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THE DUAL GROUP OF A SPHERICAL VARIETY

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To Ernest B. Vinberg
on the occasion of his 80th birthday

Abstract. Let X be a spherical variety for a connected reductive group G. Work of Gaitsgory and Nadler strongly suggests that the Langlands dual group $G^\vee$ of G has a subgroup whose Weyl group is the little Weyl group of X. Sakellaridis and Venkatesh defined a refined dual group $G_X^\vee$ and verified in many cases that there exists an isogeny φ from $G_X^\vee$ to $G^\vee$. In this paper, we establish the existence of φ in full generality. Our approach is purely combinatorial and works (despite the title) for arbitrary G-varieties.

1. Introduction

Let G be a connected reductive group defined over an algebraically closed field k of characteristic zero. It has been known for a while that the large scale geometry of a G-variety X is controlled by a root system Φ_X attached to it. For spherical varieties (i.e., varieties where a Borel subgroup of G has an open orbit) this was observed by Brion [Bri90]. For the general case see [Kno94a].

Root systems classify reductive groups. So it is tempting to ask whether the group G_X with root system Φ_X has any geometric significance. In particular, it would be desirable to have a natural homomorphism from G_X to G. Unfortunately, examples show that this is not possible. The simplest is probably $X = G/H$ where $G = \mathrm{Sp}(4, \mathbb{C})$ and $H = \mathbf{G}_m \times \mathrm{Sp}(2, \mathbb{C})$. Here Φ_X consists of the *short* roots of G, hence does not correspond to a subgroup of G.

For spherical varieties a solution to this problem was proposed by Gaitsgory and Nadler in [GN10]: instead of finding a map from G_X to G, one should look at the Langlands dual groups and try to find a homomorphism $G_X^\vee \to G^\vee$ between them. In fact, using the Tannakian formalism, they were able to construct a subgroup $G_{X,GN}^\vee$ of $G^\vee$ which seems to have the right properties but the fact that the root system of $G_{X,GN}^\vee$ is $\Phi_X^\vee$ remains conjectural.

Later, Sakellaridis and Venkatesh, [SV12], refined the notion of the dual group $G_X^\vee$ and used the hypothetical homomorphism $\varphi\colon G_X^\vee \to G^\vee$ to formulate a Plancherel theorem for spherical varieties over p-adic fields. The homomorphism φ was described in terms of what they call *associated roots*. They proved the uniqueness of φ in general and its existence in many cases.

The purpose of the present paper is to prove the existence of $\varphi\colon G_X^\vee \to G^\vee$ (in the sense of [SV12]) in full generality (Theorem 7.7). Our approach is completely combinatorial. More precisely, we use a classification of rank-1 spherical varieties due to Akhiezer [Ahi83]

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and, to a certain extent, the classification of the rank-2 varieties by Wasserman [Was96] (verified by Bravi [Bra13]).

Towards proving the existence of φ, we show that the associated roots of Sakellaridis and Venkatesh are the simple roots of a subgroup $G_X^\wedge \subseteq G^\vee$ (the associated group of X, Theorem 7.3) and that φ should map $G_X^\vee$ to $G_X^\wedge$. Observe that, by construction, $G_X^\wedge$ is of maximal rank, i.e., it contains the maximal torus $T^\vee$ of $G^\vee$. Subgroups of this type have been classified by Borel and de Siebenthal in [BDS49].

Now we show that $\varphi\colon G_X^\vee \to G_X^\wedge$ can be obtained by a process which we call *folding*. This is a slight generalization of the usual folding by a graph automorphism.

It is curious that Ressayre, [Res10], arrived at the same folding procedure in his classification of minimal rank spherical varieties. This means, in particular, that the homogeneous variety $G_X^\wedge/\varphi(G_X^\vee)$ is of an extremely special type, namely it is affine, spherical, and of minimal rank (Corollary 4.7).

Next we give, in the spirit of the Langlands philosophy, a reformulation of the main results of [Kno94a] (on moment maps) and [Kno94b] (on invariant differential operators) in terms of the dual group (§ 8).

The theory of Sakellaridis and Venkatesh also calls for a particular homomorphism $\mathrm{SL}(2) \to G^\vee$ whose image centralizes $\varphi(G_X^\vee)$ and whose existence we prove as well (Proposition 9.10). To this end, we determine the centralizer of $\varphi(G_X^\vee)$ in $G^\vee$. More precisely, we show (Theorem 9.7) that $\varphi(G_X^\vee)$ is centralized by a finite index subgroup $L_X^\wedge$ of a fixed point group $(L_X^\vee)^{W_X}$, where $L_X^\vee \subseteq G^\vee$ is a Levi subgroup and W_X is the Weyl group of $G_X^\vee$ acting on $L_X^\vee$ in a not quite obvious way. Under some non-degeneracy condition we show (Theorem 9.12) that $L_X^\wedge$ is even the entire centralizer of $\varphi(G_X^\vee)$ in $G^\vee$.

All in all, we obtain the following poset of subgroups of $G^\vee$:

$$\begin{array}{ccccccccc} & & & & G^\vee & & & & \\ & & & \diagup & & \diagdown & & & \\ & & G_X^\wedge & & & & L_X^\vee & & \\ & \diagup & & \diagdown & & \diagup & & \diagdown & \\ G_X^\vee & & & & T^\vee & & & & L_X^\wedge. \end{array}$$

In Corollary 9.9 we see that

$$G_X^\vee \times^{Z(G_X^\vee)} L_X^\wedge \hookrightarrow G^\vee \tag{1.1}$$

is an injective homomorphism, where $Z(G_X^\vee) \subseteq G_X^\vee$ is the center.

To exemplify our results, we listed in Table 3 the Lie algebras of all relevant subgroups for $X = G/H$ in Krämer's list [Krä79], i.e., for G simple and H reductive, spherical. One case is particularly curious since it involves all six exceptional groups (counting D_4):

$$\mathfrak{g} = \mathfrak{g}^\vee = \mathsf{E}_8, \quad \mathfrak{h} = \mathsf{E}_7 + \mathfrak{sl}(2), \quad \mathfrak{g}_X^\wedge = \mathsf{E}_6 + \mathfrak{t}^2,$$
$$\mathfrak{g}_X^\vee = \mathsf{F}_4, \quad \mathfrak{l}_X^\vee = \mathsf{D}_4 + \mathfrak{t}^4, \quad \mathfrak{l}_X^\wedge = \mathsf{G}_2.$$

In § 10 we study the behavior of the dual group with respect to a (Galois) group E of automorphisms giving hopefully some hints on how to define an L-group ${}^{\mathsf{L}}G_X$ of X. In particular, we found that the action of E on $G_X^\vee$ will, in general, not be the obvious one (i.e., the one induced by diagram automorphisms).

In the final section, we discuss functoriality properties with respect to various transformations of weak spherical data, like parabolic induction and localization. We also note that the group $G^\vee$, its subgroup $G_X^\wedge$, the dual group $G_X^\vee$, and the homomorphism φ are defined over $\mathbb{Z}$. Moreover, the centralizer is, in general, defined over $\mathbb{Z}[\frac{1}{2}]$.

The setting of the paper is actually more general than described above. Instead of directly working with spherical varieties, we only study them through an intermediate combinatorial structure which we call a *weak spherical datum*. This structure is a weakening (whence the name) of the *homogeneous spherical datum* of [Lun01] which is used to classify homogeneous spherical varieties (by Bravi–Pezzini [BP16]). Additionally, one can associate a weak spherical datum to *any* G-variety (Proposition 5.4) which widens the scope of our theory to this generality.

Remark. Parts of this paper are based on the second author's PhD thesis [Sch16].

2. Notation

If Ξ is a lattice we denote its dual $\operatorname{Hom}(\Xi, \mathbb{Z})$ by $\Xi^\vee$. The pairing between Ξ and $\Xi^\vee$ will be denoted by $\langle \cdot \mid \cdot \rangle$.

In the following, let $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$ be a finite root datum and let $S \subseteq \Phi$ be a fixed basis, i.e., a set of simple roots. The quadruple $\mathcal{R} := (\Lambda, S, \Lambda^\vee, S^\vee)$ will then be called a *based root datum*.

Let $\Phi^+ \subseteq \Phi$ be the set of positive roots with respect to S. The Weyl group is denoted by W. We also fix a W-invariant scalar product $(\cdot, \cdot)$ on $\Lambda \otimes \mathbb{R}$. It will only serve auxiliary purposes and will not be considered as part of the structure.

For any algebraic group H let $\Xi(H)$ be its character group. The Lie algebra of a group G, H, L, etc. will be denoted by the corresponding fraktur letter $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{l}$, etc.

Let G be a connected reductive group defined over an algebraically closed field k of characteristic 0 whose based root datum is $\mathcal{R}$ (with respect to a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$). The 1-dimensional unipotent root subgroup corresponding to the root $\alpha \in \Phi$ will be denoted by G_α. On the other hand, $G(\alpha)$ is the semisimple rank-1-group generated by G_α and $G_{-\alpha}$.

Since the dual based datum $(\Lambda^\vee, S^\vee, \Lambda, S)$ is a based root datum as well, it is the root datum of a unique connected reductive group $G^\vee$, the *dual group* of G. In this paper we take $G^\vee$ to be defined over $\mathbb{C}$ even though most constructions work over $\mathbb{Z}$ (see Proposition 11.1). This means that G and $G^\vee$ are not necessarily defined over the same field.

A choice of generators $e_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in S$, is called a *pinning*. We fix a pinning for $G^\vee$.

3. Some basic facts concerning root systems

In this section we collect a couple of well-known criteria for root (sub)systems.

Proposition 3.1. *Let Σ be a subset of an Euclidean vector space V. Assume that Σ is contained in some open half-space and that*

$$\langle \sigma \mid \tau^\vee \rangle = \frac{2(\sigma, \tau)}{(\tau, \tau)} \in \mathbb{Z}_{\leq 0}$$

for all $\sigma \neq \tau \in \Sigma$. Then Σ is the basis of a finite root system.

Proof. By Bourbaki (Chap. V, § 3.5, Lemme 3(ii)), the two conditions imply that Σ is linearly independent. Without loss of generality we may assume that Σ is a basis of V.

Now consider the Cartan matrix $C_{\tau\sigma} := \langle \sigma \mid \tau^\vee \rangle$. It is symmetrizable and its symmetrization is positive definite. It follows from [Kac90, Prop. 4.9] that Σ is a basis of a finite root system inside $\mathbb{Z}\Sigma \subseteq V$. □

Recall that a root subsystem $\Psi \subseteq \Phi$ is additively closed if $\Psi = \Phi \cap \mathbb{Z}\Psi$.

Lemma 3.2. *Let Φ be a finite root system, $\Psi \subseteq \Phi$ a root subsystem and $\Sigma \subseteq \Psi$ a basis. Then Ψ is additively closed in Φ if and only if any two elements of Σ generate an additively closed root subsystem.*

Proof. We have to show that if $\varphi = \sum_{i=1}^{N} \psi_i \in \Phi$ with $\psi_i \in \Psi$, then $\varphi \in \Psi$. First we claim that it suffices to consider the case $N = 2$. Indeed, from

$$0 < (\varphi, \varphi) = \sum_{i=1}^{N} (\varphi, \psi_i)$$

we see that there is an i with $(\varphi, \psi_i) > 0$. Let $\varphi_0 := \varphi - \psi_i$. Then either $\varphi_0 = 0$, in which case $\varphi = \psi_i \in \Psi$, or $\varphi_0 \in \Phi$. Since then $\varphi_0 = \sum_{j \neq i} \psi_j \in \Psi$, by induction on N we see that $\varphi = \varphi_0 + \psi_i \in \Psi$ by the case $N = 2$.

So assume $\varphi = \psi_1 + \psi_2 \in \Phi$ with $\psi_i \in \Psi$. Then $\Psi' := \mathrm{span}_{\mathbb{Q}}(\psi_1, \psi_2) \cap \Psi$ is an additively closed subsystem of Ψ. Hence every basis $\psi_1', \psi_2' \in \Psi'$ can be extended to a basis $\Sigma' \subseteq \Psi$ (just choose a linear function ℓ with $0 < \ell(\psi_i') < 1$ for $i = 1, 2$ and $|\ell(\psi)| > 1$ for all $\psi \in \Psi \setminus \Psi'$ and consider the indecomposable elements of $\Psi \cap \{\ell > 0\}$). Let w be the element of the Weyl group of Ψ such that $w\Sigma' = \Sigma$. Since $w\Psi'$ is additively closed in Φ by assumption, so is Ψ'. This implies $\varphi \in \Psi$. □

Lemma 3.3. *A subset $\Sigma \subseteq \Phi^+$ is the basis of an additively closed root subsystem if and only if $\sigma - \tau \notin \Phi^+$ for all $\sigma, \tau \in \Sigma$.*

Proof. Clearly, the condition implies $\tau - \sigma \notin \Phi^+$ and therefore $\sigma - \tau \notin \Phi$ for all $\sigma \neq \tau \in \Sigma$. From this we infer $\langle \sigma \mid \tau^\vee \rangle \in \mathbb{Z}_{\leq 0}$. Also the half-space condition of Proposition 3.1 is satisfied since $\Sigma \subseteq \Phi^+$. Thus, Σ is a basis of some root system $\Psi \subseteq \Phi$. For $\sigma, \tau \in \Sigma$ let $\Phi' := \mathrm{span}_{\mathbb{Q}}(\sigma, \tau) \cap \Phi$ and $\Psi' := \mathrm{span}_{\mathbb{Q}}(\sigma, \tau) \cap \Psi$. If Ψ' were not additively closed in Φ', then Φ' would be of type B_2 or G_2 and Ψ' would be its subset of short roots; but then $\sigma - \tau \in \Phi' \subseteq \Phi$, contrary to our assumption. Now Lemma 3.2 implies that Ψ is additively closed in Φ. □

4. Folding root systems

The process of folding a based root system by a graph automorphism is well-known (see e. g. [Spr98, § 10]). We are going to need a slight generalization.

For this we start with the based root datum $(\Lambda, S, \Lambda^\vee, S^\vee)$ of the connected group G. Let $\alpha \mapsto {}^s\alpha$ be an involution on S. With ${}^s\alpha^\vee := {}^s(\alpha^\vee) := ({}^s\alpha)^\vee$, we also get an involution of $S^\vee$.

Definition 4.1. The involution s is called a *folding* if for all $\alpha, \beta \in S$:

i) $\langle \alpha \mid {}^s\alpha^\vee \rangle = 0$ whenever $\alpha \neq {}^s\alpha$ and
ii) $\langle \alpha - {}^s\alpha \mid \beta^\vee + {}^s\beta^\vee \rangle = 0$.

Observe, that we do not assume s to be an automorphism of the Dynkin diagram $\mathcal{D}$ of G. This would be equivalent to

$$\langle {}^s\alpha \mid {}^s\beta^\vee \rangle = \langle \alpha \mid \beta^\vee \rangle \tag{4.1}$$

which implies property ii) of a folding.

Example 4.2. Not all foldings are automorphisms, though. Let G be of type B_3 with roots $\alpha_1, \alpha_2, \alpha_3$. Let ${}^s\alpha_1 = \alpha_3$ and ${}^s\alpha_2 = \alpha_2$. Then s is a folding but not a diagram automorphism. Indeed, the only case which has to be verified for ii) is $\alpha = \alpha_1$ and $\beta = \alpha_2$. Then

$$\langle \alpha - {}^s\alpha \mid \beta^\vee + {}^s\beta^\vee \rangle = \langle \alpha_1 - \alpha_3 \mid 2\alpha_2^\vee \rangle = 0 \tag{4.2}$$

shows that s is a folding.

We show that this example is essentially the only folding that is not a diagram automorphism.

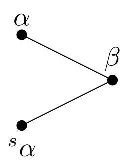
Lemma 4.3.
i) *If* $\alpha \neq {}^s\alpha \in S$*, then* α *is not connected to* ${}^s\alpha$ *in* $\mathcal{D}$*.*
ii) *For* $\alpha \neq \beta \in \mathcal{D}$ *assume that the types of the edges between* α, β *and between* ${}^s\alpha, {}^s\beta$ *are different, i.e.,* $\langle \alpha \mid \beta^\vee \rangle \neq \langle {}^s\alpha \mid {}^s\beta^\vee \rangle$ *or* $\langle \beta \mid \alpha^\vee \rangle \neq \langle {}^s\beta \mid {}^s\alpha^\vee \rangle$*. Then* $S_0 := \{\alpha, \beta, {}^s\alpha, {}^s\beta\}$ *spans a subdiagram of* $\mathcal{D}$ *which is of type* B_3*.*

Proof. Assertion i) is just the defining property i) of a folding. For ii) observe first that $\beta = {}^s\alpha$ cannot happen. So we may assume that the orbits $\{\alpha, {}^s\alpha\}$ and $\{\beta, {}^s\beta\}$ are disjoint.

Let $\mathcal{D}_0$ be the Dynkin diagram of S_0. Then, according to the number f of s-fixed points in S_0, there are three cases to be distinguished:

$f = 2$: Here s acts as identity on S_0 and ii) is trivially satisfied.

