

FROM STANDARD MONOMIAL THEORY TO SEMI-TORIC DEGENERATIONS VIA NEWTON–OKOUNKOV BODIES

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*Dedicated to Ernest Vinberg
on the occasion of his 80th birthday*

ABSTRACT. The Hodge algebra structures on the homogeneous coordinate rings of Grassmann varieties provide semi-toric degenerations of these varieties. In this paper we construct these semi-toric degenerations using quasi-valuations and triangulations of Newton–Okounkov bodies.

1. INTRODUCTION

The basic idea of this paper is to test out in the simplest (but nontrivial) case — the Grassmann variety — how to combine ideas from standard monomial theory and associated semi-toric degenerations [Chi00, DCEP82, Ses12] together with the theory of Newton–Okounkov bodies [KK12] and its associated toric degenerations [And13].

The study of flat degenerations of partial flag varieties started essentially with the work of Hodge [Hod43]. There are in general two parallel directions in the study of these degenerations: the special fibre is a toric variety or a (reduced) union of toric varieties.

In the first situation, many important developments in representation theory and discrete geometry, such as canonical bases, cluster algebras and the theory of Newton–Okounkov bodies, are applied to provide new insights in constructing different toric degenerations; see [And13, AB04, Cal02, FFL17a, GHKK14, GL96, Kav15], and for details on the (incomplete) history, see for example [FFL17b].

Flat toric degenerations whose special fibres are no longer irreducible but a union of toric varieties are called semi-toric degenerations. The quest for semi-toric degenerations arises naturally for example in case one is looking for degenerations which are compatible with certain prescribed subvarieties: a typical example for such a situation are Schubert varieties in a Grassmann variety (a nice argument why in this example one needs semi-toric degenerations can be found in [Cal02]). Semi-toric degenerations occur naturally in the work of De Concini, Eisenbud and Procesi [DCEP82] on Hodge algebras. In the case of partial flag varieties, such degenerations are constructed by Chirivì [Chi00] using Lakshmibai–Seshadri (LS) algebra structures arising from the study of standard monomial theory of partial flag varieties [Lit98].

We strongly believe that the theory of standard monomials is connected to the theory of Newton–Okounkov bodies via triangulations of the bodies. To make this vague statement more concrete, let us explain the picture we get in the case of the Grassmann variety.

The combinatorial structure connected to standard monomial theory is controlled by a partially ordered set (for short we write just poset). In the case of the Grassmann

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variety $\text{Gr}_{d,n}$, this is the set $I(d, n)$ of subsets of size d of $\{1, \dots, n\}$, with the partial order given by componentwise comparison.

Let $R = \mathbb{C}[\text{Gr}_{d,n}]$ be the homogeneous coordinate ring given by the Plücker embedding $\text{Gr}_{d,n} \hookrightarrow \mathbb{P}(\Lambda^d \mathbb{C}^n)$. For a given maximal chain \mathcal{C} in the poset $I(d, n)$, we define a valuation $\nu_{\mathcal{C}}$ on the field of rational functions $\mathbb{C}(\text{Gr}_{d,n})$, such that the associated Newton–Okounkov body \mathcal{P} is, up to unimodular equivalence, independent of the choice of the chain. In fact, \mathcal{P} is the so-called Gelfand–Tsetlin polytope. Moreover, if one looks just at the values of the standard monomials that have support on the fixed chain \mathcal{C} , this defines a simplex embedded in \mathcal{P} . Indeed, by varying the maximal chains, one gets a triangulation of \mathcal{P} such that the simplexes are in bijection with the maximal chains.

To lift this triangulation up to the level of the Grassmann variety, we pass from the set of valuations $\{\nu_{\mathcal{C}} \mid \mathcal{C} \text{ a maximal chain}\}$ to a quasi-valuation [KM16] ν by taking the minimum of them:

$$\nu: \mathbb{C}(\text{Gr}_{d,n}) \setminus \{0\} \rightarrow \mathbb{Z}^N, \quad h \mapsto \min\{\nu_{\mathcal{C}}(h) \mid \mathcal{C} \text{ is a maximal chain}\}.$$

This quasi-valuation induces a \mathbb{Z}^N -filtration \mathcal{F}_{ν} of R , such that the associated graded algebra $\text{assgrad}_{\mathcal{F}_{\nu}} R$ is the discrete Hodge algebra [DCEP82] associated to the poset $I(d, n)$. In other words, we have recovered the semi-toric degeneration of $\text{Gr}_{d,n}$ into a union of \mathbb{P}^N 's described in [DCEP82].

A geometric interpretation of the results described above is given by associating to each valuation $\nu_{\mathcal{C}}$ a toric degeneration, which is compatible with those Schubert varieties corresponding to the elements of the chain \mathcal{C} . Therefore, by passing from a family of valuations to a quasi-valuation one only gets a semi-toric degeneration, but this has the advantage of being compatible with all Schubert varieties in $\text{Gr}_{d,n}$.

The paper is organised as follows: after recalling basic notions and constructions on distributive (order) lattices and the associated Hibi varieties in § 2 and § 3, we study valuations and quasi-valuations on Hibi varieties in § 4 and § 5. In particular, we construct three different families of quasi-valuations on Hibi varieties and then apply them to construct semi-toric degenerations. The notion of an algebra governed by a lattice is introduced in § 6, and is applied to generalise the results on Hibi varieties to varieties that can be degenerated to Hibi varieties. In § 7 we show that Grassmann varieties fall into this category, and the previous constructions, once applied to these varieties, recover the Hodge algebra degeneration of Grassmann varieties. Relations to Feigin–Fourier–Littelmann–Vinberg polytopes are observed in § 8. In § 9 we discuss questions and further directions of this work.

2. DISTRIBUTIVE LATTICES

Let $(\mathcal{L}, \vee, \wedge)$ be a finite bounded distributive lattice with operations *join* \vee and *meet* \wedge . This structure induces a partial order on \mathcal{L} by:

$$p \leq q \quad \text{if } p \wedge q = p.$$

With this partial order, (\mathcal{L}, \leq) is a poset. For $p, q \in \mathcal{L}$, p is called a *decent* of q if $p < q$ and there exists no element ℓ in \mathcal{L} such that $p < \ell < q$. The unique minimal (resp. maximal) element in \mathcal{L} will be denoted by $\mathbb{0}$ (resp. $\mathbb{1}$).

Linearly ordered subsets in \mathcal{L} are called *chains*. A chain \mathcal{C} is called *maximal* if for any other chain \mathcal{C}' , $\mathcal{C} \subseteq \mathcal{C}'$ implies $\mathcal{C} = \mathcal{C}'$. Let $C(\mathcal{L})$ denote the set of all maximal chains in \mathcal{L} . The *length* $\text{len}(\mathcal{C}) = |\mathcal{C}| - 1$ of a chain \mathcal{C} is the number of steps in the chain. For a systematical introduction to lattice theory, see for example, [Grä11].

An element $m \in \mathcal{L}$ is called *join-irreducible* if $m = \ell_1 \vee \ell_2$ for some $\ell_1, \ell_2 \in \mathcal{L}$ implies $m = \ell_1$ or $m = \ell_2$. Denote by $J(\mathcal{L})$ the set of join-irreducible elements in \mathcal{L} . The partial order on \mathcal{L} induces a partial order on $J(\mathcal{L})$, making the latter a poset.

Let $\mathcal{P}(J(\mathcal{L}))$ be the *power set* of $J(\mathcal{L})$, which is itself a lattice with the union of sets “ \cup ” as join operator and the intersection of sets “ \cap ” as meet operator. A nonempty subset $\mathbf{b} \in \mathcal{P}(J(\mathcal{L}))$ is called an *order ideal* with respect to the induced partial order on $J(\mathcal{L})$ if for all $m, m' \in J(\mathcal{L})$ holds: $m \in \mathbf{b}$ and $m' < m$ implies $m' \in \mathbf{b}$. Let $\mathcal{D}(J(\mathcal{L})) \subset \mathcal{P}(J(\mathcal{L}))$ be the set of subsets consisting of order ideals with respect to the partial order.

Two lattices are called isomorphic, if there exists a bijection between them preserving the join and meet operations. Two distributive lattices are isomorphic if they are isomorphic as lattices. Notice that endowed with the operations \cup and \cap , $(\mathcal{D}(J(\mathcal{L})), \cap, \cup)$ is a distributive lattice. The following theorem can be found in [Grä11, Theorem 107].

Theorem 2.1 (Birkhoff). *The distributive lattices $(\mathcal{D}(J(\mathcal{L})), \cap, \cup)$ and $(\mathcal{L}, \vee, \wedge)$ are isomorphic.*

The isomorphism in Birkhoff’s theorem can be made explicit as follows: for $\ell \in \mathcal{L}$ we define

$$\text{Spec}(\ell) = \{m \in J(\mathcal{L}) \mid m \leq \ell\},$$

and let $\text{maxSpec}(\ell)$ be the set of maximal elements in $\text{Spec}(\ell)$.

The following map provides the isomorphism in the theorem of Birkhoff:

$$\mathcal{L} \rightarrow \mathcal{D}(J(\mathcal{L})), \quad \ell \mapsto \text{Spec}(\ell),$$

whose inverse is given by

$$\mathcal{D}(J(\mathcal{L})) \rightarrow \mathcal{L}, \quad \mathbf{b} \mapsto \bigvee_{m \in \mathbf{b}} m.$$

In the following we often identify the lattice \mathcal{L} with the lattice $\mathcal{D}(J(\mathcal{L}))$. The length of a maximal chain in \mathcal{L} is equal to the cardinality of $J(\mathcal{L}) \setminus \{\mathbb{0}\}$.

An enumeration $J(\mathcal{L}) = \{m_0, m_1, \dots, m_N\}$ of the join-irreducible elements is called an *order preserving enumeration* if $m_i < m_j$ implies $i < j$. Let $E(\mathcal{L})$ be the set of all order preserving enumerations of $J(\mathcal{L})$.

We define a map $\varphi: C(\mathcal{L}) \rightarrow E(\mathcal{L})$ as follows: starting with a maximal chain \mathcal{C} in \mathcal{L} , say

$$\mathcal{C}: \mathbb{0} = c_0 < c_1 < c_2 < \dots < c_N = \mathbb{1},$$

we associate to \mathcal{C} an enumeration of $J(\mathcal{L})$ by letting

$$m_0 = \mathbb{0} \quad \text{and for } i = 1, \dots, N, \quad \text{let } m_i \in \text{Spec}(c_i) \setminus \text{Spec}(c_{i-1})$$

be the unique new element. This defines an order preserving enumeration. Conversely, given an order preserving enumeration $\{m_0 = \mathbb{0}, m_1, \dots, m_N\}$, the associated sequence of elements

$$\mathcal{C}: m_0 < m_1 < (m_1 \vee m_2) < \dots < \bigvee_{1 \leq i \leq j} m_i < \dots < (m_1 \vee m_2 \vee \dots \vee m_N) = \mathbb{1}$$

is a maximal chain in \mathcal{L} .

Another immediate consequence of the isomorphism between \mathcal{L} and $\mathcal{D}(J(\mathcal{L}))$ is the next lemma.

Lemma 2.2. *The map $\varphi: C(\mathcal{L}) \rightarrow E(\mathcal{L})$ is a bijection.*

3. THE HIBI VARIETY $\mathcal{X}_{\mathcal{L}}$

As before, let \mathcal{L} be a finite bounded distributive lattice. The associated Hibi variety [Hib87] (or rather its projective version) is the variety $\mathcal{X}_{\mathcal{L}} \subset \mathbb{P}(\mathbb{C}^{|\mathcal{L}|})$ defined as the zero set of the homogeneous ideal

$$I(\mathcal{L}) = \langle X_{\ell_1} X_{\ell_2} - X_{\ell_1 \wedge \ell_2} X_{\ell_1 \vee \ell_2} \mid \ell_1, \ell_2 \in \mathcal{L} \text{ noncomparable} \rangle \subset \mathbb{C}[X_{\ell} \mid \ell \in \mathcal{L}].$$

The homogeneous coordinate ring $R(\mathcal{L}) := \mathbb{C}[X_\ell \mid \ell \in \mathcal{L}]/I(\mathcal{L})$ is called the Hibi ring of the lattice \mathcal{L} . Since $I(\mathcal{L})$ is homogeneous, $R(\mathcal{L})$ is naturally endowed with a grading.

