D}$, where $x_i \in \widetilde{\varpi}_i$. Again, at least two of the projections $\tilde{p}_i \colon \widetilde{V} \to \widetilde{\varpi}_i$

($i = 1, 3, 4$) are dominant. Without loss of generality, we may assume that $\tilde{p}_1$ and $\tilde{p}_3$ are dominant and then x_1 and x_3 are generic elements. Then $(\mathrm{Spin}_8)_{x_1+x_3} \simeq \mathbf{G}_2$, $(\mathbf{G}_2)_{x_4} = (\mathrm{Spin}_8)_{y_3}$, and $\widetilde{\varpi}_4|_{\mathbf{G}_2} \simeq \widehat{\varpi}_1 + \mathbb{1}$. The structure of $\mathbf{G}_2$-orbits in $\widehat{\varpi}_1$ shows that either x_4 is $\mathbf{G}_2$-generic and then $(\mathbf{G}_2)_{x_4} \simeq \mathrm{SL}_3$ or x_4 is a highest weight vector and $(\mathbf{G}_2)_{x_4} \simeq (\mathbf{G}_2)_{v_1}$; cf. Lemma 3.9.

Thus, for any G-stable divisor $D \subset V^*$, there is $\eta \in D$ such that $G_\eta = \mathrm{SL}_3$ or $(\mathbf{G}_2)_{v_1}$, and in both cases $\mathcal{S}(\mathfrak{g}_\eta)^{G_\eta}$ is generated by algebraically independent invariants of degrees 2 and 3 (see Lemma 3.9 for the latter). It follows that $F_1|_{\mathfrak{g}^*\times D}$ and $F_2|_{\mathfrak{g}^*\times D}$ are algebraically independent, and we are done. □

Combining Proposition 3.5, Theorem 3.11, and tree (3.1), we conclude that cases 3d, 3e, 2b, and 1a are good. (Note also that we have found one good case related to a representation of the classical Lie algebra $\mathcal{D}_4$.) Thus, it remains to handle only the semi-direct product $\mathcal{E}_7 \ltimes \varpi_1$ (case 4a).

Theorem 3.12. *If $\mathfrak{s} = \mathcal{E}_7 \ltimes \varpi_1$, then $\Bbbk[\mathfrak{s}^*]^S$ is not a polynomial ring.*

Proof. Here $G = \mathbf{E}_7$, $\mathfrak{g} = \mathcal{E}_7$, and $\Bbbk[V^*]^G = \Bbbk[F]$. By [Har71], a generic isotropy group for $(G : V^*)$ is connected and isomorphic to $\mathbf{E}_6$, and the null-cone $D = \{\xi \in V^* \mid F(\xi) = 0\}$ contains a dense G-orbit, say $G\cdot y$, whose isotropy group G_y is connected and isomorphic to $\mathbf{F}_4 \ltimes \varpi_1$.

Assume that $\Bbbk[\mathfrak{s}^*]^S$ is polynomial. By [Yak17, Theorem 2.8 and Lemma 3.5(i)], since $\mathsf{ind}\,\mathfrak{s} = 7$ and g.i.g. is $\mathbf{E}_6$, there are bi-homogeneous polynomials $H_1, \dots, H_6$ such that $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[F, \boldsymbol{H}_1, \dots, \boldsymbol{H}_6]$ and the $\boldsymbol{H}_i|_{\mathfrak{g}^*\times\{\xi\}}$'s yield the basic invariants of $\mathcal{E}_6$ for a generic ξ. Here we again use Proposition 2.2 and isomorphism (3.2). By [Yak17, Lemma 3.5(ii)], the restrictions $\boldsymbol{H}_i|_{\mathfrak{g}^*\times D}$, $i = 1, \dots, 6$ remain algebraically independent. On the other hand, let us consider further restrictions $\boldsymbol{H}_i|_{\mathfrak{g}^*\times\{y'\}}$, $i = 1, \dots, 6$, where y' belongs to the dense G-orbit in D. Recall that $\mathfrak{g}_{y'} \simeq \mathcal{F}_4 \ltimes \varpi_1$ and the latter is a bad $\mathbb{Z}_2$-contraction of $\mathcal{E}_6$; see Example 3.4. Moreover, the algebra of symmetric invariants of $\mathcal{F}_4 \ltimes \varpi_1$ does not have algebraically independent invariants whose degrees are the same as the degrees of basic invariants of $\mathcal{E}_6$ [Yak17, Section 6.1]. This implies that $\boldsymbol{H}_i|_{\mathfrak{g}^*\times\{y'\}}$, $i = 1, \dots, 6$ must be algebraically dependent for any $y' \in G\cdot y$.

Let $L(\boldsymbol{H}_1|_{\mathfrak{g}^*\times\{y'\}}, \dots, \boldsymbol{H}_6|_{\mathfrak{g}^*\times\{y'\}}) = 0$ be a polynomial relation for *some* y'. Since the $\boldsymbol{H}_i$'s are G-invariant, the relation with the same coefficients holds for *all* $y' \in G\cdot y$. Hence, this dependence can be lifted to $\mathfrak{g}^* \times G\cdot y$ and then carried over to $\mathfrak{g}^* \times D$. This contradiction shows that the ring $\Bbbk[\mathfrak{s}^*]^S$ cannot be polynomial. □

Summarising, we obtain the main result of the article below.

Theorem 3.13. *Let G be an exceptional algebraic group, V a (finite-dimensional rational) G-module, and $\mathfrak{s} = \mathfrak{g} \ltimes V$. Then $\Bbbk[\mathfrak{s}^*]^S$ is a polynomial ring if and only if one of the following two conditions is satisfied:* (i) *$V = \mathfrak{g}$;* (ii) *V or V^* represents one of the seven good cases in Table* 1.

Moreover, by Raïs' formula (see §2), *one has*

$$\mathrm{tr.\,deg}\,\Bbbk[\mathfrak{s}^*]^S = \mathsf{ind}\,\mathfrak{s} = \dim V^*/\!/G + \mathrm{rk}\,H.$$

In the good cases, we neither construct generators nor give their degrees. The reason is that the main results of [Yak17] as well as Theorem 2.3 are purely existence theorems. Yet, as explained in Remark 2.4, a great deal of information on the degrees is available. If ℓ is small, then using ad hoc methods one can determine all the degrees. Let us see how this can be done, for example, with $\ell = 2$. Take item 1a. Since $\mathfrak{h} = \mathfrak{sl}_3$, we have $\Bbbk[\mathfrak{s}^*]^S = \Bbbk[V^*]^G[\boldsymbol{H}_1, \boldsymbol{H}_2]$ with $\deg_{\mathfrak{g}} \boldsymbol{H}_1 = 2$, $\deg_{\mathfrak{g}} \boldsymbol{H}_2 = 3$. Set $a_i = \deg_V \boldsymbol{H}_i$. It can easily be seen that $\mathfrak{s}$ has the *codim*-2 property. The discussion in Remark 2.4 shows that

$a_1 + a_2 = 5$. Following the same strategy as in [Yak17, Prop. 3.10], one can produce an S-invariant of bi-degree $(2, 2)$. This implies that $a_1 = 2$ and $a_2 = 3$.

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