$f = 1$: Without loss of generality we may assume that $\alpha \neq {}^s\alpha$ and $\beta = {}^s\beta$. The underlying simply laced graph of $\mathcal{D}_0$ looks like this:



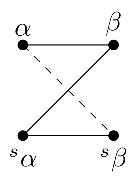
(4.3)

The second folding property ii) implies

$$a := \langle \alpha \mid \beta^\vee \rangle = \langle {}^s\alpha \mid \beta^\vee \rangle. \tag{4.4}$$

The case $a = 0$ cannot happen under the assumptions of ii). If $a \leq -2$, then $\mathcal{D}_0$ would contain two arrows, which is impossible. So assume $a = -1$. This means that β cannot be shorter than α or ${}^s\alpha$ which leaves for $\mathcal{D}_0$ only the types A_3 and B_3, confirming the assertion.

$f = 0$: We claim that s acts as an automorphisms on $\mathcal{D}_0$. Its underlying simply laced graph could a priori look like this:



(4.5)

Since $\mathcal{D}_0$ is not a cycle at least one of the diagonals is missing. Without loss of generality we may choose it to be the dashed line, i.e., we assume

$$\langle \alpha \mid {}^s\beta^\vee \rangle = \langle {}^s\beta \mid \alpha^\vee \rangle = 0. \tag{4.6}$$

Now property ii) implies

$$\langle \alpha - {}^s\alpha \mid \beta^\vee + {}^s\beta^\vee \rangle = \langle \beta - {}^s\beta \mid \alpha^\vee + {}^s\alpha^\vee \rangle = 0. \tag{4.7}$$

From this, we get the two equations

$$\begin{aligned} \langle \alpha \mid \beta^\vee \rangle &= \langle {}^s\alpha \mid \beta^\vee \rangle + \langle {}^s\alpha \mid {}^s\beta^\vee \rangle, \\ \langle \beta \mid \alpha^\vee \rangle + \langle \beta \mid {}^s\alpha^\vee \rangle &= \langle {}^s\beta \mid {}^s\alpha^\vee \rangle. \end{aligned} \tag{4.8}$$

Observe that all numbers involved are non-positive. Hence, if $\langle \alpha \mid \beta^\vee \rangle = 0$ or $\langle {}^s\beta \mid {}^s\alpha^\vee \rangle = 0$, then all other terms are 0. Then $\mathcal{D}_0$ is of type $4\mathsf{A}_1$ and the assertion is true.

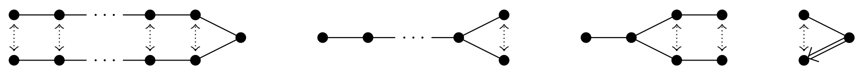
Now assume that both $\langle\alpha \mid \beta^\vee\rangle$ and $\langle {}^s\beta \mid {}^s\alpha^\vee\rangle$ are ≤ -1. If the middle terms (corresponding to the diagonal edge) were also ≤ -1, then both $\langle\alpha \mid \beta^\vee\rangle$ and $\langle {}^s\beta \mid {}^s\alpha^\vee\rangle$ were even ≤ -2. Since $\mathcal{D}_0$ does not contain two arrows this is impossible. This forces $\langle\alpha \mid {}^s\beta^\vee\rangle = \langle {}^s\beta \mid \alpha^\vee\rangle = 0$, i.e., the other diagonal edge is absent, too. But then (4.8) boils down to α, β and ${}^s\alpha, {}^s\beta$ being connected by the same type of edge proving the assertion also in this case. □

Now it is easy to classify foldings.

Corollary 4.4. *Every folding is a disjoint union of the following foldings:*

- *A component where s acts trivially.*
- *Two isomorphic components which are interchanged by s.*
- *One of the following four cases:*

(4.9)



Observe that in all cases but the last, s is a graph automorphism.

For $\alpha \in S$ we define the orbit sum

$$\bar\alpha^\vee := \begin{cases} \alpha^\vee & \text{if } \alpha = {}^s\alpha, \\ \alpha^\vee + {}^s\alpha^\vee & \text{otherwise,} \end{cases} \tag{4.10}$$

and $\bar S^\vee := \{\bar\alpha^\vee \mid \alpha \in S\}$. Then property i) of a folding implies $\langle\alpha \mid \bar\alpha^\vee\rangle = 2$ while property ii) is equivalent to $\langle\alpha \mid \bar\beta^\vee\rangle = \langle {}^s\alpha \mid \bar\beta^\vee\rangle$. We are going to show that the sets $S/\langle s\rangle$ and $\bar S^\vee$ are the roots and the coroots of a subgroup of G. More generally, we construct coverings of such a subgroup.

To this end, let Ξ be a lattice and let $r\colon \Lambda \to \Xi$ be a homomorphism with finite cokernel. Then $r^\vee\colon \Xi^\vee \to \Lambda^\vee$ will be injective which means, in particular, that Ξ and $r^\vee(\Xi^\vee)$ are still dual to each other. Let A be the torus with $\Xi(A) = \Xi$. Then r induces a homomorphism $\varphi_A\colon A \to T$ with finite kernel.

Lemma 4.5. *Let s be a folding of the root system of G. Assume that $r(\alpha - {}^s\alpha) = 0$ for all $\alpha \in S$ and that $\bar S^\vee \subseteq r^\vee(\Xi^\vee)$. Then there is a connected reductive group H with based root datum $(\Xi, r(S), r^\vee(\Xi^\vee), \bar S^\vee)$ and a homomorphism $\varphi\colon H \to G$ with finite kernel such that $\varphi|_A = \varphi_A$.*

Proof. We construct H in three stages. First, we construct a subgroup H_{ad} of the adjoint group $G_{\mathrm{ad}} := G/Z(G)$ having the root datum $(\Xi_{\mathrm{ad}}, r(S), \Xi_{\mathrm{ad}}^\vee, \bar S^\vee)$, where $\Xi_{\mathrm{ad}} := r(\mathbb{Z}\,S) = \mathbb{Z}\,r(S)$. To this end we may assume that the folding is one of the indecomposable types of Corollary 4.4. In the case s is a graph automorphism the existence of H is well known (see e.g. [Spr98, Prop. 10.3.5]): the choice of a pinning $e_\alpha \in \mathfrak{g}_\alpha$ extends the s-action to an action on G_{ad} and H_{ad} will be the connected fixed point group $(G^s_{\mathrm{ad}})^\circ$.

If s is of the last type, we have to show that $G_{\mathrm{ad}} = \mathrm{SO}(7)$ (the adjoint group of type B_3 with simple roots $\alpha_1, \alpha_2, \alpha_3$) contains a subgroup H_{ad} of rank 2 such that α_1 and α_3 restrict to the same simple root of H_{ad} and α_2 restricts to the other. Of course, such a subgroup is well known, namely $H_{\mathrm{ad}} = \mathsf{G}_2$. To see this let α_s and α_l be the two simple roots of G_2 and consider its 7-dimensional representation. It has seven weights, namely

$$2\alpha_s + \alpha_l, \alpha_s + \alpha_l, \alpha_s, 0, -\alpha_s, -\alpha_s - \alpha_l, -2\alpha_s - \alpha_l. \tag{4.11}$$

It is known that the G_2-action preserves a non-degenerate quadratic form, yielding an embedding $\mathsf{G}_2 \hookrightarrow \mathrm{SO}(7)$. The simple roots of $\mathrm{SO}(7)$ restrict to G_2 as required:

$$\begin{aligned} \operatorname{res}\alpha_1 &= (2\alpha_s+\alpha_l)-(\alpha_s+\alpha_l)=\alpha_s,\\ \operatorname{res}\alpha_2 &= (\alpha_s+\alpha_l)-\alpha_s=\alpha_l,\\ \operatorname{res}\alpha_3 &= \alpha_s. \end{aligned} \tag{4.12}$$

This establishes the existence of H_{ad} also in this case.

Now let $p\colon G\to G_{\mathrm{ad}}$ be the projection and let $H_1 = p^{-1}(H_{\mathrm{ad}})^\circ$ be the connected preimage. Then the based root datum of H_1 is $(\Xi_1, r_1(S), \Xi_1^\vee, \bar{S}^\vee)$, where

$$\Xi_1^\vee = \left\{\chi^\vee \in \Lambda^\vee \mid \langle\alpha \mid \chi^\vee\rangle = \langle {}^s\alpha \mid \chi^\vee\rangle \text{ for all } \alpha\in S\right\} \quad\text{and} \tag{4.13}$$

$$\Xi_1 = \Lambda/K \quad \text{with } K = \operatorname{span}_{\mathbb{Q}}(\alpha - {}^s\alpha \mid \alpha\in S)\cap\Lambda. \tag{4.14}$$

Moreover, $r_1\colon \Lambda\to\Xi_1$ is the projection. The conditions on Ξ and r ensure that r factors through Ξ_1 and that $\Xi^\vee$ contains the coroots inside $\Xi_1^\vee$. The isogeny theorem [Spr98, 9.6.5] then shows that there is an isogeny $H\to H_1$ inducing φ_A on $A\subseteq H$. □

Corollary 4.6. *Let $\varphi\colon H\to G$ be obtained by folding as in Lemma* 4.5. *Then:*

i) *Let $Z(G)$ be the center of a group G. Then $Z(H)=\varphi^{-1}(Z(G))$. In particular, φ induces an injective homomorphism $H_{\mathrm{ad}}\hookrightarrow G_{\mathrm{ad}}$ between adjoint groups.*
ii) *The variety $G_{\mathrm{ad}}/H_{\mathrm{ad}}$ is a product of factors isomorphic to one of the following:*

(4.15)
- K/K — *K simple, adjoint,*
- $(K\times K)/\operatorname{diag}K$ — *K simple, adjoint,*
- $\mathrm{PGL}(2n)/\mathrm{PSp}(2n)$ — $n\ge 2$,
- $\mathrm{PSO}(2n)/\mathrm{SO}(2n-1)$ — $n\ge 4$,
- $\mathsf{E}_6^{\mathrm{ad}}/\mathsf{F}_4$,
- $\mathrm{SO}(7)/\mathsf{G}_2$.

Proof. Item i) follows from the fact that the center of a reductive group is the common kernel of its simple roots and that the simple roots of H are the restrictions of the simple roots of G. Now the items of ii) correspond precisely to those of Corollary 4.4. □

The very same list of diagrams as in Corollary 4.4 already appeared in a different context. For this let $X=G/H$ be a homogeneous spherical variety. Attached to it is a lattice $\Xi(X)$ (see the paragraph before Proposition 5.4 below for a definition). Its rank is called the *rank* $\operatorname{rk}X$ *of* X. It is easy to see that the ranks of G, H, and G/H satisfy the inequality

$$\operatorname{rk}G/H \ge \operatorname{rk}G - \operatorname{rk}H. \tag{4.16}$$

Spherical varieties for which (4.16) is an equality are called *of minimal rank* and have been classified by Ressayre in [Res10]. The point is now that when G/H is affine (i.e., when H is reductive) Ressayre obtains the same list as above. Thus we obtain the following:

Corollary 4.7. *Let $\varphi\colon H\to G$ be as in Lemma* 4.5. *Then $G/\varphi(H)$ is an affine spherical variety of minimal rank.*

It is recommended to consult [Res10] for further properties of minimal rank varieties.

5. Weak spherical data

A G-variety X is called *spherical* if B has an open orbit in G. Homogeneous spherical varieties have been classified by Luna and Bravi and Pezzini, [Lun01, BP16], in terms of a combinatorial structure called a *homogeneous spherical datum*. In addition to the based root datum of G, it consists of a quintuple $(\Xi, \Sigma, \mathcal{D}, c, M)$ where Ξ is a subgroup of Λ (the weight lattice), Σ is a finite subset of Ξ (the spherical roots), $\mathcal{D}$ is a finite set (the colors), c is a map $\mathcal{D} \to \Xi^\vee$, and M is a subset of $\mathcal{D} \times S$. These objects are subject to a number of axioms (see e.g. [Lun01, § 2.2]) which we will not repeat.

In practice, it is useful to work with a structure which contains less information than a homogeneous spherical datum. It is obtained by discarding most information on $\mathcal{D}$ and renormalizing the elements of Σ. There are at least two reasons for doing so: first, these weaker structures are much easier to handle while retaining most essential information of a homogeneous spherical datum. Second, unlike homogeneous spherical data, it is possible to assign this weaker structure to any G-variety (spherical or not, see Proposition 5.4). So they have a much wider scope.

Definition 5.1. A *weak spherical datum (with respect to G or $\mathcal{R}$)* is a triple (Ξ, Σ, S^p), where $\Xi \subseteq \Lambda$ is a subgroup and $\Sigma \subseteq \Xi$, $S^p \subseteq S$ are subsets such that the following axioms are satisfied:

i) For every $\sigma \in \Sigma$ there is a subset $|\sigma| \subseteq S$ (its *support*) such that

$$\sigma = \sum_{\alpha \in |\sigma|} n_\alpha \alpha$$

with $n_\alpha \geq 1$ and such that the triple $(|\sigma|, n_*, |\sigma| \cap S^p)$ appears in Table 1 (where the n_α's are the labels and the elements of S^p are the black vertices).

ii) Let $\alpha \in S^p$. Then $\langle \Xi \mid \alpha^\vee \rangle = 0$.

iii) Let $\sigma = \alpha + \beta \in \Sigma$ be of type D_2, i.e., $\alpha, \beta \in S$ with $\alpha \perp \beta$. Then $\langle \Xi \mid \alpha^\vee - \beta^\vee \rangle = 0$.

iv) Let $\alpha, \beta \in S$ with $\alpha, \alpha + \beta \in \Sigma$. Then $\langle \beta \mid \alpha^\vee \rangle \neq -1$.

Table 1 is derived from Akhiezer's classification [Ahi83] of spherical varieties of rank 1. More precisely, that classification yields a list of all σ which can be an element of Σ for some homogeneous spherical datum. That list is, e.g., reproduced in [Kno14b]. An inspection shows that it contains entries which are multiples of each other. Then Table 1 is obtained by only picking those spherical roots which are primitive in the root lattice of G. More precisely, for every spherical root σ there is a unique factor $c \in \{\frac{1}{2}, 1, 2\}$ such that $\sigma_{\mathrm{norm}} := c\sigma$ is a weak spherical root.

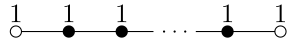
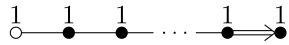
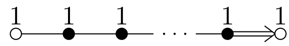
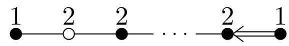
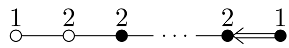
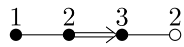
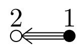
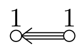
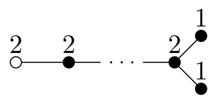
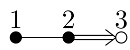
Remark 5.2. The normalization of spherical roots through primitive elements in the root lattice was proposed in [SV12].

This generalizes: let $\widetilde{\mathcal{S}} = (\widetilde{\Xi}, \widetilde{\Sigma}, \mathcal{D}, c, M)$ be a homogeneous spherical datum. Then it is easy to deduce from the axioms satisfied by $\widetilde{\mathcal{S}}$ that $\mathcal{S} = (\Xi, \Sigma, S^p)$ is a weak spherical datum, where

$$\Sigma = \{\sigma_{\mathrm{norm}} \mid \sigma \in \widetilde{\Sigma}\}, \quad \Xi = \widetilde{\Xi} + \mathbb{Z}\Sigma, \qquad (5.1)$$
$$S^p = \{\alpha \in S \mid \text{there is no } D \in \mathcal{D} \text{ with } (D, \alpha) \in M\}.$$

Remark 5.3. Observe that $\widetilde{\Xi} \subseteq \Xi \subseteq \frac{1}{2}\widetilde{\Xi}$ which means that the quotient $\Xi/\widetilde{\Xi}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ for some $m \geq 0$. Note that $m \leq \operatorname{rk} G$. The upper bound is reached when G is of adjoint type and $X = G/H$ is a symmetric variety of the same rank as G (here H is

TABLE 1. The weak spherical roots

$\lvert\sigma\rvert$	σ and $\lvert\sigma\rvert \cap S^p$
A_1	
$\mathsf{A}_n,\ n \geq 2$	
$\mathsf{B}_n,\ n \geq 2$	
$\mathsf{B}_n,\ n \geq 2$	
$\mathsf{C}_n,\ n \geq 3$	
$\mathsf{C}_n,\ n \geq 3$	
F_4	
G_2	
G_2	
D_2	
$\mathsf{D}_n,\ n \geq 3$	
B_3	

the full fixed point subgroup of an involution). Then $\widetilde{\Sigma} = 2S$ and $\widetilde{\Xi} = \mathbb{Z}\widetilde{\Sigma}$. Thus $\Sigma = S$ and therefore

$$\Xi/\widetilde{\Xi} = \mathbb{Z}S/\mathbb{Z}(2S) \cong (\mathbb{Z}/2\mathbb{Z})^{\operatorname{rk} G}.$$

By way of passing from $\widetilde{\mathcal{S}}$ to $\mathcal{S}$ one loses not only the information on the multiplier c but also all information on $\mathcal{D}$ except for which simple roots occur in M. On the other hand, it is possible to define a weak spherical datum for an arbitrary possibly non-spherical G-variety X. More precisely, $\widetilde{\Xi}(X)$, the set of characters χ_f where f is a B-semi-invariant rational function on X, makes sense for any X. Furthermore, there are several ways to attach a set $\widetilde{\Sigma}(X)$ of spherical roots to X (e.g., two of them in [Kno96, § 6]) which all differ just by the length of their roots. So the set Σ of normalized roots will be well defined. To characterize S^p let $P_\alpha \subseteq G$ be the minimal parabolic attached to the simple root $\alpha \in S$. Then we get the following.

