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QUANTIZING MISHCHENKO–FOMENKO SUBALGEBRAS FOR CENTRALIZERS VIA AFFINE W-ALGEBRAS

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Dedicated to Ernest Borisovich Vinberg for his 80th birthday

ABSTRACT. We use affine W-algebras to quantize Mishchenko–Fomenko subalgebras for centralizers of nilpotent elements in finite dimensional simple Lie algebras under certain assumptions that are satisfied for all cases in type A and all minimal nilpotent cases outside type E_8.

1. INTRODUCTION

Let G be a simple algebraic group over $\mathbb{C}$ and $\mathfrak{g} = \mathrm{Lie}(G)$. Let e be a nilpotent element of $\mathfrak{g}$ and $\{e, f, h\} \subset \mathfrak{g}$ an $\mathfrak{sl}_2$-triple containing e. Given $x \in \mathfrak{g}$ we write $\mathfrak{g}^x$ for the centralizer of x in $\mathfrak{g}$. By Elashvili's conjecture, which is now a theorem, the index of $\mathfrak{g}^e$ equals $\ell := \mathrm{rk}\,\mathfrak{g}$; see [CM10] and the references therein. This means that the set of those linear functions on $\mathfrak{g}^e$ whose stabilizer in $\mathfrak{g}^e$ has dimension equal to ℓ is non-empty and Zariski open in $(\mathfrak{g}^e)^*$. The complement of this open subset in $(\mathfrak{g}^e)^*$ will be denoted by $(\mathfrak{g}^e)^*_{\mathrm{sing}}$.

We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ as G-modules by using the Killing form of $\mathfrak{g}$. Under this identification, it was shown in [PPY07] that for $P \in S(\mathfrak{g})^{\mathfrak{g}}$ the minimal degree component ${}^eP$ of the restriction $P|_{e+\mathfrak{g}^f}$ belongs to $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. A homogeneous generating system $\{P_1, \dots, P_\ell\}$ of $S(\mathfrak{g})^{\mathfrak{g}}$ is called *good* for e if

$$\sum_{i=1}^{\ell} \deg {}^eP_i = \frac{\dim \mathfrak{g}^e + \ell}{2}.$$

Theorem 1.1 ([PPY07]). *Suppose e admits a good generating system $\{P_1, \dots, P_\ell\}$ in $S(\mathfrak{g})^{\mathfrak{g}}$ and assume further that $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ has codimension ≥ 2 in $(\mathfrak{g}^e)^*$. Then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra in variables ${}^eP_1, \dots, {}^eP_\ell$.*

In [PPY07] it was also proved that the assumptions of Theorem 1.1 hold for all nilpotent elements in simple Lie algebras of type A and C. We refer to [CM16] for further results on polynomiality of $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Given $F \in S(\mathfrak{g}^e)$ and $\chi \in (\mathfrak{g}^e)^*$ we put

$$D_\chi F(x) := \frac{d}{du} F(x + u\chi)\big|_{u=0} \quad (\forall\, x \in (\mathfrak{g}^e)^*),$$

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so that

$$F(x+u\chi)=\sum_{i=0}^{\deg F}\frac{u^i}{i!}D^i_\chi F(x).$$

To each $\chi\in(\mathfrak{g}^e)^*$ we then attach its *Mishchenko–Fomenko subalgebra* $\bar{\mathcal{A}}_{e,\chi}$ of $S(\mathfrak{g}^e)$. Recall that the $\mathbb{C}$-algebra $\bar{\mathcal{A}}_{e,\chi}$ is generated by the χ-shifts $D^i_\chi(F)$ of all $F\in S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. It is well known [MF78] that $\bar{\mathcal{A}}_{e,\chi}$ is a Poisson-commutative subalgebra of the Poisson "— Lie algebra $S(\mathfrak{g}^e)$.

Theorem 1.2 ([PY08])**.** *Under the assumptions of Theorem* 1.1 *suppose further that* $(\mathfrak{g}^e)^*_{\rm sing}$ *has codimension* ≥ 3 *in* $(\mathfrak{g}^e)^*$*. Then, for a regular element* χ *of* $(\mathfrak{g}^e)^*$*, the subalgebra* $\bar{\mathcal{A}}_{e,\chi}$ *is freely generated by all* $D^j_\chi({}^eP_i)$ *with* $i=1,\dots,\ell$ *and* $j=0,\dots,\deg{}^eP_i-1$*. Moreover,* $\bar{\mathcal{A}}_{e,\chi}$ *is a maximal Poisson-commutative subalgebra of the Poisson–Lie algebra* $S(\mathfrak{g}^e)$*.*

The canonical filtration of the universal enveloping algebra $U(\mathfrak{g}^e)$ induces that on any $\mathbb{C}$-subalgebra $\mathcal{A}$ of $U(\mathfrak{g}^e)$ and we write $\operatorname{gr}\mathcal{A}$ for the corresponding graded subalgebra of $S(\mathfrak{g}^e)=\operatorname{gr}U(\mathfrak{g}^e)$. More precisely, $\operatorname{gr}\mathcal{A}$ is generated by the symbols in $S(\mathfrak{g}^e)$ of all $a\in\mathcal{A}$. In this article we prove the following:

Theorem 1.3. *Under the assumptions of Theorem* 1.1 *for any regular* χ *there exists a commutative subalgebra* $\mathcal{A}_{e,\chi}$ *of* $U(\mathfrak{g}^e)$ *such that* $\operatorname{gr}\mathcal{A}_{e,\chi}=\bar{\mathcal{A}}_{e,\chi}$*. This subalgebra is freely generated by* $(\dim\mathfrak{g}^e+\ell)/2$ *elements. If* $(\mathfrak{g}^e)^*_{\rm sing}$ *has codimension* ≥ 3 *in* $(\mathfrak{g}^e)^*$*, then* $\mathcal{A}_{e,\chi}$ *is a maximal commutative subalgebra of* $U(\mathfrak{g}^e)$*.*

If $e=0$, then $\mathfrak{g}^e=\mathfrak{g}$ and Theorem 1.3 becomes a classical result of Lie Theory. In this case, it has been proved by Tarasov [Tar00] for $\mathfrak{g}=\mathfrak{sl}_n$, by Nazarov–Olshanski [NO96] for classical Lie algebras, and by Rybnikov [Ryb06], Chervov–Falqui–Rybnikov [CFR10] and Feigin–Frenkel–Toledano Laredo [FFTL10] for any finite dimensional simple Lie algebra $\mathfrak{g}$. Related results have also been obtained by Cherednik [Che87].

If $\mathfrak{a}$ is a finite dimensional Lie algebra over $\mathbb{C}$, then the problem of quantizing Mishchenko–Fomenko subalgebras of $S(\mathfrak{a})$, i.e. lifting them to commutative subalgebras of $U(\mathfrak{a})$, was first raised in [Vin90] and is sometimes referred to as *Vinberg's problem.* In this terminology, our paper shows that Vinberg's problem has a positive solution for a large class of centralizers in simple Lie algebras. It is worth remarking that Vinberg's problem is wide open for an arbitrary Lie algebra $\mathfrak{a}$.

In Section 4 we show that all conditions of Theorem 1.3 hold in the case where e is an element of the minimal non-zero nilpotent orbit $\mathcal{O}_{\rm min}\subset\mathfrak{g}$ and $\mathfrak{g}$ is not of type E_8. In Section 5 we describe $\mathcal{A}_{e,\chi}$ explicitly in the case where $e\in\mathcal{O}_{\rm min}$ and $\mathfrak{g}$ is of type A_ℓ with $\ell\in\{2,3\}$.

The main step of the proof of Theorem 1.3 consists in establishing a chiralization of Theorem 1.1. Namely we prove the following result: let $Z(V^{\kappa_{e,c}}(\mathfrak{g}^e))$ be the center of the universal affine vertex algebra $V^{\kappa_{e,c}}(\mathfrak{g}^e)$ associated with $\mathfrak{g}^e$ at the critical level $\kappa_{e,c}$; see Section 3 for more detail. There exists a natural filtration of the vertex algebra $V^{\kappa_{e,c}}(\mathfrak{g}^e)$ such that $\operatorname{gr}V^{\kappa_{e,c}}(\mathfrak{g}^e)\cong S(\widehat{\mathfrak{g}}^e_-)$, where $\widehat{\mathfrak{g}}^e_-=\mathfrak{g}^e[t^{-1}]t^{-1}:=\mathfrak{g}^e\otimes_{\mathbb{C}}t^{-1}\mathbb{C}[t^{-1}]$. Given $a\in V^{\kappa_{e,c}}(\mathfrak{g}^e)$ we denote by $\sigma(a)$ its symbol in $S(\widehat{\mathfrak{g}}^e_-)$. We identify $S(\mathfrak{g}^e)$ with a subring of $S(\widehat{\mathfrak{g}}^e_-)$ by using the algebra homomorphism which sends any $x\in\mathfrak{g}^e$ to $xt^{-1}\in\widehat{\mathfrak{g}}^e_-$.

Theorem 1.4. *Let* $\{P_1,\dots,P_\ell\}$ *be a good generating system of* $S(\mathfrak{g})^{\mathfrak{g}}$ *for a nilpotent element* $e\in\mathfrak{g}$ *and suppose that* $(\mathfrak{g}^e)^*_{\rm sing}$ *has codimension* ≥ 2 *in* $(\mathfrak{g}^e)^*$*. Then there exist homogeneous elements* ${}^e\widehat{P}_1,\dots,{}^e\widehat{P}_\ell$ *in* $Z(V^{\kappa_{e,c}}(\mathfrak{g}^e))$ *such that* $\sigma(\widehat{P}_i)={}^eP_i\in S(\mathfrak{g}^e)\subset S(\widehat{\mathfrak{g}}^e_-)$*, and* $Z(V^{\kappa_{e,c}}(\mathfrak{g}^e))$ *is a polynomial algebra in variables* $T^j({}^e\widehat{P}_i)$ *with* $i=1,\dots,\ell$ *and* $j\geq 0$*, where* T *is the translation operator of* $V^{\kappa_{e,c}}(\mathfrak{g}^e)$*.*

Having established Theorem 1.4 we deduce Theorem 1.3 by applying the method which Rybnikov [Ryb06] used in the case where $e = 0$. It should be mentioned that in this case Theorem 1.4 is a celebrated result of Feigin and Frenkel [FF92]. In order to prove Theorem 1.4 in our situation we use the affine W-algebra $\mathcal{W}^{\kappa_c}(\mathfrak{g}, f)$ associated with $(\mathfrak{g}, f)$ at the critical level κ_c.

2. Rybnikov's construction and vertex algebras

Let $\mathfrak{q}$ be a finite dimensional Lie algebra over $\mathbb{C}$ and let $\kappa(\,\cdot\,,\,\cdot\,)$ be a symmetric invariant bilinear form of $\mathfrak{q}$. The associated affine Kac–Moody algebra $\widehat{\mathfrak{q}}_\kappa$ is a central extension of the Lie algebra $\mathfrak{q}((t)) = \mathfrak{q} \otimes \mathbb{C}((t))$ by the one-dimensional center $\mathbb{C}\mathbf{1}$:

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{q}}_\kappa \to \mathfrak{q}((t)) \to 0.$$

The commutation relations in $\widehat{\mathfrak{q}}_\kappa$ are given by

$$[x \otimes f, y \otimes g] := [x, y] \otimes fg - \kappa(x, y)\operatorname{Res}_{t=0}(f dg)\mathbf{1}$$

for all $x, y \in \mathfrak{q}$ and $f, g \in \mathbb{C}((t))$. Set

$$\widehat{\mathfrak{q}}_+ = \mathfrak{q}[[t]] \oplus \mathbb{C}\mathbf{1}, \quad \widehat{\mathfrak{q}}_- = \mathfrak{q}[t^{-1}]t^{-1} \subset \widehat{\mathfrak{q}}_\kappa,$$

so that $\widehat{\mathfrak{q}}_\kappa = \widehat{\mathfrak{q}}_- \oplus \widehat{\mathfrak{q}}_+$.

