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TWISTOR GEOMETRY AND GAUGE FIELDS

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Abstract. The main topic of this survey article is an exposition of basics of the theory of twistors and of applications of this theory to solving equations of gauge field theory, such as, e.g., Yang–Mills equations, monopole equations, etc.

Contents

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Foreword

The main goal of this course is to present basics of twistor geometry and its applications to the solution of gauge field theory equations such as Yang–Mills equations and so on.

Twistors were introduced by Roger Penrose who used them to describe solutions of conformally invariant equations of field theory in Minkowski space. The aim of his "twistor program" was to employ the twistor correspondence to associate with solutions of these equations some objects of complex analytic geometry (such as sections of holomorphic bundles, cohomology with coefficients in sheaves of holomorphic functions, and so on) in the twistor space. As it was remarked by Penrose himself, this program is in its essence close to Einstein's idea underlying the general relativity theory. By this idea the physical bodies in the metric, created by the gravity force of astronomical objects, should move along geodesics. Roughly speaking, the equations of gravity "disappear"; only Riemannian geometry remains. Similar to that, one can say that, after switching to the twistor description, the conformally invariant equations "disappear"; only complex geometry remains.

The first part of the course, devoted to the twistor geometry, starts from the construction of the twistor model of Minkowski space and continues with the description of twistor correspondence. This correspondence assigns to geometric objects in Minkowski space the associated objects of complex geometry in twistor space. Along with the twistor model we also consider the Klein model of Minkowski space in which this space is identified with a quadric in the 5-dimensional projective space $\mathbb{CP}^5$. Then we construct the twistor bundles over arbitrary Riemannian manifolds of even dimension following the well-known paper of Atiyah–Hitchin–Singer.

In the second part of the course the introduced twistor theory is applied to the study of solutions of gauge field theory equations. As the first example we consider the Yang–Mills duality equations in $\mathbb{R}^4$ and their solutions called instantons. The Atiyah–Ward theorem yields the twistor interpretation of instantons and the Atiyah–Drinfeld–Hitchin–Manin constructions, based on this theorem, allows one to completely describe the moduli space of instantons.

The next example of gauge field theory equations is provided by the monopole equations in $\mathbb{R}^3$, also called Bogomolny equations. Their twistor interpretation was proposed by Nahm.

Other examples are related to the 2-dimensional models. As a first model we consider the self-dual Yang–Mills–Higgs equations in $\mathbb{R}^2$, called otherwise the vortex equations.

The moduli space of their solutions is described by the theorem of Taubes. Another example of 2-dimensional models is provided by Hitchin equations on Riemann surfaces. These equations are closely related to the Higgs bundles given by the pairs (E, Φ) consisting of a holomorphic vector bundle E and a holomorphic section Φ (Higgs field) of the bundle of endomorphisms of E. The Hitchin–Kobayashi correspondence establishes a relation between the stable Higgs bundles and solutions of Hitchin equations.

In conclusion we deal with the 2-dimensional σ-models, or in mathematical terminology, harmonic maps of the 2-dimensional sphere into Riemannian manifolds. The twistor interpretation of such maps was studied in detail by Eells and his colleagues.

All considered, equations have a deep physical meaning, and their study is equally interesting both for physicists and mathematicians.

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Part 1. Twistor geometry

1.1. Twistor model of Minkowski space

1.1.1. Minkowski space. The *Minkowski space* M is a 4-dimensional real vector space provided with the *Lorentz metric*. The square of the length of a vector $x = (x^0, x^1, x^2, x^3) \in M$ in this metric is given by the formula

$$|x|^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

The group L of linear transformations of M, preserving the Lorentz metric, is called the *Lorentz group*.

The vectors x with zero length $|x|^2 = 0$, are of special interest. Such vectors are called *light vectors* or *null vectors*. The *light line* is a straight line with a light tangent vector. The light lines, passing through the point 0, form the *light cone* with vertex at 0:

$$C = C_0 = \{x \in M : \ |x|^2 = 0\} = \{x \in M : \ (x^0)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2\}.$$

The interior of the light cone $V = \{x \in M : \ |x|^2 > 0\}$ consists of two components: the *future cone*

$$V_+ = \{x \in M : \ |x|^2 > 0, x^0 > 0\}$$

and the *past cone*

$$V_- = \{x \in M : \ |x|^2 > 0, x^0 < 0\}.$$

The light cone C_{x_0} with vertex at an arbitrary point $x_0 \in M$ is defined in a similar way:

$$C_{x_0} = \{x \in M : \ |x - x_0|^2 = 0\}.$$

The *complex Minkowski space* $\mathbb{C}M$ is the complexification of the Minkowski space M coinciding with the 4-dimensional complex vector space consisting of vectors $z = (z^0, z^1, z^2, z^3) \in \mathbb{C}^4$. As in the real case, a vector $z \in \mathbb{C}M$ is called the *complex light vector* if

$$|z|^2 := (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2 = 0.$$

The *complex light cone* with vertex at a point $z_0 \in \mathbb{C}M$ is given by the equation $(z-z_0)^2 = 0$. The complex analogs of the future and light cones are provided by the *future tube*

$$\mathbb{C}M_+ = \{z = x + iy \in \mathbb{C}M : \ |y|^2 > 0, y^0 > 0\}$$

and *past tube*

$$\mathbb{C}M_- = \{z = x + iy \in \mathbb{C}M : \ |y|^2 > 0, y^0 < 0\}.$$

The *Euclidean space* E is a 4-dimensional real vector space in $\mathbb{C}M$ given by the equations

$$z^0 = x^0, z^1 = ix^1, z^2 = ix^2, z^3 = ix^3,$$

where x^0, x^1, x^2, x^3 are arbitrary real numbers.

1.1.2. **Spinor model of Minkowski space.** The *Pauli map* associates with a vector $x \in M$ the complex 2×2-matrix X according to the formula

$$M \ni x = (x^0, x^1, x^2, x^3) \longmapsto X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} .$$

Using the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the Pauli map may be written in the form

$$x \longmapsto X = x^0\sigma_0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3,$$

where $\sigma_0 = I$ is the identity 2×2-matrix.

The Pauli map realizes Minkowski space M as the space Herm(2) of Hermitian 2×2-matrices. The squared Lorentz norm $|x|^2$ of a vector $x \in M$ under this map is sent to $\det X$.

There is an action of the group $\mathrm{SL}(2, \mathbb{C})$ of complex 2×2-matrices with unit determinant on the space Herm(2) of Hermitian matrices by the rule

$$X \longmapsto AXA^*, \quad X \in \mathrm{Herm}(2),$$

where $A \in \mathrm{SL}(2, \mathbb{C})$, and A^* is the Hermitian conjugate matrix $A^* = \bar{A}^t$. This action preserves $\det X$ and so, by the Pauli correspondence, generates a linear transform of Minkowski space, preserving the Lorentz metric. Note, however, that matrices $\pm A$ generate the same Lorentz transform; in other words, the group $\mathrm{SL}(2, \mathbb{C})$ is a double covering of the Lorentz group L (prove the last statement!).

The *complex Pauli map*, given by the formula

$$\mathbb{C}M \ni z \longmapsto \sum_{\mu=0}^{3} z^i \sigma_i =: Z \in \mathbb{C}[2 \times 2],$$

realizes the complex Minkowski space $\mathbb{C}M$ as the space $\mathbb{C}[2 \times 2]$ of complex 2×2-matrices.

Under this map the future tube $\mathbb{C}M_+$ is transformed to the *matrix upper halfplane*

$$H_+ = \{Z \in \mathbb{C}[2 \times 2] : \ \mathrm{Im}\, Z := \frac{1}{2i}(Z - Z^*) \gg 0\}.$$

The inequality $\mathrm{Im}\, Z \gg 0$ means that the Hermitian matrix $\mathrm{Im}\, Z$ is positively definite, i.e., its eigenvalues are positive. If we apply to H_+, by analogy with the scalar case, the *Cayley transform*

$$Z \longmapsto W = (I - iZ)^{-1}(I + iZ),$$

then the matrix upper halfplane H_+ will be sent to the *matrix disk*

$$D = \{W \in \mathbb{C}[2 \times 2] : \ I - W^*W \gg 0\}.$$

It is a *classical Cartan domain of the 1st kind.* The space Herm(2) of Hermitian matrices under the Cayley map transforms into the distinguished boundary of the matrix disk D coinciding with the group U(2) of unitary 2×2-matrices.

Using the composite map from M to the *compact* group U(2), we can construct a compactification of the space M by defining it as the inverse image of U(2) under the map $M \to \mathrm{U}(2)$. It is the so-called *conformal compactification of Minkowski space* used in the general relativity (cf. [26]).

The complex vector space $\mathbb{C}^2$, on which the group $\mathrm{SL}(2,\mathbb{C})$ acts as the group of matrices, is called the *space of spinors*.

1.1.3. **Twistor model of Minkowski space.** We shall now construct the twistor model of Minkowski space. Denote by $\mathbb{T}$ the 4-dimensional complex vector space $\mathbb{C}^4$. It is convenient to write its vectors as the pairs $\zeta = (\omega, \pi)$, where $\omega, \pi \in \mathbb{C}^2$. Associate with a matrix $Z \in \mathbb{C}[2 \times 2]$ the 2-dimensional complex subspace in $\mathbb{T}$ determined by the system of two complex equations

$$\omega = Z\pi.$$

This defines an embedding of the space $\mathbb{C}[2 \times 2]$ into the Grassmann manifold $G_2(\mathbb{T})$ consisting of 2-dimensional complex subspaces in $\mathbb{T}$.

Taking the composition with the Pauli map we obtain an embedding

$$\mathbb{C}M \longrightarrow \mathbb{C}[2 \times 2] \longrightarrow G_2(\mathbb{T}) \tag{1.1}$$

of the complex Minkowski space $\mathbb{C}M$ into the Grassmann manifold $G_2(\mathbb{T})$. Since the latter manifold is compact it is natural to consider $G_2(\mathbb{T})$ as a model of the *compactified complexified Minkowski space* $\mathbb{CM}$. The space $\mathbb{T}$ itself is called the *space of twistors*. Its projectivization $\mathbb{PT}$ consists of 4-tuples $[\zeta_1 : \zeta_2 : \zeta_3 : \zeta_4]$ of complex numbers $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ defined up to proportionality, i.e.,

$$[\zeta_1 : \zeta_2 : \zeta_3 : \zeta_4] = [\lambda\zeta_1 : \lambda\zeta_2 : \lambda\zeta_3 : \lambda\zeta_4]$$

for any nonzero complex number λ. The space $\mathbb{PT}$ is called the *space of projective twistors*. The Grassmann manifold $G_2(\mathbb{T})$ may also be considered as the Grassmann manifold $G_1(\mathbb{PT})$ of projective lines in $\mathbb{PT} = \mathbb{CP}^3$. A projective line in $\mathbb{PT}$ is determined by the pair of *homogeneous* equations in the space of twistors $\mathbb{T}$.

We now consider the "ideal elements" of $\mathbb{CM}$, i.e., the points of $G_2(\mathbb{T})$ which do not belong to the image of the map (1.1). Denote by P_∞ the subspace of $\mathbb{T}$ at "infinity" given by the equation

$$P_\infty : \ \pi = 0.$$

The 2-subspaces in $\mathbb{T}$, which do not belong to the image of the map (1.1), should have nonzero intersection with P_∞. Any 2-subspace in $\mathbb{T}$ is given by the system of equations

$$Z_1\omega = Z_2\pi,$$

where the 2×2-matrices Z_1, Z_2 are defined up to multiplication from the left by a nondegenerate 2×2-matrix. Such a subspace has nonzero intersection with P_∞ iff $\det Z_1 = 0$. As we have pointed out before, the equation $\det Z_2 = 0$ determines the complex light cone in $\mathbb{C}M$ at the origin. So the set of solutions of the equation $\det Z_1 = 0$ may be interpreted as the complex light cone "at infinity". Hence, the "ideal" set $\mathbb{CM} \setminus \mathbb{C}M$ is identified with the complex light cone "at infinity".

The constructed mapping (1.1). $\mathbb{C}M \to G_2(\mathbb{T}) = G_1(\mathbb{PT})$, is called the *twistor correspondence* or the *Penrose correspondence*.

1.2. Twistor correspondence

1.2.1. **Twistor correspondence in the case of complex Minkowski space.** By the definition of twistor correspondence

$$\{\text{point of } \mathbb{C}M\} \longrightarrow \{\text{projective line in } \mathbb{PT}\}.$$

Identifying a point in $\mathbb{PT}$ with the bundle of projective lines passing through this point, we obtain that a point in $\mathbb{PT}$ corresponds to a 2-dimensional null plane in $\mathbb{C}M$ called the *α-plane*:

$$\left\{\begin{matrix}\text{2-domensional complex null} \\ \text{plane} \equiv \alpha\text{-plane}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{point in } \mathbb{PT} \equiv \text{bundle of projective lines} \\ \text{passing through this point}\end{matrix}\right\}.$$

A plane in $\mathbb{C}M$ is called *null* or *isotropic* if it is generated by two linearly independent light vectors. The dual type of isotropic planes in $\mathbb{C}M$, called the *β-planes*, corresponds to the dual object in $\mathbb{PT}$, namely, to a projective plane identified with the system of projective lines lying in this plane:

$$\left\{\begin{matrix}\text{2-dimensional complex null} \\ \text{plane} \equiv \beta\text{-plane}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{projective plane in } \mathbb{PT} \equiv \text{system of} \\ \text{projective lines lying in this plane}\end{matrix}\right\}.$$

Taking the intersection of the last two diagrams, we find the twistor image of a complex light line

$$\{\text{complex light line in } \mathbb{C}M\} \longrightarrow \left\{\begin{matrix}(0,2)\text{-flag in } \mathbb{PT} \equiv (\text{point of } \mathbb{PT}, \text{projective plane} \\ \text{containing this point}) \equiv \text{bundle of projective lines} \\ \text{lying in this plane and passing through this point}\end{matrix}\right\}.$$

The last assertions imply that

$$\left\{\begin{matrix}\text{complex light cone in } \mathbb{C}M \equiv \text{bundle of} \\ \text{complex light lines passing through a} \\ \text{fixed point of } \mathbb{C}M\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{projective line in } \mathbb{PT} \equiv \text{family} \\ \text{of } (0,1,2)\text{-flags in } \mathbb{PT} \text{ with fixed} \\ \text{projective line}\end{matrix}\right\}.$$

1.2.2. **Twistor correspondence in the case of real Minkowski space.** The *twistor norm* of an element $\zeta = (\omega, \pi) \in \mathbb{T}$ is by definition equal to

$$\Phi(\zeta) = \operatorname{Im}\langle \omega, \pi \rangle,$$

where $\langle \omega, \pi \rangle$ is the Hermitian inner product of vectors $\omega = (\omega_1, \omega_2)$ and $\pi = (\pi_1, \pi_2)$ in $\mathbb{C}^2$:

$$\langle \omega, \pi \rangle = \omega_1 \bar{\pi}_1 + \omega_2 \bar{\pi}_2.$$

Denote by $\mathbb{N}$ the quadric in $\mathbb{T}$ given by the equation

$$\mathbb{N}: \ \Phi(\zeta) = 0,$$

and by $\mathbb{PN}$ the corresponding projective quadric.

Under the twistor correspondence the points of M are sent to the projective lines belonging to $\mathbb{PN}$:

$$\{\text{point of } M\} \longrightarrow \{\text{projective line lying in } \mathbb{PN}\}.$$

The image of a light line in M under the twistor correspondence coincides with a point in $\mathbb{PN}$ which is identified with the bundle of projective lines lying in the intersection of the complex tangent plane to $\mathbb{PN}$ in a fixed point with the quadric $\mathbb{PN}$:

$$\{\text{light line in } M\} \longrightarrow \left\{\begin{matrix}\text{point of } \mathbb{PN} \equiv (\text{point of } \mathbb{PN}, \text{complex tangent plane to } \mathbb{PN} \text{ at} \\ \text{this point}) \equiv \text{bundle of projective lines lying in the complex} \\ \text{tangent plane and passing through the fixed point}\end{matrix}\right\}.$$

A light cone in M is identified with the projective line in $\mathbb{PN}$:

$$\left\{\begin{matrix}\text{light cone in } M \equiv \text{bundle of} \\ \text{light lines passing through a} \\ \text{fived point of } M\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{projective line in } \mathbb{PN} \equiv \text{intersection of } \mathbb{PN} \text{ with} \\ \text{the complex tangent plane at every point of} \\ \text{the fixed projective line}\end{matrix}\right\}.$$

Hence, in the case of real Minkowski space the twistor correspondence determines a duality of the following type:

$$\{\text{points of } M\} \longrightarrow \{\text{projective lines in } \mathbb{PN}\},$$

$$\{\text{light lines in } M\} \longrightarrow \{\text{points of } \mathbb{PN}\}.$$

So the light lines, which can intersect each other in M split into separate points of $\mathbb{PN}$. This fact is of fundamental importance for the whole twistor theory.

The quadric $\mathbb{N}$ divides the twistor space $\mathbb{T}$ into two parts—the *space of positive twistors* $\mathbb{T}_+ = \{\zeta \in \mathbb{T} : \ \Phi(\zeta) > 0\}$ and the *space of negative twistors* $\mathbb{T}_- = \{\zeta \in \mathbb{T} : \ \Phi(\zeta) < 0\}$. The restriction of the twistor correspondence to the future and past tubes yields:

$$\{\text{point of } \mathbb{C}M_+\} \longrightarrow \{\text{projective line lying in } \mathbb{PT}_+\},$$

$$\{\text{point of } \mathbb{C}M_-\} \longrightarrow \{\text{projective line lying in } \mathbb{PT}_-\}.$$

The quadric $\mathbb{N} = \{\zeta \in \mathbb{T} : \ \Phi(\zeta) = 0\}$ has the signature (2,2), so in the appropriate basis of the space $\mathbb{T}$ it can be written in the form

$$\tilde{\Phi}(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2.$$

The group $\mathrm{SU}(2,2)$ of linear transformations of $\mathbb{T}$, preserving the quadric $\mathbb{N}$, generates transformations of compactified Minkowski space $\mathbb{M}$, sending light lines to light lines and light cones to light cones.