Proposition 5.4. *Let X be a G-variety. Then (Ξ, Σ, S^p) is a weak spherical datum, where*

$$\begin{gathered}\Sigma := \{\sigma_{\text{norm}} \mid \sigma \in \Sigma(X)\}, \quad \Xi := \Xi(X) + \mathbb{Z}\Sigma,\\ S^p := \{\alpha \in S \mid P_\alpha x = Bx \text{ for } x \text{ in a dense subset of } X\}.\end{gathered} \tag{5.2}$$

Proof. The assertion is well-known but the proof is somewhat scattered in the literature. Let $W(X)$ be the Weyl group of the root system generated by $\Sigma(X)$ and let $\mathcal{C}$ be its dominant Weyl chamber. Observe that Σ can be recovered from $\mathcal{C}$ alone. From [Kno94a, Thm. 7.4] it is known that $-\mathcal{C}$ coincides with the so-called central valuation cone $\mathcal{Z}(X)$. If X is not yet spherical then, using [Kno93, Kor. 9.5, Satz 7.5], one can show that there is a variety X' whose complexity is decreased by one but with $\Xi(X') \otimes \mathbb{Q} = \Xi(X) \otimes \mathbb{Q}$

and $\mathcal{Z}(X') = \mathcal{Z}(X)$. Moreover, $S^p(X') = S^p(X)$ by [Kno94a, § 2]. This reduces the assertion to spherical varieties where it is known. □

We return to abstract weak spherical data. The following important property is not at all apparent from the definition:

Lemma 5.5. *Let (Ξ, Σ, S^p) be a weak spherical datum and let $\sigma, \tau \in \Sigma$ be distinct. Then $(\sigma, \tau) \leq 0$.*

Proof. Lemma 5.2 of [Kno14b] lists all possible pairs $\sigma \neq \tau \in \Sigma$ with connected support, $(\sigma, \tau) > 0$ and satisfying axioms i), ii). Now all possibilities are excluded by axiom iv). The case where one of σ, τ is of type D_2 is treated in the first paragraph of the proof of [Kno14b, Thm. 4.5]. □

Since all $\sigma \in \Sigma$ are sums of positive roots this implies (see [Bou68, Chap. V, § 3.5, Lemme 3(ii)]):

Corollary 5.6. *Let (Ξ, Σ, S^p) be a weak spherical datum. Then Σ is linearly independent.*

There are a couple of obvious ways to produce new weak spherical data from old ones.

- Given Σ and S^p there is a minimal choice for Ξ, namely $\Xi_{\min} := \mathbb{Z}\Sigma$. A weak spherical datum with $\Xi = \Xi_{\min}$ is called *wonderful*. On the other hand, there is also a maximal choice, namely
$$\Xi_{\max} := \big\{\chi \in \Lambda \mid \langle \chi \mid \alpha^\vee \rangle = 0 \text{ for all } \alpha \in S^p \text{ and} \\ \langle \chi \mid \alpha^\vee - \beta^\vee \rangle = 0 \text{ for all } \alpha + \beta \in \Sigma \text{ of type } \mathsf{D}_2 \big\}. \tag{5.3}$$
If Ξ is any lattice with $\Xi_{\min} \subseteq \Xi \subseteq \Xi_{\max}$, then (Ξ, Σ, S^p) is a weak spherical datum. Of particular interest is the saturation $\Xi_{\text{sat}} := (\Xi \otimes \mathbb{Q}) \cap \Lambda$. It will describe the image of the dual group.
- If $\Sigma_0 \subseteq \Sigma$ is any subset, then (Ξ, Σ_0, S^p) is a weak spherical datum which is called the *localization in* Σ_0.
- For any subset $S_0 \subseteq S$ let $\mathcal{R}_0 := (\Lambda, S_0, \Lambda^\vee, S_0^\vee)$ (the based root datum corresponding to a Levi subgroup $L_0 \subseteq G$). Then (Ξ, Σ_0, S_0^p) is a weak spherical datum for $\mathcal{R}_0$ where $\Sigma_0 := \{\sigma \in \Sigma \mid |\sigma| \subseteq S_0\}$ and $S_0^p := S^p \cap S_0$. This weak spherical datum is called the *localization in* S_0.

Let $\Phi^p \subseteq \Phi$ be the root subsystem generated by S^p and let ρ, ρ^p be the half-sum of positive roots of Φ, Φ^p, respectively. Later we need the following lemma.

Lemma 5.7. $\rho - \rho^p \in \frac{1}{2}\Xi_{\max}$.

Proof. It is well-known that $2\rho, 2\rho^p \in \mathbb{Z}S$ and that
$$\langle \rho \mid \alpha^\vee \rangle = 1 \quad \text{for all } \alpha \in S \quad \text{and} \quad \langle \rho^p \mid \alpha^\vee \rangle = 1 \quad \text{for all } \alpha \in S^p. \tag{5.4}$$
Hence $\langle \rho - \rho^p \mid \alpha^\vee \rangle = 0$ for all $\alpha \in S^p$. Let $\sigma = \gamma_1 + \gamma_2 \in \Sigma$ of type D_2. Then
$$\langle \alpha \mid \gamma_i^\vee \rangle = 0 \quad \text{for all } \alpha \in S^p \tag{5.5}$$
implies $\langle \rho - \rho^p \mid \gamma_1^\vee - \gamma_2^\vee \rangle = \langle \rho \mid \gamma_1^\vee - \gamma_2^\vee \rangle - \langle \rho^p \mid \gamma_1^\vee - \gamma_2^\vee \rangle = 0$. □

Next we determine the relative position of any two spherical roots $\sigma, \tau \in \Sigma$. More precisely, since
$$(\mathbb{Z}\sigma + \mathbb{Z}\tau, \{\sigma, \tau\}, (|\sigma| \cup |\tau|) \cap S^p) \tag{5.6}$$
is a weak spherical datum we are reduced to data with $S = |\sigma| \cup |\tau|$. Since homogeneous spherical data of this type have been classified this poses the technical question of whether every weak spherical datum actually comes from a proper one. The following lemma shows that the answer is affirmative for wonderful systems.

Lemma 5.8. *Let $\mathcal{S} = (\Xi, \Sigma, S^p)$ be a weak spherical datum. Then Ξ contains a subgroup Ξ_0 of finite index such that (Ξ_0, Σ, S^p) is induced by a homogeneous spherical datum. If $\mathcal{S}$ is wonderful one can take $\Xi_0 = \Xi = \mathbb{Z}\Sigma$.*

Proof. We construct a homogeneous spherical datum $\widetilde{\mathcal{S}} = (\widetilde{\Xi}, \widetilde{\Sigma}, \mathcal{D}, c, M)$. Since we use its definition only in this proof we refrain from stating it here. Instead, we refer to the definition of a p-spherical system [Kno14a, Def. 71] for $p = 0$. There the axioms are labeled A1 through A8.

First, by Corollary 5.6, there is a subgroup $\Xi_0 \subseteq \Xi$ of finite index such that Σ is part of a basis of Ξ_0. Then put $\widetilde{\Xi} := \Xi_0$ and $\widetilde{\Sigma} = \Sigma$. Since all elements of Σ are primitive in Ξ_0, axiom A1 is satisfied. The axioms i)–iii) now imply the corresponding axioms A3, A2, and A8 for $\widetilde{\mathcal{S}}$. Also, since $2S \cap \Sigma = \varnothing$, axiom A7 is vacuously satisfied. The other axioms A4, A5, and A6 deal with elements of $S^{(a)} := \Sigma \cap S$. It has been shown in [Lun01] that it suffices to construct the set $\mathcal{D}^{(a)}$ of all $D \in \mathcal{D}^{(a)}$ for which there is $\alpha \in S^{(a)}$ such that $(D, \alpha) \in M$. To this end we define $\mathcal{D}^{(a)}$ formally as the disjoint union of pairs $\{D^+_\alpha, D^-_\alpha\}$ with $\alpha \in S^{(a)}$. For any $\alpha \in S^{(a)}$ we define $(D^+_\alpha, \beta) \in M$ if and only if $\alpha = \beta$. Finally we need to define the elements $c^\pm_\alpha := c(D^\pm_\alpha) \in \Xi^\vee$. To force A4, A5, and A5 to be true they have to satisfy

$$
\begin{aligned}
&c^+_\alpha(\chi) + c^-_\alpha(\chi) = \langle \chi \,|\, \alpha^\vee \rangle, \\
&c^\pm_\alpha(\alpha) = 1, \\
&c^\pm_\alpha(\sigma) \le 0 \quad \text{for } \sigma \in \Sigma \setminus \{\alpha\}.
\end{aligned}
\tag{5.7}
$$

Since Σ is part of a basis of Ξ_0 there is $c^+_\alpha \in \Xi_0^\vee$ with $c^+_\alpha(\sigma) = \delta_{\alpha\sigma}$ (Kronecker δ). With $c^-_\alpha := \alpha^\vee - c^+_\alpha$ the first two properties of (5.7) hold while the third follows from Lemma 5.5.

The last assertion is clear since $\mathbb{Z}\Sigma \subseteq \Xi_0 \subseteq \Xi$. □

Remark 5.9. The change of Ξ cannot be avoided. Take, e.g., $G = \mathrm{SL}(2)$ and $\mathcal{S} = (\mathbb{Z}\omega, \{\alpha\}, \varnothing)$. The corresponding homogeneous spherical datum would come from a non-horospherical, homogeneous G-variety, i.e., either G/T or $G/N(T)$. But those have $\Xi = \mathbb{Z}(2\omega)$ and $\Xi = \mathbb{Z}(4\omega)$, respectively. Note, however, that $\mathcal{S}$ does come from the non-spherical variety $X = \mathrm{SL}(2)/\{e\}$.

From this we deduce the following

Theorem 5.10. *Table* 2 *lists, up to isomorphism, all indecomposable weak spherical data of type* (5.6).

Proof. We know that all wonderful weak spherical data are induced from wonderful homogeneous spherical data. Now use Bravi's classification [Bra13, Appendix A] of all such homogeneous spherical data of rank 2. One can also use the earlier paper [Was96] by Wasserman which classifies wonderful varieties of rank 2. But then one has to rely on the non-trivial fact (proved in [BP16]) that all homogeneous spherical data come from spherical varieties. □

Remark 5.11. The underlinings and asterisks in Table 2 are used for the proof of Theorem 7.3.

6. Associated roots

Following [SV12], we are going to associate certain roots (of G) to every spherical root $\sigma \in \Sigma$. Observe that most spherical roots (above the dividing line in Table 1) are already roots. In this case, σ itself is the only root associated to σ.

For non-roots we use the following result:

Lemma 6.1. *Let $\sigma \in \Sigma \setminus \Phi$. Then there is a unique set $\{\gamma_1, \gamma_2\}$ of positive roots such that*

i) $\sigma = \gamma_1 + \gamma_2$,
ii) γ_1 *and* γ_2 *are strongly orthogonal, i.e.,* $(\mathbb{Q}\gamma_1 + \mathbb{Q}\gamma_2) \cap \Phi = \{\pm\gamma_1, \pm\gamma_2\}$,
iii) $\gamma_1^\vee - \gamma_2^\vee = \delta_1^\vee - \delta_2^\vee$ *with* $\delta_1, \delta_2 \in S$.

Proof. The existence of such a decomposition is established by the following table:

(6.1)

$\lvert\sigma\rvert$	γ_1, γ_2	$\gamma_1^\vee, \gamma_2^\vee$	$\delta_1^\vee, \delta_2^\vee$
D_2	$\alpha_1,\ \alpha_2$	$\alpha_1^\vee,\ \alpha_2^\vee$	$\alpha_1^\vee,\ \alpha_2^\vee$
D_n,	$(\alpha_1 + \cdots + \alpha_{n-2}) + \alpha_{n-1}$,	$(\alpha_1^\vee + \cdots + \alpha_{n-2}^\vee) + \alpha_{n-1}^\vee$,	$\alpha_{n-1}^\vee,\ \alpha_n^\vee$
$n \geq 3$	$(\alpha_1 + \cdots + \alpha_{n-2}) + \alpha_n$	$(\alpha_1^\vee + \cdots + \alpha_{n-2}^\vee) + \alpha_n^\vee$	
B_3	$\alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_2 + \alpha_3$	$\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee,\ 2\alpha_2^\vee + \alpha_3^\vee$	$\alpha_1^\vee,\ \alpha_2^\vee$

Uniqueness follows by an easy case-by-case consideration. □

Remark 6.2. i) It is easy to see that $\gamma_1, \gamma_2 \in \Phi$ are strongly orthogonal if and only if there is $w \in W$ such that $w\gamma_1$, $w\gamma_2$ are orthogonal simple roots. Observe that then the coroots $\gamma_1^\vee$, $\gamma_2^\vee$ are also strongly orthogonal.

ii) The possibility to decompose spherical roots into two strongly orthogonal roots was first observed by Brion and Pauer [BP87, 4.2 Prop.]. In this form, the construction is from [SV12].

The roots γ_1, γ_2 will be called *associated to* σ. We also use the following notation.

Definition 6.3. For $\sigma \in \Sigma$ let

$$\sigma^\wedge := \begin{cases} \{\sigma^\vee\} & \text{if } \sigma \in \Phi, \\ \{\gamma_1^\vee, \gamma_2^\vee\} & \text{if } \sigma, \gamma_1, \gamma_2 \text{ are as in Lemma 6.1.} \end{cases} \tag{6.2}$$

More generally, for any subset $\Sigma_0 \subseteq \Sigma$ let $\Sigma_0^\wedge := \bigcup_{\sigma \in \Sigma_0} \sigma^\wedge$.

We record some more properties of associated roots. Let γ_1 and γ_2 be associated to the non-root $\sigma \in \Sigma$. Using the W-invariant scalar product we define the coroot of σ (as usual) as

$$\sigma^\vee = \frac{2\sigma}{\|\sigma\|^2}. \tag{6.3}$$

Since γ_1 and γ_2 are orthogonal we have $\|\sigma\|^2 = \|\gamma_1\|^2 + \|\gamma_2\|^2$ and therefore $\sigma^\vee = c_1\gamma_1^\vee + c_2\gamma_2^\vee$ with

$$c_1 + c_2 = \frac{\|\gamma_1\|^2}{\|\sigma\|^2} + \frac{\|\gamma_2\|^2}{\|\sigma\|^2} = 1. \tag{6.4}$$

From this we deduce:

Lemma 6.4. *Let $\sigma = \gamma_1 + \gamma_2$ be as above and let $\chi \in \Xi$. Then*

$$(\chi, \sigma^\vee) = \langle \chi \mid \gamma_1^\vee \rangle = \langle \chi \mid \gamma_2^\vee \rangle. \tag{6.5}$$

Proof. Let $\varepsilon := \gamma_1^\vee - \gamma_2^\vee = \delta_1^\vee - \delta_2^\vee$. Then we claim that $\langle \Xi \mid \varepsilon \rangle = 0$. If σ is of type D_2, then this is axiom iii) (of Definition 5.1). Otherwise both δ_i are in S^p (see (6.1) and Table 1) and the claim follows from axiom ii). From the claim we get $\langle \chi \mid \gamma_1^\vee \rangle = \langle \chi \mid \gamma_2^\vee \rangle$ and therefore

$$(\chi, \sigma^\vee) = c_1\langle \chi \mid \gamma_1^\vee \rangle + c_2\langle \chi \mid \gamma_2^\vee \rangle = (c_1 + c_2)\langle \chi \mid \gamma_1^\vee \rangle = \langle \chi \mid \gamma_1^\vee \rangle. \tag{6.6}$$ □

Corollary 6.5. *Let $\sigma \in \Sigma$. Then $\chi \to (\chi, \sigma^\vee)$ is an element of $\Xi^\vee$ (also denoted by $\sigma^\vee$) which is independent of the choice of the scalar product.*

Proof. For $\sigma \in \Phi$ this is clear. Otherwise, the assertion follows from Lemma 6.4. □

Later on, we also need the following facts on spherical roots.

Lemma 6.6. *Let $\gamma \in \sigma^\wedge$ and $\delta \in S^p$. Then $\{\pm\gamma^\vee, \pm\delta^\vee\}$ is an additively closed root subsystem of $\Phi^\vee$ (and therefore $\langle \gamma \mid \delta^\vee \rangle = 0$) except for $\gamma = \gamma_i \in \sigma^\wedge$ with $\sigma \in \Sigma \setminus \Phi$ and $\delta = \delta_j$. Then $\langle \gamma \mid \delta^\vee \rangle = (-1)^{i+j}$ and $\langle \delta_1 + \delta_2 \mid \gamma_i^\vee \rangle = 0$.*

Proof. Let $\gamma \in \sigma^\wedge$ with $\sigma \in \Sigma$. Assume first that $\delta \notin |\sigma|$. Then axiom ii) implies that δ is orthogonal to every simple root in $|\sigma|$. It follows that δ is strongly orthogonal to every root whose support lies in $|\sigma|$. This holds for γ, in particular.

If $\delta \in |\sigma|$, then the assertion can be verified by going through Table 1 case-by-case:

Assume that $\gamma = \sigma \in \Phi$ but σ is not the second G_2-case. Since δ is orthogonal to γ in that case it suffices to show that $\sigma^\vee + \delta^\vee \notin \Phi$. But that follows from the fact that σ is always the dominant short root. Hence $\sigma^\vee$ is the highest root.