We regard $S(\mathfrak{q})$ as a subalgebra of $S(\widehat{\mathfrak{q}}_-)$ via the embedding $x \mapsto xt^{-1}$, $x \in \mathfrak{q}$. We also identify $S(\widehat{\mathfrak{q}}_-)$ with the function algebra $\mathbb{C}[\mathfrak{q}^*_\infty]$ on the infinite jet scheme $\mathfrak{q}^*_\infty$ of the affine space $\mathfrak{q}^*$. Then our embedding $S(\mathfrak{q}) \hookrightarrow S(\widehat{\mathfrak{q}}_-)$ gives rise to an inclusion $\mathbb{C}[\mathfrak{q}^*] \hookrightarrow \mathbb{C}[\mathfrak{q}^*_\infty]$. The Poisson algebra structure of $S(\mathfrak{q}) = \mathbb{C}[\mathfrak{q}^*]$ gives rise [Ara12a] to a Poisson *vertex* algebra structure on $S(\widehat{\mathfrak{q}}_-) = \mathbb{C}[\mathfrak{q}^*_\infty]$. The translation operator T on the Poisson vertex algebra $S(\widehat{\mathfrak{q}}_-)$ is the derivation of the $\mathbb{C}$-algebra $S(\widehat{\mathfrak{g}}_-)$ defined by the rule

$$Tx_{(-m)} = mx_{(-m-1)}, \quad T1 = 0,$$

where $x_{(m)} = xt^m$, $x \in \mathfrak{q}$. Let $Z(S(\widehat{\mathfrak{q}}_-))$ be the center of the Poisson vertex algebra $S(\widehat{\mathfrak{q}}_-)$. Identifying $\widehat{\mathfrak{q}}_-$ with $\widehat{\mathfrak{q}}/\widehat{\mathfrak{q}}_+$ we have that

$$Z(S(\widehat{\mathfrak{q}}_-)) = S(\widehat{\mathfrak{q}}_-)^{\widehat{\mathfrak{q}}_+}.$$

For $\chi \in \mathfrak{q}^*$ and $u \in \mathbb{C}^*$, we define a ring homomorphism

$$\bar{\Phi}_\chi(?, u)\colon S(\widehat{\mathfrak{q}}_-) \to S(\mathfrak{q}),$$

by setting

$$\bar{\Phi}_\chi(x_{(-m)}, u) = u^{-m}x + \delta_{m,1}\chi(x) \quad \text{for all } x \in \mathfrak{q} \text{ and } m \in \mathbb{N}.$$

This definition implies that

$$\bar{\Phi}_\chi(P, u)(\lambda) = u^{-\deg P}P(\lambda + u\chi) \tag{1}$$

for any homogeneous element $P \in S(\mathfrak{q}) \subset S(\widehat{\mathfrak{q}}_-)$ and any $\lambda \in \mathfrak{q}^*$. Furthermore,

$$\bar{\Phi}_\chi(Ta, u) = -\frac{d}{du}\bar{\Phi}_\chi(a, u) \tag{2}$$

for all $a \in S(\widehat{\mathfrak{q}}_-)$. We denote by $\bar{\Phi}_{\chi\, n}(a) \in S(\mathfrak{q})$ the coefficient of u^{-n} in $\bar{\Phi}_\chi(a, u)$, so that

$$\bar{\Phi}_\chi(a, u) = \sum_{n \geq 0} \bar{\Phi}_{\chi\, n}(a)u^{-n}.$$

For a homogeneous element P of $S(\mathfrak{g})$ we have that

$$\bar{\Phi}_{\chi\, n}(P) = \begin{cases} \frac{u^{\deg P - n}}{(\deg P - n)!} D_\chi^{\deg P - n}(P) & \text{if } n \leq \deg P; \\ 0 & \text{else.} \end{cases} \tag{3}$$

Lemma 2.1 ([Ryb06, FFTL10]). *The following are true:*

(1) For any T-invariant subspace U of $S(\widehat{\mathfrak{q}}_-)$ the image $\Phi_\chi(U,u)$ is independent of the choice of $u \in \mathbb{C}^*$ and spanned by the elements $\Phi_{\chi,n}(a)$ with $a \in U$.

(2) Let A be a subalgebra of $S(\mathfrak{q})^{\mathfrak{q}}$ and let A_∞ be the (T-invariant) subalgebra of $S(\widehat{\mathfrak{q}}_-)$ generated by A. Then $\bar{\Phi}_\chi(A_\infty)$ is the Mishchenko–Fomenko subalgebra of $S(\mathfrak{q})$ generated by the χ-shifts of P for all $P \in A$.

Let $V^\kappa(\mathfrak{q}) = U(\widehat{\mathfrak{q}}_\kappa) \otimes_{U(\widehat{\mathfrak{q}}_+)} \mathbb{C}$, where $\mathbb{C}$ is considered as a $\widehat{\mathfrak{q}}_+$-module on which $\mathfrak{q}[[t]]$ acts trivially and $\mathbf{1}$ acts as the identity. Note that $V^\kappa(\mathfrak{q}) \cong U(\widehat{\mathfrak{q}}_-)$ as vector spaces. As is well known (see [Kac98, FBZ04]), $V^\kappa(\mathfrak{q})$ is naturally a vertex algebra. The state-field correspondence

$$V^\kappa(\mathfrak{q}) \to \operatorname{End} V^\kappa(\mathfrak{q})[[z,z^{-1}]], \quad a \mapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

is uniquely determined by the conditions

$$(\mathbb{I})(z) = \mathrm{id}_{V^\kappa(\mathfrak{q})}, \quad (xt^{-1}\mathbb{I})(z) = x(z) := \sum_{n\in\mathbb{Z}} x_{(n)} z^{-n-1} \text{ for } x \in \mathfrak{q},$$

where $\mathbb{I} = 1 \otimes 1 \in V^\kappa(\mathfrak{q})$ and $x_{(n)} = xt^n$. The translation operator T is an endomorphism of $V^\kappa(\mathfrak{q})$ such that $T\mathbb{I} = 0$ and $[T, x(z)] = \frac{d}{dz}x(z)$ for all $x \in \mathfrak{g}$.

Given a vertex algebra V we write $V = E^0V \supset E^1V \supset E^2V \supset \dots$ for the *Li filtration* of V; see [Li05] and [Ara12a]. Recall that $E^pV^\kappa(\mathfrak{q})$ is spanned by all

$$(x_1t^{-n_1})\cdots(x_rt^{-n_r})\mathbb{I} \quad \text{with} \quad x_i \in \mathfrak{q} \quad \text{and} \quad -r + \sum_{i=1}^r n_i \geq p.$$

The associated graded space $\operatorname{gr} V = \bigoplus_p E^pV/E^{p+1}V$ is naturally a Poisson vertex algebra. The Poisson vertex algebra structure on $\operatorname{gr} V$ induces a genuine Poisson algebra structure on Zhu's C_2-algebra $R_V := V/E^1V \subset \operatorname{gr} V$. Moreover, R_V generates $\operatorname{gr} V$ as a differential graded algebra; see [Li05].

Setting $\deg xt^{-n} = n$ for all $x \in \mathfrak{q}$ and $n \geq 0$ gives $U(\widehat{\mathfrak{q}}_-) = \bigoplus_{j\in\mathbb{Z}_{\geq 0}} U(\widehat{\mathfrak{q}}_-)[j]$ a graded algebra structure. This, in turn, induces a grading on $V^\kappa(\mathfrak{q})$:

$$V^\kappa(\mathfrak{q}) = \bigoplus_{j\in\mathbb{Z}_{\geq 0}} V^\kappa(\mathfrak{q})[j], \tag{4}$$

where $V^\kappa(\mathfrak{q})[j] = (U(\widehat{\mathfrak{q}}_-)[j])\mathbb{I}$. Let $\{U_i(\widehat{\mathfrak{q}}_-) \mid i \geq 0\}$ be the canonical filtration of $U(\widehat{\mathfrak{q}})$ and put $U_i(\widehat{\mathfrak{q}}_-)[j] = U_i(\widehat{\mathfrak{q}}_-) \cap U(\widehat{\mathfrak{q}}_-)[j]$. The above then shows that

$$(E^pV^\kappa(\mathfrak{q}))[j] = (U_{j-p}(\widehat{\mathfrak{q}}_-)[j])\mathbb{I},$$

where $(E^pV^\kappa(\mathfrak{q}))[j] = E^pV^\kappa(\mathfrak{q}) \cap V^\kappa(\mathfrak{q})[j]$. It follows that

$$\operatorname{gr} V^\kappa(\mathfrak{q}) \cong S(\widehat{\mathfrak{q}}_-) \quad \text{and} \quad R_{V^\kappa(\mathfrak{q})} \cong S(\mathfrak{q}). \tag{5}$$

More precisely, $R_{V^\kappa(\mathfrak{q})}$ coincides with the subalgebra of $S(\widehat{\mathfrak{q}}_-)$ generated by all xt^{-1} with $x \in \mathfrak{q}$.

Given a vertex algebra V we denote by $Z(V)$ the *center* of V, so that

$$\begin{aligned} Z(V) = \{z \in V \mid [z_{(m)}, a_{(n)}] = 0 \text{ for all } a \in V \text{ and } m,n \in \mathbb{Z}\} \\ = \{z \in V \mid a_{(n)}z = 0 \text{ for all } a \in V \text{ and } n \geq 0\}. \end{aligned}$$

Here the last equality follows from the commutator formula

$$[a_{(m)}, b_{(n)}] = \sum_{i\geq 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}.$$

Since $Z(V)$ is a *commutative* vertex subalgebra of V, it is a differential algebra, i.e. a unital commutative $\mathbb{C}$-algebra equipped with a derivation; see [Bor86]. Specifically, the multiplication in $Z(V)$ is given by $a \cdot b := a_{(-1)}b$ for all $a, b \in Z(A)$. The translation operator of V preserves $Z(A)$ and acts on this algebra as a derivation.

When $V = V^\kappa(\mathfrak{q})$ we have that

$$Z(V^\kappa(\mathfrak{q})) = V^\kappa(\mathfrak{q})^{\widehat{\mathfrak{q}}_+} \cong \operatorname{End}_{\widehat{\mathfrak{q}}_\kappa}(V^\kappa(\mathfrak{q}))^{\mathrm{opp}}$$

as $\mathbb{C}$-algebras. We will often regard $Z(V^\kappa(\mathfrak{q}))$ as a commutative subalgebra of $U(\widehat{\mathfrak{q}}_-)$ via the above-mentioned identification $V^\kappa(\mathfrak{q}) = U(\widehat{\mathfrak{q}}_-)$.

Given $\chi \in \mathfrak{q}^*$ and $u \in \mathbb{C}^*$ we define an algebra homomorphism

$$\Phi_\chi(?, u)\colon U(\widehat{\mathfrak{q}}_-) \to U(\mathfrak{q}), \quad a \mapsto \Phi_\chi(a, u),$$

by the rule

$$\Phi_\chi(x_{(-m)}, u) = u^{-m}x + \delta_{m,1}\chi(x) \quad \text{for all } x \in \mathfrak{q} \text{ and } m \in \mathbb{Z}.$$

It is immediate from the definition that

$$\Phi_\chi(Ta, u) = -\frac{d}{du}\Phi_\chi(a, u) \tag{6}$$

for all $a \in V^\kappa(\mathfrak{q}) = U(\widehat{\mathfrak{q}}_-)$. We denote by $\Phi_{\chi, n}(a) \in U(\mathfrak{q})$ the coefficient of u^{-n} in $\Phi_\chi(a, u)$, so that

$$\Phi_\chi(a, u) = \sum_{n \geq 0} \Phi_{\chi, n}(a)u^{-n}.$$

The following result can be found in [FFTL10, Sect. 2.5].