We write x_ℓ for the image of X_ℓ in $R(\mathcal{L})$. It is known that $\mathbb{X}_{\mathcal{L}}$ is an irreducible, projectively normal embedded toric variety, and $R(\mathcal{L})$ is Cohen–Macaulay [Hib87]. In addition, $R(\mathcal{L})$ is an algebra with straightening law in the sense of De Concini, Eisenbud and Procesi [DCEP82]. This implies in particular that $R(\mathcal{L})$ has as \mathbb{C} -vector space a basis given by *standard monomials*, i.e., monomials of the form

$$x_{\ell_1} x_{\ell_2} \dots x_{\ell_r}, \quad \text{where } \ell_1 \geq \ell_2 \geq \dots \geq \ell_r.$$

Denote by $\mathbb{C}(\mathbb{X}_{\mathcal{L}})$ the field of rational functions on $\mathbb{X}_{\mathcal{L}}$, an element in $\mathbb{C}(\mathbb{X}_{\mathcal{L}})$ can always be represented as a quotient f/g , where $f, g \in R(\mathcal{L})$ are homogeneous of the same degree.

In [Hib87] one finds a second description of $R(\mathcal{L})$. Fix an order preserving enumeration $J(\mathcal{L}) = \{m_0 = \mathbb{O}, m_1, \dots, m_N\}$ of the join-irreducible elements and identify \mathcal{L} with the set of order ideals $\mathcal{D}(J(\mathcal{L}))$ (Theorem 2.1). Consider the polynomial ring $S_N = \mathbb{C}[y_0, \dots, y_N]$, and let $M_{J(\mathcal{L})} \subset S_N$ be the subset of monomials

$$M_{J(\mathcal{L})} = \left\{ \prod_{m_i \in \text{Spec}(\ell)} y_i \mid \ell \in \mathcal{L} \right\}.$$

We denote by $A(\mathcal{L})$ the subalgebra of S_N generated by the monomials in $M_{J(\mathcal{L})}$. We endow S_N with a grading by setting $\deg y_0 = 1$ and $\deg y_j = 0$ for all $j \geq 1$. The generators of $A(\mathcal{L})$ are homogeneous of degree 1, so $A(\mathcal{L})$ inherits in a natural way the structure of a graded algebra.

As a graded algebra, $R(\mathcal{L})$ is isomorphic to $A(\mathcal{L})$ [Hib87], the isomorphism is given on the generators by

$$x_\ell \mapsto \prod_{m_i \in \text{Spec}(\ell)} y_i.$$

This isomorphism provides an explicit description of the field of rational functions on $\mathbb{X}_{\mathcal{L}}$:

Lemma 3.1. *Let $\phi: \mathbb{C}(\mathbb{X}_{\mathcal{L}}) \rightarrow \mathbb{C}(y_1, \dots, y_N)$ be the map defined by: for $f, g \in A(\mathcal{L})$ homogeneous of the same degree,*

$$\frac{f}{g} \mapsto \frac{f}{y_0^{\deg f}} \Big/ \frac{g}{y_0^{\deg g}} \in \mathbb{C}(y_1, \dots, y_N).$$

Then ϕ is a field isomorphism.

Proof. An element in $\mathbb{C}(\mathbb{X}_{\mathcal{L}})$ can always be represented as a quotient $\frac{f}{g}$, where $f, g \in R(\mathcal{L}) \simeq A(\mathcal{L})$ are homogeneous of the same degree. In terms of the ring $A(\mathcal{L})$ this means f and g are divisible by the same power of y_0 and hence $\phi(\frac{f}{g}) \in \mathbb{C}(y_1, \dots, y_N)$. If $\frac{f}{g} = \frac{p}{q}$, then

$$\frac{f}{y_0^{\deg f}} \Big/ \frac{g}{y_0^{\deg g}} = \frac{p}{y_0^{\deg p}} \Big/ \frac{q}{y_0^{\deg q}},$$

so the image is independent of the choice of the representative. It follows that ϕ is well-defined and $\phi(\mathbb{C}(\mathbb{X}_{\mathcal{L}})) \subseteq \mathbb{C}(y_1, \dots, y_N)$.

Now one easily checks that ϕ is a ring homomorphism. Since the enumeration is order preserving, we have for all $i \geq 1$: $y_0 \cdots y_{i-1} y_i$ and $y_0 \cdots y_{i-1}$ are homogeneous elements in $M_{J(\mathcal{L})}$ of the same degree, and hence

$$y_i = \phi\left(\frac{y_0 \cdots y_{i-1} y_i}{y_0 \cdots y_{i-1}}\right) \in \phi(\mathbb{C}(\mathbb{X}_{\mathcal{L}})),$$

which implies $\phi(\mathbb{C}(\mathbb{X}_{\mathcal{L}})) = \mathbb{C}(y_1, \dots, y_N)$. □

Let \mathcal{C} be a maximal chain in \mathcal{L} . By Lemma 2.2, this can be identified with an order preserving enumeration of $J(\mathcal{L})$. If $\mathcal{C} = \{0 = c_0 < c_1 < \dots < c_N = 1\}$ and the corresponding enumeration is $\{m_0 = 0, m_1, \dots, m_N\}$, then $c_1 = m_1$, $c_2 = m_1 \vee m_2$, \dots , and the set of monomials associated to the elements in the chain are:

$$M_{\mathcal{C}} := \{x_0 = y_0, x_{c_1} = y_0 y_1, x_{c_2} = y_0 y_1 y_2, \dots, x_{c_N} = y_0 y_1 \dots y_N\}.$$

This description of $M_{\mathcal{C}}$ implies that the subalgebra $\mathbb{C}[\mathcal{C}]$ of $R(\mathcal{L}) \simeq A(\mathcal{L})$ generated by $M_{\mathcal{C}}$ is isomorphic to a polynomial algebra. For $\ell \in \mathcal{L} \setminus \{0\}$, we denote

$$\hat{x}_{\ell} := \prod_{m_i \in \text{Spec}(\ell)} \frac{x_{c_i}}{x_0} \in \mathbb{C}(\mathbb{X}_{\mathcal{L}}),$$

and $\hat{x}_0 = x_0$. We associate to $M_{\mathcal{C}}$ a sequence of rational functions

$$\widehat{M}_{\mathcal{C}} := \{\hat{x}_{c_1} = y_1, \hat{x}_{c_2} = y_1 y_2, \dots, \hat{x}_{c_N} = y_1 \dots y_N\} \subset \mathbb{C}(\mathbb{X}_{\mathcal{L}}).$$

We get as an immediate consequence.

Corollary 3.2. $\mathbb{C}(\mathbb{X}_{\mathcal{L}}) = \mathbb{C}(\hat{x}_{c_1}, \dots, \hat{x}_{c_N})$.

4. \mathbb{Z}^N -VALUED VALUATIONS AND QUASI-VALUATIONS

4.1. Valuations on function fields. Let $X \subset \mathbb{P}^N$ be a projective variety with field of rational functions $\mathbb{C}(X)$ and homogeneous coordinate ring $\mathbb{C}[X]$. The notions of pre-valuations, valuations and quasi-valuations are available in general situations [KK12, KM16]. We will restrict ourselves in this paper to these notions defined on the field $\mathbb{C}(X)$.

By a *lexicographic type total order* “ \geq ” on \mathbb{Z}^N we mean that “ \geq ” is either the lexicographic order or the reverse lexicographic order (see for example Chapter 2, Section 2 in [CLO15]). We fix such a total order “ $>$ ” on \mathbb{Z}^N and write $(\mathbb{Z}^N, >)$ to emphasise that \mathbb{Z}^N is endowed with a fixed total order.

A \mathbb{Z}^N -valued *pre-valuation* on $\mathbb{C}(X)$ is a map $\nu: \mathbb{C}(X) \setminus \{0\} \rightarrow \mathbb{Z}^N$ such that:

- (a) $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ for all nonzero f and g in $\mathbb{C}(X)$,
- (b) $\nu(cf) = \nu(f)$ for all nonzero f and $c \in \mathbb{C}^*$.

A pre-valuation $\nu: \mathbb{C}(X) \setminus \{0\} \rightarrow \mathbb{Z}^N$ is called a *valuation* if it satisfies the following condition (c); it is called a *quasi-valuation* if it satisfies the following condition (c’):

- (c) $\nu(fg) = \nu(f) + \nu(g)$ for all nonzero f and g ;
- (c’) $\nu(fg) \geq \nu(f) + \nu(g)$ for all nonzero f and g .

Let ν be a quasi-valuation. For $\mathbf{v} \in \mathbb{Z}^N$ we define

$$\begin{aligned} \nu_{\geq \mathbf{v}} &:= \{f \in \mathbb{C}(X) \setminus \{0\} \mid \nu(f) \geq \mathbf{v}\} \cup \{0\}, \\ \nu_{> \mathbf{v}} &:= \{f \in \mathbb{C}(X) \setminus \{0\} \mid \nu(f) > \mathbf{v}\} \cup \{0\}. \end{aligned}$$

The associated *leaf* is defined to be the quotient vector space

$$\nu_{\mathbf{v}} := \nu_{\geq \mathbf{v}} / \nu_{> \mathbf{v}}.$$

We say that ν has at most *one-dimensional leaves* if $\dim \nu_{\mathbf{v}} \leq 1$ for all $\mathbf{v} \in \mathbb{Z}^N$.

Restricting ν to the subalgebra $\mathbb{C}[X]$ gives a \mathbb{Z}^N -algebra filtration on $\mathbb{C}[X]$.

Proposition 4.1. *Let $X \subset \mathbb{P}^N$ be a projective variety and let ν_1, \dots, ν_r be \mathbb{Z}^N -valued quasi-valuations on $\mathbb{C}(X)$. Set*

$$\nu: \mathbb{C}(X) \setminus \{0\} \rightarrow \mathbb{Z}^N, \quad h \mapsto \min\{\nu_j(h) \mid j = 1, \dots, r\}.$$

Then ν is a \mathbb{Z}^N -valued quasi-valuation.

Proof. Let $f, g \in \mathbb{C}(X)$, then

$$\begin{aligned} \nu(f + g) &= \min\{\nu_j(f + g) \mid j = 1, \dots, r\} \\ &\geq \min\{\nu_j(f), \nu_j(g) \mid j = 1, \dots, r\} = \min\{\nu(f), \nu(g)\} \end{aligned}$$

for all nonzero f and g in $\mathbb{C}(X)$. Multiplying by a nonzero complex number does not change the value of the quasi-valuations, and

$$\begin{aligned} \nu(fg) &= \min\{\nu_j(fg) \mid j = 1, \dots, r\} \\ &\geq \min\{\nu_j(f) \mid j = 1, \dots, r\} + \min\{\nu_j(g) \mid j = 1, \dots, r\} = \nu(f) + \nu(g). \end{aligned}$$

It follows that ν satisfies the conditions (a), (b), and (c'), and hence ν is a quasi-valuation. \square

4.2. Valuations for Hibi varieties. Fix a maximal chain $\mathcal{C} = \{0 = c_0 < c_1 < \dots < c_N = \mathbb{1}\}$ in \mathcal{L} . Since $\mathbb{C}(\mathbb{X}_{\mathcal{L}}) = \mathbb{C}(\hat{x}_{c_1}, \dots, \hat{x}_{c_N})$ by Corollary 3.2, a given \mathbb{Z}^N -valued valuation on $\mathbb{C}(\mathbb{X}_{\mathcal{L}})$ is completely determined by its values on the generators. So one can attach to \mathcal{C} a matrix $B_{\nu, \mathcal{C}} \in \mathcal{M}_N(\mathbb{Z})$ having as columns the values of ν on the generators (see also [KM16]):

$$(\nu, \mathcal{C}) \mapsto B_{\nu, \mathcal{C}} = (\nu(\hat{x}_{c_1}) \mid \dots \mid \nu(\hat{x}_{c_N})).$$

If ν has at most one-dimensional leaves, then the columns of this matrix are \mathbb{Q} -linearly independent. Let $B \in \mathcal{M}_N(\mathbb{Z})$ be such that $\det B \neq 0$, and let $\mathbf{v}_1, \dots, \mathbf{v}_N$ be the column vectors. We define a valuation on $\mathbb{C}(\mathbb{X}_{\mathcal{L}}) = \mathbb{C}(\hat{x}_{c_1}, \dots, \hat{x}_{c_N})$ as follows.