If σ is the second G_2-case or of type D_2, then the assertion is moot since then $S^p = \varnothing$.

The remaining cases (D_n and B_3) are now easily checked one-by-one. □

Remark 6.7. In most cases, γ and δ are even strongly orthogonal but that is not always the case: if σ is of the first C_n-type and $\delta = \alpha_1$, then $\sigma + \delta$ is a root while $\sigma^\vee + \delta^\vee$ is not.

7. The dual and the associated group

One of the most important features of a weak spherical datum follows.

Theorem 7.1. *Let (Ξ, Σ, S^p) be a weak spherical datum. Then $(\Xi, \Sigma, \Xi^\vee, \Sigma^\vee)$ is a based root datum.*

Proof. Follows from Proposition 3.1 and Lemma 5.5. □

From this we get the main object of the paper.

Definition 7.2. Let $\mathcal{S} = (\Xi, \Sigma, S^p)$ be a weak spherical datum. Then the connected reductive group over $\mathbb{C}$ with based root datum $(\Xi^\vee, \Sigma^\vee, \Xi, \Sigma)$ is denoted by $G^\vee_{\mathcal{S}}$ and is called the *dual group of $\mathcal{S}$*.

Our principal goal is to embed the dual group $G^\vee_{\mathcal{S}}$ (up to isogeny) into the Langlands dual group $G^\vee$. For this, we define an intermediate group by noticing that also $\Sigma^\wedge$ is a basis of a root system. More precisely, we have the following.

Theorem 7.3. *Let $\mathcal{S} = (\Xi, \Sigma, S^p)$ be a weak spherical datum. Then there is a (unique) connected reductive subgroup $G^\wedge_{\mathcal{S}} \subseteq G^\vee$ containing $T^\vee \subseteq G^\vee$ such that $\Sigma^\wedge$ is its set of simple roots.*

Proof. A root subsystem of $\Phi^\vee$ is the root system of a connected reductive subgroup of $G^\vee$ containing $T^\vee$ if and only if it is additively closed (see [BDS49]). So, by Lemma 3.3, we have to show that $\varepsilon := \gamma_1^\vee - \gamma_2^\vee \notin (\Phi^\vee)^+$ for all $\gamma_1^\vee \neq \gamma_2^\vee \in \Sigma^\wedge$.

If both γ_i are associated to the same $\sigma \in \Sigma$, then $\varepsilon = \delta_1^\vee - \delta_2^\vee$, where the $\delta_i^\vee$ are distinct simple roots of $G^\vee$, hence $\varepsilon \notin \Phi^\vee$.

Now assume that γ_1, γ_2 are associated to distinct elements $\sigma_1, \sigma_2 \in \Sigma$, respectively. If $\varepsilon \in (\Phi^\vee)^+$, then the supports have to be contained one in another: $|\gamma_2^\vee| \subseteq |\gamma_1^\vee|$. In Table 2 these cases are marked by an asterisk (for convenience, the non-roots are underlined).

They are:

$\lvert\sigma_1\rvert \cup \lvert\sigma_2\rvert$	σ_1, σ_2	$\gamma_1^\vee, \gamma_2^\vee$	ε
B_4	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$ $\alpha_2+2\alpha_3+3\alpha_4$	$2\alpha_1^\vee+2\alpha_2^\vee+2\alpha_3^\vee+\alpha_4^\vee$ $\alpha_2^\vee+\alpha_3^\vee+\alpha_4^\vee$	$2\alpha_1^\vee+\alpha_2^\vee+\alpha_3^\vee$
B_4	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$ $\alpha_2+2\alpha_3+3\alpha_4$	$2\alpha_1^\vee+2\alpha_2^\vee+2\alpha_3^\vee+\alpha_4^\vee$ $2\alpha_3^\vee+\alpha_4^\vee$	$2\alpha_1^\vee+2\alpha_2^\vee$
C_n	$\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n$ α_1	$\alpha_1^\vee+2\alpha_2^\vee+\cdots+2\alpha_n^\vee$ $\alpha_1^\vee$	$2\alpha_2^\vee+\cdots+2\alpha_n^\vee$
$\mathsf{C}_n+\mathsf{A}_1$	$\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n$ $\alpha_1+\alpha_1'$	$\alpha_1^\vee+2\alpha_2^\vee+\cdots+2\alpha_n^\vee$ $\alpha_1^\vee$	$2\alpha_2^\vee+\cdots+2\alpha_n^\vee$
G_2	$\alpha_1+\alpha_2$ α_1	$\alpha_1^\vee+3\alpha_2^\vee$ $\alpha_1^\vee$	$3\alpha_2^\vee$

In none of the cases is ε a root of $G^\vee$. □

Remark 7.4. Observe that it is crucial to pass to the dual root system. Consider for example the first C_n-case in Table 2. Then

$$\gamma_1-\gamma_2=\sigma_1-\sigma_2=2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n$$

is actually a root of C_n. Therefore neither $\Sigma^\wedge$ nor Σ is the set of simple roots for a subgroup of G.

Let A be the torus with character group Ξ. Then the dual torus

$$A^\vee=\mathrm{Hom}(\Xi^\vee,\mathbf{G}_m)=\Xi\otimes\mathbf{G}_m \tag{7.1}$$

with character group $\Xi^\vee$ is, by construction, a maximal torus of the dual group $G^\vee_{\mathcal{S}}$. The inclusion $\Xi\to\Lambda$ induces a homomorphism $\eta\colon A^\vee\to T^\vee$.

Definition 7.5. Let $\mathcal{S}=(\Xi,\Sigma,S^p)$ be a weak spherical datum. Then a homomorphism $\varphi\colon G^\vee_{\mathcal{S}}\to G^\vee$ is called *adapted* if it factors through $G^\wedge_{\mathcal{S}}\subseteq G^\vee$ and if it is compatible with the map η between maximal tori, i.e., if the following diagram commutes:

$$\begin{array}{ccc} A^\vee & \xrightarrow{\eta} & T^\vee \\ \downarrow & & \downarrow \\ G^\vee_{\mathcal{S}} & \xrightarrow{\varphi} & G^\wedge_{\mathcal{S}}. \end{array} \tag{7.2}$$

To show the existence of adapted homomorphisms we observe that the sets $\sigma^\wedge$, $\sigma\in\Sigma$, partition $\Sigma^\wedge$ into subsets of size at most 2. Therefore, there is a unique involution s acting on $\Sigma^\wedge$ whose orbits are precisely the sets $\sigma^\wedge$. Our main observation follows.

Lemma 7.6. *The action of s on $\Sigma^\wedge$ is a folding in the sense of Definition* 4.1.

Proof. Folding property i) follows from Lemma 6.1 ii) (strong orthogonality of γ_1 and γ_2). Property ii) is trivial if $\alpha^\vee=\sigma\in\Sigma\cap\Phi$ since then $\alpha={}^s\alpha$. Otherwise, it follows from Lemma 6.4 since $\beta^\vee+{}^s\beta^\vee\in\Sigma\cup 2\Sigma\subseteq\Xi$ for all $\beta=\gamma^\vee\in\Sigma^\wedge$. □

From this, we get the following.

Theorem 7.7. *Let $\mathcal{S}=(\Xi,\Sigma,S^p)$ be a weak spherical datum. Then there exists an adapted homomorphism $\varphi\colon G^\vee_{\mathcal{S}}\to G^\vee$.*

Proof. Apply Lemma 4.5 to the s-action on $\Sigma^\wedge$ and $r=\mathrm{res}\colon\Lambda^\vee\to\Xi^\vee$. □

Remark 7.8. The kernel K of $\eta\colon A^\vee \to T^\vee$ is finite since $\Xi^\vee(A^\vee) = \Xi \to \Lambda = \Xi^\vee(T^\vee)$ is injective. Hence K is also the kernel of φ. Moreover, it is easy to see that $K \cong \Xi_{\rm sat}/\Xi$, where $\Xi_{\rm sat} = (\Xi \otimes \mathbb{Q}) \cap \Lambda$. This means that φ is injective if and only if Ξ is a direct summand of Λ or, put differently, that $\varphi(G^\vee_{\mathcal{S}}) = G^\vee_{\mathcal{S}_{\rm sat}}$, where $\mathcal{S}_{\rm sat} = (\Xi_{\rm sat}, \Sigma, S^p)$.

Corollary 4.6 now implies the next corollary.

Corollary 7.9. *Let $\varphi\colon G^\vee_{\mathcal{S}} \to G^\vee$ be adapted. Then:*

i) *The variety $G^\wedge_{\mathcal{S}}/\varphi(G^\vee_{\mathcal{S}})$ is affine spherical of minimal rank.*

ii) *φ induces an injective homomorphism $(G^\vee_{\mathcal{S}})_{\rm ad} \hookrightarrow (G^\wedge_{\mathcal{S}})_{\rm ad}$ between adjoint groups. The variety $(G^\wedge_{\mathcal{S}})_{\rm ad}/(G^\vee_{\mathcal{S}})_{\rm ad}$ is a product of varieties from the list* (4.15).

Next we address the uniqueness of adapted homomorphisms. This has been already answered in [SV12] but we need a little extension. First observe that, in general, adapted homomorphisms cannot be unique since $\mathrm{Ad}(a)\circ\varphi$ is again adapted whenever φ is adapted and $a \in T^\vee$. This $T^\vee$-action is not free though. Therefore, we consider the torus $T^\wedge_{\rm ad}$ whose character group is $\mathbb{Z}\Sigma^\wedge$. Then clearly $T^\wedge_{\rm ad}$ is the maximal torus of the adjoint quotient $(G^\wedge_{\mathcal{S}})_{\rm ad}$ and therefore acts on $G^\wedge_{\mathcal{S}}$ by automorphisms.

Theorem 7.10.

i) *Let $\mathcal{S}_0 = (\Xi, \Sigma_0, S^p)$ be the localization of $\mathcal{S}$ with respect to a subset $\Sigma_0 \subseteq \Sigma$ and let $\varphi\colon G^\vee_{\mathcal{S}} \to G^\vee$ be adapted. Then $G^\vee_{\mathcal{S}_0} \subseteq G^\vee_{\mathcal{S}}$ and* $\mathrm{res}\,\varphi\colon G^\vee_{\mathcal{S}_0} \to G^\vee$ *is adapted as well.*

ii) *For $\sigma \in \Sigma$ let $\mathcal{S}(\sigma)$ be the localization with $\Sigma_0 = \{\sigma\}$. Then every system $(\varphi_\sigma)_{\sigma\in\Sigma}$ of adapted homomorphisms $G^\vee_{\mathcal{S}(\sigma)} \to G^\vee$ can be uniquely extended to an adapted homomorphism $G^\vee_{\mathcal{S}} \to G^\vee$.*

iii) *$T^\wedge_{\rm ad}$ acts simply transitively on the set of adapted homomorphisms $\varphi\colon G^\vee_{\mathcal{S}} \to G^\vee$.*

Proof. Clearly, we have an inclusion $G^\vee_{\mathcal{S}_0} \subseteq G^\vee_{\mathcal{S}}$. It is easy to see that the folding process (Lemma 4.5) commutes by restricting to an s-invariant subset S_0 of S. This shows that $\mathrm{res}\,\varphi$ has values in $G^\wedge_{\mathcal{S}_0}$ and is therefore adapted, proving i).

An adapted homomorphism,

$$\varphi_\sigma\colon G^\vee_{\mathcal{S}(\sigma)} \to G^\vee,$$

is uniquely determined by the image of a generator $e_{\sigma^\vee} \in \mathfrak{g}^\vee_{\mathcal{S}}$ in $\bigoplus_{\gamma^\vee\in\sigma^\wedge} \mathfrak{g}^\vee_{\gamma^\vee}$. Moreover, the image vector has to have a non-zero component in every summand (since $\mathrm{res}_{A^\vee}\gamma^\vee = \sigma^\vee$ is non-trivial). Thus, the torus $T^\wedge_{\rm ad}$ acts transitively on the set of all φ_σ where the action of $t \in T^\wedge_{\rm ad}$ depends only on the character values $\gamma^\vee(t)$ with $\gamma^\vee \in \sigma^\wedge$. Since $\Sigma^\wedge$ is linearly independent, this implies that $T^\wedge_{\rm ad}$ acts simply transitively on the set of families $(\varphi_\sigma)_{\sigma\in\Sigma}$ of adapted homomorphisms. The existence of an adapted homomorphism shows that there is a family which has an extension φ, so all have. Uniqueness follows from the fact that $G^\vee_{\mathcal{S}}$ is generated by the subgroups $G^\vee_{\mathcal{S}(\sigma)}$, $\sigma \in \Sigma$ proving ii).

This, finally, implies that $T^\wedge_{\rm ad}$ acts also simply transitively on the set of adapted maps φ, proving iii). □

8. Momentum maps and invariant differential operators

Some results from [Kno94a] and [Kno94b] can be reformulated using the dual group or rather its Lie algebra. For this assume that $k = \mathbb{C}$. Then $\mathfrak{t}^\vee$ is is the same as the dual Cartan subalgebra $\mathfrak{t}^*$ which is canonically a subspace of the coadjoint representation.

Now let X be a smooth G-variety. Then X induces a weak spherical datum $\mathcal{S}$ (Proposition 5.4). In the following, we replace the index $\mathcal{S}$ by X. So, the dual group of X is $G^\vee_X$. Its Weyl group is W_X. Since we are only concerned with Lie algebras, the distinction between the lattices Ξ and $\widetilde{\Xi}$ in (5.2) is irrelevant.

Let $m\colon T_X^* \to \mathfrak{g}^*$ be the moment map on the cotangent bundle. It was shown in [Kno94a] that there is a canonical surjective G-invariant morphism $m_0\colon T_X^* \to \mathfrak{a}^*/W_X$ with irreducible generic fibers such that the right hand square of the following diagram commutes:

(8.1)
$$\begin{array}{ccccccc}
\mathfrak{g}_X^\vee & \twoheadrightarrow & \mathfrak{a}^\vee/W_X & = & \mathfrak{a}^*/W_X & \overset{m_0}{\twoheadleftarrow} & T_X^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow m \\
\mathfrak{g}^\vee & \twoheadrightarrow & \mathfrak{t}^\vee/W & = & \mathfrak{t}^*/W & \twoheadleftarrow & \mathfrak{g}^*.
\end{array}$$

The left hand square combines two applications of the Chevalley isomorphism. Therefore the whole diagram commutes.

If X is affine and spherical, then m_0 is the categorical quotient and the diagram can be interpreted as follows:

- There is a bijective correspondence between semisimple conjugacy classes of $\mathfrak{g}^\vee$ and closed coadjoint orbits in $\mathfrak{g}^*$.
- There is a bijective correspondence between semisimple conjugacy classes of $\mathfrak{g}_X^\vee$ and closed G-orbits in T_X^*.
- The moment map is compatible with these correspondences. In particular, if $o^\vee \subseteq \mathfrak{g}^\vee$ corresponds to $o^* \subseteq \mathfrak{g}^*$, then the symplectic reduction $m^{-1}(o^*)/\!/G$ (a finite set) is in bijection with $(o^\vee \cap \mathfrak{g}_X^\vee)/G_X^\vee$.

There is also a non-homogeneous version of this theorem which is closer to representation theory. It works by replacing the cotangent bundle by the ring $\mathcal{D}(X)$ of differential operators on X. In [Kno94b] a certain central subalgebra $\mathcal{Z}(X)$ of $\mathcal{D}(X)$ was constructed which, in case X is affine, even coincides with the center. On the other hand, if X is spherical (but possibly non-affine), then $\mathcal{D}(X) = \mathcal{Z}(X)$ is commutative. Let $Z_X = \operatorname{Spec}\mathcal{Z}(X)$ and $Z_G := \operatorname{Spec}\mathcal{Z}(G)$, where $\mathcal{Z}(G)$ is the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. The main result of [Kno94b] is a Harish-Chandra type isomorphism $Z(X) \xrightarrow{\sim} (\mathfrak{a}^* + \rho)/W_X$ where ρ is the half-sum of positive roots such that the right hand square of the following diagram commutes:

(8.2)
$$\begin{array}{ccccccc}
\mathfrak{g}_X^\vee + \rho & \twoheadrightarrow & (\mathfrak{a}^\vee + \rho)/W_X & = & (\mathfrak{a}^* + \rho)/W_X & \overset{\sim}{\longleftarrow} & Z_X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}^\vee & \twoheadrightarrow & \mathfrak{t}^\vee/W & = & \mathfrak{t}^*/W & \overset{\sim}{\longleftarrow} & Z_G.
\end{array}$$

For the top left map to make sense and the diagram to commute, we need the following:

Lemma 8.1. *The subspace $\mathfrak{g}_X^\vee + \mathbb{C}\rho \subseteq \mathfrak{g}^\vee$ is a reductive subalgebra. In particular, the affine hyperplane $\mathfrak{g}_X^\vee + \rho$ is $\operatorname{Ad} G_X^\vee$-invariant.*

Proof. With the notation of Lemma 5.7 we have $\rho_0 := \rho - \rho^p \in \frac{1}{2}\Xi_{\max}$. Thus, $\mathfrak{g}_X^\vee + \mathbb{C}\rho_0$ is the dual subalgebra corresponding to the character group $\Xi + \mathbb{Z}(2\rho_0)$. Since S is orthogonal to $\Xi_{\max}$ also $(\mathfrak{g}_X^\vee + \mathbb{C}\rho_0) \oplus \mathbb{C}\rho^p$ is a subalgebra which contains $\mathfrak{g}_X^\vee$ as an ideal with abelian quotient. This implies the assertion. □

If X is spherical or affine, then we have:

- There is a bijective correspondence between semisimple conjugacy classes of $\mathfrak{g}^\vee$ and central characters of $\mathcal{U}(\mathfrak{g})$.
- There is a bijective correspondence between semisimple conjugacy classes of $\mathfrak{g}_X^\vee + \rho$ and central characters of $\mathcal{D}(X)$.
- These correspondences are compatible. In particular, if $o^\vee \subseteq \mathfrak{g}^\vee$ corresponds to the central character χ of $\mathcal{U}(\mathfrak{g})$, then there is a bijective correspondence between the set of extensions of χ to a central character of $\mathcal{D}(X)$ and the set $(o^\vee \cap (\mathfrak{g}_X^\vee + \rho))/G_X^\vee$.