Lemma 2.2. *For any T-invariant subspace U of $U(\widehat{\mathfrak{q}}_-)$ the image $\Phi_\chi(U)$ is independent of the choice of $u \in \mathbb{C}^*$ and spanned by $\Phi_{\chi, n}(a)$ with $a \in U$.*

We let $\mathcal{A}_\chi$ denote the image of the subalgebra $Z(V^\kappa(\mathfrak{q}))$ of $U(\widehat{\mathfrak{q}}_-)$ under the homomorphism $\Phi(?, u)$. This is a commutative $\mathbb{C}$-subalgebra of $U(\mathfrak{q})$.

Lemma 2.3. *Let $a \in V^\kappa(\mathfrak{q})[d]$ be such that its image $\bar{a}$ in $R_{V^\kappa(\mathfrak{q})} \cong S(\mathfrak{q})$ has the property that $D_\chi^{d-i}(\bar{a}) \neq 0$ for some $i \leq d$. Then the symbol of $\Phi_{\chi, i}(a)$ in $S(\mathfrak{q}) = \operatorname{gr} U(\mathfrak{q})$ equals $\bar{\Phi}_{\chi, i}(\bar{a})$.*

Proof. The statement of the lemma is a reformulation (in a different notation) of [FFTL10, Lemma 3.13] and the proof in loc. cit. applies in our situation without any changes. □

3. Affine W-algebras, the Feigin–Frenkel center and quantization of Mishchenko–Fomenko subalgebras

Since we are going to apply the methods outlined in Section 2 to the case where $\mathfrak{q} = \mathfrak{g}^e$ we shall require an extension of [RT92, Théorèm 4.5(ii)] valid under our assumptions on the nilpotent element $e \in \mathfrak{g}$.

We first fix some notation. Let m be a non-negative integer and write T for the image of t in the truncated polynomial ring $\mathcal{O}_m := \mathbb{C}[t]/(t^m)$. Set $\mathfrak{q} := \mathfrak{g}^e$ and write $\mathfrak{q}_m$ for the Lie algebra $\mathfrak{q} \otimes \mathcal{O}_m$. As a vector space, $\mathfrak{q}_m = \bigoplus_{i=0}^{m-1} \mathfrak{q}T^i$, where T stands for the image of t in $\mathcal{O}_m$, and $[xT^i, yT^j] = [x, y]T^{i+j}$ for all $x, y \in \mathfrak{q}$ and $i, j \in \mathbb{Z}_{\geq 0}$ (to ease notation we omit all tensor product symbols). Following the conventions of [RT92] we identify $\mathfrak{q}_m^* = \bigoplus_{i=0}^{m-1} \mathfrak{q}^*T^i$ with the direct product of m copies of $\mathfrak{q}^*$. Specifically, the m-tuple $(\psi_0, \ldots, \psi_{m-1})$ with $\psi_0, \ldots, \psi_{m-1} \in \mathfrak{q}^*$ identifies with the linear function ψ on $\mathfrak{q}_m$ such that $\psi(xT^i) = \psi_i(x)$ for all $x \in \mathfrak{q}$ and $0 \leq i \leq m-1$.

Recall that the *index* of a finite dimensional Lie algebra $\mathfrak{a}$, denoted $\operatorname{ind}\mathfrak{a}$, is defined as the smallest dimension of the coadjoint stabilizers $\mathfrak{a}^\chi$ where χ runs through the linear functions on $\mathfrak{a}$. We denote by $\mathfrak{a}^*_{\rm sing}$ the Zariski closed, conical subset of $\mathfrak{a}^*$ consisting of all $\chi \in \mathfrak{a}^*$ with $\dim \mathfrak{a}^\chi > \operatorname{ind}\mathfrak{a}$. Since $\operatorname{ind}\mathfrak{q} = \operatorname{ind}\mathfrak{g} = \ell$, it follows from [RT92, 1.6] that $\operatorname{ind}\mathfrak{q}_m = m\ell$ and

$$\psi = (\psi_0, \dots, \psi_{m-1}) \in \mathfrak{q}_m^* \setminus (\mathfrak{q}_m)^*_{\rm sing} \iff \psi_{m-1} \in \mathfrak{q} \setminus (\mathfrak{q}^*)_{\rm sing}.$$

Each polynomial function $P \in S(\mathfrak{q})$ on $\mathfrak{g}^*$ gives rise to m polynomial functions $P_0, \dots, P_{m-1} \in S(\mathfrak{q}_m)$ on $\mathfrak{q}_m^*$ by the rule

$$P\Big(\sum_{i=0}^{m-1} \psi_i t^i\Big) = \sum_{i \geq 0} P_i(\psi_0, \dots, \psi_{m-1}) t^i \quad \forall (\psi_0, \dots, \psi_{m-1}) \in \mathfrak{q}_m^*$$

(we ignore the coefficients $P_i(\psi_0, \dots, \psi_{m-1})$ with $i \geq m$). It is worth mentioning that if $P \in S^r(\mathfrak{q})$, then $P_i \in S^r(\mathfrak{q}_m)$ for all $0 \leq i \leq m-1$. By [RT92, Lemma 3.5], all P_i with $0 \leq i \leq m-1$ lie in $S(\mathfrak{q}_m)^{\mathfrak{q}_m}$ whenever $P \in S(\mathfrak{q})^{\mathfrak{q}}$.

Proposition 3.1. *Let $\mathfrak{q} = \mathfrak{g}^e$ and suppose the pair $(\mathfrak{g}, e)$ satisfies the conditions of Theorem* 1.1. *If $\{P_1, \dots, P_\ell\} \subset S(\mathfrak{g})^{\mathfrak{g}}$ is a good generating system for e, then the elements $({}^eP_i)_j$ with $1 \leq j \leq \ell$ and $0 \leq j \leq m-1$ are algebraically independent and generate the $\mathbb{C}$-algebra $S(\mathfrak{q}_m)^{\mathfrak{q}_m}$.*

Proof. Since ${}^eP_i \in S(\mathfrak{q})^{\mathfrak{q}}$ the preceding remark shows that $({}^eP_i)_j \in S(\mathfrak{q}_m)^{\mathfrak{q}_m}$ for all admissible i and j. By [RT92, Lemma 3.3(i)], the differentials $\mathrm{d}({}^eP_i)_j$ with $1 \leq i \leq \ell$ and $0 \leq j \leq m-1$ are linearly independent at $\psi = (\psi_0, \dots, \psi_{m-1}) \in \mathfrak{q}_m^*$ if and only if the differentials d^eP_i are linearly independent at $\psi_{m-1} \in \mathfrak{q}^*$. Since $(\mathfrak{q}^*)_{\rm sing}$ coincides with the Jacobian locus $\mathcal{J}({}^eP_1, \dots, {}^eP_\ell)$ by [PPY07, Theorem 2.1], the latter has codimension ≥ 2 in $\mathfrak{q}^*$ (here we use our assumption on e). The above discussion then yields that the same holds for the Jacobian locus of the $({}^eP_i)_j$'s in $\mathfrak{q}_m^*$. As a consequence, these elements are algebraically independent in $S(\mathfrak{q}_m)$ and satisfy the conditions of [PPY07, Theorem 1.1] which is an extended characteristic zero version of [Skr02, Theorem 5.4]. Applying this result shows that the $\mathbb{C}$-subalgebra, R, of $S(\mathfrak{q}_m)^{\mathfrak{q}_m}$ generated by the $m\ell = \operatorname{ind}\mathfrak{q}_m$ elements $({}^eP_i)_j$ with $1 \leq i \leq \ell$ and $0 \leq j \leq m-1$ is algebraically closed in $S(\mathfrak{q}_m)$. On the other hand, it is well known that $\operatorname{ind}\mathfrak{q}_m$ coincides with the transcendence degree of the field of invariants $\mathbb{C}(\mathfrak{q}_m^*)^{\mathfrak{q}_m}$. It follows that the field of fractions of $S(\mathfrak{q}_m)^{\mathfrak{q}_m}$ is algebraic over the subfield generated by R. As R is algebraically closed in $S(\mathfrak{q}_m)^{\mathfrak{q}_m}$ we now deduce that $R = S(\mathfrak{q}_m)^{\mathfrak{q}_m}$ and the proposition follows. $\square$

As an immediate consequence we obtain the following:

Theorem 3.2. *Suppose all assumptions of Theorem* 1.1 *hold and let $\{P_1, \dots, P_\ell\} \subset S(\mathfrak{g})^{\mathfrak{g}}$ be a good generating system for e. Then $S(\widehat{\mathfrak{g}}^e_-)^{\widehat{\mathfrak{g}}^e_+}$ is a polynomial algebra in variables $T^j({}^eP_i)$ where $i = 1, \dots, \ell$ and $j \in \mathbb{Z}_{\geq 0}$.*

Proof. Let $\mathfrak{q} = \mathfrak{g}^e$ and write $\mathfrak{q}_\infty$ for the Lie algebra $\mathfrak{q} \otimes \mathbb{C}[t] = \mathfrak{q}[[t]]$. Since we identify $\widehat{\mathfrak{q}}_-$ with $\widehat{\mathfrak{q}}/\widehat{\mathfrak{q}}_+$ the invariant ring $S(\widehat{\mathfrak{q}}_-)^{\widehat{\mathfrak{q}}_+}$ gets identified with $S(\mathfrak{q}_\infty)^{\mathfrak{q}_\infty}$. Since the conditions of Proposition 3.1 are satisfied for all Lie algebras $\mathfrak{q}_{(m)} = \mathfrak{q}[[t]]/t^m\mathfrak{q}[[t]]$ with $m \geq 0$ it is straightforward to see that each invariant algebra $\mathbb{C}[(\mathfrak{g}^e)_{(m)})^*]^{\mathfrak{g}^e_{(m)}}$ is freely generated by the elements $T^j({}^eP_i)$ with $1 \leq i \leq \ell$ and $0 \leq j \leq m-1$. Note that $(\mathfrak{q}_{(m)})^*$ is the m-th jet scheme of the affine space $\mathfrak{q}^*$ and *all* polynomial functions $T^j({}^eP_i) \in \mathbb{C}[(\mathfrak{q})^*_\infty]$ are invariant under the natural action of the group $G^e(\mathbb{C}[t])$, where $G^e = Z_G(e)$. It follows that $T^j({}^eP_i) \in \mathbb{C}[(\mathfrak{q})^*_\infty]^{\mathfrak{q}_\infty}$ for all $i \leq \ell$ and $j \geq 0$. In conjunction with the above this shows that

$$S(\widehat{\mathfrak{q}}_-)^{\widehat{\mathfrak{q}}_+} = \mathbb{C}[(\mathfrak{q})^*_\infty]^{\mathfrak{q}_\infty} = \varinjlim_m \mathbb{C}[(\mathfrak{q})^*_{(m)}]^{\mathfrak{q}_{(m)}}$$

is a polynomial algebra in $T^j({}^eP_i)$ where $i = 1, \dots, \ell$ and $j \in \mathbb{Z}_{\geq 0}$. □

We now fix a good grading [KRW03, EK05]

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(i)$$

for e. Then $\mathfrak{g}^e = \bigoplus_{j\geq 0} \mathfrak{g}^e(j)$, where $\mathfrak{g}^e(i) = \mathfrak{g}^e \cap \mathfrak{g}(i)$. Let $\kappa_{e,c}$ be the invariant bilinear form of $\mathfrak{g}^e$ defined by

$$\kappa_{e,c}(x,y) = \begin{cases} -\frac{1}{2}\operatorname{tr}_{\mathfrak{g}_0}(\operatorname{ad} x \operatorname{ad} y) - \frac{1}{2}\operatorname{tr}_{\mathfrak{g}_1}(\operatorname{ad} x \operatorname{ad} y) & \text{for } x, y \in \mathfrak{g}^e(0), \\ 0 & \text{else,} \end{cases}$$

for all homogeneous $x, y \in \mathfrak{g}^e$.