Recall that a map of Minkowski space $\mathbb{M}$ is called *conformal* if it has this property. The group of conformal maps of $\mathbb{M}$ is denoted by $\mathrm{C}(1,3)$.

We have just shown that the maps from the group $\mathrm{SU}(2,2)$ generate conformal maps of the compactified Minkowski space $\mathbb{M}$. Note that the elements $\pm A$, $\pm iA$ from $\mathrm{SU}(2,2)$ generate the same transform of $\mathbb{M}$. In other words, the group $\mathrm{SU}(2,2)$ is a 4-fold covering of the conformal group $\mathrm{C}(1,3)$ of Minkowski space $\mathbb{M}$.

Consider in more detail the group structure of the twistor model $G_2(\mathbb{T})$ of Minkowski space. The group

$$G := \mathrm{SL}(4,\mathbb{C})/\{\pm I, \pm iI\}$$

acts in a natural way on $G_2(\mathbb{T})$. Fix the basis $\{e_i\}$ of the space $\mathbb{T}$ in which the quadric $\mathbb{N}$ is given by the equation

$$\tilde{\Phi}(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = 0.$$

Write an arbitrary linear transform of the twistor space $\mathbb{T}$ in the form of a block 4×4-matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.2}$$

where A, B, C, D are complex 2×2-matrices. Denote by P_0 the 2-dimensional subspace from $G_2(\mathbb{T})$ of the following form:

$$P_0 = \{z \in \mathbb{T} : \ z_3 = z_4 = 0\}.$$

The isotropy subgroup G_0 of the group G at P_0 consists of the block matrices (1.2) in which $C = 0$, $\det A \cdot \det D = 1$. So $G_2(\mathbb{T})$ may be identified with the homogeneous space of the group G of the form G/G_0.

Denote by $G^{\mathbb{R}}$ the real form of the group G defined by

$$G^{\mathbb{R}} = \mathrm{SU}(2,2)/\{\pm I, \pm iI\}.$$

The isotropy subgroup $G_0^{\mathbb{R}}$ at P_0 coincides with $G_0 \cap G^{\mathbb{R}}$.

The homogeneous space $G^{\mathbb{R}}/G_0^{\mathbb{R}}$ may be identified with the twistor model of the future tube $\mathbb{C}M_+$. Indeed, the twistor image of $\mathbb{C}M_+$ coincides with the set of 2-subspaces lying in $\mathbb{T}_+$. We shall call such subspaces *positive* and denote the set of all positive subspaces by $G_2^+(\mathbb{T})$. Since the subspace P_0 is positive, the group $G^{\mathbb{R}}$ preserves the positivity property and acts transitively on $G_2^+(\mathbb{T})$; it follows that the homogeneous space $G^{\mathbb{R}}/G_0^{\mathbb{R}}$ coincides with $G_2^+(\mathbb{T}) = \mathbb{C}M_+$.

1.2.3. **Twistor correspondence in the case of Euclidean space.** The image of a point of Euclidean space under the twistor correspondence coincides with the projective line in $\mathbb{PT}$ invariant under the map $j : [\zeta_1 : \zeta_2 : \zeta_3 : \zeta_4] \mapsto [-\zeta_2 : \zeta_1 : -\zeta_4 : \zeta_3]$:

$$\{\text{point of } E\} \longrightarrow \{\text{projective } j\text{-invariant line in } \mathbb{PN}\}.$$

These j-invariant lines do not intersect with each other. Moreover, in the considered case the twistor correspondence coincides with the *Hopf bundle*

$$\pi : \mathbb{CP}^3 \xrightarrow{\mathbb{CP}^1} \mathbb{E},$$

where $\mathbb{E}$ is the *compactified Euclidean space* equal to the sphere S^4. As in the 2-dimensional case, where the sphere S^2 is identified with the complex projective line, in the 4-dimensional case the sphere S^4 may be identified with the quaternion projective line.

In order to clarify this assertion we recall basic definitions related to quaternions. The *space of quaternions* $\mathbb{H}$ consists of the elements of the form

$$x = x_1 + ix_2 + jx_3 + kx_4,$$

where x_1, x_2, x_3, x_4 are arbitrary real numbers and i, j, k are imaginary units, i.e., $i^2 = j^2 = k^2 = -1$, subject to the relation $ij = -ji = k$. As a real vector space, $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$ with componentwise operations of addition and multiplication by real numbers. The relation given above allows us to introduce the operation of *multiplication* of quaternions.

The *conjugation* of quaternions is defined by the formula

$$\bar{x} = x_1 - ix_2 - jx_3 - kx_4.$$

Using it we can introduce the *norm* of a quaternion by

$$|x|^2 = x\bar{x} = \bar{x}x = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

From the algebraic point of view the space of quaternions is a noncommutative field since any nonzero quaternion x has its inverse:

$$x^{-1} = \bar{x}/|x|^2.$$

Quaternions are conveniently written in the complex form

$$x = z_1 + jz_2, \quad \text{where } z_1 = x_1 + ix_2,\ z_2 = x_3 + ix_4.$$

As a complex vector space, $\mathbb{H}$ is isomorphic to $\mathbb{C}^2$.

Another convenient way of writing quaternions is with the help of matrices. Namely, quaternions may be realized as complex 2×2-matrices by assigning to a quaternion $x = z_1 + jz_2$ the matrix

$$\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}.$$

Under thus identification the quaternion multiplication corresponds to the product of matrices. The “unit circle” in $\mathbb{H}$, coinciding with

$$\mathrm{Sp}(1) = \{x \in \mathbb{H} :\ |x|^2 = 1\},$$

is identified with the group $\mathrm{SU}(2)$ of unitary 2×2-matrices with determinant 1.

Now we can return to the interpretation of the sphere S^4 as the quaternion projective line. The *quaternion projective line* $\mathbb{HP}^1$ consists of pairs of quaternions $[(z_1 + jz_2) : (z_3 + jz_4)]$ defined up to multiplication from the right by nonzero quaternions.

The map $\pi : \mathbb{CP}^3 \to \mathbb{HP}^1$, mentioned above, is given by the tautological formula

$$[z_1 : z_2 : z_3 : z_4] \longmapsto [(z_1 + jz_2) : (z_3 + jz_4)],$$

where the 4-tuple $[z_1 : z_2 : z_3 : z_4]$ is defined up to the multiplication by a nonzero complex number while the pair $[(z_1 + jz_2) : (z_3 + jz_4)]$ is defined up to multiplication from the right by a nonzero quaternion.

The map

$$j : [z_1 : z_2 : z_3 : z_4] \longmapsto [-z_2 : z_1 : -z_4 : z_3]$$

corresponds to the multiplication of the pair $(z_1 + jz_2, z_3 + jz_4)$ by the imaginary unit j from the right which does not change the projective class $[(z_1 + jz_2) : (z_3 + jz_4)]$. So the fibers of the bundle π are j-invariant projective lines and the twistor correspondence in the Euclidean case coincides with the pull-back by π.

1.2.4. **Klein model of Minkowski space.** Any subspace from $G_2(\mathbb{T})$ is given, up to multiplication by a nonzero complex number, by the bivector $p = p_1 \wedge p_2$, where p_1, p_2 is a pair of linearly independent vectors lying in the considered 2-subspace. Fix an orthonormal basis $\{e_i\}$ in $\mathbb{T}$. Then bivectors $e_i \wedge e_j$, $i < j$, will form the basis of the exterior square $\bigwedge^2 \mathbb{T}$. So decomposing an arbitrary bivector p in this basis, we can represent it in the form

$$p = \sum_{i<j} p_{ij} e_i \wedge e_j .$$

In this way we can associate with any 2-subspace from $G_2(\mathbb{T})$ the collection $[p_{ij}]$ of its *Plücker coordinates* defined up to multiplication by a nonzero complex number.

Plücker coordinates satisfy the relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0 \tag{1.3}$$

which follows from the evident condition $p \wedge p = 0$.

The constructed correspondence allows us to identify the space $G_2(\mathbb{T})$ with the projective quadric $\mathbb{PQ}$ in the 5-dimensional complex projective space $\mathbb{CP}^5$ determined by the equation (1.3). This quadric is called the *Klein model* of the compactified complexified Minkowski space $\mathbb{CM}$. In appropriate coordinates $(u, v) = (u_1, u_2, u_3, v_1, v_2, v_3)$ in the space $\mathbb{C}^6$ the quadric $\mathbb{Q}$ in $\mathbb{C}^6$, given by the equation (1.3), may be written in the form

$$u_1^2 + u_2^2 + u_3^2 = v_1^2 + v_2^2 + v_3^2 \tag{1.4}$$

or, for short, $u^2 = v^2$.

The main objects of geometry of the Minkowski space $\mathbb{C}M$ admit the following interpretation in terms of the Klein model:

$$\{\text{point of } \mathbb{C}M\} \longrightarrow \{\text{point of quadric } \mathbb{PQ}\} .$$

The quadric $\mathbb{Q}$, given by equation (1.4), has two systems of straight generators, represented by 3-subspaces defined by the equations

$$u = Av, \quad \text{where } A \in \mathrm{O}(3, \mathbb{C}).$$

The group $\mathrm{O}(3, \mathbb{C})$ of linear transformations of $\mathbb{C}^3$, preserving the form $u^2 = u_1^2 + u_2^2 + u_3^2$, consists of two connected components singled out by the sign of $\det A$. The straight generators $\{u = Av\}$ with $\det A = 1$ correspond under the twistor correspondence to α-planes, while the generators $\{u = Av\}$ with $\det A = -1$ correspond to β-planes:

$$\left\{\begin{matrix}\text{complex light cone in } \mathbb{C}M \\ \text{with vertex at a given point}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{tangent cone to quadric } \mathbb{PQ} \equiv \text{intersection of} \\ \text{the tangent space to } \mathbb{PN} \text{ at a given point with} \\ \text{the quadric } \mathbb{PQ}\end{matrix}\right\} .$$

The real Minkowski space $\mathbb{M}$ and Euclidean space $\mathbb{E}$ admit the following interpretation in terms of the quadric $\mathbb{PQ}$:

$$\{\text{point of } \mathbb{M}\} \longrightarrow \begin{Bmatrix} \text{point of the real quadric } x_1^2 + x_2^2 + x_3^2 \\ + x_4^2 - x_5^2 - x_6^2 = 0 \text{ in } \mathbb{PQ} \end{Bmatrix},$$

$$\{\text{point of } \mathbb{E}\} \longrightarrow \begin{Bmatrix} \text{point of the real quadric } x_1^2 + x_2^2 + x_3^2 \\ + x_4^2 + x_5^2 - x_6^2 = 0 \text{ in } \mathbb{PQ} \end{Bmatrix}.$$

The group $G = \mathrm{SL}(4, \mathbb{C})/\{\pm I, \pm iI\}$, acting on the space $G_2(\mathbb{T})$, generates projective transforms of $\mathbb{C}^6$, preserving the quadric $\mathbb{Q}$, i.e., transforms from the group $\mathrm{O}(6, \mathbb{C})/\{\pm I\}$. Hence, we have a homomorphism

$$G \longrightarrow \mathrm{O}(6, \mathbb{C})/\{\pm I\}.$$

In an analogous way, Klein interpretation of the real Minkowski space $\mathbb{M}$ is related to the local isomorphism $\mathrm{SU}(2, 2) \cong \mathrm{SO}(4, 2)$, and Klein interpretation of the Euclidean space $\mathbb{E}$ is related to the local isomorphism $\mathrm{SL}(2, \mathbb{H}) \cong \mathrm{SO}(5, 1)$.

The *twistor program of Penrose* proclaims that the twistor correspondence should send solutions of conformally invariant equations of field theory, defined on Minkowski space $\mathbb{M}$, to the objects of complex geometry in twistor space $\mathbb{PT}$.

1.2.5. **Twistor bundles.** The Hopf bundle $\pi : \mathbb{CP}^3 \to S^4$, constructed above, admits a nice interpretation in terms of complex structures on the Euclidean space $E = \mathbb{R}^4$ proposed by Atiyah.

The map π over $\mathbb{R}^4$ coincides with the bundle

$$\pi : \mathbb{CP}^3 \setminus \mathbb{CP}^1_\infty \longrightarrow E,$$

where the omitted projective line $\mathbb{CP}^1_\infty$ is identified with the fibre $\pi^{-1}(\infty)$ of the Hopf bundle at $\infty \in S^4$.

The space $\mathbb{CP}^3 \setminus \mathbb{CP}^1_\infty$ is sliced by parallel projective planes $\mathbb{CP}^2$ intersecting in $\mathbb{CP}^3$ on the omitted projective line $\mathbb{CP}^1_\infty$. Consider the fibre $\pi^{-1}(p)$ of π over an arbitrary point $p \in E$. Through any point z of this fibre there passes the affine complex plane $\mathbb{C}^2_z$ from our family. Associate with the point z the complex structure J_z on the tangent plane $T_pE \cong \mathbb{R}^4$ by identifying T_pE with $\mathbb{C}^2_z$ with the help of the tangent map π_*. In this way the fibre $\pi^{-1}(p)$ of the twistor bundle π over the point p is identified with the space of complex structures on the tangent space T_pE. All constructed complex structures are compatible with the metric and orientation of $\mathbb{R}^4$ in the sense that operators J_z on T_pE are represented by skew-symmetric matrices with zero trace.

This construction admits an extension to arbitrary even-dimensional oriented Riemannian manifolds X. Namely, consider the bundle $\pi : \mathcal{J}(X) \to X$ of complex structures on X having the fibre at a point $p \in X$ equal to the space $\mathcal{J}(T_pX) \cong \mathcal{J}(\mathbb{R}^{2n})$ of complex structures on the tangent space T_pX compatible with Riemannian metric and orientation. Such complex structures on $T_pX \cong \mathbb{R}^{2n}$ are given by skew-symmetric linear operators J with zero trace and square $J^2 = -I$. The space of these structures is identified with the complex homogeneous space

$$\mathcal{J}(\mathbb{R}^{2n}) \cong \mathrm{SO}(2n)/\mathrm{U}(n)$$

and so has a canonical complex structure.

The bundle $\pi : \mathcal{J}(X) \to X$ is called the *twistor bundle* over X. We show that it has a natural almost complex structure. The Riemannian connection on X generates a natural connection in the principal $\mathrm{SO}(2n)$-bundle $\mathcal{SO}(X) \to X$ of orthonormal frames on X, and this connection determines the vertical-horizontal decomposition

$$T\mathcal{J}(X) = V \oplus H$$

of the associated bundle of complex structures. Introduce an almost complex structure $\mathcal{J}^1$ on $\mathcal{J}(X)$ by setting

$$\mathcal{J}^1 = \mathcal{J}^v \oplus \mathcal{J}^h.$$

The value of the vertical component $\mathcal{J}^v_z \in \mathrm{End}(V_z)$ at $z \in \mathcal{J}(X)$ coincides with the canonical complex structure on the complex homogeneous space $V_z \cong \mathrm{SO}(2n)/\mathrm{U}(n)$. The value of the horizontal component $\mathcal{J}^h_z \in \mathrm{End}(H_z)$ at z coincides with the complex structure $J(z) \leftrightarrow z$ on the space H_z identified with the tangent space $T_{\pi(z)}X$ via the tangent map π_*. Recall that the fibre $\pi^{-1}(p)$ of the bundle $\mathcal{J}(X) \to X$ at the point $p = \pi(z) \in X$ consists of the complex structures on T_pX, and we denote by $J(z)$ the complex structure on T_pX corresponding to the point $z \in \pi^{-1}(p)$.

The constructed almost complex structure $\mathcal{J}^1$ on $\mathcal{J}(X)$ makes the space $\mathcal{J}(X)$ an almost complex manifold. This structure was introduced by *Atiyah–Hitchin–Singer* in [5].

Part 2. Gauge fields

2.1. Instantons and Yang–Mills fields

2.1.1. **Yang–Mills equation.** Let X be a compact 4-dimensional Riemannian manifold, and let G be a compact Lie group called the *gauge group*.

The *gauge potential* A is a connection in a principal G-bundle $P \to X$ given by a 1-form on P with values in the Lie algebra $\mathfrak{g}$ of G. Denote by $\mathrm{ad}\, P = P \times_G \mathfrak{g}$ the associated bundle where G acts on $\mathfrak{g}$ by the adjoint representation. In terms of this bundle the gauge potential A is given by a 1-form

$$A \in \Omega^1(X, \mathrm{ad}\, P).$$

The main example of the gauge group G for us will be the group SU(2). In this case gauge potential A in local coordinates $(x^\mu) = (x^0, x^1, x^2, x^3)$ is given by a 1-form

$$A \sim \sum_{\mu=0}^{3} A_\mu(x) dx^\mu,$$

where A_μ are complex skew-Hermitian 2×2-matrices with zero trace, and the sign $\sim$ means (here and afterwards) an expression in local coordinates. In the particular case $G = \mathrm{U}(1)$ the gauge potential coincides with the usual electromagnetic vector potential (more precisely, with its Euclidean analogue).

The curvature F of a connection A is called the *gauge field* and is given by a 2-form on P with values in the Lie algebra $\mathfrak{g}$ or by the 2-form $F \in \Omega^2(X, \mathrm{ad}\, P)$ equal to

$$F = DA = dA + \frac{1}{2}[A, A],$$

where D is the operator of *exterior covariant differentiation*

$$D : \Omega^p(X, \mathrm{ad}\, P) \longrightarrow \Omega^{p+1}(X, \mathrm{ad}\, P)$$

generated by the connection A. In the case $G = \mathrm{SU}(2)$ the gauge field F is given in local coordinates (x^μ) by the 2-form

$$F \sim \sum_{\mu,\nu=0}^{3} F_{\mu\nu}(x) dx^\mu \wedge dx^\nu,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad \text{with } \partial_\mu = \partial/\partial x^\mu.$$

In the particular case $G = \mathrm{U}(1)$ the tensor $(F_{\mu\nu})$ coincides with (the Euclidean analogue of) the Maxwell tensor of electromagnetic field.