9. Centralizers

Let $\Phi^p = \Phi \cap \mathbb{Z}S^p \subseteq \Phi$ be the root subsystem generated by S^p. Then all roots in Φ^p are orthogonal to Ξ. Sometimes, a converse is true; see the definition below.

Definition 9.1. A weak spherical datum (Ξ, Σ, S^p) is *non-degenerate* if $\alpha \in \Phi$ and $\langle \Xi \mid \alpha^\vee \rangle = 0$ imply $\alpha \in \Phi^p$.

A similar condition has been considered in [Kno94a] along with the following remark.

Lemma 9.2. *Let* (Ξ, Σ, S^p) *be a weak spherical datum. Then* $(\Xi_{\max}, \Sigma, S^p)$ *is non-degenerate.*

Proof. Since $\langle \rho - \rho^p \mid \alpha^\vee \rangle = 0$ implies $\alpha \in \Phi^p$ (see (5.4)) the assertion follows from Lemma 5.7. □

Remark 9.3. The simplest example of a degenerate weak spherical datum is $\mathcal{S} = (0, \varnothing, \varnothing)$ when $S \neq \varnothing$. It corresponds to the full flag variety G/B.

Let $\mathcal{S} = (\Xi, \Sigma, S^p)$ be a weak spherical datum and let $W_{\mathcal{S}}$ be the Weyl group of the corresponding root system (Theorem 7.1). Clearly, it is also the Weyl group of the dual group $G^\vee_{\mathcal{S}}$. A priori, $W_{\mathcal{S}}$ acts only on Ξ. Next we show that this action extends in a natural way to all of Λ. To this end we define for $\sigma \in \Sigma$:

$$n_\sigma := \begin{cases} s_\sigma & \text{if } \sigma \in \Sigma \cap \Phi, \\ s_{\gamma_1} s_{\gamma_2} & \text{if } \sigma = \gamma_1 + \gamma_2 \in \Sigma \setminus \Phi. \end{cases} \tag{9.1}$$

Proposition 9.4. *The map $s_\sigma \mapsto n_\sigma$ extends (uniquely) to a homomorphism $n\colon W_{\mathcal{S}} \to W$. It has the following properties:*

- *$n(W_{\mathcal{S}})$ extends the $W_{\mathcal{S}}$-action on Ξ, i. e. $n(w)\chi = w\chi$ for all $\chi \in \Xi$ and $w \in W_{\mathcal{S}}$.*
- *$n(W_{\mathcal{S}})$ acts on $S^p \subseteq \Lambda$. More precisely, if $\sigma \in \Sigma$, then $n_\sigma = n(s_\sigma)$ acts on $\delta \in S^p$ as*

$$n_\sigma(\delta) = \begin{cases} \delta_{3-i} & \textit{if } \sigma = \gamma_1 + \gamma_2 \in \Sigma \setminus \Phi \textit{ and } \delta = \delta_i, \\ \delta & \textit{otherwise.} \end{cases} \tag{9.2}$$

Proof. Let $\sigma \in \Sigma$. If $\sigma \in \Phi$, then $n_\sigma = s_\sigma$. Hence also $n_\sigma = s_\sigma$ acts on Ξ and $n_\sigma(\delta) = \delta$ for $\delta \in S^p$ since then $\delta \perp \sigma$.

If $\sigma \notin \Phi$ let $\sigma = \gamma_1 + \gamma_2$ be the decomposition of Lemma 6.1. Let $\chi \in \Xi$. Then Lemma 6.4 implies

$$\begin{aligned} n_\sigma(\chi) = s_{\gamma_1} s_{\gamma_2}(\chi) &= \chi - \langle \chi \mid \gamma_1^\vee \rangle \gamma_1 - \langle \chi \mid \gamma_2^\vee \rangle \gamma_2 \\ &= \chi - \langle \chi \mid \sigma^\vee \rangle (\gamma_1 + \gamma_2) = s_\sigma(\chi). \end{aligned} \tag{9.3}$$

Now let $\delta \in S^p$. If $\delta \notin \{\delta_1, \delta_2\}$, then $\delta \perp \gamma_1, \gamma_2$ (see Lemma 6.6) and therefore $n_\sigma(\delta) = \delta$. On the other hand, if $\delta = \delta_1$, then $\langle \delta \mid \gamma_1^\vee \rangle = 1$, $\langle \delta \mid \gamma_2^\vee \rangle = -1$ and therefore

$$n_\sigma(\delta) = \delta_1 - \gamma_1 + \gamma_2 = \delta_2. \tag{9.4}$$

The assertion that $s_\sigma \mapsto n_\sigma$ extends to a homomorphism does not depend on the choice of Ξ. So, we choose the maximal one $\Xi = \Xi_{\max}$.

Now let $N \subseteq W$ be the subgroup of all $w \in W$ with $w\Xi = \Xi$, $\operatorname{res}_\Xi w \in W_{\mathcal{S}}$ and $wS^p = S^p$. Then res_Ξ induces homomorphism $N \to W_{\mathcal{S}}$. This homomorphism is surjective, since $n_\sigma \in N$ with $\operatorname{res}_\Xi n_\sigma = s_\sigma$ and since $W_{\mathcal{S}}$ is generated by all s_σ.

We show that this homomorphism is injective. For this, let $w \in N$ be in the kernel. Then the non-degeneracy of $\mathcal{S}$ (Lemma 9.2) implies that $w \in W_{S^p}$, the group generated by all s_δ, $\delta \in S^p$. Finally, $wS^p = S^p$ forces $w = e$.

Thus, we have shown that $\operatorname{res}_\Xi\colon N \to W_{\mathcal{S}}$ is an isomorphism. Its inverse n has all required properties. □

Our goal is to determine the centralizer of $\varphi(G_{\mathcal{S}}^\vee) \subseteq G^\vee$. Observe that S^p determines a Levi subgroup $L_{\mathcal{S}}^\vee \subseteq G^\vee$ and the action of $W_{\mathcal{S}}$ on S^p induces an action on $L_{\mathcal{S}}^\vee$ by permuting the generators $e_{\delta^\vee} \in \mathfrak{g}^\vee$, $\delta \in S^p$. It will turn out that the fixed point group $(L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}}$ is almost the centralizer of $\varphi(G_{\mathcal{S}}^\vee)$. But first, we look at its structure.

Proposition 9.5. *Up to isogeny, the inclusion $(L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}} \subseteq L_{\mathcal{S}}^\vee$ is isomorphic to a product of*

(9.5)
- $1 \subseteq \mathbf{G}_m$,
- $H \subseteq H$ *with H simple,*
- $\mathrm{SL}(2) \subseteq \mathrm{SL}(2)^n$ $n \geq 2$,
- $\mathrm{SO}(2n-1) \subseteq \mathrm{SO}(2n)$ $n \geq 3$,
- $\mathsf{G}_2 \subseteq \mathrm{SO}(8)$,
- $\mathrm{SO}(3) \subseteq \mathrm{SL}(3)$.

Proof. Let $\Sigma' \subseteq \Sigma$ be the set of all $\sigma \in \Sigma$ of type $\mathsf{D}_{n\geq 3}$ or B_3 and let $S' \subseteq S$ be the union of their supports. The complement $S^p \setminus S'$ gives rise to factors of the form $H \subseteq H$ because $W_{\mathcal{S}}$ acts trivially on it. Thus, we may assume that $\Sigma = \Sigma'$ and $S = S'$.

To check how two of the roots can be combined, we look at Table 2. There, we find only two indecomposable rank-2 data, where both roots are of type $\mathsf{D}_{n\geq 3}$ or B_3. These are supported on A_5 (with two roots of type D_3) and E_6 (with two roots of type D_5), respectively. This implies easily that every connected component of $S^\vee$ is one of the following:

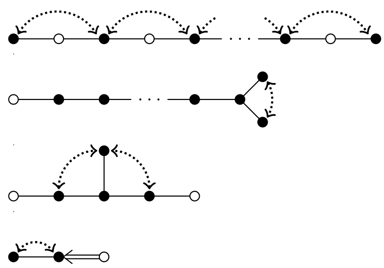
(9.6)

$\mathsf{A}_{2n-1}^\vee$ $n \geq 2$

$\mathsf{D}_n^\vee$ $n \geq 4$

$\mathsf{E}_6^\vee$

$\mathsf{B}_3^\vee$



Here the action by the simple generators $n_\sigma \in W_{\mathcal{S}}$ on S^p is indicated by dotted arrows. Now each of these diagrams gives rise to one case of (9.5). □

The group $(L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}}$ is slightly too big since, in general, not even $(T^\vee)^{W_{\mathcal{S}}}$ centralizes $G_{\mathcal{S}}^\vee$. To be precise, the character group of $(T^\vee)^{W_{\mathcal{S}}}$ is $\Xi((T^\vee)^{W_{\mathcal{S}}}) = \Lambda^\vee/\Lambda_0$ where Λ_0 is the group generated by all $\chi^\vee - w\chi^\vee$ with $\chi^\vee \in \Lambda^\vee$ and $w \in W_{\mathcal{S}}$. Then the equality

$$\chi^\vee - n_\sigma(\chi^\vee) = \begin{cases} \langle \sigma \,|\, \chi^\vee \rangle \sigma^\vee & \text{if } \sigma \in \Sigma \cap \Phi, \\ \langle \gamma_1 \,|\, \chi^\vee \rangle \gamma_1^\vee + \langle \gamma_2 \,|\, \chi^\vee \rangle \gamma_2^\vee & \text{if } \sigma = \gamma_1 + \gamma_2 \in \Sigma \setminus \Phi \end{cases} \tag{9.7}$$

shows that Λ_0 is of finite index in $\mathbb{Z}\Sigma^\wedge$. Thus, $T^{\Sigma^\wedge} \subseteq T^\vee$, the subgroup with character group $\Lambda^\vee/\mathbb{Z}\Sigma^\wedge$, is of finite index in $(T^\vee)^{W_{\mathcal{S}}}$. Clearly it equals the center of $G_{\mathcal{S}}^\wedge$ and it is also the centralizer of $\varphi(G_{\mathcal{S}}^\vee)$ in $T^\vee$.

Lemma 9.6. *There is a unique subgroup $L_{\mathcal{S}}^\wedge \subseteq (L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}}$ of finite index such that*

$$T^\vee \cap L_{\mathcal{S}}^\wedge = T^{\Sigma^\vee}. \tag{9.8}$$

Moreover, $L_{\mathcal{S}}^\wedge$ is reductive with maximal torus $(T^{\Sigma^\vee})^\circ = ((T^\vee)^{W_{\mathcal{S}}})^\circ$. Its derived subgroup L_0 is semisimple with $L_{\mathcal{S}}^\wedge = T^{\Sigma^\vee} L_0$. The simple roots of L_0 are the restrictions of the simple roots $\delta^\vee$, $\delta \in S^p$ of $L_{\mathcal{S}}^\vee$. Two restrictions are equal if and only if they lie in the same $W_{\mathcal{S}}$-orbit.

Proof. Let $\bar{L} := ((L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}})^\circ$. Then Proposition 9.5 entails that the normalizer of $\bar{L}$ is generated by $\bar{L}$ and the center of $L_{\mathcal{S}}^\vee$. This implies that $(L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}}$ itself is generated by $\bar{L}$ and $T^\vee \cap (L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}} = (T^\vee)^{W_{\mathcal{S}}}$.

Now let $\gamma^\vee \in \Lambda^\vee = \Xi(T^\vee)$ be a character and let γ_0 be its restriction to $(T^\vee)^{W_{\mathcal{S}}}$. Then γ_0 extends to a character of $\bar{L}$ if and only if $\gamma^\vee$ is trivial on the maximal torus of the semisimple part $\bar{L}'$ of $\bar{L}$. That torus is generated by the images of the simple coroots which are orbit sums $\bar{\delta} := \sum W_{\mathcal{S}}\delta$ with $\delta \in S^p$. Therefore γ_0 extends if and only if $\langle \bar{\delta} \,|\, \gamma^\vee \rangle = 0$ for all $\delta \in S^p$. We claim that this condition holds for $\gamma^\vee \in \Sigma^\wedge$. Indeed, this is clear if $\gamma \in S$. Otherwise, $\gamma^\vee = \gamma_i^\vee \in \sigma^\wedge$ for some $\sigma \in \Sigma$. Now the assertion follows from Lemma 6.6.

Thus we have shown that every element of $\mathbb{Z}\Sigma^\wedge$ extends to a character of $(L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}}$ and we can define $L_{\mathcal{S}}^\wedge \subseteq (L_{\mathcal{S}}^\vee)^{W_{\mathcal{S}}}$ to be the common kernel of these characters.

The rest of the assertions are now clear or well-known. □

Now we have the following theorem.

Theorem 9.7. *Let $\mathcal{S} = (\Xi, \Sigma, S^p)$ be a weak spherical datum. Then there is an adapted homomorphism $\varphi\colon G_{\mathcal{S}}^\vee \to G^\vee$ such that $\varphi(G_{\mathcal{S}}^\vee)$ and $L_{\mathcal{S}}^\wedge$ centralize each other.*

Proof. For $\sigma \in \Sigma$ consider the rank-1 subgroup $F_\sigma := G_{\mathcal{S}}^\vee(\sigma^\vee)$. We first show that there is an adapted homomorphism $\varphi_\sigma : F_\sigma \to G^\vee$ whose image commutes with $L_{\mathcal{S}}^\wedge$. For this let $L^\sigma := L_{\mathcal{S}_\sigma}^\wedge \subseteq L_{\mathcal{S}}^\vee$ be the subgroup corresponding to the localized system $\mathcal{S}_\sigma := (\Xi, \{\sigma\}, S^p)$. Since $L_{\mathcal{S}}^\wedge \subseteq L^\sigma$ it suffices to find φ_σ whose image commutes with L^σ. We already know that $\varphi_\sigma(F_\sigma)$ commutes with $T^\vee \cap L^\sigma$. For $\delta \in S^p$ let $H_\delta \subseteq L^\sigma$ be the semisimple rank-1 subgroup whose positive root is the restriction $\bar{\delta}^\vee$ of $\delta^\vee$ to $T^\vee \cap L^\sigma$ (see Lemma 9.6). Thus we have to show that $\varphi_\sigma(F_\sigma)$ commutes with H_δ.

Assume first that either $\sigma \in \Phi$ or that $\delta \neq \delta_i$ if $\sigma \notin \Phi$. Then H_δ commutes with all rank-1 subgroups $G^\vee(\gamma^\vee)$ with $\gamma^\vee \in \sigma^\wedge$ by Lemma 6.6. Since

$$\varphi_\sigma(F_\sigma) \subseteq \prod_{\gamma^\vee \in \sigma^\wedge} G^\vee(\gamma^\vee),$$

this implies that $\varphi_\sigma(F_\sigma)$ commutes with H_δ.

So assume that $\sigma = \gamma_1 + \gamma_2 \notin \Phi$ and $\delta = \delta_i \in S^p$. Then there are the following two cases:

The spherical root σ is of type D_n with $n \geq 3$: then one verifies that the root subsystem of $\Phi^\vee$ which is generated by $\gamma_1^\vee$, $\gamma_2^\vee$, $\delta_1^\vee$, and $\delta_2^\vee$ is of type A_3 with simple roots $\delta_1^\vee$, $\beta^\vee := \gamma_1^\vee - \delta_1^\vee = \gamma_2^\vee - \delta_2^\vee$, and $\delta_2^\vee$. This root system is additively closed in $\Phi^\vee$. Thus, it corresponds to a subgroup J of $G^\vee$ which is isogenous to $\mathrm{SL}(4)$. Let $U = \mathrm{span}_{\mathbb{C}}(u_1, u_2)$ and $V = \mathrm{span}_{\mathbb{C}}(v_1, v_2)$ be two copies of the defining representation of $\mathrm{SL}(2)$. Then the $\mathrm{SL}(2) \times \mathrm{SL}(2)$-action on $U \otimes V$ defines a homomorphism $\mathrm{SL}(2) \times \mathrm{SL}(2) \to \mathrm{SL}(4)$. More precisely, this homomorphism depends on the choice of an ordered basis which we take $(u_1 \otimes v_1,\, u_1 \otimes v_2,\, u_2 \otimes v_1,\, u_2 \otimes v_2)$. Now one checks that the first factor is mapped to the product

$$J(\gamma_1^\vee)J(\gamma_2^\vee) = J(\varepsilon_1 - \varepsilon_3)J(\varepsilon_2 - \varepsilon_4)$$

(where the ε_i denote the canonical weights of the $\mathrm{SL}(4)$-module $\mathbb{C}^4$). So this homomorphism is adapted. The second factor is mapped to the diagonal of $J(\delta_1^\vee)J(\delta_2^\vee) = J(\varepsilon_1 - \varepsilon_2)J(\varepsilon_3 - \varepsilon_4)$ which therefore equals H_δ. Both factors commute, which proves the assertion.