Let $\mathcal{W}^{\kappa_c}(\mathfrak{g}, f)$ be the affine W-algebra [KRW03] associated with $(\mathfrak{g}, e)$ at the *critical level* κ_c. Here

$$\kappa_c(x,y) = -\frac{1}{2}\operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} x \operatorname{ad} y) \quad \text{for } x, y \in \mathfrak{g}.$$

The following assertion was proved in [KW04] (see also [Ara05, Theorem 5.5.1]).

Theorem 3.3 ([KW04])**.** *There exists a filtration $F_\bullet \mathcal{W}^{\kappa_c}(\mathfrak{g}, e)$ of the vertex algebra $\mathcal{W}^{\kappa_c}(\mathfrak{g}, e)$ such that*

$$\operatorname{gr}_F \mathcal{W}^{\kappa_c}(\mathfrak{g}, e) \cong V^{\kappa_{e,c}}(\mathfrak{g}^e)$$

as vertex algebras.

This result enables us to attach to every element $a \in \mathcal{W}^{\kappa_c}(\mathfrak{g}, e)$ its F-*symbol* $\sigma_F(a) \in V^{\kappa_{e,c}}(\mathfrak{g}^e)$. Let $\mathcal{S}_e := e + \mathfrak{g}^f$, the Slodowy slice to the adjoint G-orbit of e.

Theorem 3.4 ([Ara15, Theorem 4.4.7])**.** *There is a natural isomorphism*

$$\operatorname{gr} \mathcal{W}^{\kappa_c}(\mathfrak{g}, f) \cong \mathbb{C}[(\mathcal{S}_e)_\infty],$$

where $\operatorname{gr} \mathcal{W}^{\kappa_c}(\mathfrak{g}, f)$ *denotes the graded Poisson vertex algebra associated with the Li filtration of* $\mathcal{W}^{\kappa_c}(\mathfrak{g}, f)$*.*

Let $\mathfrak{q} = \mathfrak{g}^e$. The F-filtration also induces a filtration of the associated graded vertex Poisson algebra $\operatorname{gr} \mathcal{W}^{\kappa_c}(\mathfrak{g}, e)$, which we identify with $\mathbb{C}[(\mathcal{S}_e)_\infty]$ by Theorem 3.4. Using the BRST realization of $\mathbb{C}[(\mathcal{S}_e)_\infty]$ ([Ara15]), we find in the same way as Theorem 3.3 that

$$\operatorname{gr}_F(\mathbb{C}[(\mathcal{S}_e)_\infty]) \cong \mathbb{C}[(\mathfrak{q}^*)_\infty], \tag{7}$$

which restricts to an isomorphism $\operatorname{gr}_F \mathbb{C}[\mathcal{S}_e] \cong \mathbb{C}[\mathfrak{q}^*]$. For $a \in \mathbb{C}[(\mathcal{S}_e)_\infty]$, let $\sigma_F(a) \in \mathbb{C}[(\mathfrak{q}^*)_\infty]$ denote its F-symbol. It will be crucial in what follows that for $a \in \mathbb{C}[\mathcal{S}_e] \subset \mathbb{C}[(\mathcal{S}_e)_\infty]$ the F-symbol $\sigma_F(a) \in \mathbb{C}[\mathfrak{q}^*]$ coincides with the initial term of a in the sense of [PPY07].

Theorem 3.3 in conjunction with (5) and (7) implies that

$$\sigma_F(\sigma(a)) = \sigma(\sigma_F(a)) \quad \text{for all } a \in \mathcal{W}^{\kappa_c}(\mathfrak{g}, e). \tag{8}$$

Theorem 3.5 ([Ara12b])**.** *There is a natural isomorphism of vertex algebras*

$$Z(V^{\kappa_c}(\mathfrak{g})) \xrightarrow{\sim} Z(\mathcal{W}^{\kappa_c}(\mathfrak{g}, e)),$$

which induces an embedding

$$S(\widehat{\mathfrak{g}}_-)^{\widehat{\mathfrak{g}}_+} \hookrightarrow \mathbb{C}[(\mathcal{S}_e)_\infty], \quad P \mapsto P|_{(\mathcal{S}_e)_\infty},$$

of associated graded vertex Poisson algebras.

The image of $\widehat{P}_i \in Z(V^{\kappa_c}(\mathfrak{g}))$ in $Z(\mathcal{W}^{\kappa_c}(\mathfrak{g}, e))$ will also be denoted by $\widehat{P}_i$. The following result is well known.

Theorem 3.6 (Feigin and Frenkel [FF92, Fre05]). *Each set $\{P_1, \dots, P_\ell\}$ of homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ admits a lift $\{\widehat{P}_1, \dots, \widehat{P}_\ell\} \subset Z(V^{\kappa_c}(\mathfrak{g}))$ such that the symbol of $\widehat{P}_i$ equals to $P_i \in S(\mathfrak{g})^{\mathfrak{g}} \subset S(\widehat{\mathfrak{g}}_-)^{\widehat{\mathfrak{g}}_+}$ for each $i \leq \ell$. Furthermore, $Z(V^{\kappa_c}(\mathfrak{g}))$ is a polynomial algebra in variables $T^i\widehat{P}_j$ with $i \in \mathbb{Z}_{\geq 0}$ and $j = 1, \dots, \ell$.*

We may assume that each $\widehat{P}_i$ is homogeneous with respect to the grading (4).

Proof of Theorem 1.4. If $a \in Z(\mathcal{W}^{\kappa_c}(\mathfrak{g}, e))$, then $\sigma_F(a) \in Z(V^{\kappa_{e,c}}(\mathfrak{g}^e))$. We set

$$ {}^e\widehat{P}_i := \sigma_F(\widehat{P}_i) \in Z(V^{\kappa_{e,c}}(\mathfrak{g}^e)). $$

Then, by (8),

$$ \sigma({}^e\widehat{P}_i) = \sigma(\sigma_F(\widehat{P}_i)) = \sigma_F(\sigma(\widehat{P}_i)) = \sigma_F(P_i) = {}^eP_i $$

for each $i = 1, \dots, \ell$. Thus, ${}^e\widehat{P}_i \in Z(V^{\kappa_{e,c}}(\mathfrak{g}^e))$ is a lift of ${}^eP_i \in S(\widehat{\mathfrak{g}}^e_-)^{\widehat{\mathfrak{g}}^e_+}$. We have $\sigma(T^j({}^e\widehat{P}_i)) = T^j({}^eP_i)$ for all $j \geq 0$. The above-mentioned identification $\operatorname{gr} V^{\kappa_{e,c}}(\mathfrak{g}^e) = S(\widehat{\mathfrak{g}}^e_-)$ then induces an embedding

$$ \operatorname{gr} Z(V^{\kappa_{e,c}}(\mathfrak{g}^e)) = \operatorname{gr}(V^{\kappa_{e,c}}(\mathfrak{g}^e)^{\widehat{\mathfrak{g}}_+}) \hookrightarrow S(\widehat{\mathfrak{g}}^e_-)^{\widehat{\mathfrak{g}}^e_+}. $$

In view of Theorem 3.2 the assertion follows. □

Proof of Theorem 1.3. Since Φ_χ is an algebra homomorphism, setting in Lemma 2.2 $U = Z(V^{\kappa_{e,c}}(\mathfrak{g}^e))$ and applying Theorem 1.4 one observes that the subalgebra $\mathcal{A}_{e,\chi}$ of $U(\mathfrak{g}^e)$ generated by all $\Phi_{\chi,n}({}^e\widehat{P}_i)$ with $1 \leq i \leq \ell$ and $0 \leq n \leq \deg {}^eP_i - 1$ is commutative. Since $\mathfrak{g}^e$ satisfies the conditions of Theorem 1.1 and χ is regular in $(\mathfrak{g}^e)^*$, it follows from [PY08, Theorem 3.2(i)] that the χ-shifts $D^j_\chi({}^eP_i)$ with $1 \leq i \leq \ell$ and $0 \leq j \leq \deg {}^eP_i - 1$ are algebraically independent in $S(\mathfrak{g}^e)$ and hence non-zero. Applying Lemma 2.3 now yields that the symbol of every $\Phi_{\chi,n}({}^e\widehat{P}_i)$ in $S(\mathfrak{g}^e)$ equals $\bar{\Phi}_{\chi,n}({}^eP_i)$. In view of (2), (6) and (3) this means that the Poisson-commutative subalgebra $\operatorname{gr}\mathcal{A}_{e,\chi}$ contains $\bar{\mathcal{A}}_{e,\chi}$.

Conversely, if $a \in \mathcal{A}_{e,\chi}$, then Lemma 2.3 combined with [PY08, Theorem 3.2(i)] yields that the symbol of a in $S(\mathfrak{g}^e)$ lies in the subalgebra generated by $D^j_\chi({}^eP_i)$; see [FFTL10, Theorem 3.14] for a similar argument. Hence $\operatorname{gr}\mathcal{A}_{e,\chi} = \bar{\mathcal{A}}_{e,\chi}$. If $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ has codimension ≥ 3 in $(\mathfrak{g}^e)^*$, then Theorem 1.2 says that $\bar{\mathcal{A}}_{e,\chi}$ is a maximal Poisson-commutative subalgebra of $S(\mathfrak{g}^e)$. This implies that $\operatorname{gr}\mathcal{A}_{e,\chi}$ is a maximal commutative subalgebra of $U(\mathfrak{g}^e)$, completing the proof. □

Remark 3.7. We see that if e satisfies the conditions of Theorem 1.1 and χ is a regular linear function on $\mathfrak{g}^e$, then the commutative subalgebra $\mathcal{A}_{e,\chi}$ of $U(\mathfrak{g}^e)$ is a quantization of the Mishchenko–Fomenko subalgebra $\bar{\mathcal{A}}_{e,\chi}$. So Vinberg's problem has a positive solution in this case. However, when $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ has codimension 2 in $(\mathfrak{g}^e)^*$ the maximality of the commutative subalgebra $\mathcal{A}_{e,\chi}$ in $U(\mathfrak{g}^e)$ is not guaranteed.

Remark 3.7 brings to our attention the problem of classifying those nilpotent centralizers $\mathfrak{g}^e$ for which $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ has codimension ≥ 3 in $(\mathfrak{g}^e)^*$. In the next section we show that this property holds in the case where $\ell \geq 2$ and e lies in the minimal non-zero nilpotent orbit of $\mathfrak{g}$.

4. The singular locus in the minimal nilpotent case

If G is a group of type A, then it follows from [Yak09, Theorem 5.4] that $(\mathfrak{g}^e)^*_{\rm sing}$ has codimension ≥ 3 in $(\mathfrak{g}^e)^*$ for any non-regular nilpotent element $e \in \mathfrak{g}$. On the other hand, the example in [PPY07, 3.9] shows that outside type A there exist nilpotent elements $e \in \mathfrak{g}$ for which $(\mathfrak{g}^e)^*_{\rm sing}$ has codimension 1 in $(\mathfrak{g}^e)^*$. Let $\mathcal{O}_{\rm min}$ denote the minimal non-zero nilpotent G-orbit in $\mathfrak{g}$ (it consists of all non-zero elements $e \in \mathfrak{g}$ such that $[e,[e,\mathfrak{g}]] = \mathbb{C}e$). In this section we are going to prove the following:

Theorem 4.1. *If $\mathfrak{g}$ is a finite dimensional simple Lie algebra of rank > 1 over $\mathbb{C}$ and $e \in \mathcal{O}_{\rm min}$, then $(\mathfrak{g}^e)^*_{\rm sing}$ has codimension ≥ 3 in $(\mathfrak{g}^e)^*$.*

In conjunction with [PPY07, Theorem 04] this shows that our main result (Theorem 1.3) is applicable to the minimal nilpotent centralizers $\mathfrak{g}^e$ outside type E_8.