The *gauge transform* is a fiberwise diffeomorphism $g : P \to P$ which is G-*equivariant* in the sense that

$$g(hp) = gh(p)$$

for any $h \in G$, $p \in P$. In other words, g is a section of the bundle $P \times_G G$. Locally, the gauge transform is given by a smooth function $g(x)$ on X with values in the group G, and its action on gauge potential A and gauge field F is defined by the formula

$$A_g = (\operatorname{Ad} g^{-1})dg + (\operatorname{Ad} g^{-1})A, \quad F_g = (\operatorname{Ad} g^{-1})F,$$

where Ad is the adjoint action of the group G on the Lie algebra $\mathfrak{g}$.

In the case of the group $G = \mathrm{SU}(2)$ these formulas may be rewritten in the form

$$A_g = g^{-1}dg + g^{-1}Ag, \quad F_g = g^{-1}Fg.$$

In the particular case $G = \mathrm{U}(1)$ the gauge map coincides with the phase transform $g(x) = e^{i\theta(x)}$ which acts on A as the gradient transform $A \mapsto A + id\theta$ while the gauge field F is not changed.

The *Yang–Mills action* functional is defined by the formula

$$S_{\mathrm{YM}}(A) = \frac{1}{2}\int_X \|F\|^2 \mathrm{vol}\,,$$

where the norm $\|\cdot\|$ is determined by the inner product on the space of forms, generated by the Riemannian metric on X and invariant inner product tr on the Lie algebra $\mathfrak{g}$, and whose vol is the volume element on X.

In the case of the group $G = \mathrm{SU}(2)$ this formula may be rewritten, using the Hodge $*$-operator, in the following form:

$$S_{\mathrm{YM}}(A) = \frac{1}{2}\int_X \mathrm{tr}(F \wedge *F).$$

The critical points of this functional are called the *Yang–Mills fields.* They satisfy the *Euler–Lagrange equation*

$$D^*F = 0,$$

where

$$D^* = *D* : \Omega^{p+1}(X, \operatorname{ad} P) \longrightarrow \Omega^p(X, \operatorname{ad} P)$$

is the operator conjugate to the operator D. This equation is called the *Yang–Mills equation* and is often written in the form

$$D(*F) = 0.$$

2.1.2. **Instantons.** A gauge field F is called *selfdual* (resp., *anti-selfdual*) if

$$*F = F \text{ (resp.,} * F = -F).$$

By the Bianchi identity $DF = 0$, implied by the relation $F = DA$, the gauge fields, subject to the *duality equations* $*F = \pm F$, automatically satisfy the Yang–Mills equation.

Setting $F_\pm = \frac{1}{2}(*F \pm F)$, we can represent the field F in the form

$$F = F_+ + F_-, \quad \text{where } * F_\pm = \pm F_\pm.$$

(Note that the fields $F_\pm$ are not obliged to satisfy the Bianchi identity, hence also the Yang–Mills equations.) In these terms the Yang–Mills functional may be rewritten in the form

$$S_{\mathrm{YM}}(A) = \frac{1}{2}\int_X \left(\|F_+\|^2 + \|F_-\|^2\right) \mathrm{vol}.$$

Denote by $E \to X$ the vector bundle of rank n associated with the principal bundle $P \to X$. Assign to E a topological invariant, coinciding with the *1st Pontryagin class*, which is computed by the formula

$$p_1(E) = \frac{1}{8\pi^2} \int_X \left(\|F_+\|^2 - \|F_-\|^2\right) \mathrm{vol} = \frac{1}{8\pi^2} \int_X \mathrm{tr}(F \wedge F) \tag{2.1}$$

and called the *topological charge* of F.

It is evident that

$$S_{\mathrm{YM}}(A) \geq 4\pi^2 |p_1(E)|,$$

and the equality here is attained precisely on the solutions of the duality equations. In other words, these solutions determine the local minima of the functional $S_{\mathrm{YM}}(A)$.

In physical papers the anti-selfdual (ASD)-solutions of Yang–Mills equations are called the *instantons*, while in mathematical literature it is usual to deal with the selfdual (SD)-solutions which are naturally called the *anti-instantons*.

We are mostly interested in the study of the *moduli space of instantons*:

$$\{\text{moduli space of instantons}\} \quad = \quad \frac{\{\text{instantons}\}}{\{\text{gauge transforms}\}}\,.$$

2.1.3. **Yang–Mills fields on $\mathbb{R}^4$.** Any Yang–Mills fields on $\mathbb{R}^4$ with finite Yang–Mills action by the Uhlenbeck theorem [24] may be extended to a Yang–Mills field on S^4 with values in some principal bundle $P \to S^4$ in the sense that the restriction of this field to $\mathbb{R}^4$ is gauge equivalent to the original Yang–Mills field. The spherical metric on $S^4 \setminus \{\infty\}$ is conformally equivalent to the Euclidean matric on $\mathbb{R}^4$, so the extension of a Yang–Mills field from $\mathbb{R}^4$ to S^4 is reduced to finding appropriate asymptotic conditions at infinity which will define the required principal bundle $P \to S^4$. This argument shows that the problem of the description of Yang–Mills fields on $\mathbb{R}^4$ with finite Yang–Mills action may be considered as a part of the general problem of studying Yang–Mills fields on compact 4-dimensional Riemannian manifolds.

Let A be a gauge potential on $\mathbb{R}^4$ with gauge group G. To guarantee the finiteness of the Yang–Mills action we impose on A an asymptotic condition by requiring that the potential A should tend to a trivial one (i.e., pure gauge potential) at infinity. In other words, we shall suppose that $A(x)$ is gauge equivalent to a potential of the form $g(x)^{-1}dg(x)$ for $|x| \to \infty$. If this condition is satisfied, then, by restricting g^{-1} to the sphere S^3_R of sufficiently large radius R, we shall obtain a smooth map

$$g^{-1} : S^3_R \longrightarrow G,$$

determining the homotopy class $[S^3_R, G]$. In the case of the group $G = \mathrm{SU}(2)$ it gives one more definition of the topological charge introduced earlier. Namely, this charge coincides with the degree of the map $g^{-1} : S^3_R \to \mathrm{SU}(2) \cong S^3$.

The finiteness of the Yang–Mills action on $\mathbb{R}^4$ for the instanton means "physically" that it is localized in space $\mathbb{R}^3 \subset \mathbb{R}^4$ as well as in "time" $(x^0) \subset \mathbb{R}^4$, which explains its name.

The dimension of the moduli space $\mathcal{M}_k$ of SU(2)-instantons on $\mathbb{R}^4$, having topological charge $-k$, k a positive integer, may be found with the help of the Atiyah–Singer index theorem (cf. [3]), and is equal to $8k - 3$.

Consider the case $k = 1$ in more detail. Identify the space $\mathbb{R}^4$ with the space of quaternions $\mathbb{H}$, and the group SU(2) with the group Sp(1) of quaternions with modulus 1. The Lie algebra of this group coincides with the algebra of pure imaginary quaternions, so the gauge potential on $\mathbb{R}^4$ is given in this case by a 1-form on $\mathbb{H}$ with coefficients given by pure imaginary quaternions.

The first example of 1-instantons was constructed by Belavin–Polyakov–Schwarz–Tyupkin [7]. In quaternion notation it is given by the gauge potential of the form

$$A(x) = \mathrm{Im}\left\{\frac{\bar{x}dx}{1+|x|^2}\right\} = \frac{\bar{x}dx - d\bar{x}\cdot x}{2(1+|x|^2)},$$

where $x = x^0 + ix_1 + jx_2 + kx_3$. The corresponding gauge field has the form

$$F(x) = \frac{d\bar{x}\wedge dx}{(1+|x|^2)^2} = \mathrm{Im}\left\{\frac{d\bar{x}\wedge dx}{(1+|x|^2)^2}\right\}$$

and is anti-selfdual.

Its topological charge is equal to -1. Indeed, for $|x|\to\infty$ we have

$$A(x) \sim \mathrm{Im}\left\{\frac{\bar{x}dx}{|x|^2}\right\} = \mathrm{Im}\,\{x^{-1}dx\}.$$

But for $x\neq 0$ the latter potential is pure gauge since

$$\mathrm{Im}\,\{x^{-1}dx\} = g(x)^{-1}dg(x), \quad \text{where } g(x) = \frac{x}{|x|}.$$

Hence the topological charge of the field F coincides with the degree of the map $g^{-1} : S^3 \to S^3$ acting by the formula

$$g^{-1}(x) = \frac{\bar{x}}{|x|}$$

which has degree -1. So the gauge potential A does define a 1-instanton on $\mathbb{R}^4$.

In order to obtain an SD-solution with charge $+1$ it is sufficient to replace the formula for $A(x)$ by

$$A(x) = \mathrm{Im}\left\{\frac{xd\bar{x}}{1+|x|^2}\right\}.$$

Let us construct a principal Sp(1)-bundle over S^4 corresponding to this instanton. For that apply to A the gauge field g^{-1}. We get

$$A_{g^{-1}}(x) = A(y), \quad \text{where } y := x^{-1}.$$

Now consider the standard covering of $\mathbb{HP}^1$ by open subsets

$$U_0 = \{[x:y]\in\mathbb{HP}^1 : \ y\neq 0\} \quad \text{and} \quad U_\infty = \{[x:y]\in\mathbb{HP}^1 : \ x\neq 0\}$$

and introduce the transition function

$$g_{0\infty} : U_0\cap U_\infty \longrightarrow \mathrm{Sp}(1)$$

by setting $g_{0\infty}(x) = g(x)^{-1} = \bar{x}/|x|$. Thus, we have constructed a principal Sp(1)-bundle $P\to S^4$ and 1-form A, equal to $A(x)$ on U_0 and $A(y)$ on U_∞ with $y = x^{-1}$, determining an ASD-connection in the bundle P.

An arbitrary 1-instanton on $\mathbb{R}^4$ is given by gauge potential of the form

$$A(x) = \mathrm{Im}\left\{\frac{(\bar{x}-\bar{x}_0)dx}{\lambda^2+|x-x_0|^2}\right\}, \quad \text{where } x_0\in\mathbb{H}, \lambda\in\mathbb{R},$$

depending on 5 real parameters.

Generalizing this method of construction of 1-instantons, we look for an arbitrary gauge potential on $\mathbb{R}^4$ given by the following Ansatz:

$$A(x) = \mathrm{Im}\{\varphi(x)^{-1}\partial\varphi(x)\},$$

where $\varphi(x)$ is an arbitrary smooth real-valued function of $x\in\mathbb{H}$, and

$$\partial\varphi(x) := \frac{\partial\varphi}{\partial x}dx.$$

This potential A defines an ASD-connection if it satisfies the following equation:

$$\partial\bar{\partial}\varphi = \Delta\varphi = 0.$$

We obtain a nontrivial *t'Hooft solution* if we set in this Ansatz

$$\varphi(x) = 1 + \sum_{j=1}^{k} \frac{\lambda_j^2}{|x - x_j|^2},$$

where $(x_1, \ldots, x_k)$ is a collection of different points in $\mathbb{R}^4$ and $(\lambda_1, \ldots, \lambda_k)$ is a collection of nonzero real parameters. This function corresponds to the gauge potential A with singularities of the form $\operatorname{Im}\{(x - x_j)^{-1}dx\}$ at points x_j. If we apply to $A(x)$, as in the case of 1-instanton, the gauge transform

$$g_j(x) := \frac{x - x_j}{|x - x_j|}$$

in the truncated neighborhood of the point x_j, we obtain the gauge equivalent potential of the form

$$A_j(x) = \operatorname{Im}\left\{\frac{(\bar{x} - \bar{x}_j)dx}{\lambda_j^2 + |x - x_j|^2}\right\} + \cdots$$

which already has no singularity at $x = x_j$.

By construction, the gauge potential A determines an ASD-connection outside the points $\{x_j\}$. In order to associate with the connection A an instanton on S^4, we have to consider the covering of S^4 by open balls U_j with centers at points x_j, which do not contain the points x_k with $k \neq j$, and the complement U_∞ to the union of these points in S^4. The desired bundle over S^4 is given by the transition functions g_j on intersections $U_j \cap U_\infty$ and $g_{jk} = g_j g_k^{-1}$ on intersections $U_j \cap U_k$. The forms A_j on U_j and A on U_∞ define an ASD-connection in this bundle.

The constructed solution depends on $5k$ real parameters which for $k \gg 1$ is much less than the number $8k - 3$ of real parameters of the moduli space of k-instantons. In the next section we shall give a construction which allows us to construct the whole family of k-instantons for any k.

2.2. Atiyah–Ward theorem and ADHM-construction

2.2.1. **Atiyah–Ward theorem.** This theorem gives the twistor description of G-instantons in principal G-bundles $P \to S^4$.

We start from the twistor description of instantons in a principal SU(2)-bundle $P \to S^4$. Denote by $E \to S^4$ the complex vector bundle of rank 2 associated with the principal bundle $P \to S^4$. We suppose that E is provided with a Hermitian structure and A is a connection compatible with the Hermitian structure. It means that in any unitary frame $A^* = -A$.

If E is a holomorphic vector bundle we can also consider connections compatible with the holomorphic structure. A connection is called *holomorphic* if its potential A has type (1,0) in any holomorphic frame.

There is a natural relation between Hermitian and holomorphic connections established in the following way. Let E be a holomorphic vector bundle provided with a Hermitian structure. Then there exists a unique connection on E compatible with both structures. The curvature of this connection has type (1,1).

The converse of this result is also true.

Theorem 1 (Atiyah–Hitchin–Singer). *Let E be a holomorphic vector bundle over a complex manifold X provided with a Hermitian structure. If E has a Hermitian connection* *with curvature of type* (1,1), *then there exists a unique holomorphic structure on* E *such that this connection is compatible with it.*

We return now to the vector bundle $E \to S^4$ provided with the Hermitian connection A. Consider its restriction to $\mathbb{R}^4$. Then the following assertion is true (cf. [2]). The connection A is an ASD-connection if and only if its curvature has type (1,1) with respect to any complex structure on $\mathbb{R}^4$ compatible with metric and orientation.

This assertion has in fact the infinitesimal character and is proved by direct computation in local coordinates.

Now consider the twistor bundle constructed above:

$$\pi : \mathbb{CP}^3 \longrightarrow S^4.$$

Denote by $\tilde{E} := \pi^* E$ the pull-back of the bundle E to $\mathbb{CP}^3$ via the map π and by $\tilde{\nabla} = \nabla_{\tilde{A}}$ the pull-back of the covariant derivative $\nabla = \nabla_A$ to the bundle $\tilde{E}$. If the connection A is an ASD-connection, then the assertion above implies that its pull-back $\tilde{A}$ to $\tilde{E}$ defines a holomorphic structure on $\tilde{E}$, i.e., the curvature of $\tilde{A}$ is of type (1,1).

The obtained holomorphic bundle $\tilde{E} \to \mathbb{CP}^3$ is by contruction holomorphically trivial on j-invariant projective lines in $\mathbb{CP}^3$ being the fibers of the map π.

Next consider how the Hermitian structure on E behaves under the constructed correspondence between the bundles E over S^4 and $\tilde{E}$ over $\mathbb{CP}^3$. The introduction of this structure is equivalent to the introduction of an anti-linear isomorphism $\tau : E \to E^*$ such that the form $(\xi, \tau\eta)$ is positively definite. By pulling up this isomorphism to $\tilde{E}$ we shall obtain a Hermitian structure on $\tilde{E}$, i.e., an anti-linear isomorphism $\tilde{\tau} : \tilde{E} \to \tilde{E}^*$ covering the map j on $P \to S^4$. This isomorphism has the following property:

$$(\xi, \tilde{\tau}\eta) = \overline{(\xi, \tilde{\tau}\eta)},$$

i.e., defines a positive real form on $\tilde{E}$.

Theorem 2 (Atiyah–Ward theorem [6]). *There exists a bijective correspondence between*

$$\left\{\begin{matrix}\textit{moduli space of}\\ \mathrm{SU}(2)\textit{-instantons}\\ \textit{on } S^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{holomorphic vector bundles of rank 2 over } \mathbb{CP}^3\\ \textit{holomorphically trivial on } \pi\textit{-fibers and pro-}\\ \textit{vided with a positive real form}\end{matrix}\right\}.$$

For Hermitian vector bundles $E \to S^4$ of rank n we shall obtain by the Atiyah–Ward correspondence holomorphic vector bundles $\tilde{E} \to S^4$ of rank n which are holomorphically trivial on π-fibers and provided with a positive real form.

The Atiyah–Ward theorem can also be extended to arbitrary $\mathrm{Sp}(n)$-instantons where $\mathrm{Sp}(n)$ is the group of invertible quaternion matrices preserving the standard Hermitian form $\langle x, y\rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$ on $\mathbb{H}^n$. By the Atiyah–Ward correspondence they correspond to holomorphic vector bundles of rank $2n$ on $\mathbb{CP}^3$ having additional quaternionic structure. Let E be, as above, a vector bundle of rank $2n$ with a Hermitian connection. The *quaternion structure* on E is given by a skew-symmetric isomorphism α compatible with connection. By pulling up this isomorphism to $\tilde{E}$, we shall obtain a skew-symmetric holomorphic isomorphism $\tilde{\tau} : \tilde{E} \to \tilde{E}^*$. This skew-symmetric holomorphic isomorphism determines a nondegenerate skew-symmetric form on $\tilde{E}$. Combining it with the anti-linear isomorphism $\tilde{\tau}^{-1}$, we get an anti-linear isomorphism $\tilde{j} : \tilde{E} \to \tilde{E}$ covering the map j on $P \to S^4$.

We have the following variant of the Atiyah–Ward theorem for $\mathrm{Sp}(n)$-instantons.