We use the abbreviation $\hat{x}^{\mathbf{n}}$ for $\hat{x}_{c_1}^{n_1} \dots \hat{x}_{c_N}^{n_N}$. For a monomial $\hat{x}^{\mathbf{n}} \in \mathbb{C}[\hat{x}_{c_1}, \dots, \hat{x}_{c_N}]$, we define

$$\nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{n}}) := \sum_{j=1}^N n_j \mathbf{v}_j \in \mathbb{Z}^N;$$

for polynomials in $\mathbb{C}[\hat{x}_{c_1}, \dots, \hat{x}_{c_N}]$, we define

$$\nu_{B, \mathcal{C}}\left(\sum_{\mathbf{n}} c_{\mathbf{n}} \hat{x}^{\mathbf{n}}\right) := \min\{\nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{n}}) \mid c_{\mathbf{n}} \neq 0\}.$$

By construction, $\nu_{B, \mathcal{C}}$ satisfies the conditions (a) and (b) in Section 4.1. Moreover, $\nu_{B, \mathcal{C}}$ is additive on the product of monomials, i.e.,

$$\nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{n}} \hat{x}^{\mathbf{q}}) = \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{n}+\mathbf{q}}) = \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{n}}) + \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{q}}).$$

Since a lexicographic-type order has been fixed on \mathbb{Z}^N , we have in addition the following property: if $\nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{n}}) > \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{p}})$, then for any $\mathbf{q} \in \mathbb{Z}^N$,

$$(1) \quad \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{n}+\mathbf{q}}) = \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{n}}) + \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{q}}) > \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{p}}) + \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{q}}) = \nu_{B, \mathcal{C}}(\hat{x}^{\mathbf{p}+\mathbf{q}}).$$

It follows that $\nu_{B, \mathcal{C}}(fg) = \nu_{B, \mathcal{C}}(f) + \nu_{B, \mathcal{C}}(g)$ for all $f, g \in \mathbb{C}[\hat{x}_{c_1}, \dots, \hat{x}_{c_N}]$. Extending the map $\nu_{B, \mathcal{C}}$ to the fraction field by setting

$$\nu_{B, \mathcal{C}}\left(\frac{f}{g}\right) = \nu_{B, \mathcal{C}}(f) - \nu_{B, \mathcal{C}}(g),$$

we obtain a valuation on $\mathbb{C}(X)$. The linear independence of the column vectors of B implies that $\nu_{B, \mathcal{C}}$ is a valuation with at most one-dimensional leaves.

5. EXAMPLES FOR QUASI-VALUATIONS ON HIBI VARIETIES

We provide in this section some examples of quasi-valuations on Hibi varieties.

As before, let \mathcal{L} be a finite bounded distributive lattice. By the *height* $\text{ht}(\ell)$ of an element $\ell \in \mathcal{L}$ we mean the length of a chain joining ℓ with the unique minimal element.

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ (resp. $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_N\}$) be the standard basis of \mathbb{Z}^N (resp. \mathbb{Z}^{N+1}). For this section we fix as a total order on \mathbb{Z}^N the *reverse lexicographic order*. On \mathbb{Z}^{N+1} we fix the *graded reverse lexicographic order*, where the degree is provided by the coefficient of \mathbf{e}_0 .

5.1. The support quasi-valuation. We fix a maximal chain $\mathcal{C} = \{0 = c_0 < c_1 < \dots < c_N = 1\}$ in \mathcal{L} , in order to identify $\mathbb{C}(\mathbb{X}_{\mathcal{L}})$ with $\mathbb{C}(\hat{x}_{c_1}, \dots, \hat{x}_{c_N})$. Let $J(\mathcal{L}) = \{m_0 = 0, m_1, m_2, \dots, m_N\}$ be the associated order preserving enumeration of the join-irreducible elements. For $\ell \in \mathcal{L}$, let $\text{Spec}(\ell)^* = \text{Spec}(\ell) \setminus \{0\}$. By the arguments in Section 4.2, the map which associates to \hat{x}_{c_j} the vector

$$\sum_{m_i \in \text{Spec}(c_j)^*} \mathbf{e}_i \in \mathbb{Z}^N$$

can be extended to a valuation

$$\nu_{\mathcal{C}, \text{Spec}}: \mathbb{C}(\mathbb{X}_{\mathcal{L}}) \setminus \{0\} \rightarrow \mathbb{Z}^N.$$

Now let $\mathbb{X}_{\mathcal{L}} \subset \mathbb{P}(\mathbb{C}^{|\mathcal{L}|})$ be the embedded Hibi variety and denote by

$$R(\mathcal{L}) = \bigoplus_{i \geq 0} R_i$$

the homogeneous coordinate ring (see §3). We use the valuation $\nu_{\mathcal{C}, \text{Spec}}$ to define the valuation monoid associated to $R(\mathcal{L})$ by

$$\Gamma_{\nu_{\mathcal{C}, \text{Spec}}}(R(\mathcal{L})) = \bigoplus_{i \geq 0} \Gamma_{\nu_{\mathcal{C}, \text{Spec}}}(R_i),$$

where

$$\Gamma_{\nu_{\mathcal{C}, \text{Spec}}}(R_i) = \left\{ i\mathbf{e}_0 + \nu_{\mathcal{C}, \text{Spec}}\left(\frac{h}{x_0^i}\right) \mid h \in R_i \right\}.$$

The associated Newton–Okounkov body is the closure of the convex hull:

$$\text{NO}_{\mathcal{C}}(\mathcal{L}) = \overline{\text{conv}\left(\bigcup_{j \geq 1} \left\{ \frac{1}{j}v \mid v \in \Gamma_{\nu_{\mathcal{C}, \text{Spec}}}(R_j) \right\}\right)}.$$

By Proposition 4.1, we define the quasi-valuation ν_{Spec} as follows:

$$\nu_{\text{Spec}}: \mathbb{C}(\mathbb{X}_{\mathcal{L}}) \setminus \{0\} \rightarrow \mathbb{Z}^N, \quad \nu_{\text{Spec}}(h) := \min\{\nu_{\mathcal{C}, \text{Spec}}(h) \mid \mathcal{C} \in \mathcal{C}(\mathcal{L})\}.$$

Theorem 5.1. *Let $\ell_1, \dots, \ell_k \in \mathcal{L}$ and n_1, \dots, n_k be non-zero natural numbers. The quasi-valuation $\nu_{\text{Spec}}: \mathbb{C}(\mathbb{X}_{\mathcal{L}}) \setminus \{0\} \rightarrow \mathbb{Z}^N$ satisfies the following properties:*

(1) *For any maximal chain \mathcal{C} ,*

$$\nu_{\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) \leq \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}),$$

the equality holds only if $\{\ell_1, \dots, \ell_k\} \subset \mathcal{C}$.

(2) *If the monomial $\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}$ is standard, and \mathcal{C} is a maximal chain containing $\{\ell_1, \dots, \ell_k\}$, then*

$$\nu_{\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) = \sum_{j=1}^k n_j \nu_{\text{Spec}}(\hat{x}_{\ell_j}) = \sum_{j=1}^k n_j \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{\ell_j}).$$

(3) If the monomial $\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}$ is not standard, then

$$\nu_{\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) > \sum_{j=1}^k n_j \nu_{\text{Spec}}(\hat{x}_{\ell_j}).$$

Proof. For a fixed maximal chain $\mathcal{C} = \{0 = c_0 < c_1 < \cdots < c_N = \mathbb{1}\}$ we have by definition:

$$\nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{c_j}) = \mathbf{e}_j + \sum_{i=1}^{j-1} \mathbf{e}_i.$$

Recall that the index j of \mathbf{e}_j is also the height of c_j . If $\ell \notin \mathcal{C}$ and x_ℓ corresponds to $\prod_{m_i \in \text{Spec}(\ell)} y_i = y_{i_1} \cdots y_{i_r}$, where $1 \leq i_1 < \cdots < i_r \leq N$, hence

$$\hat{x}_\ell = \frac{\hat{x}_{c_{i_1}}}{\hat{x}_{c_{i_1-1}}} \cdots \frac{\hat{x}_{c_{i_r}}}{\hat{x}_{c_{i_r-1}}}.$$

This presentation is not unique, there might be cancellations, but the term $\hat{x}_{c_{i_r}}$ shows up in the nominator and not in the denominator in any presentation. It follows that

$$\nu_{\mathcal{C}, \text{Spec}}(\hat{x}_\ell) = \mathbf{e}_{i_r} + \sum_{j=1}^{i_r-1} \lambda_j \mathbf{e}_j,$$

where $\lambda_j \in \{0, 1\}$. Now $\ell = \bigvee_{m \in \text{Spec}(\ell)} m$ and $\text{Spec}(\ell) \subset \{m_1, \dots, m_{i_r}\}$, so it follows that $c_{i_r} = m_1 \vee m_2 \vee \dots \vee m_{i_r}$ is larger or equal to ℓ . By assumption, $\ell \notin \mathcal{C}$, so we have $c_{i_r} > \ell$ and hence $i_r = \text{ht}(c_{i_r}) > \text{ht}(\ell)$. By the definition of $\text{ht}(\ell)$, there exists another maximal chain \mathcal{C}' such that $\nu_{\mathcal{C}', \text{Spec}}(\hat{x}_\ell) < \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_\ell)$.

We prove the statement (1): the first statement holds by definition. If $\{\ell_1, \dots, \ell_k\}$ is not contained in \mathcal{C} , then there exists a smallest s such that $\ell_s \notin \mathcal{C}$. The argument above shows that

$$\nu_{\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) < \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}).$$

To prove the statement (2), notice that $\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}$ is a standard monomial implies that $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k$. We extend it to a maximal chain \mathcal{C} in \mathcal{L} and apply the first part of the theorem.

For the statement (3), $\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}$ is not a standard monomial implies that there is no maximal chain containing all ℓ_1, \dots, ℓ_k , so we have

$$\nu_{\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) = \min \left\{ \sum_{j=1}^k n_j \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{\ell_j}) \mid \mathcal{C} \in C(\mathcal{L}) \right\} > \sum_{j=1}^k n_j \nu_{\text{Spec}}(\hat{x}_{\ell_j}). \quad \square$$

5.2. The maximal support quasi-valuation. For $\ell \in \mathcal{L}$, instead of the entire $\text{Spec}(\ell)$, one can use the maximal elements $\text{maxSpec}(\ell)$ to define a family of valuations and a quasi-valuation. Fix a maximal chain $\mathcal{C} = \{0 = c_0 < c_1 < \cdots < c_N = \mathbb{1}\}$ in \mathcal{L} . Let $J(\mathcal{L}) = \{m_0 = 0, m_1, m_2, \dots, m_N\}$ be the associated order preserving enumeration of the join-irreducible elements. Let $\nu_{\mathcal{C}, \text{maxSpec}}$ be the map associating to \hat{x}_{c_j} for $j = 1, \dots, N$ the vector

$$\sum_{m_i \in \text{maxSpec}(c_j)} \mathbf{e}_i \in \mathbb{Z}^N.$$

By the argument in Section 4.2, it can be extended to a \mathbb{Z}^N -valued valuation

$$\nu_{\mathcal{C}, \text{maxSpec}} : \mathbb{C}(\mathbb{X}_{\mathcal{L}}) \setminus \{0\} \rightarrow \mathbb{Z}^N.$$

By Proposition 4.1, we define the quasi-valuation ν_{maxSpec} as follows:

$$\nu_{\text{maxSpec}} : \mathbb{C}(\mathbb{X}_L) \setminus \{0\} \rightarrow \mathbb{Z}^N, \quad \nu_{\text{maxSpec}}(h) := \min \{ \nu_{\mathcal{C}, \text{maxSpec}}(h) \mid \mathcal{C} \in C(\mathcal{L}) \}.$$

The proof of the following theorem is similar to that of Theorem 5.1.

Theorem 5.2. *Let $\ell_1, \dots, \ell_k \in \mathcal{L}$ and n_1, \dots, n_k be nonzero natural numbers. The quasi-valuation $\nu_{\max\text{Spec}}: \mathbb{C}(\mathcal{X}_{\mathcal{L}}) \setminus \{0\} \rightarrow \mathbb{Z}^N$ satisfies the following properties:*

(1) *For any maximal chain \mathcal{C} ,*

$$\nu_{\max\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) \leq \nu_{\mathcal{C}, \max\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}),$$

the equality holds only if $\{\ell_1, \dots, \ell_k\} \subset \mathcal{C}$.

(2) *If the monomial $\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}$ is standard, and \mathcal{C} is a maximal chain containing $\{\ell_1, \dots, \ell_k\}$, then*

$$\nu_{\max\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) = \sum_{j=1}^k n_j \nu_{\max\text{Spec}}(\hat{x}_{\ell_j}) = \sum_{j=1}^k n_j \nu_{\mathcal{C}, \max\text{Spec}}(\hat{x}_{\ell_j}).$$

(3) *If the monomial $\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}$ is not standard, then*

$$\nu_{\max\text{Spec}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) > \sum_{j=1}^k n_j \nu_{\max\text{Spec}}(\hat{x}_{\ell_j}).$$

5.3. The height quasi-valuation. One can construct another quasi-valuation using the height. Fix a maximal chain $\mathcal{C} = \{0 = c_0 < c_1 < \dots < c_N = \mathbb{1}\}$ in \mathcal{L} and let $\nu_{\mathcal{C}, \text{ht}}$ be the map associating to \hat{x}_{c_j} for $j = 1, \dots, N$ the vector \mathbf{e}_j . By the argument in Section 4.2, it can be extended to a \mathbb{Z}^N -valued valuation

$$\nu_{\mathcal{C}, \text{ht}}: \mathbb{C}(\mathcal{X}_{\mathcal{L}}) \setminus \{0\} \rightarrow \mathbb{Z}^N.$$

By Proposition 4.1, we define the quasi-valuation ν_{ht} as follows:

$$\nu_{\text{ht}}: \mathbb{C}(\mathcal{X}_{\mathcal{L}}) \setminus \{0\} \rightarrow \mathbb{Z}^N, \quad \nu_{\text{ht}}(h) := \min\{\nu_{\mathcal{C}, \text{ht}}(h) \mid \mathcal{C} \in C(\mathcal{L})\}.$$

The proof of the following theorem is similar to that of Theorem 5.1.