The second case is when σ is of type B_3. The additively closed subsystem of $\Phi^\vee$ which is generated by $\gamma_1^\vee$, $\gamma_2^\vee$, $\delta_1^\vee$, and $\delta_2^\vee$ is the dual root system C_3. Thus $G^\vee$ contains a subgroup J which is isogenous to $\mathrm{Sp}(6)$. Now consider the 2-dimensional $\mathrm{SL}(2)$-representation $U = \mathrm{span}_{\mathbb{C}}(u_1, u_2)$ and the 3-dimensional $\mathrm{SO}(3)$-representation

$V = \mathrm{span}_{\mathbb{C}}(v_1, v_2, v_3)$ (leaving invariant the quadratic form $q(x_1, x_2, x_3) = 2x_1x_3 - x_2^2$). The choice of the basis

$$u_1 \otimes v_1,\ u_1 \otimes v_2,\ u_1 \otimes v_3,\ u_2 \otimes v_1,\ u_2 \otimes v_2,\ u_2 \otimes v_3 \in U \otimes V \tag{9.9}$$

defines a homomorphism $\mathrm{SL}(2) \times \mathrm{SO}(3) \to \mathrm{Sp}(6) \subseteq \mathrm{SL}(6)$, where the symplectic group is defined with respect to the skew-symmetric matrix $\mathrm{antidiag}(1, -1, 1, -1, 1, -1)$. Now one checks again that the first factor is mapped to the product

$$J(\gamma_1^\vee)J(\gamma_2^\vee) = J(\varepsilon_1 + \varepsilon_3)J(2\varepsilon_2)$$

(with ε_i being weights of $\mathrm{Sp}(6)$) while the second factor goes to $H_\delta \subseteq \mathrm{GL}(3) \subseteq \mathrm{Sp}(6)$.

This finishes the proof of the assertion that for every σ there is φ_σ such that $\varphi_\sigma(F_\sigma)$ commutes with $L_{\mathcal{S}}^\wedge$. But, as we showed in Theorem 7.10 ii), any family of adapted homomorphisms $(\varphi_\sigma)_{\sigma\in\Sigma}$ can be extended to a unique adapted homomorphism φ. Then $L_{\mathcal{S}}^\wedge$ commutes with all subgroups $\varphi(F_\sigma)$ and therefore with $\varphi(G_{\mathcal{S}}^\vee)$. □

Definition 9.8. A homomorphism satisfying the assertion of Theorem 9.7 will be called *very adapted.*

It is easy to see from the proof that all other very adapted homomorphisms are of the form $\mathrm{Ad}(t) \circ \varphi$, where $t \in T_{\mathrm{ad}}^\vee$ with $\gamma_1(t) = \gamma_2(t)$ for all roots $\sigma = \gamma_1 + \gamma_2$ of type $\mathsf{D}_{n\geq 3}$ or B_3.

Corollary 9.9. *Let $\varphi\colon G_{\mathcal{S}}^\vee \to G^\vee$ be a very adapted homomorphism. Then the map*

$$G_{\mathcal{S}}^\vee \times^{Z(G_{\mathcal{S}}^\vee)} L_{\mathcal{S}}^\wedge \to G^\vee\colon\ [g, l] \mapsto \varphi(g)l \tag{9.10}$$

is an injective homomorphism where $Z(G_{\mathcal{S}}^\vee)$ is the center of $G_{\mathcal{S}}^\vee$ (with character group $\Xi^\vee/\mathbb{Z}\Sigma^\vee$).

Proof. Follows from Theorem 9.7 and Corollary 4.6 i). □

In [SV12], Sakellaridis and Venkatesh conjecture that the image of the dual group is centralized by the so-called *principal* $\mathrm{SL}(2)$ of $L_{\mathcal{S}}^\vee$. To establish this, recall that Φ^p is the root system of $L_{\mathcal{S}}$ and that $\rho^p \in \Lambda$ is the half-sum of positive roots. Since $2\rho^p \in \Lambda$ we may regard it as a 1-parameter subgroup $\mathbf{G}_m \to T^\vee$. It is well-known that there is a homomorphism $\psi\colon \mathrm{SL}(2) \to L_{\mathcal{S}}^\vee$, called the principal $\mathrm{SL}(2)$, such that $\mathrm{res}_{\mathbf{G}_m} \psi = 2\rho^p$. This homomorphism is unique up to composition with $\mathrm{Ad}(t)$, $t \in T^\vee$. It will be normalized by requiring that ψ maps $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2)$ to $e_L := \sum_{\delta\in S^p} e_{\delta^\vee}$.

Proposition 9.10. *Let $\psi\colon \mathrm{SL}(2) \to L_{\mathcal{S}}^\vee$ be the principal $\mathrm{SL}(2)$ and let $\varphi\colon G_{\mathcal{S}}^\vee \to G^\vee$ be adapted. Then*

$$G_{\mathcal{S}}^\vee \times \mathrm{SL}(2) \to G^\vee\colon (g, l) \mapsto \varphi(g)\psi(l) \tag{9.11}$$

is a group homomorphism if and only if φ is very adapted.

Proof. Clearly both $2\rho^p$ and e_L are $W_{\mathcal{S}}$-invariant which implies that ψ factors through $L_{\mathcal{S}}^\wedge$. Hence, if φ is very adapted, the assertion holds by definition.

Conversely, assume that $\varphi(G_{\mathcal{S}}^\vee)$ and $\psi(\mathrm{SL}(2))$ commute with each other. We have to show that then $\varphi(G_{\mathcal{S}}^\vee)$ commutes with $L_{\mathcal{S}}^\wedge$. Since $\varphi(G_{\mathcal{S}}^\vee)$ centralizes $T^{\Sigma^\vee} = T^\vee \cap L_{\mathcal{S}}^\wedge$ it suffices to show that it centralizes the semisimple part $L_0 := (L_{\mathcal{S}}^\wedge)'$, as well. From the construction of $L_{\mathcal{S}}^\wedge$ it follows that the simple root vectors of L_0 are $W_{\mathcal{S}}$-orbit sums. Their sum is therefore the sum of all simple root vectors of $L_{\mathcal{S}}$. This implies that ψ is also a principal $\mathrm{SL}(2)$ with respect to L_0.

Now let β be a simple root of L_0. Then there is $n \geq 1$ and a coweight $\omega^\vee\colon \mathbf{G}_m \to T^\vee \cap L_0$ which is n times a fundamental coweight, i.e., with $\langle \beta' | \omega_\beta \rangle = n\delta_{\beta\,\beta'}$ for all simple roots β' of L_0. Let $e := \sum_\beta e_\beta$. Then $\varphi(G_{\mathcal{S}}^\vee)$ will centralize $t^{-n}\omega^\vee(t)e$ for all $t \in \mathbf{G}_m$

and therefore its limit for $t \to 0$ which is e_β. The same argument works for the negative simple root vectors e_β. Hence $\varphi(G^\vee_\mathcal{S})$ centralizes the Lie algebra of L_0 and therefore L_0, itself. □

Remark 9.11. Let $\psi\colon \mathrm{SL}(2) \to L^\vee_\mathcal{S}$ be any homomorphism with $\psi(\mathbf{G}_m) \subseteq T^\vee$ and let $\rho \in \Lambda$ be the corresponding 1-parameter subgroup. Then ψ is uniquely determined by the numbers $(\langle\rho|\delta^\vee\rangle)_{\delta\in S^p}$, the so-called Dynkin characteristic of ψ. The characteristic of the principal SL(2) is 2 for all δ. Now clearly the same argument works for any ψ whose characteristic is $W_\mathcal{S}$-invariant.

Next we address the problem of when $L^\wedge_\mathcal{S}$ is the full centralizer of $G^\vee_\mathcal{S}$ in $G^\vee$. For this we need the non-degeneracy condition (cf. Definition 9.1).

Theorem 9.12. *Let $\mathcal{S} = (\Xi, \Sigma, S^p)$ be a non-degenerate weak spherical datum and let $\varphi\colon G^\vee_\mathcal{S} \to G^\vee$ be very adapted. Then:*

i) *The centralizer of $\varphi(A^\vee)$ in $G^\vee$ is $L^\vee_\mathcal{S}$.*
ii) *The centralizer of $\varphi(G^\vee_\mathcal{S})$ in $G^\vee$ is $L^\wedge_\mathcal{S}$.*

Proof. Part i) is more or less the definition of non-degeneracy. Now let $C \subseteq G^\vee$ be the centralizer of $\varphi(G^\vee_\mathcal{S})$. Then $C \subseteq L^\vee_\mathcal{S}$ by i). Let $\sigma \in \Sigma$ and let $\tilde{s}_\sigma$ be any lift of $s_\sigma \in W_\mathcal{S}$ to $G^\vee_\mathcal{S}$. The image $\tilde{n}_\sigma := \varphi(\tilde{s}_\sigma)$ lies diagonally in the product of all subgroups $G^\vee(\gamma^\vee)$ with $\gamma^\vee \in \sigma^\wedge$. This implies that $\tilde{n}_\sigma$ is in fact also a lift of n_σ in $G^\vee$. Therefore, $\mathrm{Ad}\,\tilde{n}_\sigma$ permutes S^p and hence normalizes $L^\vee_\mathcal{S}$. On the other hand, $\mathrm{Ad}\,\tilde{n}_\sigma$ centralizes $L^\wedge_\mathcal{S}$. In particular, it centralizes all orbit sums $\sum_{\bar\delta\in W_\mathcal{S}\delta} e_{\bar\delta^\vee}$ with $\delta \in S^p$. This shows that $\mathrm{Ad}\,\tilde{n}_\sigma$ acts in fact as a graph automorphism on $L^\vee_\mathcal{S}$. Applying this to all $\sigma \in \Sigma$ we see that $C \subseteq (L^\vee_\mathcal{S})^{W_\mathcal{S}}$. Finally, $C = L^\wedge_\mathcal{S}$ follows from the fact that the centralizer of $\varphi(G^\vee_\mathcal{S})$ in $T^\vee$ is $T^{\Sigma^\vee}$. □

10. L-groups

If the base field k is not algebraically closed, then it is not really the dual group $G^\vee$ itself but its semidirect product ${}^{\mathsf{L}}G = G^\vee \rtimes \mathrm{Gal}(\bar{k}|k)$ with the Galois group, the so-called *L-group of G*, which is of representation theoretic significance. See, e.g., Borel's introduction [Bor79].

There is evidence that also a spherical variety X should have an L-group ${}^{\mathsf{L}}G_X$ attached to it. We will not wager a precise definition but we are going to give some constraints. In particular, the existence of equivariant (very) adapted homomorphisms determines the Galois action on $G^\vee_X$ to a large extent.

Let, more generally, $\mathcal{S}$ be a weak spherical datum. Let, moreover, E be an abstract group acting on $\mathcal{R}$ and leaving $\mathcal{S}$ invariant. Clearly, the main example is furnished by a G-variety X which is defined over k and E is the absolute Galois group. Then E acts on the root datum $\mathcal{R} = (\Lambda, S, \Lambda^\vee, S^\vee)$ by the so-called $*$-action. Moreover, it can be shown that then E leaves the weak spherical datum of X invariant (see [KK16, 9.2 i) and paragraphs before 10.5] for details).

We start with a general discussion of E-actions on a connected reductive group G where we assume E to fix B and T. Then E will act on the based root system $\mathcal{R} = (\Lambda, S, \Lambda^\vee, S^\vee)$. Conversely, assume that an E-action on $\mathcal{R}$ is given. If $(e_\alpha)_{\alpha\in S}$ is a pinning then this action lifts to a unique E-action on G preserving this pinning, i.e., with $u(e_\alpha) = e_{u\alpha}$ for all $u \in E$. An action of this type will be called *standard*.

Any two automorphism of G inducing the same automorphism of $\mathcal{R}$ differ by an automorphism of the form $\mathrm{Ad}(t)$ where t is a unique element of the adjoint torus $T_{\mathrm{ad}} := T/Z(G)$. Thus, isomorphism classes of E-actions on G which are compatible with the E-action on $\mathcal{R}$ are parameterized by the cohomology group $H^1(E, T_{\mathrm{ad}})$. Since S is a $\mathbb{Z}$-basis of $\Xi(T_{\mathrm{ad}})$, the action of E on $\Xi(T_{\mathrm{ad}})$ is a permutation representation. Hence

$$H^1(E,T_{\mathrm{ad}}) = \bigoplus_{\alpha\in S/E} \Xi(E_\alpha) \tag{10.1}$$

where $\alpha \in S$ runs through a set of representatives of the E-orbits. This means that an E-action on G is determined by a system $(\chi_\alpha)_{\alpha\in S/E}$ of characters χ_α of E_α in such a way that

$$u(e_\alpha) = \chi_\alpha(u)e_\alpha \quad \text{for } u \in E \text{ and } \alpha \in S \text{ with } u\alpha = \alpha. \tag{10.2}$$

The E-action is called *standard in* α if χ_α is trivial. Clearly, the action is standard if and only it is standard in all α.

The dual group $G^\vee$ comes with a pinning which we use to equip it with a standard E-action. Let, moreover, $\mathcal{S} = (\Xi, \Sigma, S^p)$ be an E-invariant weak spherical datum. Since E stabilizes the associated roots $\Sigma^\wedge$ as well, the associated group $G^\wedge_{\mathcal{S}}$ is an E-stable subgroup of $G^\vee$ and therefore carries an induced E-action. This action is given by a system of characters $(\chi^\wedge_\gamma)_{\gamma^\vee\in\Sigma^\wedge/E}$ which we are going to determine.

Lemma 10.1. *Let $\gamma^\vee \in \Sigma^\wedge$ and $u \in E$ with $u\gamma^\vee = \gamma^\vee$. Then*

$$\chi^\wedge_\gamma(u) = \begin{cases} -1 & \text{if } \gamma \in \Sigma \text{ is of type } \mathsf{A}_{2n}\ (n \geq 1) \text{ and } \mathrm{res}_{|\gamma|}\, u \neq \mathrm{id}, \\ 1 & \text{otherwise.} \end{cases} \tag{10.3}$$

Proof. Assume first that u acts as identity on $|\gamma|$. Then $u(e_{\alpha^\vee}) = e_{\alpha^\vee}$ for all $\alpha \in |\gamma|$. Since $e_{\gamma^\vee}$ appears in the subalgebra of $\mathfrak{g}^\vee$ generated by $(e_{\alpha^\vee})_{\alpha\in|\gamma|}$ we also have $u(e_{\gamma^\vee}) = e_{\gamma^\vee}$ and therefore $\chi^\wedge_\gamma(u) = 1$.

Now assume $\mathrm{res}_{|\gamma|}\, u \neq \mathrm{id}$. If $\gamma^\vee \in \sigma^\wedge$, then u fixes σ and acts non-trivially on $|\sigma|$. Thus σ must be of type A_n with $n \geq 2$ or D_n with $n \geq 2$. The latter possibility is excluded, since otherwise u would not fix γ.

Thus, σ is of type A_n, $n \geq 2$. We settle this case by an explicit computation. The support $|\sigma|$ corresponds to a subalgebra of $\mathfrak{g}^\vee$ which is isomorphic to $\mathfrak{sl}(N)$ with $N = n+1$. For $i \neq j$ let $E_{ij} \in \mathfrak{sl}(N)$ be the corresponding elementary matrix. One checks that the standard action of u on $\mathfrak{sl}(N)$ is $u(A) = -JA^tJ^{-1}$, where $J = \mathrm{antidiag}(1,-1,1,-1,\ldots)$. This implies

$$u(E_{ij}) = (-1)^{i+j-1}E_{N+1-j,\,N+1-i}. \tag{10.4}$$

Now the root space for γ is spanned by $E_{1\,N}$. Thus, the assertion follows from $u(E_{1\,N}) = (-1)^{n+1}E_{1\,N}$. ☐

Remark 10.2. The lemma shows that the E-action on $G^\wedge_{\mathcal{S}}$ may be non-standard. Nevertheless, this phenomenon seems to be quite rare. The tables in [BP15] show that if $\mathcal{S}$ is induced by a spherical variety $X = G/H$ with G simple and H reductive, then up to isogeny there are only two series:

$$X = \mathrm{SL}(2n+1)/S(\mathrm{GL}(m)\,\mathrm{GL}(2n+1-m)) \quad \text{with } 1 \leq m \leq n$$

and

$$X = \mathrm{SL}(2n+1)/\mathbf{G}_m\,\mathrm{Sp}(2n) \quad \text{with } n \geq 1.$$

Next, we treat the dual group. There is a slight difference in that $G^\vee_{\mathcal{S}}$ is defined abstractly by its root system and not as a subgroup of $G^\vee$. So we have to formulate the result a bit differently:

Lemma 10.3. *Let E act on $G^\vee_\mathcal{S}$ by means of a system of characters $(\chi^\vee_\sigma)_{\sigma\in\Sigma/E}$. Then there exists an adapted E-equivariant homomorphism $G^\vee_\mathcal{S}\to G^\vee$ if and only if for all $\sigma\in\Sigma$ and $u\in E$ with $u\sigma=\sigma$:*

$$\chi^\vee_\sigma(u)=\begin{cases}-1 & \text{if } \sigma\in\Sigma \text{ is of type } \mathsf{A}_{2n}\ (n\ge 1) \text{ and } \operatorname{res}_{|\sigma|}u\ne\operatorname{id},\\ \pm1 & \text{if } \sigma\in\Sigma \text{ is of type } \mathsf{D}_n\ (n\ge 2) \text{ and } \operatorname{res}_{|\sigma|}u\ne\operatorname{id},\\ 1 & \text{otherwise.}\end{cases}\tag{10.5}$$

Proof. Because of (10.3) one can choose a pinning $e_{\gamma^\vee}$ of $G^\wedge_\mathcal{S}$ such that $u(e_{\gamma^\vee})=e_{u\gamma^\vee}$ whenever $\gamma\in\Sigma^\wedge$ is not of type $\mathsf{A}_{2n\ge2}$. Let $\varphi_0\colon G^\vee_\mathcal{S}\to G^\wedge_\mathcal{S}$ be the adapted homomorphism which is obtained by folding. This induces a pinning $e^\vee_\sigma$ of $G^\vee_\mathcal{S}$. Observe that φ_0 is E-equivariant.