Theorem 3 (cf. [2]). *There is a bijective correspondence between*

$$\left\{\begin{matrix}\textit{moduli space of}\\ \mathrm{Sp}(n)\textit{-instantons}\\ \textit{on } S^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{holomorphic vector bundles of rank } 2n \textit{ over } \mathbb{CP}^3 \textit{ with}\\ \textit{nondegenerate holomorphic skew-symmetric form com-}\\ \textit{patible with the anti-linear isomorphism } \tilde{j} : \tilde{E} \to \tilde{E}\end{matrix}\right\}.$$

The compatibility with skew-symmetric form means that

$$(\tilde{j}\xi, \tilde{j}\eta) = \overline{(\xi, \eta)}.$$

Note that holomorphic vector bundles $\tilde{E}$ on $\mathbb{CP}^3$ should be holomorphically trivial on π-fibers. Moreover, restriction of the Hermitian form $(\xi, \tilde{j}\eta)$ to the π-fibers should be positive definite.

There is also a purely complex generalization of this theorem. Consider it first for the future tube $\mathbb{C}M_+$. Let E be a holomorphic vector bundle of rank n over $\mathbb{C}M_+$, and let $\nabla = \nabla_A$ be the holomorphic covariant derivative acting on sections of E which is generated by a holomorphic connection A. We call this connection *anti-selfdual* (ASD) if its curvature vanishes on all α-planes. The *complex variant of the Atiyah–Ward theorem* asserts that there exists a bijective correspondence between

$$\left\{\begin{matrix}\text{moduli space of holomorphic}\\ \text{ASD-connections on } \mathbb{C}M_+\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{holomorphic vector bundles of rank } n\\ \text{on } \mathbb{PT}_+ \text{ holomorphically trivial on pro-}\\ \text{jective lines lying in } \mathbb{PT}_+\end{matrix}\right\}.$$

This theorem is based on the following *Ward construction.* Let $\tilde{E}$ be a holomorphic vector bundle over $\mathbb{PT}_+$ which is holomorphically trivial on projective lines in $\mathbb{PT}_+$. The fiber E_z of the corresponding holomorphic vector bundle $E \to \mathbb{C}M_+$ at a point $z \in \mathbb{C}M_+$ consists by definition of holomorphic sections of the bundle $\tilde{E}$ over the projective line $\mathbb{CP}^1_z$ corresponding to the point z. If two projective lines $\mathbb{CP}^1_z$ and $\mathbb{CP}^1_{z'}$ intersect, i.e., the points z and z' lie on the same complex light line, then we can identify the fibers E_z and $E_{z'}$ with each other. In this way we define on E the parallel transport along complex light lines in $\mathbb{C}M_+$ generating a holomorphic connection in E. Since the α-plane in $\mathbb{C}M_+$ corresponds to the bundle of projective lines passing through a fixed point of $\mathbb{CP}^3$, the constructed connection is automatically anti-selfdual.

For the inverse construction (from E to $\tilde{E}$) it is convenient to use the double diagram

$$\begin{array}{ccc} & \mathbb{F}_+ & \\ {\scriptstyle\mu}\swarrow & & \searrow{\scriptstyle\nu} \\ \mathbb{PT}_+ & & \mathbb{C}M_+ \end{array}$$

where $\mathbb{F}_+$ is the space of $(0,1)$-flags in $\mathbb{PT}_+$, i.e., pairs (point of $\mathbb{PT}_+$, projective line in $\mathbb{PT}_+$ containing this point). The space $\mathbb{C}M_+$ is identified here with the Grassmann manifold $G_1(\mathbb{PT}_+)$ of projective lines lying in $\mathbb{PT}_+$, and μ, ν are natural projections. Denote by E' the pull-back of E to a bundle over $\mathbb{F}_+$ via the map ν and by ∇' the pull-back of the connection ∇ to the bundle E'. Define the fibre of the bundle $\tilde{E} \to \mathbb{PT}_+$ at a point $\zeta \in \mathbb{PT}_+$ as the space of holomorphic sections $s' \in \Gamma(\mu^{-1}(\zeta), E')$ satisfying the equation

$$\nabla'_\mu s' = 0,$$

where ∇'_μ is the component of ∇' acting along the fibers of the map μ. In other words, the fibre $\tilde{E}_\zeta$ consists of horizontal holomorphic sections of E' over $\mu^{-1}(Z)$. This definition is correct due to the anti-selfduality of ∇.

The given complex version of the Atiyah–Ward theorem remains true if we replace $\mathbb{PT}_+$ in this theorem by a domain $\tilde{D}$ in $\mathbb{CP}^3$ such that projective lines lying in it correspond to the points of some domain D in $\mathbb{C}M$. This domain should have an additional property that the intersection of any complex light line with this domain is connected and simply connected.

2.2.2. **ADHM-construction.** The *ADHM-construction* yields a description of instantons on S^4. We shall present it for the case of $\mathrm{Sp}(n)$-instantons. The Atiyah–Ward theorem reduces the problem of the description of instantons on S^4 to the problem of the classification of holomorphic vector bundles on $\mathbb{CP}^3$ which are holomorphically trivial on j-invariant projective lines.

In the case of $\mathrm{Sp}(n)$-instantons it is given a quaternion vector bundle $E \to S^4$ with fibre $\mathbb{H}^n$ provided with an ASD-connection with associated covariant derivative ∇. Suppose that the Pontryagin number of this bundle is equal to $p_1(E) = -k$ for a natural k.

Denote by $[x : y]$ the quaternionic homogeneous coordinates on $S^4 = \mathbb{HP}^1$ and consider a homogeneous matrix function on $\mathbb{H}^2$ of the form

$$\Delta(x, y) = xC + yD,$$

where C, D are quaternion $k \times (k+n)$-matrices. Suppose that $\Delta(x, y)$ has maximal rank for all $(x, y) \neq (0, 0)$. Then Δ will define a nondegenerate linear transform

$$\Delta : \mathbb{H}^2 \otimes_{\mathbb{R}} W \longrightarrow V,$$

where V is the $(k+n)$-dimensional quaternion vector space and W is its k-dimensional subspace. Then the space

$$E_{(x,y)} = \operatorname{Ker} \Delta^*(x, y),$$

having for fixed (x, y) quaternion dimension n, is the fiber of the desired quaternion vector bundle.

Denote by $P_{(x,y)} : V \to E_{(x,y)}$ the operator of orthogonal projection and provide E with the standard Levi-Civita covariant derivative ∇. If we restrict E to the Euclidean space $\mathbb{R}^4 \subset S^4$ by replacing $[x : y]$ with $x := [x : 1]$, then the covariant derivative ∇ in E will be given by the formula $\nabla = Pd/dx$ and its curvature F will be equal to

$$F = PC^* d\bar{x} \left[\Delta(x)\Delta^*(x)\right]^{-2} dxCP.$$

If the matrix $\Delta(x)\Delta^*(x)$ is real for all $x \in \mathbb{H}$, then the matrix $\left[\Delta(x)\Delta^*(x)\right]^{-2}$ will commute with quaternion $d\bar{x}$ and the expression for F will contain the only form $d\bar{x} \wedge dx$ which is ASD, i.e., the form F will be anti-selfdual. It can be shown that the topological charge of the constructed connection is equal to $-k$.

The given construction of instantons admits a transparent geometric interpretation. Namely, the bundle $E \to S^4$ coincides with the preimage of the classifying bundle for the appropriate choice of the map f from S^4 to the Grassmann manifold. In more detail, consider on the Grassmann manifold $G_n(\mathbb{H}^{n+k})$ of n-dimensional subspaces in $\mathbb{H}^{n+k}$ the standard tautological bundle. It is provided with the canonical $\mathrm{Sp}(n+k)$-invariant connection determined by the orthogonal projection. The constructed bundle $E \to S^4$ is the preimage of this classifying bundle under the map $f : S^4 \to G_n(\mathbb{H}^{n+k})$ given by the matrix function $\Delta(x, y)$. Moreover, the connection ∇ on E coincides with the connection induced by the canonical connection via the map f.

In particular, the t'Hooft solution, constructed above, can be described in these terms as the $\mathrm{Sp}(1)$-bundle with its connection on S^4 coinciding with the inverse image of the classifying $\mathrm{Sp}(1)$-bundle over $\mathbb{HP}^k$ under the following map: its restriction to $\mathbb{R}^4 = \mathbb{H}$ is given by the formula

$$x \longmapsto [1 : (x - x_1)^{-1} : \dots : (x - x_k)^{-1}].$$

To define it for $x = x_j$ one should multiply its components by the quaternion $(x - x_j)$, which does not change the image of the map in homogeneous coordinates.

According to Donaldson [10], the reality condition, imposed on the matrix function $\Delta(x)$, may be rewritten in the form of commutation relations for the components of its matrix coefficients. On the other hand, the duality equations on $\mathbb{R}^4$ may also be written in the form of commutation relations on the components of the connection $\nabla(x)$. So the ADHM-construction may be considered as a transformation between the commutation relations for matrix functions on $\mathbb{R}^4$ and commutation relations for differential operators of the first order on $\mathbb{R}^4$.

2.3. Monopoles and Nahm equations

2.3.1. **Bogomolny equations.** Let G be a compact Lie group, and let A be a G-connection on the Euclidean space $\mathbb{R}^4$. Suppose that the connection A is *static* in the sense that translation in "time" x^0 generates a gauge transform of A. Such a connection may be given by the 1-form of type

$$A = \Phi dx^0 + \sum_{j=1}^{3} A_i dx^i$$

that has the coefficients which take values in the Lie algebra $\mathfrak{g}$ of G and does not depend on x^0.

The duality equations for such a form look like

$$D'\Phi = \pm *' F',$$

where A' is a G-connection on the Euclidean space $\mathbb{R}^3$, $F' = F_{A'}$ is its curvature, $D' = D_{A'}$ is the exterior covariant derivative associated with connection A', and $*'$ is the Hodge operator on $\mathbb{R}^3$. Further on we omit the primes since we shall deal only with connections on $\mathbb{R}^3$.

So starting from this point A is a G-connection in the (trivial) principal G-bundle $P \to \mathbb{R}^3$, Φ is the section of the adjoint bundle $\operatorname{ad} P$, and we are interested in the solutions of the equation

$$D_A\Phi = \pm * F_A$$

called the *Bogomolny equation* [8]. Denoting the form $*F_A$ by B and omitting the subindex A, we can rewrite this equation as

$$D\Phi = \pm B.$$

In physical language Φ is called the *Higgs field* and the form B is interpreted as a magnetic field.

We introduce the *Yang–Mills–Higgs action* functional

$$S_{\mathrm{YMH}}(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left(\|F\|^2 + \|D\Phi\|^2\right) d^3x.$$

The critical points of this functional are called the *Yang–Mills–Higgs fields* and satisfy the following *Euler–Lagrange equation*:

$$\begin{cases} *D(*F) = [D\Phi, \Phi], \\ \qquad \Box\Phi = 0, \end{cases}$$

where $\Box\Phi = *D(*D\Phi)$.

In order to guarantee the finiteness of the action S_{YMH} we impose on the considered fields the following asymptotic conditions otherwise called the *Prasad-Sommerfield limit* [19]:

$$\|\Phi\| \longrightarrow 1, \quad \|D\Phi\| \longrightarrow 0, \quad \|F\| \longrightarrow 0$$

uniformly for $|x| \to \infty$.

Consider in more detail the case $G = \mathrm{SU}(2)$. Assign to the Yang–Mills–Higgs field a *topological invariant* given by the formula

$$k = \frac{1}{4\pi}\int_{\mathbb{R}^3} \mathrm{tr}(F \wedge D\Phi) = \lim_{R\to\infty} \frac{1}{4\pi}\int_{S^2_R} \mathrm{tr}(\Phi F).$$

This invariant coincides with the degree of the map of the sphere S^2_R of sufficiently large radius R to the Lie algebra su(2) given by the Higgs field

$$\Phi : S^2_R \longrightarrow \{\Phi \in \mathrm{su}(2) : \|\Phi\| \approx 1\} = S^2.$$

This invariant, called the *topological charge* k, may also be computed by the formula

$$k = -\frac{1}{4\pi} \lim_{R\to\infty} \int_{S^2_R} \mathrm{tr}(\Phi d\Phi \wedge d\Phi).$$

The Yang–Mills–Higgs action may be rewritten in the form

$$S_{\mathrm{YMH}}(A, \Phi) = \frac{1}{2}\int_{\mathbb{R}^3} \|F \mp D\Phi\|^2 d^3x \pm 4\pi.$$

This is the so-called *Bogomolny transform* (cf. [8]). The last formula implies that

$$S_{\mathrm{YMH}}(A, \Phi) \geq 4\pi|k|,$$

and the equality here is attained only on solutions of the *Bogomolny equation*

$$D\Phi = \pm * F.$$

In other words, solutions of the Bogomolny equation with finite action realize local minima of the action S_{YMH}. These solutions are called *monopoles* (or *BPS-monopoles* in honour of Bogomolny–Prasad–Sommerfield) because of their close relation to the *Dirac monopole*.

For monopoles with charge k the asymptotic conditions for the Higgs field Φ may be written in a more precise form:

$$\|\Phi\| = 1 - \frac{k}{r} + O(\frac{1}{r^2}) \quad \text{for } r = |x| \to \infty.$$

Apart from monopoles, the functional S_{YMH} also has other critical points found by Taubes [23]. All of them are not stable (i.e., they are saddle points) and have sufficiently large Morse index (namely, the index μ of a nonminimal critical point of the functional $S_{\mathrm{YMH}}(A, \Phi)$ for a nonminimal Yang–Mills–Higgs field with topological charge k is greater than $|k| + 1$).

We have introduced monopoles as solutions of static duality equations on $\mathbb{R}^4$. They can also be obtained from the axis-symmetric solutions of the duality equations in $\mathbb{R}^4$ with topological charge k by taking the limit of such solutions for $k \to \infty$.

2.3.2. **Examples of monopoles.** Identify $\mathbb{R}^3$ with the space of pure imaginary quaternions so that

$$x = (x^1, x^2, x^3) \in \mathbb{R}^3 \longleftrightarrow x = ix^1 + jx^2 + kx^3 \in \mathrm{Im}\,\mathbb{H}.$$

The monopole with charge ± 1, constructed by Prasad and Sommerfield in [19], has the form

$$A = \left(\frac{1}{|x|} - \frac{1}{\mathrm{sh}|x|}\right) \mathrm{Im}\left\{\frac{dx \cdot x}{|x|}\right\}, \quad \Phi = \pm\left(\frac{1}{|x|} - \frac{1}{\mathrm{th}|x|}\right)\frac{x}{|x|}. \tag{2.2}$$

An arbitrary (± 1)-monopole may be obtained from the one above by making in the last formula a change of variables $x \mapsto x - x_0$, where x_0 is an arbitrary point of $\mathbb{R}^3$. The obtained solution will depend on 3 real parameters.

It may be shown that the dimension of the moduli space $\mathcal{M}_k$ of monopoles with charge $-k$ is equal to $4k-1$. There is a construction of Taubes [22] which allows us to construct a family of monopoles depending on $3k$ real parameters. Namely, according to Taubes's theorem there exists a positive constant d such that for any collection of points $\{x_1,\dots,x_k\}$ in $\mathbb{R}^3$ with distance between them greater than d there exists a monopole (A,Φ) with topological charge $-k$. The Taubes solution looks approximately like the sum of BPS-monopoles with centers at given points $x_1,\dots,x_k$ in the sense that the zeros of the Higgs field Φ are close to the points $x_1,\dots,x_k$ and the local topological charge of Φ in these zeros is equal to -1.

2.3.3. **Nahm–Hitchin construction.** We can associate with any monopole its spectral curve. In order to construct it we use the twistor considerations by taking for the twistor space the tangent bundle $T\mathbb{P}^1$ of the Riemann sphere. This bundle may be identified with the space of oriented lines in $\mathbb{R}^3$ if we parameterize such a line by its tangent vector u and shortest distance vector v.

Associate with a point $x\in\mathbb{R}^3$ the bundle of oriented lines passing through this point. It may be identified with the holomorphic section of the tangent bundle $T\mathbb{P}^1\to\mathbb{P}^1$ which is real (invariant) with respect to the real structure given by the change of orientation of every line in $\mathbb{R}^3$ to the opposite one. In this case we have the following analog of twistor correspondence:

$$\left\{\text{point of } \mathbb{R}^3\right\}\longrightarrow\left\{\begin{matrix}\text{bundle of oriented lines passing through this point}\\ \equiv \text{holomorphic real section of } T\mathbb{P}^1\to\mathbb{P}^1\end{matrix}\right\}.$$

On the other hand,

$$\left\{\text{oriented line in } \mathbb{R}^3\right\}\longrightarrow\left\{\text{point of } T\mathbb{P}^1\right\}.$$

In contrast with the twistor space $\mathbb{CP}^3$ the space $T\mathbb{P}^1$ is not compact. But it may be compactified by replacing the line bundle $T\mathbb{P}^1\to\mathbb{P}^1$ with the bundle $\widehat{T\mathbb{P}^1}\to\mathbb{P}^1$ of tangent projective lines.

The constructed twistor correspondence allows us to apply to monopoles the ideas and methods developed for the instantons. In particular, the Atiyah–Ward theorem in the case of monopoles acquires the following form:

$$\frac{\left\{\begin{matrix}\text{solutions } (A,\Phi)\\ \text{of Bogomolny}\\ \text{equations on } \mathbb{R}^3\end{matrix}\right\}}{\{\text{gauge equivalence}\}}\longleftrightarrow\left\{\begin{matrix}\text{equivalence classes of holomorphic vector bundles}\\ \text{of rank 2 on } T\mathbb{P}^1 \text{ holomorphically trivial on real}\\ \text{holomorphic sections and provided with a positive}\\ \text{real form}\end{matrix}\right\}.$$

The construction of this correspondence is close to the original Ward construction. Denote by $E\to\mathbb{R}^3$ the vector bundle of rank 2 associated with the principal bundle $P\to\mathbb{R}^3$. Let $\nabla=\nabla_A$ be the covariant derivative generated by the connection A, acting on smooth sections of E. Then the fiber $\tilde{E}_z$ of the bundle $\tilde{E}$ of rank 2 over $T\mathbb{P}^1$ at a point $z\in T\mathbb{P}^1$ is defined in the following way. Denote by γ_z the oriented line in $\mathbb{R}^3$ corresponding to the point z. The fiber $\tilde{E}_z$ consists, by definition, of smooth sections $s\in\Gamma(\gamma_z,E)$ of E over the line γ_z satisfying the equation

$$(\nabla_\gamma - i\Phi)s=0\,, \tag{2.3}$$

where ∇_γ is component of ∇ acting along γ_z.