Theorem 5.3. *Let $\ell_1, \dots, \ell_k \in \mathcal{L}$ and n_1, \dots, n_k be nonzero natural numbers. The quasi-valuation $\nu_{\text{ht}}: \mathbb{C}(\mathcal{X}_{\mathcal{L}}) \setminus \{0\} \rightarrow \mathbb{Z}^N$ satisfies the following properties:*

(1) *For any maximal chain \mathcal{C} ,*

$$\nu_{\text{ht}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) \leq \nu_{\mathcal{C}, \text{ht}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}),$$

the equality holds only if $\{\ell_1, \dots, \ell_k\} \subset \mathcal{C}$.

(2) *If the monomial $\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}$ is standard, and \mathcal{C} is a maximal chain containing $\{\ell_1, \dots, \ell_k\}$, then*

$$\nu_{\text{ht}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) = \sum_{j=1}^k n_j \nu_{\text{ht}}(\hat{x}_{\ell_j}) = \sum_{j=1}^k n_j \nu_{\mathcal{C}, \text{ht}}(\hat{x}_{\ell_j}).$$

(3) *If the monomial $\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}$ is not standard, then*

$$\nu_{\text{ht}}(\hat{x}_{\ell_1}^{n_1} \cdots \hat{x}_{\ell_k}^{n_k}) > \sum_{j=1}^k n_j \nu_{\text{ht}}(\hat{x}_{\ell_j}).$$

5.4. Applications to semi-toric degenerations. Let ν be one of the quasi-valuations ν_{Spec} , $\nu_{\max\text{Spec}}$ or ν_{ht} defined on $\mathbb{C}(\mathcal{X}_{\mathcal{L}}) \setminus \{0\}$ above.

Restricting ν to the homogeneous coordinate ring $R(\mathcal{L}) = \mathbb{C}[\mathcal{X}_{\mathcal{L}}]$ gives a \mathbb{Z}^N -filtration of algebra on $R(\mathcal{L})$. Let $\text{gr}_{\nu}(R(\mathcal{L}))$ denote the associated graded algebra. The following corollary is a consequence of Theorem 5.1, 5.2 and 5.3.

Corollary 5.4. *The graded algebra $\text{gr}_\nu(R(\mathcal{L}))$ is the algebra generated by X_ℓ with $\ell \in \mathcal{L}$ and the following relations:*

$$\text{if } \ell_1 \text{ and } \ell_2 \text{ are not comparable in } \mathcal{L}, \text{ then } X_{\ell_1}X_{\ell_2} = 0.$$

Moreover, the images of standard monomials in $R(\mathcal{L})$ form a basis of $\text{gr}_\nu(R(\mathcal{L}))$.

Using standard arguments (see for example [DCEP82]), one obtains from this construction a flat degeneration of the projective toric variety $\mathbb{X}_{\mathcal{L}}$ into a union of toric varieties, such that each irreducible component is isomorphic to \mathbb{P}^N .

6. A LIFT TO SOME NON-TORIC CASES

We want to extend the construction of the previous subsections to varieties, which are not necessarily toric varieties and to construct in this way semi-toric degenerations. The following construction is inspired by the theory of Hodge algebras by De Concini, Eisenbud and Procesi [DCEP82]. Let $X \subset \mathbb{P}(V)$ be an embedded projective variety with homogeneous coordinate ring $R = \bigoplus_{i \geq 0} R_i$. Let \mathcal{L} be a finite bounded distributive lattice and let $\psi: \mathcal{L} \rightarrow R_1$ an injective map of sets. We write x_ℓ for the image $\psi(\ell) \in R_1$.

Definition 6.1. We say that R is *governed by \mathcal{L}* , if the set of *standard monomials*,

$$\text{Smon} := \{x_{k_1} \cdots x_{k_r} \mid k_1 \geq \dots \geq k_r \in \mathcal{L}, r \in \mathbb{Z}_{\geq 0}\},$$

forms a vector space basis for R , and if $\ell_1, \ell_2 \in \mathcal{L}$ are not comparable and

$$(2) \quad x_{\ell_1}x_{\ell_2} = \sum_{k_1 \geq k_2} a_{k_1, k_2} x_{k_1}x_{k_2}$$

is the unique expression of $x_{\ell_1}x_{\ell_2}$ as a linear combination of standard monomials, then

(a) $a_{\ell_1 \vee \ell_2, \ell_1 \wedge \ell_2} = 1$;

(b) if for some $(k_1, k_2) \neq (\ell_1 \vee \ell_2, \ell_1 \wedge \ell_2)$, $a_{k_1, k_2} \neq 0$, then for every pair (m_1, m_2) where $m_1 \in \text{maxSpec}(\ell_1 \vee \ell_2)$ and $m_2 \in \text{maxSpec}(\ell_1 \wedge \ell_2)$ such that $m_1 \geq m_2$, one of the following statements holds:

- there exists $h \in \text{maxSpec}(k_1)$ such that $h > m_1$;
- the statement above does not hold, and there exist $h \neq h' \in \text{maxSpec}(k_1)$ such that $h = m_1$ and $h' > m_2$;
- the statements above do not hold, and there exist $h \in \text{maxSpec}(k_1)$, $h' \in \text{maxSpec}(k_2)$ such that $h = m_1$ and $h' \geq m_2$.

Remark 6.2. Compared to the Hodge algebra defined in [DCEP82], some requirements in the notion of an algebra governed by a distributive lattice are stronger, for example: the relations in (2) are quadratic; the leading coefficient $a_{\ell_1 \vee \ell_2, \ell_1 \wedge \ell_2}$ is 1. However, the last two conditions in part (b) of the above definition are not apparently comparable with the conditions in a Hodge algebra. Nevertheless, we expect that if R is an algebra governed by a distributive lattice \mathcal{L} , then R admits a Hodge algebra structure generated by $\psi(\mathcal{L})$.

Fix a maximal chain $\mathcal{C} = \{0 < c_1 < \dots < c_N = 1\}$ in \mathcal{L} and let $J(\mathcal{L}) = \{m_0 = 0, m_1, m_2, \dots, m_N\}$ be the associated enumeration of the join-irreducible elements. We define a map from the set of standard monomials SMon to \mathbb{Z}^{N+1} using the valuation $\nu_{\mathcal{C}, \text{Spec}}$ defined in Section 5.1. Let $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_N\} \subset \mathbb{Z}^{N+1}$ be the canonical basis. We define

$$\mu_{\mathcal{C}, \text{Spec}}: \text{Smon} \rightarrow \mathbb{Z}^{N+1}, \quad x_0^{a_0} x_{c_1}^{a_1} \dots x_{c_N}^{a_N} \mapsto \left(\sum_{i=0}^N a_i \right) \mathbf{e}_0 + \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{c_1}^{a_1} \dots \hat{x}_{c_N}^{a_N}).$$

The coefficient of \mathbf{e}_0 is the total degree of the monomial.

Theorem 6.3. *If R is governed by \mathcal{L} , then the map $\mu_{\mathcal{C},\text{Spec}}$ extends to a valuation $\mu_{\mathcal{C},\text{Spec}}: R \setminus \{0\} \rightarrow \mathbb{Z}^{N+1}$.*

Proof. We begin with two remarks:

(i) There are two orders on $J(\mathcal{L})$:

- the induced partial order $>$ from \mathcal{L} (which is independent of the choice of \mathcal{C});
- choosing a maximal chain \mathcal{C} in \mathcal{L} provides an order preserving enumeration of $J(\mathcal{L})$, by taking the associated reverse lexicographic order we obtain the total order \succ on it.

It is clear that \succ is a refinement of $>$, i.e., for $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 > \ell_2$ implies $\ell_1 \succ \ell_2$.

(ii) Given $\ell_1, \ell_2 \in \mathcal{L}$ and $\mathcal{C} \in C(\mathcal{L})$, for the valuation on the field of rational functions of the Hibi variety we have:

$$\nu_{\mathcal{C},\text{Spec}}(\hat{x}_{\ell_1}\hat{x}_{\ell_2}) = \nu_{\mathcal{C},\text{Spec}}(\hat{x}_{\ell_1 \vee \ell_2}\hat{x}_{\ell_1 \wedge \ell_2}).$$

The map $\mu_{\mathcal{C},\text{Spec}}$ is defined on the linear basis SMon of R , we extend the map to linear combinations by taking the minimum:

$$\mu_{\mathcal{C},\text{Spec}}\left(\sum_{\mathbf{m} \in \text{Smon}} c_{\mathbf{m}}\mathbf{m}\right) = \min\{\mu_{\mathcal{C},\text{Spec}}(\mathbf{m}) \mid c_{\mathbf{m}} \neq 0\}.$$

This defines a pre-valuation on R . The valuation $\nu_{\mathcal{C},\text{Spec}}$ is defined on all monomials, so for two noncomparable elements $\ell_1, \ell_2 \in \mathcal{L}$ (notice that none of them is equal to \emptyset) we set

$$(3) \quad \mu_{\mathcal{C},\text{Spec}}(x_{\ell_1}x_{\ell_2}) = 2\mathbf{e}_0 + \nu_{\mathcal{C},\text{Spec}}(\hat{x}_{\ell_1}\hat{x}_{\ell_2}).$$

We have to show that this convention does not contradict the definition via sums of standard monomials given before:

$$\begin{aligned} \mu_{\mathcal{C},\text{Spec}}(x_{\ell_1}x_{\ell_2}) &= \mu_{\mathcal{C},\text{Spec}}\left(\sum_{k_1 \geq k_2} a_{k_1, k_2}x_{k_1}x_{k_2}\right) \\ &= \min\{\mu_{\mathcal{C},\text{Spec}}(x_{k_1}x_{k_2}) \mid a_{k_1, k_2} \neq 0\} = \min\{2\mathbf{e}_0 + \nu_{\mathcal{C},\text{Spec}}(\hat{x}_{k_1}\hat{x}_{k_2}) \mid a_{k_1, k_2} \neq 0\} \end{aligned}$$

by the formula (2). By the second remark above, it remains to show that

$$\nu_{\mathcal{C},\text{Spec}}(\hat{x}_{k_1}\hat{x}_{k_2}) > \nu_{\mathcal{C},\text{Spec}}(\hat{x}_{\ell_1 \vee \ell_2}\hat{x}_{\ell_1 \wedge \ell_2})$$

for all other terms showing up in (2) with nonzero coefficients.

Let $k_1, k_2 \in \mathcal{L}$ be two different elements such that $a_{k_1, k_2} \neq 0$ in (2) and $(k_1, k_2) \neq (\ell_1 \vee \ell_2, \ell_1 \wedge \ell_2)$.

We first assume that $n_1 \in \max\text{Spec}(\ell_1 \vee \ell_2)$ is in addition the maximal element with respect to the total order \succ , and $n_2 \in \max\text{Spec}(\ell_1 \wedge \ell_2)$ is, furthermore, the maximal element with respect to \succ among those elements n such that $n_1 \succeq n$. If there exists $h \in \max\text{Spec}(k_1)$ such that $h > n_1$, then $h \succ n_1$ and hence

$$\nu_{\mathcal{C},\text{Spec}}(\hat{x}_{k_1}\hat{x}_{k_2}) > \nu_{\mathcal{C},\text{Spec}}(\hat{x}_{\ell_1 \vee \ell_2}\hat{x}_{\ell_1 \wedge \ell_2}).$$

Suppose that this is not the case, then by Definition 6.1, there exists $h_1 \in \max\text{Spec}(k_1)$ and h_2 in either $\max\text{Spec}(k_1)$ (in this case $h_1 \neq h_2$) or $\max\text{Spec}(k_2)$, such that $h_1 = n_1$ and $h_2 \geq n_2$.

If one can find an $n'_1 \in \text{Spec}(\ell_1 \vee \ell_2)$ such that $n'_1 \succ n_2$, then proceed with the pair (n'_1, n'_2) satisfying: $n'_1 \in \max\text{Spec}(\ell_1 \vee \ell_2)$ and $n'_2 \in \max\text{Spec}(\ell_1 \wedge \ell_2)$ such that $n'_1 \succ n'_2$, and $n'_1 \prec n_1$ is maximal with this property, and n'_2 is maximal with respect to “ \succ ” among those n satisfying $n'_1 \geq n$.