Let $\sigma\in\Sigma$. If $\sigma\in\Phi$, then $\varphi_0\colon\mathfrak{g}^\vee_{\sigma^\vee}\to\mathfrak{g}^\wedge_{\sigma^\vee}$ is an isomorphism and we have to define $\chi^\vee_\sigma=\chi^\wedge_\sigma$. Thus assume $\sigma=\gamma_1+\gamma_2\notin\Phi$. Then φ_0 maps $\mathfrak{g}^\vee_{\sigma^\vee}$ into $U:=\mathfrak{g}^\wedge_{\gamma_1^\vee}\oplus\mathfrak{g}^\wedge_{\gamma_2^\vee}$. If $\operatorname{res}_{|\sigma|}u=\operatorname{id}$, then u acts trivially on U, forcing $\chi^\vee_\sigma(u)=1$. Otherwise, u interchanges the two pinning elements $e_{\gamma_i^\vee}$. Then U contains two u-stable 1-dimensional subspaces which are spanned by $e_\pm:=e_{\gamma_1^\vee}\pm e_{\gamma_2^\vee}$. Since $\varphi(\mathfrak{g}_{\sigma^\vee})$ is one of them, we see that $\chi^\vee_\sigma=\pm1$. On the other side, clearly for any choice of $e_\pm$ there is an E-equivariant adapted φ with $\varphi(e_{\sigma^\vee})=e_\pm$. ☐

Next, we study compatibility with the principal SL(2) of $L^\vee_\mathcal{S}$. The action of E on $L^\vee_\mathcal{S}$ is clearly standard. Since the $W_\mathcal{S}$-action on $L^\vee_\mathcal{S}$ is defined to be standard, we see that E acts on $L^\wedge_\mathcal{S}$. The fixed point group $L^\mathsf{L}_\mathcal{S}:=(L^\wedge_\mathcal{S})^E$ is of finite index in the fixed point group $(L^\vee_\mathcal{S})^{^\mathsf{L}W_\mathcal{S}}$, where $^\mathsf{L}W_\mathcal{S}:=W_\mathcal{S}\rtimes E$. Note that the principal SL(2) has values in $L^\mathsf{L}_\mathcal{S}$. Recall from Proposition 9.10 that for an adapted φ to commute with a principal SL(2) it necessarily has to be very adapted.

Lemma 10.4. *The following are equivalent:*

i) *There exists a very adapted E-equivariant homomorphism $\varphi\colon G^\vee_\mathcal{S}\to G^\vee$.*
ii) *For all $\sigma\in\Sigma$ and $u\in E$ with $u\sigma=\sigma$:*

$$\chi^\vee_\sigma(u)=\begin{cases}-1 & \text{if } \sigma\in\Sigma \text{ is of type } \mathsf{A}_{2n}\ (n\ge 1)\\ & \quad\text{or } \mathsf{D}_n\ (n\ge 3) \text{ and } \operatorname{res}_{|\sigma|}u\ne\operatorname{id},\\ \pm1 & \text{if } \sigma\in\Sigma \text{ is of type } \mathsf{D}_2 \text{ and } \operatorname{res}_{|\sigma|}u\ne\operatorname{id},\\ 1 & \text{otherwise.}\end{cases}\tag{10.6}$$

Proof. Compatibility with ψ creates no new constraints for $\sigma\in\Sigma\cap\Phi$ or for σ of type D_2 since in that case $\sigma^\wedge$ is orthogonal to S^p. So let $\sigma=\gamma_1+\gamma_2$ be of type $\mathsf{D}_{n\ge3}$ and $u\in E$ with $u\sigma=\sigma$ and $\operatorname{res}_{|\sigma|}\ne\operatorname{id}$. Consider, as in the proof of Theorem 9.7, the subalgebra of $\mathfrak{g}^\vee$ whose simple roots are $\delta_1^\vee$, $\beta^\vee:=\gamma_1^\vee-\delta_1^\vee=\gamma_2^\vee-\delta_2^\vee$, and $\delta_2^\vee$. It is isomorphic to $\mathfrak{sl}(4)$ with basis vectors E_{ij}. Since $\mathfrak{g}^\vee_{\beta^\vee}$ is contained in the subalgebra spanned by $\mathfrak{g}^\vee_{\alpha_1^\vee},\ldots,\mathfrak{g}^\vee_{\alpha_{n-2}^\vee}$, the action of u on $\mathfrak{g}^\vee_{\beta^\vee}$ is trivial. Moreover, the two pinning vectors $e_{\delta_1^\vee}$ and $e_{\delta_2^\vee}$ are interchanged by u. This shows that the action of u on $\mathfrak{sl}(4)$ is standard. The root space $(\mathfrak{g}^\vee_\mathcal{S})_{\sigma^\vee}$ is spanned by a vector of the form $e:=xE_{13}+yE_{24}$. If φ is very adapted, then e should commute with the u-invariant vector $c:=e_{\delta_1^\vee}+e_{\delta_2^\vee}=E_{12}+E_{34}$ which forces $x=y$. But then we have $u(e)=-e$ proving $\chi^\vee_\sigma(u)=-1$. ☐

To determine the "correct" character $\chi^\vee_\sigma$ when σ is of type D_2 (if there is any) one needs input from representation theory. Yiannis Sakellaridis has informed us that the phenomenon of non-standard E-actions is related to the notion of so-called *unstable base change maps*. This connection can be seen as follows: The E-action on $G^\vee_\mathcal{S}$ is determined by an element $c\in H^1(E,A^\vee_{\mathrm{ad}})$, where $A^\vee_{\mathrm{ad}}=A^\vee/Z$ and $Z:=Z(G^\vee_\mathcal{S})$. Let $G^\vee_\mathcal{S}\rtimes_c E$

be the corresponding semidirect product. If c can be lifted to $\tilde{c} \in H^1(E, A^\vee)$, then $(g, u) \mapsto (\tilde{c}(u)g\tilde{c}(u)^{-1}, u)$ defines an isomorphism

$$G^\vee_{\mathcal{S}} \rtimes_0 E \xrightarrow{\sim} G^\vee_{\mathcal{S}} \rtimes_c E, \tag{10.7}$$

where the left hand side denotes the semidirect product with respect to the standard action. The obstruction for the existence of $\tilde{c}$ is the image c_2 of c in $H^2(E, Z)$. It can be killed by extending the group E, e.g., by replacing it with the central extension defined by c_2. Another possibility is the Weil group.

Lemma 10.5. *Let k be a p-adic field. Then W_k, its Weil group, acts on $\mathcal{S}$ via its projection to the Galois group of k. Assume that Z is connected. Then*

$$G^\vee_{\mathcal{S}} \rtimes_0 W_k \xrightarrow{\sim} G^\vee_{\mathcal{S}} \rtimes_c W_k. \tag{10.8}$$

Proof. Indeed, $H^2(W_k, Z) = 0$ by [Kar13]. □

Now the unstable base change map is the composition of 10.8 with an adapted W_k-equivariant homomorphism φ. Thus it is a homomorphism

$$G^\vee_{\mathcal{S}} \rtimes_0 W_k \to G^\vee \rtimes_0 W_k. \tag{10.9}$$

The investigation of distinguished representations for $X = \mathrm{GL}(2, K)/\mathrm{GL}(2, k)$ with $[K : k] = 2$ (see, e.g., Flicker [Fli91]) indicates that the action of E is non-standard in this case. Here $S = \{\alpha, \bar{\alpha}\}$ and Σ contains a single root $\alpha + \bar{\alpha}$ which is of type D_2. If one also assumes that the E-action is compatible with localization, then the correct action of E on $G^\vee_{\mathcal{S}}$ would be given by

$$\chi^\vee_\sigma(u) = \begin{cases} -1 & \text{if } \sigma \in \Sigma \text{ is of type } \mathsf{A}_{2n}\ (n \geq 1) \\ & \quad \text{or } \mathsf{D}_n\ (n \geq 2) \text{ and } \mathrm{res}_{|\sigma|}\, u \neq \mathrm{id}, \\ 1 & \text{otherwise.} \end{cases} \tag{10.10}$$

There is one more piece of evidence for this which is more in line with our setup. Consider $G = \mathrm{SO}(2n)$ with $n \geq 3$ and $H = \mathrm{SO}(2n-1) \subset G$. Then $X = G/H$ has one spherical root which is of type D_n. Now consider the $(n-2)$-nd maximal parabolic subgroup P_{n-2} of $\mathrm{SO}(2n-1)$. Then one can show that $Y = G/P_{n-2}$ is spherical with spherical roots $\sigma_1 = \alpha_1 + \cdots + \alpha_{n-2}$ and $\tau = \alpha_{n-1} + \alpha_n$ (see [Was96]). So τ is of type D_2. Because there is a surjective map $Y \to X$, it is expected that $G^\vee_X$ is a subgroup $G^\vee_Y$. Now consider the outer automorphism u of G. Then the action of u on $G^\vee_X$ is non-standard. An easy calculation shows that $G^\vee_X$ is an u-stable subgroup of $G^\vee_Y$ only if the u-action on $G^\vee_Y$ is non-standard, as well. Thus τ should be non-standard for u.

11. Concluding remarks

In § 5, we mentioned a couple of production procedures for weak spherical data. Here we discuss the way they affect dual groups and centralizers. For this, we fix a weak spherical datum $\mathcal{S} = (\Xi, \Sigma, S^p)$.

• *Change of Ξ:* Let $\Xi_0 \subseteq \Xi$ be any sublattice with $\Sigma \subseteq \Xi_0$. Then $\mathcal{S}_0 := (\Xi_0, \Sigma, S^p)$ is another weak spherical datum. There is a canonical homomorphism $\iota\colon G^\vee_{\mathcal{S}_0} \to G^\vee_{\mathcal{S}}$ with finite kernel such that $\varphi_0 := \varphi \circ \iota$ is (very) adapted for $\mathcal{S}_0$ if φ is (very) adapted for $\mathcal{S}$. Since $L^\wedge_{\mathcal{S}}$ does not depend on Ξ we get the following diagram:

$$\begin{array}{ccccc} G^\vee_{\mathcal{S}_0} & \longrightarrow & G^\vee & \hookleftarrow & L^\wedge_{\mathcal{S}_0} \\ \downarrow & & \| & & \| \\ G^\vee_{\mathcal{S}} & \longrightarrow & G^\vee & \hookleftarrow & L^\wedge_{\mathcal{S}}. \end{array} \tag{11.1}$$

Geometrically, this corresponds to an isogeny $X \to X_0$ of G-varieties.

• *Localization in* Σ*:* This case has been partially dealt with in Theorem 7.10. Let $\Sigma_0 \subseteq \Sigma$ be a subset. Then $\mathcal{S}_0 := (\Xi, \Sigma_0, S^p)$ is a weak spherical datum (a *boundary degeneration* in the parlance of [SV12]). In this case, $G^\vee_{\mathcal{S}_0} \subseteq G^\vee_{\mathcal{S}}$ is a Levi subgroup. Moreover, every adapted φ restricts to an adapted φ_0. For the centralizers we have $L^\wedge_{\mathcal{S}_0} \supseteq L^\wedge_{\mathcal{S}}$:

(11.2)
$$\begin{array}{ccccc} G^\vee_{\mathcal{S}_0} & \longrightarrow & G^\vee & \longleftarrow & L^\wedge_{\mathcal{S}_0} \\ \downarrow & & \| & & \uparrow \\ G^\vee_{\mathcal{S}} & \xrightarrow[\varphi]{} & G^\vee & \longleftarrow & L^\wedge_{\mathcal{S}}. \end{array}$$

Geometrically, this procedure corresponds to replacing a G-variety by one of its "boundary components" in a suitable compactification (see [SV12]).

• *Parabolic induction:* Let $S_0 \subseteq S$ be a subset and let $\mathcal{S}_0 = (\Xi, \Sigma, S^p)$ be a weak spherical datum with respect to (the root subsystem generated by) S_0. Then $\mathcal{S} = \mathcal{S}_0$ is a weak spherical datum also with respect to S and is called a *parabolic induction.* Observe that, conversely, $\mathcal{S}$ is parabolically induced from S_0 if and only if

(11.3)
$$S^p \cup \bigcup_{\sigma \in \Sigma} |\sigma| \subseteq S_0.$$

The subset S_0 corresponds to a Levi subgroup $G^\vee_0 \subseteq G^\vee$ while dual group and centralizer stay the same:

(11.4)
$$\begin{array}{ccccc} G^\vee_{\mathcal{S}_0} & \xrightarrow{\varphi} & G^\vee_0 & \hookleftarrow & L^\wedge_{\mathcal{S}_0} \\ \| & & \downarrow & & \| \\ G^\vee_{\mathcal{S}} & \longrightarrow & G^\vee & \hookleftarrow & L^\wedge_{\mathcal{S}}. \end{array}$$

Geometrically, the parabolic induction is the variety $G \times^{P^-} Y$, where $P^- = LU^-$ is a parabolic opposite to B with Levi part L and Y is an L-variety.

• *Removal of compact factors:* Let $S^p_0 \subseteq S^p$ with $|\sigma| \cap S^p \subseteq S^p_0$ for all $\sigma \in \Sigma$. Then $\mathcal{S}_0 = (\Xi, \Sigma, S^p_0)$ is a weak spherical datum:

(11.5)
$$\begin{array}{ccccc} G^\vee_{\mathcal{S}_0} & \longrightarrow & G^\vee & \hookleftarrow & L^\wedge_{\mathcal{S}_0} \\ \| & & \| & & \downarrow \\ G^\vee_{\mathcal{S}} & \longrightarrow & G^\vee & \hookleftarrow & L^\wedge_{\mathcal{S}}. \end{array}$$

Observe that this process is, as opposed to the previous ones, *not* compatible with principal SL(2)'s. Geometrically, this process leads to a fibration $X \to X_0$ whose fibers are flag varieties.

• *Localization in* S*:* Let $S_0 \subseteq S$ be a subset. Put $\Sigma_0 := \{\sigma \in \Sigma \mid |\sigma| \subseteq S_0\}$ and $S^p_0 := S_0 \cap S^p$. Then $\mathcal{S}_0 = (\Xi, \Sigma_0, S^p_0)$ and $\mathcal{S}_1 = (\Xi, \Sigma_0, S^p)$ are weak spherical data with respect to S_0 and S, respectively. This process is the concatenation of the previous three processes. This turns out to be quite neat on the dual group side but is slightly messy for centralizers:

(11.6)
$$\begin{array}{ccccccc} G^\vee_{\mathcal{S}_0} & \longrightarrow & G^\vee_0 & \hookleftarrow & L^\wedge_{\mathcal{S}_0} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ G^\vee_{\mathcal{S}} & \longrightarrow & G^\vee & \hookleftarrow & L^\wedge_{\mathcal{S}_1} & \hookleftarrow & L^\wedge_{\mathcal{S}}. \end{array}$$

Geometrically, localization in S corresponds to looking at a certain open Białynicki-Birula cell (see e.g. [Kno14a]).

We conclude this paper with a remark on integrality. It is well-known that, due to its combinatorial construction, the Langlands dual group $G^\vee$ is defined and split over

$\mathbb{Z}$. Similarly, the dual group $G^\vee_\mathcal{S}$ and the associated group $G^\wedge_\mathcal{S}$ are also defined and split over $\mathbb{Z}$. There is a slight difficulty with the centralizer $L^\wedge_\mathcal{S}$ due to the appearance of $\mathrm{SO}(3) \subseteq \mathrm{SL}(3)$ which is only well-behaved outside the prime 2. Then the following is easy to verify.