Thus, the Bogomolny equation reduces to the family of ordinary differential equations of the form (2.3) on lines in $\mathbb{R}^3$. The main characteristic of this family is its *spectral curve* consisting of the points $z\in T\mathbb{P}^1$ for which the equation (2.3) has an L^2-solution along the line γ_z.

We have described the transition from the monopoles to spectral curves. Now we study the relation between monopoles and Nahm equations. This relation is established with the help of an infinite-dimensional analogue of ADHM-construction.

The *Nahm equations* [16] are a system of ordinary differential equations on matrix functions T_1, T_2, T_3 of a variable $t \in [0,2]$ of the form

$$\frac{dT_1}{dt} = [T_2, T_3], \quad \frac{dT_2}{dt} = [T_3, T_1], \quad \frac{dT_3}{dt} = [T_1, T_2]. \tag{2.4}$$

It is assumed that the functions $T_i(t)$ extend to meromorphic matrix functions defined in a complex neighborhood of the segment $[0,2]$ with only simple poles at the points $t = 0, 2$. Moreover, we shall impose on them the *reality* conditions

$$T_i(z) + \bar{T}_i(2-z) = 0, \quad T_i^*(z) + T_i(z) = 0 \tag{2.5}$$

and the *nondegeneracy* condition: the representations of the group SU(2), determined by the residues of the functions $T_i(z)$ in the poles, should be irreducible.

Now consider, as in ADHM-construction, the quaternion matrix function $\Delta(x,y)$ of homogeneous quaternion coordinates x, y. Its restriction to the space $\mathbb{H} = \mathbb{R}^4$ has the form $\Delta(x) = xC + D$. The operator $\Delta(x) : W \to V$, mapping a real vector space W into a quaternion vector space V, in the case of monopoles is an ordinary differential operator, and the spaces W and V are infinite-dimensional.

Now describe the Nahm construction in more detail. Since the Bogomolny equations coincide with the duality equations for static Yang–Mills fields which do not depend on the variable x^0, our map $\Delta(x)$ should satisfy the following condition:

1) $\Delta(x + y^0) = U(y^0)^{-1}\Delta(x)U(y^0)$, where $y^0 \mapsto U(y^0)$ is a representation of the group $\mathbb{R}$ in the group of quaternion unitary transformations of the space V.

Moreover, in the case of monopoles the same conditions, as in the case of instantons, should be satisfied, namely:

2) the map $\Delta^*(x)\Delta(x)$ should be real for all $x \in \mathbb{H}$;
3) the map $\Delta^*(x)\Delta(x)$ should be invertible for all $x \in \mathbb{H}$;
4) the kernel of the map $\Delta^*(x)$ should have quaternion dimension 1 for all $x \in \mathbb{H}$.

Now introduce the space V. Denote by H^0 the space $L^2(0,2)$ and define a real structure on H^0 by the formula $\sigma(f)(z) := \bar{f}(2-z)$. The space V, equal to

$$V = H^0 \otimes \mathbb{C}^k \otimes \mathbb{H},$$

is a quaternion vector space. For the real subspace W we take

$$W = \{f \in H^1 \otimes \mathbb{R}^k : \ f(0) = f(2) = 0\},$$

where H^1 is the Sobolev space $H^1(0,2)$.

Denote by e_1, e_2, e_3 the operators of left multiplication by imaginary units i, j, k, respectively, and set $e_0 = 1$. Define the map $\Delta(x) : W \to V$ as a differential operator of the form

$$\Delta(x)f = \left(\sum_{j=0}^{3} x^j e_j\right) f + i\frac{df}{dz} + i\sum_{j=1}^{3} T_j(z)e_j f,$$

where $T_j(z)$ are $(k \times k)$-matrix functions which are meromorphic in z in a complex neighborhood of the segment $[0,2]$ with unique simple poles at its ends. This operator has the desired form $xC + D$, where $C = I$ and $D = id/dz + i\sum_{j=1}^{3} T_j e_j$. The constructed operator $\Delta(x)$ satisfies conditions 1)–4), imposed on it earlier, if the matrix functions T_j satisfy the Nahm equations together with reality and nondegeneracy conditions. Then the ADHM-construction, being applied to the operator $\Delta(x)$, will give a solution of Bogomolny SU(2)-equations.

As it was remarked earlier the Bogomolny equations coincide with the duality equations for static Yang–Mills fields in $\mathbb{R}^4$, i.e., the fields not depending on the variable x^0. On the other hand, the Nahm equations (2.4) are equivalent to the duality equations for the connection

$$\sum_{\mu=0}^{4} T_\mu dx^\mu,$$

where $T_0 = 0$, and T_1, T_2, T_3 depend only on the variable $x^0 = t$. Hence we can consider the Nahm construction as a transform relating solutions of duality equations, depending on one variable, to the solutions of duality equations, depending on three variables. Recall that ADHM-construction is also a transform relating matrices, which are solutions of a system of commutation relations, with solutions of the duality equations depending on four variables. So both constructions may be considered as nontrivial duality transformations between different kinds of commutation relations.

Part 3. 2-dimensional models

3.1. 2-dimensional Yang–Mills–Higgs model

3.1.1. **Yang–Mills–Higgs model on $\mathbb{R}^2$.** Consider the *Yang–Mills–Higgs action* on $\mathbb{R}^2$ with parameter $\lambda > 0$ of the following form:

$$S^\lambda_{\text{YMH}}(A, \Phi) = \frac{1}{2}\int_{\mathbb{R}^2} \left\{ \|F\|^2 + \|D\Phi\|^2 + \frac{\lambda}{4}(\|\Phi\|^2 - 1)^2 \right\} d^2x.$$

Impose again the asymptotic conditions

$$\|\Phi\| \longrightarrow 1, \quad \|D\Phi\| \longrightarrow 0, \quad \|F\| \longrightarrow 0$$

uniformly for $|x| \to \infty$.

We restrict first to the Abelian case, i.e., we shall assume that

$$A = -iA_0 dx^0 - iA_1 dx^1$$

is a 1-form on $\mathbb{R}^2$ with smooth real-valued coefficients A_0, A_1 and that Φ is a complex scalar field given by a smooth complex-valued function on $\mathbb{R}^2$. Introduce *topological charge* given by the formula

$$k = \frac{1}{2\pi}\int_{\mathbb{R}^2} F.$$

It can also be defined as the degree of the map of the circle S^1_R of sufficiently large radius R into the topological circle coinciding with the image of $\Phi(S^1_R)$.

The *Euler–Lagrange equations* for the action S^λ_{YMH} have the form

$$\begin{cases} *d(*F) = \bar{\Phi}D\Phi - \Phi\overline{D\Phi}, \\ \Box\Phi = \dfrac{\lambda}{2}(|\Phi|^2 - 1)\Phi, \end{cases}$$

where $\Box\Phi = *D(*D\Phi)$.

In the selfdual case ($\lambda = 1$) the functional S^1_{YMH} may be rewritten, using the *Bogomolny transform*, in the form

$$S^1_{\text{YMH}}(A, \Phi) = \frac{1}{2}\int_{\mathbb{R}^2} \left\{ \|D\Phi \mp i(*D\Phi)\|^2 + \left| *F \pm \frac{1}{2}(|\Phi|^2 - 1)^2 \right|^2 \right\} d^2x,$$

which implies that

$$S^1_{\text{YMH}}(A, \Phi) \geq \pi|k|,$$

and the equality is attained here if and only if the terms in the brackets vanish. Rewrite them in the complex form by setting $z = x^0 + ix^1$, $\alpha = \frac{1}{2}(A_0 - iA_1)$, $\bar{\alpha} = \frac{1}{2}(A_0 + iA_1)$. Then the local minima of the functional S^1_{YMH} for $k \geq 0$ will satisfy the equations

$$\begin{cases} \bar{\partial}_\alpha \Phi = 0\,, \\ F_{01} + \dfrac{1}{2}(|\Phi|^2 - 1) = 0\,, \end{cases}$$

where $F_{01} = \partial_0 A_1 - \partial_1 A_0$, $\bar{\partial}_\alpha = \bar{\partial} - i\bar{\alpha}$, $\bar{\partial} = \partial/\partial\bar{z}$.

For $k < 0$ we obtain analogous equations

$$\begin{cases} \partial_\alpha \Phi = 0\,, \\ F_{01} - \dfrac{1}{2}(|\Phi|^2 - 1) = 0\,. \end{cases}$$

Solutions of the first system of equations are called the *vortices*, and solutions of the second system are called the *anti-vortices*. In contrast with the Yang–Mills–Higgs equations in $\mathbb{R}^3$, the local minima of S^1_{YMH} in $\mathbb{R}^2$ exhaust all its critical points. It is an effect of the 2-dimensionality of the considered model.

3.1.2. **Theorem of Taubes.** A description of solutions of vortex equations was given by Taubes (cf. [15]). Assume first that $k \geq 0$ and $\{z_1, \dots, z_k\}$ is an arbitrary collection of k points in the complex plane some of which may coincide. Denote by k_j the multiplicity of the point z_j in the collection $\{z_1, \dots, z_k\}$. Then there exists a unique (up to gauge equivalence) C^∞-smooth solution (A, Φ) of vortex equations having the following properties:

1) the set of zeros of Φ coincides precisely with the collections of points $\{z_1, \dots, z_k\}$ (with the same multiplicities) and in a neighborhood of the point z_j,

$$\Phi \sim c_j(z - z_j)^{k_j},\ c_j \neq 0;$$

2) the topological charge k of the solution (A, Φ) is equal to the sum $\sum k_j$ over all distinct points in the collection $\{z_1, \dots, z_k\}$.

For $k < 0$ the result is formulated in a similar way; one should only replace Φ with $\bar{\Phi}$ in the first condition and set k equal to $-\sum k_j$.

Any solution of Euler–Lagrange equations with finite action is gauge equivalent either to some k-vortex, or $|k|$-anti-vortex solution depending on the sign of k (cf. [15]). In this case there is only a Taubes conjecture asserting that for the Abelian Yang–Mills–Higgs model, governed by the functional S^λ_{YMH} on $\mathbb{R}^2$, in the case $\lambda < 1$ it should exist for any charge k a unique (up to gauge equivalence and translations of $\mathbb{R}^2$) critical point of this functional which is a local minimum. Moreover, all topological charges will be concentrated in the unique zero of the function Φ and the solution (A, Φ) will be central symmetric with respect to this zero. For $\lambda > 1$ the functional S^λ_{YMH} should have a unique critical point which is stable if and only if the topological charge is equal to $k = 0, \pm 1$.

From some physical considerations we can expect that there exists a duality between the critical points of the functional S^λ_{YMH} and (probably singular) solutions of the Euler–Lagrange equations for the functional $S^{1/\lambda}_{\mathrm{YMH}}$.

3.2. Higgs bundles and Hitchin equations

3.2.1. **Hitchin equations.** Consider the duality equations in $\mathbb{R}^2$ which are obtained from the duality equations in $\mathbb{R}^4$ under the condition that the coefficients of the connection do not depend on two variables.

Let

$$\mathcal{A} = \sum_{j=0}^{3} A_j dx^j$$

be a G-connection on $\mathbb{R}^4$ with coefficients not depending on the variables x^2 and x^3. Denote by A the forms

$$A = A_0 dx^0 + A_1 dx^1$$

and

$$\varphi_1 := A_2,\ \varphi_2 := A_3,\ \varphi := \varphi_1 - i\varphi_2.$$

Then the selfduality equations for the connection $\mathcal{A}$ will rewrite in the form

$$\begin{cases} [\nabla_0 + i\nabla_1, \varphi] = 0, \\ F_A = \dfrac{i}{2}[\varphi, \varphi^*], \end{cases}$$

where F_A is the curvature of A on $\mathbb{R}^2$ and ∇ is the covariant derivative generated by the connection A.

Introduce the complex coordinate $z = x^0 + ix^1$ on $\mathbb{R}^2$ and set

$$\Phi = \frac{1}{2}\varphi dz,\ \Phi^* = \frac{1}{2}\varphi^* d\bar{z}.$$

Then the selfduality equations will take the form

$$\begin{cases} \bar{\partial}_A \Phi = 0\,, \\ F_A + [\Phi, \Phi^*] = 0\,. \end{cases}$$

Here A is a connection in the principal G-bundle $P \to \mathbb{C}$, Φ is a smooth $(1,0)$-form on $\mathbb{C}$ with values in the complexified adjoint bundle $\mathrm{ad}^{\mathbb{C}} P$, and $\bar{\partial}_A$ is the $\bar{\partial}$-operator of exterior covariant derivation generated by the $(0,1)$-component $A^{0,1}$ of A. The above equations, called the *Hitchin equations*, are conformally invariant so one considers them on an arbitrary Riemann surface M (however, from now on we shall restrict to the case of compact Riemann surfaces).

Let $G = \mathrm{SU}(2)$, and let $E \to M$ be a complex vector bundle of rank 2 associated with the principal SU(2)-bundle $P \to M$. The Hitchin equations for Riemann surfaces M of genus 0 and 1 have no nontrivial solutions. On the other hand, such solutions do exist for Riemann surfaces M of genus $g > 1$ and will be studied later on in detail.

We close this section with the following remarks. Suppose that the genus of M is strictly greater than 1 and the bundle E is *decomposable*, i.e., $E = L \oplus L^*$ for some holomorphic line bundle L. Then the Hitchin equations take the form of the *vortex equation*

$$F_1 + 2(1 - \|\alpha\|^2)\omega = 0\,,$$

where α is a quadratic differential on M, ω is the Kähler form on M normalized by the condition $\int_M \omega = 2\pi$, and F_1 is the curvature of a U(1)-connection on L. A unique for given α solution of the last equation determines on M the metric of constant negative curvature -4. Moreover, the space of quadratic differentials on M, parameterizing the set of all solutions of this equation, is naturally diffeomorphic to the Teichmüller space of metrics of constant negative curvature on M.

We do not know if there is an analog of ADHM-construction for Hitchin equations. If such constructions do exist, then, by analogy with the 4-dimensional and 3-dimensional cases, it should yield a nontrivial duality transformation between solutions of Hitchin equations.

3.2.2. **Higgs bundles.** Let M be a compact Riemann surface of genus $g \geq 2$, and let $E \to M$ be a Hermitian vector bundle provided with a smooth Hermitian metric H. Suppose that E is provided with a *holomorphic structure* determined by the $\bar{\partial}$-operator $\bar{\partial}_E$. To emphasize the availability of the holomorphic structure we shall denote this holomorphic bundle by $(E, \bar{\partial}_E)$ and the sheaf of its holomorphic sections by $\mathcal{E}$. We shall often identify $(E, \bar{\partial}_E)$ with the sheaf $\mathcal{E}$.

If $\mathcal{S} \subset \mathcal{E}$ is a holomorphic subbundle with quotient sheaf $\mathcal{Q}$, then the smooth decomposition $E = S \oplus Q$ allows us to represent $\bar{\partial}_E$ in the form

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_S & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix}, \tag{3.1}$$

where $\beta \in \Omega^{0,1}(M, \mathrm{Hom}(Q, S))$ is called the *2nd fundamental form of subbundle S*. In this case S can be given by the orthogonal projection $\pi : E \to S$ having the following properties:

$$\pi^2 = \pi, \ \pi^* = \pi \quad \text{and} \quad (I - \pi)\bar{\partial}_E = 0. \tag{3.2}$$

These conditions imply that $\operatorname{tr} \pi = \text{const}$ and $\beta = -\bar{\partial}_E \pi$. So we have a bijective correspondence between

$$\left\{\begin{matrix}\text{holomorphic sub-}\\ \text{bundles in } \mathcal{E}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{orthogonal projectors in } E \text{ satisfying}\\ \text{conditions (3.2)}\end{matrix}\right\}.$$

Suppose that the bundle E is provided with a connection A with associated covariant derivative $\nabla \equiv \nabla_A$ compatible with the Hermitian structure. Such a connection is called *Hermitian* and satisfies the condition

$$d\langle s_1, s_2 \rangle_H = \langle d_A s_1, s_2 \rangle_H + \langle s_1, d_A s_2 \rangle_H,$$

where d_A is the exterior covariant differential generated by the connection A and s_1, s_2 are smooth sections of E. The curvature F_A of the Hermitian connection A is given by a 2-form $F_A \in \Omega^2(M, \operatorname{ad} E)$, where $\operatorname{ad} E$ denotes the bundle of Hermitian endomorphisms of E. If the connection A induces a fixed connection in the bundle $\det E$ (which is often assumed in the sequel), then $\mathrm{ad}_0 E$ (resp., $\mathrm{ad}_0^{\mathbb{C}} E$) denotes the bundle of traceless skew-Hermitian (resp., complex traceless) endomorphisms of E.

Holomorphic line bundles $L \to M$ are determined, as it is known, by the *divisors* of the form

$$\mathcal{D} = \sum_{i=1}^{N} m_i z_i,$$

where m_i, $i = 1, \dots, N$, are integers, and $z_1, \dots, z_N$ are points of M. The complex line bundle, determined by the divisor $\mathcal{D}$, is denoted by $\mathcal{L} = \mathcal{O}(\mathcal{D})$, and its *degree* $\deg L$, equal to $c_1(L)$, coincides with the degree of the divisor $\deg \mathcal{D} = \sum_{i=1}^{N} m_i$. The *degree of a vector bundle E* is by definition

$$\deg E := \deg(\det E).$$

We call by the *slope of a holomorphic vector bundle $\mathcal{E}$* the quantity

$$\mu(E) = \deg E / \operatorname{rank} E.$$

In the case when the line bundle $\mathcal{L} = \mathcal{O}(\mathcal{D})$ has a nonzero holomorphic section, the corresponding divisor is linearly equivalent to an *effective divisor* (for which all $m_i \geq 0$), so $\deg L \geq 0$.