Proof of Theorem 4.1. Let $\{e,h,f\}$ be an $\mathfrak{sl}_2$-triple with $e \in \mathcal{O}_{\rm min}$. The action of $\operatorname{ad} h$ on $\mathfrak{g}$ gives rise to a $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$$

such that $\mathfrak{g}(-2) = \mathbb{C}f$ and $\mathfrak{g}(2) = \mathbb{C}e$. Also, $\mathfrak{g}^e = \mathfrak{g}^e(0) \oplus \mathfrak{g}^e(1) \oplus \mathfrak{g}^e(2)$ where $\mathfrak{g}^e(0)$ is an ideal of codimension 1 in $\mathfrak{g}(0)$. We may assume that our symmetric invariant bilinear form κ on $\mathfrak{g}$ has the property that $\kappa(e,f) = 1$. Then $[x,y] = \langle x,y\rangle e$ for all $x,y \in \mathfrak{g}(1)$, where $\langle x,y\rangle = \kappa(f,[x,y])$. Since $\mathfrak{g}^f \cap \mathfrak{g}(1) = \{0\}$ by the $\mathfrak{sl}_2$–theory, the skew-symmetric form $\langle\cdot\,,\,\cdot\rangle$ on $\mathfrak{g}(1)$ is non-degenerate and $\mathfrak{g}(1) \oplus \mathfrak{g}(2)$ is isomorphic to a Heisenberg Lie algebra.

The centralizer $G_0 := Z_G(h)$ is a Levi subgroup of G with $\mathrm{Lie}(G_0) = \mathfrak{g}(0)$ and one can choose a maximal torus T in G_0 and a basis of simple roots Π in the root system Φ of G with respect to T in such a way that $e \in \mathfrak{g}_\theta$ where θ is the highest root of the positive system $\Phi^+(\Pi)$. Thanks to Yakimova's result mentioned earlier we may assume that G is not of type A. In this case it is well known (and easily seen) that $\mathfrak{g}^e(0) = [\mathfrak{g}(0),\mathfrak{g}(0)]$ is a semisimple group and $\mathfrak{g}(1)$ is an irreducible $\mathfrak{g}^e(0)$-module.

The bilinear form κ enables us to identify the coadjoint $\mathfrak{g}^e$-module $(\mathfrak{g}^e)^*$ with the factor-module $\mathfrak{g}/(\mathbb{C}h \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2))$. Since the latter identifies with $\mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}^e(0)$ as vector spaces and G_0-modules, we shall often express linear functions $\chi \in (\mathfrak{g}^e)^*$ in the form

$$\chi = \alpha_\chi f + n_\chi + h_\chi \quad \text{with} \quad \alpha_\chi \in \mathbb{C},\ n_\chi \in \mathfrak{g}(-1) \quad \text{and} \quad h_\chi \in \mathfrak{g}^e(0).$$

By [PPY07, p. 366], the hyperplane $\mathcal{H} := \mathfrak{g}(-1) \oplus \mathfrak{g}^e(0)$ of $(\mathfrak{g}^e)^*$ given by the equation $\alpha_\chi = 0$ is not contained in $(\mathfrak{g}^e)^*_{\rm sing}$.

Suppose for a contradiction that $(\mathfrak{g}^e)^*_{\rm sing}$ contains an irreducible component X of codimension ≤ 2 in $(\mathfrak{g}^e)^*$. Let N be the unipotent radical of the centralizer G^e. If $X \not\subseteq \mathcal{H}$, then the coadjoint action of N on the the non-empty principal Zariski open subset $X^\circ := \{\chi \in X \mid \chi(e) \neq 0\}$ induces an isomorphism of affine algebraic varieties

$$X^\circ \cong (N/(N,N)) \times (X^\circ \cap \operatorname{Ann}\mathfrak{g}(1));$$

see [PPY07, p. 366] for details. Identifying $(\mathfrak{g}^e)^*$ with $\mathfrak{g}/(\mathbb{C}h \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2))$ we have that $\operatorname{Ann}\mathfrak{g}(1) = \mathbb{C}f + \mathfrak{g}^e(0)$ and $X^\circ \cap \operatorname{Ann}\mathfrak{g}(1) = \mathbb{C}^\times f \oplus X^\circ(0)$ for some non-empty locally closed subset $X^\circ(0)$ of $\mathfrak{g}^e(0)$. Since $\dim N/(N,N) = \dim \mathfrak{g}^e(1)$, this subset has codimension ≥ 2 in $\mathfrak{g}^e(0)$. Consequently, $X^\circ(0)$ contains a regular element of the semisimple Lie algebra $\mathfrak{g}^e(0)$, say r. But if $\chi \in X$ is such that $\alpha_\chi \neq 0$, $n_\chi = 0$ and $h_\chi = r$, then $(\mathfrak{g}^e)^\chi$ consists of all $x_0 + \lambda e$ with $\lambda \in \mathbb{C}$ and $x_0 \in \mathfrak{g}^e(0)$ such that $[x_0, r] = 0$. So the stabilizer $(\mathfrak{g}^e)^\chi$ has dimension $1 + (\ell - 1) = \ell$. This contradiction shows that each $\chi \in X$ vanishes on e, i.e. X is a proper subset of $\mathcal{H}$. As we are assuming that $(\mathfrak{g}^e)^*_{\rm sing}$ has codimension ≤ 2 in $(\mathfrak{g}^e)^*$ this means that X is a hypersurface in $\mathcal{H}$.

Let $\pi\colon X \to \mathfrak{g}(-1)$ denote the morphism induced by the canonical projection $\mathcal{H} \twoheadrightarrow \mathfrak{g}(-1)$. The theorem on fiber dimensions of a morphism then implies that π is either dominant or the Zariski closure of $\pi(\mathcal{H}_{\rm sing})$ has codimension 1 in $\mathfrak{g}(-1)$. Moreover, in the latter case the generic fibers of π project onto $\mathfrak{g}^e(0)$ while in the former case they project onto hypersurfaces in $\mathfrak{g}^e(0)$. Since G_0 is a connected group and $\mathcal{H}_{\rm sing}$ is G_0-stable, the sets X and $\pi(X)$ are G_0-stable, too.

If $\chi = n_\chi + h_\chi \in \mathcal{H}$, then the stabilizer of χ in $\mathfrak{g}^e$ consists of all $x = x_0 + x_1 + x_2$ with $x_i \in \mathfrak{g}^e(i)$ such that

$$x_0 \in \mathfrak{g}^{n_\chi}(0) \quad \text{and} \quad [x_1, n_\chi] + [x_0, h_\chi] \in \mathbb{C}h. \tag{9}$$

We first consider the case where $\mathfrak{g}$ is not isomorphic to a Lie algebra of type C (in particular, we exclude the case where $\mathfrak{g}$ is of type B_2). Applying [Pan03, Theorem 4.2] or analyzing the Dynkin labels of nilpotent G-orbits as presented in [Car85, pp. 394–407] one observes that in this case the cocharacter $2\theta^\vee\colon \mathbb{C}^\times \to T$ is optimal in the sense of the Kempf–Rousseau theory for a non-empty Zariski open subset of $\mathfrak{g}(1)$. It follows that there exists an $\mathfrak{sl}_2$-triple $\{\tilde{e}, 2h, \tilde{f}\} \subset \mathfrak{g}$ such that $\tilde{f} \in \mathfrak{g}(-1)$, $\tilde{e} \in \mathfrak{g}(1)$ and the adjoint G_0 orbits of $\tilde{e}$ and $\tilde{f}$ are Zariski open in $\mathfrak{g}(1)$ and $\mathfrak{g}(-1)$, respectively.

(A) Suppose $\pi\colon X \to \mathfrak{g}(-1)$ is a dominant morphism. Then $\pi(X)$ contains a Richardson element of the parabolic subalgebra $\mathfrak{g}(0) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$ of $\mathfrak{g}$, say $\tilde{f}$. Since all such elements in $\mathfrak{g}(-1)$ are conjugate under the adjoint action of G_0, our earlier remark shows that there exists $\tilde{e} \in \mathfrak{g}(1)$ such that $\{\tilde{e}, 2h, \tilde{f}\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. By the $\mathfrak{sl}_2$–theory, the Lie algebra $\mathfrak{g}^{\tilde{f}}(0) = \mathfrak{g}^{\tilde{f}} \cap \mathfrak{g}^h$ is reductive and

$$\mathfrak{g}(0) = \mathfrak{g}^{\tilde{f}}(0) \oplus [\tilde{f}, \mathfrak{g}(1)]. \tag{10}$$

Since non-isomorphic irreducible $\mathfrak{sl}_2$-submodules of $\mathfrak{g}$ are orthogonal to each other with respect to κ, it must be that $\kappa(z, h) = 0$ for all $z \in \mathfrak{g}^{\tilde{f}}(0)$. From this it follows that $\mathfrak{g}^{\tilde{f}}(0) \subseteq [\mathfrak{g}(0), \mathfrak{g}(0)] = \mathfrak{g}^e(0)$.

Our standing hypothesis yields that X contains an element χ with $n_\chi = \tilde{f}$ and $\pi^{-1}(\tilde{f}) = \tilde{f} + Y$ for some hypersurface Y in $\mathfrak{g}^e(0)$. Applying to χ a suitable automorphism $\exp(\operatorname{ad}^* y) \in \operatorname{Ad}^* G^e$ with $y \in \mathfrak{g}(1)$ and taking (10) into account we may assume further that $h_\chi \in \mathfrak{g}^{\tilde{f}}(0)$. This entails that Y intersects with the subalgebra $\mathfrak{g}^{\tilde{f}}(0)$ of $\mathfrak{g}^e(0)$. By the affine dimension theorem, all irreducible components of that intersection have codimension ≤ 1 in $\mathfrak{g}^{\tilde{f}}(0)$.

The structure of the Lie algebra $\mathfrak{g}^{\tilde{f}}(0)$ is described in [PPY07, Tables 2 and 3]. If $\mathfrak{g}$ has type other than G_2, B_3 or D_4, then the semisimple rank of $\mathfrak{g}^{\tilde{f}}(0)$ is ≥ 1 and in the three excluded cases $\mathfrak{g}^{\tilde{f}}(0)$ is a torus of dimension 0, 1 and 2, respectively. In any event, $Y \cap \mathfrak{g}^{\tilde{f}}(0)$ contains a regular element of $\mathfrak{g}^{\tilde{f}}(0)$; we call it r' (if $\mathfrak{g}$ has type G_2, then necessarily $r' = 0$).

Taking $\chi \in X$ with $n_\chi = \tilde{f}$ and $h_\chi = r'$ and using (9) it is straightforward to see that $x_0 + x_1 + x_2 \in (\mathfrak{g}^e)^\chi$ if and only if x_0 lies in the centralizer of r' in $\mathfrak{g}^{\tilde{f}}(0)$ and $[x_1, \tilde{f}] \in \mathbb{C}h$. Since the map $\operatorname{ad} \tilde{f}\colon \mathfrak{g}(1) \to \mathfrak{g}(0)$ is injective, this gives $x_1 \in \mathbb{C}\tilde{e}$. Since $\mathfrak{g}^{\tilde{f}}(0)$ has rank $\ell - 2$ in all cases (by [PPY07, 3.9]) and $x_2 \in \mathbb{C}e$ we now conclude that the stabilizer of χ in $\mathfrak{g}^e$ has dimension ℓ. As this contradicts our assumption that $\chi \in (\mathfrak{g}^e)^*_{\rm sing}$ we thus deduce that the present case cannot occur.