If such an n'_1 cannot be found, then for any $m \in \text{Spec}(\ell_1 \vee \ell_2) \setminus \{n_1, n_2\}$, we have $n_2 \succ m$. There exist two possibilities: if $h_2 > n_2$, then $h_2 \succ n_2$ and hence again

$$\nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{k_1} \hat{x}_{k_2}) > \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{\ell_1 \vee \ell_2} \hat{x}_{\ell_1 \wedge \ell_2}).$$

If the equality $h_2 = n_2$ holds, then one proceeds with the next pair (n'_1, n'_2) , where $n'_1 \in \text{maxSpec}(\ell_1 \vee \ell_2)$ and $n'_2 \in \text{maxSpec}(\ell_1 \wedge \ell_2)$ such that $n'_1 > n'_2$, and $n'_1 \prec n_1$ is maximal with this property, and n'_2 is maximal with respect to “ \succ ” among those elements n satisfying $n'_1 \geq n$.

Since the maximal elements completely determine the value of $\nu_{\mathcal{C}, \text{Spec}}$, one obtains inductively that after a finite number of steps, either

$$\nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{k_1} \hat{x}_{k_2}) > \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{\ell_1 \vee \ell_2} \hat{x}_{\ell_1 \wedge \ell_2})$$

or

$$\nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{k_1} \hat{x}_{k_2}) = \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{\ell_1 \vee \ell_2} \hat{x}_{\ell_1 \wedge \ell_2})$$

holds. But the latter only occurs when all the maximal elements of k_1 and $\ell_1 \vee \ell_2$ agree, which can only happen when $k_1 = \ell_1 \vee \ell_2$ and $k_2 = \ell_1 \wedge \ell_2$.

Since every nonstandard monomial can be rewritten in a finite number of steps using (2) into a linear combination of standard monomials, applying (3) in each step shows that if we define

$$\mu_{\mathcal{C}, \text{Spec}}(x_{c_1} \dots x_{c_r}) := r \mathbf{e}_0 + \nu_{\mathcal{C}, \text{Spec}}(\hat{x}_{c_1} \dots \hat{x}_{c_r}),$$

then this coincides with the value of $\mu_{\mathcal{C}, \text{Spec}}$ on the minimal term in the expression of the monomial in terms of standard monomials. This implies that if we have two sums of standard monomials, then the product is a priori no longer a sum of standard monomials, but the value of $\mu_{\mathcal{C}, \text{Spec}}$ on the sum is the value of $\mu_{\mathcal{C}, \text{Spec}}$ on the product of the two minimal summands. It follows that $\mu_{\mathcal{C}, \text{Spec}}$ is a \mathbb{Z}^{N+1} -valued valuation. \square

Now one can proceed as in the case of the Hibi variety: in the same way as in Proposition 4.1, one shows that the map

$$\mu_{\text{Spec}} : R \setminus \{0\} \rightarrow \mathbb{Z}^{N+1}, \quad h \mapsto \min\{\mu_{\mathcal{C}, \text{Spec}}(h) \mid \mathcal{C} \in C(\mathcal{L})\},$$

is a \mathbb{Z}^{N+1} -valued quasi-valuation.

The proof of the following theorem is similar to Theorem 5.1.

Theorem 6.4. *Let $\ell_1, \dots, \ell_k \in \mathcal{L}$ and n_1, \dots, n_k be nonzero natural numbers. The quasi-valuation $\mu_{\text{Spec}} : R \setminus \{0\} \rightarrow \mathbb{Z}^{N+1}$ satisfies the following properties:*

(1) *For any maximal chain \mathcal{C} ,*

$$\mu_{\text{Spec}}(x_{\ell_1}^{n_1} \dots x_{\ell_k}^{n_k}) \leq \mu_{\mathcal{C}, \text{Spec}}(x_{\ell_1}^{n_1} \dots x_{\ell_k}^{n_k}),$$

the equality holds only if $\{\ell_1, \dots, \ell_k\} \subset \mathcal{C}$.

(2) *If the monomial $x_{\ell_1}^{n_1} \dots x_{\ell_k}^{n_k}$ is standard, and \mathcal{C} is a maximal chain containing $\{\ell_1, \dots, \ell_k\}$, then*

$$\mu_{\text{Spec}}(x_{\ell_1}^{n_1} \dots x_{\ell_k}^{n_k}) = \sum_{j=1}^k n_j \mu_{\text{Spec}}(x_{\ell_j}) = \sum_{j=1}^k n_j \mu_{\mathcal{C}, \text{Spec}}(x_{\ell_j}).$$

(3) *If the monomial $x_{\ell_1}^{n_1} \dots x_{\ell_k}^{n_k}$ is not standard, then*

$$\mu_{\text{Spec}}(x_{\ell_1}^{n_1} \dots x_{\ell_k}^{n_k}) > \sum_{j=1}^k n_j \mu_{\text{Spec}}(x_{\ell_j}).$$

By applying Theorem 6.4, a similar result to Corollary 5.4 can be proved for algebras R governed by \mathcal{L} .

The quasi-valuation μ_{Spec} induces a \mathbb{Z}^{N+1} -filtration of algebra on R , we let $\text{gr}_\mu(R)$ denote the associated graded algebra.

Corollary 6.5. *The graded algebra $\text{gr}_\mu(R)$ is the algebra generated by x_ℓ for $\ell \in \mathcal{L}$ and the following relations:*

$$\text{if } \ell_1 \text{ and } \ell_2 \text{ are not comparable in } \mathcal{L}, \text{ then } x_{\ell_1}x_{\ell_2} = 0.$$

Moreover, the images of standard monomials in R form a basis of $\text{gr}_\mu(R)$.

7. APPLICATIONS TO GRASSMANN VARIETIES

The results of the last sections will be applied to study semi-toric degenerations of Grassmann varieties.

7.1. The distributive lattice $I(d, n)$. Let $n \geq 1$ be a positive integer. For $1 \leq d \leq n$ we define

$$I(d, n) := \{\mathbf{I} = [i_1, \dots, i_d] \mid 1 \leq i_1 < \dots < i_d \leq n\}.$$

The *meet* and *join* operations are defined by

$$\begin{aligned} \mathbf{I} \wedge \mathbf{J} &:= [\min\{i_1, j_1\}, \dots, \min\{i_d, j_d\}], \\ \mathbf{I} \vee \mathbf{J} &:= [\max\{i_1, j_1\}, \dots, \max\{i_d, j_d\}]. \end{aligned}$$

These operations make $I(d, n)$ into a finite distributive lattice, the induced partial order on $I(d, n)$ is exactly the following natural order:

$$\mathbf{I} = [i_1, \dots, i_d] \geq \mathbf{J} = [j_1, \dots, j_d] \quad \text{if and only if } i_1 \geq j_1, \dots, i_d \geq j_d.$$

The lattice $I(d, n)$ is bounded with a unique minimal element $\mathbf{0} = [1, 2, \dots, d]$ and a unique maximal element $\mathbf{1} := [n - d + 1, \dots, n]$.

For $\mathcal{L} = I(d, n)$, recall that $J(\mathcal{L})$ is the set of join-irreducible elements in \mathcal{L} . The elements of $J(\mathcal{L})$ can be divided into two families:

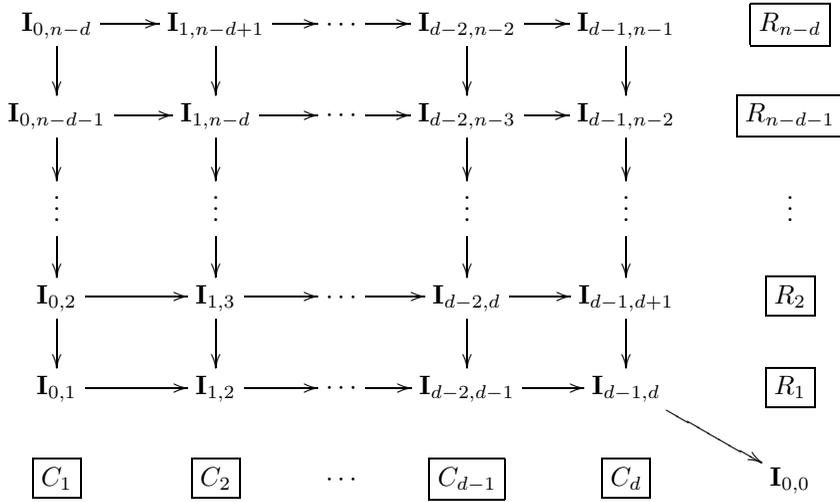
- (1) the consecutive family: for $k = 0, \dots, n - d$,

$$\mathbf{I}_{0,k} = [k + 1, k + 2, \dots, k + d];$$

- (2) the one descent family: for $1 \leq s \leq d - 1$ and $t > s - 1$,

$$\mathbf{I}_{s,t} = [1, \dots, s, t + 1, \dots, t + d - s].$$

The subposet $J(\mathcal{L})$ of \mathcal{L} looks like a block which can be presented in the following way (see for example [BL10]):

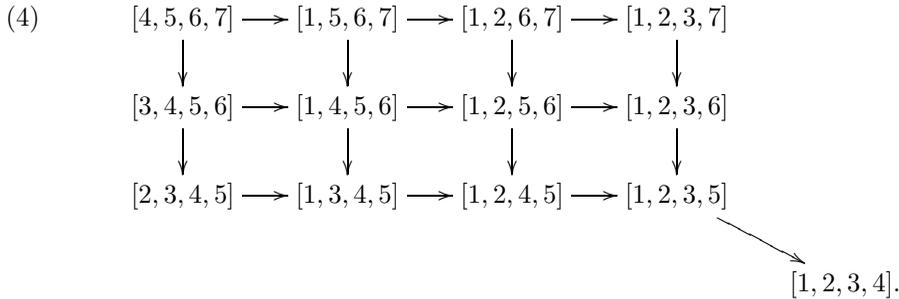


In the diagram, arrows stand for the descents, and $C_1, \dots, C_d, R_1, \dots, R_{n-d}$ are the corresponding columns and rows:

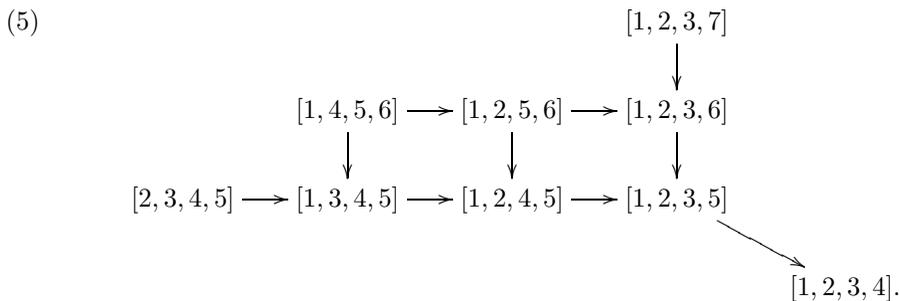
$$C_k = \{\mathbf{I}_{k-1,k+1}, \mathbf{I}_{k-1,k+2}, \dots, \mathbf{I}_{k-1,k+d}\},$$

$$R_k = \{\mathbf{I}_{0,k}, \mathbf{I}_{1,k+1}, \dots, \mathbf{I}_{d-1,k+d-1}\}.$$

Example 7.1. We provide in this example $J(\mathcal{L})$ in the case $d = 4$ and $n = 7$.



We order the join-irreducible elements in $J(\mathcal{L})$ in a rectangle as in (4). An element $\mathbf{I} \in I(d, n)$ corresponds in this picture to a subset of $J(\mathcal{L})$ below a staircase (mounting from left to right), for example, $\mathbf{I} = [2, 4, 5, 7]$ corresponds to



There exists a weight structure on $I(d, n)$. We fix a basis $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ of \mathbb{R}^n . For $1 \leq i \leq j \leq n$, let $\alpha_{i,j} = \varepsilon_{j+1} - \varepsilon_i$ and $\alpha_i = \alpha_{i,i}$. Then $\alpha_1, \dots, \alpha_{n-1}$ is a basis of

$$H = \left\{ \sum_{i=1}^n x_i \varepsilon_i \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0 \right\}.$$

The *weight* of an element $[i_1, \dots, i_d] \in I(d, n)$ is given by:

$$\text{wt}([i_1, \dots, i_d]) = \varepsilon_{i_1} + \varepsilon_{i_2} + \dots + \varepsilon_{i_d}.$$

We define a map $\omega: J(\mathcal{L}) \rightarrow \mathbb{R}^n$ as follows: $\omega(\mathbf{I}_{0,0}) = 0$ and for $(s, t) \neq (0, 0)$,

$$\omega(\mathbf{I}_{s,t}) = \varepsilon_{t+1} - \varepsilon_t.$$

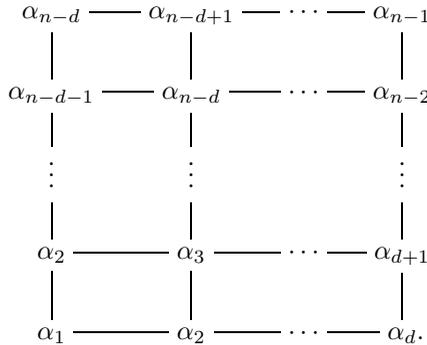
The map ω induces a map $\omega: \mathcal{P}(J(\mathcal{L})) \rightarrow \mathbb{R}^n$: for a subset S of $J(\mathcal{L})$,

$$\omega(S) := \sum_{\mathbf{I} \in S} \omega(\mathbf{I}).$$

We attach to the set $J(\mathcal{L})^* := J(\mathcal{L}) \setminus \{0\}$ the following graph $S_{d,n}$:

- (1) for each $\mathbf{I} \in J(\mathcal{L})^*$, there exists a vertex in $S_{d,n}$ labelled by $\omega(\mathbf{I})$;
- (2) there exists an edge between two vertices if and only if one vertex is the descent of the other.