We introduce the *contraction operator* $\Lambda : \Omega^2(M) \to \Omega^0(M)$ determined by the equality

$$\Lambda(f\omega) = f$$

for any smooth function f on M. This definition is extended to forms from $\Omega^2(M, \operatorname{ad} E)$.

If $\mathcal{S}$ is a holomorphic vector subbundle of a Hermitian holomorphic vector bundle $\mathcal{E}$, given by the orthogonal projector π, then there is an explicit formula for its degree, analogous to Chern–Weil formula

$$\deg \mathcal{S} = \frac{1}{2\pi} \int_M \operatorname{tr} \left(\pi i \Lambda F_{(\bar{\partial}_E, H)} \right) \omega - \frac{1}{2\pi} \int_M |\beta|^2 \omega. \tag{3.3}$$

Definition 1. A holomorphic vector bundle $\mathcal{E}$ is called *stable* (resp., *semistable*) if for any holomorphic vector subbundle $\mathcal{S} \subset \mathcal{E}$ of rank $0 < \operatorname{rank} \mathcal{S} < \operatorname{rank} \mathcal{E}$ the following inequality holds:

$$\mu(\mathcal{S}) < \mu(\mathcal{E}) \text{ (resp., } \mu(\mathcal{S}) \leq \mu(\mathcal{E})).$$

The bundle $\mathcal{E}$ is called *polystable* if it is the direct sum of stable bundles with the same slope.

It is evident that all holomorphic line bundles are stable. Moreover, if a holomorphic vector bundle $\mathcal{E}$ is (semi)stable and $\mathcal{L}$ is a holomorphic line bundle, then the bundle $\mathcal{E} \otimes \mathcal{L}$ is also (semi)stable.

The *extension* of a holomorphic vector bundle $\mathcal{S}$ *via* a holomorphic subbundle $\mathcal{Q}$ is a holomorphic vector bundle $\mathcal{E}$ which can be included into the exact sequence of sheaf homomorphisms

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{3.4}$$

The sequence (3.4) *splits* if there exists a map $\mathcal{Q} \to \mathcal{E}$ which is the right inverse to the projection $\mathcal{E} \to \mathcal{Q}$.

A connection ∇ is called *projectively flat* if

$$i \Lambda F_\nabla = \mu I,$$

where $\mu = \text{const}$. In this case the relation $\mu = \mu(E)$ holds.

Theorem 4 (Narasimhan–Seshadri [17])**.** *A holomorphic vector bundle $\mathcal{E} \to M$ admits a projectively flat connection if and only if $\mathcal{E}$ is polystable.*

Definition 2. A *Higgs bundle* is a pair $(\mathcal{E}, \Phi)$ consisting of a holomorphic vector bundle $\mathcal{E}$ and holomorphic section Φ of the bundle $K \otimes \operatorname{ad} E$, where K is the canonical bundle of the manifold M. A pair $(\mathcal{E}, \Phi)$ is called *stable* (resp., *semistable*) if for any Φ-invariant holomorphic subbundle $\mathcal{S} \subset \mathcal{E}$ of rank $0 < \operatorname{rank} \mathcal{S} < \operatorname{rank} \mathcal{E}$ the following inequality holds:

$$\mu(\mathcal{S}) < \mu(\mathcal{E}) \text{ (resp., } \mu(\mathcal{S}) \leq \mu(\mathcal{E})).$$

A Higgs bundle $(\mathcal{E}, \Phi)$ is called *polystable* if it is the direct sum of stable Higgs bundles with the same slope.

Problem 1. Let $f : (\mathcal{E}_1, \Phi_1) \to (\mathcal{E}_2, \Phi_2)$ be a holomorphic homomorphism of Higgs bundles, i.e., the relation $\Phi_2 f = f \Phi_1$ holds. Suppose that the bundles $(\mathcal{E}_i, \Phi_i)$, $i = 1, 2$, are semistable and $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2)$. Then $f \equiv 0$. If we have the equality $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$ and one of the bundles is stable, then either $f \equiv 0$ or f is an isomorphism.

A *Higgs subbundle* in a Higgs bundle $(\mathcal{E}, \Phi)$ is a Φ-invariant holomorphic subbundle $\mathcal{S} \subset \mathcal{E}$. The restriction $\Phi_S := \Phi|_S$ converts this subbundle into a Higgs bundle $(\mathcal{S}, \Phi_S)$ for which the embedding $\mathcal{S} \hookrightarrow \mathcal{E}$ is a map of Higgs bundles. In an analogous way one can define the structure of Higgs bundle on the quotient $\mathcal{Q} = \mathcal{E}/\mathcal{S}$.

Definition 3. Let $(\mathcal{E}, \Phi)$ be a Higgs bundle. The *Harder–Narasimhan filtration* on $(\mathcal{E}, \Phi)$ (for short, HN-filtration) is a filtration by Higgs subbundles of the form

$$0 = (\mathcal{E}_0, \Phi_0) \subset (\mathcal{E}_1, \Phi_1) \subset \cdots \subset (\mathcal{E}_l, \Phi_l) = (\mathcal{E}, \Phi),$$

in which the quotients $(\mathcal{Q}_i, \Phi_{Q_i}) = (\mathcal{E}_i, \Phi_i)/(\mathcal{E}_{i-1}, \Phi_{i-1})$ are semistable. It is also required that the following inequalities hold:

$$\mu(\mathcal{Q}_i) > \mu(\mathcal{Q}_{i-1}).$$

The associated graded object

$$\mathrm{gr}_{HN}(\mathcal{E}, \Phi) = \bigoplus_{i=1}^{l} (\mathcal{Q}_i, \Phi_{Q_i})$$

in this case is uniquely determined by the isomorphism class of the bundle $(\mathcal{E}, \Phi)$.

The collection $\vec{\mu}(\mathcal{E}, \Phi) = (\mu_1, \dots, \mu_n)$ of n numbers, where each of the μ_i's is repeated as many times as the rank of Q_i, is called the *HN-type* (*Harder–Narasimhan type*) of the Higgs bundle $(\mathcal{E}, \Phi)$. It is an important invariant of Higgs bundles.

3.2.3. **The moduli spaces of Higgs bundles.** Denote by $\mathcal{A}_E$ the space of Hermitian connections in a Hermitian vector bundle $E \to M$ of rank n. It is an infinite-dimensional affine space with local model $\Omega^1(M, \operatorname{ad} E)$.

The *group of gauge transformations* is by definition

$$\mathcal{G}_E = \{g \in \Omega^0(M, \operatorname{End} E) : \ gg^* = I\}$$

(in the case when the bundle $\det E$ is fixed we impose on $\mathcal{G}_E$ the additional condition $\det g = 1$). This group acts on $\mathcal{A}_E$ by sending the covariant differential d_A to the new covariant differential

$$d_{g(A)} = g \circ d_A \circ g^{-1}.$$

The space $\mathcal{A}_E$ may also be considered as the space of complex structures on $E \to M$. Indeed, from every Hermitian connection on $E \to M$ we can construct a $\bar{\partial}$-operator given by the (0,1)-component of the connection. This operator determines a complex structure on E since the (0,2)-component of the curvature vanishes in the case of Riemann surfaces. On the other hand, a $\bar{\partial}$-operator on $E \to M$ determines a unique Hermitian connection on E with the (0,1)-component equal to the original $\bar{\partial}$-operator. Such a connection is called the *Chern connection.* The corresponding covariant differential d_A decomposes into the sum of two operators d'_A and d''_A sending sections of E to forms from $\Omega^{1,0}(M, E)$ and $\Omega^{0,1}(M, E)$, respectively.

Considering $\mathcal{A}_E$ as the space of complex structures on $E \to M$, we can define an action of the *complexified group of gauge transformations* $\mathcal{G}_E^{\mathbb{C}}$ on $\mathcal{A}_E$. Namely, if the original connection corresponds to the $\bar{\partial}$-operator $\bar{\partial}_E = d''_A$, then the transformed connection $g(A)$ will correspond to the $\bar{\partial}$-operator $g \circ \bar{\partial}_E \circ g^{-1}$.

The space of Higgs bundles, by definition, is identified with

$$\mathcal{B}_E = \{(A, \Phi) \in \mathcal{A}_E \times \Omega^0(M, K \otimes \operatorname{ad}^{\mathbb{C}} E) : \ d''_A \Phi = 0\},$$

and its subspace, consisting of semistable Higgs bundles, is denoted by $\mathcal{B}_E^{ss}$.

Definition 4. The *moduli space of semistable Higgs bundles* of rank n (with fixed $\det E$) on M is identified with the categoric quotient

$$\mathfrak{M}_E^{(n)} = \mathcal{B}_E^{ss} // \mathcal{G}_E^{\mathbb{C}}.$$

Recall the definition of the categorical quotient. Let X be a complex manifold provided with a holomorphic action of a complex Lie group $G^{\mathbb{C}}$. Introduce on X the following equivalence relation: $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$ for all holomorphic functions f invariant under the action of the group $G^{\mathbb{C}}$. Denote by $\pi : X \to X/\sim$ the natural projection. We refer to the *categorical quotient* $X//G^{\mathbb{C}}$ as the Hausdorff topological space $X/\sim$ provided with the structure sheaf $\mathcal{O}(X//G^{\mathbb{C}})$ defined in the following way.

For the arbitrary open subset $U \subset X/\sim$ the algebra $\mathcal{O}(X//G^{\mathbb{C}})(U)$ consists of continuous complex-valued functions on U which pull-back by the map π to holomorphic $G^{\mathbb{C}}$-invariant functions on $\pi^{-1}(U)$.

In the case when X is a Stein space and the group $G^{\mathbb{C}}$ is reductive (i.e., it coincides with the compexification of a real compact Lie group), the space $X//G^{\mathbb{C}}$ is also Stein and the projection π is an open holomorphic map. Moreover, every fibre of π is connected and contains a unique closed orbit. Note that these assertions are, generally speaking, not true for the usual quotient coinciding with the space of orbits $X/\sim$.

The quotient $\mathcal{G}_E^{\mathbb{C}}/\mathcal{G}_E$ may be identified with the space of Hermitian metrics on E. Hence we can study the behaviour of various functionals on the orbits of the group $\mathcal{G}_E^{\mathbb{C}}$ in $\mathcal{A}_E/\mathcal{G}_E$ by two methods: either by changing the complex structure $\bar{\partial}_E$, simultaneously fixing the Hermitian metric H, or by changing the Hermitian metric H, simultaneously fixing the complex structure $\bar{\partial}_E$.

We introduce the following notation:

$$D'' = d''_A + \Phi, \quad D' = d'_A + \Phi^*.$$

The Kähler form ω and Hermitian metric H on E determine an L^2-inner product on E and $\operatorname{End} E$. For this inner product (in the case $\Phi = 0$) we have the following *Kähler identities*:

$$(D'')^* = -i[\Lambda, D'],\ (D')^* = i[\Lambda, D''].$$

The infinitesimal structure of the moduli space is determined by the deformation complex $C(A,\Phi)$ which is obtained by the differentiation of the condition $d''_A\Phi = 0$ and the action of the group of gauge transformations

$$0 \longrightarrow \Omega^0(M, \operatorname{ad}^{\mathbb{C}} E) \xrightarrow{D''} \Omega^{1,0}(M, \operatorname{ad}^{\mathbb{C}} E) \oplus \Omega^{0,1}(M, \operatorname{ad}^{\mathbb{C}} E) \xrightarrow{D''} \Omega^{1,1}(M, \operatorname{ad}^{\mathbb{C}} E) \longrightarrow 0.$$

The vanishing $(D'')^2 = 0$ is provided by the condition $d''_A\Phi = 0$.

A Higgs bundle is called *simple* if $H^0(C(A,\Phi)) \cong \mathbb{C}$ (or zero in the case of a fixed bundle $\det E$). Note that by Serre duality $H^0(C(A,\Phi)) \cong H^2(C(A,\Phi))$. A stable Higgs bundle is necessarily simple.

Proposition 1. *For any simple Higgs bundle $(\mathcal{E},\Phi)$, provided with a Hermitian connection A, the moduli space $\mathfrak{M}_E^{(n)}$ at a point (A,Φ) is a smooth complex manifold of dimension $(n^2-1)(2g-2)$, and its tangent space at this point is identified with*

$$H^1(C(A,\Phi)) \cong \{(\varphi,\beta):\ d''_A\varphi = -[\Phi,\beta],\ (d''_A)^*\beta = i\Lambda[\Phi^*,\varphi]\}.$$

For a given Higgs bundle $(\mathcal{E},\Phi)$ the coefficient of λ^{n-i} in the decomposition $\det(\lambda+\Phi)$ is a holomorphic section of the bundle $\mathcal{K}^i$, $i = 1,\dots,n$. (In the case of a fixed $\det E$ we have $\operatorname{tr}\Phi = 0$, so the decomposition starts from $i = 2$). These sections are invariant under the action of the group $\mathcal{G}_E^{\mathbb{C}}$ by conjugations, so the *Hitchin map*

$$h : \mathfrak{M}_E^{(n)} \longrightarrow \bigoplus_{i=1}^{n} H^0(M,\mathcal{K}^i)$$

is correctly defined and is a proper map.

3.2.4. **Hitchin–Kobayashi correspondence.** The *Hitchin equation* for a Higgs bundle $(\mathcal{E},\Phi)$ with trivial bundle E has the form

$$F_A + [\Phi,\Phi^*] = 0\,, \tag{3.5}$$

where Φ is a (1,0)-form with values in $\operatorname{End} E$. In the case of the bundles E of nonzero degree this equation takes on the form

$$f_{(A,\Phi)} := i\Lambda(F_A + [\Phi, \Phi^*]) = \mu, \tag{3.6}$$

where $\mu = \mu(E)$.

As we have pointed out before, equation (3.5) may be considered from two points of view: either as an equation on the Hermitian metric H with foxed complex structure $\bar\partial_E$, or as an equation on the complex structure $\bar\partial_E$ with fixed metric H.

Equation (3.5) is an equation on the minima of the *Yang–Mills–Higgs functional* given on holomorphic pairs (A, Φ) by the formula

$$YMH(A,\Phi) = \int_M \|F_A + [\Phi, \Phi^*]\|^2\omega.$$

The Euler–Lagrange equations for this functional have the form

$$d_A f_{(A,\Phi)} = 0, \quad [\Phi, f_{(A,\Phi)}] = 0. \tag{3.7}$$

The metric, for which these equations hold, is called *critical*. For such a metric the bundle $(\mathcal{E}, \Phi)$ splits into the direct sum of Higgs bundles being the solutions of equation (3.5) with different slopes.

Proposition 2. *If a Higgs bundle* $(\mathcal{E}, \Phi)$ *admits a metric satisfying equation* (3.5)*, it is polystable.*

Proof. Suppose that $\mathcal{S} \subset \mathcal{E}$ is a proper Φ-invariant subbundle. Denote by π the operator of an orthogonal projection to $\mathcal{S}$ and by $\beta = -\bar\partial_E\pi$ its 2nd fundamental form. Since $\mathcal{S}$ is Φ-invariant we have $(I-\pi)\Phi\pi = 0$, i.e., $\Phi\pi = \pi\Phi\pi$ and $\pi\Phi^* = \pi\Phi^*\pi$. It implies, in particular, that

$$\begin{aligned}\operatorname{tr}(\pi[\Phi,\Phi^*]) &= \operatorname{tr}(\pi\Phi\Phi^*) - \operatorname{tr}(\pi\Phi^*\Phi) = \operatorname{tr}(\pi\Phi\Phi^*) - \operatorname{tr}(\Phi\pi\Phi^*)\\ &= \operatorname{tr}(\pi\Phi\Phi^*\pi) - \operatorname{tr}(\Phi\pi\Phi^*\pi) = \operatorname{tr}(\pi\Phi\Phi^*\pi) - \operatorname{tr}(\pi\Phi\pi\Phi^*\pi)\\ &= \operatorname{tr}(\pi\Phi(I-\pi)\Phi^*\pi) = \operatorname{tr}(\pi\Phi(I-\pi)(I-\pi)\Phi^*\pi) = \operatorname{tr}(\pi\Phi(I-\pi)(\pi\Phi(I-\pi)^*),\end{aligned}$$

whence $\operatorname{tr}(\pi i\Lambda[\Phi,\Phi^*]) = |\pi\Phi(I-\pi)|^2$. Now from equation (3.5) and formula (3.3) for the degree we get

$$\deg\mathcal{S} = \operatorname{rank}(\mathcal{S})\mu(\mathcal{E}) - \frac{1}{2\pi}\left(\|\pi\Phi(I-\pi)\|^2 + \|\beta\|^2\right),$$

which implies that $\mu(\mathcal{S}) \le \mu(\mathcal{E})$. Moreover, the equality here is possible if and only if the two last terms from the right in the last formula vanish; in other words, if the holomorphic structure and Higgs field split. We prove the assertion of the proposition by continuing this process. □

Theorem 5 (Hitchin–Simpson)**.** *If a Higgs bundle* $(\mathcal{E}, \Phi)$ *is polystable, then it admits a metric satisfying equation* (3.5).

Note that in the case of line bundles $\mathcal{L}$ the result is proved sufficiently easy. Indeed, in this case the term $[\Phi, \Phi^*]$ vanishes so the equation (3.6) is equivalent to the condition of existence of a metric of constant curvature on L. Let H be a Hermitian metric on E. Consider the conformally equivalent metric $H_\varphi = e^\varphi H$. For it

$$F_{(\bar\partial_L, H_\varphi)} = F_{(\bar\partial_L, H)} + \partial\bar\partial\varphi,$$

and the problem of determination of the desired metric is reduced to the problem of finding a function φ satisfying the equation

$$\Delta\varphi = 2i\Lambda F_{(\bar\partial_L, H)} - 2\deg L.$$

It has a solution if and only if the integral of the right-hand side vanishes, which is evidently true in the considered case.

The proof of the theorem in the general case uses the following argument due to Donaldson [9]. Introduce for a Hermitian endomorphism φ the quantities

$$\nu(\varphi) = \sum_{j=1}^{n} |\lambda_j|, \quad N^2(\varphi) + \int_M \nu^2(\varphi)\frac{\omega}{2\pi},$$

where $\{\lambda_j\}$ are the eigenvalues of φ. Consider the functional

$$J(A,\Phi) = N\left(f_{(A,\Phi)} - \mu(E)\right).$$

The main role in the proof of Hitchin–Simpson theorem is played by the following lemma.