(B) Now suppose that the Zariski closure of $\pi(X)$ has codimension 1 in $\mathfrak{g}(-1)$. This case is more complicated, and we are fortunate that some related work has been done by Panyushev in [Pan03, Sect. 4]. Given a subspace V in $\mathfrak{g}(i)$ we denote by $V^\perp(-i)$ the orthogonal complement of V in $\mathfrak{g}(-i)$ with respect to κ. Since $2\theta^\vee$ is an optimal cocharacter for a non-empty Zariski open subset of $\mathfrak{g}(-1)$ the coordinate ring $\mathbb{C}[\mathfrak{g}(-1)]$

contains a homogeneous function of positive degree which is semi-invariant under the adjoint action of G_0; we call it φ. Since G_0 acts on $\mathfrak{g}(-1)$ with finitely many orbits, the zero locus of φ contains an open G_0-orbit $\mathcal{O}'_0$. This orbit is actually unique by [Pan03, Theorem 4.2(ii)]. We pick an element $f' \in \mathcal{O}'_0$. Since $[\mathfrak{g}(0), f']$ has codimension 1 in $\mathfrak{g}(-1)$ by our choice of e' there is a non-zero $u \in \mathfrak{g}^{f'}(1)$ such that $\mathfrak{g}^{f'}(1) = [\mathfrak{g}(0), f']^{\perp}(1) = \mathbb{C}u$. Since $\mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$ is a Heisenberg Lie algebra, $\mathfrak{g}^{f'}(-1)$ has codimension 1 in $\mathfrak{g}(-1)$. Also, $\dim \mathfrak{g}^{f'}(0) = \dim \mathfrak{g}(0) - \dim \mathfrak{g}(-1) + 1$ and $\mathfrak{g}^{f'}(2) = \{0\}$. From this it is immediate that $\mathfrak{g}^{f'} = \mathfrak{g}(-2) \oplus \mathfrak{g}^{f'}(-1) \oplus \mathfrak{g}^{f'}(0) \oplus \mathbb{C}u$ and

$$\dim \mathfrak{g}^{f'} = \dim \mathfrak{g}(0) + 2 = (\dim \mathfrak{g}^{\tilde{f}}) + 2.$$

Since the semisimple element h normalizes the line $\mathbb{C}f'$ it lies in a Cartan subalgebra of $\mathfrak{g}$ contained in the optimal parabolic subalgebra of the G-unstable vector $e' \in \mathfrak{g}$. By the Kempf–Rousseau theory, this Cartan subalgebra contains an element h' with the property that $\{e', h', f'\}$ is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Since $[h, h'] = 0$ we have that $h' \in \mathfrak{g}(0)$. Writing $e' = \sum_i e'_i$ with $e'_i \in \mathfrak{g}(i)$ and taking into account the fact that $[h', e'] = 2e'$ and $[h, \mathfrak{g}(i)] \subseteq \mathfrak{g}(i)$ for all i, one observes that $[h', e'_1] = 2e'_1$. As $[e', f'] = h = [e'_1, f']$ it follows that $\{e'_1, h', f'\}$ is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. So we may assume without loss that $e' \in \mathfrak{g}(1)$.

For $i, j \in \mathbb{Z}$ define $\mathfrak{g}(i, j) = \{x \in \mathfrak{g}(j) \mid [h', x] = ix\}$. By the $\mathfrak{sl}_2$–theory, the Lie algebra $\mathfrak{g}$ thus acquires a bi-grading $\mathfrak{g} = \bigoplus_{i,\, j \in \mathbb{Z}} \mathfrak{g}(i, j)$. By [Pan03, 4.6], it endows $\mathfrak{g}$ with a symmetry of type G_2. Specifically, the set Σ' of all $(i, j) \neq (0, 0)$ with $\mathfrak{g}(i, j) \neq \{0\}$ forms a root system of type G_2 in $\mathbb{R}^2$. More precisely,

$$\mathfrak{g}(\pm 2) = \mathfrak{g}(\pm 3, \pm 2), \quad \mathfrak{g}(0) = \mathfrak{g}(-1, 0) \oplus \mathfrak{g}(0, 0) \oplus \mathfrak{g}(1, 0),$$

and

$$\mathfrak{g}(\pm 1) = \mathfrak{g}(\pm 3, \pm 1) \oplus \mathfrak{g}(\pm 2, \pm 1) \oplus \mathfrak{g}(\pm 1, \pm 1) \oplus \mathfrak{g}(0, \pm 1).$$

Furthermore, the subspaces $\mathfrak{g}(i, j)$ corresponding to the long roots of Σ' have dimension 1 whereas all $\mathfrak{g}(i, j)$'s labelled by the short roots of Σ' have the same dimension a depending on the type of $\mathfrak{g}$. We shall see later that $a + 3 = h^{\vee}$, the dual Coxeter number of $\mathfrak{g}$.

Note that $\mathfrak{g}(0, 1) = \mathfrak{g}^{f'}(1) = \mathbb{C}u$ and $\mathfrak{g}(0, -1) = \mathfrak{g}^{e'}(-1)$. Also,

$$\mathfrak{g}^{f'}(0) \cap \mathfrak{g}^{e'}(0) = \mathfrak{g}^{f'}(0, 0) = \mathfrak{g}^{e'}(0, 0)$$

and

$$\mathfrak{g}(0) = \mathfrak{g}^{f'}(0, 0) \oplus [e', \mathfrak{g}(-2, -1)]$$

by the $\mathfrak{sl}_2$–theory. We set $\mathfrak{k} := \mathfrak{g}^{f'}(0, 0)$ and $\mathfrak{p} := [e', \mathfrak{g}(-2, -1)]$. As $(\operatorname{ad} e')^3$ annihilates the abelian subalgebra $\mathfrak{g}(-2, -1))$ of $\mathfrak{g}$, we have that

$$\begin{aligned} 0 = \tfrac{1}{3}(\operatorname{ad} e')^3([x, y]) = [(\operatorname{ad} e')^2(x), (\operatorname{ad} e')(y)] + [(\operatorname{ad} e')(x), (\operatorname{ad} e')^2(y)] \\ = \big[e', [[e', x], [e', y]]\big] = [e', [\mathfrak{p}, \mathfrak{p}]] \end{aligned}$$

for all $x, y \in \mathfrak{g}(-2, -1)$. Consequently, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ implying that $\mathfrak{g}(0) = \mathfrak{k} \oplus \mathfrak{p}$ is a symmetric decomposition of the Levi subalgebra $\mathfrak{g}(0)$ of $\mathfrak{g}$. Note that the restriction of κ to $\mathfrak{k}$ is non-degenerate and $\mathfrak{p} = [e', \mathfrak{g}(-2, -1)] = \mathfrak{k}^{\perp} = [f', \mathfrak{g}(2, 1)]$ by dimension reasons. Since $\operatorname{ad} f'$ is injective on $\mathfrak{g}(1, 0)$ by the $\mathfrak{sl}_2$–theory and $[f', \mathfrak{g}(-1, 0)] = \mathfrak{g}(-3, -1)$ is one-dimensional, the ideal $\mathfrak{g}^{f'}(-1, 0)$ of $\mathfrak{g}^{f'}(0)$ has codimension 1 in $\mathfrak{g}(-1, 0)$ and

$$\mathfrak{g}^{f'}(0) = \mathfrak{k} \oplus \mathfrak{g}^{f'}(-1, 0). \tag{11}$$

Recall that in the present case we can take $\chi \in X$ with $n_\chi = f'$ while our choice of $h_\chi \in \mathfrak{g}^e(0)$ is unconstrained. Since $\mathfrak{g}(3, 1)$ is spanned by $(\operatorname{ad} e')^3(f)$, our earlier remarks in this part yield

$$[f', \mathfrak{g}(1)] = \mathbb{C}[e', [e', f]] \oplus \mathfrak{g}(-1, 0) \oplus \mathfrak{p}.$$

Applying to $\chi = f' + h_\chi$ a suitable automorphism $\exp(\mathrm{ad}^*\, y) \in \mathrm{Ad}^*\, G^e$ with $y \in \mathfrak{g}(1)$ we may thus assume that $h_\chi = h_{\chi,0} + h_{\chi,1}$ for some $h_{\chi,0} \in \mathfrak{k}$ and $h_{\chi,1} \in \mathfrak{g}^{e'}(1,0)$. Note that κ gives rise to a perfect pairing between $\mathfrak{g}^{f'}(0) = \mathfrak{k} \oplus \mathfrak{g}^{f'}(-1,0)$ and $\mathfrak{g}^{e'}(0) = \mathfrak{k} \oplus \mathfrak{g}^{e'}(1,0)$.

Set $\mathfrak{m} := \mathfrak{k}^e \oplus \mathfrak{g}^{f'}(-1,0) = \mathfrak{g}^{f'}(0) \cap \mathfrak{g}^e(0)$, an ideal of codimension 1 in $\mathfrak{g}^{f'}(0)$, and denote by $\bar\chi$ the linear function on $\mathfrak{m}$ given by $\bar\chi(y) = \kappa(h_\chi, y)$ for all $y \in \mathfrak{m}$. Our preceding remarks imply that h_χ can be chosen in such a way that $\dim(\mathfrak{m}^*)^{\bar\chi} = \mathrm{ind}\,\mathfrak{m}$ where, as before, $\mathrm{ind}\,\mathfrak{m}$ denotes the index of the Lie algebra $\mathfrak{m}$. If $x = x_0 + x_1 + x_2$ with $x_i \in \mathfrak{g}^e(i)$ lies $(\mathfrak{g}^e)^\chi$, then (9) yields that $x_0 = x_{0,0} + x_{-1,0}$ for some $x_{0,0} \in \mathfrak{k}^e$ and $x_{1,-0} \in \mathfrak{g}^{f'}(-1)$. Moreover,

$$[h_{\chi,0}, x_{0,0}] + [h_{\chi,1}, x_{-1,0}] + [h_{\chi,0}, x_{-1,0}] + [h_{\chi,1}, x_{0,0}] + [f', x_1] \in \mathbb{C}h.$$

Then

$$\kappa([h_{\chi,0} + h_{\chi,1}, x_{0,0} + x_{0,-1}], y_{0,0} + y_{0,-1}) = 0$$

for all $y_{0,0} \in \mathfrak{k}^e$ and $y_{-1,0} \in \mathfrak{g}^{f'}(-1,0)$ forcing $x_0 = x_{0,0} + x_{0,-1} \in \mathfrak{m}^{\bar\chi}$. Since $\mathfrak{g}(-1,0) \subset [f', \mathfrak{g}(1)]$, this also shows that for any $x_0 \in \mathfrak{m}^{\bar\chi}$ there is $x_1 \in \mathfrak{g}(1)$ such that $x_0 + x_1 \in (\mathfrak{g}^e)^\chi$. As a consequence, the canonical projection $\mathfrak{g}^e \twoheadrightarrow \mathfrak{g}^e(0)$ gives rise to a surjective Lie algebra homomorphism $\psi\colon (\mathfrak{g}^e)^\chi \twoheadrightarrow \mathfrak{m}^{\bar\chi}$ whose kernel consists of all $x_1 + x_2$ with $x_i \in \mathfrak{g}(i)$ and $[f', x_1] \in \mathbb{C}h$.

Let $K = 2h^\vee\kappa$ be the Killing form of $\mathfrak{g}$. It is immediate from our earlier remarks that

$$4h^\vee = K(h,h) = 2(2a+2) + 8 = 4(a+3)$$

and

$$K(h,h') = 2(3a+3) + 12 = 6(a+3) = 6h^\vee$$

(here $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$). Therefore, $\kappa(h',h) = 3$. Similarly,

$$K(h',h') = 2(5a+9) + 2a + 18 = 12(a+3),$$

so that $\kappa(h',h') = 6$. If $h = [f', v]$ for some $v \in \mathfrak{g}$, then

$$\kappa(h,h) = \kappa([h,f'],v) = -\kappa(f',v)$$

and

$$\kappa(h',h) = \kappa([h',f'],v]) = -2\kappa(f',v)$$

implying $\kappa(h',h) = 4$. This contradiction shows that

$$\mathrm{Ker}\,\psi = \mathfrak{g}^{f'}(1) \oplus \mathfrak{g}(2) = \mathbb{C}u \oplus \mathbb{C}e$$

is two-dimensional. As a result,

$$\dim(\mathfrak{g}^e)^\chi = \mathrm{ind}\,\mathfrak{m} + 2. \tag{12}$$

Next we observe that f' lies in $\mathfrak{g}^{h'-2h} = \mathfrak{g}(-2,-1) \oplus \mathfrak{g}(0,0) \oplus \mathfrak{g}(2,1)$, a Levi subalgebra of $\mathfrak{g}$. Let $\mathfrak{l}$ denote the orthogonal complement of $h' - 2h$ in $\mathfrak{g}^{h'-2h}$. As $\kappa(h'-2h, h'-2h) = 6 - 12 + 8 = 2$ we have that $\mathfrak{g}^{h'-2h} = \mathfrak{l} \oplus \mathbb{C}(h'-2h)$. In particular, $\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}$ of rank $\ell - 1$. In order to compute the index of $\mathfrak{m}$ in a case-free fashion we shall identify $\mathfrak{m}$ with a subalgebra of $\mathfrak{l}^{f'}$.