The graph $S_{d,n}$ can be presented as follows:



For each ordered ideal \mathbf{b} in $J(\mathcal{L})^*$, one can associate the full subgraph $\mathcal{G}_{\mathbf{b}}$ in $S_{d,n}$ containing vertices corresponding to $\mathbf{I} \in \mathbf{b}$.

Lemma 7.2. For $\mathbf{I} \in \mathcal{L}$, $\omega(\text{Spec}(\mathbf{I})) = \text{wt}(\mathbf{I}) - \text{wt}(\mathbf{I}_{0,0}) \in H$.

Proof. Let $\mathbf{I} = [i_1, \dots, i_d]$ with $i_1 < \dots < i_d$.

We claim that for any $t = 1, 2, \dots, d$, $\omega(\text{Spec}(\mathbf{I}) \cap C_t) = \varepsilon_{i_t} - \varepsilon_t$. Indeed,

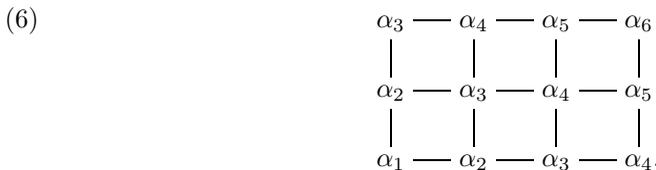
$$\text{Spec}(\mathbf{I}) \cap C_t = \{\mathbf{I}_{t-1,t}, \dots, \mathbf{I}_{t-1,i_t-1}\},$$

hence $\omega(\text{Spec}(\mathbf{I}) \cap C_t) = \varepsilon_{i_t} - \varepsilon_t$.

As $\text{Spec}(\mathbf{I})$ is the disjoint union of $\text{Spec}(\mathbf{I}) \cap C_t$ for $t = 1, \dots, d$, we obtain:

$$\omega(\text{Spec}(\mathbf{I})) = \varepsilon_{i_1} + \dots + \varepsilon_{i_d} - \varepsilon_1 - \dots - \varepsilon_d = \text{wt}(\mathbf{I}) - \text{wt}(\mathbf{I}_{0,0}). \quad \square$$

Example 7.3. We continue Example 7.1 to study $I(4, 7)$. In this case, the graph $S_{4,7}$ looks like:



Let $\mathbf{I} = [2, 4, 5, 7]$. The order ideal $\text{Spec}(\mathbf{I})$ is given in (5). The corresponding subgraph $\mathcal{G}_{\text{Spec}(\mathbf{I})}$ in the graph $S_{4,7}$ looks like:

$$(7) \quad \begin{array}{ccccccc} & & & & & & \alpha_6 \\ & & & & & & | \\ & & & & & & \alpha_5 \\ & & \alpha_3 & \text{---} & \alpha_4 & \text{---} & \\ & & | & & | & & \\ \alpha_1 & \text{---} & \alpha_2 & \text{---} & \alpha_3 & \text{---} & \alpha_4. \end{array}$$

Summing up all roots in the above graph gives:

$$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 = (\varepsilon_2 + \varepsilon_4 + \varepsilon_5 + \varepsilon_7) - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4).$$

7.2. Grassmann varieties. For more details on Grassmann varieties, see for example [LB15].

Let $n \geq 1$ be an integer. For $1 \leq d \leq n$, the Grassmannian $\text{Gr}_{d,n}$ is the set of d -dimensional subspaces in \mathbb{C}^n . The projective variety structure on $\text{Gr}_{d,n}$ is given by the Plücker embedding $\text{Gr}_{d,n} \hookrightarrow \mathbb{P}(\Lambda^d \mathbb{C}^n)$, sending a d -dimensional subspace $\text{span}\{v_1, \dots, v_k\} \subset \mathbb{C}^n$ to the point $[v_1 \wedge \dots \wedge v_k] \in \mathbb{P}(\Lambda^d \mathbb{C}^n)$. The homogeneous coordinate ring $R := \mathbb{C}[\text{Gr}_{d,n}]$ then inherits from the embedding a grading $R = \bigoplus_{i \geq 0} R_i$.

Let e_1, \dots, e_n be the standard basis of \mathbb{C}^n . For $\mathbf{I} = [i_1, \dots, i_d]$ with $1 \leq i_1 < \dots < i_d \leq n$, let $p_{\mathbf{I}} \in (\Lambda^d \mathbb{C}^n)^*$ denote the dual basis of $e_{i_1} \wedge \dots \wedge e_{i_d}$. These $p_{\mathbf{I}}$ are called Plücker coordinates of $\text{Gr}(d, n)$.

Let $1 \leq t \leq n$, $\sigma \in \mathfrak{S}_{d+1}/(\mathfrak{S}_t \times \mathfrak{S}_{d+1-t})$ be a shuffle,

$$\mathbf{I} = [i_1, \dots, i_d] \quad \text{and} \quad \mathbf{J} = [j_1, \dots, j_d].$$

We define

$$\begin{aligned} \mathbf{I}^\sigma &= (\sigma^{-1}(i_1), \dots, \sigma^{-1}(i_t), i_{t+1}, \dots, i_d), \\ \mathbf{J}^\sigma &= (j_1, \dots, j_{t-1}, \sigma^{-1}(j_t), \dots, \sigma^{-1}(j_d)). \end{aligned}$$

The homogeneous ideal $I_{d,n} \subset \mathbb{C}[p_{\mathbf{I}} \mid \mathbf{I} \in I(d, n)]$ generated by the Plücker relations

$$(8) \quad \left\{ \sum_{\sigma \in \mathfrak{S}_{d+1}/(\mathfrak{S}_t \times \mathfrak{S}_{d+1-t})} \text{sign}(\sigma) p_{\mathbf{I}^\sigma} p_{\mathbf{J}^\sigma} \mid \mathbf{I}, \mathbf{J} \in I(d, n), 1 \leq t \leq n \right\}$$

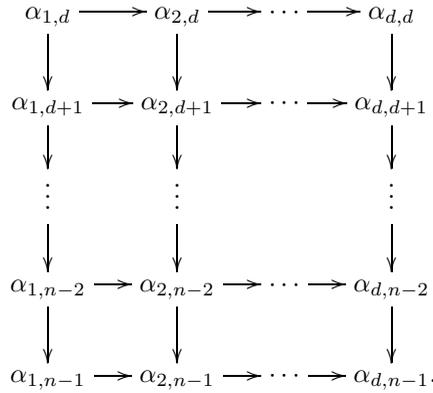
defines the Plücker embedding $\text{Gr}_{d,n} \hookrightarrow \mathbb{P}(\Lambda^d \mathbb{C}^n)$ of the Grassmann variety, i.e. the homogeneous coordinate ring $\mathbb{C}[\text{Gr}_{d,n}]$ is isomorphic to $\mathbb{C}[p_{\mathbf{I}} \mid \mathbf{I} \in I(d, n)]/I_{d,n}$ (see for example [Ses14, Section 1.3]).

Another way of describing the homogeneous coordinate ring is using the Hodge algebra [DCEP82, Hod43]. Let $\psi: I(d, n) \rightarrow R_1$ be the map sending $\mathbf{I} \in I(d, n)$ to the Plücker coordinate $p_{\mathbf{I}}$. It is known that $\mathbb{C}[\text{Gr}_{d,n}]$ has as basis the standard monomials, i.e., monomials of the form $p_{\mathbf{I}_1} p_{\mathbf{I}_2} \cdots p_{\mathbf{I}_r}$, where $\mathbf{I}_1 \geq \mathbf{I}_2 \geq \dots \geq \mathbf{I}_r$. If $\mathbf{I}_1, \mathbf{I}_2$ are not comparable, then the Plücker relations can be used to find an expression of the product

$$(9) \quad p_{\mathbf{I}_1} p_{\mathbf{I}_2} = p_{\mathbf{I}_1 \vee \mathbf{I}_2} p_{\mathbf{I}_1 \wedge \mathbf{I}_2} + \sum_{\mathbf{K}_1 > \mathbf{I}_1 \vee \mathbf{I}_2 > \mathbf{I}_1 \wedge \mathbf{I}_2 > \mathbf{K}_2} a_{\mathbf{K}_1, \mathbf{K}_2} p_{\mathbf{K}_1} p_{\mathbf{K}_2}$$

as a linear combination of standard monomials of degree 2, where the coefficient 1 in the leading term is provided by [GL96, Lemma 7.32].

7.3. **Root poset $R_{d,n}$.** We consider the following root poset $R_{d,n}$ for $\text{Gr}_{d,n}$, realized as SL_n/P_d where P_d is the maximal parabolic subgroup associated to the simple root α_d :



This poset will be used in Proposition 7.6 and §8.

7.4. **Semi-toric degenerations of Grassmann varieties.** We first show that the algebra $R = \mathbb{C}[\text{Gr}_{d,n}]$ is governed by $\mathcal{L} = I(d, n)$.

Proposition 7.4. *The homogeneous coordinate ring $\mathbb{C}[\text{Gr}_{d,n}]$ of the Grassmann variety is governed by the finite bounded distributive lattice $I(d, n)$.*

Proof. We have to show that if $a_{\mathbf{K}_1, \mathbf{K}_2} \neq 0$ in (9), then for every

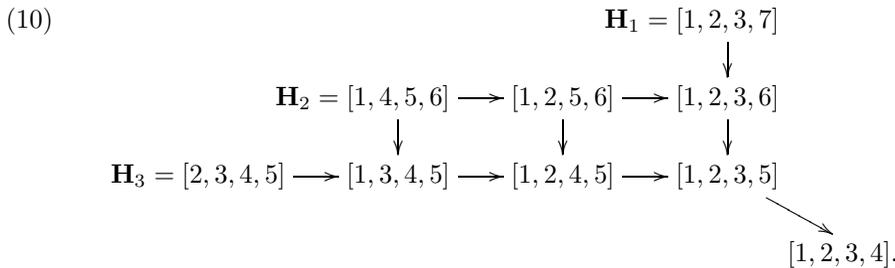
$$\mathbf{M} \in \max\text{Spec}(\mathbf{I}_1 \vee \mathbf{I}_2),$$

- there exists $\mathbf{H} \in \max\text{Spec}(\mathbf{K}_1)$ such that $\mathbf{H} > \mathbf{M}$;
- if one cannot find such an element, then there exist two different elements $\mathbf{H}, \mathbf{H}' \in \max\text{Spec}(\mathbf{K}_1)$ such that $\mathbf{H} = \mathbf{M}$ and $\mathbf{H}' > \mathbf{M}'$ for any maximal element $\mathbf{M}' \in \max\text{Spec}(\mathbf{I}_1 \wedge \mathbf{I}_2)$ which is smaller or equal to \mathbf{M} ;
- if such a pair does not exist, then there exist

$$\mathbf{H} \in \max\text{Spec}(\mathbf{K}_1), \quad \mathbf{H}' \in \max\text{Spec}(\mathbf{K}_2)$$

such that $\mathbf{H} = \mathbf{M}$ and $\mathbf{H}' \geq \mathbf{M}'$ for any $\mathbf{M}' \in \max\text{Spec}(\mathbf{I}_1 \wedge \mathbf{I}_2)$ which is smaller or equal to \mathbf{M} .

First notice that elements in the set $\max\text{Spec}(\mathbf{K}_1)$ correspond exactly to the corners of the staircase in the associated order ideal in $J(I(d, n))$; see (5) for an example. We enumerate the maximal elements (or the corners) from right to left: in the above example the enumeration is given by



Note that the elements in $\max\text{Spec}(\mathbf{I}_1 \vee \mathbf{I}_2)$ correspond to the corners of its associated staircase, which lies below the staircase associated to $\text{Spec}(\mathbf{K}_1)$.