Lemma 1. *In every orbit of the complex group $\mathcal{J}_E^{\mathbb{C}}$ of gauge transformations there exists a sequence of points $\{A_j, \Phi_j\}$ having the following properties:*

1) *the sequence $\{A_j, \Phi_j\}$ is minimizing for the functional J;*
2) $\sup |f_{(A_j,\Phi_j)}|$ *are bounded uniformly with respect to j;*
3) *L^2-norms $\|d_{A_j} f_{(A_j,\Phi_j)}\|_{L^2}$ and $\|[f_{(A_j,\Phi_j)}, \Phi_j]\|_{L^2}$ tend to zero for $j \to \infty$.*

Using this lemma and the Uhlenbeck compactness theorem [25] we can construct a Higgs bundle with the metric satisfying Hitchin equation (3.5).

The proof of this lemma theorem employs the flow generated by the Yang–Mills–Higgs functional.

Definition 5. The *Yang–Mills–Higgs flow* for a pair (A,Φ) is the flow deteremined by the system of equations

$$\begin{cases} \dfrac{\partial A}{\partial t} = -d_A^*(F_A + [\Phi, \Phi^*]), \\ \dfrac{\partial \Phi}{\partial t} = [\Phi, i\Lambda(F_A + [\Phi, \Phi^*])]. \end{cases}$$

These equations should be supplemented by the condition $d''_A\Phi = 0$ which plays the role of constraint for the given system since it is preserved under the action of the complex group of gauge transforms. The above equations define the L^2-gradient flow for the Yang–Mills–Higgs functional. Moreover, we have the following lemma.

Lemma 2. *For all $t \geq 0$,*

$$\frac{d}{dt} YMH(A,\Phi) = -2\|d_A f_{(A,\Phi)}\|^2_{L^2} - 4\|[\Phi, f_{(A,\Phi)}]\|^2_{L^2}.$$

This lemma implies that the Yang–Mills–Higgs functional decreases along the flow; moreover, the following inequality holds:

$$\int_0^\infty dt \left\{2\|d_A f_{(A,\Phi)}\|^2_{L^2} + 4\|[\Phi, f_{(A,\Phi)}]\|^2_{L^2}\right\} \leq YMH(A_0,\Phi_0).$$

Denote by $\mathcal{B}_E^{\min}$ the set of Higgs bundles satisfying Hitchin equation (3.5). The introduced Yang–Mills–Higgs flow determines an infinite-dimensional Morse theory in which the points $\mathcal{B}_E^{\min}$ correspond to the minima of the Yang–Mills–Higgs functional and critical metrics to the critical points of higher Morse indices. In fact we have the following,

Theorem 6 (Wilkin [29])**.** *The Yang–Mills–Higgs functional determines a $\mathcal{G}_E$-invariant deformation retraction of the space $\mathcal{B}_E^{ss}$ to the space $\mathcal{B}_E^{min}$.*

3.3. Harmonic maps and σ-models

3.3.1. **Harmonic maps.** Let M^m and N^n be Riemannian manifolds provided with Riemannian metrics g and h, respectively. Consider a smooth map $\varphi : M \to N$. Its *energy* is the functional of the form

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi(p)|^2 \mathrm{vol},$$

where $d\varphi$ is the differential of the map φ and vol is the volume element of the Riemannian metric g.

Choose the local coordinates (x^i) at a point $p \in M$ and the local coordinates (u^α) at its image $q = \varphi(p) \in N$. In these coordinates the local expression for $|d\varphi(p)|^2$ will have the form

$$|d\varphi(p)|^2 = \sum_{i,j}\sum_{\alpha,\beta} g^{ij}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\, h_{\alpha\beta},$$

where $\varphi^\alpha = \varphi^\alpha(x)$ are the components of the map φ and where g^{ij} is the matrix inverse to the matrix (g_{ij}) of the metric tensor g. The volume element vol is given in the chosen local coordinates by the formula

$$\mathrm{vol} \sim \sqrt{|\det(g_{ij})|}dx^1 \wedge \cdots \wedge dx^n.$$

The differential of the map $\varphi : M \to N$ may also be defined in a more invariant way as a section $d\varphi$ of the bundle

$$T^*M \otimes \varphi^{-1}(TN) \longrightarrow M,$$

where $\varphi^{-1}(TN)$ is the inverse image of the tangent bundle TN under the map φ. By definition, the fiber $\varphi^{-1}(TN)_p$ at a point $p \in M$ is the tangent space $T_{\varphi(p)}N$ to N at $q = \varphi(p)$.

The bundle $T^*M \otimes \varphi^{-1}(TN)$ is provided with a natural Riemannian metric induced by the metrics g and h.

Problem 2. Find an explicit expression for this metric in local coordinates.

In the case when M and N are open subsets of Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, the norm of the differential of the map $\varphi = (\varphi^1, \dots, \varphi^n) : M \to N$ is given by the expression

$$|d\varphi(x)|^2 = \sum_{i=1}^m\sum_{\alpha=1}^n \left|\frac{\partial\varphi^\alpha}{\partial x^i}\right|^2 = \sum_{i=1}^m \left|\frac{\partial\varphi}{\partial x^i}\right|^2 ,$$

while the energy $E(\varphi)$ is given by the *Dirichlet integral*

$$E(\varphi) = \frac{1}{2}\int_M \sum_{i=1}^m \left|\frac{\partial\varphi}{\partial x^i}\right|^2 dx^1 \wedge \cdots \wedge dx^m.$$

The extremals of this functional coincide with the maps $\varphi = (\varphi^\alpha)$ with components φ^α being harmonic functions.

A smooth map $\varphi : M \to N$ of Riemannian manifolds is called *harmonic* if it is extremal for the energy functional $E(\varphi)$ with respect to smooth variations of φ with compact support.

We shall now find the Euler–Lagrange equations for the functional $E(\varphi)$. Write them first in the local coordinates (x^i) at a point $p \in M$ and (u^α) at the point $q = \varphi(p) \in N$. Suppose that the Riemannian connections ${}^M\nabla$ of the manifold M and ${}^N\nabla$ of the manifold N are given in these coordinates by the *Kristoffel symbols*

$${}^M\nabla \sim {}^M\Gamma^k_{ij} \quad \text{and} \quad {}^N\nabla \sim {}^N\Gamma^\gamma_{\alpha\beta},$$

respectively. In these coordinates the *Euler–Lagrange equations* for the functional $E(\varphi)$ take on the form

$$\sum_{i,j} g^{ij} \left\{ \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - \sum_k {}^M\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k} + \sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} \right\}$$

$$= \Delta_M \varphi^\gamma + \sum_{i,j} g^{ij} \sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} = 0, \quad \gamma = 1, \ldots, n.$$

The operator

$$\Delta_M = \sum_{i,j} g^{ij} \left(\frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - \sum_k {}^M\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k} \right)$$

is called the *Laplace–Beltrami operator* of the manifold M determined by the metric g. It is a *linear* differential operator of the 2nd order in φ^γ. The term

$$\sum_{i,j} g^{ij} \sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j},$$

entering Euler–Lagrange equations, depends on the geometry of the manifold N, i.e., on the geometry of the image of the map φ, and is given by the expression *quadratic* in derivatives of the map φ.

For $N = \mathbb{R}^n$ the Euler–Lagrange equations, written above, convert into the system of Laplace–Beltrami equations on the components φ^γ of the map φ with solutions being harmonic functions φ^γ on M.

We now write the Euler–Lagrange equations for the energy of a map $\varphi : M \to N$ in a more invariant way. Recall that the differential $d\varphi$ may be considered as a section of the bundle

$$T^*M \otimes \varphi^{-1}(TN) \longrightarrow M.$$

The Riemannian connections ${}^M\nabla$ and ${}^N\nabla$ generate a natural connection ∇ in this bundle. In its terms the Euler–Lagrange equations may be written in a concise form

$$\mathrm{tr}(\nabla d\varphi) = 0.$$

The vector field $\tau_\varphi := \mathrm{tr}(\nabla d\varphi)$ is called the *stress field* of φ.

We now turn to the case of almost complex manifolds which is more important to us. We shall assume that the Riemannian metric g on the almost complex manifold (M, J) is *Hermitian*, i.e., it is compatible with the almost complex structure J in the sense that $g(JX, JY) = g(X, Y)$ for any vector fields $X, Y \in TM$. An almost complex manifold (M, J), provided with the Hermitian metric g, is called *almost Hermitian*. In the case when the almost complex structure J is integrable, such a manifold is called *Hermitian*.

We introduce in the almost Hermitian manifold (M, g, J) the form ω by setting $\omega(X, Y) = g(JX, Y)$ for $X, Y \in TM$. A manifold M is called *almost Kähler* if the form ω is closed. In this case ω is called the *Kähler form*. If the form ω is also nondegenerate (in this case ω determines a symplectic structure on M) and the almost complex structure is integrable, then such a manifold (M, g, J, ω) is called *Kähler*.

Let $\varphi : M \to N$ be a smooth map of almost complex manifolds. It is called *almost holomorphic* or *pseudoholomorphic* if its tangent map $\varphi_* : TM \to TN$ commutes with almost complex structures, i.e.,

$$\varphi_* \circ {}^MJ = {}^NJ \circ \varphi_*,$$

where ${}^MJ$ (resp., ${}^NJ$) is an almost complex structure on M (resp., N). The map φ is called *almost anti-holomorphic* if φ_* anti-commutes with almost complex structures, i.e.,

$$\varphi_* \circ {}^MJ = -\,{}^NJ \circ \varphi_* .$$

Let $\varphi : M \to N$ be a smooth map of almost complex manifolds. We extend its tangent map $\varphi_* : TM \to TN$ complex-linearly to a map $\varphi_* : T^{\mathbb{C}}M \to T^{\mathbb{C}}N$ of complexified tangent bundles. The obtained map, in accordance with decompositions

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \quad T^{\mathbb{C}}N = T^{1,0}N \oplus T^{0,1}N,$$

may be represented in the block form with blocks given by four operators:

$$\partial'\varphi : T^{1,0}M \longrightarrow T^{1,0}N, \quad \partial''\varphi : T^{0,1}M \longrightarrow T^{1,0}N,$$
$$\partial'\bar\varphi = \overline{\partial''\varphi} : T^{1,0}M \longrightarrow T^{0,1}N, \quad \partial''\bar\varphi = \overline{\partial'\varphi} : T^{0,1}M \longrightarrow T^{0,1}N.$$

If we identify φ_* with differential $d\varphi$, considered as a section of the bundle

$$T^{*,\mathbb{C}}M \otimes \varphi^{-1}(T^{\mathbb{C}}N) \longrightarrow M,$$

then the introduced operators will admit an analogous interpretation as sections of the corresponding subbundles of the above bundle. For example, the operator $\partial'\varphi$ may be identified with a section of the bundle

$$\Lambda^{1,0}M \otimes \varphi^{-1}(T^{1,0}N).$$

In terms of the introduced operators the map φ is almost holomorphic (resp., almost anti-holomorphic) if

$$\partial''\varphi = 0 \text{ (resp., } \partial'\varphi = 0).$$

In the case when the manifolds M and N are almost Hermitian the energy of a smooth map $\varphi : M \to N$ is represented as the sum

$$E(\varphi) = E'(\varphi) + E''(\varphi),$$

where

$$E'(\varphi) = \int_M |\partial'\varphi|^2 \mathrm{vol}, \quad E''(\varphi) = \int_M |\partial''\varphi|^2 \mathrm{vol}.$$

Using this decomposition, the criterion of holomorphicity of the map φ may be reformulated in the following way: φ is *holomorphic* (resp., *anti-holomorphic*) $\iff E''(\varphi) = 0$ (resp., $E'(\varphi) = 0$).

We can ask if the (anti)holomorphic maps of almost Hermitian manifolds are automatically harmonic. The answer to this question is positive for compact almost Kähler manifolds.

Let $\varphi : M \to N$ be a smooth map of compact almost Kähler manifolds. Then the quantity

$$k(\varphi) = E'(\varphi) - E''(\varphi)$$

depends only on the homotopy class of the map φ. Since

$$E(\varphi) = 2E'(\varphi) - k(\varphi) = 2E''(\varphi) + k(\varphi)\,,$$

it implies that the critical points of the functionals $E(\varphi), E'(\varphi)$, and $E''(\varphi)$ in this case coincide and

$$E(\varphi) \geq |k(\varphi)|.$$

Hence, (anti)holomorphic maps φ realize absolute minima of the energy $E(\varphi)$ in a given topological class: for $k(\varphi) \geq 0$ the minima are realized on almost holomorphic maps with $E''(\varphi) = 0$; for $k(\varphi) \leq 0$ they are realized on almost anti-holomorphic maps with $E'(\varphi) = 0$.

In conclusion we consider in more detail the case of harmonic maps from Riemann surfaces to Riemannian manifolds. Let $\varphi : M \to N$ be a smooth map from a Riemann surface M into a Riemannian manifold N. The tangent map $\varphi_* : TM \to TN$ may be extended complex-linearly to a map $\varphi_* : T^{\mathbb{C}}M \to T^{\mathbb{C}}N$ of complexified tangent bundles and identified with the section $d\varphi$ of the bundle

$$T^{*,\mathbb{C}}M \otimes \varphi^{-1}(T^{\mathbb{C}}N) \longrightarrow M.$$

So the differential $d\varphi$ may be represented as the sum

$$d\varphi = \delta\varphi + \bar{\delta}\varphi,$$

where $\delta\varphi$ is a section of the bundle $\Lambda^{1,0}M \otimes \varphi^{-1}(T^{\mathbb{C}}N)$ and where $\bar{\delta}\varphi$ is a section of the bundle $\Lambda^{0,1}M \otimes \varphi^{-1}(T^{\mathbb{C}}N)$.

Denote, as before, by ∇ the natural connection on the bundle $T^*M \otimes \varphi^{-1}(TN)$, generated by the Riemannian connections ${}^M\nabla$ and ${}^N\nabla$, and extend it complex-linearly to the complexified bundle $T^{*,\mathbb{C}}M \otimes \varphi^{-1}(T^{\mathbb{C}}N)$. Introduce the operators, acting on sections of this bundle, which in terms of the local complex coordinate z on M are defined in the following way:

$$\delta := \nabla_{\partial/\partial z}\ , \quad \bar{\delta} := \nabla_{\partial/\partial \bar{z}}\ .$$

Then the condition of harmonicity of the map $\varphi : M \to N$ will be written in the form

$$\bar{\delta}\delta\varphi = \nabla_{\partial/\partial\bar{z}}(\delta\varphi) = \nabla_{\partial/\partial\bar{z}}(\nabla_{\partial/\partial z}\varphi) = 0$$

or in the equivalent form

$$\delta\bar{\delta}\varphi = \nabla_{\partial/\partial z}(\bar{\delta}\varphi) = \nabla\partial/\partial z(\nabla_{\partial/\partial\bar{z}}\varphi) = 0.$$

In the case when the manifold N is Kähler, the obtained harmonicity conditions may be further simplified by using the relations

$$\delta\varphi = \partial'\varphi + \overline{\partial''\varphi}, \quad \bar{\delta}\varphi = \partial''\varphi + \overline{\partial'\varphi}.$$

Since for a Kähler manifold N the connection ${}^N\nabla$ preserves the decomposition $T^{\mathbb{C}}N = T^{1,0}N \oplus T^{0,1}N$ into the direct sum of $(1,0)$- and $(0,1)$-subspace (why?), the harmonicity condition can be rewritten in the form

$$\bar{\delta}\partial'\varphi = 0 \Leftrightarrow \delta\partial''\varphi = 0.$$

3.3.2. **Example: Harmonic maps of the Riemann sphere into itself.** We start with the following problem arising in the theory of ferromagnetism. Suppose that at any point $x = (x_1, x_2)$ of the Euclidean plane $\mathbb{R}^2$ it is given a vector $\varphi(x) \in \mathbb{R}^3$ of the unit length smoothly depending on x. In other words, it is given a smooth map $\varphi : \mathbb{R}^2 \to S^2$, $x \mapsto \varphi(x)$, of the plane $\mathbb{R}^2$ into the unit sphere $S^2 \subset \mathbb{R}^3$. The energy of the map φ is given by the Dirichlet integral

$$E(\varphi) = \frac{1}{2}\int_{\mathbb{R}^2} |d\varphi|^2 dx_1 dx_2,$$

where $|d\varphi|^2 = \left|\frac{\partial\varphi}{\partial x_1}\right|^2 + \left|\frac{\partial\varphi}{\partial x_2}\right|^2$.

In order to guarantee the finiteness of the energy $E(\varphi) < \infty$ it is natural to impose on φ the *asymptotic condition*

$$\varphi(x) \longrightarrow \varphi_0 \quad \text{uniformly for } |x| \to \infty,$$

where φ_0 is a fixed point of S^2. Under this condition the map $\varphi : \mathbb{R}^2 \to S^2$ will extend to a continuous map

$$\varphi : S^2 = \mathbb{R}^2 \cup \{\infty\} \longrightarrow S^2.$$

Such maps $\varphi : S^2 \to S^2$ have a topological invariant, namely the *degree of the map* given by the formula

$$\deg\varphi = \int_{\mathbb{R}^2} \varphi^*\omega,$$

where ω is the normalized volume form on the sphere: $\int_{S^2}\omega = 1$ and $\varphi^*\omega$ is the preimage of ω under the map φ.

Consider the following problem: find all extremals of the functional $E(\varphi)$ in the class of smooth maps $\varphi : \mathbb{R}^2 \to S^2$ with finite energy and given degree $k = \deg\varphi$.