Proposition 11.1. *Let $\mathcal{S} = (\Xi, \Sigma, S^p)$ be a weak spherical datum.*

i) *The associated subgroup $G^\wedge_\mathcal{S} \subseteq G^\vee$ is defined over $\mathbb{Z}$.*
ii) *There exist adapted homomorphisms $\varphi\colon G^\vee_\mathcal{S} \to G^\vee$ which are defined over $\mathbb{Z}$. Moreover, the group $T^\wedge_{\mathrm{ad}}(\mathbb{Z})$ ($\cong \{\pm 1\}^r$ with $r = |\Sigma^\wedge|$) acts simply transitively on these adapted homomorphisms.*
iii) *The subgroup $L^\wedge_\mathcal{S} \subseteq G^\vee$ is defined and smooth over $\mathbb{Z}[\frac{1}{2}]$.*
iv) *There exist very adapted homomorphisms $\varphi\colon G^\vee_\mathcal{S} \to G^\vee$ which are defined over $\mathbb{Z}[\frac{1}{2}]$.*

12. Tables

Table 2. Weak spherical data of rank 2

$S := \lvert\sigma\rvert \cup \lvert\tau\rvert$	σ, τ	$S \setminus S^p$	
Case A. For A_3 see also D_3.			
$\mathsf{A}_n, n \ge l+1 \ge 2$	$\alpha_1 + \ldots + \alpha_l, \alpha_{l+1} + \ldots + \alpha_n$	$\{\alpha_1, \alpha_l, \alpha_{l+1}, \alpha_n\}$	
$\mathsf{A}_n, n \ge 4$	$\underline{\alpha_1 + \alpha_n}, \alpha_2 + \ldots + \alpha_{n-1}$	$\{\alpha_1, \alpha_2, \alpha_{n-1}, \alpha_n\}$	
A_5	$\underline{\alpha_1 + 2\alpha_2 + \alpha_3}, \underline{\alpha_3 + 2\alpha_4 + \alpha_5}$	$\{\alpha_2, \alpha_4\}$	
$\mathsf{A}_2 +$ sf A_2	$\underline{\alpha_1 + \alpha'_1}, \underline{\alpha_2 + \alpha'_2}$	S	
Case B. For B_2 see also C_2.			
$\mathsf{B}_n, n \ge p+1 \ge 2$	$\alpha_1 + \ldots + \alpha_p, \alpha_{p+1} + \ldots + \alpha_n$	$\{\alpha_1, \alpha_p, \alpha_{p+1}, (\alpha_n)\}$	
B_3	$\alpha_1 + \alpha_2, \alpha_2 + \alpha_3$	S	
$S := \lvert\sigma\rvert \cup \lvert\tau\rvert$	σ, τ	$S \setminus S^p$	
B_4	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \underline{\alpha_2 + 2\alpha_3 + 3\alpha_4}$	$\{\alpha_1, \alpha_4\}$	$**$
Case C.			
$\mathsf{C}_n, n \ge 2$	$\alpha_1, \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$	$\{\alpha_1, \alpha_2\}$	$*$
$\mathsf{C}_n, n \ge 3$	$\begin{cases} \alpha_1 + \alpha_2 \\ \alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{n-1} + \alpha_n \end{cases}$	$\{\alpha_1, \alpha_2, \alpha_3\}$	
$\mathsf{C}_n, n \ge 4$	$\begin{cases} \underline{\alpha_1 + 2\alpha_2 + \alpha_3} \\ \alpha_3 + 2\alpha_4 + \ldots + 2\alpha_{n-1} + \alpha_n \end{cases}$	$\{\alpha_2, \alpha_4\}$	
$\mathsf{C}_n, n \ge p+2 \ge 3$	$\begin{cases} \alpha_1 + \ldots + \alpha_p \\ \alpha_{p+1} + 2\alpha_{p+2} + \ldots + 2\alpha_{n-1} + \alpha_n \end{cases}$	$\{\alpha_1, \alpha_p, \alpha_{p+1}, \alpha_{p+2}\}$	
$\mathsf{C}_n + \mathsf{A}_1, n \ge 2$	$\begin{cases} \underline{\alpha_1 + \alpha'_1} \\ \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n \end{cases}$	$\{\alpha_1, \alpha_2, \alpha'_1\}$	$*$
$\mathsf{C}_n, n \ge 2$	$\alpha_1 + \ldots + \alpha_{n-1}, \alpha_n$	$\{\alpha_1, \alpha_{n-1}, \alpha_n\}$	
$\mathsf{C}_n, n \ge 3$	$\underline{\alpha_1 + \alpha_n}, \alpha_2 + \ldots + \alpha_{n-1}$	$\{\alpha_1, \alpha_2, \alpha_{n-1}, \alpha_n\}$	

$\mathsf{C}_2 + \mathsf{C}_2$	$\underline{\alpha_1 + \alpha_1'}, \underline{\alpha_2 + \alpha_2'}$	S	
Case D. For D_3 see also A_3.			
$\mathsf{D}_n, n \geq p+3 \geq 4$	$\begin{cases} \alpha_1 + \ldots + \alpha_p \\ \underline{2\alpha_{p+1} + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n} \end{cases}$	$\{\alpha_1, \alpha_p, \alpha_{p+1}\}$	
$\mathsf{D}_n, n \geq 3$	$\alpha_1 + \ldots + \alpha_{n-2}, \underline{\alpha_{n-1} + \alpha_n}$	$\{\alpha_1, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$	
D_5	$\underline{\alpha_1 + 2\alpha_2 + \alpha_3}, \alpha_3 + \alpha_4 + \alpha_5$	$\{\alpha_2, \alpha_4, \alpha_5\}$	
$\mathsf{D}_n, n \geq 3$	$\begin{cases} \alpha_1 + \ldots + \alpha_{n-2} + \alpha_{n-1} \\ \alpha_1 + \ldots + \alpha_{n-2} + \alpha_n \end{cases}$	$\{\alpha_1, \alpha_{n-1}, \alpha_n\}$	
$S := \lvert\sigma\rvert \cup \lvert\tau\rvert$	σ, τ	$S \setminus S^p$	
Case EFG.			
E_6	$\begin{cases} \underline{2\alpha_1 + 2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5} \\ \underline{2\alpha_6 + 2\alpha_5 + 2\alpha_4 + \alpha_3 + \alpha_2} \end{cases}$	$\{\alpha_1, \alpha_6\}$	
E_6	$\begin{cases} \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\ \underline{2\alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5} \end{cases}$	$\{\alpha_1, \alpha_2, \alpha_6\}$	
F_4	$\alpha_1 + \alpha_2, \alpha_3 + \alpha_4$	S	
F_4	$\alpha_1 + \alpha_2 + \alpha_3, \alpha_4$	$\{\alpha_1, \alpha_3, \alpha_4\}$	
F_4	$\alpha_1 + \alpha_2 + \alpha_3, \alpha_4 + 2\alpha_3 + \alpha_2$	$\{\alpha_1, \alpha_3, \alpha_4\}$	
F_4	$\underline{\alpha_1 + \alpha_4}, \alpha_2 + \alpha_3$	S	
F_4	$\underline{\alpha_1 + 2\alpha_2 + 3\alpha_3}, \alpha_4$	$\{\alpha_3, \alpha_4\}$	
G_2	α_1, α_2	S	
G_2	$\alpha_1, \alpha_1 + \alpha_2$	S	*
$\mathsf{G}_2 + \mathsf{G}_2$	$\underline{\alpha_1 + \alpha_1'}, \underline{\alpha_2 + \alpha_2'}$	S	

TABLE 3. Lie algebras of dual groups for $X = G/H$ spherical, G simple, H reductive (see [Krä79, BP15]).

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}^\vee$	$\mathfrak{g}_X^\wedge$	$\mathfrak{g}_X^\vee$	$\mathfrak{l}_X^\vee$	$\mathfrak{l}_X^\wedge$	
$\mathfrak{sl}(n)$	$\mathfrak{s}(\mathfrak{gl}(m)+\mathfrak{gl}(n-m))$	$\mathfrak{sl}(n)$	$\mathfrak{s}(\mathfrak{gl}(2m)+\mathfrak{t}^{n-2m})$	$\mathfrak{sp}(2m)$	$\mathfrak{s}(\mathfrak{gl}(n-2m)+\mathfrak{t}^{2m})$	$\mathfrak{gl}(n-2m)$	$n \geq 2m \geq 0$
$\mathfrak{sl}(n)$	$\mathfrak{so}(n)$	$\mathfrak{sl}(n)$	$\mathfrak{sl}(n)$	$\mathfrak{sl}(n)$	$\mathfrak{t}^{n-1}$	0	$n \geq 1$
$\mathfrak{sl}(2n)$	$\mathfrak{sp}(2n)$	$\mathfrak{sl}(2n)$	$\mathfrak{s}(\mathfrak{gl}(n)+\mathfrak{gl}(n))$	$\mathfrak{sl}(n)$	$\mathfrak{s}(\mathfrak{gl}(2)^n)$	$\mathfrak{sl}(2)$	$n \geq 1$
$\mathfrak{sl}(2n+1)$	$\mathfrak{sp}(2n)+\mathfrak{t}^1$	$\mathfrak{sl}(2n+1)$	$\mathfrak{s}(\mathfrak{gl}(n+1)+\mathfrak{gl}(n))$	$\mathfrak{sl}(n+1)+\mathfrak{sl}(n)$	$\mathfrak{t}^{2n}$	$\mathfrak{t}^1$	$n \geq 1$
$\mathfrak{sl}(2n+1)$	$\mathfrak{sp}(2n)$	$\mathfrak{sl}(2n+1)$	$\mathfrak{s}(\mathfrak{gl}(n+1)+\mathfrak{gl}(n))$	$\mathfrak{s}(\mathfrak{gl}(n+1)+\mathfrak{gl}(n))$	$\mathfrak{t}^{2n}$	$\mathfrak{t}^1$	$n \geq 1$
$\mathfrak{so}(2n+1)$	$\mathfrak{so}(m)+\mathfrak{so}(2n+1-m)$	$\mathfrak{sp}(2n)$	$\mathfrak{sp}(2m)+\mathfrak{t}^{n-m}$	$\mathfrak{sp}(2m)$	$\mathfrak{sp}(2n-2m)+\mathfrak{t}^m$	$\mathfrak{sp}(2n-2m)$	$n \geq m \geq 0$
$\mathfrak{so}(4n+1)$	$\mathfrak{gl}(2n)$	$\mathfrak{sp}(4n)$	$\mathfrak{sp}(2n)+\mathfrak{sp}(2n)$	$\mathfrak{sp}(2n)+\mathfrak{sp}(2n)$	$\mathfrak{t}^{2n}$	0	$n \geq 0$
$\mathfrak{so}(4n+3)$	$\mathfrak{gl}(2n+1)$	$\mathfrak{sp}(4n+2)$	$\mathfrak{sp}(2n+2)+\mathfrak{sp}(2n)$	$\mathfrak{sp}(2n+2)+\mathfrak{sp}(2n)$	$\mathfrak{t}^{2n+1}$	0	$n \geq 0$
$\mathfrak{so}(7)$	G_2	$\mathfrak{sp}(6)$	$\mathfrak{sl}(2)^2+\mathfrak{t}^1$	$\mathfrak{sl}(2)$	$\mathfrak{gl}(3)$	$\mathfrak{so}(3)$	
$\mathfrak{so}(9)$	$\mathfrak{spin}(7)$	$\mathfrak{sp}(8)$	$\mathfrak{sl}(2)^3+\mathfrak{t}^1$	$\mathfrak{sl}(2)^2$	$\mathfrak{gl}(3)+\mathfrak{t}^1$	$\mathfrak{so}(3)$	
$\mathfrak{sp}(2n)$	$\mathfrak{sp}(2m)+\mathfrak{sp}(2n-2m)$	$\mathfrak{so}(2n+1)$	$\mathfrak{gl}(2m)+\mathfrak{t}^{n-2m}$	$\mathfrak{sp}(2m)$	$\mathfrak{gl}(2)^m+\mathfrak{so}(2n-4m+1)$	$\mathfrak{sl}(2)+\mathfrak{so}(2n-4m+1)$	$n \geq 2m \geq 2$
$\mathfrak{sp}(2n)$	$\mathfrak{gl}(n)$	$\mathfrak{so}(2n+1)$	$\mathfrak{so}(2n+1)$	$\mathfrak{so}(2n+1)$	$\mathfrak{t}^n$	0	$n \geq 0$
$\mathfrak{sp}(2n)$	$\mathfrak{sp}(2n-2)+\mathfrak{t}^1$	$\mathfrak{so}(2n+1)$	$\mathfrak{so}(4)+\mathfrak{t}^{n-2}$	$\mathfrak{so}(4)$	$\mathfrak{so}(2n-3)+\mathfrak{t}^2$	$\mathfrak{so}(2n-3)$	$n \geq 2$
$\mathfrak{so}(2n)$	$\mathfrak{so}(m)+\mathfrak{so}(2n-m)$	$\mathfrak{so}(2n)$	$\mathfrak{so}(2m+2)+\mathfrak{t}^{n-m-1}$	$\mathfrak{so}(2m+1)$	$\mathfrak{so}(2n-2m)+\mathfrak{t}^m$	$\mathfrak{so}(2n-2m-1)$	$n > m \geq 1$
$\mathfrak{so}(2n)$	$\mathfrak{so}(n)+\mathfrak{so}(n)$	$\mathfrak{so}(2n)$	$\mathfrak{so}(2n)$	$\mathfrak{so}(2n)$	$\mathfrak{t}^n$	0	$n \geq 2$
$\mathfrak{so}(4n)$	$\mathfrak{gl}(2n)$	$\mathfrak{so}(4n)$	$\mathfrak{gl}(2n)$	$\mathfrak{sp}(2n)$	$\mathfrak{gl}(2)^n$	$\mathfrak{sl}(2)$	$n \geq 1$
$\mathfrak{so}(4n+2)$	$\mathfrak{gl}(2n+1)$	$\mathfrak{so}(4n+2)$	$\mathfrak{gl}(2n)+\mathfrak{so}(2)$	$\mathfrak{sp}(2n)$	$\mathfrak{gl}(2)^n+\mathfrak{so}(2)$	$\mathfrak{sl}(2)+\mathfrak{so}(2)$	$n \geq 1$
$\mathfrak{so}(8)$	G_2	$\mathfrak{so}(8)$	$\mathfrak{gl}(2)+\mathfrak{so}(4)$	$\mathfrak{sl}(2)+\mathfrak{so}(4)$	$\mathfrak{sl}(2)+\mathfrak{t}^3$	$\mathfrak{sl}(2)$	
$\mathfrak{so}(10)$	$\mathfrak{spin}(7)+\mathfrak{t}^1$	$\mathfrak{so}(10)$	$\mathfrak{gl}(2)+\mathfrak{so}(6)$	$\mathfrak{sl}(2)+\mathfrak{so}(6)$	$\mathfrak{sl}(2)+\mathfrak{t}^4$	$\mathfrak{sl}(2)$	
E_6	F_4	E_6	$\mathfrak{gl}(3)+\mathfrak{gl}(3)$	$\mathfrak{sl}(3)$	$\mathfrak{so}(8)+\mathfrak{t}^2$	G_2	
E_6	$\mathfrak{so}(10)+\mathfrak{t}^1$	E_6	$\mathfrak{sl}(4)+\mathfrak{t}^3$	$\mathfrak{sp}(4)$	$\mathfrak{sl}(4)+\mathfrak{t}^3$	$\mathfrak{sp}(4)+\mathfrak{t}^1$	
E_6	$\mathfrak{so}(10)$	E_6	$\mathfrak{sl}(4)+\mathfrak{t}^3$	$\mathfrak{sp}(4)+\mathfrak{t}^1$	$\mathfrak{sl}(4)+\mathfrak{t}^3$	$\mathfrak{sp}(4)+\mathfrak{t}^1$	
E_6	$\mathfrak{sl}(6)+\mathfrak{sl}(2)$	E_6	E_6	F_4	$\mathfrak{t}^6$	0	
E_6	$\mathfrak{sp}(8)$	E_6	E_6	E_6	$\mathfrak{t}^6$	0	
E_7	$\mathsf{E}_6+\mathfrak{t}^1$	E_7	$\mathfrak{sl}(6)+\mathfrak{t}^2$	$\mathfrak{sp}(6)$	$\mathfrak{so}(8)+\mathfrak{t}^3$	G_2	
E_7	$\mathfrak{so}(12)+\mathfrak{sl}(2)$	E_7	$\mathsf{E}_6+\mathfrak{t}^1$	F_4	$\mathfrak{sl}(2)^3+\mathfrak{t}^4$	$\mathfrak{sl}(2)$	
E_7	$\mathfrak{sl}(8)$	E_7	E_7	E_7	$\mathfrak{t}^7$	0	
E_8	$\mathsf{E}_7+\mathfrak{sl}(2)$	E_8	$\mathsf{E}_6+\mathfrak{t}^2$	F_4	$\mathfrak{so}(8)+\mathfrak{t}^4$	G_2	
E_8	$\mathfrak{so}(16)$	E_8	E_8	E_8	$\mathfrak{t}^8$	0	
F_4	$\mathfrak{so}(9)$	F_4	$\mathfrak{sl}(2)+\mathfrak{t}^3$	$\mathfrak{sl}(2)$	$\mathfrak{sp}(6)+\mathfrak{t}^1$	$\mathfrak{sp}(6)$	
F_4	$\mathfrak{sp}(6)+\mathfrak{sl}(2)$	F_4	F_4	F_4	$\mathfrak{t}^4$	0	
G_2	$\mathfrak{sl}(3)$	G_2	$\mathfrak{gl}(2)$	$\mathfrak{sl}(2)$	$\mathfrak{gl}(2)$	$\mathfrak{sl}(2)$	
G_2	$\mathfrak{sl}(2)+\mathfrak{sl}(2)$	G_2	G_2	G_2	$\mathfrak{t}^2$	0	

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