Since $\mathbf{K}_1 > \mathbf{I}_1 \vee \mathbf{I}_2$ one has $\text{Spec}(\mathbf{I}_1 \vee \mathbf{I}_2) \subset \text{Spec}(\mathbf{K}_1)$, so for every

$$\mathbf{M} \in \max\text{Spec}(\mathbf{I}_1 \vee \mathbf{I}_2)$$

there exists $\mathbf{H} \in \max\text{Spec}(\mathbf{K}_1)$ such that $\mathbf{H} \geq \mathbf{M}$. Suppose now one can find only such an \mathbf{H} so that $\mathbf{H} = \mathbf{M}$. Having equality $\mathbf{H} = \mathbf{M}$ means that two staircases share a common corner, so there exists a j such that $\mathbf{H} = \mathbf{M} = \mathbf{H}_j$. Let $\mathbf{M}' \in \max\text{Spec}(\mathbf{I}_1 \wedge \mathbf{I}_2)$ be an element which is smaller or equal to \mathbf{M} . So \mathbf{M}' lies in the staircase below and to the right of \mathbf{H}_j . If \mathbf{H}_{j-1} or \mathbf{H}_{j+1} exists and one of the two is strictly larger than \mathbf{M}' (equality is not possible in this case since $\mathbf{M}' < \mathbf{M}$), then we are done. It remains to consider the case where neither $\mathbf{H}_{j-1} > \mathbf{M}'$ nor $\mathbf{H}_{j+1} > \mathbf{M}'$ (if they exist). In this case \mathbf{M}' lies in the rectangle formed by the columns where $\mathbf{M} = \mathbf{H}_j$ is an entry and the first column to the left of \mathbf{H}_{j-1} (respectively the last row if $j = 1$), and the rows containing $\mathbf{M} = \mathbf{H}_j$, respectively, the row just above \mathbf{H}_{j+1} (respectively the bottom row if \mathbf{H}_{j+1} does not exist). We have to find an element $\mathbf{H}' \in \text{Spec}(\mathbf{K}_2)$ such that $\mathbf{H}' \geq \mathbf{M}'$.

Let $\alpha_i = \omega(\mathbf{M}')$. Since $\mathbf{H} = \mathbf{M}$ and because of the special location of \mathbf{M}' in the rectangle described above, the α_i -component of $\omega(\text{Spec}(\mathbf{K}_1))$ and $\omega(\text{Spec}(\mathbf{I}_1 \vee \mathbf{I}_2))$ coincide. Now for weight reasons, the α_i -component of

$$\omega(\text{Spec}(\mathbf{K}_2)) \quad \text{and} \quad \omega(\text{Spec}(\mathbf{I}_1 \wedge \mathbf{I}_2))$$

also have to coincide. But this implies that the staircase associated to \mathbf{K}_2 has to include \mathbf{M}' . More precisely, \mathbf{M}' has to be an element in the tread of a stairstep. So the next corner to the left of the staircase is a maximal element $\mathbf{H}' \in \mathbf{K}_2$, which is larger or equal to \mathbf{M}' , finishing the proof. \square

Then by results in § 6, for each maximal chain $\mathcal{C} \in C(\mathcal{L})$, we have the valuation $\mu_{\mathcal{C}, \text{Spec}}$ on $\mathbb{C}(\text{Gr}_{d,n}) \setminus \{0\}$. By taking the minimum we obtain a quasi-valuation μ_{Spec} . We are at the point to apply Theorem 6.4 and Corollary 6.5, as well as the construction of Hodge algebras in [DCEP82].

Corollary 7.5. *There exists a flat degeneration of $\text{Gr}_{d,n}$ into a union of toric varieties, such that the defining ideal of the initial scheme is generated by the monomials $p_{\mathbf{I}p_{\mathbf{J}}}$ for all noncomparable pairs (\mathbf{I}, \mathbf{J}) in \mathcal{L} . The initial scheme is a union of projective spaces, one for each maximal chain in \mathcal{L} .*

Moreover, for a maximal chain \mathcal{C} , we can identify the corresponding Newton–Okounkov body.

Proposition 7.6. *For any maximal chain $\mathcal{C} \in C(\mathcal{L})$, the Newton–Okounkov body $\text{NO}_{\mathcal{C}}(\mathcal{L})$ is unimodularly equivalent to the Gelfand–Tsetlin polytope.*

Proof. The Gelfand–Tsetlin polytope associated to $\text{Gr}(d, n)$ is by definition the order polytope associated to the poset $R_{d,n}$. In the case of $R_{d,n}$, lattice points in the order polytopes are vertices, which are the characteristic functions of the order ideals in the poset [Sta86]. Choosing a maximal chain \mathcal{C} identifies $\mathbb{R}^{R_{d,n}}$ with \mathbb{R}^M .

We fix a maximal chain \mathcal{C} and show that $\text{NO}_{\mathcal{C}}(\mathcal{L})$ is the order polytope embedded in \mathbb{R}^M by the above identification. By definition the order polytope coincides with $\Gamma_{\nu_{\mathcal{C}, \text{Spec}}}(R_1)$ hence contained in $\text{NO}_{\mathcal{C}}(\mathcal{L})$, and the other inclusion is guaranteed by the Minkowski property of the Gelfand–Tsetlin polytopes. \square

As a conclusion, we constructed for each maximal chain \mathcal{C} in $I(d, n)$ a valuation $\mu_{\mathcal{C}, \text{Spec}}$ on $\mathbb{C}(\text{Gr}_{d,n})$, such that the associated Newton–Okounkov body is unimodularly equivalent to the Gelfand–Tsetlin polytope.

The subspace spanned by the standard monomials supported on \mathcal{C} is a polynomial algebra. By the definition of the valuation $\nu_{\mathcal{C}, \text{Spec}}$ in Section 5.1, the image of this polynomial subalgebra under $\mu_{\mathcal{C}, \text{Spec}}$ in the Newton–Okounkov body is clearly unimodularly equivalent to the standard simplex $\text{conv}(0, \mathbf{e}_1, \dots, \mathbf{e}_N)$.

By taking the minimum of these valuations $\mu_{\mathcal{C}, \text{Spec}}$ w.r.t. all maximal chains, we pass to a quasi-valuation μ_{Spec} . Using the argument above, as well as Theorem 6.4, one obtains by varying maximal chains in $I(d, n)$ a triangulation of a Newton–Okounkov body such that the simplexes are parametrised by the maximal chains.

8. RELATION TO FEIGIN–FOURIER–LITTELMANN–VINBERG POLYTOPES

8.1. **Feigin–Fourier–Littelmann–Vinberg (FFLV) polytopes.** A *Dyck path* in the poset $R_{d,n}$ is a chain $\mathbf{p} = \{\beta_1, \beta_2, \dots, \beta_s\}$ in $R_{d,n}$ satisfying the following conditions:

- (1) $\beta_1 = \alpha_{k,d}$ for $1 \leq k \leq d$; $\beta_s = \alpha_{d,t}$ for $d \leq t \leq n - 1$;
- (2) if $\beta_k = \alpha_{r,s}$, then β_{k+1} is either $\alpha_{r,s+1}$ or $\alpha_{r+1,s}$.

The set of all Dyck paths will be denoted by $\mathbb{D}_{d,n}$.

Let $(x_{i,j})_{1 \leq i \leq d \leq j \leq n}$ be the coordinates in the real space \mathbb{R}^M with $M = d(n - d)$. The FFLV-polytope $\text{FFLV}_{d,n}$ associated to $\text{Gr}(d, n)$ [FFL11] is the polytope in \mathbb{R}^M defined by the following inequalities: for $1 \leq i \leq d \leq j \leq n$,

$$x_{i,j} \geq 0 \quad \text{for any } \mathbf{p} \in \mathbb{D}_{d,n}, \quad \sum_{\alpha_{i,j} \in \mathbf{p}} x_{i,j} \leq 1.$$

These polytopes come from the study of PBW filtrations on Lie algebras [FFL11], which parametrise monomial bases of irreducible representations of SL_n . These polytopes are identified in [ABS11] as marked chain polytopes, in the case of Grassmann varieties, $\text{FFLV}_{d,n}$ is the chain polytope $\mathcal{C}(R_{d,n})$ of Stanley [Sta86] associated to the poset $R_{d,n}$.

We provide a bijection between the set of order ideal $\mathcal{D}(J(\mathcal{L}))$ in $J(\mathcal{L})$ and lattice points in $\text{FFLV}_{d,n}$ by constructing for each order ideal a path partition of the corresponding subgraph in $S_{d,n}$.

For $\mathbf{I} = [i_1, \dots, i_d] \in \mathcal{L}$, we want to define a path partition of $\text{Spec}(\mathbf{I})$. We first define the pairing map

$$p: \mathcal{L} \setminus \{\mathbf{I}_{0,0}\} \rightarrow \mathcal{L}, \quad \mathbf{I} \mapsto p(\mathbf{I})$$

as follows: Let $\mathbf{I} = [1, \dots, s, i_{s+1}, \dots, i_d]$, where $s \geq 0$ and $i_{s+1} \neq s + 1$. The element $p(\mathbf{I})$ is defined to be $[1, \dots, s, s + 1, i_{s+1}, \dots, i_{d-1}]$.

For any $\mathbf{I} \in \mathcal{L} \setminus \{\mathbf{I}_{0,0}\}$, the pairing map gives a sequence $\mathbf{I}_0, \mathbf{I}_1, \dots, \mathbf{I}_k$, where

- (1) $\mathbf{I}_0 = \mathbf{I}$, $\mathbf{I}_k = \mathbf{I}_{0,0}$ and $\mathbf{I}_{k-1} \neq \mathbf{I}_{0,0}$;
- (2) for any $s = 1, \dots, k$, $\mathbf{I}_s = p(\mathbf{I}_{s-1})$.

The subsets $\text{Spec}(\mathbf{I}_{s-1}) \setminus \text{Spec}(\mathbf{I}_s)$ for $s = 1, \dots, k$ form a partition of $\text{Spec}(\mathbf{I})$. In the graph $S_{d,n}$, each part corresponds to a saturated Dyck path starting from the bottom row and end up with the rightmost column. They give a partition of $\mathcal{G}_{\text{Spec}(\mathbf{I})}$.

For $s = 1, \dots, k$, we define $\beta_s := \omega(\text{Spec}(\mathbf{I}_{s-1}) \setminus \text{Spec}(\mathbf{I}_s))$ and $\beta(\mathbf{I}) = \{\beta_1, \dots, \beta_k\}$. We set $\beta(\mathbf{I}_{0,0}) = \emptyset$.

Proposition 8.1. *The following statements hold:*

- (1) For $s = 1, \dots, k$, β_s is a positive root in $R_{d,n}$.
- (2) The set $\beta(\mathbf{I})$ is an anti-chain in the root poset $R_{d,n}$.
- (3) The characteristic function $\chi_{\beta(\mathbf{I})}$ is a lattice point in $\text{FFLV}_{d,n}$.
- (4) The lattice points in $\text{FFLV}_{d,n}$ are $\{\chi_{\beta(\mathbf{I})} \mid \mathbf{I} \in I(d, n)\}$.

Proof. The statement (1) is clear by definition.

Anti-chains in $R_{d,n}$ are of the following form: $\{\alpha_{i_1, j_1}, \alpha_{i_2, j_2}, \dots, \alpha_{i_s, j_s}\}$, where $i_1 < i_2 < \dots < i_s$ and $j_1 > j_2 > \dots > j_s$. If $\mathbf{I} = [1, \dots, s, i_{s+1}, \dots, i_d]$, where $s \geq 0$ and $i_{s+1} \neq s + 1$, then $\omega(\text{Spec}(\mathbf{I}) \setminus \text{Spec}(p(\mathbf{I}))) = \varepsilon_{i_d} - \varepsilon_{s+1} = \alpha_{s+1, i_d-1}$. Therefore in the sequence $\{\beta_s = \alpha_{p_s, q_s} \mid s = 1, \dots, k\}$, $p_s < p_{s+1}$, and $q_s > q_{s+1}$, this proves (2).

Recall that $\text{FFLV}_{d,n}$ coincides with the chain polytope $\mathcal{C}(R_{d,n})$. By [Sta86, Theorem 2.2], characteristic functions of anti-chains are vertices in the chain polytope, the statement (3) is a consequence of (2).

For weight reasons, the functions $\chi_{\beta(\mathbf{I})}$ are distinct. To show part (4), it suffices to prove that in the chain polytope $\mathcal{C}(R_{d,n})$, all lattice points are vertices, which is an easy counting. \square

Let $V(\varpi_d)$ be the d th fundamental representation of SL_n and for $1 \leq i < j \leq n$, $f_{i,j}$ be a root vector corresponding to the negative root $-\alpha_{i,j}$. Each lattice point χ_A for $A = \{\alpha_{i_1, j_1}, \alpha_{i_2, j_2}, \dots, \alpha_{i_s, j_s}\}$ in $FFLV_{n,d}$ parametrises a basis element $f_{i_s, j_s} \cdots f_{i_1, j_1} \cdot v_{\varpi_d}$ in $V(\varpi_d)$.