To solve this problem it is convenient to introduce the complex coordinate $z = x_1 + ix_2$ in the definition domain $\mathbb{R}^2 \approx \mathbb{C}$ and stereographic complex coordinate w in the image $S^2 \setminus \{\infty\}$. In these coordinates the expression for the energy of the map $\varphi = w(z)$ will take the form

$$E(\varphi) = 2\int_{\mathbb{C}} \frac{|\partial w/\partial z|^2 + |\partial w/\partial \bar{z}|^2}{(1+|w|^2)^2}\,|dz\wedge d\bar{z}|,$$

while the formula for the degree φ converts into

$$\deg\varphi = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{|\partial w/\partial z|^2 - |\partial w/\partial \bar{z}|^2}{(1+|w|^2)^2}\,|dz\wedge d\bar{z}|.$$

Comparing the last two formulas we see that

$$E(\varphi) \geq 4\pi|\deg\varphi|.$$

Moreover, the equality here can be attained only in the following cases:

- if $k = \deg\varphi \geq 0$, then for $\partial w/\partial\bar{z} \equiv 0$, i.e., on holomorphic functions $\varphi = w(z)$;
- if $k = \deg\varphi < 0$, then for $\partial w/\partial z \equiv 0$, i.e., on anti-holomorphic functions $\varphi = w(z)$.

Hence, holomorphic functions $\varphi = w(z)$ realize minima of the energy $E(\varphi)$ in topological classes with $k \geq 0$, while anti-holomorphic maps $\varphi = w(z)$ realize minima of the energy $E(\varphi)$ in topological classes with $k < 0$. For minimizing maps φ the value of the energy $E(\varphi)$ is equal to $4\pi|k|$.

Let us find concrete formulas for minimizing maps. Suppose for definiteness that $k = \deg\varphi > 0$. Using the invariance of $E(\varphi)$ with respect to rotations of the sphere S^2 in the image, fix the asymptotic value φ_0 setting it equal to $\varphi_0 = w_0 = 1$.

We have to describe the holomorphic maps of the Riemann sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ into itself having degree k and equal to 1 at infinity. Such maps are given by rational functions of the form

$$\varphi = w(z) = \prod_{j=1}^{k} \frac{z-a_j}{z-b_j}\,,$$

where $a_j \neq b_j$ are arbitrary complex numbers. Analogous descriptions admit the anti-holomorphic maps minimizing $E(\varphi)$ for $k < 0$.

Note that the space of solutions of our problem depends on $4k$ real parameters (or $4k+2$ real parameters if we add rotations of the sphere S^2 in the image).

We have described all local minima of the energy functional $E(\varphi)$.

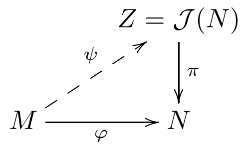
Problem 3. Prove that the energy functional $E(\varphi)$ has no other extremals apart from local minima. It is the effect of the 2-dimensionality of the considered problem.

3.3.3. **Twistor interpretation of harmonic maps.** In Section 1.2.5 we have constructed for arbitrary even-dimensional Riemannian manifold N the twistor bundle

$$\pi : Z = \mathcal{J}(N) \longrightarrow N$$

and provided the twistor space Z with almost complex structure $\mathcal{J}^1$. In this section we demonstrate how one can use this twistor bundle to solve the problem of construction of harmonic maps from compact Riemann surfaces into Riemannian manifolds.

Recall that according to the Penrose twistor program [18] any problem of Riemannian geometry on manifold N should reduce to some problem of complex geometry on its twistor space $Z = \mathcal{J}(N)$. If we believe in this Penrose thesis, then we can suppose that harmonic maps $\varphi : M \to N$ from a compact Riemann surface M to N should arise from the pseudoholomorphic maps $\psi : M \to (Z, \mathcal{J}^1)$ as projections of the latter maps to N, i.e., $\varphi = \pi \circ \psi$:



This is almost true. It turns out that the projections of pseudoholomorphic maps $\psi : M \to (Z, \mathcal{J}^1)$ to N indeed satisfy differential equations of 2nd order on N. However, these equations are not harmonic but are *ultrahyperbolic*, i.e., equations of harmonic type but with the "wrong" signature (n, n) instead of the required signature $(2n, 0)$.

So in order to construct harmonic maps $\varphi : M \to N$ as projections of pseudoholomorphic maps $\psi : M \to Z$, we should change the definition of the almost complex structure on the twistor space $Z = \mathcal{J}(N)$.

Namely, in terms of the vertical-horizontal decomposition

$$T\mathcal{J}(N) = V \oplus H$$

the required almost complex structure $\mathcal{J}^2$ on $\mathcal{J}(N)$ should be defined as

$$\mathcal{J}^2 = (-\mathcal{J}^v) \oplus \mathcal{J}^h.$$

This almost complex structure on $\mathcal{J}(N)$ was introduced by Eells and Salamon [11], and precisely this structure, as we shall see, is responsible for the twistor interpretation of harmonic maps.

Before we switch to the construction of harmonic maps as projections of pseudoholmorphic ones, consider the problem of integrability of the introduced almost complex structures $\mathcal{J}^1$ and $\mathcal{J}^2$.

We have the following *Rawnsley theorem* [20]: the almost complex structure $\mathcal{J}^1$ on the bundle $\mathcal{J}(N)$ is integrable if and only if N is conformally flat, i.e., N is conformally equivalent to a flat space. Recall that a map $\varphi : (M, g) \to (N, h)$ of Riemannian manifolds is called *conformal* if the induced metric $\varphi^* h$ on M is conformally equivalent to the Riemannian metric g of the manifold M, i.e., $\varphi^* h = \lambda g$ for some smooth positive function λ on M.

Concerning the almost complex structure $\mathcal{J}^2$ on $\mathcal{J}(N)$, it is never integrable. We can explain this fact in the following way. Using the definition of the almost complex structure $\mathcal{J}^2$, it is not difficult to show that if it would be integrable, then all local $\mathcal{J}^2$-holomorphic curves $f : U \to \mathcal{J}(N)$ should be horizontal, i.e., their tangent spaces should belong to the horizontal distribution H. On the other hand, if $(\mathcal{J}(N), \mathcal{J}^2)$ would be a complex manifold, then it should be possible to issue a local holomorphic curve on it in any complex tangent direction.

Taking into account the nonintegrability of the almost complex structure $\mathcal{J}^2$ there might be doubts if it could be useful for the description of harmonic maps. Indeed, the nonintegrable almost complex structures may be quite "bizarre"—for example, they may have even locally no nonconstant holomorphic functions. However, in the considered problem we have to deal, fortunately, not with holomorphic functions $f : Z \to \mathbb{C}$ on the twistor space Z, but with a dual object, namely holomorphic maps $\psi : M \to Z$ from Riemann surfaces M to Z. Such a map ψ is holomorphic with respect to the almost complex structure $\mathcal{J}^2$ on Z if and only if it satisfies the Cauchy–Riemann equation $\bar{\partial}_J \psi = 0$ with respect to the induced almost complex structure $J := \psi^*(\mathcal{J}^2)$ on M. But on a Riemann surface any almost complex structure is integrable (why?). In particular, the Cauchy–Riemann equation above has many local solutions.

The next theorem lies in the basis of the twistor approach to the construction of harmonic maps.

Theorem 7 (Eells–Salamon theorem [11]). *The twistor bundle*

$$\pi : (\mathcal{J}(N), \mathcal{J}^2) \longrightarrow N$$

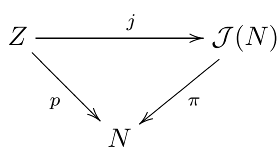
has the following property: the projection $\varphi = \pi \circ \psi$ of an arbitrary $\mathcal{J}^2$-holomorphic map $\psi : M \to \mathcal{J}(M)$ to N is a harmonic map.

Since the projection of any $\mathcal{J}^2$-holomorphic curve $\psi : M \to \mathcal{J}(M)$ is a harmonic map, one can use these pseudoholomorphic curves to construct harmonic maps $\varphi : M \to N$. Is it possible to construct in this way all harmonic maps of this type? In other words, when a given harmonic map $\varphi : M \to N$ is the projection of some $\mathcal{J}^2$-holomorphic curve $\psi : M \to \mathcal{J}(M)$? It turns out that if a map $\varphi : M \to N$ is obtained as the projection of some $\mathcal{J}^2$-holomorphic curve in $\mathcal{J}(M)$, then it should not only be harmonic but also conformal.

Conversely, any harmonic conformal map $\varphi : M \to N$ from a compact Riemann surface M to an oriented Riemannian manifold N is locally the projection of some $\mathcal{J}^2$-holomorphic curve $\psi : M \to \mathcal{J}(M)$.

The considered bundle $\mathcal{J}(M) \to N$ of Hermitian structures on N is not a unique twistor bundle which can be used for the construction of harmonic bundles. Starting from the bundle $\mathcal{J}(M) \to N$, one can also define other twistor bundles $Z \to N$ with the help of the following method proposed by Rawnsley [20].

Let $p : Z \to N$ be a smooth bundle having the fibers which are complex manifolds with complex structures smoothly depending on the point $q \in N$:



Suppose that we have a fiberwise map $j : Z \to \mathcal{J}(N)$ which is holomorphic on the fibers. We also assume that we have on the bundle $p : Z \to N$ a smooth horizontal distribution ${}^Z H$ which is sent by the map j_* to the horizontal distribution H on $\mathcal{J}(N)$. Then on ${}^Z H$ we shall have an almost complex structure ${}^Z\mathcal{J}^h$ given by the preimage of the almost complex structure $\mathcal{J}^h$ on H under the map j. Using this horizontal almost complex structure ${}^Z\mathcal{J}^h$ on ${}^Z H$ and given vertical complex structure on the fibers of the bundle $p : Z \to N$, we can introduce on Z almost complex structures ${}^Z\mathcal{J}^1$ and ${}^Z\mathcal{J}^2$ in the same way as in the case of the bundle $\pi : \mathcal{J}(N) \to N$. It is clear that the map j is almost holomorphic with respect to both introduced structures so that $p : Z \to N$ is the twistor bundle over N in the same sense as $\pi : \mathcal{J}(N) \to N$.

Let us give a concrete example of an application of the described method. Let N be a Kähler manifold of dimension m. Denote by

$$Z := G_r(T^{1,0}N) \longrightarrow N$$

the *complex Grassmannian bundle* with the fiber at $q \in N$ given by the Grassmann manifold $G_r(T^{1,0}_q N)$ of complex subspaces of dimension r in the complex vector space

$T_q^{1,0}N$. If we denote by $\mathcal{U}(N) \to N$ the principal U(m)-bundle of unitary frames on N, then

$$Z = \mathcal{U}(N) \otimes_{\mathrm{U}(m)} G_r(\mathbb{C}^m).$$

In the case of a Kähler manifold N the Riemannian connection ${}^N\nabla$ determines a connection in the bundle $\mathcal{U}(N)$ and so defines a horizontal distribution on the space Z. A complex structure on the fibers of $Z \to N$ is induced by the natural complex structure on the Grassmann manifold $G_r(\mathbb{C}^m)$. We now construct the map

$$j : Z \longrightarrow \mathcal{J}(N)$$

by setting for a subspace $W \in G_r(T_q^{1,0}N)$:

$$j(W) = \begin{cases} {}^N J & \text{on } (W \oplus \overline{W}) \cap T_q N\ , \\ -{}^N J & \text{on } \left[(W \oplus \overline{W}) \cap T_q N\right]^{\perp}\ . \end{cases}$$

The constructed map $j : Z \to \mathcal{J}(N)$ satisfies the conditions of the Rawnsley method which implies that the Grassmannian bundle $G_r(T^{1,0}N) \to N$ is a twistor bundle, meaning that the projection of any $\mathcal{J}^2$-holomorphic map $\psi : M \to G_r(T^{1,0}N)$ from a compact Riemann surface M to the manifold N is a harmonic map $\varphi : M \to N$. As we have pointed out before such a map is necessarily conformal.

In the case $r = 1$ it is possible to invert the given twistor construction, in other words, to construct for an arbitrary conformal harmonic map $\varphi : M \to N$ its twistor pull-back to a $\mathcal{J}^2$-holomorphic map $\psi : M \to G_1(T^{1,0}N)$. Note that the Grassmannian bundle $G_1(T^{1,0}N) \to N$ coincides with the projectivization $\mathbb{P}(T^{1,0}N) \to N$ of the bundle $T^{1,0}N \to N$.

Suppose that it is given a conformal harmonic map $\varphi : M \to N$ which is not anti-holomorphic (for anti-holomorphic, as well as holomorphic, maps the problem of construction of their twistor pull-backs is of no interest). Its differential $\delta\varphi$ is written in the form

$$\delta\varphi = \partial'\varphi + \overline{\partial''\varphi}.$$

If the map φ is not anti-holomorphic, then $\partial'\varphi(\partial/\partial z)$ defines a section of the bundle $\varphi^{-1}(T^{1,0}N)$ which is not identically zero and holomorphic with respect to the complex structure on this bundle induced by the Riemannian connection ${}^N\nabla$. This section can have only isolated zeros, and outside these zeros the twistor pull-back $\psi : M \to \mathbb{P}(T^{1,0}N)$ is given by the formula

$$\psi = [\partial'\varphi(\partial/\partial z)]\,.$$

In other words, the value $\psi(p)$ of the map ψ at a point $p \in M$ coincides with the complex line in $T^{1,0}_{\varphi(p)}N$ generated by the $(1,0)$-component of the vector $\varphi_*(\partial/\partial z)$. Using the holomorphicity of the constructed line subbundle in the bundle $\varphi^{-1}(T^{1,0}N)$, we can extend it to the isolated zeros of the section $\partial'\varphi(\partial/\partial z)$ (a variant of the Riemann theorem on the cancellation of isolated singularities of holomorphic functions), thus obtaining the desired map

$$\psi : M \to \mathbb{P}(T^{1,0}N).$$

The constructed map ψ is $\mathcal{J}^2$-holomorphic if φ is conformal.

Restricting the class of admissible Riemannian manifolds (as in the example where we have considered the class of Kähler manifolds N), we can construct new examples of twistor spaces using the Rawnsley method. The general idea is to choose for every class of Riemannian manifolds N as an appropriate twistor bundle the bundle of complex structures which are related to the geometry of the manifolds from the considered class.

3.3.4. **Harmonic spheres conjecture.** In Section 2.2.2 we have given the ADHM-construction which allows us to completely describe the moduli space of instantons on $\mathbb{R}^4$. This construction has a 2-dimensional reduction proposed by Donaldson [10].

The Donaldson theorem asserts that there exists a natural bijective correspondence between the moduli space of G-instantons on $\mathbb{R}^4$ and the set of based equivalence classes of holomorphic $G^{\mathbb{C}}$-bundles on $\mathbb{CP}^2$ which are holomorphically trivial on the projective line $\mathbb{CP}^1_\infty$ at "infinity". The *based equivalence* means that we consider only isomorphisms of holomorphic bundles which are equal to the identity at the based point on $\mathbb{CP}^1_\infty$.

For us it is more convenient to use another formulation of the Donaldson theorem given in Atiyah's paper [1]. In this formulation the moduli space of G-instantons on $\mathbb{R}^4$ is identified with the set of classes of based equivalence of holomorphic $G^{\mathbb{C}}$-bundles on the product $\mathbb{CP}^1 \times \mathbb{CP}^1$ which are holomorphically trivial on the union $\mathbb{CP}^1_\infty \cup \mathbb{CP}^1_\infty$ of projective lines at "infinity":

$$\left\{\begin{matrix}\text{moduli space of } G\text{-}\\ \text{instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{equivalence classes of holomorphic } G^{\mathbb{C}}\text{-bundles}\\ \text{on } \mathbb{CP}^1 \times \mathbb{CP}^1 \text{ holomorphically trivial on}\\ \mathbb{CP}^1_\infty \cup \mathbb{CP}^1_\infty\end{matrix}\right\}.$$

The role of the based point in the definition of based equivalence in this case is played by the intersection point of projective lines at "infinity".

The set of equivalence classes on the right-hand side of this correspondence may be identified by the Atiyah theorem with the set of based holomorphic maps $f : \mathbb{CP}^1 \to \Omega G$ sending $\infty \in \mathbb{CP}^1$ to the origin $o \in \Omega G$.

Indeed, fix some point $z \in \mathbb{CP}^1$. The restriction of a given holomorphic $G^{\mathbb{C}}$-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ to the projective line $\mathbb{CP}^1_z := \mathbb{CP}^1 \times \{z\}$ is determined by the transition function

$$F_z : S^1 \subset \mathbb{CP}^1_z \longrightarrow G^{\mathbb{C}},$$

which extends to some neighborhood U of the equator S^1 in $\mathbb{CP}^1_z$ to a holomorphic map $F_z : U \subset \mathbb{CP}^1_z \to G^{\mathbb{C}}$. The function $F_z : S^1 \to G^{\mathbb{C}}$ may be considered as an element of the loop group $LG^{\mathbb{C}}$, so we obtain a map

$$F : \mathbb{CP}^1 \ni z \longmapsto F_z \in LG^{\mathbb{C}}.$$

In composition with the projection $LG^{\mathbb{C}} \longrightarrow \Omega G^{\mathbb{C}} = LG^{\mathbb{C}}/L_+G^{\mathbb{C}}$ it gives a map

$$f : \mathbb{CP}^1 \longrightarrow \Omega G.$$

The constructed map f is based and holomorphic if the original $G^{\mathbb{C}}$-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ was holomorphic and trivial on $\mathbb{CP}^1_\infty \cup \mathbb{CP}^1_\infty$. The Atiyah theorem asserts that there is a bijective correspondence

$$\left\{\begin{matrix}\text{moduli space of } G\text{-}\\ \text{instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{space of based holomorphic maps}\\ f : \mathbb{CP}^1 \to \Omega G\end{matrix}\right\}.$$

Having the above result of Atiyah–Donaldson it is natural to propose a conjecture [21] obtained by the "realification" of the given correspondence. According to this conjecture, there should exist the following bijective correspondence:

$$\left\{\begin{matrix}\text{moduli space of Yang–}\\ \text{Mills } G\text{-fields on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{space of based harmonic maps}\\ h : \mathbb{CP}^1 \to \Omega G\end{matrix}\right\}.$$

The formulated conjecture remains yet unproved. The main difficulty is that there is no "real" analogue of the Donaldson theorem. The Donaldson proof is based on the monad method and is purely holomorphic.

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