We first recall that $\mathfrak{g}^{f'}(-1,0)$ is an abelian ideal of $\mathfrak{m}$, so that $\mathfrak{m} \cong \mathfrak{k}^e \ltimes \mathfrak{g}^{f'}(-1,0)$ as Lie algebras. Since $\mathfrak{k}^e = \mathfrak{k} \cap [\mathfrak{g}(0), \mathfrak{g}(0)]$ is orthogonal to both h' and h by the $\mathfrak{sl}_2$–theory, it is a subalgebra of $\mathfrak{l}^{f'}(0,0)$. From this it is immediate that $\mathfrak{l} = \mathfrak{k}^e \oplus \mathfrak{g}(-2,-1)$. Since $[f, [e', \mathfrak{g}(-1,0)]] = \mathfrak{g}(-2,-1)$ by the $\mathfrak{sl}_2$–theory and both f and e' commute with $\mathfrak{k}^e$, the $\mathfrak{k}^e$-modules $\mathfrak{g}(-1,0)$ and $\mathfrak{g}(-2,-1)$ are isomorphic. If $f' \in [f, [e', \mathfrak{g}^{f'}(-1,0)]]$, then

$$3 = \tfrac{1}{2}\kappa(h',h') = \kappa(e',f') = \kappa(e',[f,[e',w]]) = -\kappa([e',[e',f]],w)$$

for some $w \in \mathfrak{g}^{f'}$. But since $[h', e] = 3e$ and $[e', e] = 0$ the $\mathfrak{sl}_2$–theory also yields $[e', [e', f]] \in \Im \operatorname{ad} f'$. Then $\kappa([e', [e', f]], w) = 0$. This contradiction shows that $\mathfrak{g}(-2, -1) = [f, [e', \mathfrak{g}^{f'}(-1, 0)]] \oplus \mathbb{C}f'$ as $\mathfrak{k}^e$-modules and

$$\mathfrak{m} \cong \mathfrak{k}^e \ltimes \mathfrak{g}^{f'}(-1, 0) \cong \mathfrak{k}^e \ltimes [f, [e', \mathfrak{g}^{f'}(-1, 0)]]$$

identifies with an ideal of codimansion 1 in $\mathfrak{l}^{f'}$ complementary to $\mathbb{C}f'$; we call it $\mathfrak{m}'$. Since $\operatorname{ind} \mathfrak{l}^{f'} = \operatorname{rk} \mathfrak{l} = \ell - 1$ by [Pan03, Theorem 3.5], for example, and $\mathfrak{l}^{f'} = \mathfrak{m}' \oplus \mathbb{C}f'$, we now deduce that $\operatorname{ind} \mathfrak{m} = \operatorname{ind} \mathfrak{m}' = \ell - 2$. But then (12) gives $\dim(\mathfrak{g}^e)^\chi = \ell$ contrary to our choice of χ. This contradiction proves the theorem for all simple Lie algebras of type other than C.

(C) Finally, let G be of type C_ℓ and suppose first that $\ell = 2$. In this case it follows from the main results of [PPY07] that the $\mathbb{C}$-algebra $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is freely generated by two homogeneous invariants ${}^eF_1$ and ${}^eF_2$ of degree 1 and 3, and $(\mathfrak{g}^e)^*_{\text{sing}}$ coincides with their Jacobian locus $\mathcal{J}({}^eF_1, {}^eF_2)$. The centralizer $\mathfrak{g}^e$ has a $\mathbb{C}$–basis $\{E, H, F, u, v, e\}$ such that $\{E, H, F\}$ is an $\mathfrak{sl}_2$–triple in $\mathfrak{g}^e(0)$ and $u, v \in \mathfrak{g}^e(1)$ have the property that $[E, v] = u$, $[F, u] = v$ and $[u, v] = e$. Using this basis it is not hard to describe the invariant ring $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ explicitly. Namely, we may assume without loss of generality that ${}^eF_1 = e$ and

$${}^eF_2 = (4EF + H^2)e + 2Ev^2 + 2uvH - 2Fu^2.$$

As $\deg({}^eF_1) = 1$, it is then straightforward to see that $\mathcal{J}({}^eF_1, {}^eF_2)$ is isomorphic to the intersections of five quadrics

$$2yt + q^2 = 0, \quad 2xt - p^2 = 0, \quad qz - 2yp = 0, \quad 2xq + pz = 0, \quad zt + pq = 0$$

in the affine space $\mathbb{A}^6$ with coordinates x, y, z, p, q, t. This variety is nothing but

$$\Big\{\Big(\frac{p^2}{2t}, \frac{-q^2}{2t}, \frac{-pq}{t}, p, q, t\Big) \mid p, q \in \mathbb{C},\, t \in \mathbb{C}^\times\Big\} \sqcup \{(x, y, z, 0, 0, 0) \mid x, y, z \in \mathbb{C}\}.$$

We thus deduce that $(\mathfrak{g}^e)^*_{\text{sing}}$ has two irreducible components both of which are three-dimensional. So the statement holds for $\ell = 2$.

Now suppose $\ell \geq 3$. At the beginning of this proof we have assumed for a contardiction that an irreducible hypersurface X of $\mathcal{H}$ is contained in $(\mathfrak{g}^e)^*_{\text{sing}}$. Recall that N denotes the unipotent radical of G^e and $\pi\colon X \to \mathfrak{g}(-1)$ is the map induced by the second projection $\mathfrak{g}^e(0) \oplus \mathfrak{g}(-1) \twoheadrightarrow \mathfrak{g}(-1)$. Since in the present case the derived subgroup of G_0 acts transitively on the non-zero vectors of the $2(\ell-1)$-dimensional vector space $\mathfrak{g}(-1)$ the morphism π is surjective and $\dim \pi^{-1}(v) = (\dim X) - 2(\ell-1)$ for any non-zero $v \in \mathfrak{g}(-1)$.

Using the standard realization of the root system Φ we may assume that $\theta = 2\varepsilon_1$ and the Lie algebra $\mathfrak{g}^e(0)$ is generated by all root vectors e_γ and f_γ, where $\gamma = \varepsilon_i \pm \varepsilon_j$ and $2 \leq i < j \leq \ell$. We may (and will) take for v a short root vector $f_{\varepsilon_1 - \varepsilon_2}$. Set $\mathfrak{l} = \mathfrak{g}^e(0)$, a simple Lie algebra of type $\mathrm{C}_{\ell-1}$. The adjoint action of $h_v := h_{\varepsilon_2 - \varepsilon_1}$ on $\mathfrak{l}$ gives rise to a $\mathbb{Z}$-grading

$$\mathfrak{l} = \mathfrak{l}(-2) \oplus \mathfrak{l}(-1) \oplus \mathfrak{l}(0) \oplus \mathfrak{l}(1) \oplus \mathfrak{l}(2)$$

such that $\mathfrak{l}(2) = \mathbb{C}e_{2\varepsilon_2}$ and $[v, \mathfrak{g}(1)] = \mathbb{C}h_v \oplus \mathfrak{l}(1) \oplus \mathfrak{l}(2)$. As $h_{\varepsilon_2 - \varepsilon_1} - h_{2\varepsilon_2} = -h$, the same grading of $\mathfrak{l}$ is induced by the adjoint action of $h_{2\varepsilon_2}$. More importantly, $v = f_{\varepsilon_1 - \varepsilon_2}$ and $e_0 := e_{2\varepsilon_2}$ have the same centralizer in $\mathfrak{l}$, namely, $[\mathfrak{l}(0), \mathfrak{l}(0)] \oplus \mathfrak{l}(1) \oplus \mathfrak{l}(2)$.

Let S be the set of all linear functions on $\mathfrak{g}^e$ that vanish on $\mathfrak{g}(2) \oplus \mathfrak{g}(1) \oplus \mathfrak{l}(-2) \oplus \mathfrak{l}(-1) \oplus \mathbb{C}h_{2\varepsilon_2}$. This subspace of $(\mathfrak{g}^e)^*$ is canonically isomorphic to the dual space of the minimal nilpotent centralizer $\mathfrak{l}^{e_0}$ of type $\mathrm{C}_{\ell-1}$, Our discussion in the previous paragraph implies that the coadjoint action of N gives rise to an isomorphism

$$\pi^{-1}(v) \cong (N/(N, N)) \times (v + V \cap X)$$

of affine algebraic varieties. As a consequence,

$$\begin{aligned}\dim(S\cap X) &= \dim \pi^{-1}(v) - 2(\ell-1) = \dim X - 4(\ell-1)\\ &= \dim \mathfrak{g}^e - 4(\ell-1) - 2 = \dim \mathfrak{l} - 2(\ell-2) - 3\\ &= \dim \mathfrak{l}(0) + \dim \mathfrak{l}(1) + \dim \mathfrak{l}(2) - 2 = \dim \mathfrak{l}^{e_0} - 1.\end{aligned}$$

Therefore, $S\cap X$ has codimension 1 in $(\mathfrak{g}^{e_0})^*$. By [PPY07, Theorem 04], there exists $\xi\in S\cap X$ with $\dim(\mathfrak{l}^{e_0})^{\bar\xi} = \ell-1$, where $\bar\xi$ is the restriction of ξ to $\mathfrak{l}^{e_0}$ (here we use the fact that $\ell\geq 3$). Since $\mathfrak{l}^{e_0} = \mathfrak{l}^v$ it follows from the definition of V and the preceding discussion that $x = x_0+x_1+x_2$ with $x_i\in\mathfrak{g}^e(i)$ lies in $(\mathfrak{g}^e)^\xi$ if and only if $x_0\in(\mathfrak{l}^{e_0})^{\bar\xi}$ and $x_1 = 0$. But then $\dim(\mathfrak{g}^e)^\xi = (\ell-1)+1 = \ell$ violating our assumption that $X\subseteq(\mathfrak{g}^e)^*_{\rm sing}$. This contardiction completes the proof of Theorem 4.1. □

Remark 4.2. If $\mathfrak{g}$ has type other than A or E_8, then combining Theorem 4.1 with [PPY07, Theorem 04] and [OVdB10, Proposition 1.6] one obtains that *all* irreducible components of $(\mathfrak{g}^e)^*_{\rm sing}$ have codimension 3 in $(\mathfrak{g}^e)^*$. We stress that [OVdB10, Proposition 1.6] applies because outside type A the Lie algebra $\mathfrak{g}^e$ is perfect and hence all semi-invariants of $S(\mathfrak{g}^e)$ under the action of $\mathfrak{g}^e$ lie in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. If $\mathfrak{g}$ is of type E_8, then [PPY07, Theorem 04] and [OVdB10, Proposition 1.6] are no longer applicable, but according to an unpublished result of Yakimova it is still true that $(\mathfrak{g}^e)^*_{\rm sing}$ has codimension 3 in $(\mathfrak{g}^e)^*$.