Example 8.2. We continue studying Examples 7.1 and 7.3. Let $\mathbf{I} = [2, 4, 5, 7] \in I(4, 7)$. The path partition of $\text{Spec}(\mathbf{I})$ induces a partition of $\mathcal{G}_{\text{Spec}(\mathbf{I})}$, which is given as follows:

$$(11) \quad \begin{array}{c} \alpha_6 \\ \parallel \\ \alpha_3 = \alpha_4 = \alpha_5 \\ \parallel \\ \alpha_1 = \alpha_2 \quad \alpha_3 = \alpha_4. \end{array}$$

In this case $\beta(\mathbf{I}) = \{\alpha_{1,6}, \alpha_{3,4}\}$. The corresponding characteristic function provides an element in $FFLV_{4,7}$, which parametrises the basis element $f_{3,4}f_{1,6} \cdot v_{\varpi_4} \in V(\varpi_4)$ as a representation of SL_7 .

These results shed light on the study of other Hodge algebra structures on $\mathbb{C}[\text{Gr}_{d,n}]$, such that when a maximal chain is fixed, the associated toric variety is the one that appeared in [FFL17], for details, see Section 9.2.

9. REMARKS AND OUTLOOKS

9.1. Besides join-irreducible elements, it is also possible to do the above construction using meet-irreducible elements. There exists a bijection η between $I(d, n)$ and $I(n-d, n)$, sending a d -element subset of $\{1, 2, \dots, n\}$ to its complement.

Let $\overline{I(n-d, n)}$ be the distributive lattice having the same elements as $I(n-d, n)$ whose join operation (resp. meet operation) is the meet operation (resp. join operation) in $I(n-d, n)$. Then $\eta: I(d, n) \rightarrow \overline{I(n-d, n)}$ provides an isomorphism of distributive lattices. Meet-irreducible elements in $I(d, n)$ are meet-irreducible elements in $\overline{I(n-d, n)}$ hence join-irreducible elements in $I(n-d, n)$.

As projective varieties, $\text{Gr}_{d,n}$ is isomorphic to $\text{Gr}_{n-d,n}$. Therefore the construction using meet-irreducible elements provides nothing new.

9.2. One of the leading ideas of this paper is to get an interpretation and construction of standard monomial theory (in the sense of Lakshmibai, Seshadri et al. [LR08]) using filtrations obtained by valuations. Implicitly, the idea to use vanishing multiplicities to define and index standard monomials can already be found in [LS86, Section 7], in the filtration of the ideal sheaf associated to $H(\tau)_{\text{red}}$, where $H(\tau) \subset X(\tau)$ is the zero set of the extremal weight section p_τ in the Schubert variety $X(\tau) \subset G/P \subset \mathbb{P}(V(\varpi))$ for a *classical type* fundamental weight flag variety. This geometric connection also leads to the definition of LS-paths; see, for example, [Ses12].

Now in the case discussed in this paper, the fact that the degenerate algebra is a discrete Hogde algebra implies that the semi-toric degeneration is a union of \mathbb{P}^N 's, which in turn implies that one gets a triangulation of the Newton–Okounkov body we started with. The latter is unimodularly equivalent to the Gelfand–Tsetlin polytope (Proposition 7.6). It is not expected that this nice feature still holds in general. Indeed (see also Section 9.4), it is expected that the standard monomial theory developed in [Lit98] will lead in the general case to a decomposition of the Newton–Okounkov body into the polytopes described in [Deh98].

9.3. In the setting of Stanley [Sta86], one can associate to the poset $I(d, n)$ two polytopes, the order polytope and the chain polytope. The first is realised in our setting as a Newton–Okounkov body, it is unimodularly equivalent to the Gelfand–Tsetlin polytope. The FFLV-polytope can also be realised as a Newton–Okounkov body [FFL17a]. Now Stanley has described a piecewise linear map between the two polytopes, which induces a bijection on the set of lattice points. It can be shown, that the map restricted to the simplexes (see Section 9.2) is an affine linear map, so the triangulation of the Gelfand–Tsetlin polytope induces naturally a triangulation of the FFLV-polytope. It is expected that this triangulation has a similar standard monomial theory interpretation as in the Gelfand–Tsetlin case. Indeed, the results in [HL15] can be used to define a different Hodge algebra structure on the homogeneous coordinate ring of $\text{Gr}_{d,n}$, such that the construction described above leads to the FFLV-polytope as Newton–Okounkov body and the triangulation induced by the discrete Hodge algebra is the image by the transfer map of the triangulation of the Gelfand–Tsetlin polytope. It would be interesting to “detropicalize” Stanley’s transfer map.

9.4. As explained in Section 9.2, to go from the case of Grassmann varieties to partial flag varieties, the Hodge algebra is needed to be upgraded to the LS-algebra to deal with the higher multiplicity phenomenon. Using LS-algebras, Chirivì [Chi00] constructed semi-toric degenerations of partial flag varieties. In view of the construction in the current paper, it is natural to ask for a generalisation to partial flag varieties, that is to say, construct quasi-valuations to recover Chirivì degenerations.

9.5. Let \mathcal{L} be a distributive lattice and let $\alpha, \beta: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be two associative operations on \mathcal{L} satisfying for any $\ell_1, \ell_2 \in \mathcal{L}$, $\alpha(\ell_1, \ell_2) \geq \beta(\ell_1, \ell_2)$. We say the pair (α, β) is compatible if in the straightening relation (2) with ℓ_1, ℓ_2 non-comparable:

$$x_{\ell_1} x_{\ell_2} = \sum_{k_1 \geq k_2} a_{k_1, k_2} x_{k_1} x_{k_2},$$

- (a) the coefficient $a_{\alpha(\ell_1, \ell_2), \beta(\ell_1, \ell_2)} = 1$;
- (b) if for some $(k_1, k_2) \neq (\alpha(\ell_1, \ell_2), \beta(\ell_1, \ell_2))$, $a_{k_1, k_2} \neq 0$, then for every pair (m_1, m_2) where $m_1 \in \text{maxSpec}(\alpha(\ell_1, \ell_2))$ and $m_2 \in \text{maxSpec}(\beta(\ell_1, \ell_2))$ such that $m_1 \geq m_2$, one of the following statements holds:

- there exists $h \in \text{maxSpec}(k_1)$ such that $h > m_1$;
- the statement above does not hold, and there exist $h \neq h' \in \text{maxSpec}(k_1)$ such that $h = m_1$ and $h' > m_2$;
- the statements above do not hold, and there exist $h \in \text{maxSpec}(k_1)$, $h' \in \text{maxSpec}(k_2)$ such that $h = m_1$ and $h' \geq m_2$.

It would be interesting to study under what conditions on α and β , the pair (α, β) is compatible. The motivation of this question is to figure out how the FFLV-polytopes for $\text{Gr}(d, n)$ can be applied to construct semi-toric degenerations (see Section 9.3), i.e., the FFLV-polytopes appear as the Newton–Okounkov body when a maximal chain in $I(d, n)$ is fixed (see for example [HL15]).

We finish this section with the following inverse problem: let $X \subset \mathbb{P}^N$ be a projective variety with homogeneous coordinate ring $\mathbb{C}[X]$. Let $\nu: \mathbb{C}[X] \rightarrow \mathbb{Z}^N$ be a full rank valuation and let $\text{NO}_\nu(X)$ be the associated Newton–Okounkov body. Assume that $\text{NO}_\nu(X)$ is a lattice polytope with a triangulation \mathcal{T} . Can one construct a Hodge algebra structure on $\mathbb{C}[X]$ such that the triangulation arising from the standard monomials coincides with \mathcal{T} ?

REFERENCES

- [And13] D. Anderson, *Okounkov bodies and toric degenerations*, Math. Ann. **356** (2013), 1183–1202. MR3063911
- [AB04] V. Alexeev and M. Brion, *Toric degenerations of spherical varieties*, Selecta Math. (N. S.) **10** (2004), no. 4, 453–478. MR2134452
- [ABS11] F. Ardila, T. Bliem, and D. Salazar, *Gelfand–Tsetlin polytopes and Feigin–Fourier–Littelmann–Vinberg polytopes as marked poset polytopes*, J. Combin. Theory. Ser. A **118** (2011), no. 8, 2454–2462. MR2834187
- [BL10] J. Brown and V. Lakshmibai, *Singular loci of Grassmann–Hibi toric varieties*, Michigan Math. J. **59** (2010), 243–267. MR2677619
- [Cal02] P. Caldero, *Toric degenerations of Schubert varieties*, Transform. Groups **7** (2002), 51–60. MR1888475
- [Chi00] R. Chirivì, *LS algebras and application to Schubert varieties*, Transform. Groups **5** (2000), no. 3, 245–264. MR1780934
- [CLO15] D. Cox, J. Little, and D. O’Shea, *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra*, 4th ed., Undergraduate Texts in Math., Springer, Cham, 2015. MR3330490
- [DCEP82] C. De Concini, D. Eisenbud, and C. Procesi, *Hodge algebras*, Astérisque, vol. 91, Société Mathématique de France, Paris, 1982. MR680936
- [Deh98] R. Dehy, *Combinatorial results on Demazure modules*, J. Algebra **205** (1998), no. 2, 505–524. MR1632757
- [FFL17a] X. Fang, G. Fourier, and P. Littelmann, *Essential bases and toric degenerations arising from birational sequences*, Adv. Math. **312** (2017), 107–149. MR3635807
- [FFL17b] X. Fang, G. Fourier, and P. Littelmann, *On toric degenerations of flag varieties*, Representation Theory — Current Trends and Perspectives, Series of Congress Reports, EMS, 2017. MR3644794
- [FFL11] E. Feigin, G. Fourier, and P. Littelmann, *PBW filtration and bases for irreducible modules in type A_n* , Transform. Groups **16** (2011), no. 1, 71–89. MR2785495
- [FFL17] E. Feigin, G. Fourier, and P. Littelmann, *Favourable modules: filtrations, polytopes, Newton–Okounkov bodies and flat degenerations*, Transform. Groups **22** (2017), no. 2, 321–352. MR3649457
- [GHKK14] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras* (2014, preprint).
- [GL96] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transform. Groups **1** (1996), no. 3, 215–248. MR1417711
- [Grä11] G. Grätzer, *Lattice theory: foundation*, Birkhäuser/Springer Basel AG, Basel, 2011. MR2768581
- [Hib87] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 93–109. MR951198
- [HL15] T. Hibi and N. Li, *Chain polytopes and algebras with straightening laws*, Acta Math. Vietnam **40** (2015), no. 3, 447–452. MR3395648
- [Hod43] W. Hodge, *Some enumerative results in the theory of forms*, Proc. Cambridge Philos. Soc. **39** (1943), 22–30. MR0007739
- [Kav15] K. Kaveh, *Crystal bases and Newton–Okounkov bodies*, Duke Math. J. **164** (2015), 2461–2506. MR3405591
- [KK12] K. Kaveh and A. G. Khovanskii, *Newton–Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), no. 2, 925–978. MR2950767
- [KM16] K. Kaveh and C. Manon, *Khovanskii bases, higher rank valuations and tropical geometry* (2016, preprint).
- [LB15] V. Lakshmibai and J. Brown, *The Grassmannian variety: Geometric and representation-theoretic aspects*, Developments in Mathematics, vol. 42, Springer, New York, 2015. MR3408060
- [LR08] V. Lakshmibai and K. N. Raghavan, *Standard monomial theory: Invariant theoretic approach*, Encyclopaedia of Mathematical Sciences (Invariant Theory and Alg. Transform. Groups VIII), vol. 137, Springer-Verlag, Berlin, 2008. MR2388163
- [LS86] V. Lakshmibai and C. S. Seshadri, *Geometry of G/P* . V, J. Algebra **100** (1986), 462–557. MR840589

- [LS91] V. Lakshmibai and C. S. Seshadri, *Standard monomial theory*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, 1991, pp. 279–322. MR1131317
- [Lit98] P. Littelmann, *Contracting modules and standard monomial theory for symmetrizable Kac–Moody algebras*, J. Amer. Math. Soc. **11** (1998), no. 3, 551–567. MR1603862
- [Ses14] C. S. Seshadri, *Introduction to the theory of standard monomials*, 2nd ed., Texts and Readings in Mathematics, vol. 46, Hindustan Book Agency, New Delhi, 2014. MR3308481
- [Ses12] C. S. Seshadri, *Standard monomials—a historical account*, Collected papers of C. S. Seshadri, Vol. 2, Manoj Prakashan, Madras, 2012. MR2905898
- [Sta86] R. P. Stanley, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23. MR824105

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