5. Some explicit formulae in type A: The minimal nilpotent case

Let $\mathfrak{g} = \mathfrak{gl}_n$, and let e_{ij} be a standard basis element of $\mathfrak{g}$. Let $\widehat{\mathfrak{g}}_-\oplus\mathbb{C}\tau$ be the extended Lie algebra with τ acting on $\widehat{\mathfrak{g}}$ by the rule

$$[\tau, x_{(-m)}] = m x_{(-m)}.$$

For an $n\times n$ matrix $A = (a_{ij})$ with entries in an associative ring, denote by $\operatorname{cdet} A$ the column determinant defined by the formula

$$\operatorname{cdet} A = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1}\dots a_{\sigma(n)n}.$$

Let $\tau + E_{(-1)}$ be the $n\times n$ matrix with entries in $\widehat{\mathfrak{g}}_-\otimes\mathbb{C}\tau$ given by

$$\tau + E_{(-1)} = \begin{pmatrix} \tau+(e_{11})_{-1} & (e_{12})_{(-1)} & \dots & (e_{1n})_{(-1)}\\ (e_{21})_{-1} & \tau+(e_{22})_{(-1)} & \dots & (e_{2n})_{(-1)}\\ \vdots & \dots & \ddots & \vdots\\ (e_{n1})_{-1} & (e_{n2})_{(-1)} & \dots & \tau+(e_{nn})_{(-1)}\end{pmatrix}.$$

Let $e = e_{n,n-1}$, a minimal nilpotent element of $\mathfrak{g}$, and put

$$\mathfrak{g}_2 = \operatorname{span}\{e_{n,i}; 1\leq i\leq n-1\},\quad \mathfrak{g}_{-2} = \operatorname{span}\{e_{i,n}; 1\leq i\leq n-1\},$$

and $\mathfrak{g}_0 = \operatorname{span}\{e_{ij}; 1\leq i,j\leq n-1\} + \operatorname{span}\{e_{n,n}\}$, Then $\mathfrak{g} = \mathfrak{g}_{-2}\oplus\mathfrak{g}\oplus\mathfrak{g}_2$ defines a good even grading for e. In particular, $\mathfrak{g}^e = \mathfrak{g}_0^e\oplus\mathfrak{g}_2$. Denote by $\kappa_{e,c}$ the invariant symmetric bilinear form on $\mathfrak{g}^e$ such that

$$\kappa_{e,c}(x,y) = \begin{cases} -\operatorname{tr}_{\mathfrak{g}_0}(\operatorname{ad} x \operatorname{ad} y) & \text{for } x,y\in\mathfrak{g}_0^e,\\ 0 & \text{else.}\end{cases}$$

Then $\kappa_{e,c}$ coincides with the form defined in Theorem 3.3 on $\mathfrak{sl}_n^e$. We have

$$V^{\kappa_{e,c}}(\mathfrak{g}^e) = V^{\kappa_{e,c}}(\mathfrak{sl}_n^e)\otimes\pi,$$

where π is the commutative vertex algebra that is freely generated by one element $I_{(-1)}\mathbb{I}$, $I := e_{11} + e_{22} + e_{33} + \cdots + e_{nn}$. Therefore,

$$Z(V^{\kappa_{e,c}}(\mathfrak{g}^e)) = Z(V^{\kappa_{e,c}}(\mathfrak{sl}_n)) \otimes \pi$$

and there is no harm in replacing $\mathfrak{sl}_n$ with $\mathfrak{g}$.

Note that $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 = (n-1)^2 + 1$. We have

$$\mathfrak{g}^e = \mathrm{span}\big\{I,\ e_{i,j}\ (1 \leq i \leq n-2, 1 \leq j \leq n-1),\ e_{n,j}\ (1 \leq j \leq n-1)\big\}.$$

Consider the $(n-1) \times (n-1)$ matrix Z obtained from $\tau + E_{(-1)}$ by removing its $(n-1)$-th row and n-th column. Then

$$Z = \begin{pmatrix} \tau + (e_{11})_{(-1)} & (e_{12})_{(-1)} & \cdots & (e_{1\ n-2})_{(-1)} & (e_{1\ n-1})_{(-1)} \\ (e_{21})_{(-1)} & \tau + (e_{22})_{(-1)} & \cdots & (e_{2\ n-2})_{(-1)} & (e_{2\ n-1})_{(-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (e_{n-2\ 1})_{(-1)} & (e_{n-2\ 2})_{(-1)} & \cdots & \tau + (e_{n-2\ n-2})_{(-1)} & (e_{n-2\ n-1})_{(-1)} \\ (e_{n\ 1})_{(-1)} & (e_{n\ 2})_{(-1)} & \cdots & (e_{n\ n-2})_{(-1)} & (e_{n\ n-1})_{(-1)} \end{pmatrix}.$$

The entries of Z belong to the subalgebra $\widehat{\mathfrak{g}}^e_- \oplus \mathbb{C}\tau$ of $\widehat{\mathfrak{g}}_- \oplus \mathbb{C}\tau$, and so its column determinant $\mathrm{cdet}\, Z$ is an element of $U(\widehat{\mathfrak{g}}^e_- \oplus \mathbb{C}\tau)$. We may also regard it as an element of $V^{\kappa_{e,c}}(\mathfrak{g}^e) \otimes \mathbb{C}[\tau]$.

Arguing as in [CM09] one proves the following assertion:

Theorem 5.1. *The $\mathbb{C}[T]$-module $Z(V^{\kappa_{e,c}}(\mathfrak{g}^e))$ is freely generated by $I_{(-1)}$ and the coefficient $Q_1, \ldots, Q_{n-1}$ of the polynomial*

$$\mathrm{cdet}(Z) = Q_1\tau^{n-2} + Q_2\tau^{n-3} + \cdots + Q_{n-2}\tau + Q_{n-1},$$

so that

$$Z(V^{\kappa_{e,c}}(\mathfrak{g}^e)) = \mathbb{C}[T^j I_{(-1)}, T^j Q_i; 1 \leq i \leq n-1, j \in \mathbb{Z}_{\geq 0}].$$

Note that $Q_1 = e_{(-1)}$.

Example 5.2. If $n = 3$, then $Q_1 = (e_{32})_{(-1)}$ and

$$Q_2 = (e_{11})_{(-1)}(e_{32})_{(-1)} - (e_{31})_{(-1)}(e_{12})_{(-1)} + (e_{32})_{(-2)}.$$

If $n = 4$, then $Q_1 = (e_{43})_{(-1)}$ and

$$Q_2 = \begin{vmatrix} (e_{11})_{(-1)} & (e_{13})_{(-1)} \\ (e_{41})_{(-1)} & (e_{43})_{(-1)} \end{vmatrix} + \begin{vmatrix} (e_{22})_{(-1)} & (e_{23})_{(-1)} \\ (e_{42})_{(-1)} & (e_{43})_{(-1)} \end{vmatrix} + 2(e_{43})_{(-2)},$$

$$\begin{aligned} Q_3 = &\begin{vmatrix} (e_{11})_{(-1)} & (e_{12})_{(-1)} & (e_{13})_{(-1)} \\ (e_{21})_{(-1)} & (e_{22})_{(-1)} & (e_{23})_{(-1)} \\ (e_{41})_{(-1)} & (e_{42})_{(-1)} & (e_{43})_{(-1)} \end{vmatrix} + \begin{vmatrix} (e_{11})_{(-1)} & (e_{13})_{(-2)} \\ (e_{41})_{(-1)} & (e_{43})_{(-2)} \end{vmatrix} \\ &+ \begin{vmatrix} (e_{22})_{(-2)} & (e_{23})_{(-1)} \\ (e_{42})_{(-2)} & (e_{43})_{(-1)} \end{vmatrix} + \begin{vmatrix} (e_{22})_{(-1)} & (e_{23})_{(-2)} \\ (e_{42})_{(-1)} & (e_{43})_{(-2)} \end{vmatrix} + 2(e_{43})_{(-3)}. \end{aligned}$$

Here we use vertical brackets as shorthand notation for column determinants in non-commutative rings.

Let $\chi \in (\mathfrak{g}^e)^*$, and set $\chi_{ij} := \chi(e_{ij})$. Also, put $\partial_u := \frac{d}{du}$. Let A be the $(n-1)\times(n-1)$ matrix with entries in $U(\mathfrak{g}^e)\otimes\mathbb{C}[u^{-1}]\otimes\mathbb{C}[\partial_u]$ obtained by replacing $(e_{ij})_{(-1)}$ by $u^{-1}e_{ij} + \chi_{ij}$ and τ by $-\partial_u$: Its column determinant $\mathrm{cdet}(A)$ is an element of $U(\mathfrak{g}^e)\otimes\mathbb{C}[u^{-1}]\otimes\mathbb{C}[\partial_u]$. Write

$$\mathrm{cdet}(A) = A_1(-\partial_u)^{n-2} + A_2(-\partial_u)^{n-1} + \cdots + A_{n-1}(-\partial_u) + A_{n-1},$$

$$A_i = A_i^{(0)}u^{-i} + A_i^{(1)}u^{1-i} + \cdots + A_i^{(i-1)}u^{-1} + A_i^{(i)},$$

and note that $A_1^{(0)} = e$.

Theorem 5.3. *For a regular $\chi \in (\mathfrak{g}^e)^*$ we have*

$$\mathcal{A}_{e,\chi} = \mathbb{C}[I,\ A_i^{(j)};\ i = 1, \dots, n-1,\ j = 0, 1, \dots, i-1].$$

Example 5.4. If $n = 3$, then

$$A = \begin{vmatrix} -\partial_u + u^{-1}e_{11} + \chi_{11} & u^{-1}e_{12} + \chi_{12} \\ u^{-1}e_{31} + \chi_{31} & u^{-1}e_{32} + \chi_{32} \end{vmatrix},$$

$$A_1 = e_{32}u^{-1} + \chi_{32},$$

$$A_2 = \begin{vmatrix} e_{11}u^{-1} + \chi_{11} & e_{12}u^{-1} + \chi_{12} \\ e_{31}u^{-1} + \chi_{31} & e_{32}u^{-1} + \chi_{32} \end{vmatrix} + e_{31}u^{-2}.$$

If $n = 4$, then

$$A = \begin{vmatrix} -\partial_u + u^{-1}e_{11} + \chi_{11} & u^{-1}e_{12} + \chi_{12} & u^{-1}e_{13} + \chi_{13} \\ u^{-1}e_{21} + \chi_{21} & -\partial_u + u^{-1}e_{22} + \chi_{22} & u^{-1}e_{23} + \chi_{23} \\ u^{-1}e_{41} + \chi_{41} & u^{-1}e_{42} + \chi_{41} & u^{-1}e_{43} + \chi_{43} \end{vmatrix},$$

$$A_1 = e_{43}u^{-1} + \chi_{43},$$

$$A_2 = \begin{vmatrix} e_{11}u^{-1} + \chi_{11} & e_{13}u^{-1} + \chi_{13} \\ e_{41}u^{-1} + \chi_{41} & e_{43}u^{-1} + \chi_{43} \end{vmatrix} + \begin{vmatrix} e_{22}u^{-1} + \chi_{22} & e_{23}u^{-1} + \chi_{23} \\ e_{42}u^{-1} + \chi_{41} & e_{43}u^{-1} + \chi_{43} \end{vmatrix} + 2e_{43}u^{-2},$$

$$A_3 = \begin{vmatrix} e_{11}u^{-1} + \chi_{11} & e_{12}u^{-1} + \chi_{12} & e_{13}u^{-1} + \chi_{13} \\ e_{21}u^{-1} + \chi_{21} & e_{22}u^{-1} + \chi_{22} & e_{23}u^{-1} + \chi_{23} \\ e_{41}u^{-1} + \chi_{41} & e_{42}u^{-1} + \chi_{42} & e_{43}u^{-1} + \chi_{43} \end{vmatrix} + \begin{vmatrix} e_{11}u^{-1} + \chi_{11} & e_{13}u^{-2} \\ e_{41}u^{-1} + \chi_{41} & e_{43}u^{-2} \end{vmatrix}$$
$$+ \begin{vmatrix} e_{22}u^{-2} & e_{23}u^{-1} + \chi_{23} \\ e_{42}u^{-2} & e_{43}u^{-1} + \chi_{43} \end{vmatrix} + \begin{vmatrix} e_{22}u^{-1} + \chi_{22} & e_{23}u^{-2} \\ e_{42}u^{-1} + \chi_{42} & e_{43}u^{-2} \end{vmatrix} + 2e_{43}u^{-3}.$$

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